ON ROTATING DOUBLY CONNECTED VORTICES

TAOUFIK HMIDI, JOAN MATEU, AND JOAN VERDERA

Abstract. In this paper we consider rotating doubly connected vortex patches for the Euler equations in the plane. When the inner interface is an ellipse we show that the exterior interface must be a confocal ellipse. We then discuss some relations, first found by Flierl and Polvani, between the parameters of the ellipses, the velocity of rotation and the value of the vorticity in the domain enclosed by the inner ellipse.

Contents

1. Introduction

The motion of an incompressible ideal fluid with constant density is described by the Euler equations

$$\begin{cases} \partial_t u(t,x) + u(t,x)\cdot\nabla u(t,x) + \nabla P(t,x) = 0, & x\in\mathbb{R}^d, t>0,\\ \text{div } u(t,x) = 0, \\ u(0,x) = u_0(x), \end{cases}$$

where $u:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$, $T>0$, denotes the velocity field of the fluid particles and the scalar function P stands for the pressure. The incompressibility condition is an immediate consequence of the continuity equation when the density is assumed to be constant. The mathematical theory for this classical system is very reach. Many results were devoted to the local well-posedness problem in different function spaces. Roughly speaking, it is well-known that the Cauchy problem is locally well-posed if the initial velocity is above the scaling of C^1 functions, for example, if $u_0\in H^s$, $s>\frac{d}{2}+1$. For more details about this we refer the reader

2000 *Mathematics Subject Classification.* 76B03 ; 35Q35.

Key words and phrases. 2D incompressible Euler equations, rotating patch, V-state, potential theory.

to [3, 5, 11] and the references therein. Whether these solutions develop singularities in finite time is still an open problem, apparently very difficult. In dimension 2 global regularity was proved long ago [23] and extensions to the context of axisymmetric flows without swirl were obtained in [20]. In these cases the global existence follows from a special structure of the vorticity, which yields some strong conservation laws.

In this paper we shall focus on the vorticity dynamics in the plane. In this case the vorticity can be identified with the scalar function $\omega = \partial_1 u^2 - \partial_2 u^1$ and its evolution is governed by the transport equation

$$\partial_t \omega + u \cdot \nabla \omega = 0, \tag{1}$$

which amounts to saying that vorticity is conserved along particle trajectories. This yields the conservation laws

$$\|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}, \quad p \in [1, \infty].$$

The preservation of vorticity along trajectories allows to go beyond the limitations inherent to the general theory of the hyperbolic systems and to show existence and uniqueness of a global weak solution under the assumption that $\omega_0 \in L^1 \cap L^\infty$. This remarkable result was proved by Yudovich in [25]. Uniqueness follows from the fact that the gradient of the velocity belongs to all L^p spaces and its L^p norm obeys the slow growth condition

$$\sup_{p \geq 2} \frac{\|\nabla v(t)\|_{L^p}}{p} < \infty.$$

This framework offers new perspectives for study and allows for example to deal rigorously with vortex patch structures in which the vorticity takes finitely many values over a bounded region. More precisely, we say that the initial vorticity is a patch if it takes a non-zero constant value c_0 on a bounded domain D and vanishes elsewhere. In fact, we normalize so that $c_0 = 1$ and the initial vorticity is the characteristic function of the domain D. Since vorticity is preserved on particle trajectories, it can be recovered by the formula

$$\omega(t) = \chi_{D_t}, \quad D_t \triangleq \psi(t, D)$$

where ψ is the flow associated with the velocity field u, that is, the solution of the ODE

$$\psi(t, x) = x + \int_0^t u(\tau, \psi(\tau, x)) d\tau.$$

Remark that a vortex patch can be seen as a desingularization of a point vortex and provides a suitable mathematical model to study the effects of finite vortex cores.

In the vortex patch problem the dynamics of the vorticity is reduced to the motion of the one-dimensional boundary curve ∂D_t according to the nonlocal equation

$$\partial_t z = \frac{1}{2\pi} \oint_{\partial D_t} \log |z - \xi| \, d\xi, \tag{2}$$

which is referred to in the literature as the contour dynamics equation. The problem of global existence of smooth solutions to the contour dynamics equation was solved by Chemin in [5] (see [2] for a short proof). He proved that if the initial boundary ∂D belongs to the Hölder class $C^s, 1 < s < 2$, then the boundary ∂D_t remains in the same class for any positive time. In particular, singularities like corners or cusps cannot appear in finite time if the boundary of the initial domain D is smooth.

In general, the motion of the boundary of the patch is extremely complex, due to the nonlinear effects of the induced velocity. There is in the literature only one explicit solution

in the simply connected case (that is, the case in which D is a simply connected domain). It is the Kirchhoff elliptical vortex, in which D is an ellipse with semi-axes a and b. The motion of this vortex patch is simply a rotation around the center of mass of D with angular velocity $\Omega = \frac{ab}{(a+b)^2}$. When $a = b$ one obtains the circular steady solution known as the Rankine vortex. See [3, 13, 14].
The behavior of elliptical patches in an external field was first studied by Chaplygin in [6] for a pure shear. He proved that the vortex retains its elliptical shape, rotates with variable angular velocity and pulsates according to a certain law. This result was extended by Kida [12] and Neu [16] for a uniform straining field and it was found that the vortex exhibits various types of motion depending on the magnitudes of the strain. For example, for weaker straining field the vortex can rotate or nutate. However, for strong strain the vortex elongates indefinitely. Comprehensive and up-to-date surveys of the analytical techniques are provided in [17, 19].

A vortex patch that rotates, like the Kirchhoff ellipses, is called a rotating vortex patch or a V-state (for vortex state). This terminology was introduced by Deem and Zabusky in [7], where the contour dynamics equation (2) was solved numerically to show existence of V-states having an m-fold symmetry for any $m = 2, 3, \ldots$ ($m = 2$ are the Kirchhoff ellipses). The reader is urged to consult [24] where pictures of m-fold symmetric V-states and their "limiting" shapes are shown. A rigorous study including a proof of existence of non circular m-fold symmetric V-states was peformed by Burbea in [4]. He used conformal mappings combined with a bifurcation analysis. The authors showed recently in [10] that close to the circle of bifurcation the V-states are convex and have C^∞ boundaries. Global bifurcation in this context has not been studied.

The evolution of a system of N disjoint patches is in general very complicated to analyze and each individual patch varies in response to the self-induced velocity field and to that of other patches. Thus it seems to be very difficult to find explicit solutions as for the single rotating patches. The most common approximate model used to track the vortex dynamics is the *moment model* of Melander, Zabusky and Styczek [15] leading to a self-consistent system of ordinary differential equations governing the local geometric moments. Its truncated model is highly effective to treat. For example, the interaction between several Kirchhoff ellipses which are far apart can be dealt with. For a valuable discussion about this subject see [17]. A general review about vortex dynamics can be found in [1].

The main goal of this paper is to study the rigid-body motion (that is, rotation with constant angular velocity) for a linear superposition of finitely many increasing patches. For the sake of clarity and simplicity we only deal with the case of two bounded simply connected domains D_2 and D_1 such that the closure of D_2 is contained in D_1. The initial vorticity is of the form

$$\omega_0 = \chi_{D_1} + (\alpha - 1)\chi_{D_2}, \tag{3}$$

so that the parameter α represents the magnitude of the initial vorticity in the interior domain D_2. Clearly the initial vorticity is 1 on $D_1 \setminus \overline{D_2}$ and 0 off D_1. By the conservation of vorticity along trajectories the vorticity ω_t at time t is of the form

$$\omega_t = \chi_{D_{1t}} + (\alpha - 1)\chi_{D_{2t}}, \tag{4}$$

for some domains D_{1t} and D_{2t}. We say that the solution ω_t of (1) with initial datum (3) rotates uniformly if ω_t is a uniform rotation of the initial vorticity, namely

$$\omega_t(z-c) = \omega_0(e^{-it\Omega}(z-c)), \quad z \in \mathbb{C},$$

where

$$c = \frac{1}{|D_1|}\int_{D_1} z\,\omega_0(z)\,dA(z)$$

is the center of mass of ω_0 and Ω is some real number that has to be found. The problem we consider consists in finding the couples of domains D_1 and D_2 such that the multi-vortex ω_t rotates. Notice that for $\alpha = 0$ we are considering an initial doubly connected vortex patch $D_1 \setminus \overline{D_2}$ and we are asking under what conditions this patch rotates uniformly (around its center of mass). The question was raised by Luis Vega and was the initial motivation for this work. An annulus is the only known explicit solution of the doubly connected vortex patch problem.

When the domains are confocal ellipses Flierl and Polvani found in [9] the complete solutions to this problem by using elliptical coordinates. In that work the authors dealt with finitely many ellipses and special attention was devoted to the stability condition in the case of two confocal ellipses, thus generalizing the known result of Love for the Kirchhoff elliptical vortex.

It seems that no other explicit solutions of the form under consideration can be found in the literature. The aim in this paper is to solve completely the problem of the rigid-body motion in the particular case when the interior interface ∂D_2 is an ellipse. We shall prove that under this constraint the vortices of Flierl and Polvani are the only solutions. Our first result concerns Rankine vortices and reads as follows. Let $\Gamma_j = \partial D_j$, $j = 1, 2$.

Theorem 1. *Let ω_0 be an initial vorticity of the form (3) and assume that the solution ω_t (4) rotates uniformly. If Γ_1 or Γ_2 is a circle then necessarily the other curve must be a circle with the same center.*

Accordingly, if one of the curves is a circle and the second one is not then there is no rotation and the dynamics of the vorticity is not easy to track.

The second result deals with the generalized Kirchhoff vortices. Before stating it we need to introduce a piece of notation. For an ellipse with semi-axes a and b define

$$Q \triangleq \frac{a-b}{a+b}\cdot$$

Theorem 2. *Let ω_0 be an initial vorticity of the form (3). Assume that the interior curve Γ_2 is an ellipse and Γ_1 is a Jordan curve of class C^1. Then the solution ω_t rotates uniformly if and only if the following two conditions are satisfied.*

(1) *The curve Γ_1 is an ellipse with the same foci as Γ_2.*
(2) *The numbers Q_1, Q_2, α and the angular velocity Ω satisfy,*

$$\Omega = \alpha\,\frac{Q_2^2-1}{4Q_2^2}, \quad Q_1 = Q_2\left(\frac{\alpha}{Q_2^2}+1-\alpha\right) \quad \textit{and} \quad \frac{Q_2^2}{Q_2^2-1} < \alpha < 0.$$

Before giving a brief account of the proofs some remarks are in order.

Remarks. (1) For the doubly connected patches ($\alpha = 0$), if the interior curve is a non degenerate ellipse (different from a circle) then there is no rotation.

(2) We believe that Theorem 2 holds when the exterior curve is an ellipse. That is, if we assume that the exterior interface is an ellipse, then one should conclude that the interior interface is an ellipse too. This depends on an inverse problem that we have not ben able to solve.
(3) The constraints on the parameters detailed in (2) of Theorem 2 coincide with the ones given in [9].
(4) We can easily check from the expression of Q_1 that $0 < Q_1 < Q_2$. This is consistent with the fact that the ellipses are confocal and Γ_1 lies outside the domain D_2 enclosed by Γ_2.

We present now an outline of the proofs of our two results. We first derive the equations governing the motion of the boundaries in the general framework considered here. The uniform rotation condition is shown to be equivalent to a system of two steady nonlocal equations of nonlinear type coupling the Cauchy transforms of the domains D_1 and D_2. It is hopeless to solve completely this system because of its higher degree of complexity. Nevertheless, when the interior boundary Γ_2 is an ellipse we obtain an explicit formula for the Cauchy transform of the unknown domain D_1. This leads to an inverse problem of the following type: one knows the Cauchy transform of a domain and one wants to determine the domain. It is well-known that this is not always possible [21]. It is, however, possible in our special situation by using Schwarz functions and the maximum principle for harmonic functions. Once we know that Γ_1 is an ellipse we come back to the system in order to find the compatibility conditions which will in turn fix all the involved parameters.
The paper is structured as follows. In section 2 we gather some general facts about rotating vortices. In section 3 we derive the equations of motion of the boundaries via the Cauchy transforms of the domains D_1 and D_2. In section 4 we review some useful tools from complex analysis and potential theory and we discuss some inverse problems. The last section is devoted to the proofs of the main results.

2. Preliminaries on rotating vortices

In this section we discuss some elementary facts on vortex dynamics for incompressible Euler equations. Recall that vorticity $\omega = \partial_1 v^2 - \partial_2 v^1$ satisfies the transport equation (1).
We shall focus on the vortices whose dynamics undergoes a planar rigid-body motion. In this case the motion can be described by a combination of translations and rotations. For the sake of simplicity we restrict the study to the group of rotations.

Definition 1. *Let $\omega_0 \in L^1 \cap L^\infty$. We say that ω_0 is a rotating vorticity if the solution ω of equation (1) with initial condition ω_0 is given by*

$$\omega(t,x) = \omega_0(\mathbf{R}_{x_0,-\theta(t)}x), \qquad x \in \mathbb{R}^2.$$

Here we denote by $\mathbf{R}_{x_0,\theta(t)}$ the planar rotation of center x_0 and angle $\theta(t)$. Moreover we assume that the function $t \mapsto \theta(t)$ is smooth and non-constant.

In the vortex patch class this definition reduces to the following: $\omega_0 = \chi_D$, with D a bounded domain, is a rotating vorticity (or, equivalently, D is a rotating vortex patch or V-state) if and only if

$$\omega(t) = \chi_{D_t} \quad \text{and} \quad D_t \triangleq \mathbf{R}_{x_0,\theta(t)}D.$$

In the preceding definition the vorticity ω_0 is assumed to be bounded and integrable in order to get a unique global solution according to Yudovich's theorem. The velocity dynamics in the framework of rotating vortices is described as follows.

Proposition 1. *Let ω_0 be a rotating vorticity as in Definition 1. Then the velocity $v(t)$ can be recovered from the initial velocity v_0 according to the formula*

$$v(t,x)=\mathbf{R}_{x_0,\theta(t)}v_0(\mathbf{R}_{x_0,-\theta(t)}x).$$

Proof. We shall use the formula

$$\Delta v(t,x)=\nabla^\perp\omega(t,x).$$

Performing some algebraic computations we get

$$\begin{aligned}\nabla^\perp\omega(t,x) &= \mathbf{R}_{x_0,\theta(t)}\nabla^\perp\omega_0(\mathbf{R}_{x_0,-\theta(t)}x)\\ &= \mathbf{R}_{x_0,\theta(t)}\Delta v_0(\mathbf{R}_{x_0,-\theta(t)}x)\\ &= \Delta\big(\mathbf{R}_{x_0,\theta(t)}v_0(\mathbf{R}_{x_0,-\theta(t)}x)\big).\end{aligned}$$

Thus the velocity fields $x\mapsto\mathbf{R}_{x_0,\theta(t)}v_0(\mathbf{R}_{x_0,-\theta(t)}x)$ and v differ by a harmonic function and both decay at infinity. Hence they are equal. □

Definition 2. *Let ω be a compactly supported solution of (1) with non-zero total mass*

$$m(t)\triangleq\int_{\mathbb{R}^2}\omega(t,x)dx.$$

Define the center of mass of ω as

$$X(t)=\frac{1}{m(t)}\int_{\mathbb{R}^2}x\,\omega(t,x)\,dx.$$

The total mass and the center of mass are invariants of the motion. We include a short proof of this classical fact for the sake of the reader.

Proposition 2. *Let ω_0 be a smooth compactly supported initial vorticity with non zero total mass. Then for any positive time t*

$$m(t)=m(0)\quad\textit{and}\quad X(t)=X(0).$$

Proof. The conservation of mass follows easily from the characteristics method. The vorticity $\omega(t)$ can be expressed in terms of its initial value and the flow ψ according to the formula

$$\omega(t,x)=\omega_0(\psi^{-1}(t,x)).$$

The incompressibility condition entails that the flow preserves Lebesgue measure and thus the mass is conserved in time.

The invariance of the center of mass follows from the constancy of the functions

$$f_j(t)\triangleq\int_{\mathbb{R}^2}x_j\,\omega(t,x)\,dx,\quad j=1,2.$$

Differentiation of the functions f_j with respect to the time variable combined with the vorticity equation (1) yields

$$\begin{aligned} f_j'(t) &= \int_{\mathbb{R}^2} x_j\,\partial_t\omega(t,x)\,dx \\ &= -\int_{\mathbb{R}^2} x_j\,(v\cdot\nabla\omega)(t,x)\,dx \\ &= \int_{\mathbb{R}^2} v^j(t,x)\,\omega(t,x)\,dx. \end{aligned}$$

Since $\omega = \partial_1 v^2 - \partial_2 v^1$, an integration by parts yields

$$\begin{aligned} f_1'(t) &= \int_{\mathbb{R}^2} v^1(t,x)\,(\partial_1 v^2(t,x) - \partial_2 v^1(t,x))\,dx \\ &= \int_{\mathbb{R}^2} v^1(t,x)\,\partial_1 v^2(t,x)\,dx \\ &= -\int_{\mathbb{R}^2} \partial_1 v^1(t,x)\,v^2(t,x)\,dx \\ &= \int_{\mathbb{R}^2} \partial_2 v^2(t,x)\,v^2(t,x)\,dx \\ &= 0. \end{aligned}$$

□

We now explore the relationship between center of rotation and center of mass.

Proposition 3. *Let $\omega_0 = \chi_D$ be a vortex patch with non zero total mass, which rotates around the point x_0, . Then necessarily x_0 is the center of mass of the domain D.*

Proof. By a change of variables

$$\begin{aligned} X(t) &= \frac{1}{m(0)}\int_{\mathbb{R}^2} x\,\omega_0(\mathbf{R}_{-\theta(t),x_0}x)dx \\ &= \frac{1}{m(0)}\int_{\mathbb{R}^2} (\mathbf{R}_{\theta(t),x_0}x)\omega_0(x)dx \\ &= \frac{1}{m(0)}\mathbf{R}_{\theta(t),x_0}\left(\int_{\mathbb{R}^2} x\omega_0(x)dx\right) \\ &= \mathbf{R}_{\theta(t),x_0}X(0). \end{aligned}$$

Since $X(t) = X(0)$ by Proposition (2), $X(0)$ is fixed by the rotation and thus $X(0) = x_0$, as claimed. □

The last result of this section is the nontrivial fact that any rotating patch must have a constant angular velocity.

Proposition 4. *Let $\omega_0 = \chi_D$ be a rotating vortex patch different from the Rankine vortex. Then the angular velocity is necessarily constant, that is,*

$$\theta(t) = t\Omega + \theta_0, \quad t \geq 0,$$

for some constants Ω and θ_0.

Proof. Let $s \mapsto \gamma_t(s)$ be a parametrization of the boundary of the patch $D_t \triangleq \mathbf{R}_{x_0,\theta(t)}D$. Then, as we will prove in the next section, the motion of the boundary ∂D_t satisfies equation (9), namely

$$\mathrm{Im}\big((\partial_t\gamma_t - v(t,\gamma_t)\overline{\gamma_t'}\big) = 0.$$

Here the prime denotes derivative with respect to the s variable. This equation leads to (10), that is,

$$\dot\theta(t)\mathrm{Re}\left(\gamma_0\overline{\gamma_0'}\right) = \mathrm{Im}\left(v_0(\gamma_0)\,\overline{\gamma_0'}\right),$$

which is equivalent to

$$\frac{\dot\theta(t)}{2}\frac{d}{ds}|\gamma_0(s)|^2 = \mathrm{Im}\left(v_0(\gamma_0)\,\overline{\gamma_0'}\right).$$

If there exists some s with $\frac{d}{ds}|\gamma_0(s)|^2 \neq 0$ then, since the right-hand side does not depend on the time variable, we conclude that $\dot\theta(t)$ is constant. Otherwise, $\frac{d}{ds}|\gamma_0(s)|^2$ vanishes everywhere, which tells us that the initial domain is a disc. Thus our vortex is the Rankine vortex, which rotates with any angular velocity. $\square$

3. Boundary motion

We shall in what follows describe the motion of a piecewise constant vorticity in the plane. Let $D_j,\ 1 \le j \le n$, be a family of simply connected domains such that for each j the closure of D_{j+1} is contained in D_j. Assume moreover that the boundary Γ_j of D_j is a Jordan curve of class $C^1,\ 1 \le j \le n$. We set $E_j = D_j\backslash\overline{D_{j+1}}$ for $1 \le j \le n-1$ and $E_n = D_n$. Let $\alpha_j, 1 \le j \le n$, be a family of real numbers such that $\alpha_1 \neq 0$ and $\alpha_j \neq \alpha_{j+1}$ for $1 \le j \le n-1$. Now take an initial vorticity of the form

$$\omega_0 = \sum_{j=1}^n \alpha_j\,\chi_{E_j}, \tag{5}$$

where χ_{E_j} denotes the characteristic function of E_j. Since the vorticity is conserved along the particle trajectories, the initial structure of the vorticity is preserved in time. Thus the vorticity at time t has the form

$$\omega(t) = \sum_{j=1}^n \alpha_j\,\chi_{E_{j,t}}, \quad E_{j,t} = \psi(t,E_j), \tag{6}$$

where ψ is the flow map

$$\psi(t,x) = x + \int_0^t v(\tau,\psi(\tau,x))d\tau \tag{7}$$

associated with the velocity v. We will describe the dynamics of the interfaces $\Gamma_{j,t} \triangleq \psi(t,\Gamma_j), 1 \le j \le n$. In particular, the case $n = 1$ gives the equation for the boundary motion of a simply connected vortex patch and the case $n = 2$ and $\alpha_1 = 1, \alpha_2 = 0$, provides the system of two equations for the boundary of a doubly connected rotating vortex patch. There are at least two natural ways to derive the equations of the boundary.

3.1. **First approach.** The motion of the interfaces $\Gamma_{j,t},\ 1\leq j\leq n$, is subject to the kinematic constraint that the boundary is transported with the flow. In particular, it is a material surface and thus there is no flux matter across the boundary. Since we assume that the interfaces Γ_j are C^1-smooth we can express Γ_j, for each fixed j satisfying $1\leq j\leq n$, as

$$\Gamma_j=\{x\in\mathbb{R}^2:\varphi_j(x)=0\},$$

where φ_j is a real function of class C^1 on the plane, such that $\nabla\varphi_j(x)\neq 0,\ x\in\Gamma_j$, $\varphi_j<0$ on D_j and $\varphi_j>0$ on $\mathbb{R}^2\setminus\overline{D_j}$. One says that φ_j is a defining function for Γ_j. Set

$$F_j(t,x)=\varphi_j(\psi^{-1}(t,x)),$$

where ψ is the flow (7). Then $x\to F_j(t,x)$ is a defining function for $\Gamma_{j,t}=\psi(t,\Gamma_j)$. Since by definition $F_j(t,x)$ is transported by the flow , it satisfies the transport equation

$$\partial_t F+v\cdot\nabla F=0.$$

Now, let $\gamma_t(s)$ be a parametrization of $\Gamma_{1,t}$, continuously differentiable in t, and let $\vec{n}_t$ be the unit outward normal vector to $\Gamma_{1,t}$. Differentiating the equation $F(t,\gamma_t(s))=0$ with respect to t yields

$$\partial_t F+\partial_t\gamma_t\cdot\nabla F=0.$$

Since for $x\in\Gamma_{j,t}$ the vector $\nabla F(t,x)$ is perpendicular to $\Gamma_{j,t}$, we obtain

$$(\partial_t\gamma_t-v(t,\gamma_t))\cdot\vec{n}_t=0. \tag{8}$$

The meaning of (8) is that the velocity of the boundary and the the velocity of the fluid particle occupying the same position have the same normal components. We observe that equation (8) can be written in the complex form

$$\mathrm{Im}\Big\{(\partial_t\gamma_t-v(t,\gamma_t))\overline{\gamma_t'}\Big\}=0 \tag{9}$$

where the "prime" denotes derivative with respect to the s variable.
We now take a closer look at the case of a rotating doubly connected vortex patch. Assume that the two interfaces rotate with the same angular velocity $\dot\theta(t)$ around some point, which can be assumed to be the origin. Denote by γ_0 a parametrization of one of the initial interfaces. Then $\gamma_t(s)=e^{i\theta(t)}\gamma_0(s)$ is a parametrization of the transported interface at time t and, on one hand, we get

$$\begin{aligned}\mathrm{Im}(\partial_t\gamma_t\overline{\gamma_t'})&=\dot\theta(t)\,\mathrm{Re}(\gamma_t\overline{\gamma_t'})\\&=\dot\theta(t)\,\mathrm{Re}(\gamma_0\overline{\gamma_0'}).\end{aligned}$$

By Proposition 1

$$\begin{aligned}v(t,\gamma_t)&=e^{i\theta(t)}v_0(e^{-i\theta(t)}\gamma_t)\\&=e^{i\theta(t)}v_0(\gamma_0).\end{aligned}$$

Hence, on the other hand,

$$\mathrm{Im}(v(t,\gamma_t)\overline{\gamma_t'})=\mathrm{Im}\big(v_0(\gamma_0)\overline{\gamma_0'}\,\big).$$

Therefore (9) becomes

$$\mathrm{Im}\big(v_0(\gamma_0)\overline{\gamma_0'}\,\big)=\mathrm{Im}(v(t,\gamma_t)\overline{\gamma_t'})=\mathrm{Im}(\partial_t\gamma_t\overline{\gamma_t'})=\dot\theta(t)\mathrm{Re}(\gamma_0\overline{\gamma_0'}). \tag{10}$$

It follows from the identity above, as we remarked before, that the angular velocity $\dot{\theta}(t) \equiv \Omega$ is constant. Recall that $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and let Ψ stand for the stream function at time 0, namely,

$$\Psi(z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \omega_0(\xi) \log|z-\xi|\, d\xi, \quad z \in \mathbb{C}.$$

Then $v_0(z) = 2i\partial_z \Psi(\gamma_0)$ and so

$$\mathrm{Im}\big(v_0(\gamma_0)\overline{\gamma_0'}\big) = 2\mathrm{Re}\big(\partial_z\Psi(\gamma_0)\gamma_0'\big). \tag{11}$$

Combining (10) and (11) we conclude that the initial interfaces $\Gamma \triangleq \cup_{j=1}^n \Gamma_j$ satisfy the system of n equations

$$2\mathrm{Re}\big\{\partial_z\Psi(z)\, z'\big\} = \Omega\,\mathrm{Re}\big\{\overline{z}\, z'\big\}, \quad z \in \Gamma, \tag{12}$$

where z' denotes a tangent vector to the boundary at the point z.

3.2. **Second approach.** We will give another way to derive the equation (12), which consists in analyzing directly the vorticity equation. According to the Definition 1 and assuming that the center of rotation is the origin a rotating vorticity has the structure $\omega(t,x) = \omega_0(\mathbf{R}_{-\theta(t)}x)$. Straightforward computations show that

$$\nabla\omega(t,x) = \mathbf{R}_{\theta(t)}(\nabla\omega_0(\mathbf{R}_{-\theta(t)}x)).$$

Combining this identity with Proposition 1 yields

$$\begin{aligned} v\cdot\nabla\omega(t,x) &= \langle \mathbf{R}_{\theta(t)}v_0(\mathbf{R}_{-\theta(t)}x), \mathbf{R}_{\theta(t)}(\nabla\omega_0(\mathbf{R}_{-\theta(t)}x))\rangle \\ &= \langle v_0(\mathbf{R}_{-\theta(t)}x), \nabla\omega_0(\mathbf{R}_{-\theta(t)}x)\rangle \\ &= (v_0\cdot\nabla\omega_0)(\mathbf{R}_{-\theta(t)}x). \end{aligned}$$

We have used the symbol $\langle,\rangle$ to denote the usual scalar product in the plane and in the second identity the fact that rotations preserve the scalar product. A simple calculation yields

$$\begin{aligned} \partial_t\omega(t,x) &= -\dot{\theta}(t)\Big\{\big(-x_2\partial_1 + x_1\partial_2\big)\omega_0\Big\}(\mathbf{R}_{-\theta(t)}x) \\ &= -\dot{\theta}(t)(x^\perp\cdot\nabla\omega_0)(\mathbf{R}_{-\theta(t)}x). \end{aligned}$$

Consequently, the vorticity equation becomes

$$\Big(v_0(x) - \dot{\theta}(t)x^\perp\Big)\cdot\nabla\omega_0(x) = 0. \tag{13}$$

Recall that for a smoothly bounded domain D

$$\nabla\chi_D = -\vec{n}\, d\sigma,$$

where $d\sigma$ is the arc-length measure on ∂D and $\vec{n}$ the exterior unit normal. Then, for an initial vorticity as in (5), we get

$$-\nabla\omega_0 = \alpha_1\vec{n}\, d\sigma_1 + \sum_{j=1}^{n-1}(\alpha_{j+1}-\alpha_j)\vec{n}\, d\sigma_{j+1},$$

with $d\sigma_j$ the the arc-length measure on the curve Γ_j. Since by the assumption α_1 and $\alpha_{j+1}-\alpha_j$ do not vanish, equation (13) is equivalent to

$$\big(v_0(x) - \dot{\theta}(t)x^\perp\big)\cdot\vec{n}(x) = 0, \quad x\in\Gamma_j, \quad 1\le j\le n. \tag{14}$$

We conclude from (14) that the only way in which $\dot{\theta}(t)$ may be non-constant is that $x^\perp \cdot \vec{n}(x) = 0$ on the union of the interfaces $\Gamma = \cup_{j=1}^n \Gamma_j$. If this is the case then the interfaces must be concentric circles. Denote by z' a tangent vector at the point $x = z$ of Γ. Using the identities

$$v_0 \cdot \vec{n} = \nabla \Psi \cdot z' = 2\mathrm{Re}\big(\partial_z \Psi\, z'\big) \quad \text{and} \quad x^\perp \cdot \vec{n} = \mathrm{Re}\big(\overline{z}\, z'\big)$$

we finally obtain

$$2\mathrm{Re}\big(\partial_z \Psi\, z'\big) = \dot{\theta}(t)\mathrm{Re}\big(\overline{z}\, z'\big), \quad z \in \Gamma,$$

which is (12) after setting $\dot{\theta}(t) \equiv \Omega$.
We close this subsection by noticing that a rotating vortex appears as a stationary solution for the vorticity equation for the Euler system in the presence of the linear external velocity $v_e = \dot{\theta} x^\perp$, namely,

$$\partial_t \omega + (v - \dot{\theta}(t) x^\perp) \cdot \nabla \omega = 0.$$

This follows easily from (13).

3.3. **The role of the Cauchy transform.** In this subsection we describe the motion of a rotating vortex patch of the form (6) by means of the Cauchy transforms of the domains D_j. Without loss of generality and in order to simplify the presentation we shall restrict our attention to the case of two interfaces. Hence the initial vorticity has the form

$$\omega_0 = \chi_{D_1} + (\alpha - 1)\chi_{D_2}.$$

One usually defines the stream function as the logarithmic potential of the vorticity at time t, that is,

$$\Psi_t(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \omega(t,\xi) \log|z - \xi|\, dA(\xi), \quad z \in \mathbb{C}, \tag{15}$$

where dA is Lebesgue measure on the plane. Differentiating (15) with respect to the variable z yields

$$\begin{aligned} \partial_z \Psi(z) &= \frac{1}{4\pi} \int_{D_1} \frac{1}{z - \xi} dA(\xi) + (\alpha - 1) \frac{1}{4\pi} \int_{D_2} \frac{1}{z - \xi} dA(\xi) \\ &= \frac{1}{4} \mathcal{C}(\chi_{D_1})(z) + (\alpha - 1) \frac{1}{4} \mathcal{C}(\chi_{D_2})(z), \end{aligned} \tag{16}$$

where

$$\mathcal{C}(\chi_D)(z) = \frac{1}{\pi} \int_D \frac{1}{z - \xi} dA(\xi), \quad z \in \mathbb{C}, \tag{17}$$

denotes the Cauchy transform of the domain D (actually, of the characteristic function of D). It is well-known and easy to check that the Cauchy transform is continuous on $\mathbb{C}$, holomorphic off $\overline{D}$ and has zero limit at infinity. If D is a bounded domain with boundary of class C^1, there is a formula for the Cauchy transform of D, which we proceed to describe below, involving only integrals over the boundary $\Gamma = \partial D$. The Cauchy integral of the function $\overline{z}$ on D is

$$\gamma^+(z) = \fint_\Gamma \frac{\overline{\xi}}{\xi - z}\, d\xi, \quad z \in D,$$

where we have used the notation $\fint_\Gamma = \frac{1}{2\pi i} \int_\Gamma$. Similarly the Cauchy integral of $\overline{z}$ on $\mathbb{C} \setminus \overline{D}$ is

$$\gamma^-(z) = \fint_\Gamma \frac{\overline{\xi}}{\xi - z}\, d\xi, \quad z \in \mathbb{C} \setminus \overline{D}.$$

It is plain that the previous functions are holomorphic in their domains of definition. They can be extended continuously up to the boundary of D. This follows easily from dominated convergence and the identity

$$\gamma^{\pm}(z)=\fint_{\Gamma}\frac{\overline{\xi}-\overline{z}}{\xi-z}\,d\xi+\overline{z}\,\chi_D(z),$$

which holds in the domains of definition of $\gamma^{\pm}$. For the sake of simple notations the one-sided limit at the boundary will be denoted by $\gamma^{\pm}(z)$ as well. The Plemelj-Sokhotskiì formulae (see [22, p. 143]) for the function $\overline{z}$ are the identities

$$\begin{aligned}\gamma^{+}(z) &= \text{p.v.}\fint_{\Gamma}\frac{\overline{\xi}}{\xi-z}\,d\xi+\frac{\overline{z}}{2}, \quad z\in\Gamma,\\ \gamma^{-}(z) &= \text{p.v.}\fint_{\Gamma}\frac{\overline{\xi}}{\xi-z}\,d\xi-\frac{\overline{z}}{2}, \quad z\in\Gamma,\end{aligned}$$

where the boundary integrals are understood in the principal value sense. Subtracting one gets the jump formula

$$\overline{z}=\gamma^{+}(z)-\gamma^{-}(z), \quad z\in\Gamma. \tag{18}$$

The Cauchy transform of D can be reconstructed from the functions $\gamma^{\pm}$. According to the Cauchy-Pompeiu formula for the function $\overline{z}$ (see(25) below) one has

$$\mathcal{C}(\chi_D)(z)=\overline{z}-\gamma^{+}(z), \quad z\in\overline{D} \tag{19}$$

and

$$\mathcal{C}(\chi_D)(z)=-\,\gamma^{-}(z), \quad z\notin D. \tag{20}$$

We emphasize that these formulae hold also on the boundary Γ.
We now come back to the formula (16) for the stream function. Denote by $\gamma_j^{\pm}$ the Cauchy integrals of $\overline{z}$ for the domain $D_j,\ i=1,2$. The identities (19) and (20) combined with (16) yield

$$4\partial_z\Psi(z)=\overline{z}-\gamma_1^{+}(z)+(1-\alpha)\gamma_2^{-}(z), \quad z\in\Gamma_1 \tag{21}$$

and

$$4\,\partial_z\Psi=\overline{z}-\gamma_1^{+}(z)+(1-\alpha)\gamma_2^{-}(z), \quad z\in\Gamma_2. \tag{22}$$

Putting together (21), (22) and (12) we obtain the nonlinear system of two equations

$$\text{Re}\Big\{\Big(\lambda\overline{z}+(1-\alpha)\gamma_2^{-}(z)-\gamma_1^{+}(z)\Big)z'\Big\}=0, \quad z\in\Gamma_1\cup\Gamma_2, \tag{23}$$

with $\lambda=1-2\Omega$.
We mention for future reference that on Γ_2 the preceding equation can also be written in the form

$$\text{Re}\Big\{\Big((\alpha-2\Omega)\overline{z}+(1-\alpha)\gamma_2^{+}(z)-\gamma_1^{+}(z)\Big)z'\Big\}=0, \quad z\in\Gamma_2. \tag{24}$$

4. Tools from potential theory

4.1. **Preliminaries on complex analysis.** We begin by recalling a classical results about complex functions. The derivatives of a smooth function $\varphi : \mathbb{C} \to \mathbb{C}$ with respect to z and $\overline{z}$ are defined as

$$\frac{\partial\varphi}{\partial z} = \frac{1}{2}\Big(\frac{\partial\varphi}{\partial x} - i\frac{\partial\varphi}{\partial y}\Big) \qquad \text{and} \qquad \frac{\partial\varphi}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial\varphi}{\partial x} + i\frac{\partial\varphi}{\partial y}\Big).$$

Let D be a finitely connected domain bounded by finitely many smooth Jordan curves and let Γ be ∂D endowed with the positive orientation. Then the Cauchy-Pompeiu formula reads as

$$\varphi(z)\chi_{\overline{D}}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\xi)}{\xi - z}d\xi - \frac{1}{\pi}\int_D \frac{\partial\varphi}{\partial\overline{\xi}}(\xi)\frac{1}{\xi - z}\,dA(\xi), \qquad z \in \mathbb{C}.$$

For $z \in \partial D$ the boundary integral has to be understood as the limit from D of the same integral. Taking $\varphi(z) = \overline{z}$ we obtain

$$\overline{z}\chi_{\overline{D}}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\xi}}{\xi - z}d\xi + \mathcal{C}(\chi_D)(z), \quad z \in \mathbb{C}. \tag{25}$$

4.2. **Cauchy transform.** We intend to compute the Cauchy transform (17) of discs and ellipses. This can be done rather easily using (19) and (20), namely,

$$\mathcal{C}(\chi_D)(z) = \begin{cases} \overline{z} - \gamma^+(z), & z \in D \\ -\gamma^-(z), & z \notin D. \end{cases}$$

• *The discs.* To begin with we consider the unit disc. Since $\xi\overline{\xi} = 1$ on ∂D,

$$\gamma^+(z) = \fint_{|\xi|=1} \frac{1}{\xi(\xi - z)}d\xi = 0, \quad z \in D \quad \text{and} \quad \gamma^-(z) = -\frac{1}{z}, \quad z \notin \overline{D}.$$

Therefore

$$\mathcal{C}(\chi_D)(z) = \begin{cases} \overline{z}, & z \in D \\ \frac{1}{z}, & z \notin D. \end{cases}$$

For a disc of center z_0 and radius r translating and dilating the previous result gives

$$\mathcal{C}(\chi_D)(z) = \begin{cases} \overline{z} - \overline{z}_0, & z \in D \\ \frac{r^2}{z - z_0}, & z \notin D. \end{cases}$$

• *The ellipses.* Let D be the domain enclosed by the ellipse $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$. We set $c^2 = a^2 - b^2$. If the major semi-axis is a then the foci of the ellipse are $\pm\sqrt{a^2 - b^2}$. Otherwise the foci are $\pm i\sqrt{b^2 - a^2}$. Rewriting the cartesian equation of the ellipse in terms of the variables z and $\overline{z}$ and solving for $\overline{z}$ leads to

$$\overline{z} = Qz + F(z), \quad z \in \partial D,$$

where

$$Q = \frac{a - b}{a + b} \qquad \text{and} \qquad F(z) = \frac{2ab}{z\Big(1 + \sqrt{1 - \frac{c^2}{z^2}}\Big)}.$$

By Cauchy's Integral Formula

$$\fint_{\partial D} \frac{Q\xi}{\xi - z}d\xi = Qz,\ z \in D.$$

Since $\xi\mapsto\frac{F(\xi)}{\xi-z}$ is holomorphic off $\overline{D}$ and has a double zero at infinity,

$$\fint_{\partial D}\frac{F(\xi)}{\xi-z}d\xi=0,\; z\in D.$$

Hence

$$\gamma^+(z)=Qz,\quad z\in D.$$

To compute the function γ^- we use Cauchy's Integral Formula in $\mathbb{C}\setminus\overline{D}$ to get

$$\begin{aligned}\gamma^-(z)&=Q\fint_{\partial D}\frac{\xi}{\xi-z}d\xi+\fint_{\partial D}\frac{F(\xi)}{\xi-z}d\xi\\&=\fint_{\partial D}\frac{F(\xi)}{\xi-z}d\xi=-F(z).\end{aligned}$$

Therefore

$$\gamma^+(z)=Qz,\quad z\in D,\qquad\quad \gamma^-(z)=-F(z),\quad z\notin D,\tag{26}$$

and

$$\mathcal{C}(\chi_D)(z)=\left\{\begin{array}{l}\overline{z}-Qz,\quad z\in D\\ \frac{2ab}{z\left(1+\sqrt{1-\frac{c^2}{z^2}}\right)},\quad z\notin D.\end{array}\right.\tag{27}$$

Remark that γ^- satisfies the equation

$$c^2\{\gamma^-(z)\}^2+4ab\,z\,\gamma^-(z)+4a^2b^2=0.\tag{28}$$

For the general case where the ellipse is centered at z_0 and its major axis makes an angle θ with the horizontal axis one has

$$\gamma^+(z)=e^{-2i\theta}Q(z-z_0)+\overline{z_0},\quad z\in D.$$

4.3. **Inverse problems.** We shall see along this paper that the equations governing the interfaces of the rotating patch can be solved in some cases and allow to get explicitly the Cauchy transforms of the involved domains. It is a general fact that the knowledge of the Cauchy transform outside a domain D is equivalent to the knowledge of the geometrical moments $(m_n)_{n\in\mathbb{N}}$ defined by $m_n=\frac{1}{\pi}\int_D z^n dA(z)$ since it is a generating function of these moments. Now the problem is to see whether the shape of the domain is encoded by its Cauchy transform. This is an inverse problem of potential theory which appears in several contexts like celestial mechanics or geophysics: earth's shape, gravitational lensing [8], Hele-Shaw flows [21],... The inverse problem is not uniquely solvable in general as some counter-examples show (see for instance [21]). However uniqueness can be established for example under the assumption that the domains are starlike with respect to a common point ([18]). In our context, the Cauchy transform has a special algebraic form and, as we shall see, this determines uniquely the shape giving rise to the Cauchy transform at hand. There are two kinds of problems that we are led to deal with. In the first one we discuss the case where the Cauchy transform is known inside the domain and given by a first order polynomial function. In the second one the Cauchy transform is known outside the domain and this case seems to be trickier. For the first case we prove the following result.

Proposition 5. *Let Γ be a Jordan curve of class C^1 enclosing a bounded domain D. Assume that there exist $Q \in \mathbb{R}$ and $z_0 \in \mathbb{C}$ such that*

$$\gamma^+(z) \triangleq \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\xi}}{\xi - z}d\xi = Q(z-z_0) + \overline{z_0}, \quad z \in D.$$

Then the curve Γ is an ellipse of center z_0 with semi-axes a and b satisfying

$$Q = \frac{a-b}{a+b}\cdot$$

Remark. It is a surprising consequence of the Proposition that $|Q|$ must be strictly less than one. In other words, it is not possible to find a Jordan curve satisfying the hypotheses of Proposition 5 with $|Q| \geq 1$.

Proof. According to the jump formula (18) we have on Γ the decomposition

$$\begin{aligned}\overline{z} &= \gamma^+(z) - \gamma^-(z)\\ &= Q(z-z_0) + \overline{z_0} - \gamma^-(z),\end{aligned}$$

with γ^- holomorphic on $\mathbb{C}_\infty \backslash \overline{D}$ and decaying at infinity like $\frac{1}{z}$. It follows that for any $z \in \Gamma$

$$(\overline{z} - \overline{z_0})^2 + (z-z_0)^2 = (1+Q^2)(z-z_0)^2 - 2Q(z-z_0)\gamma^-(z) + \{\gamma^-(z)\}^2$$

and

$$|z-z_0|^2 = Q(z-z_0)^2 - (z-z_0)\gamma^-(z).$$

Let A and B be two real numbers that will be chosen later on. For $z \in \Gamma$ one has

$$-A\Big((\overline{z}-\overline{z_0})^2 + (z-z_0)^2\Big) + B|z-z_0|^2 = \big(BQ - A(1+Q^2)\big)(z-z_0)^2 + g(z),$$

with

$$g(z) \triangleq \big(2AQ - B\big)(z-z_0)\gamma^-(z) - A\{\gamma^-(z)\}^2.$$

Now choose A and B such that $BQ - A(1+Q^2) = 0$ in order to kill the quadratic term. For example we can take

$$A = Q \quad \text{and} \quad B = 1 + Q^2.$$

Hence

$$-Q\Big((\overline{z}-\overline{z_0})^2 + (z-z_0)^2\Big) + (1+Q^2)|z-z_0|^2 = g(z), \quad z \in \Gamma.$$

The function g is clearly holomorphic on $\mathbb{C}\backslash\overline{D}$ and has a limit at infinity given by

$$\begin{aligned}\lim_{z\to\infty} g(z) &= (2AQ - B)\lim_{z\to\infty}\frac{1}{2\pi i}\int_\Gamma \frac{\overline{\xi}z}{\xi - z}d\xi\\ &= (1-Q^2)\frac{1}{2\pi i}\int_\Gamma \overline{\xi}\, d\xi\\ &= \frac{1-Q^2}{\pi}|D|,\end{aligned}$$

where we applied Green-Stokes in the last identity. Notice that g has a continuous extension up to the boundary Γ and takes real values on this set. Then the imaginary part of g is a harmonic function on the exterior domain $\mathbb{C}\backslash\overline{D}$, continuous up to the boundary and satisfying

$$\operatorname{Im} g(z) = 0, \quad z \in \Gamma \quad \text{and} \quad \lim_{z\to\infty}\operatorname{Im} g(z) = 0.$$

By the maximum principle we conclude that Im g is identically zero on $\mathbb{C}\setminus\overline{D}$. Thus the holomorphic function g is real on $\mathbb{C}\setminus\overline{D}$ and consequently must be constant. This means that

$$-Q\Big((\overline{z}-\overline{z_0})^2+(z-z_0)^2\Big)+(1+Q^2)|z-z_0|^2=C,\quad z\in\Gamma$$

with C a constant. Set $X=\mathrm{Re}(z-z_0)$ and $Y=\mathrm{Im}\,(z-z_0)$ then

$$(1-Q)^2X^2+(1+Q)^2Y^2=C,\quad \text{on}\quad \Gamma.$$

This is an equation for the curve Γ in the cartesian coordinates X and Y. For $Q\notin\{-1,1\}$ the curve Γ is an ellipse. For $Q\in\{-1,1\}$ the curve reduces to a segment, which is not possible by the assumptions. The proof of the desired result is complete. $\square$

Next we shall consider the case where the Cauchy transform is prescribed outside the domain. We will prove the following result.

Proposition 6. *Let Γ be a Jordan curve of class C^1 enclosing a domain D and let z_1 be a point in D such that*

$$\frac{1}{2\pi i}\int_\Gamma\frac{\overline{\xi}}{\xi-z}d\xi=\frac{a}{z-z_1}+\frac{b}{(z-z_1)^2},\quad z\notin\overline{D},$$

with a and b real constants. Then there exists a constant c such that curve Γ is contained in the set

$$|z-z_1|^4+a|z-z_1|^2+2b\,Re\,z=c.$$

Proof. From the jump formula (18) we have

$$\overline{z}=\gamma^+(z)-\gamma^-(z),\quad z\in\Gamma,$$

with

$$\gamma^+(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\overline{\xi}}{\xi-z}\,d\xi,\quad z\in D,\quad\text{and}\quad \gamma^-(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\overline{\xi}}{\xi-z}\,d\xi,\quad z\notin\overline{D}.$$

By assumption

$$\overline{z}=\gamma^+(z)-\frac{a}{z-z_1}-\frac{b}{(z-z_1)^2},\quad z\in\Gamma.$$

Set $w=z-z_1$, $\widetilde{\Gamma}\triangleq\Gamma-z_1$ and $\widetilde{D}=D-z_1$. Then the preceding identity can be written as

$$\overline{w}=\phi(w)-\frac{a}{w}-\frac{b}{w^2};\quad w\in\widetilde{\Gamma},\tag{29}$$

with $\phi(w)\triangleq\gamma^+(z_1+w)-z_1$, which is holomorphic in $\widetilde{D}$. By (29)

$$w+\overline{w}=w+\phi(w)-\frac{a}{w}-\frac{b}{w^2},\quad w\in\widetilde{\Gamma},$$

$$w\overline{w}=w\phi(w)-a-\frac{b}{w},\quad w\in\widetilde{\Gamma},$$

and

$$(w\overline{w})^2=\frac{b^2}{w^2}+\frac{2ab}{w}+\phi_1(w),\quad w\in\widetilde{\Gamma},$$

where ϕ_1 is holomorphic in $\widetilde{D}$. Taking the appropriate linear combination of the previous three identities we kill the singularity at the origin, that is,

$$(w\overline{w})^2+aw\overline{w}+b(w+\overline{w})=b(w+\phi(w)+aw\phi(w)-a^2+\phi_1(w)\triangleq\phi_2(w),\quad w\in\widetilde{\Gamma}.$$

It is plain that ϕ_2 is holomorphic in $\widetilde{D}$ and the function in the left-hand side is real-valued. Thus ϕ_2 is constant and so

$$|w|^4+a|w|^2+b(w+\overline{w})=c,\quad w\in\widetilde{\Gamma}. \tag{30}$$

This completes the proof. □

5. Proofs of the main results

In this section we prove Theorem 1 and Theorem 2. Recall that we are dealing with a rotating vorticity of the form (5) with only two interfaces, that is,

$$\omega_0=\chi_{D_1}+(\alpha-1)\chi_{D_2},\quad \alpha\in\mathbb{R},$$

where D_1 and D_2 are simply connected domains satisfying $\overline{D_2}\subset D_1$. As we have already seen the description of the rotating vorticity in this special case is governed by the equations (23). Owing to the complicated structure of this system, which is strongly nonlinear and nonlocal, a description of the full set of solutions seems to be out of reach. However, as we stated in Theorem 1 we can show that if one of the interfaces is a circle then the patch is necessarily trivial, in the sense that it is an annulus. In Theorem 2 we completely solve the system assuming that the inner interface is an ellipse. Then the exterior interface is a confocal ellipse and certain relations (introduced in [9]) between the angular velocity of rotation, the inner vorticity α and the parameters of the ellipses must be satisfied. Likewise the result should also hold under the assumption that the exterior interface is an ellipse, but we have not been able to solve the corresponding inverse problem.

5.1. Circular interfaces : the proof of Theorem 1.

Proof of Theorem 1. The proof relies on equation (24) combined with the inverse problem results established in the previous section. We first study the case in which the inner interface Γ_2 is a circle, which is easier than the other one.

Case 1 : Γ_2 a circle.

Let Γ_2 be a circle centered at z_2. Assume, without loss of generality, that z_2 lies in the real axis. Then $\gamma_2^+(z)=z_2$ and thus equation (24) reduces to

$$\mathrm{Re}\Big\{\Big((\alpha-2\Omega)\overline{z}+(1-\alpha)z_2-\gamma_1^+(z)\Big)z'\Big\}=0,\quad z\in\Gamma_2.$$

Since $z'=i(z-z_2)$ is a tangent vector at the point $z\in\Gamma_2$,

$$\mathrm{Re}\Big\{\Big((\alpha-2\Omega)\overline{z}+(1-\alpha)z_2-\gamma_1^+(z)\Big)i(z-z_2)\Big\}=0,\quad z\in\Gamma_2.$$

Observe that $\gamma_1^+(z)=(1-2\Omega)z_2$ is a solution of the above equation. We will show that this is the only solution. Set $\varphi(z)=\gamma_1^+(z)-(1-2\Omega)z_2$, so that

$$\mathrm{Im}\big\{\varphi(z)(z-z_2)\big\}=0,\quad z\in\Gamma_2.$$

It is plain that $z\mapsto\varphi(z)(z-z_2)$ is holomorphic in D_1, which contains $\overline{D}_2$, and its imaginary part is a harmonic function vanishing on the boundary Γ_2. By the maximum principle

$\mathrm{Im}\big\{\varphi(z)(z-z_2)\big\}=0$ in D_2 and so $\varphi(z)(z-z_2)$ is constant in D_2. Evaluating at z_2 we see that this constant must be zero and then that φ vanishes identically on D_2 and hence on D_1 by holomorphic continuation. Therefore

$$\gamma_1^+(z)=(1-2\Omega)z_2,\quad z\in D_1. \tag{31}$$

Now in view of Proposition 5 the function γ_1^+ determines the shape of the boundary Γ_1, which turns out to be a circle centered at the point $z_1\triangleq(1-2\Omega)z_2$. The next step is to show that the two circles have the same center. With this in mind we substitute in equation (23) the expression (31) for γ_1^+ and the identity

$$\gamma_2^-(z)=\frac{r^2}{z_2-z},\quad z\notin D_2,$$

r being the radius of the circle Γ_2. We conclude that

$$\mathrm{Im}\Big\{\Big(\lambda(\overline{z}-z_2)+(1-\alpha)\frac{r^2}{z_2-z}\Big)(z-z_1)\Big\}=0,\quad z\in\Gamma_1.$$

Since z_1 and z_2 are real,

$$\lambda(z_1-z_2)\,\mathrm{Im}(z-z_1)=(1-\alpha)r^2\,\mathrm{Im}\Big\{\frac{z-z_1}{z-z_2}\Big\},\quad z\in\Gamma_1.$$

Set $w=z-z_1$ and $z_0=z_1-z_2$. Write the preceding equation in terms of w and z_0, replace w by $-w$ and add the two equations. We obtain

$$0=(1-\alpha)r^2\mathrm{Im}\Big\{\frac{w^2}{w^2-z_0^2}\Big\},\quad w\in -z_1+\Gamma_1.$$

As $\alpha\neq 1$ we obtain $z_0=0$, namely $z_1=z_2$. Then the interfaces are concentric circles and there is no restriction on the parameters α and Ω. This is coherent with the fact that in this case the vorticity is radial and therefore the flow is stationary.

Case 2 : Γ_1 a circle.

Let z_1 be the center of Γ_1. Without loss of generality we may assume that $z_1\in\mathbb{R}$ and that the center of mass is the origin. This implies that the center of mass z_2 of D_2 is real. Our goal is to prove that Γ_2 must be a circle centered at z_1. Equation (23) on Γ_1 takes the form

$$\mathrm{Im}\Big\{\Big(\lambda\overline{z}-z_1+(1-\alpha)\gamma_2^-(z)\Big)(z-z_1)\Big\}=0,\quad z\in\Gamma_1,$$

which is clearly equivalent to

$$\mathrm{Im}\Big\{\Big((\lambda-1)z_1+(1-\alpha)\gamma_2^-(z)\Big)(z-z_1)\Big\}=0,\quad z\in\Gamma_1.$$

Set $w=z-z_1$, $\tilde\Gamma_j=-z_1+\Gamma_j$ and $\tilde D_j=-z_1+D_j, i=1,2$. Thus the preceding equation becomes

$$\mathrm{Im}\Big\{\Big((\lambda-1)z_1+(1-\alpha)\gamma_2^-(z_1+w)\Big)w\Big\}=0,\quad w\in\tilde\Gamma_1,$$

Since $|\frac{\xi}{w}|<1$ for $\xi\in\tilde\Gamma_2$ and $w\in\tilde\Gamma_1$, one has

$$\begin{aligned}\gamma_2^-(z_1+w)&=\frac{1}{2\pi i}\int_{\tilde\Gamma_2}\frac{\overline{\xi}}{\xi-w}d\xi\\&=\sum_{n\geq 0}\frac{a_n}{w^{n+1}},\end{aligned}$$

where

$$a_n = -\frac{1}{2\pi i}\int_{\tilde{\Gamma}_2} \xi^n \overline{\xi}\, d\xi.$$

Therefore

$$\Big((\lambda-1)z_1 + (1-\alpha)\gamma_2^-(z_1+w)\Big)w = (\lambda-1)z_1 w + (1-\alpha)\sum_{n\geq 0}\frac{a_n}{w^n}$$

and

$$\mathrm{Im}\Big\{(\lambda-1)z_1 w + (1-\alpha)\sum_{n\geq 0}\frac{a_n}{w^n}\Big\} = 0, \quad w\in\tilde{\Gamma}_1. \tag{32}$$

By Green-Stokes

$$a_0 = \frac{1}{2\pi i}\int_{\tilde{\Gamma}_2}\overline{\xi}d\xi = \frac{1}{\pi}|D_2|.$$

Now since $\alpha\neq 1$ and $a_0\in\mathbb{R}$, equation (32) holds true if and only if

$$(\lambda-1)z_1 r_1^2 = (1-\alpha)a_1 \quad \text{and} \quad a_n = 0, \quad n\geq 2,$$

r_1 being the radius of Γ_1. Thus we obtain the following expression for γ_2

$$\gamma_2^-(z) = \frac{a_0}{z-z_1} + \frac{a_1}{(z-z_1)^2}, \quad z\in\Gamma_1. \tag{33}$$

Since γ_2^- is continuous in $\mathbb{C}\backslash D_2$, the pole z_1 of γ_2^- must be in D_2. Now we will evaluate the coefficient a_1. Using Green-Stokes

$$\begin{aligned} a_1 &= \frac{1}{2\pi i}\int_{-z_1+\Gamma_2}|\xi|^2 d\,\xi \\ &= \frac{1}{\pi}\int_{-z_1+D_2}\xi\, dA(\xi) \\ &= \frac{1}{\pi}\int_{D_2}(\xi - z_1)\, dA(\xi). \end{aligned}$$

Let z_2 be the center of mass of D_2. Thus

$$\int_{D_2}(\xi - z_2)dA(\xi) = 0,$$

and so

$$a_1 = \frac{1}{\pi}(z_2 - z_1)|D_2|.$$

If we knew that $z_1 = z_2$, then $a_1 = 0$ and therefore Γ_2 would be a circle of center z_1 by Proposition 6.
It remains to show that a_1 vanishes. Combining equations (23) and (33) we get

$$\mathrm{Re}\Big\{\Big((\lambda-1)z_1 + \lambda\overline{w} + \frac{a_0(1-\alpha)}{w} + \frac{a_1(1-\alpha)}{w^2}\Big)w'\Big\} = 0, \quad w\in\tilde{\Gamma}_2, \tag{34}$$

Our next task is to find a useful expression for a tangent vector w' to $\tilde{\Gamma}_2$ at the point w. Recall that by Proposition 6 the curve $\tilde{\Gamma}_2$ is defined in Cartesian coordinates by

$$P(x,y) \triangleq (x^2+y^2)^2 + a_0(x^2+y^2) + 2a_1 x = c.$$

A tangent vector is then given by

$$\begin{aligned} w' &= -\partial_y P + i\,\partial_x P \\ &= 4|w|^2 iw + 2ia_0 w + 2ia_1. \end{aligned}$$

Substituting this expression for w' in equation (34) one gets

$$\operatorname{Im}\left\{(\lambda-1)z_1 w(2|w|^2+a_0)+\lambda a_1\overline{w}+\frac{2A}{w}(|w|^2+a_0)+\frac{a_1\,A}{w^2}\right\}=0,\quad w\in\tilde{\Gamma}_2,$$

where we have set $A=a_1(1-\alpha)$. This gives

$$(1-\lambda)z_1(2|w|^2+a_0)+\lambda a_1+\frac{2A}{|w|^2}(|w|^2+a_0)+a_1\,A\frac{w+\overline{w}}{|w|^4}=0,\quad w\in\tilde{\Gamma}_2,$$

which is equivalent to

$$2(1-\lambda)z_1|w|^6+\big(2A+\lambda a_1+(1-\lambda)a_0z_1\big)|w|^4+2a_0A|w|^2+a_1\,A(w+\overline{w})=0,\quad w\in\widetilde{\Gamma}_2.$$

Using (30) with a replaced by a_0 and b by a_1 we find

$$2(1-\lambda)z_1|w|^6+\big(A+\lambda a_1+(1-\lambda)a_0z_1\big)|w|^4+a_0A|w|^2+Ac=0,\quad w\in\widetilde{\Gamma}_2. \tag{35}$$

Since $\widetilde{\Gamma}_2$ is connected, either $|w|$ is constant on $\widetilde{\Gamma}_2$ and we are done, or $|w|$ takes a continuum of values on $\widetilde{\Gamma}_2$. In the second case the polynomial obtained by replacing in the left-hand side of (35) $|w|$ by the real variable t has infinitely many zeroes and hence is the zero polynomial. Thus the coefficient a_0A must be zero. Since $\pi a_0=|D_2|\neq 0$ and $\alpha\neq 1$, $a_1=0$ and the proof is complete. □

5.2. **Elliptical interfaces : the proof of Theorem 2.** We turn now to the proof of Theorem 2. First we will prove that if the interior curve is an ellipse then the rigid motion of the interfaces will force the domains to have the same center of mass. The equations (23) and the explicit form of the function γ_2^+ will lead to the identification of γ_1^+, via the maximum principle. At this stage we are led to understand the link between the geometry of the domain and its inside Cauchy transform γ^+. This is a kind of inverse problem of two-dimensional potential theory that we have already discussed in the previous section in the context of circles. In the case at hand we show that the exterior curve Γ_1 is an ellipse and we find some information on its shape. Armed with this precious information we solve explicitly the equations (23).

5.2.1. **First reduction.**

Lemma 1. *Assume that $\omega_0=\chi_{D_1}+(\alpha-1)\chi_{D_2}$, $\alpha\in\mathbb{R}$, is a rotating vorticity around the origin and that Γ_2 is an ellipse. Let z_2 be the center of D_2. If $z_2\neq 0$ then the line through the origin and z_2 is an axis of the ellipse Γ_2. Moreover Γ_1 is an ellipse.*

Proof. First of all we can assume without loss of generality that z_2 is a positive real number. Let θ denote the angle between the major axis of the ellipse Γ_2 and the real axis. We have to show that $2\theta\equiv 0\,[\pi]$.
Recall that by (24) the equation that describes rotation with angular velocity Ω on Γ_2 is

$$\operatorname{Re}\Big\{\Big((\alpha-2\Omega)\overline{z}+(1-\alpha)\gamma_2^+(z)-\gamma_1^+(z)\Big)z'\Big\}=0,\quad z\in\Gamma_2,$$

where z' denotes a tangent vector to Γ_2 at the point z. For $j \in \{1,2\}$, let ϕ_j denote a complex primitive of γ_j^+ on the domain D_j. This primitive is well-defined since γ_j^+ is holomorphic on the simply connected domain D_j. Consequently equation (24) is equivalent to

$$\big(\frac{\alpha}{2} - \Omega\big)|z|^2 + \mathrm{Re}\Big((1-\alpha)\phi_2(z) - \phi_1(z)\Big) = C, \quad z \in \Gamma_2, \tag{36}$$

for some constant C.

Let ψ be the solution of the Dirichlet problem on D_2 with boundary data $|z|^2$, $z \in \Gamma_2$. Since ψ is harmonic in D_2 and D_2 is simply connected, there exists a holomorphic H function on D_2 such that ψ is the real part of H. Hence equation (36) becomes

$$\mathrm{Re}\Big(\big(\frac{\alpha}{2} - \Omega\big)H(z) + (1-\alpha)\phi_2(z) - \phi_1(z) - C\Big) = 0, \quad z \in \Gamma_2.$$

The function in the left-hand side of the preceding identity is harmonic in D_2 and continuous up to the boundary. By the maximum principle this function is identically zero in the domain D_2. Therefore, since holomorphic functions that take real values on a domain are constant,

$$\big(\frac{\alpha}{2} - \Omega\big)H'(z) + (1-\alpha)\phi_2'(z) - \phi_1'(z) = 0, \quad z \in D_2,$$

where prime denotes derivative with respect to z. Hence

$$\gamma_1^+(z) = \big(\alpha - 2\Omega\big)\partial_z\psi(z) + (1-\alpha)\gamma_2^+(z), \quad z \in D_2, \tag{37}$$

because $2\partial_z\psi(z) = H'(z)$. This determines completely the function γ_1^+ in D_2 and thus in D_1, by analytic continuation. To take full advantage of (37) we need to have explicit expressions for γ_2^+ and $2\partial_z\psi$. For γ_2^+ it is just a matter of applying a translation and a rotation to (26). We get

$$\gamma_2^+(z) = Q_2 e^{-2i\theta}(z - z_2) + z_2, \quad z \in D_2. \tag{38}$$

For $\partial_z\psi$ we solve explicitly the Dirichlet problem defining ψ and then we take a derivative with respect to z. In order to do so we need to write the equation of the boundary Γ_2 in the variables z and $\overline{z}$. Consider first the ellipse $\mathcal{E} = \{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$. Expressing x and y in terms of z and $\overline{z}$ we find for $\mathcal{E}$ the equation

$$A(z^2 + \overline{z}^2) + B|z|^2 = 1, \quad A = \frac{1}{4}\Big(\frac{1}{a^2} - \frac{1}{b^2}\Big) \quad \text{and} \quad B = \frac{1}{2}\Big(\frac{1}{a^2} + \frac{1}{b^2}\Big).$$

Assume now that a and b are the semi-axes of the ellipse Γ_2. Then an equation for Γ_2 is

$$A\Big(e^{-2i\theta}(z - z_2)^2 + e^{2i\theta}(\overline{z} - z_2)^2\Big) + B|z|^2 + B{z_2}^2 - Bz_2(z + \overline{z}) = 1.$$

Solving for $|z|^2$ and remarking that the function which gives the solution is harmonic in D_2 we conclude that

$$\psi(z) = \frac{1}{B} - z_2^2 + z_2(z + \overline{z}) - \frac{A}{B}\Big(e^{-2i\theta}(z - z_2)^2 + e^{2i\theta}(\overline{z} - z_2)^2\Big).$$

and

$$\partial_z\psi(z) = z_2 - 2\frac{A}{B}e^{-2i\theta}(z - z_2). \tag{39}$$

Inserting (39) and (38) in (37) yields

$$\gamma_1^+(z) = (1 - 2\Omega)z_2 + Q_1 e^{-2i\theta}(z - z_2), \quad z \in D_1, \tag{40}$$

where

$$Q_1 = (2\Omega - \alpha)\frac{2A}{B} + (1-\alpha)Q_2.$$

We have proved that γ_1^+ is a first degree polynomial and we are almost done. First we remark that the assumption that the center of mass of the initial vorticity is the origin implies that the centers of mass z_j of $D_j, j = 1, 2$, satisfy

$$z_1 = (1-\alpha)\frac{|D_2|}{|D_1|}z_2. \tag{41}$$

In particular z_1 is a real number.
It is a general fact that if D is any bounded domain then

$$\int_D \overline{z}\, dA(z) = \int_D \gamma^+(z)\, dA(z). \tag{42}$$

This follows from (19) and the observation that

$$\int_D \int_D \frac{1}{\zeta - z}\, dA(\zeta) dA(z) = 0$$

because the Cauchy kernel is odd. Taking the mean value of γ_1^+ on D_1 and using (40) and (42) one obtains

$$z_1 = (1-2\Omega)z_2 + Q_1 e^{-2\theta i}(z_1 - z_2).$$

Thus 2θ is an integer multiple of π and so the line through the origin and z_2 is an axis of Γ_2. We are left with the task of showing that Γ_1 is an ellipse, which is easy. Indeed, setting

$$z_1 = \frac{\lambda \pm Q_1}{1 \pm Q_1} z_2,$$

where the minus sign corresponds to $\theta = 0$ and the plus sign to $\theta = \pi/2$, we obtain, rewriting (40),

$$\gamma_1^+(z) = \pm Q_1(z - z_1) + z_1, \quad z \in D_1.$$

Here the plus sign corresponds to $\theta = 0$ and the minus sign to $\theta = \pi/2$. Since z_1 is real an application of Proposition 5 shows that Γ_1 is an ellipse, which completes the proof of Lemma 1. □

5.2.2. **Second reduction.** We know now that if $\omega_0 = \chi_{D_1} + (\alpha-1)\chi_{D_2}$ is a rotating vorticity and the interior curve is an ellipse, then the exterior curve is also an ellipse and its axes are parallel to those of the interior ellipse. However, the only information we have up to now about the relative position of the center of mass of ω_0 and the centers of the ellipses is that they lie on a straight line.

Lemma 2. *Assume that $\omega_0 = \chi_{D_1} + (\alpha-1)\chi_{D_2}$ is a rotating vorticity and that Γ_1 and Γ_2 are ellipses. Then the ellipses are centered at the center of mass of ω_0.*

Proof. Assume, without loss of generality, that the center of mass of ω_0 is the origin. Let z_j be the center of Γ_j, $j = 1, 2$. We may also assume that z_2 is real. Then z_1 is also real because of (41). By (23) the equation that describes rotation with angular velocity Ω on Γ_1 is

$$\operatorname{Re}\Big\{\Big(\lambda\overline{z} + (1-\alpha)\gamma_2^-(z) - \gamma_1^+(z)\Big)z'\Big\} = 0, \quad z \in \Gamma_1,$$

with $\lambda = 1 - 2\Omega$. Since z_1 is real, by the first reduction $\gamma_1^+(z) = \pm Q_1(z - z_1) + z_1$. Assume that the sign in front of Q_1 is plus (the same argument will work with the minus sign). Then

$$\mathrm{Re}\Big\{\Big(\lambda\overline{z} + (1-\alpha)\gamma_2^-(z) - Q_1(z-z_1) - z_1\Big)z'\Big\} = 0, \quad z \in \Gamma_1.$$

Setting $w = z - z_1$ the preceding equation becomes

$$\mathrm{Re}\Big\{\Big(\lambda\overline{w} + (1-\alpha)\gamma_2^-(z_1+w) - Q_1 w + (\lambda-1)z_1\Big)i(A_1 w + B_1\overline{w})\Big\} = 0, \quad w \in -z_1 + \Gamma_1. \tag{43}$$

Here $i(A_1 w + B_1\overline{w})$ is the tangent vector to the ellipse $-z_1+\Gamma_1$ at the point w. The expression of A_1 and B_1 in terms of the length of the semi-axes a_1 and b_1 of Γ_1 can be obtained by using the standard parametrization of an ellipse. One gets

$$A_1 = \frac{a_1^2 + b_1^2}{2a_1b_1} \qquad \text{and} \qquad B_1 = \frac{b_1^2 - a_1^2}{2a_1b_1}.$$

The ellipse $\widetilde{\Gamma}_2 = -z_2 + \Gamma_2$ is centered at the origin and its axes lies along the coordinate axes. Set

$$h_2(z) = \frac{1}{2\pi i}\int_{\widetilde{\Gamma}_2} \frac{\overline{\xi}}{\xi - z}d\xi, \quad z \in \mathbb{C} \setminus \overline{D_2}.$$

As we mentioned in (26) h_2 is given by

$$h_2(z) = -\frac{2a_2b_2}{z\Big(1 + \sqrt{1 - \frac{c_2^2}{z^2}}\Big)}, \quad z \in \mathbb{C} \setminus \overline{D_2}.$$

Translating we see that

$$\gamma_2^-(z) = h_2(z - z_2), \quad z \notin \overline{D}_2.$$

Let $d = z_1 - z_2$. Then equation (43) can be rewritten as

$$\mathrm{Re}\Big\{\Big(\lambda\overline{w} + (1-\alpha)h_2(w+d) - Q_1 w + (\lambda-1)z_1\Big)i(A_1 w + B_1\overline{w})\Big\} = 0, \quad w \in -z_1 + \Gamma_1.$$

The ellipse $-z_1+\Gamma_1$ is centered at zero and invariant under the mapping $w \to -w$. Therefore writing the above equation for $-w$ and subtracting both equations yields, for $w \in -z_1 + \Gamma_1$,

$$2(\lambda-1)z_1\mathrm{Im}(A_1 w + B_1\overline{w}) + (1-\alpha)\mathrm{Im}\Big\{\Big(h_2(w+d) + h_2(-w+d)\Big)(A_1 w + B_1\overline{w})\Big\} = 0. \tag{44}$$

Denote by $\mathbb{C}_\infty$ the extended complex plane (or Riemann sphere). Let U be the domain enclosed by the ellipse $-z_1 + \Gamma_1$. Our next task is to find a solution to the Dirichlet problem in the domain $\mathbb{C}_\infty \setminus \overline{U}$ in the Riemann sphere with boundary data $\mathrm{Im}(A_1 w + B_1\overline{w})$. By (27)

$$\overline{w} = Q_1 w - h_1(w), \quad w \in -z_1 + \Gamma_1, \tag{45}$$

with

$$h_1(w) = -\frac{2a_1b_1}{w\Big(1 + \sqrt{1 - \frac{c_1^2}{w^2}}\Big)}, \quad z \in \mathbb{C} \setminus U.$$

Hence

$$\begin{aligned}\mathrm{Im}(\overline{w}) &= Q_1\mathrm{Im}(w) - \mathrm{Im}(h_1(w)) \\ &= -Q_1\mathrm{Im}(\overline{w}) - \mathrm{Im}(h_1(w)), \quad w \in -z_1 + \Gamma_1,\end{aligned}$$

and so

$$\mathrm{Im}(\overline{w}) = -\frac{1}{1+Q_1}\mathrm{Im}(h_1(w)), \quad w \in -z_1 + \Gamma_1,$$

and

$$\begin{aligned}\mathrm{Im}(A_1 w + B_1 \overline{w}) &= (B_1 - A_1)\mathrm{Im}(\overline{w}) \\ &= \frac{A_1 - B_1}{1+Q_1}\mathrm{Im}(h_1(w)), \quad w \in -z_1 + \Gamma_1.\end{aligned}$$

The right-hand side above is harmonic in $\mathbb{C}_\infty \setminus \overline{U}$ and then is the solution of the Dirichlet problem in $\mathbb{C}_\infty \setminus \overline{U}$ with boundary data the left-hand side. Inserting this identity into (44) and using (45) and the relation $A_1 + B_1 Q_1 = 1$ one gets

$$\mathrm{Im}\left\{\mathcal{A}\, z_1\, h_1(w) + (1-\alpha)\Big(h_2(w+d) + h_2(-w+d)\Big)\big(w - B_1 h_1(w)\big)\right\} = 0, \quad w \in -z_1 + \Gamma_1,$$

where $\mathcal{A}$ stands for $2\frac{A_1 - B_1}{1+Q_1}(\lambda - 1)$. In the left-hand side of the preceding identity one is taking the imaginary part of a holomorphic function in $\mathbb{C} \setminus \overline{U}$. Hence, for some constant C,

$$\mathcal{A}\, z_1\, h_1(w) + (1-\alpha)\Big(h_2(w+d) + h_2(-w+d)\Big)\big(w - B_1 h_1(w)\big) = C, \quad w \in \mathbb{C} \setminus \overline{U}. \tag{46}$$

Observe that

$$C = (1-\alpha) \lim_{w \to \infty} w\big(h_2(w+d) + h_2(-w+d)\big) = 0.$$

Computing the coefficient of $\frac{1}{w}$ in the expansion at ∞ of the left-hand side of (46) we get the relation

$$\mathcal{A} z_1 a_1 b_1 = 2(1-\alpha) a_2 b_2 d. \tag{47}$$

Set

$$F_j(w) = \frac{1}{w\Big(1 + \sqrt{1 - \frac{c_j^2}{w^2}}\Big)}, \quad j = 1, 2, \quad w \in \mathbb{C} \setminus \overline{U}.$$

An easy argument based on (46) and (47) gives

$$2d\, F_1(w) + \Big(F_2(w+d) + F_2(-w+d)\Big)\big(w - c_1^2 F_1(w)\big) = 0, \quad w \in \mathbb{C} \setminus \overline{U}. \tag{48}$$

The function in the left-hand side above is odd and holomorphic at ∞ so that in its expansion in powers of $1/w$ the even powers vanish identically. The coefficient of $1/w$ also vanishes identically. Instead, the fact that the other odd powers vanish, because of (48), provides a countable family of equations in the parameters d, c_1 and c_2. Our goal is to show that $d = 0$ using the equations corresponding to the coefficients of $1/w^3, 1/w^5$ and $1/w^7$. Recall that then $z_1 = z_2 = 0$ and we are done.
The coefficient of $1/w^3$ is

$$d\left(\frac{3}{4}c_1^2 - \frac{3}{4}c_2^2 - d^2\right).$$

Hence either $d = 0$ or

$$d^2 = \frac{3}{4}\left(c_1^2 - c_2^2\right). \tag{49}$$

The coefficient of $1/w^5$ is

$$\frac{1}{8}d\left(2c_1^4 - 20c_2^2 d^2 - 5c_2^4 - 8d^4 + 4c_1^2 d^2 + 3c_1^2 c_2^2\right). \tag{50}$$

As before, if $d = 0$ we are done and so we can assume that this is not the case. If $c_j = 0$ for $j = 1$ or $j = 2$, then Γ_j is a circle and this case has been dealt with in subsection 5.1.

Dividing in (50) by c_2^4, eliminating the even powers of d by means of (49) and setting $q = c_1/c_2$ we get $q^2 - 12q + 11 = 0$, which yields $q = 1$ or $q = 11$. If $q = 1$ then $d = 0$ by (49). Let $q = 11$. The coefficient of $1/w^7$ turns out to be

$$\frac{1}{64} d\left(-366c_2^2d^4 - 280c_2^4d^2 - 35c_2^6 + 9c_1^6 - 64d^6 + 6c_1^4c_2^2 + 80c_1^2c_2^2d^2 + 20c_1^2c_2^4 + 8c_1^2d^4\right). \tag{51}$$

Eliminating the even powers of d in (51) by means of (49) and setting $c_1^2 = 11c_2^2$ we obtain

$$\frac{-1450}{64} d\, c_2^6 = 0.$$

Since we are in the case $c_2 \neq 0$, we conclude that $d = 0$, which completes the proof. □

5.2.3. **Resolution of the boundary equations**. Up to now we have shown that if the interior curve is an ellipse then necessarily the exterior curve is an ellipse with the same center and parallel axes. Our next target is to give a complete description of the parameters λ, α, Q_1 and Q_2 in order to get a uniform rotation. This will complete the proof of Theorem 2.

We start by investigating the equation on the interior curve Γ_2.

Equation on Γ_2. Recall that the equation (24) that describes rotation with angular velocity Ω on Γ_2 is

$$\mathrm{Re}\Big\{\Big((\alpha - 2\Omega)\overline{z} + \big[(1-\alpha)Q_2 - Q_1\big]z\Big)z'\Big\} = 0, \quad z \in \Gamma_2.$$

We have used the fact that $\gamma_j^+(z) = Q_j\, z$, $j = 1, 2$. As we mentioned before, a straightforward computation shows that a tangent vector to the ellipse Γ_j at the point z is given by

$$z' = i(A_j z + B_j\overline{z}), \quad A_j = \frac{a_j^2 + b_j^2}{2a_jb_j}, \quad B_j = \frac{b_j^2 - a_j^2}{2a_jb_j}.$$

Recall that $c_j^2 = a_j^2 - b_j^2$ gives the foci of the ellipse. Hence we obtain

$$\mathrm{Re}\Big\{i\Big(\big[(1-\alpha)Q_2 - Q_1\big]A_2z^2 + (\alpha - 2\Omega)B_2\overline{z}^2\Big)\Big\} = 0, \quad z \in \Gamma_2,$$

which is equivalent to

$$\mathrm{Re}\Big\{i\Big(\big[(1-\alpha)Q_2 - Q_1\big]A_2 - (\alpha - 2\Omega)B_2\Big)z^2\Big\} = 0, \quad z \in \Gamma_2,$$

This condition is satisfied only when

$$\big[(1-\alpha)Q_2 - Q_1\big]A_2 + (2\Omega - \alpha)B_2 = 0. \tag{52}$$

We would to write this equation in terms of Q_1, Q_2 and Ω only. From the elementary identities

$$A_j + B_jQ_j = 1 \quad \text{and} \quad A_j^2 - B_j^2 = 1$$

we get

$$A_j = \frac{1 + Q_j^2}{1 - Q_j^2} \qquad \text{and} \qquad B_j = \frac{-2Q_j}{1 - Q_j^2}. \tag{53}$$

Thus (52) becomes

$$\big[(\alpha - 1)Q_2 + Q_1\big](1 + Q_2^2) = 2(\alpha - 2\Omega)Q_2. \tag{54}$$

Equation on Γ_1. Using equation (23) on Γ_1, we get

$$\mathrm{Re}\big\{i\lambda B_1\overline{z}^2+\Big((1-\alpha)\gamma_2^-(z)-Q_1z\Big)i(A_1z+B_1\overline{z})\big\}=0,\quad z\in\Gamma_1.$$

Since $-\mathrm{Re}\{iQ_1A_1z^2\}=\mathrm{Re}\{iQ_1A_1\overline{z}^2\}$,

$$\mathrm{Re}\Big\{i(\lambda B_1+Q_1A_1)\overline{z}^2+(1-\alpha)\gamma_2^-(z)i(A_1z+B_1\overline{z})\Big\}=0,\quad z\in\Gamma_1. \tag{55}$$

Let us introduce the function

$$G(z)=(\lambda B_1+Q_1A_1)\overline{z}^2+(1-\alpha)\gamma_2^-(z)(A_1z+B_1\overline{z}).$$

Since on Γ_1 we have $\overline{z}=Q_1z-\gamma_1^-(z)$, $G(z)$ can be written as

$$G(z)=(\lambda B_1+A_1Q_1)\overline{z}^2+(1-\alpha)(A_1+B_1Q_1)z\gamma_2^-(z)-(1-\alpha)B_1\gamma_1^-(z)\gamma_2^-(z).$$

Setting $M=\lambda B_1+Q_1A_1$ and using the identity $A_1+B_1Q_1=1$ we find

$$G(z)=M\overline{z}^2+(1-\alpha)z\gamma_2^-(z)-(1-\alpha)B_1\gamma_1^-(z)\gamma_2^-(z).$$

Thus equation (55) on Γ_1 becomes

$$\mathrm{Im}\left\{M\overline{z}^2+(1-\alpha)z\gamma_2^-(z)-(1-\alpha)B_1\gamma_1^-(z)\gamma_2^-(z)\right\}=0,\quad z\in\Gamma_1. \tag{56}$$

The next step is to solve the Dirichlet problem on the domain $\mathbb{C}_\infty\setminus\overline{D_1}$ of the Riemann sphere $\mathbb{C}_\infty$ with boundary data $\mathrm{Im}(\overline{z}^2)$. With this goal in mind recall the identity

$$\overline{z}=Q_1z-\gamma_1^-(z),\quad z\in\Gamma_1, \tag{57}$$

where

$$\gamma_1^-(z)=\frac{-2a_1b_1}{z\Big(1+\sqrt{1-\frac{c_1^2}{z^2}}\Big)},\quad z\in\mathbb{C}\setminus\overline{D_1}.$$

Squaring (57) we obtain

$$\overline{z}^2=Q_1^2z^2-2Q_1z\gamma_1^-(z)+\{\gamma_1^-(z)\}^2,\quad z\in\Gamma_1.$$

By (28)

$$\begin{aligned}\overline{z}^2&=Q_1^2z^2+\frac{Q_1}{2a_1b_1}\big(c_1^2\{\gamma_1^-(z)\}^2+4a_1^2b_1^2\big)+\{\gamma_1^-(z)\}^2\\&=Q_1^2z^2+2Q_1a_1b_1+\big(1+\frac{Q_1c_1^2}{2a_1b_1}\big)\{\gamma_1^-(z)\}^2,\quad z\in\Gamma_1.\end{aligned}$$

It is easy to check that

$$1+\frac{Q_1c_1^2}{2a_1b_1}=\frac{a_1^2+b_1^2}{2a_1b_1}=A_1.$$

Consequently

$$\begin{aligned}\mathrm{Im}\;\overline{z}^2&=Q_1^2\,\mathrm{Im}\;z^2+A_1\,\mathrm{Im}\,\{\gamma_1^-(z)\}^2\\&=-Q_1^2\,\mathrm{Im}\;\overline{z}^2+A_1\,\mathrm{Im}\,\{\gamma_1^-(z)\}^2,\quad z\in\Gamma_1.\end{aligned}$$

This gives

$$\mathrm{Im}\;\overline{z}^2=\frac{A_1}{1+Q_1^2}\,\mathrm{Im}\,\{\gamma_1^-(z)\}^2,\qquad z\in\Gamma_1,$$

which tells us that the function on the right-hand side is the solution of the Dirichlet problem in $\mathbb{C}_\infty \setminus \overline{D_1}$ with boundary data given by the left-hand side. Inserting this into (56) yields

$$\operatorname{Im}\Big(\frac{MA_1}{1+Q_1^2}\{\gamma_1^-(z)\}^2 + (1-\alpha)z\gamma_2^-(z) - (1-\alpha)B_1\gamma_1^-(z)\gamma_2^-(z)\Big) = 0, \quad z \in \Gamma_1.$$

Since the function inside the imaginary part in the preceding identity is holomorphic on $\mathbb{C}_\infty \setminus \overline{D_1}$, it is constant. In other words, for some constant C,

$$\frac{MA_1}{1+Q_1^2}\{\gamma_1^-(z)\}^2 + (1-\alpha)z\gamma_2^-(z) - (1-\alpha)B_1\gamma_1^-(z)\gamma_2^-(z) = C, \qquad z \in \mathbb{C}\backslash D_1.$$

In view of (28) we obtain

$$-\frac{MA_1}{1+Q_1^2}\{\gamma_1^-(z)\}^2 + (1-\alpha)B_1\gamma_1^-(z)\gamma_2^-(z) + (1-\alpha)\frac{c_2^2}{4a_2b_2}\{\gamma_2^-(z)\}^2 = 0, \qquad z \in \mathbb{C}\backslash D_1.$$

and recalling that $B_2 = -c_2^2/2a_2b_2$

$$2\frac{MA_1}{1+Q_1^2}\{\gamma_1^-(z)\}^2 - 2(1-\alpha)B_1\gamma_1^-(z)\gamma_2^-(z) + (1-\alpha)B_2\{\gamma_2^-(z)\}^2 = 0, \qquad z \in \mathbb{C}\backslash D_1.$$

Dividing this equation by $\{\gamma_2^-(z)\}^2$ we get a second degree polynomial equation in the unknown γ_1^-/γ_2^-, namely,

$$2\frac{MA_1}{1+Q_1^2}\Big(\frac{\gamma_1^-(z)}{\gamma_2^-(z)}\Big)^2 - 2(1-\alpha)B_1\frac{\gamma_1^-(z)}{\gamma_2^-(z)} + (1-\alpha)B_2 = 0, \qquad z \in \mathbb{C}\backslash D_1.$$

This implies that $\frac{\gamma_1^-(z)}{\gamma_2^-(z)} = \mu$ with μ a constant. Consequently,

$$c_1 = c_2 \quad \text{and} \quad \mu = \frac{a_1b_1}{a_2b_2}.$$

In particular the ellipses Γ_1 and Γ_2 are confocal. Moreover

$$2\frac{MA_1}{1+Q_1^2}\big(\frac{a_1b_1}{a_2b_2}\big)^2 - 2(1-\alpha)B_1\frac{a_1b_1}{a_2b_2} + (1-\alpha)B_2 = 0.$$

One can easily check that

$$\frac{a_1b_1}{a_2b_2} = \frac{B_2}{B_1} \qquad \text{and} \qquad -2B_1\frac{a_1b_1}{a_2b_2} + B_2 = -B_2.$$

This yields

$$2\frac{MA_1}{1+Q_1^2}B_2 = (1-\alpha)B_1^2. \tag{58}$$

We will rewrite this equation in terms of Q_1, Q_2 and λ. By (53) equation (58) reduces to

$$-\frac{MQ_2}{1-Q_2^2} = (1-\alpha)\frac{Q_1^2}{1-Q_1^2}. \tag{59}$$

Now M can be expressed as

$$\begin{aligned} M &= \lambda B_1 + Q_1A_1 \\ &= (\lambda-1)B_1 + B_1 + Q_1A_1 \\ &= (1-\lambda)\frac{2Q_1}{1-Q_1^2} - Q_1. \end{aligned}$$

Thus equation(59) becomes, if $Q_1 \neq 0$,

$$Q_1 Q_2\big(Q_1 + (\alpha-1)Q_2\big) \quad = \quad (2\lambda-1)Q_2 + (\alpha-1)Q_1,$$

which is equivalent to

$$\big((1-\alpha) + Q_1Q_2\big)\big(Q_1 + (\alpha-1)Q_2\big) = \big(2\lambda - 1 - (1-\alpha)^2\big)Q_2.$$

Combining this equation with (54) we get the system

$$\left\{ \begin{array}{l} \big((1-\alpha) + Q_1Q_2\big)\big(Q_1 + (\alpha-1)Q_2\big) = \big(2\lambda - 1 - (1-\alpha)^2\big)Q_2 \\ (1+Q_2^2)\big(Q_1 + (\alpha-1)Q_2\big) = 2(\lambda+\alpha-1)Q_2. \end{array} \right. \tag{60}$$

To solve this system we distinguish two cases.

Case 1: $Q_1 + (\alpha-1)Q_2 = 0$. Since the ellipse Γ_2 is not a circle then $Q_2 \neq 0$ and the second equation of the preceding system gives necessary $\lambda = 1-\alpha$. Inserting this value into the first equation of (60) yields $\alpha = 0$ and so $Q_1 = Q_2$. The latter condition is impossible because the ellipses are confocal and different.

Case 2: $Q_1 + (\alpha-1)Q_2 \neq 0$. Dividing the first equation in (60) by the second we get

$$\frac{1-\alpha+Q_1Q_2}{1+Q_2^2} = \frac{2\lambda-1-(1-\alpha)^2}{2(\lambda+\alpha-1)} \triangleq C.$$

Hence

$$1-\alpha+Q_1Q_2 = C(1+Q_2^2).$$

Multiplying the second equation of (60) by Q_2 and using the previous identity we see that

$$\big(C+\alpha-1\big)\big(1+Q_2^2\big)^2 = 2(\lambda+\alpha-1)Q_2^2.$$

Thus

$$\frac{Q_2^2}{(1+Q_2^2)^2} = \frac{C+\alpha-1}{2(\lambda+\alpha-1)}. \tag{61}$$

Recalling that $1-\lambda = 2\Omega$ elementary arithmetics leads to

$$\begin{aligned} \frac{C+\alpha-1}{2(\lambda+\alpha-1)} &= \frac{\alpha^2+2\alpha(\lambda-1)}{4(\lambda+\alpha-1)^2} \\ &= \frac{\alpha^2-4\alpha\Omega}{4(\lambda-1+\alpha)^2}. \end{aligned}$$

Set $\rho = \frac{4Q_2^2}{(1+Q_2^2)^2}$. Then equation (61) is

$$4\rho\,\Omega^2 + 4\alpha(1-\rho)\Omega + \alpha^2(\rho-1) = 0.$$

The solutions of the quadratic equation above are

$$\Omega_\pm = \alpha\frac{(\rho-1) \pm \sqrt{1-\rho}}{2\rho},$$

which can be readily written as

$$\Omega_+ = \alpha\,\frac{1-Q_2^2}{4}, \quad \Omega_- = \alpha\frac{Q_2^2-1}{4Q_2^2}.$$

From the second equation in (60) the Q_1 associated to Ω_+ is given by

$$\begin{aligned} Q_1 &= Q_2\left(\frac{2\alpha-4\Omega_+}{1+Q_2^2}+1-\alpha\right) \\ &= Q_2. \end{aligned}$$

Since the ellipses are confocal this means that they are the same, which is not the case. The value of Q_1 associated to Ω_- is given by

$$Q_1 = Q_2\left(\frac{\alpha}{Q_2^2}+1-\alpha\right).$$

Recall that the ellipses are confocal and $\overline{D_2}\subset D_1$. Then $0<Q_1/Q_2<1$, which is equivalent to

$$-\frac{Q_2^2}{1-Q_2^2}<\alpha<0.$$

In conclusion, the ellipses rotate with the same angular velocity Ω if and only if we have the relations

$$\Omega=\alpha\,\frac{Q_2^2-1}{4Q_2^2},\quad Q_1=Q_2\left(\frac{\alpha}{Q_2^2}+1-\alpha\right)\quad\text{and}\quad -\frac{Q_2^2}{1-Q_2^2}<\alpha<0,$$

which completes the proof of Theorem 2.

Acknowledgements. The authors are grateful to Joan Josep Carmona for an illuminating conversation concerning the proof of Theorem 2 and to Luis Vega for suggesting that the study of doubly connected V-states might be interesting. J.Mateu and J.Verdera acknowledge generous support from the grants 2009SGR420 (Generalitat de Catalunya) and MTM2010-15657 (Ministerio de Ciencia e Innovación).

References

[1] H. Aref, *Integrable, chaotic, and turbulent vortex motion in two-dimensional flows*, Ann. Rev. Fluid Mech., **15** (1983), 345–389.

[2] A.L. Bertozzi and P. Constantin, *Global regularity for vortex patches*, Commun. Math. Phys. **152** (1993), 19–28.

[3] A.L. Bertozzi and A.J. Majda, *Vorticity and Incompressible Flow*, Cambridge texts in applied Mathematics, Cambridge University Press, Cambridge, (2002).

[4] J. Burbea, *Motions of vortex patches*, Lett. Math. Phys. **6** (1982), 1–16.

[5] J.Y. Chemin, *Fluides parfaits incompressibles*, Astérisque **230**, Société Mathématique de France (1995).

[6] S. A. Chaplygin, *On a pulsating cylindrical vortex.* Translated from the 1899 Russian original by G. Krichevets, edited by D. Blackmore and with comments by V. V. Meleshko. Regul. Chaotic Dyn. 12 (2007), no. 1, 101–116.

[7] G.S. Deem and N. J. Zabusky, *Vortex waves : Stationary "V-states", Interactions, Recurrence, and Breaking*, Phys. Rev. Lett. **40** 13 (1978), 859–862.

[8] C.D. Fassnacht, C. R. Keeton, D. Khavinson, *Gravitational lensing by elliptical galaxies, and the Schwarz function.* Analysis and mathematical physics, 115–129, Trends Math., Birkhäuser, Basel, 2009.

[9] G. R. Flierl, L. M. Polvani, *Generalized Kirchhoff vortices*, Phys. Fluids 29, 2376–2379 (1986).

[10] T. Hmidi, J. Mateu, J. Verdera, *Boundary Regularity of Rotating Vortex Patches.* Arch. Ration. Mech. Anal. 209 (2013), no. 1, 171–208.

[11] T. Kato and G. Ponce, *Well-posedness of the Euler and Navier-Stokes equations in the Lebesgue spaces* $L^p_s(\mathbb{R}^2)$. Rev. Mat. Iberoamericana **2** (1986), no. 1-2, 73–88.

[12] S. Kida, *Motion of an elliptical vortex in a uniform shear flow,* J. Phys. Soc. Japan. 50, 3517-3520 (1981)

[13] G. Kirchhoff, Vorlesungen uber mathematische Physik (Leipzig, 1874).

[14] H. Lamb, *Hydrodynamics*, Dover Publications, New York, (1945).

[15] M. V. Melander,, A. S. Styczek, N. J. Zabusky, *A moment model for vortex interactions of the two-dimensional Euler equations. Part 1. Computational validation of a Hamiltonian elliptical representation.* Journal of Fluid Mechanics , 167 (1986) 95-115.

[16] J. Neu, *The dynamics of columnar vortex in an imposed strain,* Phys. Fluids 27, 12397–2402 (1984).

[17] P. K. Newton, *The N-Vortex Problem. Analytical Techniques,* (Springer, New York, 2001).

[18] P. S. Novikov, *On the uniqueness for the inverse problem of potential theory*, Dokl. Akad. Nauk SSSR **18** (1938), 165-168.

[19] P. G. Saffman, *Vortex dynamics.* Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, New York, 1992.

[20] M. R. Ukhovskii, V. I. Iudovich, *Axially symmetric flows of ideal and viscous fluids filling the whole space*, Prikl. Mat. Meh. **32** (1968), no. 1, 59-69.

[21] A. N. Varchenko, P. I. Etingof, *Why the boundary of a round drop becomes a curve of order four.* University Lecture Series, 3. American Mathematical Society, Providence, RI, 1992.

[22] J. Verdera, *L^2 boundedness of the Cauchy Integral and Menger curvature,* Contemporary Mathematics **277** (2001), 139–158.

[23] W. Wolibner, *Un théorème sur l'existence du mouvement plan d'un fluide parfait homogène, incompressible, pendant un temps infiniment long,* Math. Z, Vol. 37, 1933, pp. 698–627.

[24] H. M. Wu, E. A. Overman II and N. J. Zabusky *Steady-state solutions of the Euler equations in two dimensions : rotating and translating V-states with limiting cases I. Algorithms ans results*, J. Comput. Phys. **53** (1984), 42–71.

[25] Y. Yudovich, *Nonstationary flow of an ideal incompressible liquid.* Zh. Vych. Mat., **3** (1963), 1032–1066.

IRMAR, Université de Rennes 1, Campus de Beaulieu, 35 042 Rennes cedex, France
E-mail address: thmidi@univ-rennes1.fr

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia
E-mail address: mateu@mat.uab.cat

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia
E-mail address: jvm@mat.uab.cat