Beltrami equations with coefficient in the fractional Sobolev space $W^{\theta,\frac{2}{\theta}}$

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Abstract

In this paper, we look at quasiconformal solutions $\phi : \mathbb{C} \to \mathbb{C}$ of Beltrami equations

$$\partial_{\overline{z}}\phi(z) = \mu(z)\, \partial_z\phi(z).$$

where $\mu \in L^\infty(\mathbb{C})$ is compactly supported on $\mathbb{D}$, $\|\mu\|_\infty < 1$ and belongs to the fractional Sobolev space $W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$. Our main result states that

$$\log \partial_z\phi \in W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$$

whenever $\alpha > \frac{1}{2}$. Our method relies on an n-dimensional result, which asserts the compactness of the commutator

$$[b, (-\Delta)^{\frac{\beta}{2}}] : L^{\frac{np}{n-\beta p}}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

between the fractional laplacian $(-\Delta)^{\frac{\beta}{2}}$ and any symbol $b \in W^{\beta,\frac{n}{\beta}}(\mathbb{R}^n)$, provided that $1 < p < \frac{n}{\beta}$.

1 Introduction

A Beltrami coefficient is a function $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty < 1$. By the well-known Measurable Riemann Mapping Theorem, to every compactly supported Beltrami coefficient μ one can associate a unique homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ in the local Sobolev class $W^{1,2}_{loc}$ such that the *Beltrami equation*

$$\partial_{\overline{z}}\phi(z) = \mu(z)\, \partial_z\phi(z)$$

holds for almost every $z \in \mathbb{C}$, and at the same time, $|\phi(z) - z| \to 0$ as $|z| \to \infty$. One usually calls ϕ the principal solution, and it is known to be a K-quasiconformal map with $K = \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty}$, since

$$|\partial_{\overline{z}}\phi(z)| \leq \frac{K-1}{K+1}|\partial_z\phi(z)| \qquad \text{at almost every } z \in \mathbb{C}.$$

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Recent works have shown an interest in describing the Sobolev smoothness of ϕ in terms of that of μ. As noticed already at [5], remarkable differences are appreciated under the assumption $\mu \in W^{\alpha,p}$, depending on if $\alpha p < 2$, $\alpha p = 2$ or $\alpha p > 2$. In this paper, we focus our attention on the case $\alpha p = 2$.

It was proven at [5] that if $\mu \in W^{1,2}$ then ϕ belongs to the local Sobolev space $W^{2,2-\epsilon}_{loc}$ for each $\epsilon > 0$ (and further one cannot take $\epsilon = 0$ in general). The proof was based on the elementary fact that

$$\mu \in W^{1,2} \qquad \Rightarrow \qquad \log(\partial_z \phi) \in W^{1,2}. \tag{1}$$

In particular, $\log \partial_z \phi$ enjoys a slightly better degree of smoothness than $\partial_z \phi$ itself. It is a remarkable fact that this better regularity cannot be deduced only from the fact that $\partial_z \phi \in W^{1,2-\epsilon}_{loc}$ for every $\epsilon > 0$. Somehow, this means that $\log \partial_z \phi$ contains more information than $\partial_z \phi$.

Similar phenomenon had been observed much earlier in the work of Hamilton [6], where it is shown that

$$\mu \in VMO \qquad \Rightarrow \qquad \log(\partial_z \phi) \in VMO. \tag{2}$$

Again, the VMO smoothness of $\log(\partial_z \phi)$ cannot be completely transferred to $\partial_z \phi$ itself. Indeed, the example $\phi(z) = z\,(\log|z| - 1)$, in a neighbourhood of the origin, has VMO Beltrami coefficient (at least locally) but clearly $D\phi \notin VMO$.
The VMO setting is interesting in our context since it can be seen as the limiting space of $W^{\alpha,\frac{2}{\alpha}}$. Certainly, the complex method of interpolation shows that

$$[VMO, W^{1,2}]_\alpha = W^{\alpha,\frac{2}{\alpha}}, \qquad 0 < \alpha < 1$$

(see for instance [12]). Thus, it is natural to ask if a counterpart to implication (1) holds in $W^{\alpha,\frac{2}{\alpha}}$. In the present paper, we prove the following theorem.

Theorem 1. *Let $\alpha \in (\frac{1}{2}, 1)$. Let μ be a Beltrami coefficient with compact support and such that $\mu \in W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$. Let ϕ be the principal solution to the $\mathbb{C}$-linear Beltrami equation*

$$\partial_{\overline{z}} \phi \,=\, \mu\, \partial_z \phi\,.$$

Then, $\log(\partial\phi) \in W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$.

The proof of Theorem 1 is based on two facts. The first one is the following a priori estimate for linear Beltrami equations with coefficients belonging to $W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$.

Theorem 2. *Let $\alpha \in (0,1)$ and $1 < p < \frac{2}{\alpha}$. Let μ, ν be a pair of Beltrami coefficients with compact support, such that $\| |\mu| + |\nu| \|_\infty \le k < 1$ and $\mu, \nu \in W^{\alpha, \frac{2}{\alpha}}(\mathbb{C})$. For every $g \in W^{\alpha,p}(\mathbb{C})$ the equation*

$$\partial_{\bar z} f - \mu\, \partial_z f - \nu\, \overline{\partial_z f} = g$$

admits a solution f with $Df \in W^{\alpha p}(\mathbb{C})$, unique modulo constants, and such that the estimate

$$\|Df\|_{W^{\alpha,p}(\mathbb{C})} \le C\, \|g\|_{W^{\alpha,p}(\mathbb{C})}$$

holds for a constant C depending only on k, $\|\mu\|_{W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})}$ and $\|\nu\|_{W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})}$.

Theorem 2 is sharp, in the sense that one cannot take $p = \frac{2}{\alpha}$. Thus, Theorem 1 shows that $\log \partial_z \phi$ enjoys better regularity than $\partial_z \phi$ itself.

The study of logarithms of derivatives of quasiconformal maps goes back to the work of Reimann [11], where it was shown that the real-valued logarithm $\log |\partial_z \phi| \in BMO$ whenever $\|\mu\|_\infty < 1$. References involving the complex logarithm $\log \partial_z \phi$ also lead to [1]. More recently, in [3] the authors obtained sharp bounds for the BMO norm of $\log \partial_z \phi$ also with the only assumption $\|\mu\|_\infty < 1$.

The second main ingredient in the proof of Theorem 1 is a compactness result for commutators of pointwise multipliers and the fractional laplacian, which holds in higher dimensions and has independent interest. In order to state it, given a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ we denote

$$D^\beta u(x) := \lim_{\epsilon \to 0} C_{n,\beta} \int_{|x-y|>\epsilon} \frac{u(x) - u(y)}{|x-y|^{n+\beta}}\, dy. \tag{3}$$

This is a principal value representation of the fractional laplacian $(-\Delta)^{\frac{\beta}{2}}$, whose symbol at the Fourier side is

$$\widehat{D^\beta u}(\xi) = \widehat{(-\Delta)^{\frac{\beta}{2}} u}(\xi) = |\xi|^\beta\, \hat{u}(\xi).$$

The operator D^β can also be seen as the formal inverse of I_β, the classical Riesz potential of order β, which can be represented as

$$\widehat{I_\beta u}(\xi) = |\xi|^{-\beta}\, \hat{u}(\xi).$$

With this notation, a function u belongs to $W^{\beta,p}$, $1 < p < \infty$, if and only if u and $D^\beta u$ belong to L^p, with the corresponding equivalent norm. Analogously, $u \in \dot{W}^{\beta,p}$ if and only if $D^\beta u \in L^p$.

Let us remind that if T and S are two operators, one usually calls $[T,S] = T \circ S - S \circ T$ the *commutator* of T and S.

Theorem 3. *Let $\beta \in (0,1)$ and $b \in W^{\beta,\frac{n}{\beta}}(\mathbb{R}^n)$. Then, the commutator*

$$[b, D^\beta] : L^{\frac{np}{n-\beta p}}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

is bounded and compact whenever $1 < p < \frac{n}{\beta}$.

The boundedness of the commutator can be seen as a consequence of fractional versions of the Leibnitz rule. For the compactness, the Fréchet-Kolmogorov characterization of compact subsets of L^p is combined with the cancellation properties of the kernel of the commutator. Also, in the proof of Theorem 1 one uses Theorem 3 with $\beta = 1 - \alpha$. This explains the restriction $\alpha > \frac{1}{2}$ in Theorem 1, as what one really uses is that $\mu \in W^{1-\alpha,\frac{2}{1-\alpha}}(\mathbb{C})$. Note that this space contains $W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$ if and only if $\alpha > \frac{1}{2}$.

A detailed proof of Theorem 3 is provided at Section 2. In Section 3, we find a priori estimates for generalized Beltrami equations with coefficients in $W^{\theta,\frac{2}{\theta}}$, and prove Theorem 1 and Theorem 2.

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2 Proof of Theorem 3

The proof of Theorem 3 we present here is based on classical ideas, see for instance [10]. We will need the following auxilliary result about the Leibnitz rule for fractional derivatives.

Proposition 4. (Kenig-Ponce-Vega's Inequality [8])
Let $\beta \in (0,1)$ and $1 < p < \frac{n}{\beta}$. Then the inequality

$$\|D^\beta(f\, g) - f\, D^\beta g\|_p \leq C\, \|D^\beta f\|_{\frac{n}{\beta}}\, \|g\|_{\frac{np}{n-\beta p}}.$$

holds whenever $f, g \in \mathcal{C}^\infty_c(\mathbb{R}^n)$.

With this result at hand, we immediately get that the commutator

$$[b, D^\beta] : L^{\frac{np}{n-\beta p}}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

admits a unique bounded extension. Remarkably,

$$\|[b, D^\beta]\|_{L^{\frac{np}{n-\beta p}}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \leq C\, \|b\|_{\dot{W}^{\beta,\frac{n}{\beta}}(\mathbb{R}^n)}.$$

As a consequence, if $b_n \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is such that

$$\lim_{n\to\infty} \|b_n - b\|_{\dot{W}^{\beta,\frac{n}{\beta}}(\mathbb{R}^n)} = 0$$

then

$$\lim_{n\to\infty} \|[b_n, D^\beta] - [b, D^\beta]\|_{L^{\frac{np}{n-\beta p}}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = 0$$

Thus, we are reduced to prove Theorem 3 with the extra assumption $b \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. To this end, we observe that the commutator $C_b = [b, D^\beta]$ can be represented as an integral operator

$$\begin{aligned} C_b f(x) &= b(x)\, P.V. \int K(x,y)\,(f(x) - f(y))\,dy - P.V. \int K(x,y)\,(f(x)\,b(x) - b(y)\,f(y))\,dy \\ &= P.V. \int K(x,y)\,(b(y) - b(x))\,f(y)\,dy \\ &= \int \mathcal{K}(x,y)\,f(y)\,dy \end{aligned}$$

where

$$\mathcal{K}(x,y) = C_{n,\beta}\,\frac{(b(y) - b(x))}{|y - x|^{n+\beta}}$$

and the principal value has been removed from the last integral because the smoothness of b ensures that $x \mapsto \mathcal{K}(x,y)$ is integrable. For C_b to be compact, we need to prove that the image under C_b of the unit ball of $L^{\frac{np}{n-\beta p}}(\mathbb{R}^n)$ is compact in $L^p(\mathbb{R}^n)$. To this end, we denote

$$\mathcal{F} = \{C_b f :\ \|f\|_{L^{\frac{np}{n-\beta p}}(\mathbb{R}^n)} \le 1\}.$$

The classical Fréchet-Kolmogorov's Theorem asserts that $\mathcal{F}$ is relatively compact if and only if the following conditions hold:

(i) $\mathcal{F}$ is uniformly bounded, i.e. $\sup_{\psi\in\mathcal{F}} \|\psi\|_{L^p(\mathbb{R}^n)} < \infty$.

(ii) $\mathcal{F}$ vanishes uniformly at ∞, i.e. $\sup_{\psi\in\mathcal{F}} \|\psi\,\chi_{|x|>R}\|_{L^p(\mathbb{R}^n)} \to 0$ as $R \to \infty$.

(iii) $\mathcal{F}$ is uniformly equicontinuous, i.e. $\sup_{\psi\in\mathcal{F}} \|\psi(\cdot + h) - \psi(\cdot)\|_{L^p(\mathbb{R}^n)} \to 0$ as $|h| \to 0$.

In our particular case, every element $\psi \in \mathcal{F}$ has the form $\psi = C_b f$ with $\|f\|_{L^{\frac{np}{n-\beta p}}(\mathbb{R}^n)} \le 1$. Thus (i) follows automatically from the boundedness of $[b, D^\beta] : L^{\frac{np}{n-\beta p}}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$.

To prove (ii), let $R_0 > 0$ be such that $\mathrm{supp}(b) \subset B(0, R_0)$. At points x with $|x| > 3R_0$ we have

$$|C_b f(x)| \le \int \frac{|f(y)\,b(y)|}{|x-y|^{n+\beta}}\,dy \le C\frac{\|b\|_\infty}{|x|^{n+\beta}} \int_{B(0,R_0)} |f(y)|\,dy \le C\frac{\|b\|_\infty}{|x|^{n+\beta}} \|f\|_q R_0^{n\frac{q-1}{q}}. \tag{4}$$

Thus, if $R > 3R_0$ then

$$\int_{|x|>R} |C_b f(x)|^p \, dx \le C_R \|b\|_\infty^p \|f\|^p_{\frac{np}{n-\beta p}} \int_{|x|>R} |x|^{-p(n+\beta)} \, dx \to 0 \qquad \text{as } R \to \infty$$

as needed.

For the proof of (iii), we could proceed as usually, which means to regularize the kernel $\mathcal{K}$ in the diagonal $\{x = y\}$. Then we would prove the compactness of this regularization and finally the limit of compact operators would give us the result. However, a more direct approach is available, since $\|\mathcal{K}(x,\cdot)\|_{L^1(\mathbb{R}^n)}$ is uniformly bounded.

Lemma 5. *One has*

$$\lim_{h\to 0} \sup_{f\neq 0} \frac{\|C_b f(\cdot + h) - C_b f(\cdot)\|_{L^q(\mathbb{R}^n)}}{\|f\|_{L^q(\mathbb{R}^n)}} = 0 \tag{5}$$

whenever $1 \le q \le \infty$.

Proof. We start by observing that

$$\begin{aligned}
\|\mathcal{K}(x,\cdot)\|_{L^1(\mathbb{R}^n)} &= \int_{|x-y|\le 1} |\mathcal{K}(x,y)| \, dy + \int_{|x-y|>1} |\mathcal{K}(x,y)| \, dy \\
&\le C\|\nabla b\|_\infty \int_{|x-y|\le 1} |x-y|^{-n-\beta+1} \, dy + C\|b\|_\infty \int_{|x-y|>1} |x-y|^{-n-\beta} \, dy \\
&\le C\left\{ \frac{\|\nabla b\|_\infty}{1-\beta} + \frac{\|b\|_\infty}{\beta} \right\} := A
\end{aligned}$$

As a consequence, the behavior of $C_b f$ is like the convolution of the function f with a L^1-kernel. In particular, by Jensen's inequality one gets

$$\|C_b f\|_q \le A\|f\|_q, \qquad 1 \le q \le \infty, \tag{6}$$

so that $C_b : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, $1 \le q \le \infty$.

Towards (5), we need to estimate the translates of C_b. Clearly,

$$\begin{aligned}
\|C_b f(\cdot + h) - C_b f(\cdot)\|_q^q &= \int \left| \int f(y)(\mathcal{K}(x+h,y) - \mathcal{K}(x,y)) \, dy \right|^q dx \\
&\le \int \left(\int |f(y)|^q \, |\mathcal{K}(x+h,y) - \mathcal{K}(x,y)| \, dy \right) \left(\int |\mathcal{K}(x+h,y) - \mathcal{K}(x,y)| \, dy \right)^{\frac{q}{q'}} dx \\
&\le (2A)^{q-1} \int \left(\int |\mathcal{K}(x+h,y) - \mathcal{K}(x,y)| \, dx \right) |f(y)|^q \, dy \\
&= (2A)^{q-1} \, B(h) \int |f(y)|^q dy
\end{aligned}$$

where $B(h) = \sup_y \|\mathcal{K}(\cdot + h, y) - \mathcal{K}(\cdot, y)\|_{L^1(\mathbb{R}^n)}$. In order to find estimates for $B(h)$, we choose an arbitrary $\rho > 0$ and write

$$\int |\mathcal{K}(x+h,y) - \mathcal{K}(x,y)|\, dx = \int_{|x-y|\leq\rho} \cdots + \int_{|x-y|>\rho} \cdots := I + II.$$

The integrability of $\mathcal{K}$ gives that I is small if ρ is small enough. Indeed,

$$\int_{|x-y|\leq\rho} |\mathcal{K}(x,y)|\, dx \leq \|\nabla b\|_\infty \int_{|x-y|\leq\rho} |x-y|^{-n-\beta+1}\, dx = C\frac{\|\nabla b\|_\infty}{1-\beta}\rho^{1-\beta}.$$

Moreover, if $x \in B(y,\rho)$ then $x + h \in B(y, \rho + |h|)$ so that

$$\int_{|x-y|\leq\rho} |\mathcal{K}(x+h,y)|\, dx \leq \int_{|x-(y-h)|\leq 2\rho} |\mathcal{K}(x+h,y)|\, dx \leq C\frac{\|\nabla b\|_\infty}{1-\beta}(\rho+|h|)^{1-\beta}.$$

Therefore, there exists $\rho_0 > 0$ such that if $\rho < \rho_0$ and $|h| < \rho_0/2$ then $I \leq \varepsilon/((2A)^{q-1})$. Let us then fix $\rho = \rho_0/2$, and take care of II. Note that, since $|h| < \rho_0/2$ and $|x - y| > \rho$, we have

$$\begin{aligned}|\mathcal{K}(x,+hy) - \mathcal{K}(x,y)| &= \Bigg|(b(y) - b(x+h))\left(\frac{1}{|x+h-y|^{n+\beta}} - \frac{1}{|x-y|^{n+\beta}}\right) \\ &\quad + \frac{1}{|x-y|^{n+\beta}}(b(x) - b(x+h))\Bigg| \\ &\leq 2\|b\|_\infty \frac{C|h|}{|x-y|^{n+\beta+1}} + \|\nabla b\|_\infty \frac{|h|}{|x-y|^{n+\beta}}\end{aligned}$$

Then, since we fixed $\rho = \rho_0/2$,

$$\begin{aligned}II &\leq C\|b\|_\infty |h| \int_{|x-y|>\rho} \frac{dx}{|x-y|^{n+\beta+1}} + C\|\nabla b\|_\infty |h| \int_{|x-y|>\rho} \frac{dx}{|x-y|^{n+\beta}} \\ &\leq C\frac{|h|}{\beta}\left(\frac{\|b\|_\infty}{\rho_0^{1+\beta}} + \frac{\|\nabla b\|_\infty}{\rho_0^{\beta}}\right).\end{aligned}$$

Thus, by taking $|h|$ sufficiently small, we see that $II \leq \varepsilon/((2A)^{q-1})$. Hence $B(h) \to 0$ as $|h| \to 0$, and thus (5) follows. $\square$

With the above Lemma, the proof of (iii) is almost immediate. Indeed, by (4) we see that

$$\begin{aligned}\|C_b f(\cdot + h) - C_b f(\cdot)\|_p^p &= \int_{|x|\leq R} |C_b f(x+h) - C_b f(x)|^p\, dx \\ &\quad + \int_{|x|>R} |C_b f(x+h) - C_b f(x)|^p\, dx \\ &\leq \|C_b f(\cdot + h) - C_b f(\cdot)\|^p_{\frac{np}{n-\beta p}} R^{\beta p} \\ &\quad + C_R \|b\|_\infty^p \|f\|^p_{\frac{np}{n-\beta p}} \int_{|x|>R} |x|^{-p(n+\beta)}\, dx.\end{aligned}$$

at least for $R > 3R_0$. In particular, the last term is small if R is large enough. But for this particular R, and using (5), the penultimate term is also small if $|h|$ is small. Therefore (iii) follows. Theorem 3 is proved.

3 Beltrami operators in fractional Sobolev spaces

The regularity theory for Beltrami equations relies on the behavior of the Beurling operator, which is formally defined as a principal value operator,

$$\mathcal{B}f(z) = -\frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{C}} f(z-w)\frac{1}{w^2}\,dA(w).$$

This operator intertwines the ∂_z and $\partial_{\overline{z}}$ derivatives. More precisely, its Fourier representation

$$\widehat{\mathcal{B}f}(\xi) = \frac{\overline{\xi}}{\xi}\,\hat{f}(\xi).$$

makes it clear that $\mathcal{B}(\partial_{\overline{z}}f) = \partial_z f$, at least when f is smooth and compactly supported. Furthermore, $\mathcal{B}$ is an isometry on $L^2(\mathbb{C})$, and as a Calderón-Zygmund operator, it can be boundedly extended to $L^p(\mathbb{C})$ whenever $1 < p < \infty$.

Before proving Theorem 1, we first state and prove the following fact about generalized Beltrami equations. Let us recall that $\overline{\mathcal{B}}$ denotes the composition of $\mathcal{B}$ with the complex conjugation operator, that is, $\overline{\mathcal{B}}(f) = \overline{\mathcal{B}(f)}$.

Proposition 6. *Let $\alpha \in (0,1)$. Let $\mu, \nu \in W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$ be compactly supported Beltrami coefficients, with $\||\mu| + |\nu|\|_\infty \le k < 1$. Then the generalized Beltrami operators*

$$\mathbf{Id} - \mu\,\mathcal{B} - \nu\overline{\mathcal{B}} : \dot{W}^{\alpha,p}(\mathbb{C}) \to \dot{W}^{\alpha,p}(\mathbb{C})$$

are bounded and boundedly invertible if $1 < p < \frac{2}{\alpha}$.

Proof. The operators $\mathbf{Id} - \mu\,\mathcal{B} - \nu\,\overline{\mathcal{B}}$ are clearly bounded in $\dot{W}^{\alpha,p}(\mathbb{C})$, since $\mathcal{B}$ preserves $\dot{W}^{\alpha,p}(\mathbb{C})$ (recall that we are assuming $1 < p < \frac{2}{\alpha}$) and also because if $\mu \in L^\infty(\mathbb{C}) \cap W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$ then μ is a pointwise multiplier of $\dot{W}^{\alpha,p}(\mathbb{C})$ (similarly for ν). This fact follows directly working on the expression (3) for D^α or see [13, p. 250]. Also, the operator $\mathbf{Id} - \mu\,\mathcal{B} - \nu\,\overline{\mathcal{B}}$ is clearly injective in $\dot{W}^{\alpha,p}(\mathbb{C})$, as its kernel is a subset of $L^{\frac{2p}{2-\alpha p}}(\mathbb{C})$ were we already know it is injective (see [7] for a proof in the $\mathbb{C}$-linear setting, and [9] or also [4] for a proof in the general case). Thus, in order to get the surjectivity (and finish the proof by the Open Mapping Theorem) we will prove that $\mathbf{Id} - \mu\,\mathcal{B} - \nu\,\overline{\mathcal{B}}$ is a Fredholm operator on $\dot{W}^{\alpha,p}(\mathbb{C})$ with index 0. To do this, it is sufficient if we prove that

$$D^\alpha(\mathbf{Id} - \mu\,\mathcal{B} - \nu\,\overline{\mathcal{B}})I_\alpha : L^p(\mathbb{C}) \to L^p(\mathbb{C})$$

is a Fredholm operator of index 0, since both properties stay invariant under the topological isomorphisms

$$D^\alpha : \dot{W}^{\alpha,p}(\mathbb{C}) \to L^p(\mathbb{C}),$$
$$I_\alpha : L^p(\mathbb{C}) \to \dot{W}^{\alpha,p}(\mathbb{C}).$$

But this follows easily. Indeed,

$$\begin{aligned}D^{\alpha}(\mathbf{Id}-\mu\,\mathcal{B}-\nu\,\overline{\mathcal{B}})I_{\alpha}&=\mathbf{Id}-D^{\alpha}(\mu\,\mathcal{B}+\nu\,\overline{\mathcal{B}})I_{\alpha}\\&=\mathbf{Id}-\mu\,\mathcal{B}-\nu\,\overline{\mathcal{B}}-[D^{\alpha},\mu]\,\mathcal{B}\,I_{\alpha}-[D^{\alpha},\nu]\,\overline{\mathcal{B}}\,I_{\alpha}\end{aligned}$$

Above, $\mathbf{Id}-\mu\,\mathcal{B}-\nu\,\overline{\mathcal{B}}$ is invertible in $L^p(\mathbb{C})$ by [7]. Also, $[D^{\alpha},\mu]\,\mathcal{B}\,I_{\alpha}$ is the composition of the bounded operators $I_{\alpha}:L^p(\mathbb{C})\to L^{\frac{2p}{2-\alpha p}}(\mathbb{C})$ and $\mathcal{B}:L^{\frac{2p}{2-\alpha p}}(\mathbb{C})\to L^{\frac{2p}{2-\alpha p}}(\mathbb{C})$ with the operator $[D^{\alpha},\mu]:L^{\frac{2p}{2-\alpha p}}(\mathbb{C})\to L^p(\mathbb{C})$, which is compact by Theorem 3. Hence $[D^{\alpha},\mu]\,\mathcal{B}\,I_{\alpha}:L^p(\mathbb{C})\to L^p(\mathbb{C})$ is compact, and the same happens to $[D^{\alpha},\nu]\,\overline{\mathcal{B}}\,I_{\alpha}$. Thus the term on the right hand side is the sum of an invertible operator with two compact operators. Hence it is a Fredholm operator. The claim follows. □

Corollary 7. *Let $\alpha\in(0,1)$ and $1<p<\frac{2}{\alpha}$, and let $\mu,\nu\in W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$ be compactly supported Beltrami coefficients with $\||\mu|+|\nu|\|_{\infty}\le k<1$. Then the equation*

$$\partial_{\bar z}f-\mu\,\partial_z f-\nu\overline{\partial_z f}=g \tag{7}$$

has, for each $g\in\dot W^{\alpha,p}(\mathbb{C})$, a unique solution f such that $Df\in\dot W^{\alpha,p}(\mathbb{C})$ and

$$\|Df\|_{W^{\alpha,p}(\mathbb{C})}\le C\,\|g\|_{W^{\alpha,p}(\mathbb{C})}$$

where C depends only on k, $\|\mu\|_{W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})}$ and $\|\nu\|_{W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})}$.

Proof. By simplicity, we assume that $\nu=0$. Otherwise, the proof follows similarly. First of all, let us observe that if $g\in\dot W^{\alpha,p}(\mathbb{C})$ and $\alpha p<2$ then automatically $g\in L^{\frac{2p}{2-\alpha p}}(\mathbb{C})$ by the Sobolev embedding. On the other hand, and since $W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})\subset VMO$, we know from [7] that a solution $f\in\dot W^{1,\frac{2p}{2-\alpha p}}(\mathbb{C})$ exists, and moreover

$$\|Df\|_{L^{\frac{2p}{2-\alpha p}}(\mathbb{C})}\le C\,\|g\|_{L^{\frac{2p}{2-\alpha p}}(\mathbb{C})}\le C\,\|g\|_{\dot W^{\alpha,p}(\mathbb{C})}.$$

Our goal consists of replacing the term on the left hand side by $\|Df\|_{\dot W^{\alpha,p}(\mathbb{C})}$.

To do this, we first note that $\partial_z f=\mathcal{B}(\partial_{\bar z}f)$, since $f\in\dot W^{1,\frac{2p}{2-\alpha p}}$. Thus (7) is equivalent to

$$(\mathbf{Id}-\mu\,\mathcal{B})(\partial_{\bar z}f)=g$$

Now, from Proposition 6 and our assumption $g\in\dot W^{\alpha,p}(\mathbb{C})$, we also know that there is a unique $F\in\dot W^{\alpha,p}(\mathbb{C})$ such that

$$(\mathbf{Id}-\mu\,\mathcal{B})F=g \tag{8}$$

for which we know the estimate $\|F\|_{\dot W^{\alpha,p}(\mathbb{C})}\le C\,\|g\|_{\dot W^{\alpha,p}(\mathbb{C})}$ holds. Of course, by the Sobolev embedding, $F\in L^{\frac{2p}{2-\alpha p}}(\mathbb{C})$. From the invertibility of $\mathbf{Id}-\mu\,\mathcal{B}$ on $L^{\frac{2p}{2-\alpha p}}(\mathbb{C})$, we immediately get that $F = \partial_{\bar{z}} f$ almost everywhere, and therefore $\partial_{\bar{z}} f \in \dot{W}^{\alpha,p}(\mathbb{C})$. Proving that $\partial_z f \in \dot{W}^{\alpha,p}(\mathbb{C})$ is very easy, as we already knew that $f \in \dot{W}^{1,\frac{2p}{2-\alpha p}}(\mathbb{C})$ and so we can be sure that $\partial_z f = \mathcal{B}(\partial_{\bar{z}} f)$. Thus, $Df \in \dot{W}^{\alpha,p}(\mathbb{C})$ and certainly

$$\|Df\|_{\dot{W}^{\alpha,p}(\mathbb{C})} \leq C \, \|F\|_{\dot{W}^{\alpha,p}(\mathbb{C})} \leq C \, \|g\|_{\dot{W}^{\alpha,p}(\mathbb{C})}$$

as desired. $\square$

Towards the proof of Theorem 1, we denote by $\mathsf{C}(h)$ the solid Cauchy transform,

$$\mathsf{C}\, h(z) = \frac{1}{\pi} \int_{\mathbb{C}} h(z-w) \frac{1}{w} \, dA(w). \tag{9}$$

This operator appears naturally as a formal inverse to the $\partial_{\bar{z}}$ derivative, that is, the formula $\partial_{\bar{z}} \mathsf{C}(h) = h$ holds if $h \in L^p(\mathbb{C})$ and $1 < p < \infty$. Another important feature about the Cauchy transform is that $\partial \mathsf{C} = \mathcal{B}$. The Cauchy and Beurling transforms allow for a nice representation of the principal solution ϕ of the Beltrami equation $\partial_{\bar{z}} \phi = \mu \, \partial_z \phi$,

$$\phi(z) = z + \mathsf{C}(h)(z),$$

see for instance [2, p. 165]. In this representation, h is a solution to the integral equation

$$(\mathbf{Id} - \mu \, \mathcal{B})(h) = \mu.$$

As a consequence, the invertibility of the Beltrami operators $\mathbf{Id} - \mu \, \mathcal{B}$ also plays a central role in determining the smoothness of ϕ. In particular, by applying Proposition 6 with $\mu \in W^{\alpha,\frac{2}{\alpha}}$, we see that $Dh \in W^{\alpha,p}$ provided that $p < \frac{2}{\alpha}$, whence $D\phi \in W^{\alpha,p}_{loc}$. As a consequence, by Stoilow's Factorization Theorem (e.g., [2, section 5.5]), the same conclusion holds for any quasiregular solution f of $\partial_{\bar{z}} f - \mu \, \partial_z f = 0$. However, this is not enough for Theorem 1, which we prove now.

Proof of Theorem 1. We will first prove that if $\mu \in W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$ is a compactly supported Beltrami coefficient and $\alpha > \frac{1}{2}$ (this is the point where we use that restriction) the operator

$$T_\mu := I_{1-\alpha} \, (\mathbf{Id} - \mu \, \mathcal{B}) \, D^{1-\alpha} : L^{\frac{2}{\alpha}} \, (\mathbb{C}) \longmapsto L^{\frac{2}{\alpha}} \, (\mathbb{C})$$

is continuously invertible, with lower bounds depending only on $\|\mu\|_{L^\infty(\mathbb{C})}$ and $\|\mu\|_{W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})}$. To do this, we proceed as usually,

$$\begin{aligned} T_\mu = I_{1-\alpha}(\mathbf{Id} - \mu \, \mathcal{B}) D^{1-\alpha} &= \mathbf{Id} - I_{1-\alpha} \mu \mathcal{B} D^{1-\alpha} \\ &= \mathbf{Id} - \mu \, \mathcal{B} + I_{1-\alpha} \, [D^{1-\alpha}, \mu] \, \mathcal{B}. \end{aligned}$$

Here, the term $\mathbf{Id} - \mu \, \mathcal{B}$ is bounded and continuously invertible in $L^{\frac{2}{\alpha}}(\mathbb{C})$ by [7]. Concerning the second term on the right hand side, from $\mu \in W^{\alpha,\frac{2}{\alpha}}(\mathbb{C}) \cap L^\infty(\mathbb{C})$ and $\frac{1}{2} < \alpha$ we easily get that $\mu \in W^{1-\alpha,\frac{2}{1-\alpha}}(\mathbb{C})$. Thus we are legitimate to use Theorem 3 with $\beta = 1-\alpha$ and $p = \frac{2}{\alpha}$ and get that $[\mu, D^{1-\alpha}]$ is a compact operator from $L^{\frac{2}{\alpha}}(\mathbb{C})$ into $L^2(\mathbb{C})$. As a consequence, we obtain that T_μ is a Fredholm operator from $L^{\frac{2}{\alpha}}(\mathbb{C})$ into itself, which clearly has index 0. So the desired lower bounds will be automatic if we see that it is injective.

Let $F \in L^{\frac{2}{\alpha}}$ such that $T_\mu(F) = 0$. We want to show that $F = 0$. First, if $F \in \dot{W}^{1-\alpha,2}(\mathbb{C})$ then the result follows easily. Indeed, we can then write $F := I_{1-\alpha} f$ for some $f \in L^2$ and write the equation in terms of f. We get $I_{1-\alpha}(\mathbf{Id} - \mu\,\mathcal{B})f = 0$. From the classical L^2 theory, we have that $f = 0$ and hence $F = 0$. For a general $F \in L^{\frac{2}{\alpha}}$ satisfying $T_\mu(F) = 0$ we will prove that necessarily $F \in \dot{W}^{1-\alpha,2}(\mathbb{C})$, and therefore $F = 0$. To do this, again we decompose T_μ in terms of the commutator,

$$(\mathbf{Id} - \mu\,\mathcal{B})F = I_{1-\alpha}[\mu, D^{1-\alpha}]\mathcal{B}F.$$

Then by Theorem 3 the term on the right hand side above belongs to $\dot{W}^{1-\alpha,2}(\mathbb{C})$, because $F \in L^{\frac{2}{\alpha}}(\mathbb{C})$. Using again that $\alpha > \frac{1}{2}$ one has $\mu \in W^{1-\alpha,\frac{2}{1-\alpha}}(\mathbb{C})$, and therefore we can use Proposition 6 to get that $\mathbf{Id} - \mu\,\mathcal{B} : \dot{W}^{1-\alpha,2}(\mathbb{C}) \to \dot{W}^{1-\alpha,2}(\mathbb{C})$ is continuously invertible. Hence

$$F = (\mathbf{Id} - \mu\,\mathcal{B})^{-1} I_{1-\alpha}[\mu, D^{1-\alpha}]\mathcal{B}F$$

belongs to $\dot{W}^{1-\alpha,2}(\mathbb{C})$. The claim follows.

We now finish the proof. Given $\mu \in W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$, we approximate it by $\mu_n \in \mathcal{C}_c^\infty(\mathbb{C})$ in the $W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$ topology, in such a way that $\|\mu_n\|_{L^\infty(\mathbb{C})} \le \|\mu\|_{L^\infty(\mathbb{C})}$. Then every μ_n admits a principal quasiconformal map ϕ_n, for which the function $g_n = \log \partial_z \phi_n$ is well defined and solves

$$\partial_{\bar{z}} g_n - \mu_n\,\partial_z g_n = \partial_z \mu_n.$$

Therefore

$$(\mathbf{Id} - \mu_n \mathcal{B})\partial_{\bar{z}} g_n = \partial_z \mu_n.$$

We use the Fourier representation of the classical Riesz transforms in $\mathbb{R}^2$,

$$\widehat{\mathcal{R}_j u}\,(\xi) = -i\frac{\xi_j}{|\xi|}\widehat{u}\,(\xi) \qquad j = 1, 2$$

to represent

$$\partial_{\bar{z}} g = -\pi D^{1-\alpha}(\mathcal{R}_1 + i\mathcal{R}_2)(D^\alpha g)$$
$$\partial_z g = -\pi D^{1-\alpha}(\mathcal{R}_1 - i\mathcal{R}_2)(D^\alpha g).$$

As a consequence, we obtain

$$(\mathbf{Id} - \mu_n\,\mathcal{B})D^{1-\alpha}(\mathcal{R}_1 + i\mathcal{R}_2)(D^\alpha g_n) = D^{1-\alpha}(\mathcal{R}_1 - i\mathcal{R}_2)(D^\alpha \mu_n),$$

and therefore

$$T_{\mu_n}(\mathcal{R}_1 + i\mathcal{R}_2)(D^\alpha g_n) = (\mathcal{R}_1 - i\mathcal{R}_2)(D^\alpha \mu_n).$$

We recall that both $\mathcal{R}_1 + i\mathcal{R}_2$ and $\mathcal{R}_1 - i\mathcal{R}_2$ are bounded and continuously invertible operators in $L^p(\mathbb{C})$, $1 < p < \infty$. Moreover, we have just seen that T_{μ_n} is boundedly invertible in $L^{\frac{2}{\alpha}}(\mathbb{C})$ with bounds depending only on $\|\mu_n\|_{L^\infty(\mathbb{C})}$ and $\|\mu_n\|_{W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})}$. However, each $\|\mu_n\|_\infty$ (and respectively $\|\mu_n\|_{W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})}$) is bounded by a constant multiple of $\|\mu\|_\infty$ (respectively $\|\mu\|_{W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})}$). Hence

$$\begin{aligned}
\|g_n\|_{\dot{W}^{\alpha,\frac{2}{\alpha}}(\mathbb{C})} &= \|D^\alpha g_n\|_{L^{\frac{2}{\alpha}}(\mathbb{C})} \\
&\leq C(\alpha)\, \|(\mathcal{R}_1 + i\mathcal{R}_2) D^\alpha g_n\|_{L^{\frac{2}{\alpha}}(\mathbb{C})} \\
&\leq C\left(\alpha, \|\mu\|_{L^\infty(\mathbb{C})}, \|\mu\|_{W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})}\right) \|T_{\mu_n}(\mathcal{R}_1 + i\mathcal{R}_2)(D^\alpha g_n)\|_{L^{\frac{2}{\alpha}}(\mathbb{C})} \\
&\leq C\left(\alpha, \|\mu\|_{L^\infty(\mathbb{C})}, \|\mu\|_{W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})}\right) \|(\mathcal{R}_1 - i\mathcal{R}_2) D^\alpha \mu_n\|_{L^{\frac{2}{\alpha}}(\mathbb{C})} \\
&\leq C\left(\alpha, \|\mu\|_{L^\infty(\mathbb{C})}, \|\mu\|_{W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})}\right).
\end{aligned}$$

It then follows that g_n is a bounded sequence in $\dot{W}^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$. By the Banach-Alaoglu theorem there exists $h \in \dot{W}^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$ such that

$$\lim_{n\to\infty} \langle g_n, \varphi \rangle = \langle h, \varphi \rangle$$

for each $\varphi \in W^{-\alpha,\frac{2}{2-\alpha}}(\mathbb{C})$. Remarkably, by the weak lower semicontinuity of the norm,

$$\|h\|_{\dot{W}^{\alpha,\frac{2}{\alpha}}(\mathbb{C})} = \|D^\alpha h\|_{L^{\frac{2}{\alpha}}(\mathbb{C})} \leq \liminf_{n\to\infty} \|D^\alpha g_n\|_{L^{\frac{2}{\alpha}}(\mathbb{C})} \leq C\left(\alpha, \|\mu\|_{L^\infty(\mathbb{C})}, \|\mu\|_{W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})}\right).$$

Incidentally, we already knew from the classical theory that ϕ_n converges in $W^{1,p}_{loc}(\mathbb{C})$ to the principal quasiconformal map ϕ associated to μ. In particular, modulo subsequences, $\partial_z \phi_n$ converges to $\partial_z \phi$ almost everywhere. But then g_n converges almost everywhere to $\log(\partial_z \phi)$. It then follows that $\log(\partial_z \phi) = h$ and so we deduce that $\log(\partial_z \phi)$ belongs to $\dot{W}^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$, with the same bound than h. The theorem follows. □

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