LINEAR COMBINATIONS OF HYPERSURFACES IN HYPERBOLIC SPACE

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Abstract. In this article we introduce and investigate linear combinations of hypersurfaces in hyperbolic space. For this purpose we use some linear structure in the space of horospheres.

1. Introduction

In Euclidean space, when we fix a point as an origin we obtain a linear structure. This enables to build linear combinations of geometric objects. Changing the origin translates the results, therefore this construction is invariant up to translations. For the particular case of convex bodies this gives the classical Minkowski linear combinations. They play a fundamental role in Convex Geometry leading for example to the notion of mixed volumes.

The analogue procedure does not work in hyperbolic space $\mathbb{H}^n$ because there is no such underlying linear structure. Previous attempts to define a Minkowski addition in hyperbolic space were based on the choice of an origin, but changing the origin did not lead to isometric results (cf. [Lei03]).

The aim of this work is to investigate invariant operations with geometric objects in $\mathbb{H}^n$. For this purpose we use horospheres, i.e. spheres centered at infinity, because the space of horospheres $\mathcal{H}$ is a half-cone with an invariant linear structure on each generator. We deal with geometric objects in $\mathbb{H}^n$ which are envelopes of horospheres. These enveloping horospheres define a subset of the half-cone $\mathcal{H}$ which we call the associated support images (cf. Definition 3.1). In this way we use

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the linear structure in $\mathcal{H}$ in order to build linear combinations in the hyperbolic space $\mathbb{H}^n$.

We define a scalar multiplication, the *h-dilation*, and two additions, the *sum* and the *harmonic sum*. These new constructions are invariant with respect to the group of hyperbolic motions. They only depend on the geometric objects involved and their relative position to one another.

The *h-dilation* is just scalar multiplication by positive factors of the support image along the rays of the half-cone $\mathcal{H}$. Its geometrical meaning is related to the classical parallel sets.

The *sum* is given by the addition of the support images in the rays of the half-cone $\mathcal{H}$.

The *harmonic sum* is given essentially by the harmonic mean of the support images in the rays of $\mathcal{H}$.

Next we outline some of the results. In the two dimensional situation, we get for instance the length L of the sum $c_1 + c_2$ of two curves:

$$L(c_1 + c_2) = \frac{1}{2}\left(-W(c_1, c_1 + c_2) + L(c_1) - TC(c_1) + L(c_2) - TC(c_2)\right)$$

where TC is the total curvature and W is a mixed term depending on the relative position of the two curves (cf. Proposition 4.2).

Similar formulas hold for the harmonic sum. Furthermore we obtain the following property: the harmonic sum of two spheres is a sphere (cf. Proposition 4.3). Moreover the harmonic sum of two h-convex bodies is again h-convex (cf. Proposition 4.4).

So far we deal with invariant constructions. These operations behave rather differently from the euclidean case. To compare the new operations with the classical euclidean case, we fix a point and use associated support functions. Adding support functions then defines an operation which is in some sense closer to the classical Minkowski sum but not invariant (cf. Section 4.4).

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2. Preliminaries

We use the Lorentz space model for the Hyperbolic Geometry. The model lives in the Lorentz space $\mathbb{R}^{n+1}_1$ with its Lorentz product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n - x_{n+1}y_{n+1}\,.$$

Therein the n-dimensional hyperbolic space $\mathbb{H}^n$ is realized as

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = -1 \ \wedge \ x_{n+1} > 0\}\,,$$

which is the upper half of a two-sheeted hyperboloid with the light cone $\mathcal{C}^n = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = 0\}$ as asymptotic cone. The group G of hyperbolic motions of $\mathbb{H}^n$ is given by the subgroup of the Lorentz group leaving invariant $\mathbb{H}^n$.
The infinite or ideal boundary $\mathbb{H}^n_\infty$ of $\mathbb{H}^n$ is realized as the boundary of the projective closure of $\mathbb{H}^n$, or equivalently the boundary of the projective closure of $\mathcal{C}^n$ in the projective enlargement of $\mathbb{R}^{n+1}_1$. It is a $(n-1)$-dimensional sphere and it inherits a conformal structure invariant with respect to G.

Horospheres in $\mathbb{H}^n$ may be seen as limits of distance spheres through some given point which centers run on a geodesic towards infinity. In the model distance spheres are realized by intersections of $\mathbb{H}^n$ with space-like affine hyperplanes. Therefore horospheres are realized by intersections of $\mathbb{H}^n$ with affine hyperplanes parallel to tangent hyperplanes of $\mathcal{C}^n$.
The space $\mathcal{H}$ of horospheres of $\mathbb{H}^n$ is represented by the upper half of the light cone, i.e.

$$\mathcal{H} = \mathcal{C}^n_+ = \{x \in \mathbb{R}^{n+1}_1 : \langle x, x\rangle = 0 \ \wedge \ x_{n+1} > 0\,.\}$$

Indeed, given $\theta \in \mathcal{C}^n_+$, the affine hyperplane $\Theta = \{x \in \mathbb{R}^{n+1}_1 : \langle x, \theta\rangle = -1\}$ is parallel to the tangent hyperplane $T_\theta\mathcal{C}^n_+ = \{x \in \mathbb{R}^{n+1}_1 : \langle x, \theta\rangle = 0\}$ of $\mathcal{C}^n_+$ at θ. Therefore Θ intersects $\mathbb{H}^n$ in a horosphere which we also denote by Θ. Vice versa, given a horosphere Θ as the intersection of $\mathbb{H}^n$ with an affine hyperplane Θ parallel to a hyperplane tangent to $\mathcal{C}^n_+$ along a half light-ray, there exists exactly one θ in this half light-ray such that $\Theta = \{x \in \mathbb{R}^{n+1}_1 : \langle x, \theta\rangle = -1\}$. (In the following we shall always denote horospheres in $\mathbb{H}^n$, or the underlying affine hyperplanes respectively, by capital Greek letters and the vectors in $\mathcal{C}^n_+$ representing them by the corresponding small Greek letters.) The correspondence between θ and the hyperplane Θ comes exactly from the polarity relation with respect to the quadric $\pm\mathbb{H}^n \subset \mathbb{R}^{n+1}_1$. The Lorentz product induces a degenerated product (isotropic metric) on $\mathcal{C}^n_+$.
The light-rays in the cone $\mathcal{C}^n_+$ represent the pencils of "parallel" horospheres. Two parallel horospheres Θ_1 and Θ_2 touch one another at a point at infinity, and they lie in constant hyperbolic distance to each other. A little computation in the model shows that this distance is equal to $|\ln\lambda|$, where $\lambda \in \mathbb{R}^+$ is given by $\theta_2 = \lambda\theta_1$. Here we shall use the signed distance from Θ_1 to Θ_2 given by

$$\mathrm{d}(\Theta_1, \Theta_2) = -\ln\lambda\,. \tag{2.1}$$

For fixed Θ_1, as $\lambda \to +\infty$ the horospheres Θ_2 "shrink" to the common point at infinity whereas the signed distance $\mathrm{d}(\Theta_1, \Theta_2) \to -\infty$. On the other side, if $\lambda \to 0$, then Θ_2 expands over the whole $\mathbb{H}^n$ and $\mathrm{d}(\Theta_1, \Theta_2) \to +\infty$.

To complete the geometric picture, let us look at horospheres tangent to a given distance sphere or a given equidistant respectively.
First, the horospheres through a given point $p \in \mathbb{H}^n$ are represented by the intersection of $T_p\mathbb{H}^n$ with $\mathcal{C}_+^n$. Or equivalently, through polarity p and θ are linked by $\langle p, \theta \rangle = -1$. Now a distance sphere Σ in $\mathbb{H}^n$ with center p is the intersection of $\mathbb{H}^n$ with an affine space-like hyperplane (also denoted by Σ) parallel to $T_p\mathbb{H}^n$. Then the bundle of horospheres tangent to Σ is represented by the intersection of $\mathcal{C}_+^n$ with two suitable hyperplanes Θ_1, Θ_2 parallel to Σ. More precisely, one of these hyperplanes does not intersect $\mathbb{H}^n$, and the associated horospheres envelope Σ in such a way that Σ lies in their convex sides. The other hyperplane intersects $\mathbb{H}^n$, and the associated horospheres envelope Σ in such a way that Σ lies in their concave sides. Equivalently through polarity, Θ_1 and Θ_2 are determined by the intersection of $\mathcal{C}_+^n$ with the cone tangent to $\mathbb{H}^n$ along Σ (the vertex of this tangent cone is the pole of the hyperplane Σ with respect to $\mathbb{H}^n$). To see this, one may look at Σ in the special position with center $(0, ..., 0, 1)$ using symmetry arguments; the general case then follows by the transitivity of the group of hyperbolic motions. Vice versa, given a space-like hyperplane intersecting $\mathcal{C}_+^n$, the corresponding horospheres envelope a distance sphere in $\mathbb{H}^n$. Moreover by the limit procedure, horospheres tangent to a given horosphere Θ are represented by the intersection of $\mathcal{C}_+^n$ with a suitable hyperplane parallel to Θ. Vice versa, given a hyperplane of degenerated type intersecting $\mathcal{C}_+^n$, the associated horospheres envelope a horosphere.
Second, an equidistant hypersurface E to some totally geodesic $\mathbb{H}^{n-1}$ in $\mathbb{H}^n$ is given by the intersection of $\mathbb{H}^n$ with an affine time-like hyperplane (also denoted by E). Here, $\mathbb{H}^{n-1}$ is determined by the intersection of $\mathbb{H}^n$ with the hyperplane parallel to E and through $(0, ..., 0)$. Then, the two bundles of horospheres enveloping E are represented by the intersection of $\mathcal{C}_+^n$ with two suitable hyperplanes Θ_1, Θ_2 parallel to E. Or equivalently through polarity, Θ_1 and Θ_2 are determined by the intersection of $\mathcal{C}_+^n$ with the cone tangent to $\mathbb{H}^n$ along E. Vice versa, given a time-like hyperplane, the associated horospheres envelope an equidistant to a totally geodesic $\mathbb{H}^{n-1}$ in $\mathbb{H}^n$.

3. Support maps and envelopes

3.1. **Support maps.** Our bridge between the point space $\mathbb{H}^n$ and the space of horospheres $\mathcal{C}^n_+$ is the following.

Definition 3.1. Let M be a smooth (i.e. C^∞-differentiable) regular hypersurface in $\mathbb{H}^n$ and $\nu(x)$, $x \in M$, a unit normal vector field along M. Then $\theta(x) = x + \nu(x) \in \mathcal{C}^n_+$ represents the horosphere $\Theta(x)$ which is tangent to M at x such that $\nu(x)$ points into its convex side. We call

$$\theta : M \longrightarrow \mathcal{C}^n_+ \quad , \quad x \mapsto x + \nu(x) \tag{3.1}$$

the *support map of M with respect to ν.*

Remark 3.1. Our definition of support maps corresponds to the notions in [Sch02] and in [IPS03] ("hyperbolic Gauss indicatrix").

If $e_1, \ldots, e_{n-1}$ denotes an orthonormal principal basis at $x \in M$, then we have $d\theta(e_i) = (1 - k_i)e_i$ where k_i is the corresponding principal curvature with respect to ν. And the area element of $\theta(M)$ is

$$dA_\theta = (1 - k_1) \cdots (1 - k_{n-1}) dA_x \tag{3.2}$$

($dA_x =$ area element of M at x). This shows that the support map is an immersion if and only if M has no principal curvature with respect to ν which is equal to one. We see also that the principal curvatures of horospheres with respect to the inner normal are all equal to one

3.2. **Envelopes.** Let $\theta(v_1, \ldots, v_{n-1})$ be a smooth immersion into the cone $\mathcal{C}^n_+$, such that it is transverse to the generators of $\mathcal{C}^n_+$. We look for an envelope x of Θ in $\mathbb{H}^n$, i.e. a smooth $x(v_1, \ldots, v_{n-1})$ with

$$\begin{aligned} \langle x, x \rangle &= -1, \\ \langle x, \theta \rangle &= -1, \\ \langle dx, \theta \rangle &= 0 \quad \text{(envelope condition)}. \end{aligned} \tag{3.3}$$

From the second equation we get by differentiation $\langle dx, \theta \rangle + \langle x, d\theta \rangle = 0$. Hence (3.3) can be rewritten

$$\begin{aligned} \langle x, x \rangle &= -1, \\ \langle x, \theta \rangle &= -1, \\ \langle x, d\theta \rangle &= 0. \end{aligned} \tag{3.4}$$

Because θ is an immersion into the cone $\mathcal{C}^n_+$ transverse to the generators of $\mathcal{C}^n_+$, the span of θ and $d\theta$ is $T_\theta \mathcal{C}^n_+$. Hence (3.4), and therefore (3.3) have a unique smooth solution x, the envelope of Θ. At points where x is regular, (3.3) implies that $\nu := \theta - x$ is a unit normal vector of x. It defines the orientation along x which asociated support map is θ.

Let $\hat{\theta}$ be the unique solution of

$$\begin{aligned} \langle\hat{\theta},\theta\rangle &= -2, \\ \langle\hat{\theta},d\theta\rangle &= 0, \\ \langle\hat{\theta},\hat{\theta}\rangle &= 0. \end{aligned} \tag{3.5}$$

Then the envelope x of Θ is given by

$$x=\frac{1}{2}\left(\theta+\hat{\theta}\right). \tag{3.6}$$

The associated geometric picture is the following: the horospheres Θ and $\hat{\Theta}$ are tangent to one another at the envelope point x, each lying in the concave side of the other. The mixed plane $\operatorname{span}(\theta,\hat{\theta})$ intersects $\mathbb{H}^n$ in the geodesic through x orthogonal to Θ and $\hat{\Theta}$.

3.2.1. *The 2-dimensional situation.* Let $\theta(\sigma)$ be smooth and parameterized by arc length σ. The envelope c of θ is given by

$$c=\frac{1}{2}\left(1+\langle\ddot{\theta},\ddot{\theta}\rangle\right)\theta+\ddot{\theta}\,. \tag{3.7}$$

In fact (3.7) fulfills (3.4) by noting $\langle\theta,\theta\rangle=0$ and $\langle\dot{\theta},\dot{\theta}\rangle=1$, hence by differentiation $\langle\dot{\theta},\theta\rangle=0$, $\langle\ddot{\theta},\theta\rangle=-1$ and $\langle\dot{\theta},\ddot{\theta}\rangle=0$.

Now (3.7) implies

$$\dot{c}=\frac{1-\langle\ddot{\theta},\ddot{\theta}\rangle}{2}\,\dot{\theta}\,. \tag{3.8}$$

To see this, we note that $\theta,\dot{\theta},\ddot{\theta}$ are linear independent because of the assumptions on θ and the geometry of the light cone. Hence we can try $\dot{c}=\alpha\theta+\beta\dot{\theta}+\gamma\ddot{\theta}$, and we use (3.7). Taking into account $\langle c,c\rangle=-1$ hence by differentiation $\langle\dot{c},c\rangle=0$, we get $\alpha+\frac{1}{2}\gamma(1-\langle\ddot{\theta},\ddot{\theta}\rangle)=0$. Taking into account $\langle c,\theta\rangle=-1$ hence by differentiation $\langle\dot{c},\theta\rangle+\langle c,\dot{\theta}\rangle=0$, we get $\gamma=0$. Therefore $\alpha=0$ and $\dot{c}=\beta\dot{\theta}$ with $\beta=\langle\dot{c},\dot{\theta}\rangle$. Taking into account $\langle c,\dot{\theta}\rangle=0$ hence by differentiation $\langle\dot{c},\dot{\theta}\rangle+\langle c,\ddot{\theta}\rangle=0$, we get $\beta-\frac{1}{2}(1-\langle\ddot{\theta},\ddot{\theta}\rangle)=0$, hence (3.8).
The envelope c is regular if and only if $\langle\ddot{\theta},\ddot{\theta}\rangle\neq 1$.

Remark 3.2. The condition $\langle\ddot{\theta},\ddot{\theta}\rangle\neq 1$ means, that the osculating plane of the curve θ in $\mathbb{R}^3_1$ is not tangent to the model $\mathbb{H}^2$. This property characterizes curves θ in $\mathcal{C}^2_+$ which envelopes are regular curves in $\mathbb{H}^2$.

Next, we compute the curvature of the envelope curve c at regular points. There $\nu:=\theta-c$ is the unit normal vector of c, and θ is the associated support map. Hence ν is the inner unit normal of the enveloping horocycle Θ. We take the Lipschitz-Killing curvature k_c of

c with respect to ν. (For a suitable chosen parameter orientation on c, the curvature k_c coincides with the geodesic curvature of c within the Frenet theory for curves on surfaces.) Then we calculate

$$k_c = -\langle \nu', c' \rangle = -\left(\frac{d\sigma}{ds}\right)^2 \langle \dot\nu, \dot c \rangle = \left(\frac{d\sigma}{ds}\right)^2 \langle \nu, \ddot c \rangle$$

where "$'$" denotes the derivative with respect to the arc length parameter s on c. Taking into account $\nu = \theta - c$, (3.7), (3.8) and $ds = |\frac{1}{2}(1 - \langle \ddot\theta, \ddot\theta \rangle)|\, d\sigma$, we get

$$k_c = -\frac{1 + \langle \ddot\theta, \ddot\theta \rangle}{1 - \langle \ddot\theta, \ddot\theta \rangle} \,. \tag{3.9}$$

Remark 3.3. The intersection of $\mathcal{C}^2_+$ with the osculating plane of θ at a fixed point $\theta(\sigma)$ represents a family osc_θ of horocycles with the following geometric meaning:
If the osculating plane is space-like, then the envelope curve c of θ owns an osculating circle osc_c at $c(\sigma)$, and osc_c is enveloped by the horocycles of the family osc_θ. In case the osculating plane intersects $\mathbb{H}^2$, the osculating circle osc_c lies in the concave sides of the horocycles of osc_θ, and we have $k_c < -1$ and $0 < \langle \ddot\theta, \ddot\theta \rangle < 1$. In case the osculating plane does not intersect $\mathbb{H}^2$, the osculating circle osc_c lies in the convex sides of the horocycles of osc_θ, and we have $1 < k_c$ and $1 < \langle \ddot\theta, \ddot\theta \rangle$.
If the osculating plane is time-like, then c owns an osculating equidistant osc_c at $c(\sigma)$, and osc_c is enveloped by the horocycles of the family osc_θ. In case $\langle \ddot\theta, \ddot\theta \rangle \le -1$, the horocycles of osc_θ lie in the convex side of the equidistant osc_c, and we have $0 \le k_c < 1$. In case $-1 \le \langle \ddot\theta, \ddot\theta \rangle < 0$, the horocycles of osc_θ lie in the concave side of osc_c, and we have $-1 < k_c \le 0$.

Now by (3.8) and (3.9), the length $L(c)$ and the total curvature $TC(c)$ of c write in terms of θ as

$$L(c) = \frac{1}{2} \int_\theta |1 - k_\theta^2|\, d\sigma \tag{3.10}$$

and

$$TC(c) = \int_c k_c\, ds = \frac{1}{2} \int_\theta (1 + k_\theta^2)\, \mathrm{sign}(k_\theta^2 - 1)\, d\sigma \,. \tag{3.11}$$

Here we write $k_\theta^2 = \langle \ddot\theta, \ddot\theta \rangle$, like in euclidean spaces the 1st Frenet curvature k_θ of the curve θ in $\mathbb{R}^3_1$, although $\langle \ddot\theta, \ddot\theta \rangle$ may be negative.

Finally, for closed curves c bounding a simply connected domain and oriented by its inner unit normal in $\mathbb{H}^2$, we put into game the Gauss-Bonnet formula

$$A(c) = \int_c k_c \, ds - 2\pi \ , \tag{3.12}$$

where $A(c)$ denotes the area enclosed by c.
For example, if c is oriented by its inner unit normal ν and if $1 < k_c$, hence $1 < k_\theta^2$ and c lies in the convex sides of the support horocycles Θ. In this situation c bounds an h-convex body (cf. definition 4.5). By (3.11) and (3.12) we get the area $A(c)$ enclosed by c in terms of θ

$$A(c) = \frac{1}{2} \int_\theta (1 + k_\theta^2) \, d\sigma - 2\pi \ . \tag{3.13}$$

3.2.2. *Curves and canals.* We consider here the envelope of a 1-parameter family of horospheres. This is special case of the classically called canal surfaces.

Proposition 3.1. *Let $\theta(t) \in \mathcal{C}^3_+$ be a smooth regular curve transverse to the generators, $t \in (t_0, t_1)$. The corresponding horospheres admit an smooth envelope given by a smooth mapping $f : (t_0, t_1) \times \mathbb{R} \to \mathbb{H}^3$ such that each $f(\{t\} \times \mathbb{R})$ is a horocycle, along which the envelope f and the horosphere $\Theta(t)$ are tangent. This mapping f is an immersion if and only if $\langle \ddot\theta(t), \ddot\theta(t) \rangle < 1$, assuming $\theta(t)$ is arc-length parametrized.*

Proof. It is convenient to consider the hyperbolic space $\mathbb{H}^3$ as a ball in the sphere $\mathbb{S}^3$. To this end, let us choose a unit space-like vector σ in $\mathbb{R}^5_1$, and let us identify $V = \sigma + (\sigma)^\perp$ with $\mathbb{R}^4_1$. Then $V \cap \mathcal{C}^4_+$ is isometric to hyperbolic space $\mathbb{H}^3$. In the projective completion of $\mathbb{R}^5_1$, we consider the projection from the origin towards the hyperplane at infinity. This projection defines a diffeomorphism p from $\mathbb{H}^3$ into an open ball of $\partial_\infty \mathbb{H}^4 = \mathbb{S}^3$.

Consider now the unit space-like vectors $\tau(t) = \sigma + \theta(t) \in \mathbb{R}^5_1$. The orthogonal space $(\tau(t))^\perp$ intersects the cone $\mathcal{C}^4_+$ transversely, and defines a 2-sphere in $\mathbb{S}^3$. Since $\tau'(t)$ is space-like, one can apply Lemma 1.8.3 in [HJ03] which constructs a certain mapping $F : (t_0, t_1) \times \mathbb{S}^1 \to \mathbb{S}^3$. Restricting F to $p(\mathbb{H}^3)$, and composing with p^{-1} gives the mapping f. □

Proposition 3.2. *Let $\theta(t) \in \mathcal{C}^3_+$ be a space-like curve, arc-length parametrized, and such that $\langle \ddot\theta(t), \ddot\theta(t) \rangle \equiv 1$. Then, the curve $c(t) = \theta(t) + \ddot\theta(t) \in \mathbb{H}^3$ has second order contact with the horocycle $H(t) = (c(t) + \theta(t)^\perp) \cap \theta(t)^\perp \cap \mathbb{H}^3$, and third order contact with $\Theta(t)$.*

Proof. Note that $(c(t)+\theta(t)^{\perp}) \cap \mathbb{H}^3 = \{x \in \mathbb{H}^3 | \langle x, \theta(t) = -1\}$ is a horosphere, and that $\dot{\theta}(t) \cap \mathbb{H}^3$ is a geodesic plane. Hence $H(t)$ is indeed a horocycle. Clearly $c(t) \in \mathbb{H}(t)$ for every t. By elementary considerations one shows that $\dot{c}, \ddot{c}$ are orthogonal to θ and $\dot{\theta}$ (the dots denoting derivation with respect to t). This easily gives that $\dddot{c}$ is orthogonal to θ. Hence, $\langle c(t), \theta(t_0)\rangle + 1$ (resp. $\langle c(t), \dot{\theta}(t_0)\rangle$) vanishes up to third (resp. second) order at $t = t_0$. $\square$

In some sense, $H(t)$ and $\Theta(t)$ are respectively the osculating circle, and the osculating sphere of $c(t)$. The family $\Theta(t)$ is a particular instance of a special class of canals which were called *drills* in [LS06].

Note that $c(t)$ has geodesic curvature $k_c(t) \equiv 1$. Reciprocally, one can show that any curve $c(t)$ in $\mathbb{H}^3$ with constant geodesic curvature $k_c(t) \equiv 1$ has third order contact with some horosphere at every point.

4. Linear combinations of support maps

Here we use the linear structure in the half-cone $\mathcal{C}^n_+$ in order to define a scalar multiplication, the h-dilation, and two additions, the *sum* and the *harmonic sum*.

4.1. The h-dilation.

Definition 4.1. Let M be a smooth regular hypersurface in $\mathbb{H}^n$ and θ its support map with respect to a unit normal vector field ν along M. For $\delta \in \mathbb{R}_+$, we call the envelope of $\delta\,\theta$ in $\mathbb{H}^n$ the *h-dilation* $\delta\, M$ *of* M.

Directly, the definition shows that $\delta\, M$ is the parallel hypersurface, irrespective the occurrence of singularities, of M at distance t in direction ν, where $t = d(\Theta, \Theta^*) = -\ln \delta$ (cf. (2.1)) and $\theta^* = \delta\,\theta$.

4.1.1. *The* 2*-dimensional situation.* For an oriented curve c in $\mathbb{H}^2$ with associated support map θ, we denote $c^* = \delta\, c$ the envelope of $\theta^* = \delta\,\theta$, $\delta \in \mathbb{R}_+$. We compute

$$\left\langle \frac{d^2\theta^*}{(d\sigma^*)^2}, \frac{d^2\theta^*}{(d\sigma^*)^2} \right\rangle = \frac{1}{\delta^2} \left\langle \frac{d^2\theta}{(d\sigma)^2}, \frac{d^2\theta}{(d\sigma)^2} \right\rangle ,$$

where σ, σ^* are the arc length parameters on θ, θ^* respectively. Therefore, at regular points of c^*, (3.9) implies

$$k_{c^*} = \frac{k_c + \tanh(t)}{1 + k_c \tanh(t)} \,. \tag{4.1}$$

Moreover, taking into account

$$\left(\frac{1}{\delta} - \delta\right)\left(\langle \ddot{\theta}, \ddot{\theta}\rangle + 1\right) + \left(\frac{1}{\delta} + \delta\right)\left(\langle \ddot{\theta}, \ddot{\theta}\rangle - 1\right) = 2\left(\frac{\langle \ddot{\theta}, \ddot{\theta}\rangle}{\delta} - \delta\right),$$

(3.10) yields

$$\begin{aligned} L(c^*) &= \frac{1}{2}\left(\frac{1}{\delta}-\delta\right)\frac{1}{2}\int_\theta \epsilon(\theta,\delta)\,\epsilon(\theta,1)\,(1+k_\theta^2)\,\mathrm{sign}(k_\theta^2-1)\,d\sigma + \\ &\quad + \frac{1}{2}\left(\frac{1}{\delta}+\delta\right)\frac{1}{2}\int_\theta \epsilon(\theta,\delta)\,\epsilon(\theta,1)\,|1-k_\theta^2|\,d\sigma = \\ &= \sinh(t)\,\frac{1}{2}\int_\theta \epsilon(\theta,\delta)\,\epsilon(\theta,1)\,(1+k_\theta^2)\,\mathrm{sign}(k_\theta^2-1)\,d\sigma + \\ &\quad + \cosh(t)\,\frac{1}{2}\int_\theta \epsilon(\theta,\delta)\,\epsilon(\theta,1)\,|1-k_\theta^2|\,d\sigma\,, \end{aligned} \tag{4.2}$$

with $\epsilon(\theta,\delta)=\mathrm{sign}(\langle\ddot\theta,\ddot\theta\rangle-\delta^2)$ and $\epsilon(\theta,1)=\mathrm{sign}(\langle\ddot\theta,\ddot\theta\rangle-1)$.
For example, if $\epsilon(\theta,1)=+1$ and $\delta<1$, then $\epsilon(\theta,\delta)=+1$. In this case c is h-convex (cf. Definition 4.5), i.e. $k_c>1$, and c^* is the outer parallel curve to c at distance $t=-\ln\delta$. By using (3.10) and (3.11) we obtain

$$L(c^*)=\sinh(t)\,TC(c)+\cosh(t)\,L(c)\,. \tag{4.3}$$

Formulas (4.1) and (4.3) are well-known in the setting of Steiner formulas for parallel curves in the hyperbolic plane, see for instance [VA47], [All48].

4.2. **The sum.**

Definition 4.2. Let M_1, M_2 be smooth regular hypersurfaces in $\mathbb{H}^n$, oriented by unit normal fields ν_1, ν_2 and with associated support maps θ_1,θ_2. Suppose that M_1 and M_2 are related to one another by parallel support horospheres, i.e. M_1 and M_2 can be parameterized by the same parameters such that $\Theta_1(u)$ and $\Theta_2(u)$ are parallel horospheres for each parameter u. Then we call the envelope of $\theta_1+\theta_2$ in $\mathbb{H}^n$ the *sum* M_1+M_2 *of* M_1 *and* M_2.

For example convex M_1 and M_2 are always related to one another by parallel support horospheres.

Below we will need the position of M_2 relative to M_1. This is detailed as follows.

Definition 4.3. Let M_1, M_2 be given as above. Then $\theta_2=\lambda\theta_1$, and the signed distance $\mathrm{d}(\Theta_1,\Theta_2)$ from Θ_1 to Θ_2 is given by $\mathrm{d}(\Theta_1,\Theta_2)=-\ln\lambda$, cf. (2.1). We call

$$w_{12}: M_1\to\mathbb{R} \quad , \quad u\mapsto -\ln\lambda(u) \tag{4.4}$$

the *relative support function of* M_2 *with respect to* M_1.

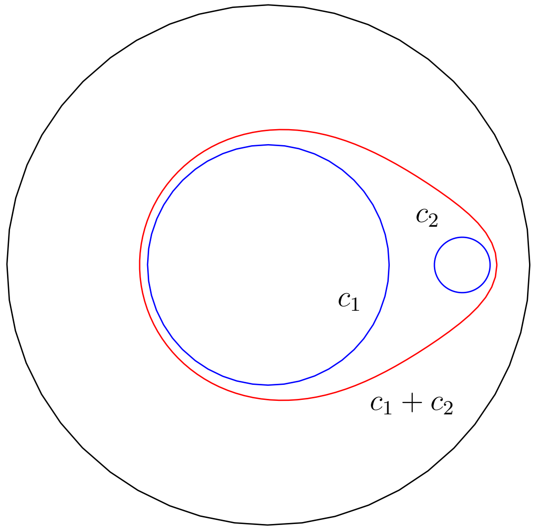

Figure 1: The sum $c_1 + c_2$ of two circles c_1, c_2 in the Poincaré disk, oriented by their outer unit normals, with radii $r_1 = 1, r_2 = 0.5$ and distance 2 between their centers

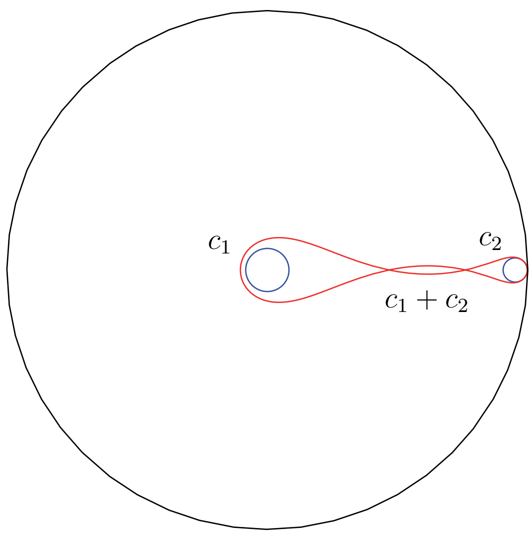

Figure 2: The sum $c_1 + c_2$ of two circles c_1, c_2 in the Poincaré disk, oriented by their outer unit normals, with radii $r_1 = 0.16, r_2 = 2$ and distance 5 between their centers

Remark 4.1. If $M_1 = M_2$ is the boundary of an h-convex body (cf. Definition4.5), and if M_1, M_2 are oppositely oriented, then the relative support function coincides with the width function with respect to horocycles considered in [GRST08].

4.2.1. *The 2-dimensional situation.* Now c_1, c_2 are smooth regular oriented curves in $\mathbb{H}^2$, related to one another by parallel support horocycles.

Lemma 4.1. *For the lengths of the support images involved in the sum the following relation holds*

$$L(\theta_1 + \theta_2) = L(\theta_1) + L(\theta_2) \ . \tag{4.5}$$

Proof. Suppose $\theta_2(\sigma_1) = \lambda(\sigma_1)\theta_1(\sigma_1)$, parameterized by the arc length parameter σ_1 on θ_1. Then we have

$$\frac{d\theta_2}{d\sigma_1} = \frac{d\lambda}{d\sigma_1}\theta_1 + \lambda\frac{d\theta_1}{d\sigma_1} \ ,$$

$$\left\langle \frac{d\theta_2}{d\sigma_1}, \frac{d\theta_2}{d\sigma_1} \right\rangle = \lambda^2 \left\langle \frac{d\theta_1}{d\sigma_1}, \frac{d\theta_1}{d\sigma_1} \right\rangle = \lambda^2 \ ,$$

hence

$$d\sigma_2 = \lambda \, d\sigma_1 \ .$$

For $\theta^* := \theta_1 + \theta_2 = (1+\lambda)\,\theta_1$, we get

$$\frac{d\theta^*}{d\sigma_1} = \frac{d\lambda}{d\sigma_1}\theta_1 + (1+\lambda)\frac{d\theta_1}{d\sigma_1}$$

and

$$\left\langle \frac{d\theta^*}{d\sigma_1}, \frac{d\theta^*}{d\sigma_1} \right\rangle = (1+\lambda)^2 \langle \frac{d\theta_1}{d\sigma_1}, \frac{d\theta_1}{d\sigma_1} \rangle = (1+\lambda)^2 \ .$$

Hence

$$d\sigma^* = (1+\lambda)\, d\sigma_1 = d\sigma_1 + d\sigma_2 \ .$$

□

Proposition 4.1. *Let c_1, c_2 be smooth regular curves in $\mathbb{H}^2$, oriented by unit normal fields ν_1, ν_2 and related to one another by parallel support horocycles. If $k_{c_1}, k_{c_2} < 1$, then the sum $\theta^* = \theta_1 + \theta_2$ envelopes a regular curve $c^* = c_1 + c_2$ in $\mathbb{H}^2$ with $k_{c^*} < 1$, i.e. c^* lies locally on the concave sides of its respective support horocycles.*

Proof. By assumption, the curves c_1, c_2 lie locally on the concave sides of their respective support horocycles, therefore the osculating planes of θ_1, θ_2 intersect $\mathbb{H}^2$ without being tangent (cf. Remark 3.2).
Now, we keep fixed an arbitrary parameter σ_1.
The osculating plane of θ_1 at σ_1 is given by

$$\theta_1(\sigma_1) + \mathrm{span}(\dot\theta_1(\sigma_1), \ddot\theta_1(\sigma_1)).$$

Let P_1 denote the parallel plane through $\theta^*(\sigma_1)$, i.e.

$$P_1 = \theta^*(\sigma_1) + \mathrm{span}(\dot\theta_1(\sigma_1), \ddot\theta_1(\sigma_1)).$$

The osculating plane of θ_1 at σ_1 intersects $\mathbb{H}^2$ without being tangent, and $\theta^* = \theta_1 + \theta_2$, therefore P_1 also intersects $\mathbb{H}^2$ without being tangent. Now $\theta_2 = \lambda\theta_1$, hence

$$\dot{\theta}_2 = \dot{\lambda}\theta_1 + \lambda\dot{\theta}_1 \quad \text{and} \quad \ddot{\theta}_2 = \ddot{\lambda}\theta_1 + 2\dot{\lambda}\dot{\theta}_1 + \lambda\ddot{\theta}_1 \tag{4.6}$$

(where the dots denote derivatives with respect to σ_1). And the osculating plane of θ_2 at σ_1 is given by

$$\theta_2(\sigma_1) + \operatorname{span}(\dot{\theta}_2(\sigma_1), \ddot{\theta}_2(\sigma_1)).$$

Let P_2 denote the parallel plane through $\theta^*(\sigma_1)$, i.e.

$$P_2 = \theta^*(\sigma_1) + \operatorname{span}(\dot{\theta}_2(\sigma_1), \ddot{\theta}_2(\sigma_1)).$$

The osculating plane of θ_2 at σ_1 intersects $\mathbb{H}^2$ without being tangent, we have $\theta^* = \theta_1 + \theta_2$, therefore P_2 also intersects $\mathbb{H}^2$ without being tangent.
The osculating plane of θ^* at σ_1 is given by

$$P^* = \theta^*(\sigma_1) + \operatorname{span}(\dot{\theta}^*(\sigma_1), \ddot{\theta}^*(\sigma_1))$$

with

$$\dot{\theta}^* = \dot{\theta}_2 + \dot{\theta}_1 \quad \text{and} \quad \ddot{\theta}^* = \ddot{\theta}_2 + \ddot{\theta}_1. \tag{4.7}$$

Let T be the tangent plane of $\mathcal{C}^2_+$ along the generator $\mathbb{R}_+ \cdot \theta_1(\sigma_1)$, i.e. $T = \theta^*(\sigma_1) + \operatorname{span}(\theta_1(\sigma_1), \dot{\theta}_1(\sigma_1))$. For $a \geq 0$ let T_a denote the plane parallel to T given by $T_a = T + a\,\ddot{\theta}_1(\sigma_1)$.
Then T_a intersects P_1 in the line

$$\ell_{1a} = \theta^*(\sigma_1) + a\,\ddot{\theta}_1(\sigma_1) + \mathbb{R}\cdot\dot{\theta}_1(\sigma_1)\,.$$

And by (4.6), T_a intersects P_2 in the line

$$\ell_{2a} = \theta^*(\sigma_1) + \frac{a}{\lambda(\sigma_1)}\,\ddot{\theta}_2(\sigma_1) + \mathbb{R}\cdot\dot{\theta}_2(\sigma_1)\,.$$

And by (4.7), T_a intersects P^* in the line

$$\ell^*_a = \theta^*(\sigma_1) + \frac{a}{1+\lambda(\sigma_1)}\Big(\ddot{\theta}_2(\sigma_1) + \ddot{\theta}_1(\sigma_1)\Big) + \mathbb{R}\cdot\Big(\dot{\theta}_2(\sigma_1) + \dot{\theta}_1(\sigma_1)\Big)\,.$$

Let g_a denote the line in T_a given by

$$g_a = \theta^*(\sigma_1) + a\,\ddot{\theta}_1(\sigma_1) + \mathbb{R}\cdot\theta_1(\sigma_1)\,.$$

Then g_a intersects ℓ_{1a} in the point

$$Q_{1a} = \theta^*(\sigma_1) + a\,\ddot{\theta}_1(\sigma_1)\,.$$

And g_a intersects ℓ_{2a} in the point

$$Q_{2a} = \theta^*(\sigma_1) + a\,\ddot{\theta}_1(\sigma_1) + \frac{a(\lambda\ddot{\lambda} - 2\dot{\lambda}^2)}{\lambda^2}\Big|_{\sigma_1}\theta_1(\sigma_1)\,.$$

And g_a intersects ℓ_a^* in the point

$$Q_a^* = \theta^*(\sigma_1) + a\,\ddot\theta_1(\sigma_1) + \frac{a((1+\lambda)\ddot\lambda - 2\dot\lambda^2)}{(1+\lambda)^2}|_{\sigma_1}\,\theta_1(\sigma_1)\,.$$

The proof now splits into two cases.
• The first case, $((1+\lambda)\ddot\lambda - 2\dot\lambda^2)|_{\sigma_1} \geq 0$:
Recall that P_1 intersects $\mathbb{H}^2$ without being tangent. Therefore there exists an $a > 0$ such that ℓ_{1a} intersects the parabola $T_a \cap \mathbb{H}^2$ without being tangent. The axis of the parabola is $g_a = \theta^*(\sigma_1) + a\,\ddot\theta_1(\sigma_1) + \mathbb{R}\cdot\theta_1(\sigma_1)$. Hence the half-ray $Q_{1a} + \mathbb{R}_+ \cdot \theta_1(\sigma_1) \subset T_a$ lies in the convex region bounded by the parabola $T_a \cap \mathbb{H}^2$. In the first case Q_a^* lies on this half-ray. Hence Q_a^* lies in the convex region bounded by the parabola $T_a \cap \mathbb{H}^2$. Hence ℓ_a^* intersects the parabola $T_a \cap \mathbb{H}^2$ without being tangent. Hence the osculating plane P^* of θ^* at σ_1 intersects $\mathbb{H}^2$ without being tangent.
• The second case, $((1+\lambda)\ddot\lambda - 2\dot\lambda^2)|_{\sigma_1} < 0$:
Recall that P_2 intersects $\mathbb{H}^2$ without being tangent. Therefore there exists an $a > 0$ such that ℓ_{2a} intersects the parabola $T_a \cap \mathbb{H}^2$ without being tangent. Hence the half-ray $Q_{2a} + \mathbb{R}_+ \cdot \theta_1(\sigma_1) \subset T_a$ lies in the convex region bounded by the parabola $T_a \cap \mathbb{H}^2$. Through the assumption in the second case we have

$$\frac{\lambda\ddot\lambda - 2\dot\lambda^2}{\lambda^2}|_{\sigma_1} \leq \frac{(1+\lambda)\ddot\lambda - 2\dot\lambda^2}{(1+\lambda)^2}|_{\sigma_1}\,.$$

Hence Q_a^* lies on this half-ray. Hence ℓ_a^* intersects the parabola $T_a \cap \mathbb{H}^2$ without being tangent. Hence the osculating plane P^* of θ^* at σ_1 intersects $\mathbb{H}^2$ without being tangent.
Altogether, this shows that the osculating planes of θ^* intersect $\mathbb{H}^2$ without being tangent. Therefore c^* is regular at σ_1, Θ^* supports c^* concave-sided, i.e. $k_{c^*} < 1$. □

Proposition 4.2. *Let c_1, c_2 be smooth regular curves in $\mathbb{H}^2$, oriented by unit normal fields ν_1, ν_2 and related to one another by parallel support horocycles. If $k_{c_1}, k_{c_2} < 1$, then the length and the total curvature of $c^* = c_1 + c_2$ write in terms of c_1, c_2 and their relative position to each other in $\mathbb{H}^2$ as follows:*

$$L(c^*) = \frac{1}{2}\left(-W(c_1, c_1 + c_2) + L(c_1) - TC(c_1) + L(c_2) - TC(c_2)\right) \tag{4.8}$$

$$TC(c^*) = \frac{1}{2}\left(-W(c_1, c_1 + c_2) - L(c_1) + TC(c_1) - L(c_2) + TC(c_2)\right), \tag{4.9}$$

with

$$\begin{aligned} W(c_1, c_1+c_2) &:= -TC(c^*) - L(c^*) = \\ &= \int_{\theta_1} e^{w_{1*}} \left((\dot{w}_{1*})^2 + 2\ddot{w}_{1*} + k_{\theta_1}^2 \right) d\sigma_1 \end{aligned} \tag{4.10}$$

and the relative support function $w_{1*}(\sigma_1) = -\ln(1+\lambda(\sigma_1))$ *of* c^* *with respect to* c_1.

Proof. By the assumptions on c_1, c_2 and by Proposition 4.1 we have for all three curves involved $k_{c_1}, k_{c_2}, k_{c^*} < 1$. Therefore by (3.2) for each of these curves we have $d\sigma = (1-k_c)\, ds$ and

$$L(\theta) = \int_\theta d\sigma = \int_c (1-k_c)\, ds = L(c) - TC(c)\,. \tag{4.11}$$

This and Lemma 4.1 (4.5) give

$$L(c^*) - TC(c^*) = L(c_1) - TC(c_1) + L(c_2) - TC(c_2)\,. \tag{4.12}$$

For all three curves c_1, c_2, c^* we have $k_\theta^2 = \langle \ddot{\theta}, \ddot{\theta} \rangle < 1$ (cf. Remark 3.3), hence by (3.8)

$$ds = \frac{1}{2}(1-k_\theta^2)\, d\sigma\,.$$

This applied to c^* yields

$$L(c^*) = \frac{1}{2} L(\theta^*) - \frac{1}{2} \int_{\theta^*} k_{\theta^*}^2\, d\sigma^*\,. \tag{4.13}$$

Now a straightforward but lengthy computation, not acted out here, starts at $\theta^* = (1+\lambda)\,\theta_1$, minds $d\sigma^* = (1+\lambda)\, d\sigma_1$, and reaches

$$\begin{aligned} k_{\theta^*}^2 = \langle \frac{d^2\theta^*}{d\sigma^{*2}}, \frac{d^2\theta^*}{d\sigma^{*2}} \rangle = \frac{1}{(1+\lambda)^2} \Bigg[\left(\frac{d}{d\sigma_1} (\ln(1+\lambda)) \right)^2 - \\ - 2 \frac{d^2}{d\sigma_1^2} (\ln(1+\lambda)) + \langle \ddot{\theta}_1, \ddot{\theta}_1 \rangle \Bigg]. \end{aligned} \tag{4.14}$$

Using the relative support function of c_* with respect to c_1, i.e. $w_{1*} = -\ln(1+\lambda)$, formula (4.14) gives

$$\int_{\theta^*} k_{\theta^*}^2\, d\sigma^* = \int_{\theta_1} e^{w_{1*}} \left((\dot{w}_{1*})^2 + 2\ddot{w}_{1*} + k_{\theta_1}^2 \right) d\sigma_1\,. \tag{4.15}$$

Hence (4.13), (4.15) and (4.11) yield

$$L(c^*) = -\frac{1}{2} \int_{\theta_1} e^{w_{1*}} \left((\dot{w}_{1*})^2 + 2\ddot{w}_{1*} + k_{\theta_1}^2 \right) d\sigma_1 + \frac{1}{2} L(c^*) - \frac{1}{2} TC(c^*)\,. \tag{4.16}$$

Finally (4.12) and (4.16) give the result. □

Remark 4.2. If c is closed, then through integration by parts (4.10) can be written as follows

$$\begin{aligned} \mathrm{W}(c_1, c_1 + c_2) &= \int_{\theta_1} e^{w_{1*}} \left(\ddot{w}_{1*} + k_{\theta_1}^2\right) d\sigma_1 = \\ &= \int_{\theta_1} e^{w_{1*}} \left(-(\dot{w}_{1*})^2 + k_{\theta_1}^2\right) d\sigma_1 \,. \end{aligned} \tag{4.17}$$

Remark 4.3. Under the conditions of Proposition 4.2, if c_1, c_2 and c^* are simple and closed, then applying the Gauss-Bonnet formula (3.12) in (4.9) we obtain for the area enclosed by c^*

$$A(c^*) = \frac{1}{2}\left(W(c_1, c_1 + c_2) + L(c_1) + L(c_2) + A(c_1) + A(c_2)\right). \tag{4.18}$$

4.3. **The harmonic sum.**

Definition 4.4. Let M_1, M_2 be smooth regular hypersurfaces in $\mathbb{H}^n$, oriented by unit normal fields ν_1, ν_2 and with associated support maps θ_1, θ_2. Suppose that M_1 and M_2 are related to one another by parallel support horospheres. Then we call

$$M_1 \# M_2 = M^*, \quad \text{given by} \quad \theta^* = \theta_1 \# \theta_2 = \frac{\lambda}{1+\lambda}\,\theta_1 \tag{4.19}$$

the *harmonic sum* $M_1 \# M_2$ *of* M_1 *and* M_2, i.e. M^* is the envelope of Θ^*.

Remark 4.4. Geometrically, this definition is induced by the vector sum of the two parallel hyperplanes Θ_1 and Θ_2 in the vector space $\mathbb{R}^{n+1}_1$. Analytically, $\frac{2\lambda}{1+\lambda}$ is the *harmonic mean* of 1 and λ, corresponding to θ_1 and θ_2.

Lemma 4.2. *Let θ_1, θ_2 be support maps as above. Then $\theta^* = \theta_1 \# \theta_2$ lies below θ_1 and θ_2 with respect to each of the generators of $\mathcal{C}^2_+$.*

Proof. We have $\theta_2 = \lambda\,\theta_1$ with $\lambda > 0$. Hence

$$\theta^* = \frac{\lambda}{1+\lambda}\theta_1 < \theta_1 \text{ , and}$$

$$\theta^* = \frac{\lambda}{1+\lambda}\theta_1 = \frac{1}{1+\lambda}\theta_2 < \theta_2 \,.$$

□

Now we specify distance spheres S in $\mathbb{H}^n$. S is determined by its center $m \in \mathbb{H}^n$ and its hyperbolic radius $|r|$. For both orientations, the associated support maps θ are given by the intersections of $\mathcal{C}^n_+$ with the space-like hyperplanes $\langle n, x\rangle = -1$, where n is time-like inside the half-cone $\mathcal{C}^n_+$, $m = n/|n|$ and $|r| = |\ln|n||$. If S is oriented by its inner unit normals, then $|n| > 1$ and we take the signed radius $r = \ln |n| > 0$. If S is oriented by its outer unit normals, then $|n| < 1$ and we take the signed radius $r = \ln |n| < 0$. In particular for circles in the hyperbolic plane, the orientation by inner unit normals is signified by counter-clockwise direction along the circle, whereas orientation by outer unit normals is signified by clockwise direction along the circle.

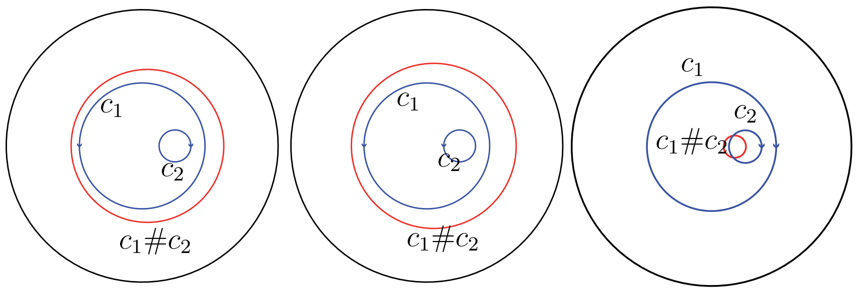

Figure 3: The harmonic sum $c_1\#c_2$ of two circles c_1, c_2 in the Poincaré disk, with signed radii $r_1 = 1, +1, -1, r_2 = -0.25, +0.25, -0.25$ and distance 0.5 between their centers

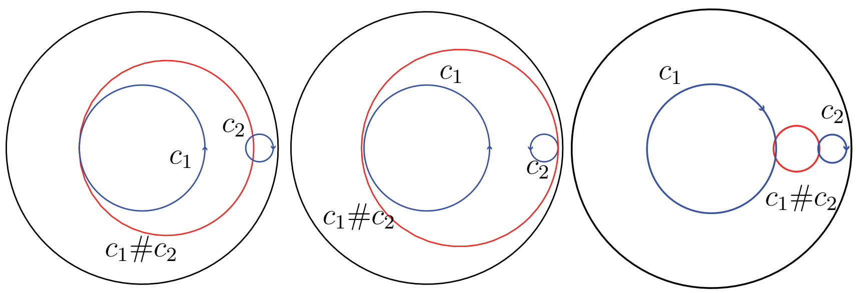

Figure 4: The harmonic sum $c_1\#c_2$ of two circles c_1, c_2 in the Poincaré disk, with signed radii $r_1 = +1, +1, -1, r_2 = -1, +1, -1$ and distance 3 between their centers

Proposition 4.3. *(1) Let S_1, S_2 be spheres or points in $\mathbb{H}^n$. Then the harmonic sum $S_1\#S_2$ of S_1 and S_2 is a sphere or a point.*

(2) Let S_1, S_2 be spheres or points in $\mathbb{H}^n$ with centers m_1, m_2 and signed radii r_1, r_2. Then the harmonic sum $S^* = S_1\#S_2$ has center m^* and signed radius r^* as follows:

$$r^* = \frac{1}{2} \ln \left(e^{2r_1} + e^{2r_2} + 2e^{r_1+r_2} \cosh(\mathrm{d}(m_1, m_2))\right) \tag{4.20}$$

where $\mathrm{d}(m_1, m_2)$ is the hyperbolic distance between m_1 and m_2, and

$$m^* = \frac{1}{|n_1 + n_2|} (n_1 + n_2) \tag{4.21}$$

with $n_1 = e^{r_1} m_1$ and $n_2 = e^{r_2} m_2$. Moreover

$$\frac{\cosh(\mathrm{d}(\mathrm{m}_1, \mathrm{m}^*))}{\cosh(\mathrm{d}(\mathrm{m}_2, \mathrm{m}^*))} = \frac{e^{r_1} + e^{r_2} \cosh(\mathrm{d}(\mathrm{m}_1, \mathrm{m}_2))}{e^{r_1} \cosh(\mathrm{d}(\mathrm{m}_1, \mathrm{m}_2)) + \mathrm{e}^{\mathrm{r}_2}}. \tag{4.22}$$

(3) *Let S_1, S_2 be spheres oriented by their inner unit normals, or points in $\mathbb{H}^n$. Then the harmonic sum $S_1 \# S_2$ of S_1 and S_2 is a sphere containing both S_1 and S_2.*

Proof. ad (1): The support maps of spheres or points are given by the intersection of $\mathcal{C}_+^n$ with space-like hyperplanes. Therefore θ_1, θ_2 are uniquely given by hyperplanes $\langle n_1, x\rangle = -1$ and $\langle n_2, x\rangle = -1$ with time-like vectors $n_1, n_2 \neq 0$ lying inside $\mathcal{C}_+^n \subset \mathbb{R}_1^{n+1}$. Then $\theta_2 = \lambda\,\theta_1$ with $\lambda = -1/\langle n_2, \theta_1\rangle$. Putting $n^* = n_1 + n_2$, we compute

$$\langle n^*, \theta^*\rangle = \langle n_1 + n_2, \frac{\lambda}{1+\lambda}\theta_1\rangle = \\ = \frac{\lambda}{1+\lambda}\langle n_1, \theta_1\rangle + \frac{\lambda}{1+\lambda}\langle n_2, \theta_1\rangle = -1\,.$$

Therefore θ^* lies in the hyperplane $\langle n^*, x\rangle = -1$, and hence it envelopes a sphere or a point.

ad (2): The support maps θ_1, θ_2 of S_1, S_2 are uniquely determined by their hyperplanes $\langle n_1, x\rangle = -1$, $\langle n_2, x\rangle = -1$ as described above. Then their centers and signed radii are given by $m_1 = n_1/|n_1|$, $m_2 = n_2/|n_1|$ and $r_1 = \ln|n_1|$, $r_2 = \ln|n_2|$. The support map θ^* is given by the hyperplane $\langle n_1 + n_2, x\rangle = -1$. Then straightforward computations give the results.

ad (3): S_1, S_2 are oriented by their inner unit normals. Hence their support maps θ_1, θ_2 support such that S_1, S_2 lie in the convex sides of Θ_1, Θ_2, i.e. their respective hyperplanes do not intersect $\mathbb{H}^n$. Then by Lemma 4.2, θ^* lies below θ_1 and θ_2 with respect to each generator of $\mathcal{C}_+^2$. Therefore the hyperplane of θ^* does not intersect $\mathbb{H}^2$. Hence $S_1 \# S_2$ lies in the convex sides of Θ^*. Moreover Θ^* contains Θ_1 and Θ_2, hence $S_1 \# S_2$ contains S_1 and S_2. □

Now we treat h-convex bodies in $\mathbb{H}^n$.

Definition 4.5. A compact subset $K \subset \mathbb{H}^n$ is called *h-convex* (*horo-convex*) if for each $x \in \partial K$ there exists at least one horosphere Θ which supports K at x such that $x \in \Theta$ and $K \subseteq B_\Theta$, B_Θ = closed convex horoball bounded by Θ.

This definition is equivalent to the following condition on K: for each pair of points in K, all the entire horocycle segments joining them are also contained in K. In the smooth case, Definition 4.5 is equivalent to the fact that all normal curvatures of ∂K with respect to inner normals are bigger or equal to 1 (cf. [GR99])

Now, each pencil of parallel horospheres has exactly one member which supports K this way. Therefore there is an associated support map $\theta_K = \theta$ which assigns to each pencil of parallel horospheres, determined by its point u at infinity, just this supporting horosphere $\Theta(u)$, i.e. we have the support map

$$\theta : \mathbb{H}^n_\infty \to \mathcal{C}^n_+ \ , \ u \mapsto \theta(u) \, . \tag{4.23}$$

It is easy to see that θ is continuous. Alternatively, the support map may be considered as a continuous map from the outer unit normal bundle $N^1\partial K$ of ∂K (for sets of positive reach in the sense of H. Federer) into $\mathcal{C}^n_+$, i.e.

$$\theta : N^1 \to \mathcal{C}^n_+ \ , \ (x, \nu) \mapsto x - \nu \tag{4.24}$$

($x \in \partial K \, , \, \nu \in N^1_x$). (Note: In the smooth regular case, one can use the inner unit normal bundle and the support map as in (3.1).) This way, for each h-convex K in $\mathbb{H}^n$ there is an associated support image via θ in $\mathcal{C}^n_+$.
Vice versa, given a continuous graph $\theta(u) \, , \, u \in \mathbb{H}^n_\infty$, in $\mathcal{C}^n_+$, we define

$$K_\theta := \bigcap_{u \in \mathbb{H}^n_\infty} B_{\Theta(u)} \ , \tag{4.25}$$

where $B_{\Theta(u)}$ is the closed horoball with boundary $\Theta(u)$, i.e. the closed convex region bounded by $\Theta(u)$.
K_θ is h-convex, because it is the intersection of horoballs. This can be shown like the analogous fact in euclidean convexity.
K_θ is compact, because it is closed and bounded. To this: It is closed, because it is the intersection of closed subsets. And it is bounded, because the graph of θ is bounded away from the vertex of $\mathcal{C}^n_+$, hence there is a space-like hyperplane in $\mathbb{R}^{n+1}_1$ separating graph and vertex, and the intersection of this hyperplane with $\mathcal{C}^n_+$ envelopes a hyperbolic distance sphere containing K_θ.

In order to characterize the support images of h-convex bodies, we need $\theta_x := T_x\mathbb{H}^n \cap \mathcal{C}^n_+$, the pencils of horospheres through the points x in $\mathbb{H}^n$.

Definition 4.6. A graph $\psi : \mathbb{H}^n_\infty \to \mathcal{C}^n_+$ is called *h-supported*, if for every $u \in \mathbb{H}^n_\infty$ there exists $x(u) \in \Psi(u)$ such that the graph ψ lies below $\theta_{x(u)}$, i.e. ψ lies in the apex-side of $\theta_{x(u)}$ in the cone $\mathcal{C}^n_+$.

Remark 4.5. If a graph in $\mathcal{C}^n_+$ is h-supported, then it is continuous.

Then the 1-1-correspondence between compact h-convex bodies in $\mathbb{H}^n$ and h-supported graphs in $\mathcal{C}^n_+$ is as follows.

Lemma 4.3. *(1) Let $K \subset \mathbb{H}^n$ be compact and h-convex, then its support image θ_K is a continuous and h-supported graph in $\mathcal{C}^n_+$.*

(2) *Let θ be a continuous and h-supported graph in $\mathcal{C}^n_+$, then the associated compact h-convex body K_θ (cf.* (4.25)*) has θ as its support image.*

Proof. ad (1). By the definition of support maps (cf. (4.23)), given $u \in \mathbb{H}^n_\infty$, there exists $x(u) \in \partial K$ with $x(u) \in \Theta_K(u)$. Furthermore, for each $u' \in \mathbb{H}^n_\infty$ the associated support horosphere $\Theta_K(u')$ contains in its convex side a parallel horosphere through $x(u)$. Therefore $\theta_K(u')$ lies below $\theta_{x(u)}(u')$ in the generator of $\mathcal{C}^n_+$ through u'. Hence θ_K lies below $\theta_{x(u)}$.

ad (2). By the definition of K_θ (cf. (4.25)) and the definition of support maps (cf. (4.23)), we see that the support image ψ of K_θ lies above θ in $\mathcal{C}^n_+$. On the other hand, pick an $u \in \mathbb{H}^n_\infty$. Because θ is h-supported, there exists $x(u) \in \Theta(u)$ such that θ lies below $\theta_{x(u)}$ in $\mathcal{C}^n_+$. Therefore $x(u) \in B_{\Theta(u')}$ for all $u' \in \mathbb{H}^n_\infty$, hence $x(u) \in K_\theta$. This implies that $\psi(u)$ lies below $\theta(u)$. Therefore we get $\psi(u) = \theta(u)$, hence $\psi = \theta$. $\square$

Given h-convex compact bodies K_1, K_2 with associated support maps θ_1, θ_2, we call $K_{\theta_1 \# \theta_2}$ (cf. (4.25)) the *harmonic sum* $K_1 \# K_2$ of K_1 and K_2.

Lemma 4.4. *Let K_1, K_2, L_1, L_2 be compact h-convex bodies in $\mathbb{H}^n$ with associated support maps $\theta_1, \theta_2, \psi_1, \psi_2$. If $K_2 \subseteq K_1$ and $L_2 \subseteq L_1$, then $K_2 \# L_2 \subseteq K_1 \# L_1$.*

Proof. $K_2 \subseteq K_1$ and $L_2 \subseteq L_1$ imply $\theta_1 \le \theta_2$ and $\psi_1 \le \psi_2$. Here the ordering is fulfilled on each generator, i.e. e.g. $\theta_1(u) \le \theta_2(u)$, $u \in \mathbb{H}^n_\infty$, with respect to the canonical ordering on the generators of $\mathcal{C}^n_+$.
Hence $\theta_2 = \lambda\, \theta_1$, $\lambda \ge 1$, and $\psi_2 = \mu\, \psi_1$, $\mu \ge 1$. If we write $\psi_1 = \rho\, \theta_1$, $\rho > 0$, then according to the definition of the harmonic sum,

$$\theta_1 \# \psi_1 = \frac{\rho}{1+\rho}\theta_1 \quad \text{, and}$$

$$\theta_2 \# \psi_2 = \frac{\frac{\mu\rho}{\lambda}}{1+\frac{\mu\rho}{\lambda}}\theta_2 = \frac{\mu\rho}{1+\frac{\mu\rho}{\lambda}}\theta_1$$

(note: $\psi_2 = \mu\psi_1 = \mu\rho\theta_1 = (\mu\rho/\lambda)\theta_2$).
Now we compute

$$\begin{aligned} & \frac{\rho}{1+\rho} \le \frac{\mu\rho}{1+\frac{\mu\rho}{\lambda}} \\ \Leftrightarrow\ & \frac{\rho}{1+\rho} \le \frac{\lambda\mu\rho}{\lambda+\mu\rho} \\ \Leftrightarrow\ & \lambda\rho + \mu\rho^2 \le \lambda\mu\rho + \lambda\mu\rho^2 \\ \Leftrightarrow\ & 0 \le \lambda\rho(\mu-1) + \mu\rho^2(\lambda-1)\ . \end{aligned}$$

The latter inequality is fulfilled, because of $\mu \geq 1$, $\lambda \geq 1$ and $\rho > 0$. Hence

$$\theta_1 \# \psi_1 \leq \theta_2 \# \psi_2 \ .$$

By (4.25), this implies

$$K_{\theta_2 \# \psi_2} \subseteq K_{\theta_1 \# \psi_1} \text{ , i.e. } K_2 \# L_2 \subseteq K_1 \# L_1 \ . \qquad \square$$

Proposition 4.4. *Let K_1, K_2 be compact h-convex bodies in $\mathbb{H}^n$ with associated support maps θ_1, θ_2. Then the harmonic sum $\theta_1 \# \theta_2$ of θ_1 and θ_2 is the support map of a compact h-convex body $K_1 \# K_2$, the harmonic sum of K_1 and K_2. Moreover $K_1, K_2 \subset K_1 \# K_2$.*

Proof. By Lemma 4.3 θ_1, θ_2 are h-supported continuous graphs in $\mathcal{C}^n_+$. So, given $u \in \mathbb{H}^n_\infty$, there exist $x_1(u), x_2(u) \in \mathbb{H}^n$ such that $\theta_1 \leq \theta_{x_1(u)}$, $\theta_2 \leq \theta_{x_2(u)}$ in $\mathcal{C}^n_+$, and $\theta_1(u) = \theta_{x_1(u)}(u)$, $\theta_2(u) = \theta_{x_2(u)}(u)$. According to the proof of Lemma 4.4, this implies $\theta_1 \# \theta_2 \leq \theta_{x_1(u)} \# \theta_{x_2(u)}$ and $(\theta_1 \# \theta_2)(u) = (\theta_{x_1(u)} \# \theta_{x_2(u)})(u)$. According to Proposition 4.3 and Lemma 4.2, $\theta_{x_1(u)} \# \theta_{x_2(u)}$ represents a sphere in $\mathbb{H}^n$, $\theta_{x_1(u)} \# \theta_{x_2(u)}$ is the intersection of an affine hyperplane E with $\mathcal{C}^n_+ \subset \mathbb{R}^{n+1}_1$, and $E \cap \mathbb{H}^n = \emptyset$. Therefore, a suitable rotation of E around $E \cap T_{(\theta_1 \# \theta_2)(u)} \mathcal{C}^n_+$ gives a tangent hyperplane $T_{y(u)} \mathbb{H}^n$ at some $y(u) \in \mathbb{H}^n$, such that $T_{y(u)} \mathbb{H}^n \cap \mathcal{C}^n_+ = \theta_{y(u)}$ and $(\theta_{x_1(u)} \# \theta_{x_2(u)})(u) \in \theta_{y(u)}$. According to this construction we have

$$\theta_1 \# \theta_2 \leq \theta_{x_1(u)} \# \theta_{x_2(u)} \leq \theta_{y(u)} \ .$$

Now, $u \in \mathbb{H}^n_\infty$ was chosen arbitrary, thus we see that $\theta_1 \# \theta_2$ is a h-supported graph in $\mathcal{C}^n_+$. Moreover by the definition of the harmonic sum, $\theta_1 \# \theta_2$ is a continuous graph. Therefore by Lemma 4.3, $\theta_1 \# \theta_2$ is the support image of a compact h-convex body $K_{\theta_1 \# \theta_2} =: K_1 \# K_2$. Finally, Lemma 4.2 implies $K_1, K_2 \subseteq K_1 \# K_2$. $\square$

4.3.1. *The 2-dimensional situation.*

Proposition 4.5. *Let c_1, c_2 be smooth regular curves in $\mathbb{H}^2$, oriented by unit normal fields ν_1, ν_2 and related to one another by parallel support horocycles. If $k_{c_1}, k_{c_2} > 1$, then the length and the total curvature of $c^* = c_1 \# c_2$ write in terms of c_1, c_2 and their relative position to each other in $\mathbb{H}^2$ as follows:*

$$L(c^*) = \frac{1}{2} \left(W(c_1, c_1 \# c_2) - V(c_1, c_1 \# c_2) \right) \tag{4.26}$$

and

$$TC(c^*) = \frac{1}{2} \left(W(c_1, c_1 \# c_2) + V(c_1, c_1 \# c_2) \right) \ , \tag{4.27}$$

where

$$\begin{aligned} W(c_1, c_1\#c_2) &:= L(c^*) + TC(c^*) = \int_{\theta^*} k_{\theta^*}^2 \, d\sigma^* = \\ &= \int_{\theta_1} e^{w_{1*}} \left((\dot{w}_{1*})^2 + 2\ddot{w}_{1*} + k_{\theta_1}^2 \right) \, d\sigma_1 \end{aligned} \tag{4.28}$$

with the relative suppport function $w_{1*}(\sigma_1) = -\ln\left(\frac{\lambda(\sigma_1)}{1+\lambda(\sigma_1)}\right)$ *of* c^* *with respect to* c_1*, and*

$$\begin{aligned} V(c_1, c_1\#c_2) &:= \int_{\theta^*} d\sigma^* = \\ &= \int_{\theta_1} e^{-w_{1*}(\sigma_1)} d\sigma_1 = \int_{\theta_1} \frac{\lambda(\sigma_1)}{1+\lambda(\sigma_1)} d\sigma_1 \, . \end{aligned} \tag{4.29}$$

Proof. We can write $\theta_2 = \lambda\theta_1$, hence by the definition of the harmonic sum $\theta^* = \frac{\lambda}{1+\lambda}\,\theta_1$, hence $d\sigma^* = \frac{\lambda}{1+\lambda}\,d\sigma_1$ (we see here that $d\sigma^*$ is essentially the harmonic mean of $d\sigma_1$ and $d\sigma_2$, cf. Remark 4.4). By the proof of Proposition 4.4 the harmonic sum c^* fulfils $k_{c^*} > 1$, hence $d\sigma^* = (k_{c^*} - 1)\, ds^*$ and

$$L(\theta^*) = \int_{\theta^*} d\sigma^* = \int_{c^*} (k_{c^*} - 1)\, ds^* = TC(c^*) - L(c^*)$$

and (cf. (3.10))

$$L(c^*) = \frac{1}{2} \int_{\theta^*} \left(k_{\theta^*}^2 - 1 \right) \, d\sigma^*.$$

These two equations yield (4.26) and (4.27). The last equality in (4.28) can be proved as (4.15). □

Remark 4.6. If c is closed, then through integration by parts W in (4.28) can be rewritten like (4.17).

4.4. **Support functions and the analogon of the Minkowski sum.** As a base for a support function, we fix a point or a sphere in $\mathbb{H}^n$, or more general a h-convex body with oriented smooth regular boundary hypersurface M_0. Let θ_0 be the associated support map.

Let M be a smooth regular hypersurface oriented by a unit normal field ν, and let $\theta(x) = x + \nu(x)$, $x \in M$, be the respective support map. Then we have $\theta = \lambda\,\theta_0$ along M, and we call, according to (2.1),

$$h = -\ln\lambda \tag{4.30}$$

the *support function* of M with respect to M_0.

Support functions using horospheres and based at points are considered in [Fil70], [San67], [San68]. Support functions using equidistants and based on geodesics through a fixed point are considered in [Lei04].

4.4.1. *The analogon of the Minkowski sum in the 2-dimensional situation.* We use support functions with respect to a point $o \in \mathbb{H}^2$. Then θ_0 is given by the intersection of $\mathcal{C}^2_+$ with the tangent plane $T_o\mathbb{H}^2$, and $k^2_{\theta_0} = 1$. Given two strongly h-convex bodies with smooth regular boundary curves c_i , $i = 1, 2$, oriented by their inner unit normal vectors. Then we have $\theta_i = \lambda_i \theta_0$ and $h_i = -\ln \lambda_i$. We consider $\theta^\diamond = \theta_1 \diamond \theta_2 = \lambda^\diamond \theta_0 = \lambda_1 \lambda_2 \theta_0$, $h^\diamond = -\ln \lambda^\diamond = h_1 + h_2$. In the light of the last equations and the classical euclidean situation, we call the envelope $c_1 \diamond c_2$ of $\theta_1 \diamond \theta_2$ the *Minkowski sum* of c_1 and c_2 with respect to o.
As in (4.15) we have

$$\begin{aligned} \int_{\theta_i} k^2_{\theta_i} d\sigma_i &= \int_{\theta_0} e^{h_i} \left((\dot{h}_i)^2 + 2\ddot{h}_i + k^2_{\theta_0} \right) d\sigma_0 = \\ &= \int_{\theta_0} e^{h_i} \left(\ddot{h}_i + k^2_{\theta_0} \right) d\sigma_0 \,. \end{aligned} \tag{4.31}$$

The c_i are strongly convex, hence $k_{c_i} > 1$ and $k^2_{\theta_i} > 1$ (cf. Remark 3.3). By (3.10), (3.11) we get

$$\begin{aligned} L(c_i) &= \frac{1}{2} \int_{\theta_i} \left(k^2_{\theta_i} - 1 \right) d\sigma_i = \\ &= \frac{1}{2} \int_{\theta_0} e^{h_i} \left(\ddot{h}_i + k^2_{\theta_0} - e^{-2h_i} \right) d\sigma_0 \end{aligned} \tag{4.32}$$

and

$$TC(c_i) = \frac{1}{2} \int_{\theta_0} e^{h_i} \left(\ddot{h}_i + k^2_{\theta_0} + e^{-2h_i} \right) d\sigma_0 \,. \tag{4.33}$$

Remark 4.7. Formula (4.32) rewritten by (4.17) is the work of [Fil70] (note: $k^2_{\theta_0} = 1$).

As in the proof of Proposition 4.5 we get

$$\begin{aligned} L(c_i) &= \frac{1}{2} \left(W(o, c_i) - V(o, c_i) \right) \\ TC(c_i) &= \frac{1}{2} \left(W(o, c_i) + V(o, c_i) \right) \end{aligned} \tag{4.34}$$

with

$$W(o, c_i) := \int_{\theta_0} e^{h_i} \left(\ddot{h}_i + k^2_{\theta_0} \right) d\sigma_0 \tag{4.35}$$

and

$$V(o, c_i) := \int_{\theta_0} e^{-h_i} \, d\sigma_0 \,. \tag{4.36}$$

If $c^\diamond = c_1 \diamond c_2$ is strongly h-convex, then similar computations arrive at

$$\begin{aligned} L(c^\diamond) &= \frac{1}{2}\left(W(o, c_1 \diamond c_2) - V(o, c_1 \diamond c_2)\right) \\ TC(c^\diamond) &= \frac{1}{2}\left(W(o, c_1 \diamond c_2) + V(o, c_1 \diamond c_2)\right), \end{aligned} \tag{4.37}$$

with

$$\begin{aligned} W(o, c_1 \diamond c_2) &:= \int_{\theta_0} e^{h_1+h_2}\left(\ddot{h}_1 + \ddot{h}_2 + k_{\theta_0}^2\right) d\sigma_0 = \\ &= \int_{\theta_0} e^{h_1}\left(\ddot{h}_1 + \frac{1}{2}k_{\theta_0}^2\right) e^{h_2}\, d\sigma_0 + \\ &\quad + \int_{\theta_0} e^{h_2}\left(\ddot{h}_2 + \frac{1}{2}k_{\theta_0}^2\right) e^{h_1}\, d\sigma_0 \end{aligned} \tag{4.38}$$

and

$$\begin{aligned} V(o, c_1 \diamond c_2) &:= \int_{\theta_0} e^{-(h_1+h_2)}\, d\sigma_0 = \\ &= \frac{1}{2}\left(\int_{\theta_2} e^{-h_1}\, d\sigma_2 + \int_{\theta_1} e^{-h_2}\, d\sigma_1\right), \end{aligned} \tag{4.39}$$

where we used $d\sigma_i = \lambda_i\, d\sigma_0 = e^{-h_i} d\sigma_0$.

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