Decay Rates for a class of Diffusive-dominated Interaction Equations

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Abstract

We analyse qualitative properties of the solutions to a mean-field equation for particles interacting through a pairwise potential while diffusing by Brownian motion. Interaction and diffusion compete with each other depending on the character of the potential. We provide sufficient conditions on the relation between the interaction potential and the initial data for diffusion to be the dominant term. We give decay rates of Sobolev norms showing that asymptotically for large times the behavior is then given by the heat equation. Moreover, we show an optimal rate of convergence in the L^1-norm towards the fundamental solution of the heat equation.

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1 Introduction

In this paper we consider the diffusive aggregation equations,

$$\partial_t \rho = \nabla \cdot (\rho(\nabla W * \rho)) + \Delta \rho \tag{1.1a}$$

$$\rho(0, x) = \rho_0, \tag{1.1b}$$

where $\rho = \rho(t, x)$ is a real function depending on time $t \geq 0$ and space $x \in \mathbb{R}^N$, $W : \mathbb{R}^N \longrightarrow \mathbb{R}$ is an *interaction potential* verifying $W(x) = W(-x)$ (without loss of generality, see [14]). These equations have received a lot of attention in the recent years because of their ubiquity in different models and areas of applied and pure mathematics. Collective behavior of animals (swarming), chemotaxis models, and granular media models are some examples, see [23, 24, 10, 5, 14] and the references therein. On the other hand, these equations have been studied in connection to entropy-entropy dissipation techniques, optimal transport, and gradient flows with respect to probability measure distances, see [14, 2] and the references therein.

Without diffusion, the continuity equation (1.1a) with a singular interaction potential W can lead to very involved dynamics where blow-up can occur, and where Dirac Delta singularities and smooth parts of the solution can coexist. More precisely, assume that we have an interaction potential which is radial, smooth away from the origin and whose gradient may be singular at the origin, with a local behavior not worse than that of $|x|$ (i.e., having at worst a Lipschitz singularity at the origin); then, blow-up in finite time of $L^1 \cap L^\infty$ solutions was reported in [6]. In fact, the almost sharp condition which determines the behavior of global existence or blow-up in $L^1 \cap L^\infty$ is the so-called *Osgood condition.* It is given in terms of the size of the gradient of the potential W, which is said to satisfy the Osgood condition if

$$\int_0^1 \frac{1}{k'(r)} dr = +\infty \tag{1.2}$$

with $W(x) = k(|x|)$. Specifically if (1.2) is satisfied, with some mild additional monotonicity conditions on k'', and also $\nabla W \in W^{1,q}$ with $q < N$ and $\rho_0 \in L^1 \cap L^p$ with $p > N/(N-1)$, $N \geq 2$, then there are global weak solutions, see [7]. Aggregation (blow-up) only happens when $t = \infty$, see [6, 7] for the L^1-L^∞ and L^1-L^p results, respectively. In fact, weak measure solutions were proved to exist after the L^∞-blow-up time in a unique way for attractive potentials, see [12] for a definition of weak measure solution and further details. In [12], the authors also illustrate the existence of weak measure solutions with very complicated patterns and Dirac Delta formations.

In this work we address the following issue: under which conditions on the interaction potential can linear diffusion prevail, leading to a diffusive-dominated behavior for large times? More precisely, we give sufficient conditions on the interaction potential and the initial data such that the competition between the possible aggregation due to an attractive interaction potential and the linear diffusion in (1.1a) is won by the latter.

Sharp conditions for separating global existence of solutions from blow-up of solutions to (1.1) have been given in several papers related to homogeneous interaction potentials or to the classical Keller-Segel model [10, 8, 4]. Also, conditions for global existence or blow-up have been given in [20] based on the L^p regularity of the gradient ∇W of the potential. Blow-up conditions have been studied in detail as well for fractional diffusions [22, 21]. However, we are not aware of many results dealing with the asymptotic behavior once the diffusion dominates over the aggregation except for [9] where the authors show that the solutions of the Keller-Segel model behave like the solution of the heat equation for small mass and the recent papers [19, 20] in which they deal precisely with this issue for the equation (1.1); we comment further on these works below.

Here we show that under suitable smallness conditions involving both the interaction potential and the initial data, see Theorem 3.1, the behavior of the solution is determined by the heat equation for large times (see Theorems 3.1, 3.3 and 4.1). In other words, we get a result of asymptotic simplification for all dimensions under some size conditions where the nonlinearity disappears and the decay rates and behavior are like the diffusive equation at least up to first order. We remark that the asymptotic simplification result in L^1 without rate and the decay rates in L^p were obtained in the one dimensional case in [19] under some smallness condition similar to one of the possibilities in Theorem 3.1 below by using scaling arguments. Also, global existence results were reported in the multidimensional case in [20] but no uniform-in-time L^∞ bounds nor decay rates under smallness size conditions were proved.

Asymptotic simplification results have been reported for problems in fluid mechanics [1, 25, 26] and in nonlinear convection-diffusion equations [13]. Here, we use the technique of Fourier splitting [27], a technique quite successful for the $N \geq 3$ dimensional Navier-Stokes equations, together with direct estimates over the bilinear integral term associated to (1.1a) via Duhamel's formula to get the optimal decay rates in Sobolev and L^p spaces in section 3. Section 2 is devoted to setting the basic well-posedness theory of global-in-time solutions with uniform-in-time L^∞ estimates. Finally, Section 4 is devoted to combining these time decay estimates with entropy-entropy dissipation arguments [28, 30, 3, 13, 14, 15, 16] to obtain decay rates in entropy in self-similar variables and in L^1 towards the self-similar heat kernel for large times.

2 Well-posedness and Global bounds

2.1 Notation

We usually omit the variables of the unknown ρ in eq. (1.1), which are understood to be (t, x). Also, we usually write $\rho_t(x) = \rho(t, x)$, which is useful when referring to the function $x \mapsto \rho(t, x)$ for a given time t (we emphasize that ρ_t is not to be confused with $\partial_t \rho$; subindex notation for partial derivatives is never used in this paper). When not specified, integrals are over all of $\mathbb{R}^N$, and in the variable x.

We use the standard multi-index notation for derivatives throughout: for a function $f : \mathbb{R}^N \to \mathbb{C}$ and a multi-index $\gamma = (\gamma_1, \gamma_2, ..., \gamma_N)$, with integers $\gamma_j \geq 0$, we denote $|\gamma| = \sum_{j=1}^N \gamma_j$ and define

$$\partial^\gamma f = \partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2} \dots \partial_{x_N}^{\gamma_N} f.$$

We let $\mathcal{S} = \mathcal{S}(\mathbb{R}^N)$ be the usual Schwartz space of rapidly decreasing functions. The Fourier and inverse Fourier transform of $f \in \mathcal{S}$ are defined by

$$\hat{f}(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} v(x)\, dx \qquad \text{and} \qquad \check{f}(x) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{ix\cdot\xi} f(\xi)\, d\xi,$$

respectively, and extended as usual to $\mathcal{S}'$. If k is a nonnegative integer, $\mathcal{W}^{k,p}(\mathbb{R}^N) = \mathcal{W}^{k,p}$ will signify, as is standard, the Sobolev space consisting of functions in $L^p(\mathbb{R}^N)$ whose generalized derivatives up to order k belong to $L^p(\mathbb{R}^N) = L^p$, $1 \leq p \leq \infty$, with norm $\|\cdot\|_{k,p}$ defined by

$$\|f\|_{k,p}^p := \sum_{|\gamma| \leq k} \|\partial^\gamma f\|_p^p \tag{2.1}$$

for any $f \in \mathcal{W}^{k,p}$, where the sum is over all multi-indices γ with $|\gamma| \leq k$. When $p = 2$, $\mathcal{W}^{k,2}(\mathbb{R}^N) = H^k(\mathbb{R}^N) = H^k$, where the space H^k is defined for all $k \in \mathbb{R}$ as the space of all $f \in \mathcal{S}'$ such that $(1 + |\xi|^2)^{k/2} \hat{f}(\xi) \in L^2$. The norm in H^k (that of $\mathcal{W}^{k,2}$ defined in (2.1)) is sometimes denoted by $\|\cdot\|_{H^k}$ instead of $\|\cdot\|_{k,2}$. The space of bounded continuous functions from an interval I to a normed space X is denoted by $BC(I, X)$.

To simplify the notation, the domain $\mathbb{R}^N$ of the above spaces will usually be omitted.

We will use the well-known fact that for any $m \geq 0$ there is a constant C depending only on m and the dimension N such that

$$\|fg\|_{m,p} \leq C\big(\|f\|_{m,p}\|g\|_\infty + \|f\|_\infty \|g\|_{m,p}\big), \tag{2.2}$$

for functions $f, g \in \mathcal{W}^{m,p} \cap L^\infty$. We also define $D^m\rho$ by $\widehat{D^m\rho} = |\xi|^m \widehat{\rho}$. Hence, by Plancherel's identity $\|D^m\rho\|_2 = \||\xi|^m\widehat{\rho}\|_2$. By C we denote arbitrary constants that can change from line to line.

We recall the Gagliardo-Nirenberg-Sobolev (GNS) inequalities, which we will be using repeatedly in the sequel, see [18, Theorem 9.3] for a proof: given $1 \le q, s \le \infty$ and integers $0 \le j < m$, there exists a number $C > 0$ depending on q, s, j, m and the dimension N such that

$$\|D^j v\|_p \le C\,\|D^m v\|_q^\theta\,\|v\|_s^{1-\theta}, \tag{2.3}$$

with

$$\frac{1}{p} = \frac{j}{N} + \theta\left(\frac{1}{q} - \frac{m}{N}\right) + (1-\theta)\frac{1}{s},$$

where $\frac{j}{m} \le \theta \le 1$, with the following exception: if $m - j - N/q$ is a nonnegative integer, then the GNS inequality (2.3) is only valid for $\frac{j}{m} \le \theta < 1$.

We will make use of the standard heat semigroup $e^{t\Delta}$, which is defined as the convolution in the x variable with the heat kernel

$$G(t,x) := \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}\,, \tag{2.4}$$

whose derivatives satisfy for all integers $m \ge 1$

$$|D^m G(t,x)| \le C\, t^{-\frac{N+m}{2}} e^{-\frac{|x|^2}{8t}} \tag{2.5}$$

for some constant $C > 0$ depending on the dimension N.

2.2 Local-in-time existence

In order to show short-time existence of solutions to the equation (1.1) we use a common fixed-point iteration as also done in [20]. Some elements in the proofs of subsections 2.2 and 2.3 are related to results reported in [20] but we prefer to include them for the sake of the reader. One can formally rewrite (1.1) by using Duhamel's formula and integrating by parts:

$$\rho_t = e^{t\Delta}\rho_0 - \int_0^t \nabla e^{(t-s)\Delta}\big(\rho_s(\nabla W * \rho_s)\big)\,ds, \tag{2.6}$$

where $\nabla e^{t\Delta}$ denotes the convolution in x with the gradient of the heat kernel (2.4). As it is common, we define a mild solution of (1.1) as one that has some reasonable regularity for (2.6) to make sense, and satisfies (2.6). For this definition, and for the short-time existence result 2.4, we do not assume that ρ is nonnegative, as it is not needed for the argument. In the sequel we work in any dimension $N \ge 1$.

Definition 2.1. *Take $T \in (0, +\infty]$, $p \in [1, +\infty]$, and $\rho_0 \in L^p$. Assume that $\nabla W \in (L^q)^N$. A* mild L^p solution *to equation* (1.1) *on* $[0,T)$ *with initial condition ρ_0 is a function $\rho \in L^1_{loc}([0,T), L^p)$ such that* (2.6) *holds for all $t \in (0,T)$.*

Remark 2.2. *Note that $\nabla W * \rho \in L^1_{loc}([0,T), (L^\infty)^N)$, so if $\rho \in L^p$ the product $\rho(\nabla W * \rho)$ is in $\in L^1_{loc}([0,T), (L^p)^N)$ and the integral in* (2.6) *makes sense.*

Remark 2.3. *It is easy to see that whenever a mild solution ρ has enough regularity to be a classical solution, it is then in fact a classical solution. This is: if a mild solution ρ has continuous first-order time derivatives and continuous second-order space derivatives, then it is a classical solution.*

We will show the following short-time existence theorem:

Theorem 2.4 (Short-time existence)**.** *Take $p \in [1, +\infty]$, $m \geq 0$, and $\rho_0 \in \mathcal{W}^{m,p}$. Assume that $\nabla W \in (L^q)^N$, with $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists a maximal time $T^* \in (0, +\infty]$ and a unique mild solution $\rho \in \mathcal{C}([0,T), \mathcal{W}^{m,p})$ of problem (1.1). If $T^* < +\infty$ then*

$$\|\rho_t\|_{m,p} \to +\infty \qquad \text{as } t \to T* . \tag{2.7}$$

Let $p' \in [0, +\infty]$, $m' \geq 0$. If additionally $\rho_0 \in \mathcal{W}^{m',p'}$ and $\nabla W \in (L^{q'})^N$ with $\frac{1}{p'} + \frac{1}{q'} = 1$, then the solution given above belongs to $BC([0,T), \mathcal{W}^{m,p} \cap \mathcal{W}^{m',p'})$.

This result will follow from a standard fixed point theorem for bilinear forms which for completeness we state here, see [11]:

Lemma 2.5. *Let X be an abstract Banach space with norm $\|\cdot\|_X$ and $B : X \times X \to X$ a bilinear operator such that for any $x_1, x_2 \in X$,*

$$\|B(x_1, x_2)\|_X \leq \eta \|x_1\|_X \|x_2\|_X \tag{2.8}$$

then for any $y \in X$ such that

$$4\eta \|y\|_X < 1 \tag{2.9}$$

the equation $x = y + B(x,x)$ has a solution $x \in X$. In particular the solution satisfies $\|x\|_X \leq 2\|y\|_X$ and is the only one such that $\|x\|_X < \frac{1}{2\eta}$.

Proof of Theorem 2.4. Take $T > 0$. Following a standard strategy, we first show that for T small enough there exists a mild solution on $[0,T)$. In order to find a function ρ satisfying (2.6) we choose the bilinear form defined as

$$\mathcal{B}(\rho, \psi) = -\int_0^t \nabla e^{(t-s)\Delta} \cdot (\nabla W * \psi_s)\rho_s \, ds \qquad \text{for } t \in [0,T). \tag{2.10}$$

In order to apply Lemma 2.5 estimate this bilinear form in the space $X := BC([0,T), \mathcal{W}^{m,p})$, with norm given by

$$|||\rho|||_{m,p} := \sup_{t \in [0,T)} \|\rho_t\|_{m,p}.$$

In this proof we denote by C_m a number that depends only on m, which may change from line to line. If γ is a multi-index with $|\gamma| \leq m$, then, for $\rho, \psi \in BC([0,T), \mathcal{W}^{1,p})$, and any $t \in [0,T)$,

$$\begin{aligned}
\|\partial^\gamma \mathcal{B}(\rho,\psi)\|_p &\leq \int_0^t \left\| \nabla e^{(t-s)\Delta} \cdot \partial^\gamma \big((\nabla W * \psi_s)\rho_s \big) \right\|_p ds \\
&\leq C \int_0^t \frac{1}{\sqrt{t-s}} \left\| \partial^\gamma \big((\nabla W * \psi_s)\rho_s \big) \right\|_p ds \\
&\leq C_m \int_0^t \frac{1}{\sqrt{t-s}} \|\nabla W * \psi_s\|_{m,\infty} \|\rho_s\|_{m,p} \, ds \\
&\leq C_m \int_0^t \frac{1}{\sqrt{t-s}} \|\nabla W\|_q \|\psi_s\|_{m,p} \|\rho_s\|_{m,p} \, ds,
\end{aligned} \tag{2.11}$$

which shows, taking the supremum on $[0,T)$ and summing over all multi-indices with $|\gamma| \leq m$, that

$$|||\mathcal{B}(\rho,\psi)|||_{m,p} \leq C_m \sqrt{T} \, \|\nabla W\|_q \, |||\rho_s|||_{m,p} \, |||\psi_s|||_{m,p}.$$

In the third inequality in (2.11) we have used that, for $f \in \mathcal{W}^{m,p}$, $g \in \mathcal{W}^{m,\infty}$,

$$\|\partial^\gamma (fg)\|_p \le C_m \|f\|_{m,p} \, \|g\|_{m,\infty}, \tag{2.12}$$

for any multi-index with $|\gamma| \le m$. This can be easily seen by writing out and estimating the derivatives of the product fg. This gives the estimate (2.8) with $\eta = C_m \sqrt{T}\, \|\nabla W\|_q$. Taking $y \in BC([0,T), \mathcal{W}^{m,p})$ defined by $t \mapsto e^{t\Delta}\rho_0$ in Lemma 2.5, we can choose T small enough so that (2.9) is satisfied. Hence Lemma 2.5, yields the existence of a function $\rho \in BC([0,T), \mathcal{W}^{1,p})$ satisfying (2.6), i.e., a mild solution. This solution is a priori unique only in the set of solutions satisfying $\|\rho\|_{m,p} \le 1/(2\eta)$, but a standard argument using the continuity of ρ shows that it is in fact the unique mild solution in $BC([0,T), \mathcal{W}^{1,p})$.

The existence of a maximal time T^* and the blow-up of the solution at T^* if $T^* < +\infty$ follows now from a standard argument. The last part of Theorem 2.4 is obtained by an analogous reasoning, considering now the space $X := BC([0,T), \mathcal{W}^{m,p} \cap \mathcal{W}^{m',p'})$. $\square$

2.3 A priori time-dependent bounds and global existence

In this section we obtain L^p bounds for our solutions. Due to equation (2.7) in Theorem 2.4 these bounds imply global existence of the solutions.

Proposition 2.6.

i) Let $\rho_0 \in L^1 \cap L^p$ with $\rho_0 \ge 0$, and $\nabla W \in (L^q \cap L^\infty)^N$ with $1/p + 1/q = 1$. Let $\rho \in \mathcal{C}([0,T); L^1 \cap L^p)$ be a mild solution to (1.1) *on $[0,T)$ as obtained in Theorem* 2.4 *with initial data ρ_0. Then there is a constant $C \ge 0$ (depending only on ρ_0 and the dimension N) such that for all $t \in [0,T)$*

$$\|\rho_t\|_1 = \|\rho_0\|_1 =: M, \tag{2.13}$$

$$\|\rho_t\|_p \le \|\rho_0\|_p \exp\left(C \|\nabla W\|_\infty \|\rho_0\|_1 \sqrt{t}\right) := C_p(t)\, \|\rho_0\|_p. \tag{2.14}$$

ii) Assume that $\rho_0 \in \mathcal{W}^{m,2} \cap L^1 \cap L^\infty$ for some $m \ge 1$, and $\nabla W \in (L^1 \cap L^\infty)^N$. Then there is a time-dependent function $C_{m,p}(t)$, bounded on finite time intervals, depending only on ρ_0, $\|\nabla W\|_1$, $\|\nabla W\|_\infty$ and the dimension N, such that for all $t \in [0,T)$,

$$\|\rho_t\|_{H^m} \le C_{m,p}(t)\, \|\rho_0\|_{H^m}. \tag{2.15}$$

To prove this proposition we will use the following modified Gronwall Lemma (see [26]):

Lemma 2.7. *Let $0 \le T \le \infty$, $\delta \in (0,1)$, and let $f : [0,T] \to [0,\infty)$ be continuous and satisfy*

$$f(t) \le A + B \int_0^t (t-s)^{-\delta} f(s)\, ds$$

for all $t \in [0,T)$. Then

$$f(t) \le A\Phi(B\Gamma(1-\delta) t^{1-\delta})$$

for $t \in [0,T)$, where $\Phi : \mathbb{C} \to \mathbb{C}$ is defined by

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n(1-\delta)+1)}$$

Proof of Proposition 2.6. Inequality (2.13) follows by direct integration of equation (1.1) (or of the mild formulation (2.6)) since it has divergence form. To obtain (2.14), take the L^p norm in (2.6) to obtain

$$\begin{aligned}\|\rho_t\|_p &\le \|\rho_0\|_p + C\int_0^t \frac{1}{\sqrt{t-s}}\|\rho_s\|_p \,\|\nabla W * \rho_s\|_\infty \,ds \\ &\le \|\rho_0\|_p + C\int_0^t \frac{1}{\sqrt{t-s}}\|\rho_s\|_p \,\|\nabla W\|_\infty \|\rho_s\|_1 \,ds \\ &\le \|\rho_0\|_p + C\|\nabla W\|_\infty \|\rho_0\|_1 \int_0^t \frac{1}{\sqrt{t-s}}\|\rho_s\|_p \,ds.\end{aligned} \tag{2.16}$$

The modified Gronwall inequality yields (2.14) in Lemma 2.7.

To obtain (2.15) take any multi-index γ with $|\gamma| \le m$, apply ∂^γ to (1.1) and multiply by $\partial^\gamma \rho$ to obtain

$$\frac{1}{2}\frac{d}{dt}\int (\partial^\gamma \rho)^2 = -\int \nabla(\partial^\gamma \rho)\cdot \big(\partial^\gamma(\rho(\nabla W * \rho))\big) - \int |\nabla \partial^\gamma \rho|^2. \tag{2.17}$$

(To make this reasoning rigorous, as we are using eq. (1.1) instead of the weak formulation, we have to carry it out on approximating solutions with smooth initial data and then pass to the limit. This process is straightforward and as such we omit the details.) By Cauchy-Schwarz's inequality the first term can be bounded by

$$\left|\int \nabla(\partial^\gamma \rho)\cdot \big(\partial^\gamma(\rho(\nabla W * \rho))\big)\right| \le \left(\int |\nabla \partial^\gamma \rho|^2\right)^{1/2} \left(\int \big|\partial^\gamma(\rho(\nabla W * \rho))\big|^2\right)^{1/2}. \tag{2.18}$$

To estimate the second parentheses, use equation (2.2) and then Young's convolution inequality to obtain

$$\begin{aligned}\left(\int \big|\partial^\gamma(\rho(\nabla W * \rho))\big|^2\right)^{1/2} &\le \|\rho(\nabla W * \rho)\|_{m,2} \le C\big(\|\rho\|_{m,2}\|\nabla W * \rho\|_\infty + \|\rho\|_\infty \|\nabla W * \rho\|_{m,2}\big) \\ &\le C\|\nabla W\|_\infty \,\|\rho\|_{m,2}\big(\|\rho\|_1 + \|\rho\|_\infty\big) =: C(t)\|\rho\|_{m,2},\end{aligned}$$

with $C(t)$ a given function that involves $\|\nabla W\|_\infty$, $\|\rho_0\|_1$ and $C_\infty(t)$ from (2.14). Using this in (2.18) and applying Young's inequality we get

$$\left|\int \nabla(\partial^\gamma \rho)\cdot \big(\partial^\gamma(\rho(\nabla W * \rho))\big)\right| \le \frac{1}{2}\int |\nabla \partial^\gamma \rho|^2 + C(t)^2\|\rho\|_{m,2}^2. \tag{2.19}$$

Combining this with in (2.17) yields

$$\frac{1}{p}\frac{d}{dt}\int (\partial^\gamma \rho)^2 \le C(t)^2 \|\rho\|_{m,2}^2.$$

Adding all multi-indices γ with $|\gamma| \le m$ gives

$$\frac{d}{dt}\|\rho\|_{m,2}^2 \le C(t)^2\|\rho\|_{m,2}^2$$

for some other time-dependent function $C(t)$. Integrating this inequality over time proves the last part of the Proposition if all derivatives above are well defined. □

Now, we combine the short time existence in Theorem 2.4 with the a priori results in Proposition 2.6 to yield the global existence.

Theorem 2.8. *Under the conditions i) of Proposition* 2.6*, there exists a unique global mild solution ρ of* (1.1) *with $\rho \in \mathcal{C}([0,\infty); L^p)$. Under the conditions ii) of Proposition* 2.6*, there exists a unique global mild solution ρ of* (1.1) *with $\rho \in \mathcal{C}([0,\infty); H^m)$.*

2.4 Uniform bound for $\|\rho\|_\infty$

Consider the solution $\rho \in \mathcal{C}([0,+\infty, L^1 \cap L^\infty)$ obtained from Theorem 2.8 with $p=\infty$. In this section we prove that the L^∞ norm is actually uniformly bounded for all times:

Theorem 2.9. *Let the interaction potential W be such that $\nabla W \in (L^1 \cap L^\infty)^N$. Let $\rho_0 \in L^1 \cap L^\infty$ nonnegative, and suppose ρ is the solution constructed in Theorem 2.8 with data ρ_0. Then there exists a constant $\mathcal{C}_\infty$ depending only on N, W, and $\|\rho_0\|_1$ such that*

$$\|\rho_t\|_\infty \le \mathcal{C}_\infty \quad \text{for all } t \ge 0. \tag{2.20}$$

Proof. Choose any time $t_0 > 0$. From Proposition 2.6 there is a constant $\tilde{C}_0$ for which

$$\|\rho_t\|_\infty \le \tilde{C}_0\,, \qquad t \in [0,t_0]. \tag{2.21}$$

We will prove that, for some $\delta > 0$,

$$\|\rho_t\|_\infty \le \mathcal{C}_\infty\,, \qquad t \in [t_0-\delta,\infty).$$

This will ensure the global-in-time bound for the L^∞ norm.

Pick δ satisfying $t_0 > \delta > 0$ (further conditions on δ will be fixed below). As ρ is a mild solution, we may use Duhamel's formula (2.6) between $t-\delta$ and t to obtain, for any $t \ge \delta$,

$$\rho_t = e^{\delta\Delta}\rho_{t-\delta} - \int_{t-\delta}^{t} \nabla e^{(t-s)\Delta}\big(\rho_s(\nabla W * \rho_s)\big)\,ds. \tag{2.22}$$

For the first term in (2.22) we have

$$\begin{aligned} e^{\delta\Delta}\rho_{t-\delta} &\le \frac{1}{(4\pi\delta)^{N/2}}\int e^{-\frac{|x-y|^2}{4t}}\rho(t-\delta,y)\,dy \\ &\le \frac{1}{(4\pi\delta)^{N/2}}\|\rho_{t-\delta}\|_1 = \frac{1}{(4\pi\delta)^{N/2}}\|\rho_0\|_1. \end{aligned} \tag{2.23}$$

To bound the second term in (2.22) let $g := (\nabla W * \rho_s)\rho_s$,

$$\left|\int_{t-\delta}^{t} \nabla e^{(t-s)\Delta}\cdot g\,ds\right| \le \int_{t-\delta}^{t} \|\nabla G(t-s,\cdot) * g\|_\infty\,ds \le \int_{t-\delta}^{t} \|\nabla G(t-s,\cdot)\|_r\,\|g\|_q\,ds \tag{2.24}$$

with $\frac{1}{r}+\frac{1}{q}=1$. We use the following estimate for $\|g\|_q$:

$$\|g\|_q \le \|\rho_s\|_\infty \|\nabla W * \rho_s\|_q = \|\rho_s\|_\infty\,\|\nabla W\|_q\,\|\rho_0\|_1. \tag{2.25}$$

For $\|\nabla G(t-s,\cdot)\|_r$ we use the following standard estimate deduced from (2.5)

$$\|\nabla G(t,\cdot)\|_r \le Ct^{-\frac{\theta}{2}}\,t^{-\frac{(1-\theta)(1+N)}{2}} = Ct^{\frac{1}{2}(-1-N(1-\theta))} \tag{2.26}$$

with $\theta = \frac{1}{r}$ and $C = C(N)$. To finish the argument it is necessary that the right hand side of the last expression is integrable near 0, hence we need

$$-1 < \frac{1}{2}(-1-N(1-\theta)) \Leftrightarrow 1-\theta < \frac{1}{N} \Leftrightarrow r < \frac{N}{N-1}, \tag{2.27}$$

which means, from (2.27), that we need $N < q \le \infty$. Putting (2.23), (2.24), (2.25) and (2.26) together in (2.22) we obtain, for $t \ge \delta$,

$$\begin{aligned}
\|\rho_t\|_\infty &\le \frac{1}{(4\pi\delta)^{N/2}}\|\rho_0\|_1 + C\|\nabla W\|_q\,\|\rho_0\|_1 \int_{t-\delta}^t (t-s)^{\frac{1}{2}(-1-N(1-\theta))}\|\rho_s\|_\infty\,ds \\
&\le C\,\delta^{-N/2}\|\rho_0\|_1 + C\,C_2 \sup_{\tau\in(t-\delta,t)} \|\rho(\tau)\|_\infty \int_{t-\delta}^t (t-s)^{\frac{1}{2}(-1-N(1-\theta))}\,ds \\
&\le C\,\delta^{-N/2}\|\rho_0\|_1 + C\,C_2\,\delta^\mu \sup_{\tau\in(t-\delta,t)} \|\rho(\tau)\|_\infty
\end{aligned}$$

where $C_2 := \|\nabla W\|_q\,\|\rho_0\|_1$, C is a generic constant depending only on N and $2\mu = 1 - N(1-\theta) > 0$. Now, we take the supremum of both sides of the inequality for $t \ge t_0$ to obtain

$$\begin{aligned}
\sup_{t\ge t_0}\|\rho_t\|_\infty &\le C\,\delta^{-N/2}\|\rho_0\|_1 + C\,C_2\,\delta^\mu \sup_{t\ge t_0-\delta}\|\rho(\tau)\|_\infty \\
&\le C\,\delta^{-N/2}\|\rho_0\|_1 + C\,C_2\,\delta^\mu\,\tilde{C}_0 + C\,C_2\,\delta^\mu \sup_{t\ge t_0}\|\rho(\tau)\|_\infty
\end{aligned}$$

by using the local estimate in time (2.21) for times less than t_0. Choosing $0 < \delta < t_0$ such that $2\,C\,C_2\,\delta^\mu = 1$ we have

$$\sup_{t\ge t_0}\|\rho_t\|_\infty \le 2\,C\,\delta^{-N/2}\|\rho_0\|_1 + \tilde{C}_0 = (2\,C)^{1+N/2\mu}\,\|\nabla W\|_q^{N/(2\mu)}\,\|\rho_0\|_1^{1+N/(2\mu)} + \tilde{C}_0 := \mathcal{C}_\infty.$$

This completes the proof of the Theorem. □

3 Algebraic decay of the solution

3.1 L^2 decay

To obtain the decay in L^2 of the solutions, we first need to show that solutions satisfy an estimate of the form

$$\frac{d}{dt}\int \rho^2\,dx \le -C\int |\nabla\rho|^2\,dx, \tag{3.1}$$

for some $C > 0$. This will hold for *sufficiently small potentials.* From this, given that the solution remains in L^1, decay at the same rate as solutions to the heat equation will follow.

Theorem 3.1. *Let the potential W be such that $\nabla W \in (L^1 \cap L^\infty)^N$ and take $\rho_0 \in L^1 \cap L^\infty \cap H^1$ nonnegative. Assume additionally that one of the following smallness conditions holds:*

i) It holds that $\|\nabla W\|_\infty \|\rho_0\|_1^{\frac{N+4}{N+2}} < 1$.

ii) $W \in L^1$ and $\|W\|_1 \mathcal{C}_\infty < 1$, where $\mathcal{C}_\infty$ is such that (2.20) holds (i.e., $\|\rho_t\|_\infty \le \mathcal{C}_\infty$ for all $t \ge 0$).

iii) $W \in L^2$ and $\|W\|_2 \mathcal{C}_2 < 1$, where $\mathcal{C}_2$ is such that $\|\rho_t\|_2 \le \mathcal{C}_2$ for all $t \ge 0$.

iv) $\mathcal{C}\|\rho_0\|_1 \|[\Delta W]_+\|_{N/2} \le 1/4$ where $\mathcal{C}$ is given below in the proof and $[\Delta W]_+ := \max\{\Delta W, 0\}$.

Then there exists a number $\mathcal{K} > 0$ depending only on the constants defined in the hypotheses such that

$$\|\rho_t\|_2 \le \mathcal{K}\,(t+1)^{-N/2}. \tag{3.2}$$

Proof. We first establish (3.1) in all cases. Multiply equation (1.1a) by ρ and integrate to get

$$\frac{d}{dt}\int \rho^2 dx = \int \rho\nabla\cdot(\rho\nabla W * \rho)dx - \int |\nabla\rho|^2 dx. \tag{3.3}$$

We need to bound the first term on the right hand side of the last equation.

Case i): We use Hölder's and Young's inequalities to obtain

$$\begin{aligned}\int \rho\nabla\cdot(\rho\nabla W * \rho)\,dx = -\int \nabla\rho\cdot(\rho\nabla W * \rho)\,dx &\le \|\nabla\rho\|_2\|\rho\,(\nabla W * \rho)\|_2\\ &\le \|\nabla\rho\|_2\,\|\rho\|_2\,\|\nabla W * \rho\|_\infty \le \|\nabla\rho\|_2\,\|\rho\|_2\,\|\nabla W\|_\infty\,\|\rho\|_1\\ &\le \|\nabla\rho\|_2^2\,\|\nabla W\|_\infty\,\|\rho_0\|_1^{\frac{N+4}{N+2}},\end{aligned}$$

where we interpolated the L^2 norm of ρ between L^1 and $\dot H^1$ by means of the GNS inequality (2.3) with $j=0$, $p=q=2$ and $m=s=1$ and the conservation of mass (2.13). Combining this estimate with (3.3) yields (3.1), due to the condition i) in the hypotheses.

Case ii): By a similar reasoning we have

$$\begin{aligned}\int \rho\nabla\cdot(\rho\nabla W * \rho)dx &= -\int \nabla\rho\cdot(\rho\nabla W * \rho)dx\\ &\le \|\nabla\rho\|_2\,\|\rho\,(W * \nabla\rho)\|_2 \le \|\nabla\rho\|_2^2\,\|\rho\|_\infty\,\|W\|_1 \le \|\nabla\rho\|_2^2\,\mathcal{C}_\infty\,\|W\|_1.\end{aligned}$$

Combining this estimate with (3.3) yields (3.1), since $\mathcal{C}_\infty\,\|W\|_1 < 1$.

Case iii): Similarly,

$$\begin{aligned}\int \rho\nabla\cdot(\rho\nabla W * \rho)dx &= -\int \nabla\rho\cdot(\rho\nabla W * \rho)dx\\ &\le \|\nabla\rho\|_2\|\rho(W * \nabla\rho)\|_2 \le \|\nabla\rho\|_2\|\rho\|_2\|W * \nabla\rho\|_\infty\\ &\le \|\nabla\rho\|_2^2\,\mathcal{C}_2\,\|W\|_2.\end{aligned}$$

This gives (3.1) as before, since $\mathcal{C}_2\,\|W\|_2 < 1$ by hypothesis.

Case iv): Since the Laplacian of the interaction potential lies in a L^p space, we can write

$$\int \rho\nabla\cdot(\rho\nabla W * \rho)dx = -\int \rho\nabla\rho\cdot(\nabla W * \rho)dx = \int \frac{\rho^2}{2}(\Delta W * \rho)dx \le \int \frac{\rho^2}{2}([\Delta W]_+ * \rho) =: I\,.$$

Case iv.a): We first consider the case $N=2$. Then, choosing any $1<p<+\infty$,

$$2I \le \|\rho\|_{2p}^2\|[\Delta W]_+ * \rho\|_q \quad \text{with } \frac{1}{p}+\frac{1}{q}=1.$$

By the GNS inequality (2.3) we have

$$\|\rho\|_{2p}^2 \le C\|\nabla\rho\|_2^{2a}\|\rho\|_1^{2(1-a)}\,, \tag{3.4}$$

where $a=\frac{2p-1}{2p}$. Now we estimate the convolution by

$$\|[\Delta W]_+ * \rho\|_q \le \|[\Delta W]_+\|_1\|\rho\|_q\,. \tag{3.5}$$

Interpolating L^q between L^1 and H^1 by the GNS inequality (2.3) with $b = \frac{1}{p}$ yields the estimate

$$\|\rho\|_q \le C\|\nabla\rho\|_2^b \|\rho\|_1^{1-b} . \tag{3.6}$$

Note that $a + \frac{b}{2} = 1$. Thus combining (3.4), (3.5), and (3.6) gives

$$I \le 2C\|[\Delta W]_+\|_1 \|\nabla\rho\|_2^2 \|\rho\|_1 .$$

Note that p can be chosen to be any $p \in (1, \infty)$, and this final inequality does not depend on the choice (except for the constant C in front of the inequality). This concludes the proof of the case $N = 2$.

Case iv.b): We now consider $N \ge 3$. In this case the nonlinear term is bounded by

$$I \le \frac{1}{2}\|\rho\|_2^2 \|[\Delta W]_+ * \rho\|_\infty \le C\|\rho\|_1^{\frac{4}{N+2}} \|\nabla\rho\|_2^{\frac{2N}{N+2}} \|[\Delta W]_+ * \rho\|_\infty ,$$

by means of the GNS inequality (2.3). To estimate the term $\|[\Delta W]_+ * \rho\|_\infty$, we use Hölder's inequality

$$\|[\Delta W]_+ * \rho\|_\infty \le \|[\Delta W]_+\|_p \|\|\rho\|_q ,$$

with $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. We interpolate the L^q norm between L^1 and $\dot{H}^1$ using again the GNS inequality (2.3) to obtain

$$\|\rho\|_q \le \|\nabla\rho\|_2^a \|\rho\|_1^{1-a} , \tag{3.7}$$

with $\frac{1}{q} = a(\frac{1}{2} - \frac{1}{N}) + 1 - a$, and thus $a = \frac{2N}{p(N+2)}$. For $a \le 1$ we need $p \ge \frac{2N}{2+N}$. We want to choose p so that

$$\frac{N}{2+N} + \frac{a}{2} = \frac{N}{2+N} + \frac{N}{p(N+2)} = 1.$$

This will hold if $p = N/2$, $N \ge 3$. Notice that the GNS inequality (3.7) does not hold for $N = 2$ since we should take $a = 1$ and $q = \infty$, which is not allowed. From the above inequalities, it follows that

$$I \le C\|\rho\|_1 \|\nabla\rho\|_2^2 \|[\Delta W]_+\|_{N/2} .$$

Hence, Case iv) for $N \ge 3$ follows.

We now finish the proof of the L^2-decay in the three cases above. The GNS inequality (2.3) once more gives

$$\|\rho\|_2 \le \|\nabla\rho\|_2^{\frac{N}{N+2}} \|\rho\|_1^{\frac{2}{N+2}}, \quad \text{so} \quad \|\nabla\rho\|_2 \ge \|\rho\|_2^{\frac{N+2}{N}} \|\rho\|_1^{-\frac{2}{N}} .$$

Thus from (3.1) it follows that

$$\frac{d}{dt}\int \rho^2 dx \le -\frac{1}{2}\int |\nabla\rho|^2 dx \le -\mathcal{K}\|\rho\|_2^{\frac{N+2}{N}} .$$

Integration of this differential inequality yields the expected decay. □

Remark 3.2. *Some typical examples of interaction potentials in applications are variations of the so-called Morse potential $W(x) = 1 - e^{-|x|^\alpha}$ with $\alpha \ge 1$, see [6]. For instance, we can get decay for solutions to the aggregation equation with $W(x) = 1 - e^{-|x|^2}$ provided*

$$\mathcal{C} \le \frac{1}{4}\left(\|\rho_0\|_1 \|[\Delta W]_+\|_{N/2}\right)^{-1} ,$$

here $\|[\Delta W]_+\|_{N/2}^{N/2} = \int_{|x|\le N} (|x|^2 - N)^{N/2} e^{-\frac{N}{2}|x|^2} dx$.

3.2 H^m decay

In this subsection we consider the decay in H^m spaces. The aim is to show that for certain potentials the decay rate will be the same as the one for the solutions to the heat equation. Specifically, we show that if the L^2 decay happens at the same rate as for the heat equation (as was shown in Theorem 3.1 under some additional conditions) then the decay in H^m will happen at the same rate as for the heat equation. Our potential will satisfy the hypotheses given in Theorems 2.8 and 2.9, which ensures the existence of a unique global solution, L^∞-uniformly bounded in time. In this subsection, we remind the reader that $D^m\rho$ is defined via the Fourier transform by $\widehat{D^m\rho} = |\xi|^m\widehat{\rho}$, as remarked in Section 2.1.

Theorem 3.3. *Let the potential W satisfy $\nabla W \in (L^1 \cap L^\infty)^N$ and let $\rho_0 \in L^1 \cap L^\infty \cap H^m$, $m \geq 1$, with ρ_0 nonnegative. Consider ρ the solution to* (1.1) *given by Theorems* 2.8 *and* 2.9 *with data ρ_0 . Assume that the solution satisfies the L^2-decay estimate* (3.2)*. Then there exists a constant $C \geq 0$ which depends only on W, ρ_0, m and N such that for all $t \geq 0$*

$$\|D^m\rho\|_2 \leq C(t+1)^{-(N/4+m/2)}\,. \tag{3.8}$$

Proof. In this proof we denote by C any nonnegative number that depends only on the same quantities as the constant C in the statement. We first need to show that for t large enough

$$\frac{d}{dt}\int |D^m\rho|^2\,dx \leq -\frac{1}{2}\int |D^{m+1}\rho|^2\,dx. \tag{3.9}$$

For this, apply the operator D^m to the equation (1.1a), multiply by $D^m\rho$, and integrate in space. After reordering and integration by parts it follows that

$$\frac{d}{dt}\int |D^m\rho|^2 dx = -2\int D^m\rho\, D^m\nabla\cdot(\rho(\nabla W * \rho))dx - 2\int\sum_{j=1}^N |\partial_j D^m\rho|^2 dx\,.$$

Integrating by parts in the first integral on the right hand side yields

$$\frac{d}{dt}\int |D^m\rho|^2 dx = 2\int\sum_{j=1}^N(\partial_j D^m\rho)\, D^m(\rho(\partial_j W * \rho))dx - 2\int\sum_{j=1}^N |\partial_j D^m\rho|^2 dx\,.$$

By Hölder and Young's inequalities, we obtain

$$\frac{d}{dt}\int |D^m\rho|^2 dx \leq \int |D^m(\rho(W * \nabla\rho))|^2 dx - \int\sum_{j=1}^n |\partial_j D^m\rho|^2 dx\,.$$

It follows that

$$\frac{d}{dt}\int |D^m\rho|^2 dx \leq \int |D^m(\rho(W * \nabla\rho))|^2 dx - \int \;|D^{m+1}\rho|^2 dx\,, \tag{3.10}$$

where we have used that

$$\left(\sum_{j=1}^N |\xi_j|^2\right)^{m+1} |\widehat{u}|^2 = \left(\sum_{j=1}^N |\xi_j|^2\right)^m \sum_{k=1}^N |\xi_k|^2|\widehat{u}|^2\,.$$

We need to bound the first term on the right hand side of the inequality (3.10). For this first note that

$$\int |D^m(fg)|^2 dx = \int ||\xi|^m \widehat{fg}|^2 d\xi \leq C_m \left[\int ||\xi|^m \widehat{f} * \widehat{g} d\xi + \int |\xi|^m \widehat{g} * \widehat{f} d\xi \right].$$

Here, we used that $|\xi|^m \leq C_m(|\xi-\eta|^m + |\eta|^m)$. Let $f = \rho$ and $g = W * \nabla\rho$

$$\int |D^m(\rho(W*\nabla\rho))|^2 dx \leq C_m \left[\int |(D^m\rho)(W*\nabla\rho)|^2 dx + \int |\rho(W * D^{m+1}\rho)|^2 dx \right] =: I + II$$

To estimate I, we proceed by means of the GNS inequality (2.3) to get

$$\|D^m\rho\|_2^2 \leq C\|D^{m+1}\rho\|_2^{\frac{2m}{m+1}} \|\rho\|_2^{\frac{2}{m+1}} \qquad \text{and} \qquad \|\nabla\rho\|_2^2 \leq C\|D^{m+1}\rho\|_2^{\frac{2}{m+1}} \|\rho\|_2^{\frac{2m}{m+1}}.$$

This yields

$$I \leq C\|D^m\rho\|_2^2 \, \|(W*\nabla\rho)\|_\infty^2 \leq \|D^{m+1}\rho\|_2^2 \, \|\rho\|_2^2 \, \|W\|_2^2 .$$

Since $W \in L^2$ and $\|\rho\|_2 \to 0$ due to Theorem 3.1, it follows that there exists T_0 so that for all $t \geq T_0$,

$$I \leq 1/4 \|D^{m+1}\rho\|_2^2 . \tag{3.11}$$

To bound II, note that

$$II \leq C\|\rho\|_2^2 \|W * D^{m+1}\rho\|_\infty^2 \leq C\|D^{m+1}\rho\|_2^2 \|\rho\|_2^2 \|W\|_2^2 .$$

Just as in the case for I for $t \geq T_1$, we have

$$I \leq 1/4 \|D^{m+1}\rho\|_2^2 . \tag{3.12}$$

Combining (3.11) and (3.12), for $t \geq \mathcal{T} = \max\{T_0, T_1\}$, inequality (3.9) follows. After applying the Fourier transform and applying Plancherel's Theorem it follows from (3.9) that

$$\frac{d}{dt} \int ||\xi|^m \widehat{\rho}|^2 dx \leq -\frac{1}{2} \int ||\xi|^{m+1} \rho|^2 dx .$$

Now we proceed by Fourier splitting, see [27]. We split the frequency domain into $\mathcal{S}$ and $\mathcal{S}^c$, where

$$\mathcal{S}(t) = \{\xi : |\xi| \leq \mathcal{G}(t)\}, \;\; \mathcal{G}(t) = \left(\frac{2k}{t+1}\right)^{1/2} .$$

Hence by Plancherel we obtain

$$\begin{aligned} \frac{d}{dt} \int |\widehat{D^m\rho}|^2 dx &\leq -\frac{1}{2} \int_{\mathbb{R}^N} |\xi|^2 |\widehat{D^m\rho}|^2 d\xi \leq -\frac{k}{t+1} \int_{\mathcal{S}^c(t)} |\widehat{D^m\rho}|^2 d\xi \\ &\leq -\frac{k}{t+1} \int_{\mathbb{R}^N} |\widehat{D^m\rho}|^2 d\xi + \frac{k}{t+1} \int_{\mathcal{S}(t)} |\widehat{D^m\rho}|^2 d\xi \end{aligned}$$

Hence

$$\frac{d}{dt} \left[(t+1)^k \|D^m\rho_t\|_2^2 \right] \leq C \frac{k}{(t+1)^{1+m}} \int_{\mathcal{S}(t)} |\widehat{\rho}|^2 d\xi \leq C\|\rho_0\|_1^2 \frac{1}{(t+1)^{1+m+N/2}} ,$$

since $\rho \in L^1$. Choosing $k > N/2 + m$, it follows that

$$\frac{d}{dt} \left[(t+1)^k \|D^m\rho_t\|_2^2 \right] \leq C(t+1)^{k-1-N/2-m} ,$$

Integrating on $[\mathcal{T}, t]$ gives the desired decay rate (3.8). This together with the a priori estimate at time $t = \mathcal{T}$ obtained in Proposition 2.6 concludes the proof. □

3.3 L^p decay

We give first the decay for the L^∞ norm. The decay of the solutions in all other L^p spaces follows by interpolation.

Lemma 3.4. *Let the interaction potential W be such that $\nabla W \in (L^1 \cap L^\infty)^N$. Let $\rho_0 \in L^1 \cap L^\infty \cap H^{m+1}$, with $m > N/2$, and ρ_0 nonnegative. Consider ρ the solution to* (1.1a)-(1.1b) *with data ρ_0 given by Theorem* 2.8 *with the properties in Theorems* 2.9 *and* 3.3 *under one of the additional smallness assumptions in Theorem* 3.1. *Then there is some constant $C > 0$ such that for all $t \geq 0$ and $p \in [2, \infty)$,*

$$\|\rho_t\|_p \leq C(t+1)^{-N/2(1-1/p)} . \tag{3.13}$$

Proof. Let us start by the case $p = \infty$. Using the GNS inequality 2.3 with $j = 0$, $a = \infty$, $b = 2$ and $s = 1$, we obtain

$$\|\rho\|_\infty \leq C\|D^m\rho\|_2^{\frac{2N}{2m+N}} \|\rho\|_1^{\frac{2m-N}{2m+N}},$$

valid for any $m > N/2$. Using the decay, we have for $\|D^m\rho\|_2$,

$$\|\rho\|_\infty \leq C(1+t)^{-(\frac{m}{2}+\frac{N}{4})\frac{2N}{2m+N}} = C(1+t)^{-\frac{N}{2}}.$$

The general case $p \in [2, \infty)$ follows by interpolating L^p between L^2 and L^∞. □

3.4 Decay of $\|x\rho\|_2$

We need to study the behavior of the norm $\|x\rho\|_2$, as this norm will appear later in estimates. We begin by studying a moment of ρ:

Lemma 3.5. *Let the interaction potential such that $\nabla W \in (L^1 \cap L^\infty)^N$. Let $\rho_0 \in L^1 \cap L^\infty \cap H^1$ be nonnegative and with $|x|^2\rho_0 \in L^1$, and ρ be a solution to* (1.1a)-(1.1b) *with data ρ_0 given by Theorem* 2.8. *Moreover, assume that for some $C > 0$,*

$$\big|x \cdot \nabla W(x)\big| \leq C\,, \qquad x \in \mathbb{R}^N. \tag{3.14}$$

Then, there is a constant $C = C(\|\rho_0\|_1, N)$ such that for all $t \geq 0$

$$\int |x|^2\rho(t,x)\,dx \leq \int |x|^2\rho_0(x)\,dx + Ct \qquad \textit{and} \qquad \|x\rho_t\|_2^2 \leq C(1+t)^{1-\frac{N}{2}} . \tag{3.15}$$

Proof. Multiplying (1.1a) by $|x|^2$ and integrating, we get

$$\begin{aligned}\frac{d}{dt}\int |x|^2\rho(x)\,dx &= 2N\|\rho_0\|_1 - 2\int \rho(x\cdot(\nabla W * \rho))\,dx \\ &= 2N\|\rho_0\|_1 - \iint (x-y)\rho(x)\rho(y)\nabla W(x-y)\,dx\,dy \leq 2N\|\rho\|_1 + C\|\rho_0\|_1^2,\end{aligned}$$

which establishes (3.15). Now, using (3.13) and (3.15), we get

$$\|x\rho\|_2^2 = \int_{\mathbb{R}^N} |x|^2\rho(x)^2\,dx \leq \|\rho\|_\infty \int_{\mathbb{R}^N} |x|^2\rho(x)\,dx \leq C(1+t)^{1-\frac{N}{2}} .$$

□

4 Asymptotic simplification towards the Heat Equation

In this section we will show that the flow of the aggregation-diffusion equation (1.1) behaves like the heat equation for large times provided we are under conditions for which the L^2 decay of Theorem 3.1 holds. We prove it by two different arguments; though the results obtained by following these two strategies are similar, we have kept both of them because they give a better understanding of the behavior of the equation. The proof in section 4.1 is based on direct estimates of the bilinear form (2.10). It is quite straightforward and gives a better result in the one-dimensional ($N = 1$) case. On the other hand, the argument given in section 4.2 is based on the self-similar change of variables and entropy arguments as in [16, 13], which we briefly describe now.

We consider the standard self-similar scaling [15, 16] as is usually done for the heat equation to pass to the corresponding Fokker-Planck equation:

$$f(s,y) = e^{Ns}\rho\left(\frac{1}{2}(e^{2s}-1), e^s y\right), \tag{4.1}$$

or, equivalently,

$$\rho(t,x) = (1+2t)^{-N/2} f\left(\frac{1}{2}\log(1+2t), (1+2t)^{-1/2}x\right). \tag{4.2}$$

Then f satisfies the equivalent rescaled equation

$$\partial_s f = \nabla_y \cdot (yf) + \Delta_y f + \nabla_y \cdot (f(\nabla_y \widetilde{W} * f)), \tag{4.3}$$

with initial data $f(0,y) = \rho_0(y)$ if and only if ρ is a solution to (1.1). Here, we write

$$\widetilde{W}(s,y) := W(ye^s), \quad \text{so} \quad \nabla_y \widetilde{W}(s,y) := e^s \nabla W(ye^s). \tag{4.4}$$

We consider the entropy functional

$$H[f] := \int \left(f\log f + \frac{|y|^2}{2} f\right) dy. \tag{4.5}$$

We show below that $H[f]$ converges exponentially to 0. The convergence for ρ obtained in this way is stronger, and requires stronger conditions on the initial data. Also, observe that in self-similar variables the nonlinear term transforms to the time-dependent term involving $\widetilde{W}$, where the asymptotic simplification becomes apparent.

The results in this section are gathered in the following theorem. We recall that G is the fundamental solution of the heat equation, defined in (2.4).

Theorem 4.1. *Let the interaction potential $W \in L^1 \cap L^2$ such that $\nabla W \in (L^1 \cap L^\infty)^N$. Consider ρ the solution to* (1.1a)-(1.1b) *with data $\rho_0 \in L^1 \cap L^\infty \cap H^2$, $\rho_0 \geq 0$, constructed in Theorem* 2.8. *We denote $M := \|\rho_0\|$. Assume also one of the smallness conditions in Theorem* 3.1.

i) Then there exists $C > 0$ such that:

(a) For $N = 1$,

$$\left\|\rho_t - MG(t)\right\|_1 \leq \frac{C\log t}{\sqrt{t}} \quad \textit{for all } t \geq 1. \tag{4.6}$$

(b) For $N \geq 2$,

$$\left\|\rho_t - MG(t)\right\|_1 \leq \frac{C}{\sqrt{t}} \quad \textit{for all } t \geq 1. \tag{4.7}$$

ii) Assume further that $|x|^2\rho_0 \in L^1(\mathbb{R}^N)$. Then there exists $C > 0$ such that:

(a) For $N = 1$,

$$H[f] \leq C\, e^{-s} \quad \textit{for all } s > 0. \tag{4.8}$$

(b) For $N = 2$,

$$H[f] \leq C\,(1+s)\, e^{-2s} \quad \textit{for all } s > 0. \tag{4.9}$$

(c) For $N \geq 3$,

$$H[f] \leq C\, e^{-2s} \quad \textit{for all } s > 0. \tag{4.10}$$

Remark 4.2. *We notice that under the above hypotheses Theorems* 2.9 *and* 3.1 *apply to the solution ρ in the statement.*

Remark 4.3. *It is well-known that* (4.10) *directly implies, through the Csiszár-Kullback inequality* [30, 31] *and the change of variables* (4.1)*, that*

$$\left\|\rho_t - MG(t)\right\|_1 \leq C\, t^{-1/2} \quad \textit{for all } t \geq 0\,.$$

This gives the same order of convergence as (4.7) *for any $N \geq 3$. However, the corresponding results for $N = 1$ or 2 are weaker than* (4.6)*. In summary, point ii) above, which uses entropy/entropy dissipation tools, gives a worse rate of convergence than point i), whose proof uses a direct argument. However, one has to take into account that point ii) bounds a nonlinear quantity, the logarithmic entropy, essentially a quadratic functional at first order in expansion around the Maxwellian. Therefore, it is not surprising that we get a weaker result by estimating a quadratic functional. In fact, a direct argument in the spirit of point i) estimating the difference in L^2 of the solution with respect to the fundamental solution of the heat equation leads to similar rates of convergence as the entropy argument.*

4.1 Direct argument

Let us prove point 4.7 of Theorem 4.1. We first give a technical lemma that will be used below:

Lemma 4.4. *For $\alpha \geq 1$, we have the following bound*

$$\int_0^t (t-s)^{-1/2}(1+s)^{-\alpha}\, ds \leq \begin{cases} \frac{C}{t^{1/2}} & \textit{if } \alpha > 1 \\ \frac{C}{t^{1/2}} \log(t) & \textit{if } \alpha = 1 \end{cases} \qquad \textit{for all } t \geq 1,$$

where $C > 0$ is some number which depends only on α.

Proof.

$$\int_0^t (t-s)^{-1/2}(1+s)^{-\alpha}\, ds = \int_0^{t/2} (1+s)^{-\alpha}\, ds + \int_{t/2}^t (t-s)^{-1/2}(1+s)^{-\alpha}\, ds = I + II$$

To estimate I we proceed as follows, bounding $(t-s)^{-1/2}$ by $(t/2)^{-1/2}$:

$$I = \int_0^{t/2} (t-s)^{-1/2}(1+s)^{-\alpha}\, ds \leq \begin{cases} \frac{C}{t^{1/2}} & \text{if } \alpha > 1 \\ \frac{C}{t^{1/2}} \log(t+1) & \text{if } \alpha = 1 \end{cases} \qquad \text{for all } t \geq 1,$$

The estimate of II follows by

$$II = \int_{t/2}^{t} (t-s)^{-1/2}(1+s)^{-\alpha}\, ds \le (1+t/2)^{-\alpha} \int (t-s)^{-1/2}\, ds \le \frac{C}{t^{1/2}}$$

Adding the estimates for I and II gives the conclusion of the Lemma. □

Proof of point 4.7 of Theorem 4.1. Taking into account (2.5), we get $\|\nabla G(t)\|_1 \le C_N t^{-1/2}$, for some constant $C_N > 0$ depending only on the dimension. Using this estimate on the second term in the right-hand side of the Duhamel formula (2.6), we get

$$\begin{aligned}
h(t) &:= \int \int_0^t \left| \Big(\nabla G(t-s,\cdot) * (\rho(s,\cdot)(\nabla W * \rho)(s,\cdot)) \Big)(x) \right| ds\, dx \\
&\le \int_0^t \|\nabla G(t-s,\cdot)\|_1 \; \|\rho(s,\cdot)(\nabla W * \rho)(s,\cdot)\|_1 \, ds \\
&\le C_N \int_0^t (t-s)^{-1/2} \; \|\rho(s,\cdot)\|_2 \|(\nabla W * \rho)(s,\cdot)\|_2 \, ds \\
&\le C_N \int_0^t (t-s)^{-1/2} \; \|\rho(s,\cdot)\|_2 \|W\|_1 \|\nabla \rho(s,\cdot)\|_2 \, ds \\
&\le C \|W\|_1 \int_0^t (t-s)^{-1/2} \; (1+s)^{-(N/2+1/2)} \, ds,
\end{aligned}$$

where we have used Theorems 3.1 and 3.3 for $m = 1$. Now, using Lemma 4.4 for $N \ge 2$, we obtain

$$h(t) \le C \|W\|_1 t^{-1/2} \qquad (t \ge 1),$$

for some constant $C > 0$ which depends on the dimension N. From (2.6), this gives

$$\|\rho_t - e^{t\Delta}\rho_0\|_1 \le C\|W\|_1 t^{-1/2} \quad \text{ for all } t \ge 1.$$

Using the known asymptotic behavior of the heat equation, for instance [17], the claim of the Theorem follows. In the case $N = 1$, the same reasoning using the second bound in Lemma 4.4 gives

$$\|\rho_t - e^{t\Delta}\rho_0\|_1 \le C\|W\|_1 t^{-1/2} \log t \quad \text{ for all } t \ge 1,$$

which proves the result. □

4.2 Entropy argument

We prove the second part of Theorem 4.1:

Proof. In addition to the entropy (4.5), we define as usual the entropy dissipation of the linear Fokker-Planck equation as

$$D[f] := \int f \left| \nabla \left(\frac{|y|^2}{2} + \log f \right) \right|^2 dy\,.$$

As it is classically known [14], the evolution of the free energy for the equation (4.3) can be obtained as

$$\begin{aligned}
&\frac{d}{ds}\left[H[f] + \frac{1}{2} \iint \widetilde{W}(s, x-y) f(x) f(y)\, dx\, dy \right] \\
&\qquad = -\int f \left| \nabla \left(\frac{|y|^2}{2} + \log f + \widetilde{W} * f \right) \right|^2 dy + \frac{1}{2} \int\int \partial_s \widetilde{W}(s, x-y) f(x) f(y)\, dx\, dy,
\end{aligned}$$

and expanding part of the square in the first part yields the term $D[f]$ on the right hand side of the equation

$$\begin{aligned}\frac{d}{ds}\left[H[f]+\frac{1}{2}\iint \widetilde{W}(s,x-y)f(x)f(y)\,dx\,dy\right] &= -D[f]-\int f\left|\nabla\widetilde{W}*f\right|^2\,dy\\ &+2\int f\,\nabla\left(\frac{|y|^2}{2}+\log f\right)\cdot\left(\nabla\widetilde{W}*f\right)\,dy+\frac{1}{2}\iint \partial_s\widetilde{W}(s,x-y)f(x)f(y)\,dx\,dy\\ &=: -D[f]+T_2+T_3+T_4. \quad (4.11)\end{aligned}$$

Recall that due to the classical Logarithmic Sobolev inequality [29, 28], we have

$$2H[f]\le D[f]. \tag{4.12}$$

We show that terms other than $-D[f]$ decay in time at least like e^{-s}. For the term T_3 we have

$$T_3 = 2\int f\,y\cdot(\nabla\widetilde{W}*f)\,dy+2\int \nabla f\cdot(\nabla\widetilde{W}*f)\,dy =: T_{31}+T_{32}, \tag{4.13}$$

We prove that both T_{31} and T_{32} decay exponentially: for T_{31}, using the Cauchy-Schwarz and the Young inequalities,

$$T_{31}\le 2\,\|yf\|_2\,\|\widetilde{W}*\nabla f\|_2\le 2\,\|yf\|_2\,\|\nabla f\|_2\,\|\widetilde{W}\|_1 = 2Ce^{-Ns}\,\|yf\|_2\,\|\nabla f\|_2\le Ce^{-Ns}\,. \tag{4.14}$$

Here, we used that $\|\widetilde{W}\|_1 = e^{-Ns}\,\|W\|_1$ due to (4.4). The last step follows since both $\|yf\|_2$ and $\|\nabla f\|_2$ are uniformly bounded for all times by Theorem 3.3, (3.15) and the change of variables (4.1)-(4.2). By a similar argument for T_{32}, we have

$$T_{32} = 2\int \nabla f\cdot(\widetilde{W}*\nabla f)\le 2\,\|\nabla f\|_2^2\,\|\widetilde{W}\|_1 = 2Ce^{-Ns}\,\|\nabla f\|_2^2\le Ce^{-Ns}. \tag{4.15}$$

Using that $\|\nabla f\|_2$ is uniformly bounded in time due to Theorem 3.3 through the change of variables (4.1)-(4.2). Note that, $T_2\le 0$. For T_4 we use that $\partial_s\widetilde{W}(s,y) = y\cdot\nabla_y\widetilde{W}(s,y)$, the Cauchy-Schwarz and Young inequalities as above, to get

$$T_4 = \frac{1}{2}\int f(\partial_s\widetilde{W}*f)\,dy = \frac{1}{2}\int f\,y\cdot(\nabla\widetilde{W}*f)\,dy = \frac{1}{4}T_{31}\le Ce^{-Ns} \quad \text{for } s\ge 0. \tag{4.16}$$

On the other hand, the integral on the left hand side of (4.11) can be bounded as follows

$$\left|\frac{1}{2}\iint \widetilde{W}(s,x-y)f(x)f(y)\,dx\,dy\right|\le \frac{1}{2}\int f(|\widetilde{W}|*f)\,dy\le \frac{1}{2}\|f\|_2^2\|\widetilde{W}\|_1\le Ce^{-Ns}. \tag{4.17}$$

Hence from (4.11), using (4.12), (4.13), (4.14), (4.15), (4.16), and (4.17), we obtain for all $s\ge 0$ that

$$\begin{aligned}\frac{d}{ds}\left(H[f]+\frac{1}{2}\int f(\widetilde{W}*f)\,dy\right) &\le -2\left(H[f]+\frac{1}{2}\int f(\widetilde{W}*f)\,dy\right)+Ce^{-Ns}+\int f(\widetilde{W}*f)\,dy\\ &\le -2\left(H[f]+\frac{1}{2}\int f(\widetilde{W}*f)\,dy\right)+Ce^{-Ns}\,.\end{aligned}$$

Hence, from this differential inequality we deduce that there is some constant C which depends only on the initial condition f_0 such that

$$H[f]+\frac{1}{2}\int f(\widetilde{W}*f)\le Ce^{-\min\{N,2\}s},$$

which implies, using again (4.17),

$$H[f] \le Ce^{-\min\{N,2\}s},$$

for some other number C depending only on f_0. The final steps are just classical implications of this entropy control. Denoting the Maxwellian by

$$\mathcal{M}(y) = (2\pi)^{-N/2} \exp\left(-\frac{|y|^2}{2}\right).$$

By means of the Csiszár-Kullback inequality, see [28, 30, 31], we get

$$\|f(s) - M\mathcal{M}(s)\|_1^2 \le C(H(f(s)) - H(\mathcal{M}(s))) \le Ce^{-\min\{N,2\}s},$$

from which the announced result follows through the change of variables (4.1)-(4.2). $\square$

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