RECTIFIABLE MEASURES, SQUARE FUNCTIONS INVOLVING DENSITIES, AND THE CAUCHY TRANSFORM

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Abstract. This paper is devoted to the proof of two related results. The first one asserts that if μ is a Radon measure in $\mathbb{R}^d$ satisfying

$$\limsup_{r\to 0}\frac{\mu(B(x,r))}{r}>0 \qquad\text{and}\qquad \int_0^1\left|\frac{\mu(B(x,r))}{r}-\frac{\mu(B(x,2r))}{2r}\right|^2\frac{dr}{r}<\infty$$

for μ-a.e. $x\in\mathbb{R}^d$, then μ is rectifiable. Since the converse implication is already known to hold, this yields the following characterization of rectifiable sets: a set $E\subset\mathbb{R}^d$ with finite 1-dimensional Hausdorff measure $\mathcal{H}^1$ is rectifiable if and only

$$\int_0^1\left|\frac{\mathcal{H}^1(E\cap B(x,r))}{r}-\frac{\mathcal{H}^1(E\cap B(x,2r))}{2r}\right|^2\frac{dr}{r}<\infty \qquad \text{for } \mathcal{H}^1\text{-a.e. } x\in E.$$

The second result of the paper deals with the relationship between a similar square function in the complex plane and the Cauchy transform $\mathcal{C}_\mu f(z)=\int\frac{1}{z-\xi}\,f(\xi)\,d\mu(\xi)$. Suppose that μ has linear growth, that is, $\mu(B(z,r))\leq c\,r$ for all $z\in\mathbb{C}$ and all $r>0$. It is proved that $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$ if and only if

$$\int_{z\in Q}\int_0^\infty\left|\frac{\mu(Q\cap B(z,r))}{r}-\frac{\mu(Q\cap B(z,2r))}{2r}\right|^2\frac{dr}{r}\,d\mu(z)\leq c\,\mu(Q) \qquad \text{for every square } Q\subset\mathbb{C}.$$

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1. Introduction

A set $E \subset \mathbb{R}^d$ is called n-rectifiable if there are Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^d$, $i = 1, 2, \ldots$, such that

$$\mathcal{H}^n\left(\mathbb{R}^d \setminus \bigcup_i f_i(\mathbb{R}^n)\right) = 0, \tag{1.1}$$

where $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure. Also, one says that a a Radon measure μ on $\mathbb{R}^d$ is n-rectifiable if μ vanishes out of an n-rectifiable set $E \subset \mathbb{R}^d$ and moreover μ is absolutely continuous with respect to $\mathcal{H}^n|_E$. On the other hand, E is called purely n-unrectifiable if for $\mathcal{H}^n(F \cap E) = 0$ for any n-rectifiable set $F \subset \mathbb{R}^d$. In the case $n = 1$, instead of saying that a set or a measure is 1-rectifiable, one just says that it is rectifiable.

One of the main objectives of geometric measure theory consists in characterizing n-rectifiable sets and measures in different ways. For instance, there are characterizations in terms of the almost everywhere existence of approximate tangent planes, in terms of the size of projections on n-planes, and in terms of the existence and densities. To describe the latter characterization in detail, we need to introduce some terminology.

Given a Radon measure μ and $x \in \mathbb{R}^d$ we denote

$$\Theta^{n,*}(x,\mu) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{(2r)^n}, \qquad \Theta^n_*(x,\mu) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{(2r)^n}.$$

These are the upper and lower n-dimensional densities of μ at x. If they coincide, they are denoted by $\Theta^n(x,\mu)$. In the case when $\mu = \mathcal{H}^n|_E$ for some set $E \subset \mathbb{R}^d$, we also write $\Theta^{n,*}(x,E)$, $\Theta^n_*(x,E)$, $\Theta^n(x,E)$ instead of $\Theta^{n,*}(x,\mathcal{H}^n|_E)$, $\Theta^n_*(x,\mathcal{H}^n|_E)$, $\Theta^n(x,\mathcal{H}^n|_E)$, respectively.

The following result is due to Besicovitch for $n = 1$, $d = 2$, to Marstrand [Mar] for $n = 2$, $d = 3$, and to Mattila [Mat1] for arbitrary n, d.

Theorem A. *Let n be a positive integer and let $E \subset \mathbb{R}^d$ be $\mathcal{H}^n$-measurable with $\mathcal{H}^n(E) < \infty$. We have:*

(a) *E is n-rectifiable if and only if $\Theta^n(x,E)$ exists and equals 1 for $\mathcal{H}^n$-a.e. $x \in E$.*
(b) *E is purely n-unrectifiable if and only if $\Theta^n_*(x,E) < 1$ for $\mathcal{H}^n$-a.e. $x \in E$.*

Another fundamental result concerning the relationship between rectifiability and densities is given by the following celebrated theorem of Preiss [Pr].

Theorem B. *A Radon measure μ in $\mathbb{R}^d$ is n-rectificable if and only if the density $\Theta^n(x,\mu)$ exists and is non-zero for μ-a.e. $x \in \mathbb{R}^d$.*

In particular, for $\mu = \mathcal{H}^n|E$ with $\mathcal{H}^n(E) < \infty$, the preceding theorem ensures the n-rectifiability of E just assuming that the density $\Theta^n(x,E)$ exists and is non-zero for $\mathcal{H}^n$-a.e. $x \in E$.

Quite recently, in the works [CGLT] and [TT], the authors have obtained some results which can be considered as square function versions of Preiss theorem. In particular, in [TT] the following is proved:

Theorem C. *Let μ be a Radon measure in $\mathbb{R}^d$ such that $0 < \Theta^n_*(x,\mu) \leq \Theta^{n,*}(x,\mu) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$. Then μ is n-rectifiable if and only if*

$$\int_0^1 \left| \frac{\mu(B(x,r))}{r^n} - \frac{\mu(B(x,2r))}{(2r)^n} \right|^2 \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{1.2}$$

This theorem was preceded by the proof of a related result in [CGLT] which characterizes the so called uniform n-rectifiability in terms of a square function similar to the one in (1.2). See the next section for the precise definition of uniform rectifiability and the statement of this result.

A natural question is if the condition (1.2) above implies the n-rectifiability of E just under the assumption that $0 < \Theta^{n,*}(x,\mu) < \infty$ μ-a.e. If this were true, then we would deduce that a set $E \subset \mathbb{R}^d$ with $\mathcal{H}^n(E) < \infty$ is n-rectifiable if and only if

$$\int_0^1 \left| \frac{\mathcal{H}^n(E \cap B(x,r))}{r^n} - \frac{\mathcal{H}^n(E \cap B(x,2r))}{(2r)^n} \right|^2 \frac{dr}{r} < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in E.$$

The arguments used in [TT] make an essential use of the assumption that the lower density $\Theta_*(x,\mu)$ is positive. So different techniques are required if one wants to extend Theorem C to the case of vanishing lower density. In the present paper we solve this problem in the case $n = 1$:

Theorem 1.1. *Let μ be a Radon measure in $\mathbb{R}^d$ such that $\Theta^{1,*}(x,\mu) > 0$ for μ-a.e. $x \in \mathbb{R}^d$. Then μ is rectifiable if and only if*

$$\int_0^1 \left| \frac{\mu(B(x,r))}{r} - \frac{\mu(B(x,2r))}{2r} \right|^2 \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{1.3}$$

Corollary 1.2. *Let $E \subset \mathbb{R}^d$ be a Borel set with $\mathcal{H}^1(E) < \infty$. The set E is rectifiable if and only if*

$$\int_0^1 \left| \frac{\mathcal{H}^1(E \cap B(x,r))}{r} - \frac{\mathcal{H}^1(E \cap B(x,2r))}{2r} \right|^2 \frac{dr}{r} < \infty \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in E.$$

I do not know if the analogous result in the case $n > 1$ holds.

Note that the "only if" part of Theorem 1.1 is an immediate consequence of Theorem C above. Indeed, if μ is rectifiable, then it follows easily that $0 < \Theta^n_*(x,\mu) \leq \Theta^{n,*}(x,\mu) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$. So the assumptions of Theorem C are fulfilled and thus (1.2) holds.

In the present paper we prove the "if" implication of Theorem 1.1. This combines a compactness argument which originates from [CGLT] and constructive techniques involving stopping time conditions. One of the main difficulties, which is absent in [TT], consists in controlling the oscillations of the densities $\frac{\mu(B(x,r))}{r}$ as $r \to 0$. If the power in the integrand of (1.3) were 1 instead of 2, then this task would be significantly easier, and we could argue as in [TT] or as in [ADT].

In our arguments, a basic tool for the control of such oscillations of the density is the construction of suitable measures σ^k supported on some approximating curves Γ^k so that, for each k, σ^k has linear growth with some absolute constant and such that the $L^2(\sigma^k)$ norm of a smooth version of the square function in (1.3), with μ replaced by σ^k, is very small. The main obstacle to extend Theorem 1.1 to higher dimensions lies in the difficulty to extend this construction to the case $n > 1$.

In the final part of this paper we prove a striking connection between the boundedness in $L^2(\mu)$ of the square function

$$T\mu(x) = \left(\int_0^\infty \left|\frac{\mu(B(x,r))}{r} - \frac{\mu(B(x,2r))}{2r}\right|^2 \frac{dr}{r}\right)^{1/2}$$

and the $L^2(\mu)$ boundedness of the Cauchy transform. Recall that given a complex Radon measure ν on $\mathbb{C}$, its Cauchy transform is defined by

$$\mathcal{C}\nu(z) = \int \frac{1}{z-\xi}\,d\nu(\xi),$$

whenever the integral makes sense. For $\varepsilon > 0$, the ε-truncated Cauchy transform of ν is given by

$$\mathcal{C}_\varepsilon\nu(z) = \int_{|z-\xi|>\varepsilon} \frac{1}{z-\xi}\,d\nu(\xi).$$

Note that the last integral is absolutely convergent for all $z \in \mathbb{C}$, unlike the integral defining $\mathcal{C}\nu(z)$, in general. Given $f \in L^p(\mu)$, one denotes $\mathcal{C}_\mu f = \mathcal{C}(f\,\mu)$ and $\mathcal{C}_{\mu,\varepsilon} f = \mathcal{C}_\varepsilon(f\,\mu)$. One says that $\mathcal{C}_\mu$ is bounded in $L^p(\mu)$ if and only if the operators $\mathcal{C}_{\mu,\varepsilon}$ are bounded in $L^p(\mu)$ uniformly on $\varepsilon > 0$.

In the particular case when $\mu = \mathcal{H}^1|_E$ with $\mathcal{H}^1(E) < \infty$, by the theorem of David-Léger [Lé], the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$ implies the rectifiability of E. So it is natural to expect some relationship between the behaviors of the Cauchy transform of μ and of the square function $T\mu$. The next theorem, which is the second main result of this paper, shows that indeed there is a very strong and precise connection between the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$ and the $L^2(\mu)$ behavior of $T\mu$ for arbitrary measures μ with linear growth.

Theorem 1.3. *Let μ be a finite Radon measure in $\mathbb{C}$ satisfying the linear growth condition*

$$\mu(B(x,r)) \leq c\,r \qquad \text{for all } x \in \mathbb{C} \text{ and all } r > 0.$$

The Cauchy transform $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$ if and only if

(1.4)
$$\int_{x\in Q}\int_0^\infty \left|\frac{\mu(Q\cap B(x,r))}{r} - \frac{\mu(Q\cap B(x,2r))}{2r}\right|^2 \frac{dr}{r}\,d\mu(x) \leq c\,\mu(Q) \qquad \text{for every square } Q \subset \mathbb{C}.$$

The behavior of the square function $T\mu$ is related to the cancellation properties of the densities $\frac{\mu(B(x,r))}{r}$, $x \in \mathbb{C}$, $r > 0$. On the other hand, heuristically the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$ seems to be more connected to the behavior of the approximate tangents to μ. So it is quite remarkable (to the author's point of view) that the behavior of $T\mu$ is so strongly connected to the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$, as shown in the preceding theorem.

The proof of Theorem 1.3 uses a corona decomposition analogous to the one of [To1]. We will see in this paper that, loosely speaking, the condition (1.4) is equivalent to the existence of a corona decomposition such as the one mentioned above, which in turn is equivalent to the $L^2(\mu)$ boundedness of the Cauchy transform because of the results of [To1].

The plan of the paper is the following. In Section 2 we introduce some notation and terminology and we review some results which will be needed later. Section 3 contains a blow up argument which, roughly speaking, shows that, given a ball $B(x_0, r_0)$, if

$$\int_{\delta r_0}^{\delta^{-1} r_0} \left| \frac{\mu(B(x,r))}{r} - \frac{\mu(B(x,2r))}{2r} \right|^2 \frac{dr}{r}$$

is very small for a big proportion in measure μ of points $x \in B(x_0, \delta^{-1} r_0)$, with $\delta > 0$ sufficiently small, then the measure μ is close to a flat measure in $B(x_0, r_0)$. The argument is quite similar to the one used for the AD-regular case in [CGLT] (see Section 2 for the definition of AD-regular measures). Next, in Section 4 we review the construction of the dyadic cells from David-Mattila [DaM], which will be very useful for the proof of Theorem 1.1.

In Section 5 we state the Main Lemma 5.1. In a sense, this lemma asserts, in some quantitative way, that given a doubling dyadic cell R with side length $\ell(R)$, if

$$\int_{0}^{\delta^{-1} \ell(R)} \left| \frac{\mu(B(x,r))}{r} - \frac{\mu(B(x,2r))}{2r} \right|^2 \frac{dr}{r}$$

is very small for a big proportion in μ-measure of the points x near R and δ is small enough, then either a big proportion of the measure $\mu|_R$ is concentrated on an AD-regular curve, or $\frac{\mu(B(x,r))}{r} \ll \frac{\mu(B(x,\ell(R)))}{\ell(R)}$ for many points $x \in \operatorname{supp} \mu$ and some $r = r(x) \leq \ell(R)$. In the same section, we show how Theorem 1.1 follows from the Main Lemma 5.1 by means of a suitable corona type decomposition.

Sections 6-16 are devoted to the proof of the Main Lemma. In Sections 6-9 we introduce some stopping cells and an auxiliary measure $\widetilde{\mu}$ and we prove some related results. In Section 10 we construct some AD-regular curves Γ^k and in Section 12 we construct measures ν^k supported on Γ^k which, in a sense, approximate μ. Section 14 deals with the construction of the aforementioned auxiliary measures σ^k, which are supported on Γ^k. In this section we also obtain some suitable square function estimates involving σ^k, which will be used in the subsequent section to estimate the $L^2(\sigma^k)$ norm of the density of ν^k with respect to σ^k. This is the main ingredient used in Section 16 to show that there are very few stopping cells of high density, and to finish the proof of the Main Lemma.

Sections 17-19 deal with the proof of Theorem 1.3. By means of the Main Lemma 5.1, in Section 17 it is shown that if the condition (1.4) holds, then one can construct a corona type decomposition for μ analogous to the one of [To1], which suffices to show that the Cauchy transform is bounded in $L^2(\mu)$. In the subsequent section, some Calderón-Zygmund type results are obtained for the square function operator $T_\mu f := T(f\mu)$, which will be necessary later to show the remaining implication of Theorem 1.3, namely that the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$

implies (1.4). This is proved in the final Section 19 of this paper, relying on the corona type decomposition for μ constructed in [To1].

2. Preliminaries

In this paper the letters c, C stand for some constants which may change their values at different occurrences. On the other hand, constants with subscripts, such as c_1, do not change their values at different occurrences. The notation $A \lesssim B$ means that there is some fixed constant c (usually an absolute constant) such that $A \leq c\,B$. Further, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$. We will also write $A \approx_{c_1,c_2} B$ and $A \lesssim_{c_1,c_2} B$ if we want to make explicit the dependence on the constants c_1 and c_2 of the relationships "$\approx$" and "$\lesssim$".

2.1. **AD-regular and uniformly rectifiable measures.** A measure μ is called n-AD-regular (or just AD-regular or Ahlfors-David regular) if there exists some constant $c_0 > 0$ such that

$$c_0^{-1} r^n \leq \mu(B(x,r)) \leq c_0\, r^n \quad \text{ for all } x \in \operatorname{supp}(\mu) \text{ and } 0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu)).$$

A measure μ is uniformly n-rectifiable if it is n-AD-regular and there exist $\theta, M > 0$ such that for all $x \in \operatorname{supp}(\mu)$ and all $r > 0$ there is a Lipschitz mapping g from the ball $B_n(0,r)$ in $\mathbb{R}^n$ to $\mathbb{R}^d$ with $\operatorname{Lip}(g) \leq M$ such that

$$\mu(B(x,r) \cap g(B_n(0,r))) \geq \theta r^n.$$

In the case $n = 1$, μ is uniformly 1-rectifiable if and only if $\operatorname{supp}(\mu)$ is contained in a rectifiable curve Γ in $\mathbb{R}^d$ such that the arc length measure on Γ is 1-AD-regular.

A set $E \subset \mathbb{R}^d$ is called n-AD-regular if $\mathcal{H}^n|_E$ is n-AD-regular, and it is called uniformly n-rectifiable if $\mathcal{H}^n|_E$ is uniformly n-rectifiable.

The notion of uniform rectifiability was introduced by David and Semmes [DaS1], [DaS2]. In these works they showed that a big class of singular singular integrals with odd kernel is bounded in $L^2(\mu)$ if μ is uniformly rectifiable. See [NToV] for a recent related result in the converse direction involving the n-dimensional Riesz transforms.

In [CGLT] it is shown that uniform n-rectifiability can be characterized as follows.

Theorem 2.1. *Let μ be an n-AD-regular measure. Then μ is uniformly n-rectifiable if and only if there exists a constant c such that, for any ball $B(x_0,R)$ centered at* $\operatorname{supp}(\mu)$*,*

$$\int_{x\in B(x_0,R)} \int_0^R \left| \frac{\mu(B(x,r))}{r} - \frac{\mu(B(x,2r))}{2r} \right|^2 \frac{dr}{r}\, d\mu(x) \leq c\, R^n.$$

2.2. **The β and α coefficients.** Given a closed ball $B \subset \mathbb{R}^d$, we set

$$\beta_{\mu,1}(B) = \inf_L \frac{1}{\mu(B)} \int_B \frac{\operatorname{dist}(y,L)}{r(B)}\, d\mu(y),$$

where $r(B)$ stands for the radius of B and the infimum is taken over all the lines L. The L^∞ version is the following:

$$\beta_{\mu,\infty}(B) = \inf_L \sup_{y \in B \cap \operatorname{supp}\mu} \frac{\operatorname{dist}(y,L)}{r(B)}.$$

The analogous bilateral coefficients are defined by

$$b\beta_{\mu,1}(B) = \inf_L \left[\frac{1}{\mu(B)} \int_B \frac{\operatorname{dist}(y,L)}{r(B)}\, d\mu(y) + \frac{1}{r(B)} \int_{L\cap B} \frac{\operatorname{dist}(x, \operatorname{supp}\mu)}{r(B)}\, d\mathcal{H}^1(x) \right],$$

and

$$b\beta_{\mu,\infty}(B) = \inf_L \left[\sup_{y\in B\cap \operatorname{supp}\mu} \frac{\operatorname{dist}(y,L)}{r(B)} + \sup_{y\in L\cap B} \frac{\operatorname{dist}(x, \operatorname{supp}\mu)}{r(B)} \right].$$

Below we will use the so called α coefficients from [To2]. They are defined as follows. Given a closed ball $B \subset \mathbb{R}^d$ which intersects $\operatorname{supp}(\mu)$, and two Radon measures σ and ν in $\mathbb{R}^d$, we set

$$\operatorname{dist}_B(\sigma,\nu) := \sup\Big\{ \Big| \int f\, d\sigma - \int f\, d\nu \Big| : \operatorname{Lip}(f) \le 1,\, \operatorname{supp} f \subset B \Big\},$$

where $\operatorname{Lip}(f)$ stands for the Lipschitz constant of f. It is easy to check that this is indeed a distance in the space of finite Borel measures supported in the interior of B. See [Chapter 14, Ma] for other properties of this distance. In fact, this is a variant of the well known Wasserstein distance W_1 from mass transport. Given a subset $\mathcal{A}$ of Radon measures in $\mathbb{R}^d$, we set

$$\operatorname{dist}_B(\mu,\mathcal{A}) := \inf_{\sigma\in\mathcal{A}} \operatorname{dist}_B(\mu,\, \sigma).$$

We define

$$\alpha_\mu(B) := \frac{1}{r(B)\,\mu(B)} \inf_{c\ge 0, L} \operatorname{dist}_B(\mu,\, c\mathcal{H}^1|_L),$$

the infimum is taken over all the constants $c \ge 0$ and all the lines L. Also, we denote by c_B and L_B a constant and a line that minimize $\operatorname{dist}_{B_Q}(\mu,\, c\mathcal{H}^1|_L)$, respectively (it is easy to check that this minimum is attained). We also write $\mathcal{L}_B := c_B \mathcal{H}^1|_{L_1}$, so that

$$\alpha_\mu(B) = \frac{1}{r(B)\mu(B)} \operatorname{dist}_B(\mu,\, \mathcal{L}_B).$$

Let us remark that c_B and L_B (and so $\mathcal{L}_B$) may be not unique. Moreover, we may (and will) assume that $L_B \cap B \neq \varnothing$.

Lemma 2.2. *Let $B, B' \subset \mathbb{R}^d$ be two balls. The coefficients $\alpha_\mu(\cdot)$ satisfy the following properties:*

(a) $\alpha_\mu(B) \lesssim 1$.
(b) *If $B \subset B'$, $r(B) \approx r(B')$, and $\mu(B) \approx \mu(B')$, then $\alpha_\mu(B) \lesssim \alpha_\mu(B')$.*
(c) *If $\mu(\frac14 B) \approx \mu(B)$ and $\alpha_\mu(B) \le c_1$, where c_1 is some constant small enough, then $L_B \cap \frac12 B \neq \varnothing$ and $c_B \approx \frac{\mu(B)}{r(B)}$.*

Proof. The statements (a) and (b) are direct consequences of the definitions.

Let us turn our attention to (c). To show that $L_B \cap \frac12 B \neq \varnothing$ if c_1 is small enough, take a smooth function function φ such that $\chi_{\frac14 B} \le \varphi \le \chi_{\frac12 B}$ with $\|\nabla\varphi\|_\infty \lesssim 1/r(B)$. Then we have $\|\nabla(\varphi \operatorname{dist}(\cdot, L_B))\|_\infty \lesssim 1$, and since $\varphi \operatorname{dist}(\cdot, L_B)$ vanishes on L_B, we have

$$\left| \int \varphi(x) \operatorname{dist}(x, L_B)\, d\mu(x) \right| \lesssim \alpha_\mu(B)\, r(B)\, \mu(B).$$

On the other hand,

$$\begin{aligned}\int \varphi(x)\operatorname{dist}(x, L_B)\, d\mu(x) &\geq \operatorname{dist}(\operatorname{supp}(\varphi),\, L_B) \int \varphi\, d\mu \\ &\gtrsim \operatorname{dist}(\operatorname{supp}(\varphi),\, L_B)\, \mu(\tfrac{1}{4}B) \\ &\approx \operatorname{dist}(\operatorname{supp}(\varphi),\, L_B)\, \mu(B).\end{aligned}$$

If $\alpha_\mu(B)$ is small enough we infer that $\operatorname{dist}(\operatorname{supp}(\varphi), L_B) \leq r(B)/10$, and so $L_B \cap \frac{1}{2}B \neq \varnothing$.

Let us check now that $c_B \approx \frac{\mu(B)}{r(B)}$. Let ψ be a smooth function such that $\chi_{\frac{1}{2}B} \leq \psi \leq \chi_B$ and $\|\nabla\psi\|_\infty \lesssim 1/r(B)$. Then

$$\left|\int \psi\, d\mu - \int \psi\, d\mathcal{L}_B\right| \lesssim \alpha_\mu(B)\mu(B).$$

Thus,

$$\int \psi\, d\mu - C\alpha_\mu(B)\mu(B) \leq c_B \int \psi\, d\mathcal{H}^1|_{L_B} \leq \int \psi\, d\mu + C\alpha_\mu(B)\mu(B).$$

From the second inequality, we deduce easily that $c_B \lesssim \frac{\mu(B)}{r(B)}$. From the first one, we see that if $\alpha_\mu(B) \leq c_1$, where c_1 is small enough, then

$$c_B \int \psi\, d\mathcal{H}^1|_{L_B} \geq \frac{1}{2}\,\mu(B) - C\alpha_\mu(B)\mu(B) \geq \frac{1}{4}\mu(B),$$

which implies that $c_B \gtrsim \frac{\mu(B)}{r(B)}$. □

We have the following relationship between $\beta_{\mu,1}(B)$, $b\beta_{1,\mu}(B)$ and $\alpha_\mu(B)$:

Lemma 2.3. *Let $B \subset \mathbb{R}^d$ be a ball such that $\mu(\frac{1}{2}B) \approx \mu(2B)$. Then we have*

$$\beta_{\mu,1}(B) \leq b\beta_1(B) \lesssim \alpha_\mu(2B).$$

In fact,

$$\int_B \frac{\operatorname{dist}(y, L_B)}{r(B)\mu(B)}\, d\mu(y) + \int_{L_B\cap B} \frac{\operatorname{dist}(x, \operatorname{supp}\mu)}{r(B)^2}\, d\mathcal{H}^1|_{L_B}(x) \lesssim \alpha_\mu(2B).$$

This result has been proved in the case that μ is AD-regular in [To2]. Almost the same arguments work in the present situation.

Lemma 2.4. *Let $B, B' \subset \mathbb{R}^d$ be balls such that $B \subset B'$ which satisfy $\mu(\frac{1}{4}B) \approx \mu(\frac{1}{4}B') \approx \mu(B')$, with $r(B) \approx r(B')$. Then we have*

$$\operatorname{dist}_H\big(L_B \cap B',\, L_{B'} \cap B'\big) \leq C\alpha_\mu(B')\, r(B'), \tag{2.1}$$

where dist_H stands for the Hausdorff distance. Also,

$$|c_B - c_{B'}| \leq C\alpha_\mu(B'). \tag{2.2}$$

This result has also been proved for μ being AD-regular in [To2], and again the same arguments are valid in the present situation.

3. A compactness argument

Let $\mathcal{F}$ denote the family of 1-flat measures, that is, the family of measures σ of the form

$$\sigma = c\,\mathcal{H}^1|_L,$$

where L is a line and $c > 0$. Given $x \in \mathbb{R}^d$ and $r > 0$, we denote

$$\Delta_\mu(x,r) = \left|\frac{\mu(B(x,r))}{r} - \frac{\mu(B(x,2r))}{2r}\right|,$$

and for a ball $B \subset \mathbb{R}^d$,

$$\Theta_\mu(B) = \frac{\mu(B)}{r(B)}.$$

The main objective of this section is to prove the following result:

Lemma 3.1. *Let $\varepsilon > 0$ and let μ be a Radon measure on $\mathbb{R}^d$ and $B_0 \subset \mathbb{R}^d$ some closed ball. Given $\delta > 0$, denote by $G(B_0,\delta)$ the collection of points $x \in \mathbb{R}^d$ such that*

$$\int_{\delta r(B_0)}^{\delta^{-1} r(B_0)} |\Delta_\mu(x,r)|^2 \,\frac{dr}{r} \leq \delta^4\, \Theta_\mu(B_0)^2.$$

Suppose that $\mu(B_0 \cap G(B_0,\delta)) > 0$ and that

$$\mu(\delta^{-1}B_0 \setminus G(B_0,\delta)) \leq \delta^4\, \mu(\delta^{-1}B_0).$$

If δ is small enough, depending only on d and ε, then

$$\alpha_\mu(2B_0) < \varepsilon.$$

First we will need to prove some auxiliary results and to introduce some additional notation. For any Borel function $\varphi : \mathbb{R} \to \mathbb{R}$, let

$$\varphi_t(x) = \frac{1}{t}\varphi\left(\frac{|x|}{t}\right),\; t > 0$$

and define

$$\Delta_{\mu,\varphi}(x,t) := \int \big(\varphi_t(y-x) - \varphi_{2t}(y-x)\big)\, d\mu(y), \tag{3.1}$$

whenever the integral makes sense.

Lemma 3.2. *Let $\varphi : [0,\infty) \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function supported in $[0,2]$ which is constant in $[0,1/2]$. Let $x \in \mathbb{R}^d$ and $0 \leq r_1 < r_2$. For any $1 \leq p < \infty$ we have*

$$\int_{r_1}^{r_2} |\Delta_{\mu,\varphi}(x,r)|^p \,\frac{dr}{r} \leq c \int_{r_1/2}^{2r_2} |\Delta_\mu(x,r)|^p \,\frac{dr}{r},$$

where c depends only on φ and p.

Proof. This follows by writing φ as a suitable convex combination of functions of the form $\chi_{[0,r]}$. For completeness we show the details. For $s \geq 0$, we write

$$\frac{1}{R}\varphi\left(\frac{s}{R}\right) = -\int_0^\infty \frac{1}{R^2}\,\varphi'\left(\frac{r}{R}\right)\,\chi_{[0,r]}(s)\,dr,$$

so that, by Fubini and changing variables,

(3.2)

$$\begin{aligned}\Delta_{\mu,\varphi}(x,R) &= -\int_0^\infty \frac{1}{R^2}\,\varphi'\Big(\frac{r}{R}\Big)\,\chi_{[0,r]}(|\cdot|)*\mu(x)\,dr + \int_0^\infty \frac{1}{(2R)^2}\,\varphi'\Big(\frac{r}{2R}\Big)\,\chi_{[0,r]}(|\cdot|)*\mu(x)\,dr\\ &= -\int_0^\infty \varphi'(t)\left(\frac{1}{R}\,\chi_{[0,tR]}(|\cdot|)*\mu(x) - \frac{1}{2R}\,\chi_{[0,2tR]}(|\cdot|)*\mu(x)\right)dt\\ &= -\int_{1/2}^2 t\,\varphi'(t)\,\Delta_\mu(x,tR)\,dt,\end{aligned}$$

taking into account that φ' is supported on $[1/2,2]$ in the last identity. As a consequence, since $\int |t\,\varphi'(t)|^{p'}dt \lesssim 1$, by Cauchy-Schwarz we get

$$|\Delta_{\mu,\varphi}(x,r)|^p \leq \left|\int_{1/2}^2 t\,\varphi'(t)\,\Delta_\mu(x,tr)\,dt\right|^p \lesssim \int_{1/2}^2 |\Delta_\mu(x,tr)|^p\,dt = \int_{r/2}^{2r} |\Delta_\mu(x,s)|^p\,\frac{ds}{r}.$$

Thus

$$\int_{r_1}^{r_2} |\Delta_{\mu,\varphi}(x,r)|^p\,\frac{dr}{r} \lesssim \int_{r_1}^{r_2}\int_{r/2}^{2r} |\Delta_\mu(x,s)|^p\,ds\,\frac{dr}{r^2} \lesssim \int_{r_1/2}^{2r_2} |\Delta_\mu(x,s)|^p\,\frac{ds}{s}.$$

□

Lemma 3.3. *Let μ be a non-zero Radon measure in $\mathbb{R}^d$. Then μ is 1-flat if and only if $\Delta_\mu(x,r)=0$ for all $x\in\operatorname{supp}\mu$ and all $r>0$.*

Proof. It is clear if μ 1-flat, then $\Delta_\mu(x,r)=0$ for all $x\in\operatorname{supp}\mu$ and all $r>0$. To prove the converse implication it is enough to show that μ is 1-uniform, that is, there exists some constant $c>0$ such that

$$\mu(B(x,r)) = c\,r \qquad \text{for all } x\in\operatorname{supp}\mu \text{ and all } r>0.$$

It is well known that 1-uniform measures are 1-flat (see [Mat2, Chapter 17], for example).

We intend to apply Theorem 3.10 from [CGLT], which asserts that, if μ is AD-regular and $\Delta_{\mu,\varphi}(x,r)=0$ for all $x\in\operatorname{supp}\mu$ and all $r>0$, with $\varphi(y)=e^{-|y|^2}$, then μ is 1-flat. To prove the AD-regularity of μ, assume for simplicity that $0\in\operatorname{supp}\mu$. Since $\Delta_\mu(0,r)=0$ for all $r>0$, we deduce that $\mu(B(0,2^n)) = 2^n\,\mu(B(0,1))$ for all $n\geq 1$. For $x\in\operatorname{supp}\mu\cap B(0,n)$ and any integer $m\leq n$, using now that $\Delta_\mu(x,r)=0$ for all $r>0$, we infer that $\mu(B(x,2^m)) = 2^{m-n}\mu(B(x,2^n))$. Since $B(0,2^{n-1})\subset B(x,2^n)\subset B(0,2^{n+1})$, we have

$$2^{n-1}\mu(B(0,1)) \leq \mu(B(x,2^n)) \leq 2^{n+1}\mu(B(0,1)).$$

Thus

$$c_2\,2^{m-1} \leq \mu(B(x,2^m)) \leq c_2\,2^{m+1},$$

with $c_2=\mu(B(0,1))$. Since n can be take arbitrarily large and the preceding estimate holds for all $m\leq n$, the AD-regularity of μ follows.

On the other hand, as in (3.2), we have

$$\Delta_{\mu,\varphi}(x,r) = -\int_{1/2}^2 t\,\varphi'(t)\,\Delta_\mu(x,tr)\,dt,$$

and so $\Delta_{\mu,\varphi}(x,r)$ vanishes identically on $\operatorname{supp}\mu$ for all $r>0$, as wished. □

Lemma 3.4. *Let μ be a Radon measure in $\mathbb{R}^d$ such that $1 \leq \mu(\bar{B}(0,1)) \leq \mu(B(0,2)) \leq 9$. For all $\varepsilon > 0$ there exists $\delta > 0$ depending only on d and ε such that if*

$$\int_\delta^{\delta^{-1}} \int_{x \in B(0,\delta^{-1})} |\Delta_\mu(x,r)|\, d\mu(x)\, \frac{dr}{r} \leq \delta^{1/2},$$

then

$$\operatorname{dist}_{B(0,2)}(\mu, \mathcal{F}) < \varepsilon.$$

Proof. Suppose that there exists an $\varepsilon > 0$, and for each $m \geq 1$ there exists a Radon measure μ_m such that $1 \leq \mu_m(\bar{B}(0,1)) \leq \mu_m(B(0,2)) \leq 9$, which satisfies

$$\int_{1/m}^{m} \int_{x \in B(0,m)} |\Delta_{\mu_m}(x,r)|\, d\mu_m(x)\, \frac{dr}{r} \leq \frac{1}{m^{1/2}}, \tag{3.3}$$

and

$$\operatorname{dist}_{B(0,2)}(\mu_m, \mathcal{F}) \geq \varepsilon. \tag{3.4}$$

We will first show that the sequence $\{\mu_m\}$ has a subsequence which is weakly * convergent (i.e. when tested against compactly supported continuous functions). This follows from standard compactness once we show that μ_m is uniformly bounded on compact sets. That is, for any compact $K \subset \mathbb{R}^d$, $\sup_m \mu_m(K) < \infty$. To prove this, for $n \geq 4$, $1/4 < r < 1/2$, and $x \in B(0,1)$, we write

$$\begin{aligned}\frac{\mu_m(B(0,2^{n-3}))}{2^{n+2}} \leq \frac{\mu_m(B(x,2^n r))}{2^n r} &\leq \sum_{k=1}^{n} |\Delta_{\mu_m}(x, 2^{k-1}r)| + \frac{\mu_m(B(x,r))}{r} \\ &\leq \sum_{k=1}^{n} |\Delta_{\mu_m}(x, 2^{k-1}r)| + 4\,\mu_m(B(0,2)).\end{aligned}$$

Integrating this estimate with respect to μ on $B(0,1)$ and with respect to $r \in [1/4, 1/2]$, using (3.3) for m big enough we obtain

$$\mu_m(B(0,2^{n-3})) \leq 2^{n+2} \left[\sum_{k=1}^{n} \int_{1/4}^{1/2} \int_{B(0,1)} |\Delta_{\mu_m}(x, 2^{k-1}r)| d\mu_m(x) \frac{dr}{r} + 4\,\mu_m(B(0,2))\right] \leq c(n),$$

which proves the uniform boundedness of μ_m on compact sets.

Our next objective consists in proving that μ is a 1-flat measures. As shown in Lemma 3.3 it is enough to show that $\Delta_\mu(x,r) = 0$ for all $x \in \operatorname{supp} \mu$ and all $r > 0$. Indeed, it is easy to check that $1 \leq \mu(\bar{B}(0,1)) \leq \mu(B(0,2)) \leq 9$, and thus μ is not identically zero.

To prove that $\Delta_\mu(x,r)$ vanishes identically on $\operatorname{supp} \mu$ for all $r > 0$, we will show first that, given any $\mathcal{C}^\infty$ function $\varphi : [0,\infty) \to \mathbb{R}$ which is supported in $[0,2]$ and constant in $[0,1/2]$, we have

$$\int_0^\infty \int_{x \in \mathbb{R}^d} |\Delta_{\mu,\varphi}(x,r)|\, d\mu(x)\, \frac{dr}{r} = 0. \tag{3.5}$$

The proof of this fact is elementary. Suppose that μ_{m_j} converges weakly to μ. Fix m_0 and let $\eta > 0$. Set $K = [1/m_0, m_0] \times \bar{B}(0, 2m_0)$. Now $\{y \to \varphi_t(x-y) - \varphi_{2t}(x-y),\ (t,x) \in K\}$ is an equicontinuous family of continuous functions supported inside a fixed compact set, which implies that $(\varphi_t - \varphi_{2t}) * \mu_{m_j}(x)$ converges to $(\varphi_t - \varphi_{2t}) * \mu(x)$ uniformly on K. It therefore follows that

$$\iint_K |(\varphi_t - \varphi_{2t}) * \mu(x)| d\mu(x) \frac{dt}{t} = \lim_j \int_{1/m_0}^{m_0} \int_{x \in \bar{B}(0,m_0)} |(\varphi_t - \varphi_{2t}) * \mu_{m_j}(x)| d\mu_{m_j}(x) \frac{dt}{t} = 0,$$

by (3.3). Since this holds for any $m_0 \geq 1$, our claim (3.5) is proved.

Denote by G the subset of those points $x \in \operatorname{supp}(\mu)$ such that

$$\int_0^\infty |\Delta_{\mu,\varphi}(x,r)| \frac{dr}{r} = 0.$$

It is clear now that G has full μ-measure. By continuity, it follows that $\Delta_{\mu,\varphi}(x,r) = 0$ for all $x \in \operatorname{supp}\mu$ and all $r > 0$. Finally, by taking a suitable sequence of $\mathcal{C}^\infty$ functions φ_k which converge to $\chi_{[0,1]}$ we infer that $\Delta_\mu(x,r) = 0$ for all $x \in \operatorname{supp}\mu$ and $r > 0$, and thus μ is 1-flat.

However, by condition (3.4), letting $m \to \infty$, we have

$$\operatorname{dist}_{B(0,2)}(\mu, \mathcal{F}) \geq \varepsilon,$$

because $\operatorname{dist}_{B(0,2)}(\cdot, \mathcal{F})$ is continuous under the weak * topology, see [Mat2, Lemma 14.13]. So $\mu \notin \mathcal{F}$, which is a contradiction. □

By renormalizing the preceding lemma we get:

Lemma 3.5. *Let μ be a Radon measure in $\mathbb{R}^d$ and let $B_0 \subset \mathbb{R}^d$ be some ball such that $0 < \mu(\bar{B}_0) \leq \mu(2B_0) \leq 9\,\mu(\bar{B}_0)$. For all $\varepsilon > 0$ there exists $\delta > 0$ depending only on d and ε such that if*

$$\int_{\delta\, r(B_0)}^{\delta^{-1} r(B_0)} \int_{x \in \delta^{-1} B_0} |\Delta_\mu(x,r)|\, d\mu(x)\, \frac{dr}{r} \leq \delta^{1/2}\, \frac{\mu(\bar{B}_0)^2}{r(B_0)},$$

then

$$\operatorname{dist}_{2B_0}(\mu, \mathcal{F}) < \varepsilon\, r(B_0)\, \mu(\bar{B}_0).$$

Proof. Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an affine map which maps $\bar{B}_0$ to $\bar{B}(0,1)$. Consider the measure $\sigma = \frac{1}{\mu(\bar{B}_0)} T\#\mu$, where as usual $T\#\mu(E) := \mu(T^{-1}(E))$, and apply the preceding lemma to σ. □

Lemma 3.6. *Let μ be a Radon measure on $\mathbb{R}^d$ and let $x \in \mathbb{R}^d$, $r > 0$, be such that $\mu(B(x,r/2)) > 0$. If*

$$\int_{r/2}^{2r} \Delta_\mu(x,t)^2\, \frac{dt}{t} \leq \frac{1}{200}\, \Theta_\mu(B(x,r))^2,$$

then

$$\mu(B(x,2r)) \leq 9\,\mu(B(x,r)).$$

Proof. Observe that

$$\int_{r/2}^{2r} \Delta_\mu(x,t)^2\, \frac{dt}{t} = \int_{r/2}^{r} \left[\Delta_\mu(x,t)^2 + \Delta_\mu(x,2t)^2\right] \frac{dt}{t} \leq \frac{1}{200}\, \Theta_\mu(B(x,r))^2.$$

Denote by σ the measure dt/t on $(0,\infty)$. Then, by Chebyshev,

$$\sigma\left(\left\{t \in [r/2,r] : \left[\Delta_\mu(x,t)^2 + \Delta_\mu(x,2t)^2\right] > \lambda\right\}\right) \leq \frac{1}{200\lambda}\, \Theta_\mu(B(x,r))^2.$$

Thus, if we choose $\lambda = \Theta_\mu(B(x,r))^2/100$, then there exists some $t \in [r/2, r]$ such that

$$\Delta_\mu(x,t)^2 + \Delta_\mu(x,2t)^2 \leq \frac{1}{100}\,\Theta_\mu(B(x,r))^2,$$

taking into account that $\sigma([r/2,r]) = \log 2 > 1/2$. This implies that

$$\max\bigl(\Delta_\mu(x,t), \Delta_\mu(x,2t)\bigr) \leq \frac{1}{10}\,\Theta_\mu(B(x,r)),$$

and so

$$\bigl|\Theta_\mu(B(x,4t)) - \Theta_\mu(B(x,t))\bigr| \leq \Delta_\mu(x,t) + \Delta_\mu(x,2t) \leq \frac{1}{5}\,\Theta_\mu(B(x,r)).$$

Then we deduce that

$$\begin{aligned}\Theta_\mu(B(x,2r)) &\leq 2\,\Theta_\mu(B(x,4t)) \leq 2\,\Theta_\mu(B(x,t)) + \frac{2}{5}\,\Theta_\mu(B(x,r))\\ &\leq \left(4 + \frac{2}{5}\right)\Theta_\mu(B(x,r)) = \frac{22}{5}\,\Theta_\mu(B(x,r)),\end{aligned}$$

which is equivalent to saying that $\mu(B(x,2r)) \leq \frac{44}{5}\,\mu(B(x,r))$. □

Proof of Lemma 3.1. We set $B(x_0,r_0) := B_0$. We will assume first that $x_0 \in G(B_0,\delta) \cap \operatorname{supp}\mu$. We will show that if $\delta > 0$ is small enough, the assumptions in the lemma imply that $0 < \mu(B_0) \leq \mu(2B_0) \leq 9\,\mu(B_0)$ and

$$\int_{4\delta r_0}^{(4\delta)^{-1}r_0}\int_{x\in(4\delta)^{-1}B_0} |\Delta_\mu(x,r)|\,d\mu(x)\,\frac{dr}{r} \leq (4\delta)^{1/2}\,\frac{\mu(B_0)^2}{r(B_0)}. \tag{3.6}$$

Then the application of Lemma 3.5 finishes the proof (in the case $x_0 \in G(B_0,\delta)$).

The constant δ will be chosen smaller than $1/10$, and so Lemma 3.6 ensures that

$$0 < \mu(2B_0) \leq 9\,\mu(B_0) \leq 81\,\mu(\tfrac{1}{2}B_0). \tag{3.7}$$

For any $x \in G(B_0,\delta)$, we write

$$\begin{aligned}\int_{\delta r_0}^{\delta^{-1}r_0} |\Delta_\mu(x,r)|\,\frac{dr}{r} &\leq (2\,\log\delta^{-1})^{1/2}\left(\int_{\delta r_0}^{\delta^{-1}r_0}\Delta_\mu(x,r)^2\,\frac{dr}{r}\right)^{1/2}\\ &\leq \bigl(2\,\delta^4\,\log\delta^{-1}\bigr)^{1/2}\,\Theta_\mu(B(x,r)).\end{aligned} \tag{3.8}$$

For $x \in (4\delta)^{-1}B_0 \setminus G(B_0,\delta)$ and and $4\delta r_0 \leq r \leq (4\delta)^{-1}r_0$ we use the brutal estimate

$$|\Delta_\mu(x,r)| \leq \frac{\mu(B(x,(2\delta)^{-1}r_0))}{4\,\delta\,r_0} \leq \frac{\mu(B(x_0,\delta^{-1}r_0))}{4\,\delta\,r_0}. \tag{3.9}$$

By integrating the estimate (3.8) on $(4\delta)^{-1}B_0 \cap G(B_0,\delta)$ and (3.9) on $(4\delta)^{-1}B_0 \setminus G(B_0,\delta)$ and using that $\mu(\delta^{-1}B_0 \setminus G(B_0,\delta)) \leq \delta^4\,\mu(\delta^{-1}B_0)$, we get

$$\begin{aligned}\int_{4\delta r_0}^{(4\delta)^{-1}r_0}\int_{x\in(4\delta)^{-1}B_0} |\Delta_\mu(x,r)|\,d\mu(x)\,\frac{dr}{r} &\leq c\,\delta^2\,(\log\delta^{-1})^{1/2}\,\frac{\mu(B(x_0,r_0))}{r_0}\,\mu(B(x_0,\delta^{-1}r_0))\\ &\quad + \delta^4\,\frac{\mu(B(x_0,\delta^{-1}r_0))^2}{4\,\delta\,r_0}.\end{aligned} \tag{3.10}$$

We will estimate $\mu(B(x_0,\delta^{-1}r_0))$ now. Without loss of generality we assume that $\delta = 2^{-n}$, for some big integer n. By changing variables, we have

$$\int_{2^{-n}r_0}^{2^n r_0} \Delta_\mu(x_0,r)^2\,\frac{dr}{r} = \sum_{k=-n+1}^{n} \int_{2^{k-1}r_0}^{2^k r_0} \Delta_\mu(x_0,r)^2\,\frac{dr}{r} = \int_{r_0/2}^{r_0} \sum_{k=-n+1}^{n} \Delta_\mu(x_0,2^k r)^2\,\frac{dr}{r}.$$

Denote by σ the measure dr/r on $(0,\infty)$. Then, by Chebyshev,

$$\sigma\Big(\Big\{r\in[r_0/2,r_0] : \sum_{k=-n+1}^{n} \Delta_\mu(x_0,2^k r)^2 > \lambda\Big\}\Big) \le \frac{\delta^4}{\lambda}\,\Theta_\mu(B_0)^2.$$

Thus, if we choose, for instance $\lambda = \delta^2\,\Theta_\mu(B_0)^2$, then there exists some $t\in[r_0/2,r_0]$ such that

$$\sum_{k=-n+1}^{n} \Delta_\mu(x_0,2^k t)^2 \le \delta^2\,\Theta_\mu(B_0)^2,$$

taking into account that $\sigma([r_0/2,r_0]) = \log 2 > \delta^2$, for δ small enough. From the fact that $\Delta_\mu(x_0,2^k t) \le \delta\,\Theta_\mu(B_0)$ for $-n+1\le k\le n$, we infer that

$$\Theta_\mu(B(x_0,2^{n+1}t)) \le \Theta_\mu(B(x_0,t)) + \sum_{k=0}^{n} \Delta_\mu(x_0,2^k t) \le 2\,\Theta_\mu(B(x_0,r_0)) + (n+1)\,\delta\,\Theta_\mu(B_0).$$

Using that $n = \log(\delta^{-1})/\log 2$ and that $\Theta_\mu(B(x_0,\delta^{-1}r_0)) \le 2\,\Theta_\mu(B(x_0,2^{n+1}t))$, we get

$$\Theta_\mu(B(x_0,\delta^{-1}r_0)) \le (4 + c\,\delta\,\log\delta^{-1})\,\Theta_\mu(B_0) \le 5\,\Theta_\mu(B_0),$$

for δ small enough. This is equivalent to saying that $\mu(B(x_0,\delta^{-1}r_0)) \le 5\,\delta^{-1}\,\mu(B_0)$. Plugging this estimate into (3.10), we obtain

$$\int_{4\delta\,r_0}^{(4\delta)^{-1}\,r_0} \int_{x\in(4\delta)^{-1}B_0} |\Delta_\mu(x,r)|\,d\mu(x)\,\frac{dr}{r} \le \big(c\,\delta\,(\log\delta^{-1})^{1/2} + c\,\delta\big)\,\frac{\mu(B_0)^2}{r_0}.$$

For δ small enough the right hand side above is smaller than $\frac{(4\delta)^{1/2}\mu(B_0)^2}{r_0}$, as wished, and thus (3.6) holds and we are done.

Suppose now that $x_0 \notin G(B_0,\delta)\cap\operatorname{supp}\mu$. Let $x_1\in B_0\cap G(B_0,\delta)\cap\operatorname{supp}\mu$ and consider the ball $B_1 = B(x_1,2r_0)$. Since $\Theta_\mu(B_0)\le 2\Theta_\mu(B_1)$, every $x\in G(B_0,\delta)$ satisfies

$$\int_{4\delta r(B_1)}^{(4\delta^{-1})r(B_1)} |\Delta_\mu(x,r)|^2\,\frac{dr}{r} \le \int_{\delta r(B_0)}^{\delta^{-1}r(B_0)} |\Delta_\mu(x,r)|^2\,\frac{dr}{r} \le \delta^4\,\Theta_\mu(B_0)^2 \le (4\delta)^4\,\Theta_\mu(B_1)^2,$$

and thus $x\in G(B_1,4\delta)$. Therefore,

$$\mu((4\delta)^{-1})B_1\setminus G(B_1,4\delta)) \le \mu(\delta^{-1}B_0\setminus G(B_0,\delta)) \le \delta^4\,\mu(\delta^{-1}B_0) \le \delta^4\,\mu(\delta^{-1}B_1) \le (4\delta)^4\,\mu(\delta^{-1}B_1).$$

Thus, applying the conclusion of the lemma to the ball B_1, with δ small enough, we deduce that $\alpha_\mu(2B_1)\le\varepsilon$. Taking also into account that $\frac12 B_1\subset 2B_0$, by (3.7) applied to B_1 we have

$$\mu(2B_1) \le 81\,\mu(\tfrac12\,B_1) \le 81\,\mu(2B_0),$$

and thus we get

$$\alpha(2B_0) = \frac{1}{2r_0\,\mu(2B_0)} \inf_{c\ge0,L} \operatorname{dist}_{2B_0}(\mu,\,c\mathcal{H}^1|_L) \lesssim \frac{1}{2r_1\,\mu(2B_1)} \inf_{c\ge0,L} \operatorname{dist}_{2B_1}(\mu,\,c\mathcal{H}^1|_L) \lesssim \varepsilon.$$

□

Remark 3.7. By arguments very similar to the ones used in the preceding proof, one shows that under the assumptions of Lemma 3.1, for all $x \in G(B_0,\delta)\cap B_0$, we have

$$\mu(B(x,r)) \approx \frac{\mu(B_0)}{r_0}\,r \qquad \text{for } \delta\, r_0 \le r \le \delta^{-1} r_0,$$

assuming δ small enough.

4. The dyadic lattice of cells with small boundaries

In our proof of Theorem 1.1 we will use the dyadic lattice of cells with small boundaries constructed by David and Mattila in [DaM, Theorem 3.2]. The properties of this dyadic lattice are summarized in the next lemma.

Lemma 4.1 (David, Mattila)**.** *Let μ be a Radon measure on $\mathbb{R}^d$, $E = \operatorname{supp}\mu$, and consider two constants $C_0 > 1$ and $A_0 > 5000\,C_0$. Then there exists a sequence of partitions of E into Borel subsets Q, $Q \in \mathcal{D}_k$, with the following properties:*

- *For each integer $k \ge 0$, E is the disjoint union of the cells Q, $Q \in \mathcal{D}_k$, and if $k < l$, $Q \in \mathcal{D}_l$, and $R \in \mathcal{D}_k$, then either $Q \cap R = \varnothing$ or else $Q \subset R$.*
- *The general position of the cells Q can be described as follows. For each $k \ge 0$ and each cell $Q \in \mathcal{D}_k$, there is a ball $B(Q) = B(z_Q, r(Q))$ such that*
$$z_Q \in E, \qquad A_0^{-k} \le r(Q) \le C_0\,A_0^{-k},$$
$$E \cap B(Q) \subset Q \subset E \cap 28\,B(Q) = E \cap B(z_Q, 28r(Q)),$$
and
$$\text{the balls } 5B(Q),\ Q \in \mathcal{D}_k, \text{ are disjoint.}$$
- *The cells $Q \in \mathcal{D}_k$ have small boundaries. That is, for each $Q \in \mathcal{D}_k$ and each integer $l \ge 0$, set*
$$N_l^{ext}(Q) = \{x \in E \setminus Q:\ \operatorname{dist}(x,Q) < A_0^{-k-l}\},$$
$$N_l^{int}(Q) = \{x \in Q:\ \operatorname{dist}(x,E\setminus Q) < A_0^{-k-l}\},$$
and
$$N_l(Q) = N_l^{ext}(Q) \cup N_l^{int}(Q).$$
Then
$$\mu(N_l(Q)) \le (C^{-1}C_0^{-3d-1}A_0)^{-l}\,\mu(90B(Q)). \tag{4.1}$$
- *Denote by $\mathcal{D}_k^{db}$ the family of cells $Q \in \mathcal{D}_k$ for which*
$$\mu(100B(Q)) \le C_0\,\mu(B(Q)), \tag{4.2}$$
and set $\mathcal{B}_k = \mathcal{D}_k \setminus \mathcal{D}_k^{db}$. We have that $r(Q) = A_0^{-k}$ when $Q \in \mathcal{B}_k$ and
$$\mu(100B(Q)) \le C_0^{-l}\,\mu(100^{l+1}B(Q)) \quad \text{for all } l \ge 1 \text{ such that } 100^l \le C_0 \text{ and } Q \in \mathcal{B}_k. \tag{4.3}$$

We use the notation $\mathcal{D} = \bigcup_{k\geq 0} \mathcal{D}_k$. For $Q \in \mathcal{D}$, we set $\mathcal{D}(Q) = \{P \in \mathcal{D} : P \subset Q\}$. Given $Q \in \mathcal{D}_k$, we denote $J(Q) = k$. We set $\ell(Q) = 56\, C_0\, A_0^{-k}$ and we call it the side length of Q. Note that

$$\frac{1}{28}\, C_0^{-1}\ell(Q) \leq \operatorname{diam}(Q) \leq \ell(Q).$$

Observe that $r(Q) \approx \operatorname{diam}(Q) \approx \ell(Q)$. Also we call z_Q the center of Q, and the cell $Q' \in \mathcal{D}_{k-1}$ such that $Q' \supset Q$ the parent of Q. We set $B_Q = 28\, B(Q) = B(z_Q, 28\, r(Q))$, so that

$$E \cap \tfrac{1}{28} B_Q \subset Q \subset B_Q.$$

We assume A_0 big enough so that the constant $C^{-1} C_0^{-3d-1} A_0$ in (4.1) satisfies

$$C^{-1} C_0^{-3d-1} A_0 > A_0^{1/2} > 10.$$

Then we deduce that, for all $0 < \lambda \leq 1$,

(4.4)
$$\mu\big(\{x \in Q : \operatorname{dist}(x, E \setminus Q) \leq \lambda\, \ell(Q)\}\big) + \mu\big\{x \in 4B_Q : \operatorname{dist}(x, Q) \leq \lambda\, \ell(Q)\}\big) \leq c\, \lambda^{1/2}\, \mu(3.5B_Q).$$

We denote $\mathcal{D}^{db} = \bigcup_{k\geq 0} \mathcal{D}_k^{db}$ and $\mathcal{D}^{db}(Q) = \mathcal{D}^{db} \cap \mathcal{D}(Q)$. Note that, in particular, from (4.2) it follows that

$$\mu(100B(Q)) \leq C_0\, \mu(Q) \qquad \text{if } Q \in \mathcal{D}^{db}.$$

For this reason we will call the cells from $\mathcal{D}^{db}$ doubling.

As shown in [DaM, Lemma 5.28], any cell $R \in \mathcal{D}$ can be covered μ-a.e. by a family of doubling cells:

Lemma 4.2. *Let $R \in \mathcal{D}$. Suppose that the constants A_0 and C_0 in Lemma 4.1 are chosen suitably. Then there exists a family of doubling cells $\{Q_i\}_{i\in I} \subset \mathcal{D}^{db}$, with $Q_i \subset R$ for all i, such that their union covers μ-almost all R.*

The following result is proved in [DaM, Lemma 5.31].

Lemma 4.3. *Let $R \in \mathcal{D}$ and let $Q \subset R$ be a cell such that all the intermediate cells S, $Q \subsetneq S \subsetneq R$ are non-doubling (i.e. belong to $\bigcup_{k\geq 0} \mathcal{B}_k$). Then*

$$\mu(100B(Q)) \leq A_0^{-10(J(Q)-J(R)-1)} \mu(100B(R)). \tag{4.5}$$

Let us remark that the constant 10 in (4.5) can be replaced by any other positive constant if A_0 and C_0 are chosen suitably in Lemma 4.1, as shown in (5.30) of [DaM].

From the preceding lemma we deduce:

Lemma 4.4. *Let $Q, R \in \mathcal{D}$ be as in Lemma 4.4. Then*

$$\Theta_\mu(100B(Q)) \leq C_0\, A_0^{-9(J(Q)-J(R)-1)}\, \Theta_\mu(100B(R))$$

and

$$\sum_{S\in\mathcal{D}: Q\subset S\subset R} \Theta_\mu(100B(S)) \leq c\, \Theta_\mu(100B(R)),$$

with c depending on C_0 and A_0.

Proof. By (4.5),

$$\Theta_\mu(100\,B(Q)) \leq A_0^{-10(J(Q)-J(R)-1)}\,\frac{\mu(100\,B(R))}{r(100\,B(Q))} = A_0^{-10(J(Q)-J(R)-1)}\Theta_\mu(100\,B(R))\,\frac{r(B(R))}{r(B(Q))}.$$

The first inequality in the lemma follows from this estimate and the fact that $r(B(R)) \leq C_0\,A_0^{(J(Q)-J(R))}\,r(B(Q))$.

The second inequality in the lemma is an immediate consequence of the first one. □

From now on we will assume that C_0 and A_0 are some big fixed constants so that the results stated in the lemmas of this section hold.

5. The Main Lemma

5.1. **Statement of the Main Lemma.** Let μ the measure in Theorem 1.1 and $E = \operatorname{supp}\mu$, and consider the dyadic lattice associated with μ described in Section 4. Let $F \subset E$ be an arbitrary compact set such that

$$\int_F \int_0^1 \Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x) < \infty. \tag{5.1}$$

Given $Q \in \mathcal{D}$, we denote by $\mathsf{G}(Q,\delta,\eta)$ the set of the points $x \in \mathbb{R}^d$ such that

$$\int_{\delta\,\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2\,\frac{dr}{r} \leq \eta\,\Theta_\mu(2B_Q)^2. \tag{5.2}$$

The next lemma concentrates the main difficulties for the proof of the "if" implication of Theorem 1.1.

Main Lemma 5.1. *Let $0 < \varepsilon < 1/100$. Suppose that δ and η are small enough positive constants (depending only on ε). Let $R \in \mathcal{D}^{db}$ be a doubling cell with $\ell(R) \leq \delta$ such that*

$$\mu(R\setminus F) \leq \eta\,\mu(R), \qquad \mu(\lambda B_R \setminus F) \leq \eta\,\mu(\lambda B_R) \quad \text{for all } 2 < \lambda \leq \delta^{-1}, \tag{5.3}$$

and

$$\mu\big(\delta^{-1}B_R \cap F \setminus \mathsf{G}(R,\delta,\eta)\big) \leq \eta\,\mu(R\cap F). \tag{5.4}$$

Then there exists an AD-regular curve Γ_R (with the AD-regularity constant bounded by some absolute constant) and a family of pairwise disjoint cells $\mathsf{Stop}(R) \subset \mathcal{D}(R)\setminus\{R\}$ such that, denoting by $\mathsf{Tree}(R)$ the subfamily of the cells from $\mathcal{D}(R)$ which are not strictly contained in any cell from $\mathsf{Stop}(R)$, the following holds:

- (a) *μ-almost all $F\cap R\setminus\bigcup_{Q\in\mathsf{Stop}(R)}Q$ is contained in Γ_R and moreover $\mu|_{F\cap R\setminus\bigcup_{Q\in\mathsf{Stop}(R)}Q}$ is absolutely continuous with respect to $\mathcal{H}^1|_{\Gamma_R}$.*
- (b) *For all $Q \in \mathsf{Tree}(R)$, $\Theta(1.1B_Q) \leq A\,\Theta_\mu(1.1B_R)$, where $A \geq 100$ is some absolute constant.*

(c) *The cells from* $\mathsf{Stop}(R)$ *satisfy*

$$\sum_{Q\in\mathsf{Stop}(R)} \Theta_\mu(1.1B_Q)^2\,\mu(Q) \leq \varepsilon\,\Theta_\mu(B_R)^2\,\mu(R) + c(\varepsilon)\sum_{Q\in\mathsf{Tree}(R)}\int_{F\cap\delta^{-1}B_Q}\int_{\delta\ell(Q)}^{\delta^{-1}\ell(Q)}\Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x).$$

Let us remark that the assumption that $\ell(R)\leq\delta$ can be removed if we assume that

$$\int_F\int_0^\infty \Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x)<\infty,$$

instead of (5.1).

We will prove the Main Lemma in Sections 6–16. Before proving it, we show how Theorem 1.1 follows from this.

5.2. **Proof of Theorem 1.1 using the Main Lemma 5.1.** As remarked in the Introduction, we only have to prove the "if" implication of the theorem. First we prove the following auxiliary result, which will be used to deal with some cells $R\in\mathcal{D}$ such that (5.4) does not hold.

Lemma 5.2. *Let* $R\in\mathcal{D}$ *be a cell such that*

$$\mu\big(\delta^{-1}B_R\cap F\setminus \mathsf{G}(R,\delta,\eta)\big)>\eta\,\mu(R\cap F). \tag{5.5}$$

Then

$$\Theta_\mu(2B_R)^2\,\mu(R\cap F)\leq\frac{1}{\eta^2}\int_{\delta^{-1}B_R\cap F}\int_{\delta\ell(R)}^{\delta^{-1}\ell(R)}\Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x).$$

Proof. For all $x\in\delta^{-1}B_R\cap F\setminus\mathsf{G}(R,\delta,\eta)$ we have

$$\int_{\delta\,\ell(R)}^{\delta^{-1}\ell(R)}\Delta_\mu(x,r)^2\,\frac{dr}{r}>\eta\,\Theta_\mu(2B_R)^2.$$

Thus, integrating on $\delta^{-1}B_R\cap F\setminus\mathsf{G}(R,\delta,\eta)$ and applying (5.5), we derive

$$\begin{aligned}\int_{\delta^{-1}B_R\cap F\setminus\mathsf{G}(R,\delta,\eta)}\int_{\delta\,\ell(R)}^{\delta^{-1}\ell(R)}\Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x)&\geq\eta\,\Theta_\mu(2B_R)^2\,\mu\big(\delta^{-1}B_R\cap F\setminus\mathsf{G}(R,\delta,\eta)\big)\\&\geq\eta^2\,\Theta_\mu(2B_R)^2\,\mu(R\cap F),\end{aligned}$$

and the lemma follows. □

To prove the "if" implication of Theorem 1.1 clearly it is enough to show that $\mu|_F$ is rectifiable. To this end, let x_0 be a point of density of F and for $\eta>0$ let $B_0=B(x_0,r_0)$ be some ball such that

$$\mu(B_0\setminus F)\leq\eta^2\mu(B_0)\qquad\text{and}\qquad\mu(\tfrac12 B_0)\geq\frac{1}{2^{d+1}}\,\mu(B_0). \tag{5.6}$$

Taking into account that for μ-almost every $x_0\in F$ there exists a sequence of balls like B_0 centered at x_0 with radius tending to 0 fulfilling (5.6) (see Lemma 2.8 of [To3] for example), it suffices to prove that any ball like B_0 contains a rectifiable subset with positive μ-measure.

Denote by $\mathcal{B}^1$ the family of cells $R \in \mathcal{D}$ with $\delta^{-1}B_R \subset B_0$ such that

$$\mu(\lambda B_R \setminus F) \leq \eta\, \mu(\lambda B_R) \quad \text{for some } \lambda \text{ with } 2 < \lambda \leq \delta^{-1},$$

and by $\mathcal{B}^2$ the family of cells $R \in \mathcal{D}$ contained in B_0 such that

$$\mu(R \setminus F) \geq \eta\, \mu(R).$$

Next we show that union of the cells from $\mathcal{B}^1 \cup \mathcal{B}^2$ has very small μ-measure.

Lemma 5.3. *We have*

$$\mu\Bigg(\bigcup_{R\in\mathcal{B}^1\cup\mathcal{B}^2} R\Bigg) \leq c\,\eta\,\mu(B_0). \tag{5.7}$$

Proof. To deal with the cells from $\mathcal{B}^1$ we consider the maximal operator

$$M_* f(x) = \sup_{B \text{ ball}:\, x\in\frac12 B} \frac{1}{\mu(B)} \int_B |f|\,d\mu. \tag{5.8}$$

This operator is known to be bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. Note that for all $x \in R \in \mathcal{B}^1$,

$$M_*\chi_{B_0\setminus F}(x) \geq \eta.$$

Then, using the first estimate in (5.6) we get

$$\mu\Bigg(\bigcup_{R\in\mathcal{B}^1} R\Bigg) \leq \mu\big(\{x\in\mathbb{R}^d :\, M_*\chi_{B_0\setminus F}(x)\geq\eta\}\big) \leq c\,\frac{\mu(B_0\setminus F)}{\eta} \leq c\,\eta\,\mu(B_0),$$

as wished.

To deal with the cells from $\mathcal{B}^2$, we argue analogously, by taking the maximal dyadic operator

$$M^d f(x) = \sup_{Q\in\mathcal{D}: x\in Q} \frac{1}{\mu(Q)} \int |f|\,d\mu. \tag{5.9}$$

□

Let us continue with the proof of Theorem 1.1. From (5.7) and the fact that $\mu(B_0) \approx \mu(\frac12 B_0)$ we infer that, for η small enough, there exists some cell $R_0 \in \mathcal{D}^{db}$ satisfying $R_0 \subset \frac34 B_0$, $\ell(R_0) \leq \delta$, $\delta^{-1}B_{R_0} \subset \frac{9}{10}B_0$, and

$$\mu\Bigg(R_0 \setminus \bigcup_{Q\in\mathcal{B}^1\cup\mathcal{B}^2} Q\Bigg) > 0.$$

We are going now to construct a family of cells Top contained in R_0 inductively, by applying the Main Lemma 5.1. To this end, we need to introduce some additional notation.

Recall that the Main Lemma asserts that if $R \in \mathcal{D}^{db}$, with $\ell(R) \leq \delta$, satisfies the assumptions (5.3) and (5.4), then it generates some family of cells $\mathsf{Stop}(R)$ fulfilling the properties (a), (b) and (c). Now it is convenient to define $\mathsf{Stop}(R)$ also if the assumptions (5.3) or (5.4) do not hold. In case that R is a descendant of R_0 such that $R \in \mathcal{D}^{db} \setminus (\mathcal{B}^1 \cup \mathcal{B}^2)$ does not satisfy (5.4), that is,

$$\mu\big(\delta^{-1}B_R \cap F \setminus \mathsf{G}(R,\delta,\eta)\big) > \eta\,\mu(R\cap F),$$

we let $\mathsf{Stop}(R)$ be the family of the sons of R. In other words, for $R \in \mathcal{D}_k$, $\mathsf{Stop}(R) = \mathcal{D}_{k+1} \cap \mathcal{D}(R)$.

On the other hand, if R is a descendant of R_0 such that $R \in \mathcal{D}^{db} \cap (\mathcal{B}^1 \cup \mathcal{B}^2)$ (note that this means that some of the inequalities in (5.3) does not hold), we set $\mathsf{Stop}(R) = \varnothing$.

Given a cell $Q \in \mathcal{D}$, we denote by $\mathcal{MD}(Q)$ the family of maximal cells (with respect to inclusion) from $P \in \mathcal{D}^{db}(Q)$ such that $2B_P \subset 1.1B_Q$. Recall that, by Lemma 4.2, this family covers μ-almost all Q. Moreover, by Lemma 4.4 it follows that if $P \in \mathcal{MD}(Q)$, then $\Theta_\mu(2B_P) \leq c\,\Theta_\mu(1.1B_Q)$.

We are now ready to construct the aforementioned family Top. We will have $\mathsf{Top} = \bigcup_{k\geq 0} \mathsf{Top}_k$. First we set

$$\mathsf{Top}_0 = \{R_0\}.$$

Assuming Top_k to be defined, we set

$$\mathsf{Top}_{k+1} = \bigcup_{R\in\mathsf{Top}_k} \bigcup_{Q\in\mathsf{Stop}(R)} \mathcal{MD}(Q).$$

Note that the families $\mathcal{MD}(Q)$ with $Q \in \mathcal{S}(R)$, $R \in \mathsf{Top}_k$ are pairwise disjoint. Next we prove a key estimate.

Lemma 5.4. *If ε is chosen small enough in the Main Lemma, then*

$$\sum_{R\in\mathsf{Top}} \Theta_\mu(2B_R)^2\,\mu(R) \leq 2\,\Theta_\mu(2B_{R_0})^2\,\mu(R_0) + c(\varepsilon,\eta,\delta) \int_F \int_0^1 \Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x). \tag{5.10}$$

Proof. For $k \geq 0$ we have

$$\sum_{P\in\mathsf{Top}_{k+1}} \Theta_\mu(2B_P)^2\,\mu(P) = \sum_{R\in\mathsf{Top}_k} \sum_{Q\in\mathsf{Stop}(R)} \sum_{P\in\mathcal{MD}(R)} \Theta_\mu(2B_P)^2\,\mu(P).$$

From Lemma 4.3 we infer that any $P \in \mathcal{MD}(Q)$ satisfies $\Theta_\mu(2B_P) \leq c\,\Theta_\mu(1.1B_Q)$. So we get

$$\sum_{P\in\mathsf{Top}_{k+1}} \Theta_\mu(2B_P)^2\,\mu(P) \leq c \sum_{R\in\mathsf{Top}_k} \sum_{Q\in\mathsf{Stop}(R)} \Theta_\mu(1.1B_Q)^2\,\mu(Q). \tag{5.11}$$

If the conditions (5.3) and (5.4) hold, then (c) in the Main Lemma tells us that

$$\sum_{Q\in\mathsf{Stop}(R)} \Theta_\mu(1.1B_Q)^2\mu(Q) \leq \varepsilon\,\Theta_\mu(2B_R)^2\mu(R) + c(\varepsilon) \sum_{Q\in\mathsf{Tree}(R)} \int_{F\cap\delta^{-1}B_Q} \int_{\delta\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x). \tag{5.12}$$

In the case $R \notin \mathcal{B}^1 \cup \mathcal{B}^2$ and $\mu\big(\delta^{-1}B_R \cap F \setminus \mathsf{G}(R,\delta,\eta)\big) > \eta\,\mu(R\cap F)$, recalling that $\mathsf{Stop}(R)$ is the family of the sons of R, we derive

$$\sum_{Q\in\mathsf{Stop}(R)} \Theta_\mu(1.1B_Q)^2\,\mu(Q) \leq c\,\Theta_\mu(2B_R)^2\,\mu(R).$$

On the other hand, by Lemma 5.2

$$\Theta_\mu(2B_R)^2\,\mu(R) \leq 2\,\Theta_\mu(2B_R)^2\,\mu(R\cap F) \leq \frac{2}{\eta^2} \int_{\delta^{-1}B_R\cap F} \int_{\delta\ell(R)}^{\delta^{-1}\ell(R)} \Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x),$$

taking into account that $\mu(R) \leq 2\,\mu(R\cap F)$, as $R \notin \mathcal{B}^2$, $\eta < 1/2$. So (5.12) also holds in this case, replacing $c(\varepsilon)$ by $2/\eta^2$.

Finally, if $R \in \mathcal{B}^1 \cup \mathcal{B}^2$, by construction we have

$$\sum_{Q\in \mathsf{Stop}(R)} \Theta_\mu(1.1B_Q)^2\, \mu(Q) = 0,$$

since $\mathsf{Stop}(R) = \varnothing$.

Plugging the above estimates into (5.11) we obtain

$$\sum_{P\in \mathsf{Top}_{k+1}} \Theta_\mu(2B_P)^2\, \mu(P) \leq c_3\, \varepsilon \sum_{R\in \mathsf{Top}_k} \Theta_\mu(2B_R)^2\, \mu(R) + c(\varepsilon,\eta) \sum_{R\in \mathsf{Top}_k} \sum_{Q\in \mathsf{Tree}(R)} \int_{F\cap \delta^{-1}B_Q} \int_{\delta\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2 \,\frac{dr}{r}\, d\mu(x).$$

Choosing ε such that $c_3\, \varepsilon \leq 1/2$, we deduce that

$$\sum_{R\in \mathsf{Top}} \Theta_\mu(2B_R)^2\, \mu(R) \leq 2\, \Theta_\mu(2B_{R_0})^2 + c(\varepsilon,\eta) \sum_{R\in \mathsf{Top}} \sum_{Q\in \mathsf{Tree}(R)} \int_{F\cap \delta^{-1}B_Q} \int_{\delta\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2 \,\frac{dr}{r}\, d\mu(x). \tag{5.13}$$

By the finite overlap of the domains of the last integrals as $Q \in \mathcal{D}(R_0)$, we derive

$$\begin{aligned} \sum_{R\in \mathsf{Top}} \sum_{Q\in \mathsf{Tree}(R)} \int_{F\cap \delta^{-1}B_Q} \int_{\delta\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2 \,\frac{dr}{r}\, d\mu(x) &\leq \sum_{Q\in \mathcal{D}} \int_{F\cap \delta^{-1}B_Q} \int_{\delta\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2 \,\frac{dr}{r}\, d\mu(x) \\ &\leq c(\delta) \int_F \int_0^1 \Delta_\mu(x,r)^2 \,\frac{dr}{r}\, d\mu(x), \end{aligned}$$

which together with (5.13) yields (5.10). □

From the preceding lemma we deduce that for μ-a.e. $x \in R_0$,

$$\sum_{R\in \mathsf{Top}: x\in R} \Theta_\mu(2B_R)^2 < \infty. \tag{5.14}$$

For a given $x \in R_0 \setminus \bigcup_{Q\in \mathcal{B}^1\cup\mathcal{B}^2} Q$ such that (5.14) holds, let $R_0, R_1, R_2, \ldots$ be the cells from Top such that $x \in R_i$. Suppose that this is an infinite sequence and assume that $R_0 \supset R_1 \supset R_2 \supset \ldots$, so that for each $i \geq 0$, $R_{i+1} \in \mathcal{MD}(Q)$ for some $Q \in \mathsf{Stop}(R_i)$. From the property (b) in the Main Lemma and Lemma 4.3 it follows that

$$\Theta_\mu(B(x,r)) \leq c\, \Theta_\mu(2B_{R_i}) \qquad \text{for } \ell(R_{i+1}) \leq r \leq \ell(R_i),$$

with c depending on the constant A. As a consequence,

$$\Theta^{1,*}(x,\mu) \leq c\, \limsup_{i\to\infty} \Theta_\mu(2B_{R_i}).$$

From (5.14), we infer that the limit on the right hand side above vanishes and so $\Theta^{1,*}(x,\mu) = 0$. So we have shown that for any $x \in R_0$ satisfying (5.14), the condition $\Theta^{1,*}(x,\mu) > 0$ implies that the collection of cells $R \in \mathsf{Top}$ which contain x is finite.

Given $R \in \mathsf{Top}$, denote by $\mathsf{Top}(R)$ the collection of cells from Top which are strictly contained in R and are maximal with respect to the inclusion. That is,

$$\mathsf{Top}(R) = \bigcup_{Q \in \mathsf{Stop}(R)} \mathcal{MD}(Q).$$

Note that by the property (a) in the Main Lemma and the above construction, if $R \in \mathsf{Top} \setminus \mathcal{B}^1 \cup \mathcal{B}^2$ and (5.4) holds, then there exists a set Z_R of μ-measure 0 and a set $W_R \subset \Gamma_R$ such that

$$R \subset Z_R \cup W_R \cup \bigcup_{Q \in \mathsf{Top}(R)} Q, \tag{5.15}$$

with $\mu|_{W_R}$ being absolutely continuous with respect to $\mathcal{H}^1|_{\Gamma_R}$. On the other hand, if $R \in \mathsf{Top} \setminus \mathcal{B}^1 \cup \mathcal{B}^2$ and (5.4) does not hold, then

$$R = Z_R \cup \bigcup_{Q \in \mathsf{Top}(R)} Q \tag{5.16}$$

for some set Z_R of μ-measure 0.

Suppose now that $\Theta^{1,*}(x, \mu) > 0$, that

$$x \in R_0 \setminus \left(\bigcup_{R \in \mathsf{Top}} Z_R \bigcup_{Q \in \mathcal{B}^1 \cup \mathcal{B}^2} Q \right), \tag{5.17}$$

and that (5.14) holds. Note that the set of such points is a subset of full μ-measure of $R_0 \setminus \bigcup_{Q \in \mathcal{B}^1 \cup \mathcal{B}^2} Q$. Let R_n be the smallest cell from Top which contains x. Since $x \notin \bigcup_{Q \in \mathcal{B}^1 \cup \mathcal{B}^2} Q$, we have $R_n \notin \mathcal{B}^1 \cup \mathcal{B}^2$. So either (5.15) or (5.16) hold for R_n. Since $x \notin Z_{R_n}$ and x does not belong to any cell from $\mathsf{Top}(R_n)$ (by the choice of R_n), we infer that we are in the case (5.15) (i.e. R_n is a cell for which (5.15) holds) and $x \in W_{R_n} \subset \Gamma_{R_n}$. Thus the subset of points x with $\Theta^{1,*}(x, \mu) > 0$ satisfying (5.17) and (5.14) is contained in $\bigcup_n W_{R_n}$, which is a rectifiable set such that $\mu|_{\bigcup_n W_{R_n}}$ is absolutely continuous with respect to $\mathcal{H}^1$. □

6. The stopping cells for the proof of Main Lemma 5.1

6.1. **The good and the terminal cells.**

The remaining part of this paper, with the exception of Sections 17-19, is devoted to the proof of Main Lemma 5.1.

The main task in this section consists in the construction of the stopping cells from $\mathcal{D}$, which later will be used in the construction of the curve Γ_R of the Main Lemma.

First we introduce the notation $\mathsf{G}(Q_1, Q_2, \delta, \eta)$ for $Q_1, Q_2 \in \mathcal{D}$ and $\delta, \eta > 0$. This is the set of the points $x \in \mathbb{R}^d$ such that

$$\int_{\delta\, \ell(Q_1)}^{\delta^{-1} \ell(Q_2)} \Delta_\mu(x, r)^2 \, \frac{dr}{r} \leq \eta \, \Theta_\mu(2B_{Q_2})^2. \tag{6.1}$$

Note that $\mathsf{G}(Q, Q, \delta, \eta) = \mathsf{G}(Q, \delta, \eta)$.

Let $R \in \mathcal{D}$ be as in the Main Lemma 5.1. We denote $x_0 = z_R$ (this is the center of R) and $r_0 = r(B_R)$, so that $B(x_0, r_0) = B_R$, and thus

$$R \subset B(x_0, r_0), \qquad r_0 \approx \ell(R).$$

Now we need to define some families of stopping cells which are not good for the construction of the curve mentioned above. Let $A, \tau > 0$ be some constants to be fixed below, an We assume τ to be very small, with $\tau \leq 10^{-30}$ say, and $A \geq 100$. Moreover, we let $K \geq 100$ be some big absolute constant (probably $K = 10^4$ suffices) which depends on the ambient dimension d but not on the other constants δ, η, τ, A. The reader should think that $1 \ll K \ll \delta^{-1}$.

- A cell $Q \in \mathcal{D}$ belongs to BCF_0 if $\ell(Q) \leq \ell(R)$ and either
$$\mu(Q \setminus F) \geq \eta^{1/2}\, \mu(Q) \quad \text{or} \quad \mu(\lambda B_Q \setminus F) \geq \eta^{1/2}\, \mu(\lambda B_Q) \quad \text{for some } 1.1 \leq \lambda \leq \delta^{-1/2}.$$
- A cell $Q \in \mathcal{D}$ belongs to LD_0 if $\ell(Q) \leq \ell(R)$, $Q \notin BCF_0$, and
$$\Theta_\mu(1.1B_Q) \leq \tau\, \Theta_\mu(B_R).$$
- A cell $Q \in \mathcal{D}$ belongs to HD_0 if $\ell(Q) \leq \ell(R)$, $Q \notin BCF_0$, and
$$\Theta_\mu(1.1B_Q) \geq A\, \Theta_\mu(1.1B_R).$$
- A cell $Q \in \mathcal{D}$ belongs to BCG_0 if $Q \notin BCF_0 \cup LD_0 \cup HD_0$, $\ell(Q) \leq \ell(R)$, and
$$\mu(\delta^{-1/2} B_Q \cap F \setminus \mathsf{G}(Q, R, \delta^{1/2}, \eta)) \geq \eta\, \mu(\delta^{-1/2} B_Q \cap F).$$
- A cell $Q \in \mathcal{D}$ belongs to $BS\Delta_0$ if $Q \notin BCF_0 \cup LD_0 \cup HD_0 \cup BCG_0$, $\ell(Q) \leq \ell(R)$, and
$$\sum_{P \in \mathcal{D}: Q \subset P \subset R} \frac{1}{\mu(1.1B_P)} \int_{1.1B_P \cap F} \int_{\delta\, \ell(P)}^{\delta^{-1}\ell(P)} \Delta_\mu(x,r)^2\, \frac{dr}{r}\, d\mu \geq \eta\, \Theta_\mu(B_R)^2.$$

Next we consider the subfamily of $BCF_0 \cup LD_0 \cup HD_0 \cup BCG_0 \cup BS\Delta_0$ of the cells which are maximal with respect to inclusion (thus they are disjoint), and we call it Term. We denote by BCF the subfamily of the cells from Term which belong by BCF_0, and by LD, HD, BCG, $\mathsf{BS}\Delta$, $\mathsf{BS}\beta$ the subfamilies of the cells from Term which belong to LD_0, HD_0, BCG_0, and $BS\Delta_0$, respectively. Notice that we have the disjoint union
$$\mathsf{Term} = \mathsf{BCF} \cup \mathsf{LD} \cup \mathsf{HD} \cup \mathsf{BCG} \cup \mathsf{BS}\Delta.$$
The notations BCF, LD, HD, BCG, and $\mathsf{BS}\Delta$ stand for "big complement of F", "low density", and "high density", "big complement of G", and "big sum of Δ coefficients", respectively; and Term for "terminal".

We denote by Good the subfamily of the cells $Q \subset B(x_0, \frac{1}{10} K r_0)$ with $\ell(Q) \leq \ell(R)$ such that there does not exist any cell $Q' \in \mathsf{Term}$ with $Q' \supset Q$. Notice that $\mathsf{Term} \not\subset \mathsf{Good}$ while, on the other hand, $R \in \mathsf{Good}$.

6.2. **Some basic estimates.** The following statement is an immediate consequence of the construction.

Lemma 6.1. *If $Q \in \mathcal{D}$, $\ell(Q) \leq \ell(R)$, and Q is not contained in any cell from* Term *(and so in particular, if $Q \in$* Good*), then*
$$\tau\, \Theta_\mu(B_R) \leq \Theta_\mu(1.1B_Q) \leq A\, \Theta_\mu(B_R).$$

Lemma 6.2. *If $Q\in\mathcal{D}$, $Q\subset B(x_0,K^2r_0)$, $\ell(Q)\leq\ell(R)$, and Q is not contained in any cell from* Term *(and so in particular, if $Q\in$* Good*), then*

$$\mu(aB_Q)\leq c(a)\,\mu(Q) \tag{6.2}$$

for any $a\geq 1$ such that $r(a\,B_Q)\leq\delta^{-3/4}r(B_R)$, assuming that the constant C_0 in the construction of the lattice $\mathcal{D}$ in Lemma 4.1 is big enough.

Proof. First we will show that

$$\mu(aB_Q)\leq c(a)\,\mu(3.3B_Q) \tag{6.3}$$

for a as in the lemma. Since Q is not contained in any cell from $\mathsf{BCF}\cup\mathsf{BS\Delta}$, we have

$$\mu(1.1B_Q\backslash F)<\eta^{1/2}\,\mu(1.1B_Q)\quad\text{ and }\quad\frac{1}{\mu(1.1B_Q)}\int_{1.1B_Q\cap F}\int_{\delta\,\ell(Q)}^{\delta^{-1}\ell(Q)}\Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu\leq\eta\,\Theta_\mu(B_R)^2.$$

Thus

$$\int_{1.1B_Q\cap F}\int_{\delta\,\ell(Q)}^{\delta^{-1}\ell(Q)}\Delta_\mu(x,t)^2\,\frac{dt}{t}\,d\mu\leq\eta\,\Theta_\mu(B_R)^2\,\mu(1.1B_Q)\leq 2\,\eta\,\Theta_\mu(B_R)^2\,\mu(1.1B_Q\cap F).$$

Hence there exists $y_0\in 1.1B_Q\cap F$ such that

$$\int_{\delta\,\ell(Q)}^{\delta^{-1}\ell(Q)}\Delta_\mu(y_0,t)^2\,\frac{dt}{t}\,d\mu\leq 2\,\eta\,\Theta_\mu(B_R)^2.$$

Take r such that $2.2r(B_Q)\leq r\leq\delta^{-1}\ell(Q)/2$. For these r's we have $B(y_0,r)\supset 1.1B_Q$ and thus

$$\Theta_\mu(B(y_0,r))\geq c(\tau,\delta)\,\Theta_\mu(B_R),$$

and thus, by Lemma 3.6,

$$\mu(B(y_0,2r))\leq 9\,\mu(B(y_0,r))\quad\text{ for }\;2.2r(B_Q)\leq r\leq\delta^{-1}\ell(Q)/2. \tag{6.4}$$

Iterating this estimate we deduce that

$$\mu(B(y_0,a\,r))\leq c(a)\,\mu(B(y_0,r))\leq c(a)\,\mu(3.3B_Q)\qquad\text{for }a\,r\leq\delta^{-1}\ell(Q)/4,$$

since $B(y_0,2.2r(B_Q))\subset 3.3B_Q$. Applying this estimate also to the ancestors of Q, (6.3) follows.

To prove (6.2) it is enough to show that

$$\mu(3.3B_Q)\leq c\,\mu(Q). \tag{6.5}$$

Note that by the property (4.3) of the cells of David and Mattila, if $Q\in\mathcal{D}\setminus\mathcal{D}^{db}$, then

$$\mu(3.3B_Q)=\mu(28\cdot 3.3B(Q))\leq\mu(100B(Q))\leq C_0^{-1}\,\mu(100^2B(Q))\quad\text{ if }C_0\geq 100. \tag{6.6}$$

By (6.3), for a cell Q satisfying the assumptions in the lemma we have

$$\mu(100^2B(Q))\leq c\,\mu(3.3B_Q)$$

with c independent of C_0. Thus (6.6) does not hold if C_0 is chosen big enough. Hence $Q\in\mathcal{D}^{db}$ and then

$$\mu(3.3B_Q)\leq\mu(100B(Q))\leq C_0\,\mu(B(Q))\leq C_0\,\mu(Q),$$

and so (6.5) holds. □

Remark 6.3. Let Q be as in the preceding lemma. That is, $Q \in \mathcal{D}$, $Q \subset B(x_0, K^2 r_0)$, $\ell(Q) \leq \ell(R)$, and Q is not contained in any cell from Term. We showed in (6.4) that there exists some $y_0 \in 1.1B_Q \cap F$ such that

$$\mu(B(y_0, 2r)) \leq 9\,\mu(B(y_0, r)) \qquad 2.2r(B_Q) \leq r \leq \delta^{-1}\ell(Q)/2.$$

From this estimate it follows that

$$\mu(b\,B_Q) \leq C(a,b)\,\mu(a\,B_Q) \qquad \text{for } 3.3 \leq a \leq b \leq \delta^{-1/2}, \tag{6.7}$$

with the constant $C(a,b)$ independent of the constant C_0 from the construction of the David-Mattila cells. This fact will be very useful later. On the contrary, the constant in the inequality (6.5) depends on C_0.

Lemma 6.4. *If η is small enough (with $\eta \ll \delta$), then*

$$\mu\Bigg(\bigcup_{Q \in \mathsf{BCF}: Q \subset R} Q\Bigg) \leq c\,\eta^{1/4}\,\mu(R).$$

Proof. The arguments are similar to the ones of Lemma 5.3. Denote by $\mathcal{B}_R^1$ the family of cells $Q \in \mathcal{D}$ which are contained in R and satisfy

$$\mu(\lambda B_Q \setminus F) \geq \eta^{1/2}\,\mu(\lambda B_Q)$$

for some $1.1 \leq \lambda \leq \delta^{-1/2}$, and by $\mathcal{B}_R^2$ the family of the ones that are contained in R and satisfy

$$\mu(Q \setminus F) \geq \eta^{1/2}\,\mu(Q).$$

To deal with the cells from $\mathcal{B}_R^1$ we consider the maximal operator M_{**} (which is a variant of M_*, introduced in (5.8)):

$$M_{**}f(x) = \sup_{B \text{ ball}: x \in B} \frac{1}{\mu(1.1B)} \int_{1.1B} |f|\,d\mu.$$

Similarly to M_*, this operator is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. It turns out that for all $x \in Q \in \mathcal{B}_R^1$, $M_{**}\chi_{c\,\delta^{-1/2}B_R \setminus F}(x) \geq \eta^{1/2}$, because $\delta^{-1/2}B_Q \subset c\,\delta^{-1/2}B_R$, for some absolute constant c. So we have

$$\begin{aligned}\mu\Bigg(\bigcup_{R \in \mathcal{B}_R^1} R\Bigg) &\leq \mu\big(\{x \in \mathbb{R}^d : M_{**}\chi_{c\,\delta^{-1/2}B_R \setminus F}(x) \geq \eta^{1/2}\}\big)\\ &\leq c\,\frac{\mu(c\,\delta^{-1/2}B_R \setminus F)}{\eta^{1/2}} \leq c\,\eta^{1/2}\,\mu(c\,\delta^{-1/2}B_R).\end{aligned}$$

By Lemma 6.2, we know that

$$\mu(c\,\delta^{-1/2}B_R) \leq c(\delta)\,\mu(R).$$

Hence,

$$\mu\Bigg(\bigcup_{R \in \mathcal{B}_R^1} R\Bigg) \leq c'(\delta)\,\eta^{1/2}\,\mu(R) \leq \frac{1}{2}\,\eta^{1/4}\,\mu(R), \tag{6.8}$$

assuming η enough (depending on δ).

To deal with the cells from $\mathcal{B}_R^2$, we argue with the maximal dyadic operator M^d defined in (5.9). Indeed, since every $Q \in \mathcal{B}_R^2$ is contained in $\{x \in \mathbb{R}^d : M^d \chi_{R\setminus F}(x) \geq \eta^{1/2}\}$, we have

$$\mu\Bigg(\bigcup_{R\in\mathcal{B}_R^2} R\Bigg) \leq \mu\big(\{x \in \mathbb{R}^d : M^d\chi_{R\setminus F}(x) \geq \eta^{1/2}\}\big) \leq c\,\frac{\mu(R\setminus F)}{\eta^{1/2}} \leq c\,\eta^{1/2}\,\mu(R). \tag{6.9}$$

Adding the estimates (6.8) and (6.9) the lemma follows. □

Lemma 6.5. *For all* $Q \in \mathsf{BCG}$*,*

$$\Theta_\mu(B_R)^2\mu(\delta^{-1/2}B_Q) \leq \frac{2}{\eta^2}\int_{\delta^{-1}B_Q\cap F}\int_{\delta\ell(Q)}^{\delta^{-1}\ell(R)} \Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x).$$

Proof. Note that for all $x \in \delta^{-1}B_Q \cap F \setminus \mathsf{G}(Q,R,\delta,\eta)$ we have

$$\int_{\delta\,\ell(Q)}^{\delta^{-1}\ell(R)} \Delta_\mu(x,r)^2\,\frac{dr}{r} > \eta\,\Theta_\mu(B_R)^2.$$

Thus, integrating on $\delta^{-1}B_Q \cap F \setminus \mathsf{G}(Q,R,\delta,\eta)$ and taking into account that $Q \in \mathsf{BCG}$ we get

$$\begin{aligned}\int_{\delta^{-1}B_Q\cap F\setminus\mathsf{G}(Q,R,\delta,\eta)}\int_{\delta\,\ell(Q)}^{\delta^{-1}\ell(R)} \Delta_\mu(x,r)^2\,\frac{dr}{r} &\geq \eta\,\Theta_\mu(B_R)^2\,\mu\big(\delta^{-1}B_Q\cap F\setminus\mathsf{G}(R,\delta,\eta)\big)\\ &\geq \eta^2\,\Theta_\mu(B_R)^2\,\mu(\delta^{-1}B_Q\cap F).\end{aligned}$$

Since $Q \notin \mathsf{BCF}$, we have $\mu(\delta^{-1/2}B_Q\cap F) \geq (1-\eta^{1/2})\,\mu(\delta^{-1/2}B_Q) \geq \frac{1}{2}\mu(\delta^{-1/2}B_Q)$, and the lemma follows. □

6.3. **The regularized family** Reg **and the family** Qgood**.** The cells from Term have the inconvenient that their side lengths may change drastically even if they are close to each other. For this reason it is appropriate to introduce a regularized version of this family, which we will call Reg. The first step for the construction consists in introducing the following auxiliary function $d : \mathbb{R}^d \to [0,\infty)$:

$$d(x) = \inf_{Q\in\mathsf{Good}}\big(|x - z_Q| + \ell(Q)\big). \tag{6.10}$$

Note that $d(\cdot)$ is a 1-Lipschitz function because it is the infimum of a family of 1-Lipschitz functions.

We denote

$$W_0 = \{x \in \mathbb{R}^d : d(x) = 0\}.$$

For each $x \in E \setminus W_0$ we take the largest cell $Q_x \in \mathcal{D}$ such that $x \in Q_x$ with

$$\ell(Q_x) \leq \frac{1}{60}\inf_{y\in Q_x} d(y). \tag{6.11}$$

We consider the collection of the different cells Q_x, $x \in E \setminus W_0$, and we denote it by Reg. Also, we let Qgood (this stands for "quite good") be the family of cells $Q \in \mathcal{D}$ such that Q is contained in $B(x_0, 2Kr_0)$ and Q is not strictly contained in any cell of the family Reg. Note that $\mathsf{Reg} \subset \mathsf{Qgood}$.

Let us remark that the family $\mathsf{Stop}(R)$ described in the Main Lemma is made up of the cells from the family Reg which are contained in R. That is,

$$\mathsf{Stop}(R) := \mathcal{D}(R) \cap \mathsf{Reg}.$$

To simplify notation, from now on we will write Stop instead of $\mathsf{Stop}(R)$ and, analogously, Tree instead of $\mathsf{Tree}(R)$.

Lemma 6.6. *The cells from* Reg *are pairwise disjoint and satisfy the following properties:*

(a) *If* $P \in \mathsf{Reg}$ *and* $x \in B(z_P, 50\ell(P))$*, then* $10\,\ell(P) \leq d(x) \leq c\,\ell(P)$*, where* c *is some constant depending only on* A_0*. In particular,* $B(z_P, 50\ell(P)) \cap W_0 = \varnothing$*.*

(b) *There exists some absolute constant* c *such that if* $P, P' \in \mathsf{Reg}$ *and* $B(z_P, 50\ell(P)) \cap B(z_{P'}, 50\ell(P')) \neq \varnothing$*, then*

$$c^{-1}\ell(P) \leq \ell(P') \leq c\,\ell(P).$$

(c) *For each* $P \in \mathsf{Reg}$*, there at most* N *cells* $P' \in \mathsf{Reg}$ *such that*

$$B(z_P, 50\ell(P)) \cap B(z_{P'}, 50\ell(P')) \neq \varnothing,$$

where N *is some absolute constant.*

(d) *If* $x \notin B(x_0, \frac{1}{8}Kr_0)$*, then* $d(x) \approx |x - x_0|$*. As a consequence, if* $P \in \mathsf{Reg}$ *and* $B(z_P, 50\ell(P)) \not\subset B(x_0, \frac{1}{8}Kr_0)$*, then* $\ell(P) \gtrsim Kr_0$*.*

Proof. To prove (a), consider $x \in B(z_P, 50\ell(P))$. Since $d(\cdot)$ is 1-Lipschitz and, by definition, $d(z_P) \geq 60\,\ell(P)$, we have

$$d(x) \geq d(z_P) - |x - z_P| \geq d(z_P) - 50\,\ell(P) \geq 10\,\ell(P).$$

To prove the converse inequality, by the definition of Reg, there exists some $z' \in \widehat{P}$, the parent of P, such that

$$d(z') \leq 60\,\ell(\widehat{P}) \leq 60\,A_0\,\ell(P).$$

Also, we have

$$|x - z'| \leq |x - z_P| + |z_P - z'| \leq 50\,\ell(P) + A_0\,\ell(P).$$

Thus,

$$d(x) \leq d(z') + |x - z'| \leq (50 + 61\,A_0)\,\ell(P).$$

The statement (b) is an immediate consequence of (a), and (c) follows easily from (b).

Finally, the first assertion in (d) follows from the fact that all the cells from Good are contained in $B(x_0, \frac{1}{10}Kr_0)$, by definition. Together with (a), this yields that if $B(z_P, 50\ell(P)) \not\subset B(x_0, \frac{1}{8}Kr_0)$, then $\ell(P) \gtrsim Kr_0$. □

Lemma 6.7. *Every cell* $Q \in \mathsf{Reg}$ *with* $Q \subset B(x_0, \frac{1}{10}Kr_0)$ *is contained in some cell* $Q' \in \mathsf{Term}$*.*

Proof. Suppose that Q is not contained in such a cell Q'. This means that $Q \in \mathsf{Good}$. Then, by the definition of $d(\cdot)$ in (6.10), for every $x \in Q$ we have $d(x) \leq \operatorname{diam}(Q) + \ell(Q) \leq 2\ell(Q)$. Thus, by (6.11), $\ell(Q_x) < \ell(Q)$, and so $Q \notin \mathsf{Reg}$. □

Lemma 6.8. *There exists some absolute constant $c_4 > 2$ such that for every cell $Q \in \mathsf{Qgood}$ contained in $B(x_0, 2K\, r_0)$ there exist $Q' \in \mathsf{Good}$ such with $\ell(Q') \approx \ell(Q)$ such that $2B_{Q'} \subset c_4 B_Q$. Further the following holds:*

$$\tau\, \Theta_\mu(B_R) \lesssim \Theta_\mu(c_4 B_Q) \lesssim A\, \Theta_\mu(B_R).$$

Proof. The first statement is consequence of the construction of the family Reg. The second one follows from the first one, together with the doubling properties of $1.1 B_{Q'}$ (by Lemma 6.2) and the fact that

$$\tau\, \Theta_\mu(B_R) \leq \Theta_\mu(1.1 B_{Q'}) \leq A\, \Theta_\mu(B_R).$$

□

Lemma 6.9. *If $Q \in \mathsf{Qgood}$ and $Q \subset B(x_0, K\, r_0)$, then there exists some ball $\widetilde{B}_Q$ containing $2B_Q$, with radius $r(\widetilde{B}_Q) \leq c_5\, \ell(Q)$ (where $c_5 \geq 1$ is some absolute constant) which satisfies the following properties:*

(a) *Denote by $G(\widetilde{B}_Q)$ the subset of points $x \in \mathbb{R}^d$ such that*

$$\int_{\delta^{1/2} r(\widetilde{B}_Q))}^{\delta^{-1/2} r(\widetilde{B}_Q)} |\Delta_\mu(x,r)|^2\, \frac{dr}{r} \leq \eta^{1/4}\, \Theta_\mu(\widetilde{B}_Q)^2.$$

Then we have

$$\mu(\delta^{-1/4} \widetilde{B}_Q \setminus G(\widetilde{B}_Q)) \leq \eta^{1/4}\, \mu(\delta^{-1/4} B_{Q'}),$$

and moreover $\mu(\widetilde{B}_Q \cap G(\widetilde{B}_Q)) > 0$.

(b) *If $\varepsilon_0 > 0$ is some arbitrary (small) constant, assuming η and δ small enough (depending on ε_0), we have*

$$\alpha_\mu(2\widetilde{B}_Q) \leq \varepsilon_0.$$

(c) *For any $a \geq 1$ such that $r(a\, \widetilde{B}_Q) \leq \delta^{-3/4} r(B_R)$.*

$$\mu(a\widetilde{B}_Q) \leq c(a)\, \mu(\widetilde{B}_Q).$$

Proof. By the definition of the cells from Qgood, there exists some $Q' \in \mathsf{Good}$ such that $2B_Q \subset c\, B_{Q'}$, for some absolute constant $c \geq 2$. Since Q' is good, by construction it satisfies

$$\mu(\delta^{-1/2} B_{Q'} \setminus F) < \eta^{1/2}\, \mu(\delta^{-1/2} B_{Q'})$$

and

$$\mu(\delta^{-1/2} B_{Q'} \cap F \setminus \mathsf{G}(Q', R, \delta^{1/2}, \eta)) < \eta\, \mu(\delta^{-1/2} B_{Q'} \cap F).$$

From these two estimates we infer that

$$\mu(\delta^{-1/2} B_{Q'} \setminus \mathsf{G}(Q', R, \delta^{1/2}, \eta)) < 2\eta^{1/2}\, \mu(\delta^{-1/2} B_{Q'}).$$

Further, by Lemma 6.2, if $\eta \ll \delta$ we get $2\eta^{1/2}\, \mu(\delta^{-1/2} B_{Q'}) < \mu(B_{Q'})$, and thus

$$\mu(B_{Q'} \cap \mathsf{G}(Q', R, \delta^{1/2}, \eta)) > 0.$$

The first assertion of the lemma follows if we take $\widetilde{B}_Q = B_{Q'}$, noting that $\mathsf{G}(Q', R, \delta^{1/2}, \eta) \subset G(\widetilde{B}_Q)$ if η is small enough depending on τ (using that $\Theta_\mu(Q') \geq \tau\, \Theta_\mu(R)$). The second assertion is an immediate corollary of the first one and Lemma 3.1. The last one follows from Lemma 6.2. □

From the preceding results, we obtain the following easily. We leave the proof for the reader.

Lemma 6.10. *Given $\varepsilon_0 > 0$, assume that η and δ are small enough. If $Q \in \mathsf{Qgood}$ and $Q \subset B(x_0, 2K\,r_0)$, then there exists some absolute constant $c_6 \geq 4$ such that, for any $a \geq c_6$ such that $a\,\ell(Q) \leq \ell(R)$,*

$$\tau\,\Theta_\mu(B_R) \lesssim \Theta_\mu(aB_Q) \lesssim A\,\Theta_\mu(B_R) \tag{6.12}$$

and

$$\alpha_\mu(aB_Q) \lesssim \varepsilon_0. \tag{6.13}$$

7. The measure $\widetilde{\mu}$ and some estimates about its flatness

We consider the set

$$\widetilde{E} = B(x_0, 2Kr_0) \cap \left(W_0 \cup \bigcup_{P \in \mathsf{Reg}} \big[4B_P \cap F \cap \mathsf{G}(P, R, \delta^{1/4}, \eta)\big]\right). \tag{7.1}$$

Then we set

$$\widetilde{\mu} = \mu|_{\widetilde{E}}.$$

Our first objective consists in showing that, in a sense, $\mu(E \cap B(x_0, 2Kr_0) \setminus \widetilde{E})$ is very small.

Lemma 7.1. *If $P \in \mathsf{Reg}$, then we have*

$$\mu(4B_P \setminus \widetilde{E}) \leq \eta^{1/4}\,\mu(c_7\,B_P),$$

with $c_7 = 5c_6$ (where c_6 appears in Lemma 6.10), where c_7 is some absolute constant, and we assume η mall enough.

Proof. Note that

$$\mu(4B_P \setminus \widetilde{E}) \leq \mu\big(4B_P \setminus (F \cap \mathsf{G}(P, R, \delta^{1/4}, \eta))\big),$$

By the definition of the cells from Reg, there exists some cell $Q \in \mathsf{Good}$ with $\ell(Q) \approx \ell(P)$ such that $4\,B_P \subset c_8\,B_Q$, where c_8 is some absolute constant. Since $\ell(Q) \approx \ell(P)$, we deduce that $\mathsf{G}(Q, R, \delta^{1/2}, \eta) \subset \mathsf{G}(P, R, \delta^{1/4}, \eta)$, and thus

$$\mu(4B_P \setminus (F \cap \mathsf{G}(P, R, \delta^{1/4}, \eta))) \leq \mu(\delta^{-1/2}B_Q \setminus (F \cap \mathsf{G}(Q, R, \delta^{1/2}, \eta))). \tag{7.2}$$

To estimate the right hand side above we take into account that since $Q \notin \mathsf{BCF}$,

$$\mu(\delta^{-1/2}B_Q \setminus F) \leq \eta^{1/2}\,\mu(\delta^{-1/2}B_Q),$$

and as $Q \notin \mathsf{BCG}$,

$$\mu(\delta^{-1/2}B_Q \cap F \setminus \mathsf{G}(Q, R, \delta^{1/2}, \eta)) \leq \eta\,\mu(\delta^{-1/2}B_Q \cap F).$$

So we get

$$\begin{aligned}
\mu(\delta^{-1/2}B_Q \setminus (F \cap \mathsf{G}(Q, R, \delta^{1/2}, \eta))) &\leq \mu(\delta^{-1/2}B_Q \setminus F) + \mu(\delta^{-1/2}B_Q \cap F \setminus \mathsf{G}(Q, R, \delta^{1/2}, \eta)) \\
&\leq \eta^{1/2}\,\mu(\delta^{-1/2}B_Q) + \eta\,\mu(\delta^{-1/2}B_Q \cap F) \\
&\leq 2\,\eta^{1/2}\,\mu(\delta^{-1/2}B_Q).
\end{aligned}$$

Gathering the estimates above, we obtain

$$\mu(4B_P \setminus \widetilde{E}) \leq 2\,\eta^{1/2}\,\mu(\delta^{-1/2}B_Q).$$

By Lemma 6.2, we know that
$$\mu(\delta^{-1/2}B_Q) \leq c(\delta)\,\mu(B_Q) \leq c(\delta)\,A\,\Theta_\mu(R)\,\ell(Q).$$
On the other hand, since $c_7 > c_6$, by Lemma 6.10,
$$\mu(c_7\,B_P) \gtrsim \tau\,\Theta_\mu(R)\,\ell(P) \gtrsim \tau\,\Theta_\mu(R)\,\ell(Q).$$
Thus we derive
$$\mu(4B_P \setminus \widetilde{E}) \leq c(\delta)\,\eta^{1/2}\,A\,\tau^{-1}\,\mu(c_7\,B_P).$$
If η is small enough, we get the desired conclusion. □

Lemma 7.2. *Let $Q \in \mathsf{Good}$ and let $a \geq 2$ with $r(aB_Q) \leq K\,\ell(R)$. Denote by $\mathcal{B}(aB_Q)$ the subcollection of cells P from Reg which intersect aB_Q and satisfy*
$$\mu(4B_P) \leq \eta^{1/10}\mu(c_7\,B_P). \tag{7.3}$$
Then we have
$$\sum_{P\in\mathcal{B}(aB_Q)} \mu(P) \leq \eta^{1/4}\,\mu(aB_Q),$$

Proof. Note first that every $P \in \mathcal{B}(aB_Q)$ satisfies $r(P) \leq c\,r(aB_Q)$. In fact, Q contains either some point from W_0 or some cell $P' \in \mathsf{Reg}$, and if P were too big, we would have too close cells with very different sizes (or a cell and a point from W_0), which would contradict the properties (a) or (b) of Lemma 6.6. As a consequence of the fact that $r(P) \leq c\,r(aB_Q)$, we infer that $P \subset c'aB_Q$, for some absolute constant c'.

We consider two types of cells $P \in \mathcal{B}(aB_Q)$. We set $P \in \mathcal{B}_1(aB_Q)$ if $\mu(P \setminus F) > \frac{1}{2}\,\mu(P)$, and $P \in \mathcal{B}_2(aB_Q)$ otherwise. Taking into account that $Q \notin BF_0$ (because $Q \in \mathsf{Good}$), we derive
$$\sum_{P\in\mathcal{B}_1(aB_Q)} \mu(P) \leq 2 \sum_{P\in\mathcal{B}_1(aB_Q)} \mu(P \setminus F) \leq 2\,\mu(c'aB_Q \setminus F) \leq c\,\eta^{1/2}\mu(c'aB_Q).$$
By Lemma 6.2, we have $\mu(c'aB_Q) \leq c''\,\mu(aB_Q)$, and so we get
$$\sum_{P\in\mathcal{B}_1(aB_Q)} \mu(P) \leq c\,\eta^{1/2}\,\mu(aB_Q) \leq \frac{1}{2}\,\eta^{1/4}\,\mu(aB_Q). \tag{7.4}$$

Now turn our attention to the cells from $\mathcal{B}_2(aB_Q)$. Take $P \in \mathcal{B}_2(aB_Q)$ such that $\mu(P) > 0$. We claim that for every $x \in B_P$
$$\int_{2r(B_P)}^{4c_7 r(B_P)} \Delta_\mu(x,t)^2\,\frac{dt}{t} \gtrsim \Theta_\mu(c_7 B_P)^2. \tag{7.5}$$
To see this, note first that for such x and for $1 \leq t \leq 2$ we have $B(x,t\,r(B_P)) \subset 4B_P$. Let N be the minimal integer such that $c_7 B_P \subset B(x, 2^N r(B_Q))$ for every $x \in B_P$. Obviously, N is an absolute constant depending on c_7. We write
$$\begin{aligned} c^{-1}\Theta_\mu(c_7 B_P) - c\,\Theta_\mu(4B_P) &\leq \Theta_\mu(B(x,t2^N r(B_P))) - \Theta_\mu(B(x,t\,r(B_P))) \\ &\leq \sum_{k=1}^{N-1} \big|\Theta_\mu(B(x,t2^k r(B_P)) - \Theta_\mu(B(x,t2^{k+1} r(B_P))\big|. \end{aligned}$$

From the assumption (7.3) it turns out that the left hand above is comparable to $\Theta_\mu(c_7B_P)$. Therefore, we deduce that

$$\Theta_\mu(c_7B_P)^2 \leq c\sum_{k=1}^{N-1}\big|\Theta_\mu(B(x,t2^kr(B_P)) - \Theta_\mu(B(x,t2^{k+1}r(B_P))\big|^2 = c\sum_{k=1}^{N-1}\Delta_\mu(x,t2^kr(B_P))^2,$$

with the constant c possibly depending on N and thus on c_7. Integrating with respect to $t\in[1,2]$, (7.5) follows easily.

Integrating now (7.5) with respect to μ on $P\cap F$ and recalling that $\mu(P)\approx\mu(P\cap F)$, we get

$$\begin{aligned}\int_{P\cap F}\int_{2r(B_P)}^{4c_7r(B_P)}\Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x) &\gtrsim \Theta_\mu(c_7B_P)^2\,\mu(P\cap F)\\ &\approx \Theta_\mu(c_7B_P)^2\,\mu(P)\gtrsim c(A,\tau)\,\Theta_\mu(B_R)^2\,\mu(P).\end{aligned}$$

Consider $S(P)\in\mathsf{Good}$ such that $P\subset 4B_{S(P)}$ and $\ell(S(P))\approx\ell(P)$. Then, for δ small enough, we have

$$\begin{aligned}\int_{4B_{S(P)}\cap F}\int_{\delta\ell(S(P))}^{\delta^{-1}\ell(S(P))}\Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x) &\geq \int_{P\cap F}\int_{2r(B_P)}^{4c_7r(B_P)}\Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x)\\ &\gtrsim c(A,\tau)\,\Theta_\mu(B_R)^2\,\mu(P).\end{aligned}$$

Since $\ell(P)\approx\ell(S(P))$ and $P\subset 1.1B_{S(P)}$, for a given $S\in\mathsf{Good}$, the number of cells $P\in\mathcal{D}$ such that $S=S(P)$ does not exceed some fixed absolute constant. Moreover, it is easy to check that $S\subset caB_Q$ for some fixed $c>1$. Then we infer that

$$\Theta_\mu(B_R)^2\sum_{P\in\mathcal{B}_2(aB_Q)}\mu(P)\leq c(A,\tau)\sum_{S\in\mathsf{Good}:S\subset caB_Q}\int_{1.1B_S\cap F}\int_{\delta\ell(S)}^{\delta^{-1}\ell(S)}\Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x). \tag{7.6}$$

We estimate the right hand side above using the fact that the good cells are not in $BS\Delta_0$:

$$\begin{aligned}&\sum_{S\in\mathsf{Good}:S\subset caB_Q}\int_{1.1B_S\cap F}\int_{\delta\ell(S)}^{\delta^{-1}\ell(S)}\Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x)\\
&\quad\leq \sum_{S\in\mathsf{Good}:S\subset caB_Q}\ \sum_{\substack{T\in\mathsf{Term}:\\T\subset S}}\frac{\mu(T)}{\mu(S)}\int_{1.1B_S\cap F}\int_{\delta\ell(S)}^{\delta^{-1}\ell(S)}\Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x)\\
&\quad\leq c\sum_{\substack{T\in\mathsf{Term}:\\T\subset caB_Q}}\mu(T)\sum_{S\in\mathsf{Good}:T\subset S\subset caB_Q}\frac{1}{\mu(1.1B_S)}\int_{1.1B_S\cap F}\int_{\delta\ell(S)}^{\delta^{-1}\ell(S)}\Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x)\\
&\quad\leq c\sum_{\substack{T\in\mathsf{Term}:\\T\subset caB_Q}}\eta\,\Theta_\mu(B_R)^2\,\mu(T)\leq c\,\eta\,\Theta_\mu(B_R)^2\,\mu(aB_Q)\leq\frac{1}{2}\,\eta^{1/4}\,\Theta_\mu(B_R)^2\,\mu(aB_Q).\end{aligned}$$

This estimate, together with (7.4) and (7.6), proves the lemma. □

Lemma 7.3. *Let $Q\in\mathsf{Good}$ and let $a\geq 1$ be such that $r(aB_Q)\leq K\,\ell(R)$. Then*

$$\mu(aB_Q\setminus\widetilde{E})\leq\eta^{1/10}\,\mu(aB_Q),$$

assuming η small enough.

Proof. Denote by $\mathcal{A}(aB_Q)$ the subfamily of the cells from Reg which intersect $aB_Q \setminus \widetilde{E}$. We have

$$\mu(aB_Q \setminus \widetilde{E}) \leq \sum_{P\in\mathcal{A}(aB_Q)} \mu(P \setminus \widetilde{E}).$$

We distinguish two cases according to wether or not the cells P belong to $\mathcal{B}(aB_Q)$ (this is the family of cells introduced in Lemma 7.2). For the cells $P \in \mathcal{A}(aB_Q)$ which belong to $\mathcal{B}(aB_Q)$, we have shown that

$$\sum_{P\in\mathcal{B}(aB_Q)} \mu(P) \leq \eta^{1/4}\,\mu(aB_Q). \tag{7.7}$$

For the ones that do not belong to $\mathcal{B}(aB_Q)$, by Lemma 7.1 and the finite superposition of the balls $4B_P$, $P \in \mathsf{Reg}$, we have

$$\begin{aligned}\sum_{P\in\mathcal{A}(aB_Q)\setminus\mathcal{B}(aB_Q)} \mu(P \setminus \widetilde{E}) &\leq \eta^{1/4} \sum_{P\in\mathcal{A}(aB_Q)\setminus\mathcal{B}(aB_Q)} \mu(c_7\, B_P)\\ &\leq \eta^{1/4}\,\eta^{-1/10} \sum_{P\in\mathcal{A}(aB_Q)\setminus\mathcal{B}(aB_Q)} \mu(4B_P)\\ &\leq \eta^{3/20}\,\mu\Bigg(\bigcup_{P\in\mathcal{A}(aB_Q)\setminus\mathcal{B}(aB_Q)} 4B_P\Bigg).\end{aligned}$$

The same argument used in the previous lemma for the cells of $\mathcal{B}(aB_Q)$ shows that the cells from $\mathcal{A}(B_Q)$ are contained in $c'aB_Q$, for some absolute constant c'. Thus we have

$$\mu\Bigg(\bigcup_{P\in\mathcal{A}(aB_Q)\setminus\mathcal{B}(aB_Q)} 4B_P\Bigg) \leq c\,\eta^{3/20}\,\mu(c'aB_Q) \leq c''\,\eta^{3/20}\,\mu(aB_Q).$$

Adding this estimate and (7.7), the lemma follows, assuming η small enough. $\square$

Notice that, by the preceding lemma, we have

$$\widetilde{\mu}(aB_Q) \geq (1-\eta^{1/10})\,\mu(aB_Q) \qquad \text{for } Q \in \mathsf{Good},\ a \geq 1, \text{ with } r(a\,B_Q) \leq K\,\ell(R). \tag{7.8}$$

Lemma 7.4. *Let $Q \in \mathsf{Qgood}$ and and let $Q' \in \mathsf{Good}$ be such that $2B_Q \subset 2B_{Q'}$ and $\ell(Q) \approx \ell(Q')$. For any $y \in Q \cap \widetilde{E}$, we have*

$$\widetilde{\mu}(B(y,r)) \approx \mu(B(y,r)) \approx \frac{\mu(Q')}{\ell(Q')}\,r \qquad \text{for } \delta^{1/5}\,\ell(Q) \leq r \leq \delta^{-5}\ell(Q). \tag{7.9}$$

Further, if $\widetilde{\mu}(Q) > 0$, then

$$\widetilde{\mu}(2B_Q) \approx \mu(2B_Q) \approx \mu(Q'). \tag{7.10}$$

As a consequence, for any $Q \in \mathsf{Qgood}$ such that $\widetilde{\mu}(Q) > 0$, we have

$$\Theta_{\widetilde{\mu}}(2B_Q) \gtrsim \tau\,\Theta_\mu(B_R).$$

Proof. By the definition of $\widetilde{E}$, either $y \in W_0$ or there exists some $P \in \mathsf{Reg}$ such that

$$y \in 4B_P \cap Q \cap \mathsf{G}(P, R, \delta^{1/4}, \eta).$$

In the later case, from Lemma 6.6 it follows that $\ell(P) \lesssim \ell(Q) \approx \ell(Q')$. Together with the definition of $\mathsf{G}(P, R, \delta^{1/4}, \eta)$ in (6.1), this yields

$$\int_{c\delta^{1/4}\,\ell(Q')}^{\delta^{-1/4}\ell(R)} \Delta_\mu(y,r)^2\,\frac{dr}{r} \leq \int_{\delta^{1/4}\,\ell(P)}^{\delta^{-1/4}\ell(R)} \Delta_\mu(y,r)^2\,\frac{dr}{r} \leq \eta\,\Theta_\mu(2B_R)^2 \leq c\,\tau^{-2}\,\Theta_\mu(2B_{Q'})^2.$$

In the case that $y \in W_0$, it is immediate to check that the last estimate also holds. So in any case, by Remark 3.7, we get

$$\mu(B(y,r)) \approx \frac{\mu(2B_{Q'})}{\ell(Q')}\,r \qquad \text{for } c\,\delta^{1/4}\,\ell(Q') \leq r \leq c\,\delta^{-1/4}\ell(R), \tag{7.11}$$

assuming η and δ small enough. This proves one of the comparabilities in (7.9). For the remaining one, we apply Lemma 7.3. Indeed, for for $\delta^{1/5}\,\ell(Q) \leq r \leq \ell(Q)$ we have $B(y,r) \subset cB_{Q'}$, for some absolute constant c, and thus

$$\mu(B(y,r) \setminus \widetilde{E}) \leq \mu(cB_{Q'} \setminus \widetilde{E}) \leq \eta^{1/10}\,\mu(cB_{Q'}) \lesssim \eta^{1/10}\,\mu(2B_{Q'}).$$

Plugging (7.11), we get

$$\mu(B(y,r) \setminus \widetilde{E}) \lesssim \eta^{1/10}\,\mu(B(y,r))\,\frac{\ell(Q')}{r} \lesssim \eta^{1/10}\,\delta^{-5}\,\mu(B(y,r)).$$

Thus, assuming $\eta \ll \delta$, we deduce that $\widetilde{\mu}(B(y,r)) = \mu(B(y,r) \cap \widetilde{E}) \approx \mu(B(y,r))$ and so we are done with (7.9).

To prove (7.10), we take y as above and note that, in particular, by (7.9) $\widetilde{\mu}(B(y,r(B_Q))) \approx \mu(B(y,r(B_Q))) \approx \mu(Q')$. Since $B(y,r(B_Q)) \subset 2B_Q$, this implies that

$$\widetilde{\mu}(2B_Q) \approx \mu(2B_Q) \approx \mu(2B'_Q) \approx \mu(Q').$$

The last statement of the lemma follows from (7.10) and the fact that $\Theta_\mu(2B_{Q'}) \gtrsim \tau\,\Theta_\mu(B_R)$. □

Lemma 7.5. *For given ε_0, $\varepsilon_0' > 0$, if η and δ are taken small enough, the following holds for all $Q \in \mathsf{Good}$ and $a \geq 1$ such that $r(a\,B_Q) \leq K\,\ell(R)$:*

$$\alpha_\mu(aB_Q) \lesssim \varepsilon_0 \qquad \textit{and} \qquad \alpha_{\widetilde{\mu}}(aB_Q) \lesssim \varepsilon_0'. \tag{7.12}$$

Proof. The first estimate in (7.12) has already been seen in (6.13).

To show that $\alpha_{\widetilde{\mu}}(a\,B_Q) \lesssim \varepsilon_0'$, take a 1-Lipschitz function f supported on aB_Q. Then we have

$$\left|\int f\,d\mu - \int f\,d\widetilde{\mu}\right| \leq 2r(a\,B_Q)\,\mu(a\,B_Q \setminus \widetilde{E}) \leq 2\,\eta^{1/10}\,r(a\,B_Q)\,\mu(a\,B_Q),$$

by (7.8). Thus, $\operatorname{dist}_{a\,B_Q}(\mu, \widetilde{\mu}) \leq 2\,\eta^{1/10}\,r(a\,B_Q)\,\mu(a\,B_Q)$, and so

$$\alpha_\mu(a\,B_Q) \leq \alpha_{\widetilde{\mu}}(a\,B_Q) + 2\,\eta^{1/10} \leq \varepsilon_0',$$

if η is taken small enough. □

Remark 7.6. From the preceding proof, it follows that if c_Q, L_Q minimize $\alpha_\mu(aB_Q)$, then

$$\operatorname{dist}_{a\,B_Q}(\widetilde{\mu}, c_Q\,\mathcal{H}^1|_{L_Q}) \leq c\,\varepsilon_0'.$$

In next lemma we extend the result stated in Lemma 7.5 to the cells from Qgood.

Lemma 7.7. *For given ε_0, $\varepsilon_0' > 0$, if η and δ are taken small enough, the following holds for all $Q \in \mathsf{Qgood}$ with $\widetilde{\mu}(Q) > 0$ and $a \geq 2$ such that $r(a\,B_Q) \leq K\,\ell(R)$:*

$$\alpha_\mu(aB_Q) \lesssim \varepsilon_0 \qquad \text{and} \qquad \alpha_{\widetilde{\mu}}(aB_Q) \lesssim \varepsilon_0'. \tag{7.13}$$

Proof. Let $S \in \mathsf{Qgood}$ be such that $2B_S \subset aB_Q$ and $r(aB_Q) \approx \ell(S)$, and let $Q' \in \mathsf{Good}$ be such that $aB_Q \subset 2B_{Q'}$ and $\ell(Q') \approx r(aB_Q)$. Since $2B_S \subset 2B_{Q'}$ and $\ell(Q') \approx \ell(S)$, by Lemma 7.4 we have $\widetilde{\mu}(2B_S) \approx \mu(2B_{Q'})$. As $2B_S \subset aB_Q \subset 2B_{Q'}$, we infer that

$$\widetilde{\mu}(2B_S) \approx \widetilde{\mu}(aB_Q) \approx \mu(2B_{Q'}).$$

Then we deduce

$$\alpha_\mu(aB_Q) \leq c\,\alpha_\mu(2B_{Q'}) \qquad \text{and} \qquad \alpha_{\widetilde{\mu}}(aB_Q) \leq c\,\alpha_{\widetilde{\mu}}(2B_{Q'}),$$

and by Lemma 7.5 we are done. $\square$

We also have:

Lemma 7.8. *Let $\varepsilon_0'' > 0$ be an arbitrary (small) constant. Let $Q \in \mathsf{Qgood}$ be such that $\widetilde{\mu}(Q) > 0$. If $4 \leq a \leq \delta^{-1/5}$ and $r(a\,B_Q) \leq \frac{1}{2}K\,\ell(R)$, then*

$$b\beta_{\infty,\widetilde{\mu}}(aB_Q) \leq \varepsilon_0'',$$

assuming δ and η small enough. In fact,

$$\sup_{x \in aB_Q} \frac{\operatorname{dist}(x, L_{a,Q})}{r(a\,B_Q)} + \sup_{x \in L_{a,Q} \cap aB_Q} \frac{\operatorname{dist}(x, \operatorname{supp}\widetilde{\mu})}{r(a\,B_Q)} \leq \varepsilon_0'', \tag{7.14}$$

where $L_{a,Q}$ is the same line minimizing $\alpha_\mu(2aB_Q)$.

Proof. We can assume $\alpha_\mu(2a\,B_Q) \leq \varepsilon_0'''$, with ε_0''' as small as wished if η and δ are small enough. As shown in Lemma 2.3 this implies that

$$\frac{1}{\mu(a\,B_Q)} \int_{aB_Q} \frac{\operatorname{dist}(y, L_{a,Q})}{r(a\,B_Q)}\,d\mu(y) + \int_{L_{a,Q} \cap aB_Q} \frac{\operatorname{dist}(x, \operatorname{supp}\mu)}{r(a\,B_Q)^2}\,d\mathcal{H}^1|_{L_{a,Q}}(x) \lesssim \varepsilon_0'''. \tag{7.15}$$

From Lemma 7.5 and the subsequent remark we also have

$$\frac{1}{\mu(a\,B_Q)} \int_{aB_Q} \frac{\operatorname{dist}(y, L_{a,Q})}{r(a\,B_Q)}\,d\widetilde{\mu}(y) + \int_{L_{a,Q} \cap aB_Q} \frac{\operatorname{dist}(x, \operatorname{supp}\widetilde{\mu})}{r(a\,B_Q)^2}\,d\mathcal{H}^1|_{L_{a,Q}}(x) \lesssim \varepsilon_0'''. \tag{7.16}$$

Moreover, minor modifications in the proofs of these results show that the ball $a\,B_Q$ can be replaced by $\frac{3}{2}a\,B_Q$ in (7.15) and (7.16), at the cost of worsening the constants implicit in the "$\lesssim$" relation.

We will now estimate the first sup in the the left hand side of (7.14). To this end, recall that $\widetilde{\mu} = \mu|_{\widetilde{E}}$. Take $x \in \widetilde{E} \cap aB_Q \setminus L_{a,Q}$ and set

$$d_x := \frac{1}{2}\,\min(\operatorname{dist}(x, L_{a,Q}), r(aB_Q)) \approx \operatorname{dist}(x, L_{a,Q}).$$

Then we have $B(x,d_x)\subset \frac32 a\,B_Q$, and thus

$$\frac{1}{\mu(\frac32 a\,B_Q)}\int_{\frac32 a\,B_Q}\frac{\operatorname{dist}(y,L_{a,Q})}{r(\frac32 a\,B_Q)}\,d\mu(y)\geq \frac{d_x\,\mu(B(x,d_x))}{r(\frac32 a\,B_Q)\,\mu(\frac32 a\,B_Q)}\gtrsim \frac{d_x\,\mu(B(x,d_x))}{r(a\,B_Q)\,\mu(a\,B_Q)}. \tag{7.17}$$

By Lemma 7.4 we have

$$\mu(B(x,r))\approx \frac{\mu(2B_Q)}{\ell(Q)}\,r\gtrsim \tau\,\Theta(B_R)\,r\qquad \text{for } \delta^{1/5}\,\ell(Q)\leq r\leq \delta^{-1/5}\ell(Q).$$

So we infer that if $d_x\geq \delta^{1/5}\ell(Q)$, then $\mu(B(x,d_x))\gtrsim \tau\,\Theta_\mu(B_R)\,d_x$, and by (7.17),

$$\varepsilon_0'''\gtrsim \frac{1}{\mu(\frac32 a\,B_Q)}\int_{\frac32 a\,B_Q}\frac{\operatorname{dist}(y,L_{a,Q})}{r(\frac32 a\,B_Q)}\,d\mu(y)\gtrsim \frac{\tau\,\Theta_\mu(B_R)\,d_x^2}{r(a\,B_Q)\,\mu(a\,B_Q)}\gtrsim \frac{A^{-1}\,\tau\,d_x^2}{r(a\,B_Q)^2}.$$

Therefore,

$$d_x\lesssim (\varepsilon_0''')^{1/2}A\,\tau^{-1}\,r(a\,B_Q),$$

and then in either case

$$d_x\leq \max\bigl(\delta^{1/5}\,\ell(Q),\,c\,(\varepsilon_0''')^{1/2}A\,\tau^{-1}\,r(a\,B_Q)\bigr)\lesssim \max\bigl(\delta^{1/5},\,(\varepsilon_0''')^{1/2}A\,\tau^{-1}\bigr)\,r(a\,B_Q).$$

Taking the supremum on all $x\in\widetilde E\cap a\,B_Q$, we deduce that

$$\sup_{x\in\operatorname{supp}\widetilde\mu\cap a\,B_Q}\frac{\operatorname{dist}(x,L_{a,Q})}{r(a\,B_Q)}\lesssim \max\bigl(\delta^{1/5},\,(\varepsilon_0''')^{1/2}A\,\tau^{-1}\bigr). \tag{7.18}$$

To estimate the second sup on the left side of (7.14), take $x\in L_{a,Q}\cap aB_Q$, and let $\widetilde d_x=\operatorname{dist}(x,\operatorname{supp}\widetilde\mu)$. Then it follows that for all $y\in L_{a,Q}\cap\frac32 a\,B_Q\cap B(x,\widetilde d_x)$, $\operatorname{dist}(y,\operatorname{supp}\widetilde\mu)\geq \widetilde d_x/2$. Thus,

$$\varepsilon_0''\gtrsim\int_{L_{a,Q}\cap\frac32 a\,B_Q}\frac{\operatorname{dist}(x,\operatorname{supp}\widetilde\mu)}{r(\frac32 a\,B_Q)^2}\,d\mathcal H^1|_{L_{a,Q}}(x)\gtrsim \frac{\widetilde d_x\,\mathcal H^1(B(x,\widetilde d_x/2)\cap\frac32 a\,B_Q)}{r(\frac32 a\,B_Q)^2}\gtrsim\frac{(\widetilde d_x)^2}{r(\frac32 a\,B_Q)^2}.$$

Taking the sup on all the points $x\in L_{a,Q}\cap aB_Q$, we obtain

$$\sup_{x\in L_{a,Q}\cap a\,B_Q}\frac{\operatorname{dist}(x,\operatorname{supp}\widetilde\mu)}{r(a\,B_Q)}\leq \varepsilon_0'''^{1/2}. \tag{7.19}$$

The lemma follows from (7.18) and (7.19), assuming η and δ small enough. $\square$

From now on, we assume that for some small constant $\varepsilon_0>0$, we have

$$\alpha_\mu(a\,B_Q)\leq\varepsilon_0,\quad \alpha_{\widetilde\mu}(a\,B_Q)\leq\varepsilon_0,\quad b\beta_{\infty,\widetilde\mu}(a\,B_Q)\leq\varepsilon_0, \tag{7.20}$$

for any $Q\in\mathsf{Qgood}$ with $\widetilde\mu(Q)>0$ and for $a\geq 2$ with $r(a\,B_Q)\leq\frac12 K\,\ell(R)$. To this end, we will need the constants δ and η to be chosen small enough in the Main Lemma.

8. The measure of the cells from BCF, LD, BSΔ and BCG

To prove the property (c) stated Main Lemma 5.1, we have to estimate the sum

$$\sum_{P\in \mathsf{Stop}(R)} \Theta_\mu(1.1B_P)^2\,\mu(P).$$

According to Lemma 6.7, this sum can be split as follows:

$$\sum_{P\in\mathsf{Stop}}\ldots = \sum_{Q\in\mathcal{D}(R)\cap\mathsf{BCF}}\;\sum_{P\in\mathsf{Stop}:P\subset Q}\ldots + \sum_{Q\in\mathcal{D}(R)\cap\mathsf{LD}}\;\sum_{P\in\mathsf{Stop}:P\subset Q}\ldots + \sum_{Q\in\mathcal{D}(R)\cap\mathsf{HD}}\;\sum_{P\in\mathsf{Stop}:P\subset Q}\ldots$$
$$+ \sum_{Q\in\mathcal{D}(R)\cap\mathsf{BCG}}\;\sum_{P\in\mathsf{Stop}:P\subset Q}\ldots + \sum_{Q\in\mathcal{D}(R)\cap\mathsf{BS\Delta}}\;\sum_{P\in\mathsf{Stop}:P\subset Q}\ldots,$$

where we denoted $\ldots = \Theta_\mu(1.1B_P)^2\,\mu(P)$. In this section we will estimate all the sums on the right had side above, with the exception of the one involving the cells $Q\in\mathsf{HD}$.

Regarding the sum involving the family BCF, we have:

Lemma 8.1. *If η is small enough, we have*

$$\sum_{Q\in\mathcal{D}(R)\cap\mathsf{BCF}}\;\sum_{P\in\mathsf{Stop}:P\subset Q} \Theta_\mu(1.1B_P)^2\,\mu(P) \lesssim \eta^{1/5}\,\Theta_\mu(B_R)^2\,\mu(R).$$

Proof. Recall that, by Lemma 6.4

$$\mu\Bigg(\bigcup_{Q\in\mathsf{BCF}:Q\subset R} Q\Bigg) \leq c\,\eta^{1/4}\,\mu(R).$$

On the other hand, by Lemma 6.8, any cell $P\in\mathsf{Stop}$ satisfies $\Theta_\mu(1.1B_P)\lesssim A\,\Theta_\mu(B_R)$, and thus

$$\begin{aligned}\sum_{Q\in\mathcal{D}(R)\cap\mathsf{BCF}}\;\sum_{P\in\mathsf{Stop}:P\subset Q} \Theta_\mu(1.1B_P)^2\,\mu(P) &\lesssim A\,\Theta_\mu(B_R)^2 \sum_{Q\in\mathcal{D}(R)\cap\mathsf{BCF}} \mu(Q)\\ &\lesssim A\,\eta^{1/4}\,\Theta_\mu(B_R)^2\,\mu(R) \lesssim \eta^{1/5}\,\Theta_\mu(B_R)^2\,\mu(R).\end{aligned}$$

□

Concerning the family LD we have:

Lemma 8.2. *We have*

$$\sum_{Q\in\mathcal{D}(R)\cap\mathsf{LD}}\;\sum_{P\in\mathsf{Stop}:P\subset Q} \Theta_\mu(1.1B_P)^2\,\mu(P) \lesssim A^2\,\tau^{1/4}\,\Theta_\mu(B_R)^2\,\mu(R).$$

Proof. Let $Q\in\mathcal{D}(R)\cap\mathsf{LD}$. To estimate the sum $\sum_{P\in\mathsf{Stop}:P\subset Q}\Theta_\mu(1.1B_P)^2\,\mu(P)$ we distinguish two cases according to wether $\ell(P)>\tau^{1/2}\,\ell(Q)$ or not.

Suppose first that $\ell(P)>\tau^{1/2}\,\ell(Q)$. If the parameters A_0, C_0 in the construction of the David-Mattila cells are chosen appropriately (with $1\ll C_0\ll A_0$), then $1.1B_P\subset 1.1B_Q$ and so it follows that

$$\Theta_\mu(1.1B_P)\lesssim \frac{\ell(Q)}{\ell(P)}\,\Theta_\mu(1.1B_Q)\lesssim \tau^{-1/2}\,\tau\,\Theta_\mu(B_R)\approx \tau^{1/2}\,\Theta_\mu(B_R).$$

Therefore,

$$\sum_{P\in \mathsf{Stop}:P\subset Q} \Theta_\mu(1.1B_P)^2\,\mu(P)\lesssim \tau\,\Theta_\mu(B_R)^2\,\mu(Q). \tag{8.1}$$

For the case when $\ell(P)\leq \tau^{1/2}\,\ell(Q)$ we will use the small boundaries condition of Q. By Lemma 6.6 this implies that $d(x)\lesssim \tau^{1/2}\,\ell(Q)$ for all $x\in P$. Since $P\subset Q\not\in \mathsf{Good}$, then we deduce that

$$\operatorname{dist}(P,E\setminus Q)\lesssim \tau^{1/2}\,\ell(Q).$$

Thus, by (4.4),

$$\sum_{\substack{P\in \mathsf{Stop}:P\subset Q\\ \ell(P)\leq\tau^{1/2}\ell(Q)}} \Theta_\mu(1.1B_P)^2\,\mu(P)\lesssim A^2\,\Theta_\mu(B_R)^2 \sum_{\substack{P\in \mathsf{Stop}:P\subset Q\\ \operatorname{dist}(P,E\setminus Q)\lesssim\tau^{1/2}\,\ell(Q)}}\mu(P)\lesssim A^2\,\tau^{1/4}\,\Theta_\mu(B_R)^2\,\mu(3.5B_Q). \tag{8.2}$$

Next we claim that if there exists some cell $P\in\mathsf{Stop}$ contained in Q such that $\ell(P)\leq \tau^{1/2}\,\ell(Q)$, then Q is doubling, i.e. $Q\in\mathcal{D}^{db}$. Indeed, by the definition of Reg, assuming τ small enough, the existence of such cell P implies the existence of some cell $Q'\in\mathsf{Good}$ such that $\ell(Q')\approx\ell(P)$ and $3.3B_{Q'}\subset 1.02B_Q$. Taking a suitable ancestor of Q', we deduce that there exists some $Q''\in\mathsf{Good}$ such that $3.3B_{Q''}\subset 1.05B_Q$ and $\ell(Q'')\approx\ell(Q)$.

Let $a\geq 3.3$ be the maximal number such that $a\,B_{Q''}\subset 1.1B_Q$. Notice that $r(a\,B_{Q''})\geq r(1.1B_Q)-r(1.05B_Q)=0.05\,r(B_Q)$. By Remark 6.3, we know that $\mu(bB_{Q''})\leq C(a,b)\,\mu(aB_{Q''})$ for $3.3\leq a\leq b\leq \delta^{-1/2}\ell(Q)$, with $C(a,b)$ not depending on C_0. So we have

$$\mu(400B_Q)\leq c\,\mu(a\,B_{Q''})\leq c\,\mu(1.1B_Q), \tag{8.3}$$

with c independent of C_0. By arguments analogous to the ones of Lemma 6.2, this implies that $Q\in\mathcal{D}^{db}$. Indeed, if $Q\not\in\mathcal{D}^{db}$, we have

$$\mu(1.1B_Q)=\mu(28\cdot 1.1B(Q))\leq\mu(100B(Q))\leq C_0^{-1}\,\mu(100^2B(Q)),$$

while by (8.3),

$$\mu(100^2B(Q))=\mu(\tfrac{1}{28}\,100^2\,B_Q)\leq\mu(400B_Q)\leq c\,\mu(1.1B_Q),$$

and so we get a contradiction if C_0 is assumed big enough in the construction of the David-Mattila cells.

Since Q is doubling, we have $\mu(3.5B_Q)\leq\mu(100B(Q))\leq c\,\mu(B(Q))\leq c\,\mu(Q)$. Then, by (8.2), we deduce that

$$\sum_{\substack{P\in \mathsf{Stop}:P\subset Q\\ \ell(P)\leq\tau^{1/2}\ell(Q)}} \Theta_\mu(1.1B_P)^2\,\mu(P)\lesssim A^2\,\tau^{1/4}\,\Theta_\mu(B_R)^2\,\mu(Q).$$

Together with (8.1), this yields the desired conclusion. □

Next we will deal with the cells from $\mathsf{BS}\Delta$:

Lemma 8.3. *We have*

$$\sum_{Q\in\mathsf{BS}\Delta:Q\subset R}\Theta_\mu(B_R)^2\,\mu(Q)\lesssim\frac{1}{\eta}\sum_{Q\in\mathsf{Tree}}\int_{1.1B_Q\cap F}\int_{\delta\,\ell(Q)}^{\delta^{-1}\ell(Q)}\Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x).$$

Proof. Recall that the cells $Q \in \mathsf{BS\Delta}$ satisfy

$$\sum_{P\in\mathcal{D}:Q\subset P\subset R} \frac{1}{\mu(1.1B_P)} \int_{1.1B_P\cap F} \int_{\delta\,\ell(P)}^{\delta^{-1}\ell(P)} \Delta_\mu(x,r)^2 \,\frac{dr}{r}\, d\mu \geq \eta\, \Theta_\mu(B_R)^2.$$

So we have

$$\sum_{\substack{Q\in\mathsf{BS\Delta}:\\ Q\subset R}} \Theta_\mu(B_R)^2\, \mu(Q) \leq \frac{1}{\eta} \sum_{\substack{Q\in\mathsf{Term}:\\ Q\subset R}} \sum_{\substack{P\in\mathcal{D}:\\ Q\subset P\subset R}} \frac{\mu(Q)}{\mu(P)} \int_{1.1B_P\cap F} \int_{\delta\,\ell(P)}^{\delta^{-1}\ell(P)} \Delta_\mu(x,r)^2 \,\frac{dr}{r}\, d\mu(x).$$

Denote now $\widetilde{\mathsf{Tree}}$ the family of cells P which are contained in R and are not strictly contained in any cell from Term. By interchanging the order of summation, the term on the right hand side above equals

$$\frac{1}{\eta} \sum_{P\in\widetilde{\mathsf{Tree}}} \int_{1.1B_P\cap F} \int_{\delta\,\ell(P)}^{\delta^{-1}\ell(P)} \Delta_\mu(x,r)^2 \,\frac{dr}{r}\, d\mu(x) \sum_{\substack{Q\in\mathsf{Term}:\\ Q\subset P}} \frac{\mu(Q)}{\mu(P)}$$

$$= \frac{1}{\eta} \sum_{P\in\widetilde{\mathsf{Tree}}} \int_{1.1B_P\cap F} \int_{\delta\,\ell(P)}^{\delta^{-1}\ell(P)} \Delta_\mu(x,r)^2 \,\frac{dr}{r}\, d\mu(x).$$

Since $\widetilde{\mathsf{Tree}} \subset \mathsf{Tree}$, we are done. □

Now we turn our attention to the cells from BCG:

Lemma 8.4. *Suppose that δ is small enough. Then we have*

$$\sum_{\substack{Q\in\mathsf{BCG}:\\ Q\subset R}} \Theta_\mu(B_R)^2\, \mu(Q) \lesssim \delta^{1/2}\, \Theta_\mu(B_R)^2\, \mu(R) + \frac{1}{\eta} \sum_{Q\in\mathsf{Tree}} \int_{\delta^{-1}B_Q\cap F} \int_{\delta^5\,\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2 \,\frac{dr}{r}\, d\mu(x).$$

Proof. We need to distinguish three types of cells from BCG:

- $Q \in \mathsf{BCG}_1$ if $Q \in \mathcal{D}(R)$ and $\ell(Q) \geq \delta^4\, \ell(R)$.
- $Q \in \mathsf{BCG}_2$ if $Q \in \mathcal{D}(R)$, $\ell(P) \leq \delta^4\, \ell(R)$, and $\operatorname{dist}(Q, E\setminus R) \leq \delta\, \ell(R)$.
- $Q \in \mathsf{BCG}_3$ if $Q \in \mathcal{D}(R)$, $\ell(P) \leq \delta^4\, \ell(R)$, and $\operatorname{dist}(Q, E\setminus R) > \delta\, \ell(R)$.

First we will estimate the measure of the cells from BCG_2. To this end we will use the fact that R has "small boundaries". More precisely, recall that by (4.4) we have

$$\mu\big(\{x\in R : \operatorname{dist}(x, E\setminus R) \leq \lambda\, \ell(R)\}\big) \leq c\, \lambda^{1/2}\, \mu(R).$$

By definition, every cell $Q \in \mathsf{BCG}_2$ satisfies

$$Q \subset \{x\in R : \operatorname{dist}(x, E\setminus R) \leq (\delta + c\,\delta^4)\, \ell(R)\},$$

and thus

$$\sum_{Q\in\mathsf{BCG}_2} \mu(Q) \leq \mu\big(\{x\in R : \operatorname{dist}(x, E\setminus R) \leq (\delta + c\,\delta^4)\, \ell(R)\}\big) \lesssim (\delta + c\,\delta^4)^{1/2}\, \mu(R) \lesssim \delta^{1/2}\, \mu(R).$$

To deal with BCG_1 recall that the cells $Q \in \mathsf{BCG}$ satisfy

$$\mu(\delta^{-1/2}B_Q \cap F \setminus \mathsf{G}(Q, R, \delta^{1/2}, \eta)) \geq \eta\, \mu(\delta^{-1/2}B_Q \cap F) \geq \frac{1}{2}\, \mu(\delta^{-1/2}B_Q), \tag{8.4}$$

where we used that $Q \notin \mathsf{BCF}$ for the last inequality. Taking also into account that

$$\int_{\delta^{1/2}\ell(Q)}^{\delta^{-1/2}\ell(R)} \Delta_\mu(x,r)^2 \frac{dr}{r} \geq \eta\, \Theta_\mu(2B_R)^2 \qquad \text{for all } x \notin \mathsf{G}(Q,R,\delta^{1/2},\eta), \tag{8.5}$$

by Chebyshev we infer that

$$\mu(Q) \leq 2\,\mu(\delta^{-1/2}B_Q \cap F) \leq \frac{1}{\Theta_\mu(2B_R)^2} \int_{\delta^{-1/2}B_Q\cap F} \int_{\delta^{1/2}\ell(Q)}^{\delta^{-1/2}\ell(R)} \Delta_\mu(x,r)^2 \frac{dr}{r}\, d\mu(x).$$

Using that $\ell(Q) \geq \delta^4\,\ell(R)$ for $Q \in \mathsf{BCG}_1$, we infer that

$$\begin{aligned}\sum_{Q\in\mathsf{BCG}_1} \Theta_\mu(B_R)^2\,\mu(Q) &\lesssim \frac{1}{\eta} \sum_{Q\in\mathsf{BCG}_1} \int_{\delta^{-1/2}B_Q\cap F} \int_{\delta^{1/2}\ell(Q)}^{\delta^{-1/2}\ell(R)} \Delta_\mu(x,r)^2 \frac{dr}{r}\, d\mu(x)\\ &\lesssim \frac{1}{\eta} \int_{\delta^{-1}B_R\cap F} \int_{\delta^{5}\ell(R)}^{\delta^{-1/2}\ell(R)} \Delta_\mu(x,r)^2 \frac{dr}{r}\, d\mu(x).\end{aligned}$$

Finally we will estimate the measure of cells from BCG_3. To this end we consider the function

$$f(x) = \sum_{P\in\mathsf{Stop}} \left(\int_{\delta^2\ell(P)}^{\delta^{-1}\ell(R)} \Delta_\mu(x,r)^2 \frac{dr}{r} \right)^{1/2} \chi_{P\cap F}(x).$$

We claim that

$$Q \subset \left\{ x\in\mathbb{R}^d : M_* f(x) > \tfrac{1}{2}\,\eta^{1/2}\,\Theta_\mu(2B_R)\right\} \qquad \text{for all } Q \in \mathsf{BCG}_3, \tag{8.6}$$

where M_* is the maximal operator introduced in (5.8). To prove the claim, consider $Q \in \mathsf{BCG}_3$ and notice that by (8.5),

$$\left(\int_{\delta^{1/2}\,\ell(Q)}^{\delta^{-1/2}\ell(R)} \Delta_\mu(x,r)^2 \frac{dr}{r} \right)^{1/2} \geq \eta^{1/2}\,\Theta_\mu(2B_R) \qquad \text{for all } x \notin \mathsf{G}(Q,R,\delta^{1/2},\eta).$$

and as (8.4) also holds in this case, we infer that

$$\frac{1}{\mu(\delta^{-1/2}B_Q)} \int_{\delta^{-1/2}B_Q\cap F} \left(\int_{\delta^{1/2}\,\ell(Q)}^{\delta^{-1/2}\ell(R)} \Delta_\mu(x,r)^2 \frac{dr}{r} \right)^{1/2} d\mu(x) \geq \frac{1}{2}\,\eta^{1/2}\,\Theta_\mu(2B_R). \tag{8.7}$$

Observe now that if $P \in \mathsf{Reg}$, $P \cap \delta^{-1/2}B_Q \neq \varnothing$, then

$$\ell(P) \leq c\,\delta^{-1/2}\ell(Q) \leq \delta^{-1}\ell(Q) \leq \delta^3\,\ell(R). \tag{8.8}$$

In particular, this implies that the left hand side of (8.7) is not greater than

$$\frac{1}{\mu(\delta^{-1/2}B_Q)} \int_{\delta^{-1/2}B_Q} \left(\sum_{P\in\mathsf{Reg}} \chi_{P\cap F}(x) \int_{\delta^2\,\ell(P)}^{\delta^{-1/2}\ell(R)} \Delta_\mu(x,r)^2 \frac{dr}{r} \right)^{1/2} d\mu(x). \tag{8.9}$$

Further, from (8.8) it follows that

$$\begin{aligned}\operatorname{dist}(P, E\setminus R) &\geq \operatorname{dist}(Q, E\setminus R) - \operatorname{diam}(\delta^{-1/2}B_Q) - \operatorname{diam}(P)\\ &\geq \delta\,\ell(R) - c\,\delta^{-1/2}\,\delta^4\,\ell(R) - c\,\delta^3\,\ell(R) > 0,\end{aligned}$$

assuming δ small enough. Thus $P \subset R$, and so we can assume that the sum in (8.9) runs over $R \in \mathsf{Stop}$. Altogether, from (8.7) and the above considerations it follows that

$$\frac{1}{\mu(\delta^{-1/2}B_Q)} \int_{\delta^{-1/2}B_Q} f(x)\, d\mu(x) \geq \frac{1}{2}\, \eta^{1/2}\, \Theta_\mu(2B_R),$$

which proves (8.6).

From the claim above and Chebyshev we deduce that

$$\sum_{Q\in\mathsf{BCG}_3} \Theta_\mu(B_R)^2\, \mu(Q) \lesssim \sum_{Q\in\mathsf{BCG}_3} \frac{1}{\eta} \int_Q M_*f(x)^2\, d\mu(x) \lesssim \frac{1}{\eta} \int M_*f(x)^2\, d\mu(x) \lesssim \frac{1}{\eta} \int |f|^2\, d\mu.$$

To conclude with the family BCG_3 it just remains to note that

$$\begin{aligned} \int |f|^2\, d\mu &= \sum_{P\in\mathsf{Stop}} \int_{P\cap F} \int_{\delta^2\ell(P)}^{\delta^{-1}\ell(R)} \Delta_\mu(x,r)^2\, \frac{dr}{r}\, d\mu(x) \\ &\leq \sum_{P\in\mathsf{Stop}} \sum_{Q:P\subset Q\subset R} \int_{P\cap F} \int_{\delta^2\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2\, \frac{dr}{r}\, d\mu(x) \\ &= \sum_{Q\in\mathsf{Tree}} \int_{Q\cap F} \int_{\delta^2\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2\, \frac{dr}{r}\, d\mu(x). \end{aligned}$$

Gathering the estimates we obtained for the families BCG_1, BCG_2 and BCG_3, the lemma follows. $\square$

By combining the results obtained in Lemmas 8.1, 8.2, 8.3, and 8.4, and taking into account that $\Theta_\mu(1.1B_Q) \lesssim A\, \Theta_\mu(B_R)$ for all $Q \in \mathsf{Stop}(R)$, we get the following.

Lemma 8.5. *If η and δ are small enough, then*

$$\sum_{\substack{Q\in\mathcal{D}(R):\\ Q\subset \mathsf{BCF}\cup\mathsf{LD}\cup\mathsf{BCG}\cup\mathsf{BS}\Delta}} \Theta_\mu(1.1B_Q)^2\, \mu(Q) \lesssim A^2\, (\eta^{1/5} + \tau^{1/4} + \delta^{1/2})\, \Theta_\mu(B_R)^2\, \mu(R) + \frac{A^2}{\eta} \sum_{Q\in\mathsf{Tree}} \int_{\delta^{-1}B_Q\cap F} \int_{\delta^5\,\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2\, \frac{dr}{r}\, d\mu(x).$$

9. The new families of cells $\mathsf{BS}\beta$, NTerm, NGood, NQgood and NReg

To complete he proof of the Main Lemma 5.1, it remains to construct the curve Γ_R and to estimate the sum $\sum_{Q\in\mathcal{D}(R)\cap\mathsf{HD}} \sum_{P\in\mathsf{Stop}:P\subset Q} \Theta_\mu(1.1B_P)^2\, \mu(P)$. To this end, we need first to introduce a new type of terminal cells. Let M be some very big constant to be fixed below (in particular, $M \gg A\,\tau^{-1}$). We say that a cell $Q \in \mathcal{D}$ belongs to $BS\beta_0$ if $Q \notin BCF_0 \cup LD_0 \cup HD_0 \cup BCG_0 \cup BC\Delta_0$, $\ell(Q) \leq \ell(R)$, and

$$\sum_{P\in\mathcal{D}:Q\subset P\subset R} \beta_{\widetilde{\mu},\infty}(2B_P)^2 \geq M. \tag{9.1}$$

Next we consider the subfamily of $\mathsf{Term} \cup BS\beta_0$ of the cells which are maximal with respect to inclusion (thus they are disjoint), and we call it NTerm. We denote by $\mathsf{BS}\beta$ the subfamily of the cells from NTerm which belong by $BS\beta_0$. The notation NTerm stands for "new term", and $\mathsf{BS}\beta$ for "big sum of $\beta's$". Note that

$$\mathsf{NTerm} \subset \mathsf{Term} \cup \mathsf{BS}\beta.$$

The definition of the family $BS\beta_0$, and so of $\mathsf{BS}\beta$, depends on the measure $\widetilde{\mu}$. This is the reason why $BS\beta_0$ and $\mathsf{BS}\beta$ were not introduced in Section 6, like the others cells of Term. The introduction of the new family $\mathsf{BS}\beta$ is necessary to guaranty the lower Ahlfors-David regularity of the measures σ_k in the forthcoming Section 14.

Similarly to Section 6, we denote by NGood the subfamily of the cells $Q \subset B(x_0, \frac{1}{10}Kr_0)$ with $\ell(Q) \leq \ell(R)$ such that there does not exist any cell $Q' \in \mathsf{NTerm}$ with $Q' \supset Q$. Notice that $R \in \mathsf{NGood}$, $\mathsf{NGood} \subset \mathsf{Good}$, and $\mathsf{NTerm} \not\subset \mathsf{NGood}$.

We need now to define a regularized version of NTerm which we will call NReg. To this end, we proceed exactly as in Section 6. First we consider the auxiliary 1-Lipschitz function $\widetilde{d} : \mathbb{R}^d \to [0, \infty)$:

$$\widetilde{d}(x) = \inf_{Q \in \mathsf{NGood}} \big(|x - z_Q| + \ell(Q)\big). \tag{9.2}$$

We denote

$$NW_0 = \{x \in \mathbb{R}^d : \widetilde{d}(x) = 0\}.$$

For each $x \in E \setminus NW_0$ we take the largest cell $Q_x \in \mathcal{D}$ such that $x \in Q_x$ with

$$\ell(Q_x) \leq \frac{1}{60} \inf_{y \in Q_x} \widetilde{d}(y).$$

We denote by NReg the collection of the different cells Q_x, $x \in E \setminus W_0$. Further, we consider the subcollection of the cells from NReg with non-vanishing $\widetilde{\mu}$-measure and we relabel it as $\{Q_i\}_{i \in I}$. Also, we denote by NQgood the family of cells $Q \in \mathcal{D}$ such that Q is contained in $B(x_0, 2Kr_0)$ and Q is not strictly contained in any cell of the family NReg. Note that $\mathsf{NReg} \subset \mathsf{NQgood}$. Moreover, since $\widetilde{d}(x) \geq d(x)$ for all $x \in \mathbb{R}^d$, it follows that $\mathsf{NQgood} \subset \mathsf{Qgood}$. Thus all the properties proved in Sections 6 and 7 for the cells from Qgood also hold for the ones from NQgood.

The following result and its proof, which we omit, are analogous to the ones of Lemma 6.6.

Lemma 9.1. *The cells $\{Q_i\}_{i \in I}$ are pairwise disjoint and satisfy the following properties:*

(a) *If $x \in B(z_{Q_i}, 50\ell(Q_i))$, then $10\,\ell(Q_i) \leq \widetilde{d}(x) \leq c\,\ell(Q_i)$, where c is some constant depending only on A_0. In particular, $B(z_{Q_i}, 50\ell(Q_i)) \cap NW_0 = \varnothing$.*

(b) *There exists some constant c such that if $B(z_{Q_i}, 50\ell(Q_i)) \cap B(z_{Q_j}, 50\ell(Q_j)) \neq \varnothing$, then*

$$c^{-1}\ell(Q_i) \leq \ell(Q_j) \leq c\,\ell(Q_i).$$

(c) *For each $i \in I$, there at most N cells Q_j, $j \in I$, such that*

$$B(z_{Q_i}, 50\ell(Q_i)) \cap B(z_{Q_j}, 50\ell(Q_j)) \neq \varnothing,$$

where N is some absolute constant.

(d) *If $x \notin B(x_0, \frac{1}{8}Kr_0)$, then $\widetilde{d}(x) \approx |x - x_0|$. As a consequence, if $B(z_{Q_i}, 50\ell(Q_i)) \not\subset B(x_0, \frac{1}{8}Kr_0)$, then $\ell(Q_i) \gtrsim Kr_0$.*

10. The approximating curves Γ^k

In this section we will construct somes curves Γ^k which, in a sense, approximate $\operatorname{supp}\widetilde{\mu}$ on $B(x_0,\frac{1}{4}Kr_0)$ up to the scale of the cubes $\{Q_i\}_{i\in I}$. This curves will be used to show that the measure of the cells from HD is small.

The curves Γ^k are constructed inductively in the following way. Let

$$d_0 = \operatorname{diam}\big(\operatorname{supp}\widetilde{\mu}\cap \bar{B}(x_0,\tfrac{1}{4}Kr_0)\big),$$

and take $z_A, z_B \in \operatorname{supp}\widetilde{\mu}\cap \bar{B}(x_0,\frac{1}{4}Kr_0)$ such that $|z_A - z_B| = d_0$. The curve Γ^1 is just the segment L_1^1 with endpoints $x_0^1 \equiv z_A$ and $x_1^1 \equiv z_B$.

For $k\geq 1$ we assume that Γ^k contains points $z_A \equiv x_0^k,\, x_1^k,\ldots,x_{N_k-1}^k,\, x_{N_k}^k \equiv z_B$ from $\operatorname{supp}\widetilde{\mu}\cap \bar{B}(x_0,\frac{1}{2}Kr_0))$ and that Γ^k is the union of the segments $L_j^k := \big[x_{j-1}^k, x_j^k\big]$, for $j=1,\ldots,N_k$. Then Γ^{k+1} is constructed as follows. Each one of the segments L_j^k, $j=1,\ldots,N_k$, that constitutes Γ^k is replaced by a curve Γ_j^k with the same end points as L_j^k by the following rules:

(A) If $\mathcal{H}^1(L_j^k)\leq 2^{-(k+1)/2}\, d_0$, we set $\Gamma_j^k = L_j^k$.

(B) If there exists some cell Q_i, $i\in I$, such that $2B_{Q_i}\cap L_j^k\neq\varnothing$ and $\mathcal{H}^1(L_j^k)\leq \ell(Q_i)$, then we also set $\Gamma_j^k = L_j^k$.

(C) If the conditions in (A) and (B) do not hold, that is to say, if $\mathcal{H}^1(L_j^k) > 2^{-(k+1)/2}\, d_0$ and also $\mathcal{H}^1(L_j^k) > \ell(Q_i)$ for all $i\in I$ such that $2B_{Q_i}\cap L_j^k\neq\varnothing$, then we consider the mid point of the segment L_j^k, which we denote by z_j^k, and we take a point $p_j^k\in\operatorname{supp}\widetilde{\mu}$ such that

$$|p_j^k - z_j^k| \leq c\,\varepsilon_0\,\mathcal{H}^1(L_j^k). \tag{10.1}$$

The existence of p_j^k is ensured by the fact that the ball B centered at x_{j-1}^k (recall that the end points of L_j^k are x_{j-1}^k and x_j^k and they belong to $\operatorname{supp}\widetilde{\mu}$) satisfies $b\beta(B)\lesssim\varepsilon_0$. This follows from the fact that if $Q\in\mathcal{D}$ is the smallest cell containing x_{j-1}^k such that $\ell(Q) > \mathcal{H}^1(L_j^k)$ and x_{j-1}^k belongs to some cell Q_i, $i\in I$, then we have $Q_i\subset Q$, and so we can apply Lemma 7.8 to Q. Then we set

$$\Gamma_j^k = [x_{j-1}^k, p_j^k]\cup[p_j^k, x_j^k].$$

The points $z_A \equiv x_0^{k+1},\, x_1^{k+1},\ldots,x_{N_k}^{k+1},\, x_{N_{k+1}}^{k+1}\equiv z_B$ are obtained from the sequence

$$x_0^k,\, x_1^k,\ldots,x_{N_k-1}^k,\, x_{N_k}^k$$

just by inserting the point p_j^k between x_{j-1}^k and x_j^k when Γ_j^k is constructed as in (C), for every $j\in[1,N_k]$, and relabeling the points from the resulting sequence suitably. Note that in the cases (A) and (B), the segment L_j^k will coincide with some segment L_h^{k+1} from Γ^{k+1}, while in the case (C) L_j^k is replaced by two new segments L_h^{k+1}, L_{h+1}^{k+1}, satisfying

$$\frac{1}{3}\,\mathcal{H}^1(L_j^k) < \mathcal{H}^1(L_{h'}^{k+1}) < \frac{1}{2^{1/2}}\mathcal{H}^1(L_j^k),$$

both for $h'=h$ and $h'=h+1$. In the cases (A) and (B) we say that L_h^{k+1} is generated by L_j^k and in the case (C), that both L_h^{k+1} and L_{h+1}^{k+1} are generated by L_j^k.

We will call the points $z_A \equiv x_0^k, x_1^k, \dots, x_{N_k-1}^k, x_{N_k}^k \equiv z_B$ vertices of Γ^k.

Next we define the auxiliary map $\Pi_k : \Gamma^k \to \Gamma^{k+1}$ as follows. Given $x \in L_j^k$, we let $\Pi_k(x)$ be the unique point in $\Gamma_j^k \subset \Gamma^{k+1}$ whose orthogonal projection to L_j^k is x. In particular, note that if $\Gamma_j^k = L_j^k$, then $\Pi_k(x) = x$. To simplify notation, we denote $\ell_j^k = \mathcal{H}^1(L_j^k)$. Note that the condition (A) guaranties that

$$\ell_j^k \geq 2^{-(k+2)/2}\, d_0 \qquad \text{for all } k \geq 1,\ 1 \leq j \leq N_k. \tag{10.2}$$

We denote by ρ_j^k the line which contains L_j^k.

Also, we consider the (open) ball

$$B_j^k = B(z_j^k, \ell_j^k)$$

(recall that z_j^k stands for the mid point of L_j^k). Observe that $L_j^k \subset \frac{1}{2}\overline{B_j^k}$. By the argument just below (10.1), it is clear that $\widetilde{E} \cap \frac{1}{10} B_j^k \neq \varnothing$ is ε_0 is small enough. By Lemma 7.4 this guaranties that if $Q \in \mathsf{Good}$ fulfils $2B_Q \cap B_j^k \neq \varnothing$ and $\ell(Q) \approx r(B_j^k)$, then

$$\mu(\tfrac{1}{6} B_j^k) \approx \mu(B_j^k) \approx \mu(2B_j^k) \approx \mu(Q), \tag{10.3}$$

assuming η, δ and ε_0 small enough. That such a cell Q exists follows easily from the construction of Γ^k and (b) in the next lemma.

Lemma 10.1. *The following properties hold for all $L_j^k \subset \Gamma^k$, with $k \geq 1$:*

(a) *If $x \in L_j^k$, then*

$$|\Pi_k(x) - x| \leq c\, \varepsilon_0\, \ell_j^k.$$

(b) *If there exists some Q_{i_0}, $i_0 \in I$, such that* $\operatorname{dist}(Q_{i_0}, L_j^k) \leq 2\,\ell_j^k + 2\,\ell(Q_{i_0})$, *then*

$$\ell_j^k \approx \max\big(\ell(Q_{i_0}),\, 2^{-k/2} d_0\big).$$

(c) *If there exists some point $x \in NW_0$ (i.e. $\widetilde{d}(x) = 0$) such that* $\operatorname{dist}(x, L_j^k) \leq 2\,\ell_j^k$, *then*

$$\ell_j^k \approx 2^{-k/2} d_0.$$

(d) *If L_h^k satisfies* $\operatorname{dist}(L_j^k, L_h^k) \leq 2\,\ell_j^k$, *then $\ell_j^k \approx \ell_h^k$.*

Proof. The statement in (a) is an immediate consequence of (10.1). To prove (b), consider a sequence of segments $[z_A, z_B] = L_{j_1}^1,\ L_{j_2}^2, \dots, L_{j_k}^k = L_j^k$, so that for each m $L_{j_{m+1}}^{m+1}$ is one of the segments that form $\Gamma_{j_m}^m$ (n particular, we may have $L_{j_{m+1}}^{m+1} = L_{j_m}^m$).

Suppose first that in the construction described above, the option (B) holds for some $m = 1, \dots, k$. That is, there exists some cell Q_i, $i \in I$, such that $2B_{Q_i} \cap L_{j_m}^m \neq \varnothing$ and $\mathcal{H}^1(L_{j_m}^m) \leq \ell(Q_i)$. Take the minimal index $m \in [1, k]$ such that this holds. By construction, we have $L_{j_m}^m = L_{j_{m+1}}^{m+1} = \ldots = L_j^k$. So

$$\ell_j^k \leq \ell(Q_i). \tag{10.4}$$

Note now that $L_{j_{m-1}}^{m-1} \neq L_{j_m}^m$ (otherwise this would contradict the definition of m). Suppose that $x_{j_{m-1}}^{m-1}$ is a common endpoint both of $L_{j_m}^m$ and $L_{j_{m-1}}^{m-1}$. Then

$$\operatorname{dist}(x_{j_{m-1}}^{m-1}, 2B_{Q_i}) \leq \ell_{j_m}^m \leq \ell(Q_i),$$

which implies that $x^{m-1}_{j_{m-1}} \in B(z_{Q_i}, 50\ell(Q_i))$. From Lemma 9.1 we infer that $\widetilde{d}(x^{m-1}_{j_{m-1}}) > 0$ and that there exists some $i' \in I$ such that $x^{m-1}_{j_{m-1}} \in Q_{i'}$ with $\ell(Q'_i) \approx \ell(Q_i)$. Since the option (B) in the construction of Γ_{m-1} does not hold for $L^{m-1}_{j_{m-1}}$, we have $\mathcal{H}^1(L^{m-1}_{j_{m-1}}) > \ell(Q_{i'})$. Thus

$$\ell^k_j = \ell^m_{j_m} \approx \ell^{m-1}_{j_{m-1}} > \ell(Q_{i'}) \approx \ell(Q_i).$$

Together with (10.4), this estimate shows that

$$\ell^k_j \approx \ell(Q_i). \tag{10.5}$$

Moreover, the fact that $L^{m-1}_{j_{m-1}} \neq L^m_{j_m}$ also implies that the option (A) does not hold for $m-1$, and thus $\mathcal{H}^1(L^{m-1}_{j_{m-1}}) > 2^{-m/2}d_0$. Hence,

$$\ell^k_j = \ell^m_{j_m} \approx \ell^{m-1}_{j_{m-1}} > 2^{-m/2}d_0 \geq 2^{-k/2}d_0.$$

That is, $\ell^k_j \approx \max(\ell(Q_i),\, 2^{-k/2}d_0)$ if the option (B) of the algorithm holds for some m. Moreover, if Q_{i_0} is as in (b), by (10.4) we get

$$\begin{aligned}\operatorname{dist}(Q_{i_0}, 2B_{Q_i}) &\leq \operatorname{dist}(Q_{i_0}, L^k_j) + \ell^k_j + \operatorname{dist}(L^k_j, 2B_{Q_i})\\ &\leq 2\ell^k_j + 2\,\ell(Q_{i_0}) + \ell^k_j + 0 \leq 3\,\ell(Q_i) + 2\,\ell(Q_{i_0}),\end{aligned}$$

which implies that $B(z_{Q_i}, 50\ell(Q_i)) \cap B(z_{Q_{i_0}}, 50\ell(Q_{i_0})) \neq \varnothing$. So $\ell(Q_i) \approx \ell(Q_{i_0})$ and $\ell^k_j \approx \max(\ell(Q_{i_0}),\, 2^{-k/2}d_0)$, as wished.

If the option (B) does not hold for any $m \in [1, k]$, then we claim that

$$\ell^k_j \approx 2^{-k/2}d_0.$$

This follows easily form the fact that $\ell^1_{j_1} = d_0$, and for any m we have:

- If $\ell^m_{j_m} \leq 2^{-(m+1)/2}d_0$, then $\ell^{m+1}_{j_{m+1}} = \ell^m_{j_m}$.
- If $\ell^m_{j_m} > 2^{-(m+1)/2}d_0$, then $\frac{1}{3}\ell^m_{j_m} < \ell^{m+1}_{j_{m+1}} < \frac{1}{2^{1/2}}\ell^m_{j_m}$.

We leave the details for the reader.

To complete the proof of (b) it remains to check that $\ell(Q_{i_0}) \leq A_1\, 2^{-k/2}d_0$ for some absolute big enough constant A_1. Suppose not and let $Q_{i'}$, $i' \in I$, such that $x^{m-1}_{j_{m-1}} \in Q_{i'}$. Then we have

$$\operatorname{dist}(Q_{i_0}, Q_{i'}) \leq \ell^k_j + \operatorname{dist}(Q_{i_0}, L^k_j) \leq 3\,\ell^k_j + 2\,\ell(Q_{i_0}) \leq c\,2^{-k/2}d_0 + 2\,\ell(Q_{i_0}) \leq \Big(\frac{c}{A_1} + 2\Big)\,\ell(Q_{i_0}).$$

For A_1 big enough this tells us $B(z_{Q_i}, \ell(Q_i)) \cap B(z_{Q_{i'}}, \ell(Q_{i'})) \neq \varnothing$ and thus

$$\ell(Q_{i'}) \approx \ell(Q_i) \geq A_1\, 2^{-k/2}d_0.$$

So $\ell(Q_{i'}) > A_1\, 2^{-k/2}d_0$ for A_1 big enough, which is not possible in this case (as we assumed that the option (B) does not hold for any $m \in [1, k]$).

The statement in (c) can be considered as a particular case of the one in (b). Indeed, when $\widetilde{d}(x) = 0$, one can thing that of the point x as a cell from the family $\{Q_i\}_{i\in I}$ with side length 0. We leave the details for the reader.

Finally we turn our attention to (d). So we consider L^k_j and L^k_h such that $\operatorname{dist}(L^k_j, L^k_h) \leq 2\,\ell^k_j$ and we have to show that $\ell^k_j \approx \ell^k_h$. We intend to apply the statement just proved in (b). If $\ell^k_j \approx 2^{-k/2}d_0$ and $\ell^k_h \approx 2^{-k/2}d_0$ we clearly have $\ell^k_j \approx \ell^k_h$. Suppose now that $\ell^k_j \geq A_2\,\ell^k_h$ for some big constant A_2. By (b) this implies that $\ell_j^k \geq c\, A_2\, 2^{-k/2} d_0$ and so there exists some cell Q_i, $i \in I$ such that $\operatorname{dist}(L_j^k, Q_i) \leq 2\,\ell_j^k + 2\,\ell(Q_i)$ with $\ell(Q_i) \approx \ell_j^k$. Then we have

$$\operatorname{dist}(L_h^k, Q_i) \leq \operatorname{dist}(L_h^k, L_j^k) + \ell_j^k + \operatorname{dist}(L_j^k, Q_i) \leq 2\,\ell_j^k + 2\,\ell(Q_i) + \ell_j^k + 2\,\ell_j^k \leq c\,\ell_j^k \leq \frac{c}{A_2}\,\ell_h^k.$$

Assuming A_2 big enough again, this yields $\operatorname{dist}(L_h^k, Q_i) \leq 2\,\ell_h^k$ and then, by (b),

$$\ell_j^k \lesssim \ell(Q_i) \approx \ell_j^k.$$

So we get $\ell_h^k \approx \ell_j^k$.

If we suppose that $\ell_j^k \geq A_2\,\ell_h^k$, by interchanging the roles of j and h we derive analogously that $\ell_j^k \gtrsim \ell_h^k$, and thus $\ell_j^k \approx \ell_h^k$. □

Remark 10.2. Note that, from the statements (b) and (c) in Lemma 10.1, in particular one deduces that if $\operatorname{dist}(x, L_j^k) \leq 2\,\ell_j^k)$, then then

$$\ell_j^k \approx \max\big(\widetilde{d}(x),\, 2^{-k/2} d_0\big).$$

Lemma 10.3. *For all $k \geq 1$ and $1 \leq j \leq N_k$, $\operatorname{supp}\widetilde{\mu} \cap 2B_j^k$ is contained in the $(c\,\varepsilon_0\,\ell_j^k)$-neighborhood of the line ρ_j^k (recall that $B_j^k = B(z_j^k, \ell_j^k)$). Moreover, if L_h^k satisfies $\operatorname{dist}(L_j^k, L_h^k) \leq 2\,\ell_j^k$, then*

$$\operatorname{dist}_H(\rho_j^k \cap 2B_j^k,\; \rho_h^k \cap 2B_j^k) \lesssim \varepsilon_0\,\ell_j^k.$$

In particular,

$$\measuredangle(\rho_{j-1}^k,\, \rho_j^k) \lesssim \varepsilon_0. \tag{10.6}$$

Proof. For $k \geq 1$ and $1 \leq j \leq N_k$, consider the ball $B_j^k = B(z_j^k, \ell_j^k)$ and the segment L_j^k with endpoints $x_{j-1}^k, x_j^k \in \operatorname{supp}\widetilde{\mu}$. Suppose that $\widetilde{d}(x_{j-1}^k) > 0$. Then there exists some cell Q_i, $i \in I$, such that $x_{j-1}^k \in Q_i$. By (b) in the preceding lemma, $\ell_j^k \gtrsim \ell(Q_i)$. So there exists some cell $P \supset Q_i$ such that $4B_P \supset 2B_j^k$, with $\ell(P) \approx \ell_j^k$. By Lemma 7.8 and (7.20),

$$b\beta_{\infty,\widetilde{\mu}}(P) \leq \varepsilon_0. \tag{10.7}$$

Moreover, since the endpoints of L_j^k are both in $\operatorname{supp}\widetilde{\mu}$ and $\ell_j^k \approx \ell(P)$, it easily follows that

$$\operatorname{dist}_H(\rho_j^k \cap 4B_P, \rho_P \cap 4B_P) \leq c\,\beta_{\infty,\widetilde{\mu}}(P) \leq c\,\varepsilon_0\,\ell(P), \tag{10.8}$$

where ρ_P stands for a best approximating line for $b\beta_{\infty,\widetilde{\mu}}(P)$. From this fact and (10.7) one infers that $\operatorname{supp}\widetilde{\mu} \cap 2B_j^k$ is contained in the $(c'\varepsilon_0\,\ell_j^k)$-neighborhood of ρ_j^k. The proof is analogous if $\widetilde{d}(x_{j-1}^k) = 0$.

The second statement of the lemma follows as above, just taking the cell P big enough so that $2B_j^k \cup 2B_h^k \subset 4B_P$, still with $\ell(P) \approx \ell_j^k$. We leave the details for the reader. □

Next we intend to show that each curve Γ_k is a AD-regular, with a constant uniform on k. The lemma below is the first step.

Lemma 10.4. *For fixed $k \geq 1$ and $j \in [1, N_k]$, the only segment of the family $\{L_h^k\}_{1 \leq h \leq N_k}$ that intersects the open ball $\frac{1}{2}B_j^k = B(z_j^k, \frac{1}{2}\ell_j^k)$ is L_j^k. In other words,*

$$\tfrac{1}{2}\,B_j^k \cap \Gamma_j^k = \tfrac{1}{2}\,B_j^k \cap L_j^k.$$

Proof. Suppose that the statement above does not hold and let us argue by contradiction. Consider the least integer $k \geq 1$ such that there exists $h, j \in [1, N_k]$, with $h \neq j$, such that

$$L_h^k \cap \tfrac{1}{2}B_j^k \neq \varnothing. \tag{10.9}$$

By construction, we must have $k \geq 2$. By the preceding lemma, for each $m \in [1, N_{k-1} - 1]$, the angle between the lines ρ_m^{k-1} and ρ_{m+1}^{k-1} is bounded by $c\,\varepsilon_0$. This implies that either $\measuredangle(x_{m-1}^{k-1}, x_m^{k-1}, x_{m+1}^{k-1})$ is very close to to 0 or very close to π. Since L_m^{k-1} does not intersect $\frac{1}{2}B_{m+1}^{k-1}$, this angle must be very close to π. That is,

$$|\measuredangle(x_{m-1}^{k-1}, x_m^{k-1}, x_{m+1}^{k-1}) - \pi| \lesssim \varepsilon_0. \tag{10.10}$$

Because of the way Γ^k is generated from Γ^{k-1}, we infer that the angles $\measuredangle(x_{m-1}^k, x_m^k, x_{m+1}^k)$ are also very close to π for all $m \in [1, N_k - 1]$. As a consequence, if L_h^k and B_j^k satisfy (10.9), then $|h - j| \geq N(\varepsilon_0)$, where $N(\varepsilon_0)$ is some big integer depending only on ε_0 which tends to ∞ as $\varepsilon_0 \to 0$.

Consider the segments $L_{h'}^{k-1}$ and $L_{j'}^{k-1}$ which generate L_h^k and L_j^k respectively. Notice that

$$|h' - j'| \geq \frac{|h - j|}{2} - 1 \geq \frac{1}{4}\,N(\varepsilon_0),$$

for $N(\varepsilon_0)$ big enough (i.e. ε_0 small enough). Take $y \in L_h^k \cap B_j^k$ and $y' \in L_{h'}^{k-1}$ with $\Pi_{k-1}(y') = y$, so that

$$|y - y'| \lesssim \varepsilon_0\,\ell_{h'}^{k-1} \approx \varepsilon_0\,\ell_h^k \approx \varepsilon_0\,\ell_j^k,$$

by Lemma 10.1 (a), (c). By Lemma 10.3, we deduce that

$$\mathrm{dist}_H(\rho_h^k \cap B_j^k, \rho_j^k \cap B_j^k) \lesssim \varepsilon_0\,\ell_j^k,$$

and so $\mathrm{dist}(y, L_j^k \cap \frac{1}{2}B_j^k) \lesssim \varepsilon_0\,\ell_j^k$. Thus, there exists some $x \in L_j^k \cap \frac{1}{2}B_j^k$ such that $|x - y| \lesssim \varepsilon_0\,\ell_j^k$. We take now $x' \in L_{j'}^{k-1}$ such that $\Pi_{k-1}(x') = x$, which, in particular, implies that

$$|x - x'| \lesssim \varepsilon_0\,\ell_j^k.$$

Then we have

$$|x' - y'| \leq |x' - x| + |x - y| + |y - y'| \leq c\,\varepsilon_0\,\ell_j^k \leq \frac{1}{10}\,\ell_{j'}^{k-1},$$

assuming ε_0 small enough. Therefore,

$$L_{h'}^{k-1} \cap \tfrac{3}{4}\,B_{j'}^{k-1} \neq \varnothing.$$

From (10.10) and the fact that the lines $\rho_{h'}^{k-1}$ and $\rho_{h'}^{k-1}$ are very close we infer that that exists some $h'' \in [1, N_{k-1}]$, with $|h'' - h'| \leq c_{10}$ (where c_{10} is some absolute constant), such that

$$L_{h''}^{k-1} \cap \tfrac{1}{2}B_{j'}^{k-1} \neq \varnothing. \tag{10.11}$$

The fact that $|h'' - h'| \leq c_{10}$ and $|h' - j'| \geq N(\varepsilon_0)$ ensures that $h'' \neq j'$. This contradicts the minimality of k and proves the lemma. $\square$

Lemma 10.5. *For all $k \geq 1$ and $1 \leq j \leq N_k - 1$,*

$$|\measuredangle(x_{j-1}^{k-1}, x_j^{k-1}, x_{j+1}^{k-1}) - \pi| \lesssim \varepsilon_0. \tag{10.12}$$

Proof. This has been shown in (10.10). □

Lemma 10.6. *For every fixed $k \geq 1$, the balls $\frac{1}{6} B_j^k$, $1 \leq j \leq N_k$, are pairwise disjoint.*

Proof. Suppose not. Let $1 \leq j, h \leq N_k$ be such that $\frac{1}{6} B_j^k \cap \frac{1}{6} B_h^k \neq \varnothing$, with $h \neq j$, and $\ell_j^k \geq \ell_h^k$, say. Then $\frac{1}{6} B_h^k \subset \frac{1}{2} B_j^k$ and thus L_h^k is intersects B_j^k, which contradicts Lemma 10.4. □

Lemma 10.7. *For all $k \geq 1$, we have*

$$\operatorname{supp} \widetilde{\mu} \cap B(x_0, \tfrac{1}{4} K r_0) \subset \bigcup_{j=1}^{N_k} B_j^k.$$

Proof. We will argue by induction on k. This clearly holds for $k = 1$, taking into account Lemma 7.8. Suppose now this holds for k and let us see how this follows for $k+1$. Consider the ball B_j^k, for some $k \geq 1$ and $1 \leq j \leq N_k$. Take a segment L_h^{k+1} generated by L_j^k. By construction, we have $\ell_h^{k+1} \approx \ell_j^k$, and by Lemma 10.3,

$$\operatorname{dist}_H(\rho_h^{k+1} \cap B_j^k, \rho_j^k \cap B_j^k) \leq c\, \varepsilon_0\, \ell_j^k. \tag{10.13}$$

Consider now the maximal $m, n \geq 0$ such that all the balls

$$B_{h-m}^{k+1},\ B_{h-m+1}^{k+1}, \ldots, B_h^{k+1}, \ldots, B_{h+n-1}^{k+1},\ B_{h+n}^{k+1} \tag{10.14}$$

intersect B_j^k. By (c) from Lemma 10.1 it follows easily that $\ell_p^{k+1} \approx \ell_j^k$ for $h - m \leq p \leq h+n$ and moreover m and n are uniformly bounded. Further (10.13) also holds replacing ρ_h^{k+1} by ρ_p^{k+1} and by Lemma 10.5,

$$|\measuredangle(x_{p-1}^{k+1}, x_p^{k+1}, x_{p+1}^{k+1}) - \pi| \lesssim \varepsilon_0,$$

for all p. By elementary geometry, the segments L_{h-m}^{k+1}, $L_{h-m+1}^{k+1}, \ldots, L_h^{k+1}, \ldots, L_{h+n-1}^{k+1}$, L_{h+n}^{k+1} form a polygonal line γ such that

$$\operatorname{dist}_H(\gamma \cap B_j^k, \rho_j^k \cap B_j^k) \lesssim \varepsilon_0\, \ell_j^k.$$

Moreover, one can also verify that, for ε_0 small enough, the intersection of the $(c' \varepsilon_0\, \ell_j^k)$-neighborhood of ρ_j^k with B_j^k is contained in the union of the balls (10.14), and so

$$\operatorname{supp} \widetilde{\mu} \cap B_j^k \subset \bigcup_{p=h-m}^{h+n} B_p^{k+1},$$

which yields

$$\operatorname{supp} \widetilde{\mu} \cap \bigcup_{j=1}^{N_k} B_j^k \subset \operatorname{supp} \widetilde{\mu} \cap \bigcup_{h=1}^{N_{k+1}} B_h^{k+1}.$$

□

Lemma 10.8. *The curves Γ^k are AD-regular uniformly on k, with the AD-regularity constant bounded by $c\,A\,\tau^{-1}$.*

Proof. Since Γ^k is a curve, we only have to check the upper AD-regularity. Let $B(x,r)$ be a ball centered at some point $x \in L_j^k$. Suppose first that $r \leq 2\,\ell_j^k$. If $B(x,r)$ intersects another segment L_h^k, then $\ell_h^k \approx \ell_j^k$ because $\mathrm{dist}(L_j^k, L_h^k) \leq r \leq 2\,\ell_j^k$. Therefore, there exists some absolute constant $c \geq 1$ such that $B_h^k \subset c\,B_j^k$. Since the balls $\frac{1}{6}B_h^k$, $1 \leq h \leq N_k$, are pairwise disjoint, it follows that the number of balls B_h^k contained in $c\,B_j^k$ which satisfy $r(B_h^k) \approx r(B_j^k)$ is uniformly bounded above. Then we infer that

$$\mathcal{H}^1(\Gamma^k \cap B(x,r)) \leq \sum_{h: B_h^k \subset c\,B_j^k} \mathcal{H}^1(L_h^k \cap B(x,r)) \leq c\,r.$$

Suppose now that $r > 2\,\ell_j^k$. First we claim that if $B(x,r)$ intersects another segment L_h^k, then $\ell_h^k \leq M_0\,r$, for some absolute constant M_0. Indeed, if $r \leq \ell_h^k$, then we obtain

$$\mathrm{dist}(L_h^k, L_j^k) \leq r \leq \ell_h^k,$$

which implies that $\ell_h^k \approx \ell_j^k \leq \frac{1}{2}\,r$, and proves the claim. So we deduce that the ball B_h^k is contained in $B(x, C\,r)$, for some $C \geq 1$.

Now we write

$$\mathcal{H}^1(\Gamma^k \cap B(x,r)) \leq \sum_{h: B_h^k \subset B(x, C\,r)} \mathcal{H}^1(L_h^k).$$

Observe now that $\mu(\frac{1}{6}\,B_h^k) \gtrsim \tau\,\Theta_\mu(B_R)\,\ell_h^k$ by (10.3), and thus

$$\mathcal{H}^1(\Gamma^k \cap B(x,r)) \lesssim \frac{1}{\tau\,\Theta_\mu(B_R)} \sum_{h: B_h^k \subset B(x, C\,r)} \mu(\tfrac{1}{6}\,B_h^k). \tag{10.15}$$

Since, for a fixed k, the balls $\frac{1}{6}\,B_h^k$ are disjoint, we have

$$\sum_{h: B_h^k \subset B(x, C\,r)} \mu(\tfrac{1}{6}\,B_h^k) \leq \mu(B(x, C\,r)) \leq c\ A\,\Theta_\mu(B_R)\,r.$$

Plugging this estimate into (10.15) we obtain

$$\mathcal{H}^1(\Gamma^k \cap B(x,r)) \lesssim A\,\tau^{-1}\,r.$$

□

Remark 10.9. It is easy to check that the limit in the Hausdorff metric of the sequence of curves $\{\Gamma^k\}_k$ exists. By the preceding lemma, it is an AD-regular curve Γ with the AD-regularity constant bounded by $c\,A\,\tau^{-1}$.

The next lemma asserts that, in a sense, $\operatorname{supp}\widetilde{\mu}$ is very close to Γ^k.

Lemma 10.10. *If $x \in \operatorname{supp}\widetilde{\mu} \cap B(x_0, \frac{1}{4}Kr_0)$, then*

$$\mathrm{dist}(x, \Gamma^k) \lesssim \varepsilon_0\ \max\big(\widetilde{d}(x),\, 2^{-k/2}d_0\big),$$

for all $k \geq 1$.

Proof. By Lemma 10.7, there exists some ball B_j^k which contains x. Therefore,

$$\mathrm{dist}(x, \rho_j^k) \lesssim \varepsilon_0\, \ell_j^k.$$

By Lemmas 10.1 (d) and 10.5, we deduce that

$$\mathrm{dist}(x, \Gamma^k) \lesssim \varepsilon_0\, \ell_j^k.$$

On the other hand, by Lemma 10.1 (b), (c), since $\mathrm{dist}(x, L_j^k) \leq 2\,\ell_j^k$, we have

$$\ell_j^k \approx \max\big(\widetilde{d}(x),\, 2^{-k/2} d_0\big),$$

and thus we are done. □

Note that, in particular, from the preceding lemma one deduces that NW_0 is supported in the limiting curve Γ. So we have:

Lemma 10.11. *The set NW_0 is rectifiable.*

The next result can be understood as a kind of converse of Lemma 10.10. Roughly speaking, it asserts that for each $x \in \Gamma^k$ there exists some point from $\mathrm{supp}\,\widetilde{\mu}$ which is very close.

Lemma 10.12. *Let $k \geq 1$ and $1 \leq j \leq N_k$. For every $x \in \Gamma^k \cap B_j^k$ there exists some $x' \in \mathrm{supp}\,\widetilde{\mu}$ such that*

$$|x - x'| \lesssim \varepsilon_0\, \ell_j^k.$$

Proof. This follows from the fact that $b\beta_{\infty,\widetilde{\mu}}(2B_j^k) \lesssim \varepsilon_0$ and since $x_{j-1}^k, x_j^k \in \mathrm{supp}\,\widetilde{\mu} \cap 2B_j^k$ and $|x_{j-1}^k - x_j^k| \approx \mathrm{diam}(B_j^k)$ we infer that $\mathrm{dist}_H(\rho_j^k \cap 2B_j^k, L_{2B_j^k} \cap 2B_j^k) \lesssim \varepsilon_0$, where $L_{2B_j^k}$ is the best approximating line for $b\beta(2B_j^k)$. □

Finally we have:

Lemma 10.13. *Let $L_{j_1}^1, L_{j_2}^2, \dots, L_{j_k}^k$ be a sequence of segments such that $L_{j_{m+1}}^{m+1}$ is generated by $L_{j_m}^m$ for $m = 1, \dots, k-1$. Then*

$$\sum_{m=1}^{k-1} \measuredangle(\rho_{j_m}^m, \rho_{j_{m+1}}^{m+1})^2 \leq c\, M.$$

Proof. Is is easy to check that

$$\measuredangle(\rho_{j_m}^m, \rho_{j_{m+1}}^{m+1}) \lesssim \beta_{\infty,\widetilde{\mu}}(B_{j_m}^m).$$

Let $Q \in \mathsf{Qgood}$ be a cell such that $x_{j_k}^k \in Q$ with $\ell(Q) \approx \ell_{j_k}^k$. By the construction of the cells from $\mathsf{BS}\beta$, we have

$$\sum_{P \in \mathcal{D}: Q \subset P \subset B(x_0, K r_0)} \beta_{\infty,\widetilde{\mu}}(2B_P)^2 \lesssim M.$$

Then we deduce that

$$\sum_{m=1}^{k-1} \beta_{\infty,\widetilde{\mu}}(B_{j_m}^m)^2 \lesssim \sum_{P \in \mathcal{D}: Q \subset P \subset B(x_0, K r_0)} \beta_{\infty,\widetilde{\mu}}(2B_P)^2 \lesssim M,$$

and we are done. □

Remark 10.14. If in the construction of the curves Γ^k above we replace the function $\widetilde{d}(\cdot)$ by $d(\cdot)$ and the cells $\{Q_i\}_{i\in I}$ by the cells from the family Reg which have positive $\widetilde{\mu}$ measure, we will get curves Γ_R^k which will satisfy properties analogous to the ones of Γ^k, with the exception of the one stated in Lemma 10.13. So very similar versions of Lemmas 10.1-10.12 will hold for Γ_R^k, $k\geq 1$. Moreover, letting Γ_R be the limit in the Hausdorff metric of the curves Γ_R^k, one obtains $W_0\subset\Gamma_R$ and so W_0 is rectifiable. Using the fact that $\Theta_\mu(1.1B_Q)\lesssim A\,\Theta_\mu(B_R)$ for any $Q\in\mathcal{D}$ with $\ell(Q)\leq\ell(R)$ such that $\mu(Q\cap W_0)>0$, it follows easily that $\mu|_{W_0}$ is absolute continuous with respect to $\mathcal{H}^1|_{\Gamma_R}$.

11. The small measure $\widetilde{\mu}$ of the cells from $\mathsf{BS}\beta$

Recall that $Q\in\mathsf{BS}\beta_0$ if $Q\notin BCF_0\cup LD_0\cup HD_0\cup BCG_0\cup BC\Delta_0$, $\ell(Q)\leq\ell(R)$, and

$$\sum_{P\in\mathcal{D}:Q\subset P\subset R}\beta_{\widetilde{\mu},\infty}(2B_P)^2\geq M. \tag{11.1}$$

The cells from $\mathsf{BS}\beta$ are the ones from NTerm which belong to $\mathsf{BS}\beta_0$. We denote by $\mathsf{BS}\beta_1$ the cells from $\mathsf{BS}\beta$ which are contained in $B(x_0,\frac{1}{10}Kr_0)$.

In this section we will prove the following:

Lemma 11.1. *Assume that M is big enough (depending only A and τ). Then*

$$\widetilde{\mu}\Bigg(\bigcup_{Q\in\mathsf{BS}\beta_1}Q\Bigg)\leq\frac{c(A,\tau,K)}{M}\,\mu(R).$$

To prove the preceding result we will use the usual lattice $\mathcal{D}(\mathbb{R}^d)$ of dyadic cubes of $\mathbb{R}^d$. Given a cube $Q\in\mathcal{D}(\mathbb{R}^d)$, we denote by $\ell(Q)$ its side length and by z_Q its center. We define

$$\beta_{\Gamma^k,\infty}(Q)=\inf_L\sup_{y\in 3Q\cap\Gamma^k}\frac{\operatorname{dist}(y,L)}{\ell(Q)},$$

where $3Q$ stands for the cube concentric with Q with side length $3\ell(Q)$.

Proof of Lemma 11.1. Consider the following auxiliary curve:

$$\widetilde{\Gamma}^k=\Gamma^k\cup\bigcup_{j=1}^{N_k}\partial B_j^k.$$

Since $\mathcal{H}^1(\widetilde{\Gamma}^k)\lesssim\mathcal{H}^1(\Gamma^k)\lesssim_{A,\tau}K\,\ell(R)$, by Jones' traveling salesman theorem [Jo], [Ok], it follows that

$$\sum_{Q\in\mathcal{D}(\mathbb{R}^d)}\beta_{\infty,\widetilde{\Gamma}^k}(Q)^2\,\ell(Q)\lesssim K\,\ell(R). \tag{11.2}$$

Now we claim that

$$\sum_{\substack{P\in\mathsf{Qgood}:\\ P\subset B(x_0,\frac14 Kr_0)\\ \ell(P)\geq 2^{-k/2}d_0}}\beta_{\widetilde{\mu},\infty}(2B_P)^2\,\ell(P)\lesssim_{A,\tau}\sum_{\substack{Q\in\mathcal{D}(\Gamma^k_{ex}):\\ Q\subset B(x_0,Kr_0)}}\beta_{\infty,\widetilde{\Gamma}^k}(Q)^2\,\ell(Q).$$

To this end, recall that

$$\operatorname{supp}\widetilde{\mu}\cap B(x_0,\tfrac14 Kr_0)\subset\bigcup_{j=1}^{N_k}B_j^k.$$

Take a cell $P\in\mathsf{Qgood}$ such that $\ell(P)\geq 2^{-k/2}d_0$. To such a cell we can associate a cube $Q(P)\in\mathcal{D}(\mathbb{R}^d)$ such that $2B_P\subset 3Q(P)$ and $\ell(Q(P))\approx\ell(P)$. Then it follows that

$$\beta_{\infty,\widetilde{\mu}}(2B_P)\lesssim\beta_{\infty,\widetilde{\Gamma}^k}(Q(P)),$$

and since for a given $Q\in\mathcal{D}(\mathbb{R}^d)$, the number of cells $P\in\mathsf{Qgood}$ such that $Q=Q(P)$ does not exceed some absolute constant, the claim follows. Together with (11.2), this gives

$$\sum_{\substack{P\in\mathsf{Qgood}:\\ P\subset B(x_0,\frac14 Kr_0)\\ \ell(P)\geq 2^{-k/2}d_0}}\beta_{\widetilde{\mu},\infty}(2B_P)^2\,\ell(P)\lesssim_{A,\tau,K}\ell(R).$$

From the last estimate, taking into account that $\mathsf{BS}\beta_1\subset\mathsf{Qgood}$, by (11.1) and Chebyshev, we derive

$$\begin{aligned}\sum_{\substack{Q\in\mathsf{BS}\beta_1:\\ \ell(Q)\geq 2^{-k/2}d_0}}\widetilde{\mu}(Q)&\leq\frac{1}{M}\sum_{\substack{Q\in\mathsf{BS}\beta_1:\\ \ell(Q)\geq 2^{-k/2}d_0}}\;\sum_{P\in\mathcal{D}:Q\subset P\subset B(x_0,Kr_0)}\beta_{\infty,\widetilde{\mu}}(2B_P)^2\widetilde{\mu}(Q)\\
&\lesssim\frac{1}{M}\left(\mu(B(x_0,Kr_0))+\sum_{\substack{Q\in\mathsf{BS}\beta_1\\ \ell(Q)\geq 2^{-k/2}d_0}}\;\sum_{P\in\mathcal{D}:Q\subset P\subset B(x_0,\frac14 Kr_0)}\beta_{\infty,\widetilde{\mu}}(2B_P)^2\widetilde{\mu}(Q)\right)\\
&\lesssim_{A,\tau,K}\frac{\Theta_\mu(B_R)}{M}\left(\ell(R)+\sum_{\substack{P\in\mathsf{Qgood}:\\ P\subset B(x_0,\frac14 Kr_0)\\ \ell(P)\geq 2^{-k/2}d_0}}\beta_{\infty,\widetilde{\mu}}(2B_P)^2\,\ell(P)\right)\\
&\lesssim_{A,\tau,K}\frac{\Theta_\mu(B_R)\,\ell(R)}{M}\approx_{A,\tau,K}\frac{\mu(R)}{M}.\end{aligned}$$

Letting $k\to\infty$, the lemma follows. □

12. The approximating measure ν^k on Γ_{ex}^k

For technical reasons, it is convenient to define an **extended curve** Γ_{ex}^k. Recall that the endpoints of Γ^k coincide with the endpoints $z_A\ z_B$ of the segment L_1^1, which is contained in the line ρ_1^1. We set

$$\Gamma_{ex}^k=\Gamma^k\cup(\rho_1^1\setminus L_1^1).$$

We define analogously $\Gamma_{ex}=\Gamma\cup(\rho_1^1\setminus L_1^1)$. Notice that $\Gamma_{ex}^1=\rho_1^1$.

In this section, we will construct a measure ν^k supported on Γ_{ex}^k which will approximate $\widetilde{\mu}$ at the level of the balls B_j^k, $1\leq j\leq N_k$. Taking a weak * limit of the measures ν^k we will get a measure ν supported on Γ_{ex} which approximates $\widetilde{\mu}$ on $B(x_0,\frac14 Kr_0)$.

Consider a radial $\mathcal{C}^\infty$ function $\widetilde{\theta}$ which is supported on the ball $B(0,\frac{3}{2})$ and equals 1 on $\bar{B}(0,1)$. For $k\geq 1$ and $1\leq j\leq N_k$, we set

$$\widetilde{\theta}_j^k=\widetilde{\theta}\biggl(\frac{x-z_j^k}{\ell_j^k}\biggr).$$

Recall that $B_j^k=B(z_j^k,\ell_j^k)$ and thus $\widetilde{\theta}_j^k$ equals 1 on B_j^k and is supported on $\frac{3}{2}B_j^k$. Notice that

$$\sum_{j=1}^{N_k}\widetilde{\theta}_j^k\approx 1\qquad\text{on}\quad \textstyle\bigcup_{j=1}^{N_k}B_j^k.$$

Next we modify the functions $\widetilde{\theta}_j^k$ in order to get functions θ_j^k satisfying $\sum_{j=1}^{N_k}\theta_j^k=1$ on $\bigcup_{j=1}^{N_k}B_j^k$. For a fixed k, we define θ_j^k inductively on j as follows. First we set $\theta_1^k=\widetilde{\theta}_1^k$. Then we write

$$\theta_2^k=(1-\theta_1^k)\widetilde{\theta}_2^k.$$

In general, if $\theta_1^k,\ldots,\theta_j^k$ have already been defined, we set

$$\theta_{j+1}^k=\left(1-\sum_{h=1}^{j}\theta_j^k\right)\,\widetilde{\theta}_{j+1}^k.$$

Also, we define

$$\theta_0^k=1-\sum_{j=1}^{N_k}\theta_j^k.$$

Lemma 12.1. *For each $k\geq 1$, the functions θ_j^k, $1\leq j\leq N_k$ satisfy the following properties:*

(a) *θ_j^k is a non-negative and it is supported on $\frac{3}{2}\bar{B}_j^k$, and for all $n\geq 0$,*

$$\|\nabla^n\theta_j^k\|_\infty\leq c(n)\,\frac{1}{(\ell_j^k)^n}.$$

(b) *For all $x\in\mathbb{R}^d$,*

$$\sum_{1\leq j\leq N_k}\theta_j^k(x)\leq 1.$$

(c) *For all $x\in\bigcup_{1\leq j\leq N_k}B_j^k$,*

$$\sum_{1\leq j\leq N_k}\theta_j^k(x)=1.$$

We leave the easy proof for the reader.

Remark 12.2. Concerning the function θ_0^k, let us remark that

$$|\nabla^n\theta_j^k(x)|\leq c(n)\,\frac{1}{r_0^n}\qquad\text{for all } x\in\Gamma^k. \tag{12.1}$$

This is due to the fact that

$$\Gamma^k\cap\operatorname{supp}(\nabla\theta_0^k)\subset B(z_1^k,C\,\ell_1^k)\cup B(z_{N_k}^k,C\,\ell_{N_k}^k),$$

for some absolute constant C.

On the other hand, one should expect (12.1) to hold for all $x \in \mathbb{R}^d$. In this case, one can only ensure that

$$|\nabla^n \theta_j^k(x)| \leq c(n)\, \frac{1}{\min(\ell_j^k)^n}.$$

We are ready now to define the measures ν^k. For $1 \leq j \leq N_k$, we set

$$\nu_j^k = c_j^k\, \theta_j^k\, \mathcal{H}^1|_{\Gamma_{ex}^k}, \quad \text{with} \quad c_j^k = \frac{\int \theta_j^k\, d\mu}{\int \theta_j^k\, d\mathcal{H}^1|_{\Gamma_{ex}^k}}. \tag{12.2}$$

Also, we set $c_0^k = c_1^k$ and $\nu_0^k = c_0^k\, \theta_0^k\, \mathcal{H}^1|_{\Gamma_{ex}^k}$. Then we write

$$\nu^k = \sum_{j=0}^{N_k} \nu_j^k.$$

Lemma 12.3. *The measure ν^k is AD-regular. Indeed, there exists some constant $c = c(A,\tau)$ such that*

$$c^{-1}\Theta_\mu(B_R)\, r \leq \nu^k(B(x,r)) \leq c\,\Theta_\mu(B_R)\, r \qquad \textit{for all } x \in \Gamma^k.$$

Proof. This follows easily from the fact that

$$c_j^k \approx_{A,\tau} \Theta_\mu(B_R).$$

□

13. Square function estimates for ν^k

Let $\varphi : [0,\infty) \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function supported in $[0,2]$ which is constant in $[0,1/2]$. We denote $\psi_r(z) = \varphi_r(z) - \varphi_{2r}(z)$, with

$$\varphi_r(x) = \frac{1}{r}\varphi\left(\frac{|x|}{r}\right),\ r > 0,$$

so that we have $\Delta_{\mu,\varphi}(x,r) = \psi_r * \mu(x)$. In Lemma 3.2 we showed that, for $0 \leq r_1 < r_2$, we have

$$\int_{r_1}^{r_2} |\Delta_{\mu,\varphi}(x,r)|^2\, \frac{dr}{r} \leq c \int_{r_1/2}^{2r_2} |\Delta_\mu(x,r)|^2\, \frac{dr}{r}. \tag{13.1}$$

Recall that $\widetilde{\mu} = \mu|_{\widetilde{E}}$, with

$$\widetilde{E} = B(x_0, 2Kr_0) \cap \left(W_0 \cup \bigcup_{Q \in \mathsf{Reg}} \left[4B_Q \cap F \cap \mathsf{G}(Q,R,\delta^{1/4},\eta)\right]\right).$$

If $x \in \widetilde{E}$, then either $x \in W_0$ or there exists some some $Q' \in \mathsf{Reg}$ such that $x \in 4B_{Q'} \cap F \cap \mathsf{G}(Q',R,\delta^{1/4},\eta)$. If Q is the cell from Reg which contains x, then $\ell(Q) \approx \ell(Q')$, and by (13.1) and the definition of $\mathsf{G}(Q',R,\delta^{1/4},\eta)$ it follows that

$$\int_{c\delta^{1/4}\ell(Q)}^{c^{-1}\delta^{-1/4}\ell(R)} \Delta_{\mu,\varphi}(x,r)^2\, \frac{dr}{r} \lesssim \int_{\delta^{1/4}\ell(Q')}^{\delta^{-1/4}\ell(R)} \Delta_\mu(x,r)^2\, \frac{dr}{r} \lesssim \eta\, \Theta_\mu(B_R)^2. \tag{13.2}$$

The next objective consists in proving the following.

Lemma 13.1. *Let $k \geq 1$ and $1 \leq j \leq N_k$. For every $x \in \Gamma^k \cap B_j^k$,*

$$\int_{\ell_j^k}^{Kr_0/4} |\Delta_{\mu,\varphi}(x,r)|^2 \, \frac{dr}{r} \lesssim (\eta + \varepsilon_0^2)\, A^2 \, \Theta_\mu(B_R)^2.$$

Proof. By Lemma 10.12, there exists some $x' \in \widetilde{E}$ such that $|x - x'| \lesssim \varepsilon_0 \, \ell_j^k$. From Remark 10.2 it follows easily that there exists some cell $Q \in \mathsf{Qgood}$ with $\ell(Q) \approx \ell_j^k$ which contains x'. Together with (13.2) this gives

$$\int_{\ell_j^k}^{Kr_0} \left|\Delta_{\mu,\varphi}(x',r)\right|^2 \frac{dr}{r} \lesssim \eta \, \Theta_\mu(B_R)^2.$$

By Lemma 6.8, we know that $\mu(B(x',r)) \lesssim A\,\Theta_\mu(B_R)\, r$ for $\ell_j^k \leq r \leq Kr_0$. Next note that, for r such that $|x - x'| \leq r \leq \delta^{-1} r/4$,

$$|\psi_r * \mu(x) - \psi_r * \mu(x')| \leq |x - x'| \sup_{z \in [x,x']} |\nabla\,(\psi_r * \mu)(z)| \lesssim \frac{|x-x'|}{r^2}\, \mu(B(x',4r)) \lesssim \frac{\varepsilon_0 \, \ell_j^k}{r} A\,\Theta_\mu(B_R).$$

Therefore,

$$\left|\Delta_{\mu,\varphi}(x,r) - \Delta_{\mu,\varphi}(x',r)\right| \lesssim \frac{|x-x'|}{r} A\,\Theta_\mu(B_R)$$

for r such that $|x - x'| \leq r \leq \delta^{-1} r/4$. Thus,

$$\begin{aligned}\int_{\ell_j^k}^{Kr_0/4} |\Delta_{\mu,\varphi}(x,r)|^2 \, \frac{dr}{r} &\leq 2 \int_{\ell_j^k}^{Kr_0/4} \left|\Delta_{\mu,\varphi}(x',r)\right|^2 \frac{dr}{r} + c\left(A\,\varepsilon_0 \Theta_\mu(B_R)\right)^2 \int_{\ell_j^k}^{\infty} \frac{(\ell_j^k)^2}{r^2}\, \frac{dr}{r} \\ &\leq 2\,\eta\, A^2\, \Theta_\mu(B_R)^2 + c\,\varepsilon_0^2\, A^2\, \Theta_\mu(B_R)^2.\end{aligned}$$

□

Recall the definition of ν_j^k and c_j^k in (12.2).

Lemma 13.2. *Suppose that $2B_j^k \subset B(x_0, \frac{1}{6}Kr_0)$ for some $k \geq 1$ and $1 \leq j \leq N_k$. For all $x \in \mathbb{R}^d$ and all $r \geq c^{-1}\ell_j^k$,*

$$\left|\int \psi_r(x-y)\, d\nu_j^k(y) - \int \psi_r(x-y)\, \theta_j^k(y)\, d\widetilde{\mu}(y)\right| \lesssim \varepsilon_0 \, \frac{\widetilde{\mu}(B_j^k)\, \ell_j^k}{r^2}. \tag{13.3}$$

Let us remark that the condition $2B_j^k \subset B(x_0, \frac{1}{6}Kr_0)$ guaranties that $2B_j^k$ is far from the endpoints of Γ^k.

Proof. Taking into account that $\nu_j^k = c_j^k\, \theta_j^k \mathcal{H}^1|_{\Gamma^k}$ and that $\int d\nu_j^k = \int \theta_j^k \, d\widetilde{\mu}$, we have

$$\begin{aligned}&\int \psi_r(x-y)\, d\nu_j^k(y) - \int \psi_r(x-y)\, \theta_j^k(y)\, d\widetilde{\mu}(y) \\ &\qquad = \int \left(\psi_r(x-y) - \psi_r(x - z_j^k)\right)\, d(\nu_j^k - \theta_j^k \widetilde{\mu})(y) \\ &\qquad = \int \left(\psi_r(x-y) - \psi_r(x - z_j^k)\right) \theta_j^k(y)\, d(c_j^k \mathcal{H}^1|_{\Gamma^k} - \widetilde{\mu})(y).\end{aligned} \tag{13.4}$$

To estimate the last integral we wish to apply that $\alpha_{\widetilde{\mu}}(2B_j^k) \lesssim \varepsilon_0$. Denote by $c_{2B_j^k}$ and $L_{2B_j^k}$ the constant and the line minimizing $\alpha_{\widetilde{\mu}}(2B_j^k)$. For fixed x, denote $f(y) = \psi_r(x-y) - \psi_r(x-z_j^k)$. Then the left side of (13.4) can be written as follows

$$\begin{aligned}\int f(y)\,\theta_j^k(y)\,d(c_j^k\mathcal{H}^1|_{\Gamma^k} - \widetilde{\mu})(y) &= (c_j^k - c_{2B_j^k})\int f(y)\,\theta_j^k(y)\,d\mathcal{H}^1|_{\Gamma^k}\\ &\quad + c_{2B_j^k}\int f(y)\,\theta_j^k(y)\,d(\mathcal{H}^1|_{\Gamma^k} - \mathcal{H}^1|_{L_{2B_j^k}})(y)\\ &\quad + \int f(y)\,\theta_j^k(y)\,d(c_{2B_j^k}\mathcal{H}^1|_{L_{2B_j^k}} - \widetilde{\mu})(y)\\ &= T_1 + T_2 + T_3.\end{aligned}$$

To estimate T_2 we use (13.7) and the fact that $|c_{2B_j^k}| \lesssim \Theta(2B_j^k)$, by Lemma 2.2 (c). Then we have

$$|T_2| \lesssim \Theta_{\widetilde{\mu}}(2B_j^k)\,\mathrm{Lip}\big(f\,\theta_j^k\big)\,\mathrm{dist}_{2B_j^k}(\mathcal{H}^1|_{\Gamma^k}, \mathcal{H}^1|_{L_{2B_j^k}}). \tag{13.5}$$

Observe that

$$\|f\|_{\infty,2B_j^k} = \|\psi_r(x-\cdot) - \psi_r(x-z_j^k)\|_{\infty,2B_j^k} \lesssim \frac{\ell_j^k}{r^2}$$

and

$$\mathrm{Lip}\big(f\,\theta_j^k\big) \le \|f\|_\infty\|\theta_j^k\|_\infty + \|f\|_{\infty,2B_j^k}\,\|\nabla\theta_j^k\|_\infty \lesssim \frac{1}{r^2} + \frac{\ell_j^k}{r^2}\,\frac{1}{\ell_j^k} \approx \frac{1}{r^2}. \tag{13.6}$$

From Lemmas 7.8 and 10.5 and the construction of Γ^k, one can easily check that

$$\mathrm{dist}_H(2B_j^k \cap \Gamma^k, 2B_j^k \cap L_{2B_j^k}) \lesssim \varepsilon_0\,\ell_j^k$$

and also that

$$\mathrm{dist}_{2B_j^k}(\mathcal{H}^1|_{\Gamma^k}, \mathcal{H}^1|_{L_{2B_j^k}}) \lesssim \varepsilon_0\,(\ell_j^k)^2. \tag{13.7}$$

Therefore, by (13.5), (13.6), (13.7), and (10.3), we obtain

$$|T_2| \lesssim \Theta_{\widetilde{\mu}}(2B_j^k)\,\frac{1}{r^2}\,\varepsilon_0\,(\ell_j^k)^2 \approx \varepsilon_0\,\widetilde{\mu}(B_j^k)\,\frac{\ell_j^k}{r^2}.$$

Concerning T_3, using (13.6) and (10.3) again, we get

$$|T_3| \lesssim \mathrm{Lip}\big(f\,\theta_j^k\big)\,\alpha_{\widetilde{\mu}}(2B_j^k)\,\mu(2B_j^k)\,\ell_j^k \lesssim \alpha_{\widetilde{\mu}}(2B_j^k)\,\widetilde{\mu}(B_j^k)\,\frac{\ell_j^k}{r^2} \lesssim \varepsilon_0\,\widetilde{\mu}(B_j^k)\,\frac{\ell_j^k}{r^2}.$$

To deal with T_1 we need first to estimate $|c_j^k - c_{2B_j^k}|$. To this end, we write

$$\begin{aligned}&\left|\int \theta_j^k\,d\widetilde{\mu} - c_{2B_j^k}\int_{\Gamma^k}\theta_j^k\,d\mathcal{H}^1\right|\\ &\qquad\le \left|\int \theta_j^k\,d\widetilde{\mu} - c_{2B_j^k}\int_{L_{2B_j^k}}\theta_j^k\,d\mathcal{H}^1\right| + \left|c_{2B_j^k}\int_{L_{2B_j^k}}\theta_j^k\,d\mathcal{H}^1 - c_{2B_j^k}\int_{\Gamma^k}\theta_j^k\,d\mathcal{H}^1\right|.\end{aligned} \tag{13.8}$$

Since $\|\nabla\theta_j^k\|_\infty \lesssim 1/\ell_j^k$, the first term on the right hand side does not exceed

$$c\,\frac{1}{\ell_j^k}\,\alpha_{\widetilde\mu}(2B_j^k)\,\widetilde\mu(B_j^k)\,\ell_j^k \lesssim \varepsilon_0\,\widetilde\mu(B_j^k).$$

Arguing as in (13.5), we deduce that the last term on the right hand side of (13.8) is bounded by

$$\Theta_{\widetilde\mu}(2B_j^k)\,\mathrm{Lip}\big(\theta_j^k\big)\,\mathrm{dist}_{2B_j^k}(\mathcal{H}^1|_{\Gamma^k},\mathcal{H}^1|_{L_{2B_j^k}}) \lesssim \Theta_{\widetilde\mu}(2B_j^k)\,\frac{1}{\ell_j^k}\,\varepsilon_0\,(\ell_j^k)^2 \lesssim \varepsilon_0\,\widetilde\mu(B_j^k).$$

Then we deduce that

$$\left|\int\theta_j^k\,d\widetilde\mu - c_{2B_j^k}\int_{\Gamma^k}\theta_j^k\,d\mathcal{H}^1\right| \lesssim \varepsilon_0\,\widetilde\mu(B_j^k).$$

Recalling that $c_j^k = \frac{\int\theta_j^k\,d\mu}{\int\theta_j^k\,d\mathcal{H}^1|_{\Gamma^k}}$, we obtain

$$\big|c_j^k - c_{2B_j^k}\big| = \frac{1}{\int_{\Gamma^k}\theta_j^k\,d\mathcal{H}^1}\left|\int\theta_j^k\,d\widetilde\mu - c_{2B_j^k}\int_{\Gamma^k}\theta_j^k\,d\mathcal{H}^1\right| \lesssim \frac{\varepsilon_0\,\widetilde\mu(B_j^k)}{\int_{\Gamma^k}\theta_j^k\,d\mathcal{H}^1}.$$

Therefore, we have

$$|T_1| \lesssim |c_j^k - c_{2B_j^k}|\,\|f\|_{\infty,2B_j^k}\int_{\Gamma^k}\theta_j^k(y)\,d\mathcal{H}^1 \lesssim \frac{\varepsilon_0\,\widetilde\mu(B_j^k)}{\int_{\Gamma^k}\theta_j^k\,d\mathcal{H}^1}\,\frac{\ell_j^k}{r^2}\int_{\Gamma^k}\theta_j^k(y)\,d\mathcal{H}^1 = \frac{\varepsilon_0\,\widetilde\mu(B_j^k)\ell_j^k}{r^2}.$$

Gathering the estimates obtained for T_1, T_2 and T_3, the lemma follows. □

Lemma 13.3. *For $x\in\Gamma^k$, let $\ell^k(x)$ denote the segment L_j^k which contains x (if this is not unique, the choice does not matter). We have*

$$\int_{B(x_0,\frac18 Kr_0)}\int_{\ell^k(x)}^{Kr_0/100}|\Delta_{\widetilde\mu,\varphi}(x,r)-\Delta_{\nu^k,\varphi}(x,r)|^2\,\frac{dr}{r}\,d\nu^k(x) \lesssim_{A,\tau,K} \varepsilon_0^2\,\Theta_\mu(B_R)^2\,\mu(R). \tag{13.9}$$

Note that in the integral above $\operatorname{supp}\nu^k\cap B(x_0,\frac18 Kr_0)\subset\Gamma^k$.

Proof. Let $x\in B(x_0,\frac18 Kr_0)\cap\Gamma^k$, $r\geq\ell^k(x)$, and write

$$\Delta_{\widetilde\mu,\varphi}(x,r)-\Delta_{\nu^k,\varphi}(x,r) = \sum_{j=1}^{N_k}\left(\int\psi_r(x-y)\,d\nu_j^k(y) - \int\psi_r(x-y)\,\theta_j^k(y)\,d\widetilde\mu(y)\right).$$

Since $\operatorname{supp}\theta_j^k\subset\frac32 B_j^k$, the integral on the right hand side vanishes unless $\frac32 B_j^k$ intersects $B(x,r)$. Since $r\geq\ell^k(x)$, it follows easily that the latter condition implies $2B_j^k\subset B(x,c_{11}r)$, for some absolute constant c_{11}, by Lemma 10.1.

For a ball B_j^k such that $2B_j^k\subset B(x,c_{11}r)$, by Lemma 13.2 we have

$$\left|\int\psi_r(x-y)\,d\nu_j^k(y) - \int\psi_r(x-y)\,\theta_j^k(y)\,d\widetilde\mu(y)\right| \lesssim \varepsilon_0\,\frac{\widetilde\mu(B_j^k)\,\ell_j^k}{r^2}.$$

Hence,

$$\big|\Delta_{\widetilde\mu,\varphi}(x,r)-\Delta_{\nu^k,\varphi}(x,r)\big| \lesssim \varepsilon_0\sum_{j:2B_j^k\subset B(x,c_{11}r)}\frac{\widetilde\mu(B_j^k)\,\ell_j^k}{r^2}.$$

By Cauchy-Schwarz, from the last estimate we infer that

$$\left|\Delta_{\widetilde{\mu},\varphi}(x,r)-\Delta_{\nu^k,\varphi}(x,r)\right|^2 \lesssim \varepsilon_0^2\Bigg(\sum_{j:2B_j^k\subset B(x,c_{11}r)}\widetilde{\mu}(B_j^k)\Bigg)\Bigg(\sum_{j:2B_j^k\subset B(x,c_{11}r)}\frac{\widetilde{\mu}(B_j^k)\,(\ell_j^k)^2}{r^4}\Bigg)$$

$$\lesssim \varepsilon_0^2\,\frac{\widetilde{\mu}(B(x,c_{11}r))}{r}\sum_{j:2B_j^k\subset B(x,c_{11}r)}\frac{\widetilde{\mu}(B_j^k)\,(\ell_j^k)^2}{r^3},$$

where in the last inequality we took into account that $\mu(B_j^k)\approx\mu(\frac16 B_j^k)$ and that the balls $\frac16 B_j^k$ are pairwise disjoint for every fixed k. Since $\mu(B(x,c_{11}r))\lesssim A\,\Theta_\mu(B_R)\,r$, we obtain

$$\left|\Delta_{\widetilde{\mu},\varphi}(x,r)-\Delta_{\nu^k,\varphi}(x,r)\right|^2 \lesssim \varepsilon_0^2\,A\,\Theta_\mu(B_R)\sum_{j:2B_j^k\subset B(x,c_{11}r)}\frac{\widetilde{\mu}(B_j^k)\,(\ell_j^k)^2}{r^3}.$$

Now we use this inequality to estimate the left hand side of (13.9):

$$\int_{B(x_0,\frac18 Kr_0)}\int_{\ell^k(x)}^{Kr_0/100}|\Delta_{\widetilde{\mu},\varphi}(x,r)-\Delta_{\nu^k,\varphi}(x,r)|^2\,\frac{dr}{r}\,d\nu^k(x) \tag{13.10}$$

$$\lesssim \varepsilon_0^2\,A\,\Theta_\mu(B_R)\sum_{h=1}^{N_k}\int_{L_h^k}\int_{\ell_h^k}^{Kr_0/100}\sum_{j:2B_j^k\subset B(x,c_{11}r)}\frac{\widetilde{\mu}(B_j^k)\,(\ell_j^k)^2}{r^3}\,\frac{dr}{r}\,d\nu^k(x).$$

Note now that if $x\in B_h^k$, $r\geq\ell_h^k$ and $2B_j^k\subset B(x,c_{11}r)$, then

$$r\gtrsim \operatorname{dist}(B_h^k,B_j^k)+r(B_j^k)+r(B_h^k)=:D(B_j^k,B_h^k).$$

Then, by Fubini,

$$\int_{\ell_h^k}^{Kr_0/100}\sum_{j:2B_j^k\subset B(x,c_{11}r)}\frac{\widetilde{\mu}(B_j^k)\,(\ell_j^k)^2}{r^3}\,\frac{dr}{r}\leq\sum_{j=1}^{N_k}\widetilde{\mu}(B_j^k)(\ell_j^k)^2\int_{c^{-1}D(B_j^k,B_h^k)}^{\infty}\frac{1}{r^3}\,\frac{dr}{r}$$

$$\lesssim\sum_{j=1}^{N_k}\frac{\widetilde{\mu}(B_j^k)(\ell_j^k)^2}{D(B_j^k,B_h^k)^3}.$$

Plugging this estimate into (13.10), this gives

$$\int_{B(x_0,\frac18 Kr_0)}\int_{\ell^k(x)}^{Kr_0/100}|\Delta_{\widetilde{\mu},\varphi}(x,r)-\Delta_{\nu^k,\varphi}(x,r)|^2\,\frac{dr}{r}\,d\nu^k(x) \tag{13.11}$$

$$\lesssim \varepsilon_0^2\,A\,\Theta_\mu(B_R)\sum_{h=1}^{N_k}\nu^k(L_h^k)\sum_{j=1}^{N_k}\frac{\widetilde{\mu}(B_j^k)(\ell_j^k)^2}{D(B_j^k,B_h^k)^3}$$

$$=\varepsilon_0^2\,A\,\Theta_\mu(B_R)\sum_{j=1}^{N_k}\widetilde{\mu}(B_j^k)(\ell_j^k)^2\sum_{h=1}^{N_k}\frac{\nu^k(L_h^k)}{D(B_j^k,B_h^k)^3}.$$

Since the measure ν^k satisfies the linear growth condition

$$\nu^k(B(y,r))\lesssim_{A,\tau}\Theta_\mu(B_R)\,r\qquad\text{for all } r>0,$$

one easily gets

$$\sum_{h=1}^{N_k} \frac{\nu^k(L_h^k)}{D(B_j^k, B_h^k)^3} \lesssim_{A,\tau} \frac{\Theta_\mu(B_R)}{(\ell_j^k)^2}.$$

Then, going back to (13.11), we obtain

$$\int_{B(x_0,\frac{1}{8}Kr_0)} \int_{\ell^k(x)}^{Kr_0/100} |\Delta_{\widetilde{\mu},\varphi}(x,r) - \Delta_{\nu^k,\varphi}(x,r)|^2 \frac{dr}{r}\, d\nu^k(x)$$

$$\lesssim_{A,\tau} \varepsilon_0^2\, \Theta_\mu(B_R)^2 \sum_{j=1}^{N_k} \widetilde{\mu}(B_j^k) \leq_{A,\tau,K} \varepsilon_0^2\, \Theta_\mu(B_R)^2\, \mu(R),$$

as wished. □

Lemma 13.4. *Let H^k be the subset of those points $x \in \Gamma^k \cap B(x_0, \frac{1}{8}Kr_0)$ such that*

$$\int_{\ell^k(x)}^{Kr_0/100} \left|\Delta_{\nu^k,\varphi}(x,r)\right|^2 \frac{dr}{r} > \varepsilon_0^{1/2}\, \Theta_\mu(B_R)^2.$$

Then

$$\nu^k(H^k) \leq \varepsilon_0^{1/2}\, \nu^k(\Gamma^k),$$

assuming η small enough.

Proof. For $x \in \Gamma^k \cap B(x_0, \frac{1}{8}Kr_0)$ we write

$$\begin{aligned}\left(\int_{\ell^k(x)}^{Kr_0/100} \left|\Delta_{\nu^k,\varphi}(x,r)\right|^2 \frac{dr}{r}\right)^{1/2} &\leq \left(\int_{\ell^k(x)}^{Kr_0/100} \left|\Delta_{\nu^k,\varphi}(x,r) - \Delta_{\widetilde{\mu},\varphi}(x,r)\right|^2 \frac{dr}{r}\right)^{1/2} \\ &\quad + \left(\int_{\ell^k(x)}^{Kr_0/100} \left|\Delta_{\widetilde{\mu},\varphi}(x,r) - \Delta_{\mu,\varphi}(x,r)\right|^2 \frac{dr}{r}\right)^{1/2} \\ &\quad + \left(\int_{\ell^k(x)}^{Kr_0/100} |\Delta_{\mu,\varphi}(x,r)|^2 \frac{dr}{r}\right)^{1/2} \\ &=: I_1(x) + I_2(x) + I_3(x).\end{aligned}$$

By Lemma 13.1, if η and ε_0 are assumed small enough,

$$I_3(x) \leq \frac{\varepsilon_0^{1/2}}{3}\, \Theta_\mu(B_R) \qquad \text{for all } x \in \Gamma^k \cap B(x_0, \tfrac{1}{8}Kr_0).$$

Thus,

$$\begin{aligned}\nu^k(H^k) &\leq \nu^k\left(\left\{x \in \Gamma^k \cap B(x_0, \tfrac{1}{8}Kr_0) : I_1(x) > \tfrac{1}{3}\varepsilon_0^{1/2}\, \Theta_\mu(B_R)\right\}\right) \\ &\quad + \nu^k\left(\left\{x \in \Gamma^k \cap B(x_0, \tfrac{1}{8}Kr_0) : I_2(x) > \tfrac{1}{3}\varepsilon_0^{1/2}\, \Theta_\mu(B_R)\right\}\right).\end{aligned} \tag{13.12}$$

By Chebyshev and Lemma 13.3, the first term on the right hand side does not exceed

$$\frac{9}{\varepsilon_0\,\Theta_\mu(B_R)^2}\int_{B(x_0,\frac18 Kr_0)}\int_{\ell^k(x)}^{Kr_0/100}|\Delta_{\widetilde\mu,\varphi}(x,r)-\Delta_{\nu^k,\varphi}(x,r)|^2\,\frac{dr}{r}\,d\nu^k(x)$$
$$\leq \frac{9\,c(A,\tau,K)\,\varepsilon_0^2\,\Theta_\mu(B_R)^2}{\varepsilon_0\,\Theta_\mu(B_R)^2}\,\mu(R)\leq\frac{\varepsilon_0^{1/2}}{2}\,\nu^k(\Gamma^k),$$

where in the last inequality we took into account that $\varepsilon_0\ll c(A,\tau,K)$ and that $\mu(R)\approx_K\nu^k(\Gamma^k)$.

To estimate the last term in (13.12), we consider the operator T_φ defined as follows for a measure $\lambda\in M(\mathbb{R}^d)$:

$$T_\varphi\lambda(x)=\left(\int_0^\infty|\Delta_{\lambda,\varphi}(x,r)|^2\,\frac{dr}{r}\right)^{1/2}.$$

As shown in [TT, Theorem 5.1], T_φ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mathcal{H}^1|_{\Gamma^k})$ when Γ^k is an AD-regular curve, with the norm bounded by some constant depending only on the AD-regularity constant of Γ^k, and so on A and τ. Take the measure

$$\lambda=\chi_{B(x_0,Kr_0)}\,(\mu-\widetilde\mu).$$

Using the aforementioned boundedness of T_φ and the fact that $\nu^k\lesssim A\,\Theta_\mu(B_R)\,\mathcal{H}^1|_{\Gamma^k_{ex}}$, we deduce that the last term in (13.12) is bounded by

$$c\,A\,\Theta_\mu(B_R)\,\mathcal{H}^1\left(\left\{x\in\Gamma^k: T_\varphi\lambda(x)>\tfrac13\varepsilon_0^{1/2}\,\Theta_\mu(B_R)\right\}\right)$$
$$\leq c(A,\tau)\,\Theta_\mu(B_R)\,\frac{\|\lambda\|}{\varepsilon_0^{1/2}\,\Theta_\mu(B_R)}=\frac{c(A,\tau)\,\|\lambda\|}{\varepsilon_0^{1/2}}.$$

Note now that by Lemma 7.3

$$\|\lambda\|=\mu(B(x_0,Kr_0)\setminus\widetilde E)\leq\eta^{1/10}\,\mu(B(x_0,Kr_0))\lesssim_K\eta^{1/10}\,\mu(R)\lesssim_K\eta^{1/10}\,\nu^k(\Gamma^k).$$

Thus, for η small enough,

$$\nu^k\left(\left\{x\in\Gamma^k\cap B(x_0,\tfrac18 Kr_0): I_2(x)>\tfrac13\varepsilon_0^{1/2}\,\Theta_\mu(B_R)\right\}\right)\leq\frac{c(A,\tau,K)\,\eta^{1/10}}{\varepsilon_0^{1/2}}\,\nu^k(\Gamma^k)\leq\frac{\varepsilon_0^{1/2}}{2}\,\nu^k(\Gamma^k),$$

which completes the proof of the lemma. □

Lemma 13.5. *We have*

$$\left(\int_{\Gamma^k_{ex}}\int_{\frac{Kr_0}{100}}^\infty+\int_{\Gamma^k_{ex}}\int_0^{\ell_k(x)}+\int_{\Gamma^k_{ex}\setminus B(x_0,\frac{Kr_0}{8})}\int_0^{\frac{Kr_0}{100}}\right)\left|\Delta_{\nu^k,\varphi}(x,r)\right|^2\frac{dr}{r}\,d\mathcal{H}^1(x)\lesssim\varepsilon_0^{1/2}\Theta_\mu(B_R)^2\,\ell(R).$$

Proof. First we will estimate the integral

$$\int_{\Gamma^k_{ex}}\int_{\frac{Kr_0}{100}}^\infty\left|\Delta_{\nu^k,\varphi}(x,r)\right|^2\frac{dr}{r}\,d\mathcal{H}^1(x).$$

To this end, take $r\geq Kr_0/100$ and let χ by a $\mathcal{C}^\infty$ bump function χ which equals 1 on $B(x_0,\frac12 Kr_0)$ and vanishes on $\mathbb{R}^d\setminus B(x_0,Kr_0)$, with $\|\nabla\chi\|_\infty\leq c(\delta)/r_0$. For $x\in\rho_1^1$, we have $\psi_r*\mathcal{H}^1|_{\rho_1^1}(x)=0$, and thus

$$\Delta_{\nu^k,\varphi}(x,r)=\int\psi_r(x-y)\,d\nu^k(y)-c_0^k\int\psi_r(x-y)\,d\mathcal{H}^1|_{\rho_1^1}(y).$$

As ν^k coincides with $c_0^k\, d\mathcal{H}^1|_{\rho_1^1}$ out of a small neighborhood of $B(x_0, \frac{1}{4}Kr_0)$, we have

$$\Delta_{\nu^k,\varphi}(x,r) = \int \chi(y)\, \psi_r(x-y)\, d(\nu^k - c_0^k\, d\mathcal{H}^1|_{\rho_1^1})(y).$$

Therefore,

$$|\Delta_{\nu^k,\varphi}(x,r)| \leq \mathrm{Lip}\big(\chi\, \psi_r(x-\cdot)\big)\, \mathrm{dist}_{B(x_0,Kr_0)}(\nu^k, c_0^k\, d\mathcal{H}^1|_{\rho_1^1}).$$

By the construction of ν^k and the definition of c_0^k, it is not difficult to see that

$$\mathrm{dist}_{B(x_0,Kr_0)}(\nu^k, c_0^k\, \mathcal{H}^1|_{\rho_1^1}) \leq c(K)\, \varepsilon_0\, \nu^k(B(x_0,Kr_0))\, r_0. \tag{13.13}$$

We leave the details for the reader. Using also that $\mathrm{Lip}\big(\chi\, \psi_r(x-\cdot)\big)\big) \leq c(K)/(r\, r_0)$, we obtain

$$|\Delta_{\nu^k,\varphi}(x,r)| \leq \frac{c(K)\, \varepsilon_0\, \mu(R)\, r_0}{r\, r_0} \leq c(K)\, \varepsilon_0\, \Theta_\mu(B_R)\, \frac{r_0}{r}. \tag{13.14}$$

If $x \in \Gamma_{ex}^k \setminus \rho_1^1$, then we consider the point x' which is the orthogonal projection of x on ρ_1^1, and we write

$$\begin{aligned}|\Delta_{\nu^k,\varphi}(x,r)| &\leq |\Delta_{\nu^k,\varphi}(x',r)| + |\Delta_{\nu^k,\varphi}(x',r) - \Delta_{\nu^k,\varphi}(x,r)| \\ &\leq c(K)\, \varepsilon_0\, \Theta_\mu(B_R)\, \frac{r_0}{r} + |\Delta_{\nu^k,\varphi}(x',r) - \Delta_{\nu^k,\varphi}(x,r)|,\end{aligned}$$

by applying (13.14) to x'. To estimate the last term note that

$$|x - x'| \leq \sup_{1\leq j\leq N_k} \mathrm{dist}(x_j^k, \rho_1^1) \lesssim \beta_\infty(B(x_0,Kr_0)\, K\, r_0 \leq c(K)\, \varepsilon_0\, r_0.$$

So we have

$$|\Delta_{\nu^k,\varphi}(x',r) - \Delta_{\nu^k,\varphi}(x,r)| \leq |x-x'|\, \|\nabla(\psi_r * \nu^k)\|_\infty \leq c(K)\, \varepsilon_0\, r_0\, \frac{\Theta(B_R)}{r}.$$

Thus (13.14) also holds in this case.

Note also that $\Delta_{\nu^k,\varphi}(x,r)$ vanishes for $x \in \Gamma_{ex}^k$ such that $B(x,4r) \cap B(x_0, \frac{1}{2}Kr_0) \neq \varnothing$. So we may assume that $r \gtrsim K\, r_0 + |x - x_0|$, and thus

$$\begin{aligned}\int_{Kr_0/100}^\infty \big|\Delta_{\nu^k,\varphi}(x,r)\big|^2\, \frac{dr}{r} &\leq c(K)\, \varepsilon_0^2\, \Theta_\mu(B_R)^2 \int_{c\,Kr_0/100 + c\,|x-x_0|}^\infty \frac{r_0^2}{r^3}\, dr \\ &\leq c(K)\, \varepsilon_0^2\, \frac{r_0^2\, \Theta_\mu(B_R)^2}{r_0^2 + |x-x_0|^2}.\end{aligned}$$

From the preceding estimate, it follows immediately that

$$\int_{\Gamma_{ex}^k} \int_{\frac{Kr_0}{100}}^\infty \big|\Delta_{\nu^k,\varphi}(x,r)\big|^2\, \frac{dr}{r}\, d\mathcal{H}^1(x) \leq c(K)\, \varepsilon_0^2\, \Theta_\mu(B_R)^2\, \ell(R) \leq \varepsilon_0\, \Theta_\mu(B_R)^2\, \ell(R).$$

By arguments in the same spirit, one can show that

$$\left(\int_{\Gamma_{ex}^k} \int_0^{\ell_k(x)} + \int_{\Gamma_{ex}^k \setminus B(x_0, \frac{Kr_0}{8})} \int_0^{\frac{Kr_0}{100}}\right) \big|\Delta_{\nu^k,\varphi}(x,r)\big|^2\, \frac{dr}{r}\, d\mathcal{H}^1(x) \lesssim \varepsilon_0^{1/2}\, \Theta_\mu(B_R)^2\, \ell(R).$$

We leave the details for the reader. □

Remark 13.6. For the record, note that from (13.13) it follows easily that

$$c_0^k \approx \Theta_\mu(B_R)$$

with the comparability constant not depending on A, τ, K or M.

Now we need the following auxiliary result.

Proposition 13.7. *For $g \in L^p(\mathcal{H}^1|_{\Gamma^k_{ex}})$, consider the operator*

$$T_{\varphi,\mathcal{H}^1|_{\Gamma^k_{ex}}} g(x) = \left(\int_0^\infty \left|\psi_r * (g\,\mathcal{H}^1|_{\Gamma^k_{ex}})(x)\right|^2 \frac{dr}{r}\right)^{1/2}. \tag{13.15}$$

Then $T_{\varphi,\mathcal{H}^1|_{\Gamma^k_{ex}}}$ is bounded in $L^p(\mathcal{H}^1|_{\Gamma^k_{ex}})$ for $1 < p < \infty$.

Proof. We consider the operator

$$T_{\mathcal{H}^1|_{\Gamma^k_{ex}}} g(x) = \left(\int_0^\infty \Delta_{g\mathcal{H}^1|_{\Gamma^k_{ex}}}(x,r)^2 \frac{dr}{r}\right)^{1/2}.$$

As shown in [TT], $T_{\mathcal{H}^1|_{\Gamma^k_{ex}}}$ is bounded in $L^2(\mathcal{H}^1|_{\Gamma^k})$ and from $L^1(\mathcal{H}^1|_{\Gamma^k})$ to $L^{1,\infty}(\mathcal{H}^1|_{\Gamma^k})$. Thus by interpolation it is bounded in $L^p(\mathcal{H}^1|_{\Gamma^k})$ for $1 < p < 2$. By applying Lemma 3.2 to the measure $g\mathcal{H}^1|_{\Gamma^k_{ex}}$, it follows that

$$T_{\varphi,\mathcal{H}^1|_{\Gamma^k_{ex}}} g(x) \leq c\, T_{\mathcal{H}^1|_{\Gamma^k_{ex}}} g(x), \tag{13.16}$$

and thus $T_{\varphi,\mathcal{H}^1|_{\Gamma^k_{ex}}}$ is also bounded in $L^p(\mathcal{H}^1|_{\Gamma^k_{ex}})$ for $1 < p \leq 2$.

We will show in Proposition 18.1 that the $L^2(\mathcal{H}^1|_{\Gamma^k})$ boundedness of $T_{\mathcal{H}^1|_{\Gamma^k_{ex}}}$ implies its boundedness in $L^p(\mathcal{H}^1|_{\Gamma^k})$ for $2 < p < \infty$. Thus again by (13.16), $T_{\varphi,\mathcal{H}^1|_{\Gamma^k_{ex}}}$ is bounded in $L^p(\mathcal{H}^1|_{\Gamma^k_{ex}})$ for $2 < p < \infty$.

An alternative argument to show that $T_{\varphi,\mathcal{H}^1|_{\Gamma^k_{ex}}}$ is bounded in $L^p(\mathcal{H}^1|_{\Gamma^k_{ex}})$ for $2 < p < \infty$ consists in proving its boundedness from $L^\infty(\mathcal{H}^1|_{\Gamma^k})$ to $BMO(\mathcal{H}^1|_{\Gamma^k})$ (which follows by rather stander arguments). Then by interpolation between the pairs $(L^2(\mathcal{H}^1|_{\Gamma^k}), L^2(\mathcal{H}^1|_{\Gamma^k}))$ and $(L^\infty(\mathcal{H}^1|_{\Gamma^k}), BMO(\mathcal{H}^1|_{\Gamma^k}))$ we are done. □

Lemma 13.8. *We have*

$$\int_{\Gamma^k_{ex}} \int_0^\infty \left|\Delta_{\nu^k,\varphi}(x,r)\right|^2 \frac{dr}{r}\, d\mathcal{H}^1(x) \leq \varepsilon_0^{1/10}\, \Theta_\mu(B_R)^2\, \ell(R).$$

Proof. Let F^k be the subset of those points $x \in \Gamma^k \cap B(x_0, \frac{1}{8}Kr_0)$ such that

$$\int_0^{\frac{Kr_0}{100}} \left|\Delta_{\nu^k,\varphi}(x,r)\right|^2 \frac{dr}{r} > \varepsilon_0^{1/4}\, \Theta_\mu(B_R)^2.$$

From Lemmas 13.4 and 13.5 we deduce that

$$\mathcal{H}^1(F^k) \lesssim \varepsilon_0^{1/4}\, \ell(R),$$

assuming η small enough. Therefore,

$$\int_{\Gamma^k\cap B(x_0,\frac18 Kr_0)\setminus F^k}\int_0^{\frac{Kr_0}{100}} \big|\Delta_{\nu^k,\varphi}(x,r)\big|^2\,\frac{dr}{r}\,d\mathcal{H}^1(x) \lesssim_{A,\tau,K} \varepsilon_0^{1/4}\,\Theta_\mu(B_R)^2\ell(R). \tag{13.17}$$

Denote by g_k the density of ν^k with respect to $\mathcal{H}^1|_{\Gamma^k}$. Note that $\Delta_{\nu^k,\varphi}(x,r) = \psi_r * (\chi_{B(x_0,2Kr_0)}\nu_k)$ for $x \in \Gamma^k \cap B(x_0, \frac18 Kr_0)$ and $r \leq Kr_0/100$. Then, by Hölder's inequality and the $L^4(\mathcal{H}^1|_{\Gamma^k_{ex}})$ boundedness of the operator $T_{\varphi,\mathcal{H}^1|_{\Gamma^k_{ex}}}$ from (13.15), we get

$$\begin{aligned}\int_{F^k}\int_0^{\frac{Kr_0}{100}} \big|\Delta_{\nu^k,\varphi}(x,r)\big|^2\,\frac{dr}{r}\,d\mathcal{H}^1(x) &\leq \int_{F^k} |T_{\varphi,\mathcal{H}^1|_{\Gamma^k_{ex}}}(g_k\,\chi_{B(x_0,2Kr_0)})|^2\,d\mathcal{H}^1(x)\\ &\leq \mathcal{H}^1(F^k)^{1/2}\,\|T_{\varphi,\mathcal{H}^1|_{\Gamma^k_{ex}}}(g_k\,\chi_{B(x_0,2Kr_0)})\|^2_{L^4(\mathcal{H}^1|_{\Gamma^k_{ex}})}\\ &\lesssim_{A,\tau,K} (\varepsilon_0^{1/4}\,\ell(R))^{1/2}\,\|g_k\,\chi_{B(x_0,2Kr_0)}\|^2_{L^4(\mathcal{H}^1|_{\Gamma^k_{ex}})}\\ &\lesssim_{A,\tau,K} \varepsilon_0^{1/8}\;\Theta_\mu(B_R)^2\,\ell(R).\end{aligned}$$

From this estimate and (13.17) we deduce

$$\begin{aligned}\int_{\Gamma^k\cap B(x_0,\frac18 Kr_0)}\int_0^{\frac{Kr_0}{100}} \big|\Delta_{\nu^k,\varphi}(x,r)\big|^2\,\frac{dr}{r}\,d\mathcal{H}^1(x) &\lesssim_{A,\tau,K} (\varepsilon_0^{1/4}+\varepsilon_0^{1/8})\,\Theta_\mu(B_R)^2\ell(R)\\ &\lesssim_{A,\tau,K} \varepsilon_0^{1/8}\,\Theta_\mu(B_R)^2\ell(R).\end{aligned}$$

In combination with Lemma 13.5, this concludes the proof of the lemma. □

14. The good measure σ^k on Γ^k

In this section, for each k we will construct a measure σ^k supported on Γ^k having linear growth (with an absolute constant), so that moreover

$$\int_{\Gamma^k_{ex}}\int_0^\infty \big|\Delta_{\sigma^k,\varphi}(x,r)\big|^2\,\frac{dr}{r}\,d\mathcal{H}^1(x) \tag{14.1}$$

is very small. The measure σ^k will be used as a kind of reference measure in Section 15, where we will estimate the wavelet coefficients of the density of ν^k with respect to σ^k in terms of the square function (14.1) and of the analogous square function involving the measure ν^k. By means of these estimates we will prove later that the cells from HD have small μ-mass.

To define the measures σ^k we will use the maps $\Pi_k : \Gamma^k \to \Gamma^{k+1}$ introduced at the beginning of Section 10. Recall that, given $x \in L^k_j$, $\Pi_k(x)$ is defined by the property that the orthogonal projection of $\Pi_k(x)$ on L^k_j is x. We extend Π_k to the whole curve Γ^k_{ex} just by setting $\Pi_k(x) = x$ for $x \in \Gamma^k_{ex} \setminus \Gamma^k$. Note that $\Gamma^k_{ex} \setminus \Gamma^k = \Gamma^{k+1}_{ex} \setminus \Gamma^{k+1}$ and so the definition is correct. By abusing notation, we continue to denote by Π_k this extension.

We set $\sigma^1 = \mathcal{H}^1|_{\Gamma^1_{ex}} = \mathcal{H}^1|_{\rho^1_1}$, and then by induction, $\sigma^{k+1} = \Pi_{k,\#}(\sigma^k)$ for $k \geq 1$, where $\Pi_{k,\#}(\sigma^k)$ is the image measure of σ^k by Π_k. Note that σ^1 is just the length on the line ρ^1_1 (which coincides with Γ^1), and then for $k \geq 1$, $\sigma^k = g_k\,\mathcal{H}^1|_{\Gamma^k}$, with $\|g_k\|_\infty \leq 1$. This follows from the easily proved fact that $\|g_{k+1}\|_\infty \leq \|g_k\|_\infty$. Taking into account that Γ^k is AD regular, it follows that σ^k has linear growth with some constant depending on A and τ (analogously to

Γ^k). Next we show that the linear growth of σ^k does not depend on these constants. This fact will play an important role later.

Lemma 14.1. *There exists an absolute constant c_0 such that*

$$\sigma^k(B(x,r)) \leq c_0\, r \qquad \text{for } x \in \mathbb{R}^d \text{ and } r > 0.$$

Proof. It is enough to show that $\sigma^k|_{\Gamma^k}$ has linear growth with some absolute constant because σ^k coincides with the arc length measure on $\Gamma^k_{ex} \setminus \Gamma^k$. So it suffices to prove that

$$\sigma^k(B(x,r) \cap \Gamma^k) \leq c\, r \qquad \text{for } x \in \Gamma^k \text{ and } 0 < r \leq \operatorname{diam}(\Gamma^k). \tag{14.2}$$

Suppose first that $r \leq 2^{-k/2} d_0$ (recall that $d_0 = \ell_1^1 \approx \operatorname{diam}(\Gamma^k)$). In this case, by Lemma 10.6 it follows easily that $B(x,r)$ intersects a number of segments L_j^k bounded above by an absolute constant. Since $\sigma^k|_{\Gamma^k} = g_k\, \mathcal{H}^1|_{\Gamma^k}$ with $\|g_k\|_\infty \leq 1$, (14.2) holds in this case.

Suppose now that $r > 2^{-k/2}\, d_0$. Let $0 \leq m \leq k$ be the integer such that

$$2^{-(m-1)/2}\, d_0 \leq r \leq 2^{-m/2}\, d_0.$$

Note that

$$\sigma^k = \Pi_{k,\#}(\Pi_{k-1,\#}(\dots(\Pi_{m,\#}(\sigma^m)))).$$

Let $y \in B(x,r)$. For $m \leq n \leq k$, let x_n, y_n be such that $\Pi_{k-1}(\Pi_{k-2}(\dots(\Pi_n(x_n)))) = x$ and $\Pi_{k-1}(\Pi_{k-2}(\dots(\Pi_n(y_n)))) = y$. Since $|x_n - x_{n+1}| \lesssim \varepsilon_0\, 2^{-n/2} d_0$ for all n, writing $x = x_k$ we get

$$|x - x_m| \leq \sum_{n=m}^{k-1} |x_n - x_{n+1}| \lesssim \varepsilon_0\, 2^{-m/2}\, d_0.$$

Analogously, $|y - y_m| \lesssim \varepsilon_0\, 2^{-m/2}\, d_0$. Therefore,

$$\Pi_m^{-1}(\dots(\Pi_{k-2}^{-1}(\Pi_{k-1}^{-1}(B(x,r))))) \subset B(x_m, (1 + c\,\varepsilon_0) 2^{-m/2} d_0),$$

and so

$$\begin{aligned}\sigma^k(B(x,r) \cap \Gamma^k) &= \sigma^m(\Pi_m^{-1}(\dots(\Pi_{k-2}^{-1}(\Pi_{k-1}^{-1}(B(x,r) \cap \Gamma^m))))) \\ &\leq \sigma^m(B(x_m, (1 + c\,\varepsilon_0) 2^{-m/2} d_0) \cap \Gamma^m).\end{aligned}$$

Arguing as above, since $\ell_j^m \lesssim 2^{-m/2} d_0$ for $1 \leq j \leq N_m$ and the number of segments L_j^m that intersect $B(x_m, (1 + c\,\varepsilon_0) 2^{-m/2} d_0)$ is bounded by some absolute constant, we deduce that

$$\sigma^m(B(x_m, (1 + c\,\varepsilon_0) 2^{-m/2} d_0) \cap \Gamma^m) \leq c\, 2^{-m/2} d_0 \leq c\, r.$$

□

Next we show that σ^k is also lower AD-regular, with a constant depending on M now.

Lemma 14.2. *The density g_k of σ^k with respect to $\mathcal{H}^1|_{\Gamma^k_{ex}}$ satisfies*

$$g_k(x) \geq c(M) > 0 \qquad \text{for all } x \in \Gamma^k_{ex}.$$

Proof. For $x \in \Gamma^k_{ex} \setminus \Gamma^k$ we have $g^k(x) = 1$.

Suppose now that $x \in L_j^k \subset \Gamma^k$, and consider a the sequence of segements $L_1^1 = L_{j_1}^1, L_{j_2}^2, \dots, L_{j_k}^k = L_j^k$ such that $L_{j_{m+1}}^{m+1}$ is generated by $L_{j_m}^m$ for $m = 1, \dots, k-1$. By Lemma 10.13 (see (10.8)) we know that $\measuredangle(\rho^m_{j_m},\rho^{m+1}_{j_{m+1}})$ is bounded by the $\beta_{\infty,\widetilde{\mu}}$ coefficient of a suitable cell $P\in\mathcal{D}$ such that $2B_P$ contains $L^m_{j_m}$ and $L^{m+1}_{j_{m+1}}$. Thus we have

$$\sum_{m=1}^{k-1}(\measuredangle(\rho^m_{j_m},\rho^{m+1}_{j_{m+1}}))^2\leq c\,M. \tag{14.3}$$

Denote by $g_{(m)}$ the (constant) value of the density g_m on $L^m_{j_m}$. We claim that

$$\left|\frac{g_{(m+1)}}{g_{(m)}}-1\right|\leq(\measuredangle(\rho^m_{j_m},\rho^{m+1}_{j_{m+1}}))^2.$$

Indeed, let I_{m+1} be an arbitrary interval contained in $L^{m+1}_{j_{m+1}}$ and denote by I_m the interval from $L^m_{j_m}$ such that $\Pi_m(I_m)=I_{m+1}$, so that $\sigma^{m+1}(I_{m+1})=\sigma(I_m)$ and thus

$$g_{(m+1)}\,\mathcal{H}^1(I_{m+1})=g_{(m)}\,\mathcal{H}^1(I_m),$$

Since $\mathcal{H}^1(I_m)=\cos\measuredangle(\rho^m_{j_m},\rho^{m+1}_{j_{m+1}})\,\mathcal{H}^1(I_{m+1})$, we get

$$\frac{g_{(m+1)}}{g_{(m)}}=\frac{\mathcal{H}^1(I_m)}{\mathcal{H}^1(I_{m+1})}=\cos\measuredangle(\rho^m_{j_m},\rho^{m+1}_{j_{m+1}}).$$

Thus,

$$\left|\frac{g_{(m+1)}}{g_{(m)}}-1\right|=|\cos\measuredangle(\rho^m_{j_m},\rho^{m+1}_{j_{m+1}})-1|\leq(\sin\measuredangle(\rho^m_{j_m},\rho^{m+1}_{j_{m+1}}))^2\leq(\measuredangle(\rho^m_{j_m},\rho^{m+1}_{j_{m+1}}))^2, \tag{14.4}$$

and the claim follows.

From the previous claim and (14.3) we derive

$$\sum_{m=1}^{k-1}\left|\frac{g_{(m+1)}}{g_{(m)}}-1\right|\leq c\,M.$$

which implies that

$$C(M)^{-1}\leq\prod_{m=1}^{k-1}\frac{g_{(m+1)}}{g_{(m)}}\leq C(M).$$

As $g_{(1)}=1$, we get $g_{(k)}\geq C(M)^{-1}$, as wished. □

To estimate the integral (14.1) it is convenient to introduce a dyadic lattice over Γ^k_{ex}, which we will denote by $\mathcal{D}(\Gamma^k_{ex})$. This lattice is made up of subsets of Γ^k_{ex} and is analogous to the lattice $\mathcal{D}$ associated with μ which has been introduced in Section 4. However, since the arc-length measure on Γ^k_{ex} is AD-regular, the arguments for the construction of $\mathcal{D}(\Gamma^k_{ex})$ are easier than the ones for $\mathcal{D}$. There are many references where the reader can find such a construction. For example, see the classical works of [Ch] and [Da], or the more recent [NToV] for the precise version that we state below:

- The family $\mathcal{D}(\Gamma^k_{ex})$ is the disjoint union of families $\mathcal{D}_m(\Gamma^k_{ex})$ (families of level m cells, which are subsets of Γ^k_{ex}), $m\in\mathbb{Z}$.
- If $Q',Q''\in\mathcal{D}_m(\Gamma^k_{ex})$, then either $Q'=Q''$ or $Q'\cap Q''=\varnothing$.
- Each $Q'\in\mathcal{D}_{m+1}(\Gamma^k_{ex})$ is contained in some $Q\in\mathcal{D}_m(\Gamma^k_{ex})$ (necessarily unique due to the previous property). We say that Q' is the son of Q, and that Q is the parent of Q'.

- For each $m \in \mathbb{Z}$, $\Gamma^k_{ex} = \bigcup_{Q\in\mathcal{D}_m(\Gamma^k_{ex})} Q$.
- For each $Q \in \mathcal{D}_m(\Gamma^k_{ex})$, there exists $z_Q \in Q$ (the "center" of Q) such that $Q \subset B(z_Q, 2^{-4m+2})$ and $\mathrm{dist}(z_Q, Q') \geq 2^{-4m-3}$ for any $Q' \in \mathcal{D}_m(\Gamma^k_{ex})$ different from Q.

We write $\ell(Q) = 2^{-4m}$, and we call it the side length of Q. Also, we set

$$B_Q = B(z_Q, 4\ell(Q)),$$

so that we have

$$\Gamma^k_{ex} \cap \frac{1}{32} B_Q \subset Q \subset \Gamma^k_{ex} \cap B_Q.$$

We define

$$\beta_{\Gamma^k_{ex},\infty}(P) = \beta_{\Gamma^k_{ex},\infty}(4B_P).$$

Lemma 14.3. *We have*

$$\int_{\Gamma^k_{ex}} \int_0^\infty \left|\Delta_{\sigma^k,\varphi}(x,r)\right|^2 \frac{dr}{r}\, d\mathcal{H}^1(x) \lesssim_{A,\tau} \sum_{Q\in\mathcal{D}(\Gamma^k_{ex})} \beta_{\Gamma^k_{ex},\infty}(Q)^4\, \ell(Q). \tag{14.5}$$

Note the power 4 over $\beta_{\Gamma^k_{ex},\infty}(Q)$ in the last equation. At first sight, it may seem surprising because the usual power is 2 in most square function type estimates. The fact that we get a power larger than 2 will allow us to show that the left hand side of (14.5) is small if ε_0 is also small.

Proof of Lemma 14.3. By convenience, for $i \leq 0$, we denote $\Gamma^i_{ex} = \rho^1_1$ and $\sigma^i = \mathcal{H}^1|_{\Gamma^i_{ex}}$, and Π_i is the identity map on ρ^1_1.

The first step to prove the lemma consist in estimating $\Delta_{\sigma^k,\varphi}(x,r) = \psi_r * \sigma^k(x)$ in terms of the β coefficients of Γ^k_{ex}. Suppose first that $r \geq 2^{-(k+2)/2} d_0$. Let $m \leq k$ the maximal integer such that $2^{-(m+2)/2}\, d_0 \geq 10r$. Note that $\ell^m_j \geq 10r$ for all $1 \leq j \leq N_m$, by (10.2). Consider the sequence of points $x_m, x_{m+1}, \dots, x_k = x$ such that $x_i \in \Gamma^i_{ex}$ and $\Pi_i(x_i) = x_{i+1}$ for $i = m, m+1, \dots, k-1$. Then we write

$$|\psi_r * \sigma^k(x)| \leq |\psi_r * \sigma^k(x) - \psi_r * \sigma^m(x_m)| + |\psi_r * \sigma^m(x_m)|,$$

so that

$$\begin{aligned}\int_{\Gamma^k_{ex}} \int_{2^{-(k+2)/2}d_0}^\infty |\psi_r * \sigma^k(x)|^2 \frac{dr}{r}\, d\mathcal{H}^1(x) &\lesssim \int_{\Gamma^k_{ex}} \int_{2^{-(k+2)/2}d_0}^\infty |\psi_r * \sigma^k(x) - \psi_r * \sigma^m(x_m)|^2 \frac{dr}{r}\, d\mathcal{H}^1(x)\\ &\quad + \int_{\Gamma^k_{ex}} \int_{2^{-(k+2)/2}d_0}^\infty |\psi_r * \sigma^m(x_m)|^2 \frac{dr}{r}\, d\mathcal{H}^1(x)\\ &= (1) + (2).\end{aligned} \tag{14.6}$$

Notice that, although it is not stated explicitly, in the integrals above m depends on r, and thus x_m depends on x and r.

Estimate of (1).
We write

$$|\psi_r * \sigma^k(x) - \psi_r * \sigma^m(x_m)| \leq \sum_{i=m}^{k-1} |\psi_r * \sigma^i(x_i) - \psi_r * \sigma^{i+1}(x_{i+1})|. \tag{14.7}$$

Since $\sigma^{i+1} = \Pi_{i,\#}\sigma^i$, we have

$$\begin{aligned}|\psi_r * \sigma^i(x_i) - \psi_r * \sigma^{i+1}(x_{i+1})| &= \left|\int \psi_r(y - x_i)\, d\sigma^i(y) - \int \psi_r(y - \Pi_i(x_i))\, d\sigma^{i+1}(y)\right| \\ &= \left|\int \big[\psi_r(y - x_i) - \psi_r(\Pi_i(y) - \Pi_i(x_i))\big]\, d\sigma^i(y)\right|.\end{aligned}$$

To deal with the last integral, recall that $\psi_r(z) = \frac{1}{r}\,\varphi\Big(\frac{|z|}{r}\Big) - \frac{1}{2r}\,\varphi\Big(\frac{|z|}{2r}\Big)$ and that φ is supported on $[-2, 2]$ and constant in $[-1/2, 1/2]$. So we have

$$\big|\psi_r(\Pi_i(y) - \Pi_i(x_i)) - \psi_r(y - x_i)\big| \lesssim \frac{c}{r^2}\,\big||\Pi_i(y) - \Pi_i(x_i)| - |y - x_i|\big|,$$

and moreover the left hand side vanishes unless $|y - x_i| \approx r$ or $|\Pi_i(y) - \Pi_i(x_i)| \approx r$, which is equivalent to saying just that $|y - x_i| \approx r$ (because $|y - x_i| \approx |\Pi_i(y) - \Pi_i(x_i)|$ for $y \in \Gamma^i_{ex}$). Therefore,

$$|\psi_r * \sigma^i(x_i) - \psi_r * \sigma^{i+1}(x_{i+1})| \lesssim \frac{1}{r^2} \int_{c^{-1}r \leq |y - x_i| \leq 5\, r} \big||\Pi_i(y) - \Pi_i(x_i)| - |y - x_i|\big|\, d\sigma^i(y). \tag{14.8}$$

Now we have:

Claim 14.4. *For $m \leq i \leq k$, let $x_i, y \in \Gamma^i_{ex}$ be as in (14.8), with*

$$c^{-1} r \leq |y - x_i| \leq 5\, r. \tag{14.9}$$

Let $Q^i(x_i), Q^i(y) \in \mathcal{D}(\Gamma^k_{ex})$ be the largest cells with $\ell(Q^i(x_i)), \ell(Q^i(y)) \leq 2^{i/2}\, d_0$ such that $2B_{Q^i(x_i)}$ contains x_i and $2B_{Q^i(y)}$ contains y. Let $S \in \mathcal{D}(\Gamma^k_{ex})$ be the smallest cell such that $2B_S$ contains $Q^i(x_i)$ and $Q^i(y)$ for all $y \in \Gamma^i_{ex}$ satisfying (14.9) and $m \leq i \leq k$ (so $\operatorname{diam}(B_S) \approx r$). Then

$$\begin{aligned}\big||\Pi_i(y) - \Pi_i(x_i)| - |y - x_i|\big| \lesssim\; &\ell(Q^i(y)) \Bigg(\sum_{\substack{Q \in \mathcal{D}(\Gamma^k_{ex}):\\ Q^i(y) \subset Q \subset 2B_S}} \beta_{\Gamma^k_{ex},\infty}(Q)\Bigg)^2 \\ &+ \ell(Q^i(x_i)) \Bigg(\sum_{\substack{Q \in \mathcal{D}(\Gamma^k_{ex}):\\ Q^i(x_i) \subset Q \subset 2B_S}} \beta_{\Gamma^k_{ex},\infty}(Q)\Bigg)^2,\end{aligned} \tag{14.10}$$

for $m \leq i \leq k$.

Let us assume the claim for the moment and let us continue the proof of the lemma. Let $j(i)$ the level of the largest cells $P \in \mathcal{D}(\Gamma^k_{ex})$ such that $\ell(P) \leq 2^{i/2}\, d_0$. Plugging the above estimate into (14.8) we derive

(14.11)

$$
\begin{aligned}
&|\psi_r * \sigma^i(x_i) - \psi_r * \sigma^{i+1}(x_{i+1})| \\
&\qquad \lesssim \sum_{\substack{P\in\mathcal{D}_{j(i)}(\Gamma_{ex}^k):\\ P\subset 2B_S}} \frac{1}{\ell(S)^2}\int_{y\in 2B_P: c^{-1}r\leq|y-x_i|\leq 5\,r} \big||\Pi_i(y)-\Pi_i(x_i)| - |y-x_i|\big|\,d\sigma^i(y) \\
&\qquad \lesssim \sum_{\substack{P\in\mathcal{D}_{j(i)}(\Gamma_{ex}^k):\\ P\subset 2B_S}} \frac{\ell(P)^2}{\ell(S)^2}\Bigg(\sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset Q\subset 2B_S}} \beta_{\Gamma_{ex}^k,\infty}(Q)\Bigg)^2 \\
&\qquad\quad + \sum_{\substack{P\in\mathcal{D}_{j(i)}(\Gamma_{ex}^k):\\ P\subset 2B_S}} \frac{\ell(P)^2}{\ell(S)^2}\Bigg(\sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ Q^i(x_i)\subset Q\subset 2B_S}} \beta_{\Gamma_{ex}^k,\infty}(Q)\Bigg)^2.
\end{aligned}
$$

Note that

$$
\sum_{\substack{P\in\mathcal{D}_{j(i)}(\Gamma_{ex}^k):\\ P\subset 2B_S}} \frac{\ell(P)^2}{\ell(S)^2} \leq C(A,\tau)\frac{\ell(Q^i(x_i))}{\ell(S)}.
$$

So the last sum in (14.11) does not exceed

$$
C(A,\tau)\,\frac{\ell(Q^i(x_i))}{\ell(S)}\Bigg(\sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ Q^i(x_i)\subset Q\subset 2B_S}} \beta_{\Gamma_{ex}^k,\infty}(Q)\Bigg)^2.
$$

Going back to equation (14.7), we get

$$
\begin{aligned}
(14.12)\qquad |\psi_r * \sigma^k(x) - \psi_r * \sigma^m(x_m)| \lesssim_{A,\tau} &\sum_{\substack{P\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset 2B_S}} \frac{\ell(P)^2}{\ell(S)^2}\Bigg(\sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset Q\subset 2B_S}} \beta_{\Gamma_{ex}^k,\infty}(Q)\Bigg)^2 \\
&+ \sum_{\substack{P\in\mathcal{D}(\Gamma_{ex}^k):\\ x\in P\subset 2B_S}} \frac{\ell(P)}{\ell(S)}\Bigg(\sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset Q\subset 2B_S}} \beta_{\Gamma_{ex}^k,\infty}(Q)\Bigg)^2,
\end{aligned}
$$

where we took into account that $\#\{i\in\mathbb{Z} : j(i) = j_0\}$ is bounded independently of j_0.

To prove (14.5) we will need to square the preceding inequality. Let us deal with the first sum on the right hand side. By Cauchy-Schwarz, we obtain

$$\begin{aligned}\Bigg(\sum_{\substack{P\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset 2B_S}} \frac{\ell(P)^2}{\ell(S)^2}\Bigg(\sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset Q\subset 2B_S}} \beta_{\Gamma_{ex}^k,\infty}(Q)\Bigg)^2\Bigg)^2 \\ \leq \Bigg(\sum_{\substack{P\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset 2B_S}} \frac{\ell(P)^2}{\ell(S)^2}\Bigg(\sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset Q\subset 2B_S}} \beta_{\Gamma_{ex}^k,\infty}(Q)\Bigg)^4\Bigg)\Bigg(\sum_{\substack{P\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset 2B_S}} \frac{\ell(P)^2}{\ell(S)^2}\Bigg).\end{aligned}$$

The last factor on the right side does not exceed some constant depending on A and τ. Also, by Hölder's inequality, it easily follows that

$$\Bigg(\sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset Q\subset 2B_S}} \beta_{\Gamma_{ex}^k,\infty}(Q)\Bigg)^4 \lesssim \sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset Q\subset 2B_S}} \beta_{\Gamma_{ex}^k,\infty}(Q)^4\,\frac{\ell(Q)^{1/2}}{\ell(P)^{1/2}}. \tag{14.13}$$

So we get

$$\begin{aligned}\Bigg(\sum_{\substack{P\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset 2B_S}} \frac{\ell(P)^2}{\ell(S)^2}\Bigg(\sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset Q\subset 2B_S}} \beta_{\Gamma_{ex}^k,\infty}(Q)\Bigg)^2\Bigg)^2 \\ &\lesssim_{A,\tau} \sum_{\substack{P\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset 2B_S}} \frac{\ell(P)^2}{\ell(S)^2} \sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset Q\subset 2B_S}} \beta_{\Gamma_{ex}^k,\infty}(Q)^4\,\frac{\ell(Q)^{1/2}}{\ell(P)^{1/2}} \\ &=_{A,\tau} \sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ Q\subset 2B_S}} \beta_{\Gamma_{ex}^k,\infty}(Q)^4 \sum_{\substack{P\in\mathcal{D}(\Gamma_{ex}^k):\\ P\subset Q}} \frac{\ell(Q)^{1/2}\,\ell(P)^{3/2}}{\ell(S)^2} \\ &\lesssim_{A,\tau} \sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ Q\subset 2B_S}} \beta_{\Gamma_{ex}^k,\infty}(Q)^4\,\frac{\ell(Q)^2}{\ell(S)^2}.\end{aligned}$$

Now we turn our attention to the last sum on the right side of (14.12). By Cauchy-Schwarz, we obtain

$$\begin{aligned}&\Bigg(\sum_{\substack{P\in\mathcal{D}(\Gamma^k_{ex}):\\ x\in P\subset 2B_S}}\frac{\ell(P)}{\ell(S)}\Bigg(\sum_{\substack{Q\in\mathcal{D}(\Gamma^k_{ex}):\\ P\subset Q\subset 2B_S}}\beta_{\Gamma^k_{ex},\infty}(Q)\Bigg)^2\Bigg)^2\\
&\qquad\leq\Bigg(\sum_{\substack{P\in\mathcal{D}(\Gamma^k_{ex}):\\ x\in P\subset 2B_S}}\frac{\ell(P)}{\ell(S)}\Bigg(\sum_{\substack{Q\in\mathcal{D}(\Gamma^k_{ex}):\\ P\subset Q\subset 2B_S}}\beta_{\Gamma^k_{ex},\infty}(Q)\Bigg)^4\Bigg)\Bigg(\sum_{\substack{P\in\mathcal{D}(\Gamma^k_{ex}):\\ x\in P\subset 2B_S}}\frac{\ell(P)}{\ell(S)}\Bigg)\\
&\qquad\lesssim\sum_{\substack{P\in\mathcal{D}(\Gamma^k_{ex}):\\ x\in P\subset 2B_S}}\frac{\ell(P)}{\ell(S)}\Bigg(\sum_{\substack{Q\in\mathcal{D}(\Gamma^k_{ex}):\\ P\subset Q\subset 2B_S}}\beta_{\Gamma^k_{ex},\infty}(Q)\Bigg)^4.\end{aligned}$$

By (14.13), the right hand side above is bounded by

$$\begin{aligned}c\sum_{\substack{P\in\mathcal{D}(\Gamma^k_{ex}):\\ x\in P\subset 2B_S}}\frac{\ell(P)}{\ell(S)}\sum_{\substack{Q\in\mathcal{D}(\Gamma^k_{ex}):\\ P\subset Q\subset 2B_S}}\beta_{\Gamma^k_{ex},\infty}(Q)^4\,\frac{\ell(Q)^{1/2}}{\ell(P)^{1/2}}&\approx\sum_{\substack{Q\in\mathcal{D}(\Gamma^k_{ex}):\\ x\in Q\subset 2B_S}}\beta_{\Gamma^k_{ex},\infty}(Q)^4\sum_{\substack{P\in\mathcal{D}(\Gamma^k_{ex}):\\ x\in P\subset Q}}\frac{\ell(Q)^{1/2}\,\ell(P)^{1/2}}{\ell(S)}\\
&\lesssim\sum_{\substack{Q\in\mathcal{D}(\Gamma^k_{ex}):\\ x\in Q\subset 2B_S}}\beta_{\Gamma^k_{ex},\infty}(Q)^4\,\frac{\ell(Q)}{\ell(S)}.\end{aligned}$$

Gathering the above estimates, we obtain

$$\Big(|\psi_r*\sigma^k(x)-\psi_r*\sigma^m(x_m)|\Big)^2\lesssim_{A,\tau}\sum_{\substack{Q\in\mathcal{D}(\Gamma^k_{ex}):\\ Q\subset 2B_S}}\beta_{\Gamma^k_{ex},\infty}(Q)^4\,\frac{\ell(Q)^2}{\ell(S)^2}+\sum_{\substack{Q\in\mathcal{D}(\Gamma^k_{ex}):\\ x\in Q\subset 2B_S}}\beta_{\Gamma^k_{ex},\infty}(Q)^4\,\frac{\ell(Q)}{\ell(S)}.$$

The preceding inequality holds for all $x\in\Gamma^k$ and $r\geq 2^{-(k+2)/2}d_0$ with $S\in\mathcal{D}(\Gamma^k_{ex})$ being the smallest cell such that $2B_S$ contains $B(x,4r)$. If these conditions hold, then we write $(x,r)\in I_S$. Then it follows that

$$\begin{aligned}&\int_{\Gamma^k_{ex}}\int_{2^{-(k+2)/2}d_0}^{\infty}|\psi_r*\sigma^k(x)-\psi_r*\sigma^m(x_m)|^2\,\frac{dr}{r}\,d\mathcal{H}^1(x)\\
&\qquad\lesssim_{A,\tau}\sum_{S\in\mathcal{D}(\Gamma^k_{ex})}\iint_{(x,r)\in I_S}\sum_{\substack{Q\in\mathcal{D}(\Gamma^k_{ex}):\\ Q\subset 2B_S}}\beta_{\Gamma^k_{ex},\infty}(Q)^4\,\frac{\ell(Q)^2}{\ell(S)^2}\,\frac{dr}{r}\,d\mathcal{H}^1(x)\\
&\qquad\quad+\sum_{S\in\mathcal{D}(\Gamma^k_{ex})}\iint_{(x,r)\in I_S}\sum_{\substack{Q\in\mathcal{D}(\Gamma^k_{ex}):\\ x\in Q\subset 2B_S}}\beta_{\Gamma^k_{ex},\infty}(Q)^4\,\frac{\ell(Q)}{\ell(S)}\,\frac{dr}{r}\,d\mathcal{H}^1(x).\end{aligned}$$

Applying Fubini for the last term on the right hand side, we infer that

(14.14)

$$\int_{\Gamma_{ex}^k}\int_{2^{-(k+2)/2}d_0}^{\infty}|\psi_r*\sigma^k(x)-\psi_r*\sigma^m(x_m)|^2\,\frac{dr}{r}\,d\mathcal{H}^1(x)$$

$$\lesssim_{A,\tau}\sum_{S\in\mathcal{D}(\Gamma_{ex}^k)}\Bigg(\sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ Q\subset 2B_S}}\beta_{\Gamma_{ex}^k,\infty}(Q)^4\,\frac{\ell(Q)^2}{\ell(S)^2}\,\ell(S)+\sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ Q\subset 2B_S}}\beta_{\Gamma_{ex}^k,\infty}(Q)^4\,\frac{\ell(Q)}{\ell(S)}\,\ell(Q)\Bigg)$$

$$\lesssim_{A,\tau}\sum_{S\in\mathcal{D}(\Gamma_{ex}^k)}\sum_{\substack{Q\in\mathcal{D}(\Gamma_{ex}^k):\\ Q\subset 2B_S}}\beta_{\Gamma_{ex}^k,\infty}(Q)^4\,\frac{\ell(Q)}{\ell(S)}\,\ell(Q)$$

$$=_{A,\tau}\sum_{Q\in\mathcal{D}(\Gamma_{ex}^k)}\beta_{\Gamma_{ex}^k,\infty}(Q)^4\,\ell(Q)\sum_{\substack{S\in\mathcal{D}(\Gamma_{ex}^k):\\ 2B_S\supset Q}}\frac{\ell(Q)}{\ell(S)}$$

$$\lesssim_{A,\tau}\sum_{Q\in\mathcal{D}(\Gamma_{ex}^k)}\beta_{\Gamma_{ex}^k,\infty}(Q)^4\,\ell(Q).$$

For the record, note that the preceding estimate is also valid if we replace ψ_r by φ_r. Indeed, above we did not use any cancellation property of ψ_r. Instead, we just took into account that ψ_r is smooth, radial, supported on $B(0,4r)$, and constant on $B(0,r/2)$. All these properties are also satisfied by φ_r.

Estimate of ②.
Recall that

$$②=\int_{\Gamma_{ex}^k}\int_{2^{-(k+2)/2}d_0}^{\infty}|\psi_r*\sigma^m(x_m)|^2\,\frac{dr}{r}\,d\mathcal{H}^1(x).$$

Since $\psi_r*\sigma^m(x_m)=0$ for $m\leq 0$, we can assume that $m\geq 1$, which implies that $10r\leq \ell_1^1$ by the dependence of m on r. Recall that $\Gamma^m=\bigcup_{i=1}^{N_m}L_i^m$. For convenience, for each $m\in[1,k]$, we will consider two additional segments L_0^k, L_{N_m+1} of length $\ell_0^k=\ell_{N_m+1}^k=2^{-m/2}d_0$, so that they are contained in $\rho_1^1\setminus L_1^1$, and one of the endpoints of L_0^m is $x_0^m=z_A$ and one of the endpoints of L_{N_m+1} is $x_{N_m}^m$. So joining these segments to Γ^m we obtain a small extension of Γ^m which we denote by $\Gamma_{ex}^m{}'$ and is contained in Γ_{ex}^m. Note that $\psi_r*\sigma^m(x_m)=0$ if $x_m\not\in\Gamma_{ex}^m{}'$. For convenience again, we say that L_0^{m-1} generates L_0^m, and that $L_{N_{m-1}+1}^{m-1}$ generates $L_{N_m+1}^m$.

On each segment L_j^m, $0\leq j\leq N_m+1$, σ^m equals some constant multiple of the arc length measure. So it turns out that, for $x\in\Gamma^m$, $\psi_r*\sigma^m(x_m)$ vanishes unless $\operatorname{supp}\psi_r(x_m-\cdot)$ intersects more than one segment L_j^m. Recall also that $\ell_j^m\geq 10r$ for all $j\in[1,N_m]$ and that $\operatorname{supp}\psi_r(x_m-\cdot)\subset\bar{B}(x_m,4r)$. As a consequence, by Lemma 10.4 it follows easily that $\operatorname{supp}\psi_r(x_m-\cdot)$ can intersect at most two segments L_j^m, L_{j+1}^m. We have:

Claim 14.5. *Let $x_m\in L_j^m\subset\Gamma_{ex}^m{}'$ be such that $B(x_m,4r)\cap L_{j+1}^m\neq\varnothing$. Denote by g_j^m and g_{j+1}^m the constant densities of σ^m on L_j^m and L_{j+1}^m respectively. Then*

$$|\psi_r*\sigma^m(x_m)|\lesssim\measuredangle(\rho_j^m,\rho_{j+1}^m)^2+|g_j^m-g_{j+1}^m|. \tag{14.15}$$

We will assume the claim for the moment and we will continue the proof of the lemma. To this end, denote by $L_1^1, L_a^2, L_a^3, \dots, L_a^{m-1}$ the ancestors of L_j^m. That is to say, for each $1 \le i \le m-1$, L_a^i is one of the segments L_h^i that constitutes Γ^i and L_a^i generates L_a^{i+1} (with $L_a^m = L_j^m$).

Analogously, let $L_1^1, L_b^2, L_b^3, \dots, L_b^{m-1}$ be the ancestors of L_{j+1}^m. Let n be maximal integer such that $L_a^n = L_b^n$. That is, $L_a^n = L_b^n$ is the closest common ancestor of L_j^m and L_{j+1}^m. We denote by g_a^i and g_b^i the constant density of σ^i on L_a^i and L_b^i, respectively. Then we write

$$|g_j^m - g_{j+1}^m| \le \sum_{i=n}^{m-1} |g_a^i - g_a^{i+1}| + \sum_{i=n}^{m-1} |g_b^i - g_b^{i+1}|. \tag{14.16}$$

As in (14.4), for each i we have

$$|g_a^i - g_a^{i+1}| \lesssim \measuredangle(\rho_a^i, \rho_a^{i+1})^2 \lesssim \beta_{\Gamma_{ex}^k,\infty}(4B_a^i)^2. \tag{14.17}$$

So from (14.15), (14.16) and (14.17) we deduce that

$$|\psi_r * \sigma^m(x_m)| \lesssim \measuredangle(\rho_j^m, \rho_{j+1}^m)^2 + \sum_{i=n}^{m-1} \big(\beta_{\Gamma_{ex}^k,\infty}(4B_a^i)^2 + \beta_{\Gamma_{ex}^k,\infty}(4B_b^i)^2\big) \le \sum_{i=n}^{m-1} \beta_{\Gamma_{ex}^k,\infty}(c_{12}B_a^i)^2,$$

for some absolute constant c_{12}.

We need now to introduce some additional notation. We write $L \sim \Gamma^k$ if $L = L_j^i$ for some $1 \le i \le k$, $0 \le j \le N_i+1$. For such L, we write $\widetilde{\ell}(L) = 2^{-i/2}\, d_0$ and $\beta_{\Gamma_{ex}^k,\infty}(L) = \beta_{\Gamma_{ex}^k,\infty}(c_{12}B_j^i)$. We say that L_{j-1}^i and L_{j+1}^i are neighbors of L_j^i. Also, given $L \sim \Gamma^k$ and $L' \sim \Gamma^k$, we write $L' \prec L$ if L' is an ancestor of L such that L' is not the ancestor of all the neighbors of L.

Using the above notation, given $x_m \in L_j^m = L$, by Cauchy-Schwarz, we get

$$|\psi_r * \sigma^m(x_m)|^2 \lesssim \Big(\sum_{L'\sim\Gamma^k: L'\prec L} \beta_{\Gamma_{ex}^k,\infty}(L')^2\Big)^2 \lesssim \sum_{L'\sim\Gamma^k: L'\prec L} \beta_{\Gamma_{ex}^k,\infty}(L')^4\, \frac{\widetilde{\ell}(L')^{1/2}}{\widetilde{\ell}(L)^{1/2}}. \tag{14.18}$$

Then we deduce

$$\begin{aligned} \textcircled{2} &= \int_{\Gamma^k} \int_{2^{-(k+2)/2}d_0}^{d_0/10} |\psi_r * \sigma^m(x_m)|^2\, \frac{dr}{r}\, d\mathcal{H}^1(x) \\ &\lesssim \sum_{L\sim\Gamma^k} \sum_{L'\sim\Gamma^k: L'\prec L} \beta_{\Gamma_{ex}^k,\infty}(L')^4\, \frac{\widetilde{\ell}(L')^{1/2}}{\widetilde{\ell}(L)^{1/2}}\, \widetilde{\ell}(L) \\ &= \sum_{L'\sim\Gamma^k} \beta_{\Gamma_{ex}^k,\infty}(L')^4 \sum_{L\sim\Gamma^k: L'\prec L} \frac{\widetilde{\ell}(L')^{1/2}}{\widetilde{\ell}(L)^{1/2}}\, \widetilde{\ell}(L). \end{aligned} \tag{14.19}$$

To deal with the last sum on the right hand side, note that for any given $L' \sim \Gamma^k$ the number of segments $L_h^i \sim \Gamma^k$ such that $L' \prec L_h^i$ of a fixed generation i is at most 2. Then it follows that

$$\sum_{L\sim\Gamma^k: L'\prec L} \frac{\widetilde{\ell}(L')^{1/2}}{\widetilde{\ell}(L)^{1/2}}\, \widetilde{\ell}(L) \lesssim \widetilde{\ell}(L'),$$

So we deduce that

$$\textcircled{2} \lesssim \sum_{L'\sim\Gamma^k} \beta_{\Gamma_{ex}^k,\infty}(L')^4\, \widetilde{\ell}(L') \lesssim_{A,\tau} \sum_{Q\in\mathcal{D}(\Gamma_{ex}^k)} \beta_{\Gamma_{ex}^k,\infty}(Q)^4\, \ell(Q).$$

The remaining term.
It remains to estimate the integral

$$\text{③} = \int_{\Gamma^k_{ex}} \int_0^{2^{-(k+2)/2}d_0} |\psi_r * \sigma^k(x)|^2 \, \frac{dr}{r} \, d\mathcal{H}^1(x). \tag{14.20}$$

The arguments will be quite similar to the ones we used for ②. We will use the same notation as the one for that case.

Note that $\psi_r * \sigma^k(x) = 0$ for $x \in \Gamma^k_{ex} \setminus {\Gamma^k_{ex}}'$. In fact, $\psi_r * \sigma^k(x)$ vanishes unless x belongs to some segment L^m_j, $0 \leq j \leq N_k + 1$ and x is the at a distance at most $4r$ from one of the endpoints of L^m_j. Moreover, arguing as in (14.18) (setting $m = k$ and $x_m = x$), for $x \in L^k_j$ we get

$$|\psi_r * \sigma^k(x)|^2 \lesssim \sum_{L' \sim \Gamma^k : L' \prec L^k_j} \beta_{\Gamma^k_{ex},\infty}(L')^4 \, \frac{\widetilde{\ell}(L')^{1/2}}{\widetilde{\ell}(L^k_j)^{1/2}}.$$

As a consequence,

$$\int_{L^k_j} |\psi_r * \sigma^k(x)|^2 \, d\mathcal{H}^1(x) \lesssim r \sum_{L' \sim \Gamma^k : L' \prec L^k_j} \beta_{\Gamma^k_{ex},\infty}(L')^4 \, \frac{\widetilde{\ell}(L')^{1/2}}{\widetilde{\ell}(L^k_j)^{1/2}},$$

using also that $\psi_r * \sigma^k(x) = 0$ far from the endpoints of L^k_j, as explained above. Then we obtain

$$\begin{aligned}
\text{③} &= \sum_{j=0}^{N_{k+1}} \int_0^{2^{-(k+2)/2}d_0} \int_{L^k_j} |\psi_r * \sigma^k(x)|^2 \, d\mathcal{H}^1(x) \, \frac{dr}{r} \\
&\lesssim \sum_{j=0}^{N_{k+1}} \int_0^{2^{-(k+2)/2}d_0} r \sum_{L' \sim \Gamma^k : L' \prec L^k_j} \beta_{\Gamma^k_{ex},\infty}(L')^4 \, \frac{\widetilde{\ell}(L')^{1/2}}{\widetilde{\ell}(L^k_j)^{1/2}} \, \frac{dr}{r} \\
&= \sum_{j=0}^{N_{k+1}} \sum_{L' \sim \Gamma^k : L' \prec L^k_j} \beta_{\Gamma^k_{ex},\infty}(L')^4 \, \frac{\widetilde{\ell}(L')^{1/2}}{\widetilde{\ell}(L^k_j)^{1/2}} \, \widetilde{\ell}(L^k_j).
\end{aligned}$$

Note that the right had side above does not exceed the right hand side of (14.19). So arguing as we did for ②, we deduce

$$\text{③} \lesssim_{A,\tau} \sum_{Q \in \mathcal{D}(\Gamma^k_{ex})} \beta_{\Gamma^k_{ex},\infty}(Q)^4 \, \ell(Q).$$

□

Proof of Claim 14.4. To simplify notation, we set $y_i := y$ and $y_{i+1} := \Pi_i(y)$. In this way, the left side of (14.10) becomes $\big||y_{i+1} - x_{i+1}| - |y_i - x_i|\big|$. Denote by L the line through x_i and y_i. If $x_i \in \Gamma^i$, let $\rho^i(x_i)$ be the line which supports the segment L^i_h that contains x_i, and in the case that $x_i \in \Gamma^i_{ex} \setminus \Gamma^i$, let $\rho^i(x_i) = \rho^1_1$. Let $\rho^i(y_i)$ the analogous one that contains y_i. Denote by α_x the angle between L and $\rho^i(x_i)$, and by α_y the one between L and $\rho^i(y_i)$.

We distinguish two cases. In the first one we assume that both α_x, α_y are very small, say $\alpha_x + \alpha_y \leq 1/1000$. Consider the line L' through x_{i+1} which is parallel to L. See Figure 1.

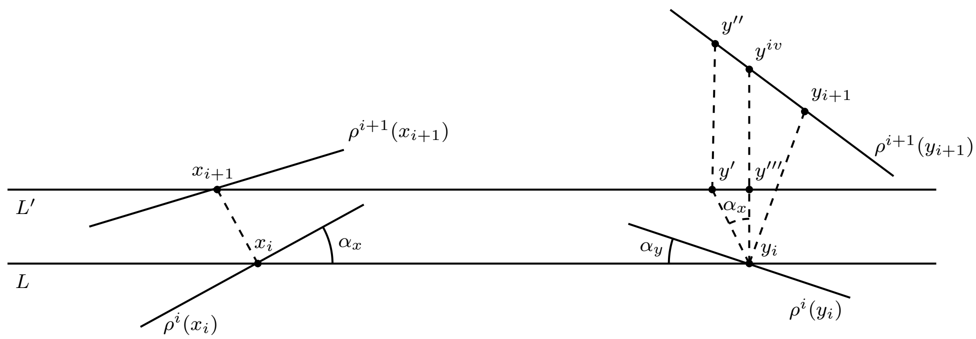

FIGURE 1. The points $x_i, x_{i+1}, y_i, y_{i+1}, y', y'', y''', y^{iv}$ and the different lines in the proof of Claim 14.4.

Let $y_i' \in L'$ be the point such that the segment $[x_i, x_{i+1}]$ is parallel to $[y_i, y']$, so that moreover $|y_i - y'| = |x_i - x_{i+1}|$ and $|y_i - x_i| = |y' - x_{i+1}|$. So we have

$$\begin{aligned}\big||y_{i+1} - x_{i+1}| - |y_i - x_i|\big| &= \big||y_{i+1} - x_{i+1}| - |y' - x_{i+1}|\big| \\ &= \frac{\big||y_{i+1} - x_{i+1}|^2 - |y' - x_{i+1}|^2\big|}{\big||y_{i+1} - x_{i+1}| + |y' - x_{i+1}|\big|} \\ &\lesssim \frac{\big||y_{i+1} - x_{i+1}|^2 - |y' - x_{i+1}|^2\big|}{r},\end{aligned} \tag{14.21}$$

since $|y' - x_{i+1}| \approx r$. Let $\rho^{i+1}(y_{i+1})$ be the line containing the segment L_j^{i+1} such that $y_{i+1} \in L_j^{i+1}$. Note that the angle between $\rho^{i+1}(y_{i+1})$ and L is small because $\alpha_y \leq 1/1000$ and $\measuredangle(\rho^{i+1}(y_{i+1}), \rho^i(y_i))$ is also small too. Let $y'' \in \rho^{i+1}(y_{i+1})$ be such that the angle between the segment $[y'', y']$ and the line L' is a right angle. Then, by Pythagoras' theorem,

$$|y' - x_{i+1}|^2 + |y' - y''|^2 = |y'' - x_{i+1}|^2.$$

Thus

$$\begin{aligned}\big||y_{i+1} - x_{i+1}|^2 - |y' - x_{i+1}|^2\big| &\leq \big||y_{i+1} - x_{i+1}|^2 - |y'' - x_{i+1}|^2\big| + \big||y'' - x_{i+1}|^2 - |y' - x_{i+1}|^2\big| \\ &= \big||y_{i+1} - x_{i+1}|^2 - |y'' - x_{i+1}|^2\big| + |y' - y''|^2.\end{aligned} \tag{14.22}$$

For the last term on the right side we set

$$|y' - y''|^2 \lesssim |y' - y_i|^2 + |y_i - y_{i+1}|^2 + |y_{i+1} - y''|^2 = |x_i - x_{i+1}|^2 + |y_i - y_{i+1}|^2 + |y_{i+1} - y''|^2.$$

Regarding the first term on the right hand side of (14.22), we have

$$\begin{aligned}\big||y_{i+1} - x_{i+1}|^2 - |y'' - x_{i+1}|^2\big| &= \big||y_{i+1} - x_{i+1}| - |y'' - x_{i+1}|\big| \cdot \big||y_{i+1} - x_{i+1}| + |y'' - x_{i+1}|\big| \\ &\leq |y_{i+1} - y''| \cdot \big||y_{i+1} - x_{i+1}| + |y'' - x_{i+1}|\big|.\end{aligned} \tag{14.23}$$

We write

$$|y_{i+1}-x_{i+1}|\leq |y_{i+1}-y_i| + |y_i - x_i| + |x_i - x_{i+1}|.$$

By the assumption (14.9), $|y_i - x_i|\lesssim r$. The other terms on the right side above are also bounded by $c\,r$ because for all $z\in\Gamma^i$, $|z-\Pi_i(z)|\lesssim \varepsilon_0\,2^{-i/2}d_0\leq \varepsilon_0\,2^{-m/2}d_0\lesssim r$. We also set

$$|y''-x_{i+1}|\leq |y''-y_{i+1}| + |y_{i+1}-x_{i+1}|\leq |y''-y_{i+1}|+c\,r.$$

Thus the left side of (14.23) does not exceed

$$|y_{i+1}-y''|^2 + c\,r\,|y_{i+1}-y''|.$$

Then, by (14.22) and the above inequalities, we obtain

$$\big||y_{i+1}-x_{i+1}|^2-|y'-x_{i+1}|^2\big|\leq |x_i-x_{i+1}|^2+|y_i-y_{i+1}|^2+c\,|y''-y_{i+1}|^2+c\,r\,|y''-y_{i+1}|. \tag{14.24}$$

Note that

$$|x_i-x_{i+1}|\lesssim \ell(Q^i(x_i))\,\beta_{\Gamma^k_{ex},\infty}(Q^i(x_i))\qquad\text{and}\qquad |y_i-y_{i+1}|\lesssim \ell(Q^i(y_i))\,\beta_{\Gamma^k_{ex},\infty}(Q^i(y_i)). \tag{14.25}$$

So it remains to estimate the term $|y''-y_{i+1}|$ from (14.24). To this end, we consider the points y''', y^{iv}, as in Figure 1. That is, we consider a hyperplane H orthogonal to L through y_i and then we put $\{y'''\}=H\cap L'$ and $\{y^{iv}\}=H\cap\rho^{i+1}(y_{i+1})$.

We write

$$|y_{i+1}-y''|\leq |y_{i+1}-y^{iv}|+|y^{iv}-y''|.$$

By elementary geometry, it follows that

$$|y_{i+1}-y^{iv}|\lesssim \sin\alpha_y\,|y_i-y_{i+1}|$$

and

$$|y^{iv}-y''|\lesssim |y'-y'''|\lesssim \sin\alpha_x\,|y_i-y'| = \sin\alpha_x\,|x_i-x_{i+1}|.$$

So we get

$$|y_{i+1}-y''|\lesssim \sin\alpha_x\,|x_i-x_{i+1}|+\sin\alpha_y\,|y_i-y_{i+1}|\lesssim r.$$

Therefore,

$$|y_{i+1}-y''|^2+r\,|y_{i+1}-y''|\lesssim r\,\big(\sin\alpha_x\,|x_i-x_{i+1}|+\sin\alpha_y\,|y_i-y_{i+1}|\big).$$

Now we take into account that

$$\sin\alpha_x\lesssim \sum_{Q\in\mathcal{D}(\Gamma^k_{ex}):Q^i(x_i)\subset Q\subset 2B_S}\beta_{\Gamma^k_{ex},\infty}(Q), \tag{14.26}$$

and analogously for $\sin\alpha_y$. Appealing to (14.25) then we deduce

$$\begin{aligned}|y_{i+1}-y''|^2+r\,|y_{i+1}-y''| &\lesssim r\,|x_i-x_{i+1}|\sum_{Q\in\mathcal{D}(\Gamma^k_{ex}):Q^i(x_i)\subset Q\subset 2B_S}\beta_{\Gamma^k_{ex},\infty}(Q)\\ &\quad + r\,|y_i-y_{i+1}|\sum_{Q\in\mathcal{D}(\Gamma^k_{ex}):Q^i(y_i)\subset Q\subset 2B_S}\beta_{\Gamma^k_{ex},\infty}(Q)\\ &\lesssim r\,\ell(Q_i(x_i))\Big(\sum_{Q\in\mathcal{D}(\Gamma^k_{ex}):Q^i(x_i)\subset Q\subset 2B_S}\beta_{\Gamma^k_{ex},\infty}(Q)\Big)^2\\ &\quad + r\,\ell(Q_i(y_i))\Big(\sum_{Q\in\mathcal{D}(\Gamma^k_{ex}):Q^i(y_i)\subset Q\subset 2B_S}\beta_{\Gamma^k_{ex},\infty}(Q)\Big)^2.\end{aligned} \tag{14.27}$$

By (14.25) again, it is also clear that both $|x_i - x_{i+1}|^2$ and $|y_i - y_{i+1}|^2$ are bounded by some constant times the right hand side of the preceding inequality. Then, from (14.24) and (14.27) it follows that

$$\begin{aligned}\frac{\big||y_{i+1}-x_{i+1}|^2 - |y'-x_{i+1}|^2\big|}{r} \lesssim\ & \ell(Q_i(x_i))\Big(\sum_{Q\in\mathcal{D}(\Gamma^k_{ex}):Q^i(x_i)\subset Q\subset 2B_S} \beta_{\Gamma^k_{ex},\infty}(Q)\Big)^2 \\ &+ \ell(Q_i(y_i))\Big(\sum_{Q\in\mathcal{D}(\Gamma^k_{ex}):Q^i(y_i)\subset Q\subset 2B_S} \beta_{\Gamma^k_{ex},\infty}(Q)\Big)^2,\end{aligned}$$

which together with (14.21) proves the claim in the case when $\alpha_x + \alpha_y \leq 1/1000$.

Suppose now that $\alpha_x + \alpha_y > 1/1000$, so that, for example, $\alpha_x > 1/2000$. Then we write

$$\begin{aligned}\big||y_{i+1}-x_{i+1}| - |y_i - x_i|\big| &\leq |y_{i+1}-y_i| + |x_{i+1}-x_i| \\ &\leq \ell(Q_i(x_i)) + \ell(Q_i(y_i)) \\ &= 2\,\ell(Q_i(x_i)) \\ &\lesssim (\sin\alpha_x)^2\,\ell(Q_i(x_i)).\end{aligned}$$

Using (14.26), we obtain

$$\big||y_{i+1}-x_{i+1}| - |y_i - x_i|\big| \lesssim \ell(Q_i(x_i))\,\big(\sin\alpha_x\big)^2 \lesssim \ell(Q_i(x_i))\left(\sum_{\substack{Q\in\mathcal{D}(\Gamma^k_{ex}):\\ Q^i(x_i)\subset Q\subset 2B_S}} \beta_{\Gamma^k_{ex},\infty}(Q)\right)^2,$$

which proves the claim in this second case. □

Proof of Claim 14.5. Note that $\operatorname{supp}\psi_r(x_m - \cdot)$ intersects L^m_j, L^m_{j+1}, and no other segments of the form L^m_h. So we have

$$\begin{aligned}\psi_r * \sigma^m(x_m) &= \int_{L^m_j} \psi_r(x_m - y)\, g^m_j\, d\mathcal{H}^1(y) + \int_{L^m_{j+1}} \psi_r(x_m - y)\, g^m_{j+1}\, d\mathcal{H}^1(y) \\ &= \left(\int_{L^m_j} \psi_r(x_m - y)\, g^m_j\, d\mathcal{H}^1(y) + \int_{L^m_{j+1}} \psi_r(x_m - y)\, g^m_j\, d\mathcal{H}^1(y)\right) \\ &\quad + \left(\int_{L^m_{j+1}} \psi_r(x_m - y)\,(g^m_{j+1} - g^m_j)\, d\mathcal{H}^1(y)\right) \\ &= \text{Ⓐ} + \text{Ⓑ}.\end{aligned}$$

It is immediate to check that $\big|\text{Ⓑ}\big| \lesssim |g^m_j - g^m_{j+1}|$.

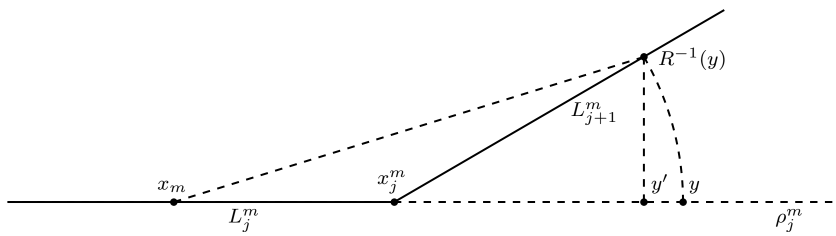

FIGURE 2. The points $x_m, x_j^m, y, y', R^{-1}(y)$ and the segments L_j^m and L_{j+1}^m in the proof of Claim 14.5.

Regarding Ⓐ, we set

$$\textcircled{A} = \left(\int_{L_j^m} \psi_r(x_m - y)\, g_j^m \, d\mathcal{H}^1(y) + \int_{\rho_j^m \setminus L_j^m} \psi_r(x_m - y)\, g_j^m \, d\mathcal{H}^1(y)\right) + \left(\int_{L_{j+1}^m} \psi_r(x_m - y)\, g_j^m \, d\mathcal{H}^1(y) - \int_{\rho_j^m \setminus L_j^m} \psi_r(x_m - y)\, g_j^m \, d\mathcal{H}^1(y)\right).$$

The first term on the right hand side vanishes (taking into account that $x_m \in \rho_m^j$), and so we only have to deal with the last one. To this end, consider a rotation R which transforms L_{j+1}^m into a segment contained in $\overline{\rho_j^m \setminus L_j^m}$, fixes $x_j^m = L_j^m \cap L_{j+1}^m$, and leaves invariant the subspace of $\mathbb{R}^d$ orthogonal to plane formed by L_j^m and L_{j+1}^m (assuming these segments to be not collinear, otherwise we let R be the identity). Since $\mathcal{H}^1|_{L_{j+1}^m} = R^{-1}\#(\mathcal{H}^1|_{R(L_{j+1}^m)})$, we have

$$\begin{aligned}\int_{L_{j+1}^m} \psi_r(x_m - y)\, g_j^m \, d\mathcal{H}^1(y) &= \int \psi_r(x_m - y)\, g_j^m \, dR^{-1}\#(\mathcal{H}^1|_{R(L_{j+1}^m)})(y) \\ &= \int \psi_r(x_m - R^{-1}(y))\, g_j^m \, d\mathcal{H}^1|_{R(L_{j+1}^m)}(y) \\ &= \int_{\rho_j^m \setminus L_j^m} \psi_r(x_m - R^{-1}(y))\, g_j^m \, d\mathcal{H}^1(y).\end{aligned}$$

Therefore,

$$\textcircled{A} = \int_{\rho_j^m \setminus L_j^m} \big[\psi_r(x_m - R^{-1}(y)) - \psi_r(x_m - y)\big]\, g_j^m \, d\mathcal{H}^1(y).$$

For $y \in \operatorname{supp}\big[\psi_r(x_m - R^{-1}(\cdot)) - \psi_r(x_m - \cdot)\big] \cap \rho_j^m$, we claim that

$$\big||x_m - R^{-1}(y)| - |x_m - y|\big| \lesssim r\, \measuredangle(\rho_j^m, \rho_{j+1}^m)^2. \tag{14.28}$$

To see this, consider the orthogonal projection y' of $R^{-1}(y)$ on ρ_j^m (see Figure 2), and set

$$\big||x_m - y| - |x_m - R^{-1}(y)|\big| \leq \big||x_m - y| - |x_m - y'|\big| + \big||x_m - y'| - |x_m - R^{-1}(y)|\big|.$$

By Pythagoras theorem, taking into account that $|x_m - y'| \approx |x_m - y| \approx r$, we get

$$\begin{aligned}\big||x_m - y'| - |x_m - R^{-1}(y)|\big| &\lesssim \frac{|R^{-1}(y) - y'|^2}{r}\\ &\lesssim \sin\big(\measuredangle(R^{-1}(y), x_m, y')\big)^2\, r \leq \sin\big(\measuredangle(R^{-1}(y), x_j^m, y')\big)^2\, r.\end{aligned}$$

On the other hand,

$$\begin{aligned}\big||x_m - y| - |x_m - y'|\big| &= |y - y'| = |x_j^m - R^{-1}(y)| - |x_j^m - y'|\\ &= \big(1 - \cos\big(\measuredangle(R^{-1}(y), x_j^m, y')\big)\big)\, |x_j^m - R^{-1}(y)| \lesssim \sin\big(\measuredangle(R^{-1}(y), x_j^m, y')\big)^2\, r,\end{aligned}$$

which completes the proof of (14.28).

Now, from (14.28) we deduce

$$\big|\psi_r(x_m - R^{-1}(y)) - \psi_r(x_m - y)\big| \lesssim \frac{\measuredangle(\rho_j^m, \rho_{j+1}^m)^2}{r}.$$

So we obtain

$$\Big|\text{Ⓐ}\Big| \lesssim \frac{\measuredangle(\rho_j^m, \rho_{j+1}^m)^2}{r}\, \mathcal{H}^1\big(\rho_j^m \cap \operatorname{supp}\big[\psi_r(x_m - R^{-1}(\cdot)) - \psi_r(x_m - \cdot)\big]\big) \lesssim \measuredangle(\rho_j^m, \rho_{j+1}^m)^2.$$

Together with the estimate we got for Ⓑ, this concludes the proof of the claim. □

Next we denote

$$\widetilde{\Delta}_{\sigma^k,\varphi}(x,r) = \frac{1}{r}\int_{|x-y|\leq 4r} \big|\varphi_r * \sigma^k(x) - \varphi_r * \sigma^k(y)\big|\, d\sigma^k(y)$$

and

$$\widetilde{\widetilde{\Delta}}_{\sigma^k,\varphi}(x,r) = \int \bigg|\int \big[\varphi_r(x-z)\, \varphi_{2r}(z-y) - \varphi_{2r}(x-z)\varphi_r(z-y)\big]\sigma^k(z)\bigg|\, d\sigma^k(y).$$

Arguing as above, we will get estimates for $\widetilde{\Delta}_{\sigma^k,\varphi}$ and $\widetilde{\widetilde{\Delta}}_{\sigma^k,\varphi}$ analogous to the ones obtained in Lemma 14.3 for $\Delta_{\sigma^k,\varphi}$. We will not give detailed proofs because the arguments are very similar to the ones for Lemma 14.3.

Lemma 14.6. *We have*

$$\int_{\Gamma_{ex}^k}\int_0^\infty \Big|\widetilde{\Delta}_{\sigma^k,\varphi}(x,r)\Big|^2 \frac{dr}{r}\, d\mathcal{H}^1(x) \leq_{A,\tau} \sum_{Q\in\mathcal{D}(\Gamma_{ex}^k)} \beta_{\Gamma_{ex}^k,\infty}(Q)^4\, \ell(Q).$$

Sketch of the proof. We will just explain the estimate of the integral

$$\int_{\Gamma_{ex}^k}\int_{2^{-(k+2)/2}d_0}^\infty |\widetilde{\Delta}_{\sigma^k,\varphi}(x,r)|^2\, \frac{dr}{r}\, d\mathcal{H}^1(x).$$

The arguments for remaining integral

$$\text{③} = \int_{\Gamma_{ex}^k}\int_0^{2^{-(k+2)/2}d_0} |\widetilde{\Delta}_{\sigma^k,\varphi}(x,r)|^2\, \frac{dr}{r}\, d\mathcal{H}^1(x).$$

are very similar to the analogous integral in (14.20) in Lemma 14.3.

Notice that by Cauchy-Schwarz we have

$$\widetilde{\Delta}_{\sigma^k,\varphi}(x,r)^2 \lesssim \frac{1}{r}\int_{|x-y|\leq 4r}\big|\varphi_r*\sigma^k(x)-\varphi_r*\sigma^k(y)\big|^2\,d\sigma^k(y).$$

Thus,

$$\int_{\Gamma_{ex}^k}\int_{2^{-(k+2)/2}d_0}^{\infty}\widetilde{\Delta}_{\sigma^k,\varphi}(x,r)^2\,\frac{dr}{r}\,d\mathcal{H}^1(x)$$
$$\lesssim \int_{2^{-(k+2)/2}d_0}^{\infty}\int_{x\in\Gamma_{ex}^k}\int_{y\in\Gamma_{ex}^k:|x-y|\leq 4r}\big|\varphi_r*\sigma^k(x)-\varphi_r*\sigma^k(y)\big|^2\,d\mathcal{H}^1(y)d\mathcal{H}^1(x)\,\frac{dr}{r^2}.$$

Given $x\in\Gamma_{ex}^k$ and $r\geq 2^{-(k+2)/2}d_0$, take the maximal integer $m\leq k$ such that $2^{-(m+2)/2}\,d_0\geq 10r$. As in the proof of Lemma 14.3, consider the points $x_m,x_{m+1},\dots,x_k=x$ such that $x_i\in\Gamma_{ex}^i$ and $\Pi_i(x_i)=x_{i+1}$ for $i=m,\,m+1,\dots,k-1$. Analogously, for $y\in\Gamma_{ex}^k$, let $y_m,y_{m+1},\dots,y_k=y$ be such that $y_i\in\Gamma_{ex}^i$ and $\Pi_i(y_i)=y_{i+1}$ for $i=m,\,m+1,\dots,k-1$. Then we set

$$\big|\varphi_r*\sigma^k(x)-\varphi_r*\sigma^k(y)\big|\leq\big|\varphi_r*\sigma^k(x)-\varphi_r*\sigma^m(x_m)\big|+\big|\varphi_r*\sigma^k(y)-\varphi_r*\sigma^m(y_m)\big|$$
$$+\big|\varphi_r*\sigma^m(x_m)-\varphi_r*\sigma^m(y_m)\big|,$$

so that

$$\int_{\Gamma_{ex}^k}\int_{2^{-(k+2)/2}d_0}^{\infty}\widetilde{\Delta}_{\sigma^k,\varphi}(x,r)^2\,\frac{dr}{r}\,d\mathcal{H}^1(x)$$
$$\lesssim \int_{2^{-(k+2)/2}d_0}^{\infty}\int_{x\in\Gamma_{ex}^k}\int_{y\in\Gamma_{ex}^k:|x-y|\leq 4r}\big|\varphi_r*\sigma^k(x)-\varphi_r*\sigma^m(x_m)\big|^2\,d\mathcal{H}^1(y)d\mathcal{H}^1(x)\,\frac{dr}{r^2}$$
$$+\int_{2^{-(k+2)/2}d_0}^{\infty}\int_{x\in\Gamma_{ex}^k}\int_{y\in\Gamma_{ex}^k:|x-y|\leq 4r}\big|\varphi_r*\sigma^k(y)-\varphi_r*\sigma^m(y_m)\big|^2\,d\mathcal{H}^1(y)d\mathcal{H}^1(x)\,\frac{dr}{r^2}$$
$$+\int_{2^{-(k+2)/2}d_0}^{\infty}\int_{x\in\Gamma_{ex}^k}\int_{y\in\Gamma_{ex}^k:|x-y|\leq 4r}\big|\varphi_r*\sigma^m(x_m)-\varphi_r*\sigma^m(y_m)\big|^2\,d\mathcal{H}^1(y)d\mathcal{H}^1(x)\,\frac{dr}{r^2}$$
$$=\text{(1a)}+\text{(1b)}+\text{(2)}.$$

To deal with (1a) observe that

$$\text{(1a)}\lesssim_{A,\tau}\int_{2^{-(k+2)/2}d_0}^{\infty}\int_{x\in\Gamma_{ex}^k}\big|\varphi_r*\sigma^k(x)-\varphi_r*\sigma^m(x_m)\big|^2\,d\mathcal{H}^1(x)\,\frac{dr}{r}.$$

By (14.14) (which also holds with ψ_r replaced by φ_r), we get

$$\text{(1a)}\lesssim_{A,\tau}\sum_{Q\in\mathcal{D}(\Gamma_{ex}^k)}\beta_{\Gamma_{ex}^k,\infty}(Q)^4\,\ell(Q).$$

By Fubini, the integral (1b) coincides with (1a). On the other hand, the estimates for (2) are also analogous to the ones for the term also denoted by (2) in the proof of Lemma 14.3 and so we omit the details again. □

Lemma 14.7. *We have*

$$\int_{\Gamma_{ex}^k}\int_0^\infty \left|\tilde{\tilde{\Delta}}_{\sigma^k,\varphi}(x,r)\right|^2 \frac{dr}{r}\,d\mathcal{H}^1(x) \lesssim_{A,\tau} \sum_{Q\in\mathcal{D}(\Gamma_{ex}^k)} \beta_{\Gamma_{ex}^k,\infty}(Q)^4\,\ell(Q).$$

Sketch of the proof. The arguments are very similar to the ones of the preceding two lemmas. So we will just explain the estimate of the integral

$$\int_{\Gamma_{ex}^k}\int_{2^{-(k+2)/2}d_0}^\infty \tilde{\tilde{\Delta}}_{\sigma^k,\varphi}(x,r)^2\,\frac{dr}{r}\,d\mathcal{H}^1(x).$$

Denote

$$F_r^k(x,y) = r\int \big(\varphi_r(x-z)\,\varphi_{2r}(z-y) - \varphi_{2r}(x-z)\,\varphi_r(z-y)\big)\,d\sigma^k(z),$$

so that

$$\tilde{\tilde{\Delta}}_{\sigma^k,\varphi}(x,r) = \frac{1}{r}\int |F_r^k(x,y)|\,d\sigma^k(y).$$

Note that F_r^k vanishes identically if σ^k coincides with the arc-length measure on some line containing x and y. Observe also that $F_r^k(x,y)=0$ if $|x-y|\geq 6r$. By Cauchy-Schwarz then we have

$$\tilde{\tilde{\Delta}}_{\sigma^k,\varphi}(x,r)^2 \lesssim \frac{1}{r}\int_{y\in\Gamma_{ex}^k:|x-y|\leq 6r} |F_r^k(x,y)|^2\,d\mathcal{H}^1(y).$$

Thus,

$$\begin{aligned}\int_{\Gamma_{ex}^k}\int_{2^{-(k+2)/2}d_0}^\infty &\tilde{\tilde{\Delta}}_{\sigma^k,\varphi}(x,r)^2\,\frac{dr}{r}\,d\mathcal{H}^1(x)\\ &\lesssim \int_{2^{-(k+2)/2}d_0}^\infty \int_{x\in\Gamma_{ex}^k}\int_{y\in\Gamma_{ex}^k:|x-y|\leq 6r} \left|F_r^k(x,y)\right|^2 d\mathcal{H}^1(y)d\mathcal{H}^1(x)\,\frac{dr}{r^2}.\end{aligned}$$

Given $x,y\in\Gamma_{ex}^k$ and $r\geq 2^{-(k+2)/2}d_0$ let $m\leq k$ the maximal integer such that $2^{-(m+2)/2}\,d_0\geq 10r$. As in the proof of Lemma 14.6, consider the points $x_m,x_{m+1},\dots,x_k=x$ such that $x_i\in\Gamma_{ex}^i$ and $\Pi_i(x_i)=x_{i+1}$ for $i=m,\,m+1,\dots,k-1$ and the analogous ones $y_m,y_{m+1},\dots,y_k=y$.

For each i, denote

$$F_{r,a}^i(x_i,y_i) = r\int \varphi_r(x_i-z)\,\varphi_{2r}(z-y_i)\,d\sigma^i(z),$$

$$F_{r,b}^i(x_i,y_i) = r\int \varphi_{2r}(x_i-z)\,\varphi_r(z-y_i)\,d\sigma^i(z),$$

so that $F_r^i(x_i,y_i) = F_{r,a}^i(x_i,y_i) - F_{r,b}^i(x_i,y_i)$. Write

$$\begin{aligned}
&\int_{\Gamma_{ex}^k}\int_{2^{-(k+2)/2}d_0}^\infty \widetilde{\widetilde{\Delta}}_{\sigma^k,\varphi}(x,r)^2\,\frac{dr}{r}\,d\mathcal{H}^1(x)\\
&\quad\lesssim \int_{2^{-(k+2)/2}d_0}^\infty \int_{x\in\Gamma_{ex}^k}\int_{y\in\Gamma_{ex}^k:|x-y|\le 6r}\big|F_{r,a}^k(x,y) - F_{r,a}^m(x_m,y_m)\big|^2\,d\mathcal{H}^1(y)d\mathcal{H}^1(x)\,\frac{dr}{r^2}\\
&\qquad+\int_{2^{-(k+2)/2}d_0}^\infty \int_{x\in\Gamma_{ex}^k}\int_{y\in\Gamma_{ex}^k:|x-y|\le 6r}\big|F_{r,b}^k(x,y) - F_{r,b}^m(x_m,y_m)\big|^2\,d\mathcal{H}^1(y)d\mathcal{H}^1(x)\,\frac{dr}{r^2}\\
&\qquad+\int_{2^{-(k+2)/2}d_0}^\infty \int_{x\in\Gamma_{ex}^k}\int_{y\in\Gamma_{ex}^k:|x-y|\le 6r}\big|F_{r}^m(x_m,y_m)\big|^2\,d\mathcal{H}^1(y)d\mathcal{H}^1(x)\,\frac{dr}{r^2}\\
&\quad= \text{(1a)} + \text{(1b)} + \text{(2)}.
\end{aligned}$$

Estimate of (1a).
By the triangle inequality,

$$\big|F_{r,a}^k(x,y) - F_{r,b}^m(x_m,y_m)\big| \le \sum_{i=m}^{k-1}\big|F_{r,a}^i(x_i,y_i) - F_{r,a}^{i+1}(x_{i+1},y_{i+1})\big|.$$

Since $\sigma^{i+1} = \Pi_{i,\#}\sigma^i$, we have

$$\begin{aligned}
&\big|F_{r,a}^i(x_i,y_i) - F_{r,a}^{i+1}(x_{i+1},y_{i+1})\big|\\
&\quad= r\left|\int \varphi_r(x_i-z)\,\varphi_{2r}(z-y_i)\,d\sigma^i(z) - \int \varphi_r(\Pi_i(x_i)-\Pi_i(z))\,\varphi_{2r}(\Pi_i(z)-\Pi_i(y_i))\,d\sigma^i(z)\right|\\
&\quad\le r\int |\varphi_r(x_i-z) - \varphi_r(\Pi_i(x_i)-\Pi_i(z))|\,\varphi_{2r}(z-y_i)\,d\sigma^i(z)\\
&\qquad+ r\int |\varphi_{2r}(z-y_i) - \varphi_{2r}(\Pi_i(z)-\Pi_i(y_i))|\,\varphi_r(\Pi_i(x_i)-\Pi_i(z))\,d\sigma^i(z).
\end{aligned}$$

As φ_r is supported on $B(0,2r)$ and constant in $B(0,r/2)$, we derive

(14.29)

$$\begin{aligned}
\big|F_{r,a}^i(x_i,y_i) - F_{r,a}^{i+1}(x_{i+1},y_{i+1})\big| &\lesssim \frac{1}{r^2}\int_{c^{-1}r\le|x_i-z|\le 6r}\big||\Pi_i(x_i)-\Pi_i(z)| - |x_i-z|\big|\,d\sigma^k(z)\\
&\quad+\frac{1}{r^2}\int_{c^{-1}r\le|y_i-z|\le 6r}\big||\Pi_i(y_i)-\Pi_i(z)| - |y_i-z|\big|\,d\sigma^k(z).
\end{aligned}$$

Observe the similarities between this estimate and the one of $|\psi_r * \sigma^i(x_i) - \psi_r * \sigma^{i+1}(x_{i+1})|$ in (14.8). Then we obtain

$$\begin{aligned}
\text{(1a)} &\lesssim \int_{2^{-(k+2)/2}d_0}^\infty \int_{x\in\Gamma_{ex}^k}\left|\sum_{i=m}^{k-1}\frac{1}{r^2}\int_{c^{-1}r\le|x_i-z|\le 6r}\big||\Pi_i(x_i)-\Pi_i(z)| - |x_i-z|\big|\,d\sigma^k(z)\right|^2 d\mathcal{H}^1(x)\,\frac{dr}{r}\\
&\quad+\int_{2^{-(k+2)/2}d_0}^\infty \int_{y\in\Gamma_{ex}^k}\left|\sum_{i=m}^{k-1}\frac{1}{r^2}\int_{c^{-1}r\le|y_i-z|\le 6r}\big||\Pi_i(y_i)-\Pi_i(z)| - |y_i-z|\big|\,d\sigma^k(z)\right|^2 d\mathcal{H}^1(y)\,\frac{dr}{r}.
\end{aligned}$$

By Fubini, both terms on the right hand side coincide. Moreover, arguing as in the estimate of the term ① in the proof of Lemma 14.3 it follows that both are bounded by

$$C(A,\tau)\sum_{Q\in\mathcal{D}(\Gamma^k_{ex})}\beta_{\Gamma^k_{ex},\infty}(Q)^4\,\ell(Q).$$

Estimate of (1b).

The arguments are almost the same as the ones for (1a).

Estimate of ②.
We will use the following.

Claim 14.8. *Let $x_m\in L^m_j\subset\Gamma^m$ be such that $B(x_m,6r)\cap L^m_{j+1}\neq\varnothing$. Denote by g^m_j and g^m_{j+1} the constant densities of σ^m on L^m_j and L^m_{j+1} respectively. Then*

$$|F^m_r(x_m,y_m)|\lesssim\measuredangle(\rho^m_j,\rho^m_{j+1})^2+|g^m_j-g^m_{j+1}|. \tag{14.30}$$

The proof is quite similar to the one of Claim 14.5, taking into account that

$$\int_L\big(\varphi_r(x-z)\,\varphi_{2r}(z-y)-\varphi_{2r}(x-z)\,\varphi_r(z-y)\big)\,d\mathcal{H}^1(z)$$

vanishes when L is a line and $x,y\in L$. For the reader's convenience we show the detailed proof below.

Arguing as we did to estimate the term denoted also by ② in the proof of Lemma 14.3, we find that

$$② \lesssim_{A,\tau}\sum_{Q\in\mathcal{D}(\Gamma^k_{ex})}\beta_{\Gamma^k_{ex},\infty}(Q)^4\,\ell(Q).$$

□

Proof of Claim 14.8. To simplify notation, we write

$$f_r(x,y,z)=r\big(\varphi_r(x-z)\,\varphi_{2r}(z-y)-\varphi_{2r}(x-z)\,\varphi_r(z-y)\big),$$

so that

$$F^m_r(x_m,y_m)=\int f_r(x_m,y_m,z)\,d\sigma^m(z).$$

Note that $f_r(x_m,y_m,z)$ vanishes unless $|x_m-y_m|\leq 6r$ and $|x_m-z|\leq 4r$. So to estimate $F^m_r(x_m,y_m)$ we may assume that $x_m\in L^m_j$ and $y_m,z\in L^m_j\cup L^m_{j+1}$. Denote by $\Pi_{\rho^m_j}$ the orthogonal projection on the line ρ^m_j, and let $\sigma^{m\prime}=\Pi_{\rho^m_j,\#}\sigma^m$ and $y_m'=\Pi_{\rho^m_j}(y_m)$. Then we have

$$F^m_r(x_m,y_m)-\int f_r(x_m,y_m',z)\,d\sigma^{m\prime}(z)=\int\big[f_r(x_m,y_m,z)-f_r(x_m,\Pi_{\rho^m_j}(y_m),\Pi_{\rho^m_j}(z))\big]\,d\sigma(z).$$

Arguing as in (14.29), we derive

$$\left|F_r^m(x_m,y_m) - \int f_r(x_m,y_m',z)\,d\sigma^{m\prime}(z)\right| \lesssim \frac{1}{r^2}\int_{c^{-1}r\leq|x_m-z|\leq 6r}\big||x_m-\Pi_{\rho_j^m}(z)|-|x_m-z|\big|\,d\sigma^m(z)$$
$$+\frac{1}{r^2}\int_{c^{-1}r\leq|y_m-z|\leq 6r}\big||\Pi_{\rho_j^m}(y_m)-\Pi_{\rho_j^m}(z)|-|y_m-z|\big|\,d\sigma^m(z).$$

By Pythagoras' theorem it follows easily that

$$\big||x_m-\Pi_{\rho_j^m}(z)|-|x_m-z|\big| \lesssim \frac{|\Pi_{\rho_j^m}(z)-z|^2}{r}\lesssim r\,\measuredangle(\rho_j^m,\rho_{j+1}^m)^2$$

and also

$$\big||\Pi_{\rho_j^m}(y_m)-\Pi_{\rho_j^m}(z)|-|y_m-z|\big| \lesssim \frac{|\Pi_{\rho_j^m}(z)-z|^2}{r}+\frac{|\Pi_{\rho_j^m}(y_m)-y_m|^2}{r}\lesssim r\,\measuredangle(\rho_j^m,\rho_{j+1}^m)^2.$$

Therefore,

$$\left|F_r^m(x_m,y_m) - \int f_r(x_m,y_m',z)\,d\sigma^{m\prime}(z)\right| \lesssim \measuredangle(\rho_j^m,\rho_{j+1}^m)^2. \tag{14.31}$$

On the other hand, it is easy to check that

$$\sigma^{m\prime}|_{B(x_m,6r)\cap L_j^m} = g_j^m\,\mathcal{H}^1|_{B(x_m,6r)\cap L_j^m}$$

and

$$\sigma^{m\prime}|_{B(x_m,6r)\cap\rho_j^m\setminus L_j^m} = \frac{g_{j+1}^m}{\cos\measuredangle(\rho_j^m,\rho_{j+1}^m)}\,\mathcal{H}^1|_{B(x_m,6r)\cap\rho_j^m\setminus L_j^m}.$$

So taking into account that $\int f_r(x_m,y_m',z)\,d\mathcal{H}^1|_{\rho_j^m}(z)=0$, we obtain

$$\left|\int f_r(x_m,y_m',z)\,d\sigma^{m\prime}(z)\right| = \left|\int f_r(x_m,y_m',z)\,d(\sigma^{m\prime}-g_j^m\,\mathcal{H}^1|_{\rho_j^m})(z)\right|$$
$$\leq \int |f_r(x_m,y_m',z)|\left|\frac{g_{j+1}^m}{\cos\measuredangle(\rho_j^m,\rho_{j+1}^m)}-g_j^m\right|d\mathcal{H}^1|_{\rho_j^m}(z).$$

Using that

$$\left|\frac{g_{j+1}^m}{\cos\measuredangle(\rho_j^m,\rho_{j+1}^m)}-g_j^m\right| \lesssim \measuredangle(\rho_j^m,\rho_{j+1}^m)^2+|g_j^m-g_{j+1}^m|,$$

and that $|f_r(x_m,y_m',z)|\lesssim 1/r$ and $\operatorname{supp} f_r(x_m,y_m',\cdot)\subset B(x_m,cr)$, we infer that

$$\left|\int f_r(x_m,y_m',z)\,d\sigma^{m\prime}(z)\right| \lesssim \measuredangle(\rho_j^m,\rho_{j+1}^m)^2+|g_j^m-g_{j+1}^m|,$$

which together with (14.31) proves the claim. □

The following is an immediate consequence of the preceding results.

Lemma 14.9. *We have*

$$\int_{\Gamma_{ex}^k}\int_0^\infty \big|\Delta_{\sigma^k,\varphi}(x,r)\big|^2\,\frac{dr}{r}\,d\mathcal{H}^1(x)\leq C(A,\tau,K)\,\varepsilon_0^2\,\ell(R). \tag{14.32}$$

The analogous estimate holds replacing $\Delta_{\sigma^k,\varphi}(x,r)$ by $\widetilde\Delta_{\sigma^k,\varphi}(x,r)$ or $\widetilde{\widetilde\Delta}_{\sigma^k,\varphi}(x,r)$.

Proof. We have shown in Lemma 14.2 that

$$\int_{\Gamma^k_{ex}} \int_0^\infty \left|\Delta_{\sigma^k,\varphi}(x,r)\right|^2 \frac{dr}{r}\, d\mathcal{H}^1(x) \le C(A,\tau) \sum_{Q\in\mathcal{D}(\Gamma^k_{ex})} \beta_{\Gamma^k_{ex},\infty}(Q)^4\, \ell(Q).$$

Since $\beta_{\Gamma^k_{ex},\infty}(Q) \lesssim \varepsilon_0$ for all $Q \in \mathcal{D}(\Gamma^k_{ex})$, we have

$$\sum_{Q\in\mathcal{D}(\Gamma^k_{ex})} \beta_{\Gamma^k_{ex},\infty}(Q)^4\, \ell(Q) \lesssim \varepsilon_0^2 \sum_{Q\in\mathcal{D}(\Gamma^k_{ex})} \beta_{\Gamma^k_{ex},\infty}(Q)^2\, \ell(Q).$$

On the other hand, by Jones' traveling salesman theorem [Jo], [Ok] , it follows easily that the sum on the right is bounded by $c\,\mathcal{H}^1(\Gamma^k)$, and so by $C(A,\tau,K)\,\ell(R)$. □

15. The $L^2(\sigma^k)$ norm of the density of ν^k with respect to σ^k

Recall that both ν^k and σ^k are AD-regular measures supported on Γ^k. In particular, they are mutually absolutely continuous with respect to $\mathcal{H}^1|_{\Gamma^k}$ and thus there exists some function f_k bounded above and away from zero such that $\nu^k = f_k\, \sigma^k$. By Lemma 12.3, the density of ν^k with respect to $\mathcal{H}^1|_{\Gamma^k_{ex}}$ satisfies

$$\frac{d\nu^k}{d\mathcal{H}^1|_{\Gamma^k_{ex}}} \approx_{A,\tau} \Theta_\mu(B_R), \tag{15.1}$$

and by Lemmas 14.1 and 14.2,

$$\frac{d\sigma^k}{d\mathcal{H}^1|_{\Gamma^k_{ex}}} \approx_M 1. \tag{15.2}$$

So we have

$$f_k = \frac{d\nu^k}{d\sigma^k} \approx_{A,\tau,M} \Theta_\mu(B_R).$$

Recall also that the density $\frac{d\nu^k}{d\mathcal{H}^1|_{\Gamma^k_{ex}}}$ is constantly equal to c_0^k far away from $B(x_0,Kr_0)$. Analogously, σ^k coincides $\mathcal{H}^1|_{\Gamma^k_{ex}}$ out of $B(x_0,Kr_0)$. So $f_k - c_0^k$ is compactly supported. The main objective of this section consists in estimating the L^2 norm of $f_k - c_0^k$ with respect to σ^k. To this end, for $r>0$, $x\in\mathbb{R}^d$, and a function $g \in L^1_{loc}(\sigma^k)$ we define

$$D_r g(x) = \frac{\varphi_r * (g\sigma^k)(x)}{\varphi_r * \sigma^k(x)} - \frac{\varphi_{2r} * (g\sigma^k)(x)}{\varphi_{2r} * \sigma^k(x)}$$

(recall that $\varphi_r(y) = \frac{1}{r}\,\varphi\Big(\frac{|y|}{r}\Big)$).

Lemma 15.1. *For $f_k = \dfrac{d\nu^k}{d\sigma^k}$, $r>0$, and $x \in \Gamma^k_{ex}$, we have*

$$\int_{\Gamma^k_{ex}} \int_0^\infty |D_r f_k|^2 \frac{dr}{r}\, d\sigma^k \lesssim_{A,\tau,K,M} \varepsilon_0^{1/10}\, \Theta_\mu(B_R)^2\, \ell(R).$$

Proof. By Lemmas 13.8 and 14.9, it is enough to show that

$$|D_r f_k(x)| \lesssim_{A,\tau,M} |\psi_r * \nu^k(x)| + \Theta_\mu(B_R)\, |\psi_r * \sigma^k(x)|. \tag{15.3}$$

Then we write

$$\begin{aligned}|D_r f_k(x)| &= \left|\frac{\varphi_r * \nu^k(x)}{\varphi_r * \sigma^k(x)} - \frac{\varphi_{2r} * \nu^k(x)}{\varphi_{2r} * \sigma^k(x)}\right| \\ &= \left|\frac{\varphi_r * \nu^k(x)\, \varphi_{2r} * \sigma^k(x) - \varphi_{2r} * \nu^k(x)\, \varphi_r * \sigma^k(x)}{\varphi_r * \sigma^k(x)\, \varphi_{2r} * \sigma^k(x)}\right| \\ &\leq \left|\frac{\varphi_r * \nu^k(x) - \varphi_{2r} * \nu^k(x)}{\varphi_r * \sigma^k(x)}\right| + \left|\frac{\big(\varphi_{2r} * \sigma^k(x) - \varphi_r * \sigma^k(x)\big)\varphi_{2r} * \nu^k(x)}{\varphi_r * \sigma^k(x)\, \varphi_{2r} * \sigma^k(x)}\right|.\end{aligned}$$

The inequality (15.3) just follows then from (15.1) and (15.2), which imply that $\varphi_r * \sigma^k(x) \gtrsim_M 1$, $\varphi_{2r} * \sigma^k(x) \gtrsim_M 1$, and $\varphi_{2r} * \nu^k(x) \lesssim_{A,\tau} \Theta_\mu(B_R)$. □

Notice that the operators D_r vanish on constant functions. That is, $D_r 1 \equiv 0$. In order to apply some quasiorthogonality arguments, we would also need their adjoints to satisfy $D_r^* 1 \equiv 0$. Unfortunately this property is not fulfilled. For this reason, we are going to introduce a variant of the operator D_r which we will denote by $\widetilde{D}_r$ that will be better suited for the quasiorthogonality techniques we intend to apply.

For a function $g \in L^1_{loc}(\sigma^k)$ and $r > 0$, we denote

$$S_r g(x) = \frac{\varphi_r * (g\sigma^k)(x)}{\varphi_r * \sigma^k(x)},$$

so that $D_r g = S_r g - S_{2r} g$. Let W_r be the operator of multiplication by $1/S_r^* 1$. Then we consider the operator

$$\widetilde{S}_r = S_r\, W_r\, S_r^*,$$

and we define $\widetilde{D}_r = \widetilde{S}_r - \widetilde{S}_{2r}$. Notice that $\widetilde{S}_r$, and thus $\widetilde{D}_r$, is self-adjoint. Moreover $\widetilde{S}_r 1 \equiv 1$, so that

$$\widetilde{D}_r 1 = \widetilde{D}_r^* 1 = 0.$$

We denote by $s_r(x,y)$ the kernel of S_r with respect to σ^k. That is, $s_r(x,y)$ is the function such that $S_r g(x) = \int s_r(x,y)\, g(y)\, d\sigma^k(y)$. Observe that this equals

$$s_r(x,y) = \frac{1}{\varphi_r * \sigma^k(x)}\, \varphi_r(x-y).$$

On the other hand, the kernel of $\widetilde{S}_k$ is the following:

$$\widetilde{s}_r(x,y) = \int s_r(x,z)\, \frac{1}{S_r^* 1(z)}\, s_r(y,z)\, d\sigma^k(z).$$

The following is, by now, a standard result from Littlewood-Paley theory in homogeneous spaces due to David, Journé, and Semmes.

Theorem 15.2. [DJS] *Let* $1 < r_0 < 2$ *and let* $g \in L^2(\sigma^k)$. *Then*

$$\|g\|^2_{L^2(\sigma^k)} \approx_{A,\tau,M} \int_{\Gamma^k_{ex}} \sum_{j\in\mathbb{Z}} |\widetilde{D}_{2^{-j}r_0} g|^2\, d\sigma^k. \tag{15.4}$$

Note that the constants involved in the estimate (15.4) do not depend on r_0. An easy consequence is the following.

Lemma 15.3. *For $g \in L^2(\sigma^k)$, we have*

$$\|g\|^2_{L^2(\sigma^k)} \approx_{A,\tau,M} \int_{\Gamma^k_{ex}} \int_0^\infty |\widetilde{D}_r g|^2 \, \frac{dr}{r} d\sigma^k.$$

Proof. Just notice that

$$\int_{\Gamma^k_{ex}} \int_0^\infty |\widetilde{D}_r g|^2 \, \frac{dr}{r} d\sigma^k = \int_{\Gamma^k_{ex}} \int_1^2 \sum_{j\in\mathbb{Z}} |\widetilde{D}_{r2^{-j}} g|^2 \, \frac{dr}{r} \, d\sigma^k,$$

and then use Fubini and (15.4). □

Lemma 15.4. *Let $f \in L^\infty(\sigma^k)$. The following estimates hold:*

$$\int_{\Gamma^k_{ex}} \int_0^\infty |S_r f - S^*_r f|^2 \, \frac{dr}{r} \, d\sigma^k \lesssim_{A,\tau,K,M} \varepsilon_0^2 \, \|f\|^2_{L^\infty(\sigma^k)} \, \ell(R), \tag{15.5}$$

and

$$\int_{\Gamma^k_{ex}} \int_0^\infty |S_r \, S_{2r} f - S_{2r} \, S_r f|^2 \, \frac{dr}{r} \, d\sigma^k \lesssim_{A,\tau,K,M} \varepsilon_0^2 \, \|f\|^2_{L^\infty(\sigma^k)} \, \ell(R). \tag{15.6}$$

Proof. To see the first estimate, we write

$$\begin{aligned} |S_r f(x) - S^*_r f(x)| &= \left| \int_{\Gamma^k_{ex}} \left(\frac{1}{\varphi_r * \sigma^k(x)} - \frac{1}{\varphi_r * \sigma^k(y)} \right) \varphi_r(x-y) \, f(y) \, d\sigma^k(y) \right| \\ &\lesssim_{A,\tau,M} \|f\|_{L^\infty(\sigma^k)} \int_{\Gamma^k_{ex}} \left|\varphi_r * \sigma^k(x) - \varphi_r * \sigma^k(y)\right| \varphi_r(x-y) \, d\sigma^k(y). \end{aligned}$$

Since $\operatorname{supp} \varphi_r(x - \cdot) \subset B(x, 2r)$, the last integral is bounded by

$$\frac{c}{r} \int_{|x-y|\leq 2r} \left|\varphi_r * \sigma^k(x) - \varphi_r * \sigma^k(y)\right| d\sigma^k(y) \leq c \, \widetilde{\Delta}_{\sigma^k,\varphi}(x, r).$$

By Lemma 14.9, we derive

$$\int_{\Gamma^k_{ex}} \int_0^\infty \left|\widetilde{\Delta}_{\sigma^k,\varphi}(x, r)\right|^2 \, \frac{dr}{r} \, d\mathcal{H}^1(x) \lesssim_{A,\tau,K,M} \varepsilon_0^2 \, \|f\|^2_{L^\infty(\sigma^k)} \, \Theta_\mu(B_R)^2 \, \ell(R),$$

and thus (15.5) follows.

To prove (15.6) we write

$$\begin{aligned} S_r \, S_{2r} f(x) - S_{2r} \, S_r f(x) &= \iint \frac{1}{\varphi_r * \sigma^k(x) \, \varphi_{2r} * \sigma^k(z)} \, \varphi_r(x-z) \, \varphi_{2r}(z-y) \, f(y) \, d\sigma^k(z) \, d\sigma^k(y) \\ &\quad - \iint \frac{1}{\varphi_{2r} * \sigma^k(x) \, \varphi_r * \sigma^k(z)} \, \varphi_{2r}(x-z) \, \varphi_r(z-y) \, f(y) \, d\sigma^k(z) \, d\sigma^k(y) \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1=\iint\left[\frac{1}{\varphi_r*\sigma^k(x)\,\varphi_{2r}*\sigma^k(z)}-\frac{1}{\varphi_{2r}*\sigma^k(x)\,\varphi_r*\sigma^k(z)}\right]\varphi_r(x-z)\,\varphi_{2r}(z-y)f(y)d\sigma^k(z)d\sigma^k(y),$$

$$I_2=\iint\left[\frac{1}{\varphi_{2r}*\sigma^k(x)\,\varphi_r*\sigma^k(z)}-\frac{1}{\varphi_r*\sigma^k(x)\,\varphi_{2r}*\sigma^k(x)}\right]$$
$$\cdot\left[\varphi_r(x-z)\,\varphi_{2r}(z-y)-\varphi_{2r}(x-z)\,\varphi_r(z-y)\right]f(y)\,d\sigma^k(z)d\sigma^k(y),$$

$$I_3=\frac{1}{\varphi_r*\sigma^k(x)\,\varphi_{2r}*\sigma^k(x)}\iint\left[\varphi_r(x-z)\,\varphi_{2r}(z-y)-\varphi_{2r}(x-z)\varphi_r(z-y)\right]f(y)\sigma^k(z)d\sigma^k(y).$$

To estimate I_1 we set

$$\left|\frac{1}{\varphi_r*\sigma^k(x)\,\varphi_{2r}*\sigma^k(z)}-\frac{1}{\varphi_{2r}*\sigma^k(x)\,\varphi_r*\sigma^k(z)}\right|\leq\frac{\left|\varphi_{2r}*\sigma^k(x)-\varphi_r*\sigma^k(x)\right|}{\varphi_r*\sigma^k(x)\,\varphi_{2r}*\sigma^k(x)\,\varphi_{2r}*\sigma^k(z)}$$
$$+\frac{\left|\varphi_r*\sigma^k(z)-\varphi_{2r}*\sigma^k(z)\right|}{\varphi_{2r}*\sigma^k(x)\,\varphi_r*\sigma^k(z)\,\varphi_{2r}*\sigma^k(z)}$$
$$\lesssim_{A,\tau,M}|\psi_r*\sigma^k(x)|+|\psi_r*\sigma^k(z)|.$$

Then we obtain

$$|I_1|\lesssim_{A,\tau,M}\|f\|_{L^\infty(\sigma^k)}\left[|\psi_r*\sigma^k(x)|+\varphi_r*(|\psi_r*\sigma^k|\sigma^k)(x)\right].$$

So writing $I_1\equiv I_{1,r}(x)$, we have

$$\|I_{1,r}\|_{L^2(\sigma^k)}\lesssim_{A,\tau,M}\|f\|_{L^\infty(\sigma^k)}\,\|\psi_r*\sigma^k\|_{L^2(\sigma^k)}. \tag{15.7}$$

Concerning I_2, we have

$$\left|\frac{1}{\varphi_{2r}*\sigma^k(x)\,\varphi_r*\sigma^k(z)}-\frac{1}{\varphi_r*\sigma^k(x)\,\varphi_{2r}*\sigma^k(x)}\right|=\frac{\left|\varphi_r*\sigma^k(x)-\varphi_r*\sigma^k(z)\right|}{\varphi_r*\sigma^k(x)\,\varphi_r*\sigma^k(z)\,\varphi_{2r}*\sigma^k(z)}$$
$$\lesssim_{A,\tau,M}\left|\varphi_r*\sigma^k(x)-\varphi_r*\sigma^k(z)\right|.$$

Notice that if z,y belong to the domain of integration of I_2, then $|x-z|\leq 4r$ and $|y-z|\leq 6r$. So we can write

$$\begin{aligned}|I_2|&\lesssim_{A,\tau,M}\frac{1}{r^2}\int_{|x-z|\leq 4r}\left|\varphi_r*\sigma^k(x)-\varphi_r*\sigma^k(z)\right|d\sigma^k(z)\int_{|x-y|\leq 6r}|f(y)|\,d\sigma^k(y)\\&\lesssim_{A,\tau,M}\|f\|_{L^\infty(\sigma^k)}\frac{1}{r}\int_{|x-z|\leq 4r}\left|\varphi_r*\sigma^k(x)-\varphi_r*\sigma^k(z)\right|d\sigma^k(z)\\&=_{A,\tau,M}\|f\|_{L^\infty(\sigma^k)}\,\widetilde{\Delta}_{\sigma^k,\varphi}(x,r).\end{aligned}$$

To deal with I_3 we just write

$$\begin{aligned}I_3&\lesssim_{A,\tau,M}\|f\|_{L^\infty(\sigma^k)}\int\left|\int\left[\varphi_r(x-z)\,\varphi_{2r}(z-y)-\varphi_{2r}(x-z)\varphi_r(z-y)\right]\sigma^k(z)\right|d\sigma^k(y)\\&=_{A,\tau,M}\|f\|_{L^\infty(\sigma^k)}\,\widetilde{\widetilde{\Delta}}_{\sigma^k,\varphi}(x,r).\end{aligned}$$

Gathering the estimates obtained for I_1, I_2 and I_3 and applying Lemma 14.9 we obtain

$$\int_{\Gamma^k_{ex}}\int_0^\infty|S_r\,S_{2r}f(x)-S_{2r}\,S_rf(x)|^2\,\frac{dr}{r}\,d\mathcal{H}^1(x)\leq_{A,\tau,K,M}\varepsilon_0^2\,\|f\|^2_{L^\infty(\sigma^k)}\,\ell(R).$$

□

Lemma 15.5. *For* $f_k = \dfrac{d\nu^k}{d\sigma^k}$*, we have*

$$\|f_k - c_0^k\|_{L^2(\sigma^k)}^2 \lesssim_{A,\tau,K,M} \varepsilon_0^{1/10}\,\Theta_\mu(B_R)^2\,\ell(R). \tag{15.8}$$

Proof. As explained at the beginning of this section, the function $f_k - c_0^k$ is compactly supported and moreover it is bounded. Thus it belongs to $L^2(\sigma^k)$. Then by Lemma 15.3,

$$\|f_k - c_0^k\|_{L^2(\sigma^k)}^2 \approx_{A,\tau,M} \int_{\Gamma_{ex}^k}\int_0^\infty |\widetilde{D}_r(f_k - c_0^k)|^2\,\frac{dr}{r}\,d\sigma^k.$$

Since $\widetilde{D}_r$ vanishes on constant functions, it turns out that $\widetilde{D}_r(f_k - c_0^k) = \widetilde{D}_r f_k$. Thus, using also Lemma 15.1, to prove (15.8) it suffices to show that

$$\int_{\Gamma_{ex}^k}\int_0^\infty |\widetilde{D}_r f_k|^2\,\frac{dr}{r}\,d\sigma^k \lesssim_{A,\tau,K,M} \int_{\Gamma_{ex}^k}\int_0^\infty |D_r f_k|^2\,\frac{dr}{r}\,d\sigma^k + \varepsilon_0^2\,\|f_k\|_{L^\infty(\sigma^k)}^2\,\ell(R). \tag{15.9}$$

We are going to show that (15.9) holds for any function $f \in L^\infty(\sigma^k)$. To this end, recall that $\widetilde{D}_r = \widetilde{S}_r - \widetilde{S}_{2r}$ and $\widetilde{S}_r = S_r\,W_r\,S_r^*$, where W_r is the operator of multiplication by $1/S_r^*1$. Note that for any $x \in \mathbb{R}^d$,

$$|\widetilde{S}_r f(x) - S_r\,S_r^* f(x)| = \Big|S_r\Big(\Big(\frac{1}{S_r^*1} - 1\Big)\,S_r^* f\Big)(x)\Big|.$$

Since, for any $y \in \mathbb{R}^d$, $|\frac{1}{S_r^*1(y)} - 1| \lesssim_{A,\tau,M} |S_r^*1(y) - 1|$ and $|S_r^* f(y)| \lesssim \|f\|_{L^\infty(\sigma^k)}$, we get

$$|\widetilde{S}_r f(x) - S_r\,S_r^* f(x)| \lesssim_{A,\tau,M} \|f\|_{L^\infty(\sigma^k)}\,\big|S_r\big(|S_r^*1 - 1|\big)(x)\big|.$$

As S_r is bounded in $L^2(\sigma^k)$ uniformly on r, we obtain

$$\|\widetilde{S}_r f - S_r\,S_r^* f\|_{L^2(\sigma^k)} \lesssim_{A,\tau,M} \|f\|_{L^\infty(\sigma^k)}\,\|S_r^*1 - 1\|_{L^2(\sigma^k)}.$$

Applying (15.5) to $f = 1$, taking into account that $S_r 1 \equiv 1$, we deduce that

$$\begin{aligned}\int_{\Gamma_{ex}^k}\int_0^\infty |\widetilde{S}_r f - S_r\,S_r^* f|^2\,\frac{dr}{r}\,d\sigma^k &\lesssim_{A,\tau,M} \|f\|_{L^\infty(\sigma^k)}^2 \int_{\Gamma_{ex}^k}\int_0^\infty |S_r^*1 - S_r 1|^2\,\frac{dr}{r}\,d\sigma^k\\ &\lesssim_{A,\tau,K,M} \varepsilon_0^2\,\|f\|_{L^\infty(\sigma^k)}^2\,\ell(R).\end{aligned}$$

So we infer that

$$\int_{\Gamma_{ex}^k}\int_0^\infty |\widetilde{D}_r f - (S_r\,S_r^* f - S_{2r}\,S_{2r}^* f)|^2\,\frac{dr}{r}\,d\sigma^k \lesssim_{A,\tau,K,M} \varepsilon_0^2\,\|f\|_{L^\infty(\sigma^k)}^2\,\ell(R).$$

As a consequence, to prove (15.9) for $f_k = f$ it is enough to show that

$$\int_{\Gamma_{ex}^k}\int_0^\infty |S_r\,S_r^* f - S_{2r}\,S_{2r}^* f|^2\,\frac{dr}{r}\,d\sigma^k \lesssim_{A,\tau,K,M} \int_{\Gamma_{ex}^k}\int_0^\infty |D_r f|^2\,\frac{dr}{r}\,d\sigma^k + \varepsilon_0^2\,\|f\|_{L^\infty(\sigma^k)}^2\,\ell(R). \tag{15.10}$$

We write

$$\begin{aligned}&\|S_r\,S_r^*f-S_{2r}\,S_{2r}^*f\|_{L^2(\sigma^k)}\\&\qquad\le\|S_r\,S_rf-S_{2r}\,S_{2r}f\|_{L^2(\sigma^k)}+\|S_r\,S_rf-S_r\,S_r^*f\|_{L^2(\sigma^k)}+\|S_{2r}\,S_{2r}f-S_{2r}\,S_{2r}^*f\|_{L^2(\sigma^k)}\\&\qquad\lesssim_{A,\tau,M}\|S_r\,S_rf-S_{2r}\,S_{2r}f\|_{L^2(\sigma^k)}+\|S_rf-S_r^*f\|_{L^2(\sigma^k)}+\|S_{2r}f-S_{2r}^*f\|_{L^2(\sigma^k)}.\end{aligned}$$

To estimate the last two terms on the right side we will use (15.5). For the first one we set

$$\begin{aligned}&\|S_r\,S_rf-S_{2r}\,S_{2r}f\|_{L^2(\sigma^k)}\\&\qquad\le\|S_r\,S_rf-S_r\,S_{2r}f\|_{L^2(\sigma^k)}+\|S_r\,S_{2r}f-S_{2r}\,S_rf\|_{L^2(\sigma^k)}+\|S_{2r}\,S_rf-S_{2r}\,S_{2r}f\|_{L^2(\sigma^k)}\\&\qquad\lesssim_{A,\tau,M}\|S_rf-S_{2r}f\|_{L^2(\sigma^k)}+\|S_r\,S_{2r}f-S_{2r}\,S_rf\|_{L^2(\sigma^k)}+\|S_rf-S_{2r}f\|_{L^2(\sigma^k)},\end{aligned}$$

because of the $L^2(\sigma^k)$ boundedness of S_r and S_{2r}. Thus,

$$\begin{aligned}&\int_{\Gamma_{ex}^k}\int_0^\infty|S_r\,S_r^*f-S_{2r}\,S_{2r}^*f|^2\,\frac{dr}{r}\,d\sigma^k\\&\qquad\lesssim_{A,\tau,M}\int_0^\infty\|S_r\,S_{2r}f-S_{2r}\,S_rf\|_{L^2(\sigma^k)}^2\,\frac{dr}{r}\\&\qquad\quad+\int_0^\infty\big(\|S_rf-S_r^*f\|_{L^2(\sigma^k)}^2+\|S_{2r}f-S_{2r}^*f\|_{L^2(\sigma^k)}^2\big)\,\frac{dr}{r}+\int_0^\infty\|S_rf-S_{2r}f\|_{L^2(\sigma^k)}^2\,\frac{dr}{r}.\end{aligned}$$

By Lemma 15.4, the first and second integrals on the right hand side do not exceed

$$C(A,\tau,K,M)\,\varepsilon_0^2\,\|f\|_{L^\infty(\sigma^k)}^2\,\ell(R),$$

while the last one equals

$$\int_{\Gamma_{ex}^k}\int_0^\infty|D_rf|^2\,\frac{dr}{r}\,d\sigma^k.$$

So (15.9) is proved for any $f\in L^\infty(\sigma^k)$ and consequently the lemma follows. □

16. The end of the proof of the Main Lemma 5.1

In this section first we will show that the measure of the union of the cells from HD which are contained in R is small. The estimate of the $L^2(\sigma^k)$ norm of $f_k-c_0^k$ will play a key role in the arguments. Afterwards we will finish the proof of the Main Lemma.

First we show a technical result:

Lemma 16.1. *Let $Q\in\mathsf{NTerm}$. There exists some cell $P\in\mathsf{NReg}$ such that*

$$P\cap 1.1B_Q\neq\varnothing,\qquad \ell(P)\approx\ell(Q),\qquad \mu(P)\gtrsim\mu(1.1B_Q). \tag{16.1}$$

If moreover $Q\in\mathsf{HD}$, then

$$\widetilde\mu(P)\approx\mu(P)\approx\mu(1.1B_Q),$$

assuming η small enough.

Proof. By Lemma 9.1, it follows easily that any cell $S\in\mathsf{NReg}$ with $S\cap 1.1B_Q$ satisfies $\ell(S)\lesssim\ell(Q)$. Let $0<t<1/100$ be some constant to be fixed below. Suppose first that all the cells $S\in\mathsf{NReg}$ which intersect $1.1B_Q$ satisfy $\ell(S)\geq t\,\ell(Q)$ and that $NW_0\cap 1.1B_Q=\varnothing$. In this case, the number of such cells is bounded above by some constant depending only on t, and so if we let P be a cell of this family with maximal μ-measure, then we have

$$\mu(P) \gtrsim_t \mu(1.1B_Q) \qquad \text{and} \qquad \ell(P) \approx_t \ell(Q). \tag{16.2}$$

Suppose now that there exists some cell $S \in \mathsf{NReg}$ which intersects $1.1B_Q$ such that $\ell(S) < t\,\ell(Q)$, or that $NW_0 \cap 1.1B_Q \neq \varnothing$. We claim that this implies that $Q \in \mathcal{D}^{db}$. To prove this, note that if there exists a cell S with $S \cap 1.1B_Q \neq \varnothing$ and $\ell(S) < t\,\ell(Q)$, then $S \subset 1.2B_Q$, and taking a suitable ancestor of S we infer that there exists some cell $S' \in \mathsf{NGood}$ with $\ell(S') \approx \ell(Q)$, $\operatorname{dist}(S', S) \lesssim \ell(S')$. The same holds in the case when $NW_0 \cap 1.1B_Q \neq \varnothing$. Further, if t is small enough, then we can assume that $3.3B_{S'} \subset 1.3B_Q$. Let $a \geq 3.3$ be the maximal number such that $a\,B_{S'} \subset 1.5B_Q$. Notice that $r(a\,B_{S'}) \geq r(1.5B_Q) - r(1.3B_Q) = 0.2\,r(B_Q)$. Since $100^2B(Q)$ is contained in $c\,a\,B_{S'}$ for some constant $c \lesssim 1$ (independent of C_0), by Remark 6.3 we deduce that

$$\mu(100^2B(Q)) \leq c_{13}\,\mu(a\,B_{S'}) \leq c_{13}\,\mu(1.5B_Q) = c_{13}\,\mu(1.5\cdot 28B(Q)) \leq c_{13}\,\mu(100B(Q)),$$

with c_{13} independent of C_0. Then (4.3) does not hold for Q if C_0 is taken big enough, which ensures that $Q \in \mathcal{D}^{db}$ as claimed.

The fact that $Q \in \mathcal{D}^{db}$ guaranties that $\mu(B(Q)) \approx \mu(1.1B_Q)$, by (4.2). Using also the small boundaries condition (4.1) we infer that, for some l big enough, the set $G(Q) = B(Q) \setminus N_l(Q)$ has μ-measure comparable to $\mu(B(Q))$, and so to $\mu(1.1B_Q)$. Since $Q \notin \mathsf{NGood}$, from the definitions of $\widetilde{d}(\cdot)$ and NReg, it follows easily that any cell $S \in \mathsf{NReg}$ which intersects $G(Q)$ satisfies $\ell(S) \approx \ell(Q)$. Thus letting P be a cell from NReg with $P \cap G(Q) \neq \varnothing$ having maximal μ-measure, as in (16.2) we deduce that

$$\mu(P) \gtrsim \mu(G(Q)) \approx \mu(1.1B_Q) \qquad \text{and} \qquad \ell(P) \approx \ell(Q).$$

It remains now to show that if $Q \in \mathsf{HD} \cap \mathsf{NTerm}$, then $\widetilde{\mu}(P) \approx \mu(P) \approx \mu(1.1B_Q)$. In this case, $\mu(1.1B_Q) \gtrsim A\,\Theta_\mu(R)\,\ell(Q)$, and thus

$$\mu(P) \lesssim A\,\Theta_\mu(B_R)\,\ell(P) \approx A\,\Theta_\mu(B_R)\,\ell(Q) \lesssim \mu(1.1B_Q),$$

and so $\mu(P) \approx \mu(1.1B_Q)$. To prove that $\widetilde{\mu}(P) \approx \mu(P)$, let $\widehat{Q}$ be the parent of Q and let $c_{14} > 0$ be such that $P \subset c_{14}B_{\widehat{Q}}$. Since $\widehat{Q} \in \mathsf{Good}$, by Lemma 7.3 we have $\mu(c_{14}B_{\widehat{Q}} \setminus \widetilde{E}) \leq \eta^{1/10}\,\mu(c_{14}B_{\widehat{Q}})$, and thus

$$\mu(P\setminus\widetilde{E}) \leq \mu(c_{14}B_{\widehat{Q}}\setminus\widetilde{E}) \leq \eta^{1/10}\,\mu(c_{14}B_{\widehat{Q}}) \lesssim \eta^{1/10}\,A\,\Theta_\mu(B_R)\ell(Q) \lesssim \eta^{1/10}\,\mu(1.1B_Q) \lesssim \eta^{1/10}\,\mu(P),$$

which ensures that $\widetilde{\mu}(P) = \mu(P \cap \widetilde{E}) \approx \mu(P)$ for η small enough. □

Lemma 16.2. *We have*

$$\mu\Bigg(\bigcup_{Q\in\mathsf{HD}\cap\mathcal{D}(R)} Q\Bigg) \lesssim \bigg(\frac{c(A,\tau,K)}{M} + c(A,\tau,K,M)\,\varepsilon_0^{1/10}\bigg)\,\mu(R).$$

Proof. Notice that

$$\mu\Bigg(\bigcup_{Q\in\mathsf{HD}\cap\mathcal{D}(R)} Q\Bigg) \leq \mu\Bigg(\bigcup_{Q\in\mathsf{BS}\beta\cap\mathcal{D}(R)} Q\Bigg) + \mu\Bigg(\bigcup_{Q\in\mathsf{HD}\cap\mathsf{NTerm}\cap\mathcal{D}(R)} Q\Bigg).$$

By Lemma 11.1, the first term on the right side does not exceed $\frac{c(A,\tau,K)}{M}\,\mu(R)$. So it is enough to show that

$$\mu\biggl(\bigcup_{Q\in\mathsf{HD}_1} Q\biggr) \leq c(A,\tau,K,M)\,\varepsilon_0^{1/10}\,\mu(R), \tag{16.3}$$

where

$$\mathsf{HD}_1 = \mathsf{HD}\cap\mathsf{NTerm}\cap\mathcal{D}(R).$$

Consider a cell $Q\in\mathsf{HD}_1$. We wish to relate the measure μ on Q to the measure ν^k on some appropriate ball $B^k_{j(Q)}$. To this end, let $P=P(Q)\in\mathsf{NReg}$ be the cell satisfying (16.1). Suppose that k is big enough so that

$$2^{-k/2}\,d_0 \leq \ell(P(Q)). \tag{16.4}$$

By Lemma 10.7 we know that $\widetilde{\mu}$-almost all $P(Q)$ is contained in the union of the balls B^k_j, $j=1,\ldots,N_k$, and by Lemma 6.6 (a), the balls B^k_j which intersect P have radii comparable to $\ell(P(Q))$. Thus the number of such balls does not exceed some absolute constant. So letting $B^k_{j(Q)}$ be the ball of this family which has maximal $\widetilde{\mu}$-measure, it turns out that

$$\widetilde{\mu}(B^k_{j(Q)}) \approx \widetilde{\mu}(P(Q)) \approx \mu(1.1B_Q) \gtrsim A\,\Theta_\mu(B_R)\,\ell(Q) \approx A\,\Theta_\mu(B_R)\,r(B^k_{j(Q)}).$$

Recall now that $\nu^k=\sum_{j=0}^{N_k}\nu^k_j$, with $\operatorname{supp}\nu^k_j\subset\frac32\bar{B}^k_j$ for $j\in[1,N_k]$. Further, if $\frac32\bar{B}^k_j\cap\bar{B}^k_{j(Q)}\neq\varnothing$, by Lemma 10.1 (d), we have $r(B^k_j)=\ell^k_j\approx\ell^k_{j(Q)}=r(B^k_{j(Q)})$ and thus $\frac32\bar{B}^k_j\subset c_{15}B^k_{j(Q)}$ for some absolute constant c_{15}. So we have

$$\nu^k\bigl(c_{15}B^k_{j(Q)}\bigr) \geq \sum_{j:\frac32\bar{B}^k_j\cap\bar{B}^k_{j(Q)}\neq\varnothing}\|\nu^k_j\|.$$

Since $\|\nu^k_j\|=\int\theta^k_j\,d\mu$ (see (12.2)) and by Lemma 12.1

$$\sum_{j:\frac32\bar{B}^k_j\cap\bar{B}^k_{j(Q)}\neq\varnothing}\theta^k_j \geq \chi_{B^k_{j(Q)}},$$

we infer that

$$\nu^k\bigl(c_{15}B^k_{j(Q)}\bigr) \geq \sum_{j:\frac32\bar{B}^k_j\cap\bar{B}^k_{j(Q)}\neq\varnothing}\int\theta^k_j\,d\mu \geq \widetilde{\mu}(B^k_{j(Q)}) \approx A\,\Theta_\mu(B_R)\,r(B^k_{j(Q)}).$$

From the preceding estimate, taking into account that $c^k_0\approx\Theta_\mu(B_R)$ (see Remark 13.6) and that σ^k has linear growth with an absolute constant (see Lemma 14.1), we obtain

$$\begin{aligned}\int_{c_{15}B^k_{j(Q)}}|f_k-c^k_0|\,d\sigma^k &\geq \nu^k(c_{15}B^k_{j(Q)}) - c^k_0\,\sigma^k(c_{15}B^k_{j(Q)})\\ &\geq c\,A\,\Theta_\mu(B_R)\,r(B^k_{j(Q)}) - c'\,\Theta_\mu(B_R)\,r(B^k_{j(Q)}) \geq \Theta_\mu(B_R)\,r(B^k_{j(Q)}),\end{aligned}$$

assuming A big enough. By Cauchy-Schwarz and the linear growth of σ^k, the left hand side is bounded above by $c\,\|f_k-c^k_0\|_{L^2(\sigma^k)}\,r(B^k_{j(Q)})^{1/2}$. Then for some constant $c_{16}>1$ big enough so that $c_{15}B^k_{j(Q)}\subset c_{16}B_Q$ we get

(16.5)

$$\|\chi_{c_{16}B_Q}\,(f_k-c^k_0)\|^2_{L^2(\sigma^k)} \geq \|\chi_{c_{15}B^k_{j(Q)}}\,(f_k-c^k_0)\|^2_{L^2(\sigma^k)} \gtrsim \Theta_\mu(B_R)^2\,r(B^k_{j(Q)}) \gtrsim_{A,\tau}\;\Theta_\mu(B_R)\,\mu(1.1B_Q).$$

Consider now a finite family $\mathsf{HD}_2 \subset \mathsf{HD}_1$ such that

$$\mu\Big(\bigcup_{Q\in\mathsf{HD}_2} Q\Big) \geq \frac{1}{2}\,\mu\Big(\bigcup_{Q\in\mathsf{HD}_1} Q\Big), \tag{16.6}$$

and take k big enough so that (16.4) holds for all the cells $P(Q)$ associated to any $Q \in \mathsf{HD}_2$ as explained above. Consider a subfamily $\mathsf{HD}_3 \subset \mathsf{HD}_2$ such that the balls $\{c_{16}B_Q\}_{Q\in\mathsf{HD}_3}$ are pairwise disjoint and

$$\bigcup_{Q\in\mathsf{HD}_2} c_{16}B_Q \subset \bigcup_{Q\in\mathsf{HD}_3} 3c_{16}B_Q.$$

Taking into account that $\mu(3c_{16}B_Q) \lesssim A\,\Theta_\mu(B_R)\,\ell(Q) \lesssim \mu(1.1B_Q)$ and using (16.6), (16.5) and (15.8), we get

$$\begin{aligned}\mu\Big(\bigcup_{Q\in\mathsf{HD}_1} Q\Big) &\leq 2\,\mu\Big(\bigcup_{Q\in\mathsf{HD}_2} c_{16}B_Q\Big) \leq 2\sum_{Q\in\mathsf{HD}_3}\mu(3c_{16}B_Q) \lesssim \sum_{Q\in\mathsf{HD}_3}\mu(1.1B_Q)\\ &\lesssim_{A,\tau} \frac{1}{\Theta_\mu(B_R)}\sum_{Q\in\mathsf{HD}_3}\|\chi_{c_{16}B_Q}\,(f_k - c_0^k)\|^2_{L^2(\sigma^k)} \leq \frac{1}{\Theta_\mu(B_R)}\,\|f_k - c_0^k\|^2_{L^2(\sigma^k)}\\ &\lesssim_{A,\tau,K,M} \varepsilon_0^{1/10}\,\Theta_\mu(B_R)\,\ell(R),\end{aligned}$$

which proves the lemma. □

The preceding lemma was the last step for the proof of Main Lemma 5.1. For the reader's convenience, we state it here again. Recall that $F \subset \operatorname{supp}\mu = E$ is an arbitrary compact set such that

$$\int_F\int_0^1 \Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x) < \infty.$$

Main Lemma. *Let $0 < \varepsilon < 1/100$. Suppose that δ and η are small enough positive constants (depending only on ε). Let $R \in \mathcal{D}^{db}$ be a doubling cell with $\ell(R) \leq \delta$ such that*

$$\mu(R\setminus F) \leq \eta\,\mu(R), \qquad \mu(\lambda B_R\setminus F) \leq \eta\,\mu(\lambda B_R) \quad \text{for all } 2 < \lambda \leq \delta^{-1},$$

and

$$\mu\big(\delta^{-1}B_R \cap F \setminus \mathsf{G}(R,\delta,\eta)\big) \leq \eta\,\mu(R\cap F).$$

Then there exists an AD-regular curve Γ_R (with the AD-regularity constant bounded by some absolute constant) and a family of pairwise disjoint cells $\mathsf{Stop}(R) \subset \mathcal{D}(R)\setminus\{R\}$ such that, denoting by $\mathsf{Tree}(R)$ the subfamily of the cells from $\mathcal{D}(R)$ which are not strictly contained in any cell from $\mathsf{Stop}(R)$, the following holds:

(a) *μ-almost all $F\cap R\setminus\bigcup_{Q\in\mathsf{Stop}(R)} Q$ is contained in Γ_R, and moreover $\mu|_{F\cap R\setminus\bigcup_{Q\in\mathsf{Stop}(R)} Q}$ is absolutely continuous with respect to $\mathcal{H}^1|_{\Gamma_R}$.*

(b) *For all $Q \in \mathsf{Tree}(R)$, $\Theta(1.1B_Q) \leq A\,\Theta_\mu(1.1B_R)$, where $A \geq 100$ is some absolute constant.*

(c) *The cells from* $\mathsf{Stop}(R)$ *satisfy*

$$\sum_{Q\in\mathsf{Stop}(R)} \Theta_\mu(1.1B_Q)^2\,\mu(Q) \leq \varepsilon\,\Theta_\mu(B_R)^2\,\mu(R) + c(\varepsilon)\sum_{Q\in\mathsf{Tree}(R)}\int_{F\cap\delta^{-1}B_Q}\int_{\delta\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x).$$

Notice that the curve Γ_R mentioned in the Main Lemma is not the limit in the Hausdorff distance of the curves Γ^k, but the limit of the curves Γ_R^k which are described in Remark 10.14. On the other hand, the statement in (b) is a consequence of Lemma 6.8, possibly after adjusting the constant A suitably.

The inequality in (c) follows from Lemmas 8.5 and 16.2. Indeed, recall that Lemma 8.5 asserts that, for η and δ are small enough,

$$\sum_{\substack{Q\in\mathcal{D}(R):\\ Q\subset\mathsf{BCF}\cup\mathsf{LD}\cup\mathsf{BCG}\cup\mathsf{BS}\Delta}} \Theta_\mu(1.1B_Q)^2\,\mu(Q) \lesssim A^2\,(\eta^{1/4}+\tau^{1/4}+\delta^{1/2})\,\Theta_\mu(B_R)^2\,\mu(R) + \frac{A^2}{\eta}\sum_{Q\in\mathsf{Tree}}\int_{\delta^{-1}B_Q\cap F}\int_{\delta^5\,\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x),$$

while from Lemma 16.2 we deduce that

$$\begin{aligned}\sum_{Q\in\mathsf{HD}\cap\mathcal{D}(R)} \Theta_\mu(Q)^2\,\mu(Q) &\lesssim A^2\left(\frac{c(A,\tau,K)}{M} + c(A,\tau,K,M)\,\varepsilon_0^{1/10}\right)\,\Theta_\mu(B_R)^2\,\mu(R)\\ &\leq \left(\frac{c'(A,\tau,K)}{M} + c'(A,\tau,K,M)\,\varepsilon_0^{1/10}\right)\,\Theta_\mu(B_R)^2\,\mu(R).\end{aligned}$$

So choosing M big enough and ε_0 (and thus η and δ) small enough, the inequality in (c) follows, replacing δ by δ^5, say.

17. Proof of Theorem 1.3: boundedness of T_μ implies boundedness of the Cauchy transform

For the reader's convenience, we state again Theorem 1.3:

Theorem. *Let μ be a finite Radon measure in $\mathbb{C}$ satisfying the linear growth condition*

$$\mu(B(x,r)) \leq c\,r \qquad \text{for all } x\in\mathbb{C} \text{ and all } r>0.$$

The Cauchy transform $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$ if and only if

(17.1)
$$\int_{x\in Q}\int_0^\infty \left|\frac{\mu(Q\cap B(x,r))}{r} - \frac{\mu(Q\cap B(x,2r))}{2r}\right|^2\frac{dr}{r}\,d\mu(x) \leq c\,\mu(Q) \qquad \text{for every square } Q\subset\mathbb{C}.$$

Given $f\in L^1_{loc}(\mu)$, we denote

$$T_\mu f(x) = \left(\int_0^\infty \left|\frac{(f\mu)(B(x,r))}{r} - \frac{(f\mu)(B(x,2r))}{2r}\right|^2\frac{dr}{r}\right)^{1/2},$$

where $(f\mu)(A) = \int_A f\,d\mu$, and we write $T\mu(x) = T_\mu 1(x)$. In this way, the condition (17.1) states that

$$\|T_\mu \chi_Q\|^2_{L^2(\mu|_Q)} \leq c\,\mu(Q) \qquad \text{for every square } Q \subset \mathbb{C}.$$

In this section we will prove that if μ has linear growth and

$$\|T_\mu \chi_Q\|_{L^2(\mu|_Q)} \leq c\,\mu(Q)^{1/2} \qquad \text{for every square } Q \subset \mathbb{C}, \tag{17.2}$$

then $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$. To prove this, we will use the relationship between the Cauchy transform of μ and the curvature

$$c^2(\mu) = \iiint \frac{1}{R(x,y,z)^2}\,d\mu(x)\,d\mu(y)\,d\mu(z),$$

where $R(x, y, z)$ stands for the radius of the circumference passing through x, y, z. If in the integral above we integrate over $\{(x, y, z) \in \mathbb{C}^3 : |x-y| > \varepsilon, |y-z| > \varepsilon, |x-z| > \varepsilon\}$, we get the ε-truncated curvature $c^2_\varepsilon(\mu)$. The following result is due to Melnikov and Verdera [MV].

Proposition 17.1. *Let μ be a finite Radon measure on $\mathbb{C}$ with c_0-linear growth. For all $\varepsilon > 0$, we have*

$$\|\mathcal{C}_\varepsilon \mu\|^2_{L^2(\mu)} = \frac{1}{6} c^2_\varepsilon(\mu) + O(\mu(\mathbb{C})), \tag{17.3}$$

with

$$|O(\mu(\mathbb{C}))| \leq c\,c_0^2\,\mu(\mathbb{C}),$$

where c is some absolute constant.

Another important tool to show that the condition (17.2) implies the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$ is the so called non-homogeneous $T1$ theorem, which in the particular case of the Cauchy transform reads as follows.

Theorem 17.2. *Let μ be a Radon measure on $\mathbb{C}$ with linear growth. The Cauchy transform $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$ if and only if for all $\varepsilon > 0$ and all the squares $Q \subset \mathbb{C}$,*

$$\|\mathcal{C}_{\mu,\varepsilon} \chi_Q\|_{L^2(\mu\lfloor Q)} \leq c\,\mu(Q)^{1/2},$$

with c independent of ε.

See Theorem 3.5 of [To3] for the proof, for example.

By Proposition 17.1 and Theorem 17.2, to prove that (17.2) implies the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$, it suffices to show that for any measure μ with linear growth

$$c^2(\mu|_Q) \leq C\,\mu(Q) + C\,\|T_\mu \chi_Q\|^2_{L^2(\mu|_Q)} \qquad \text{for every square } Q \subset \mathbb{C}.$$

Clearly, this is equivalent to proving the following.

Theorem 17.3. *Let μ be a compactly supported Radon measure on $\mathbb{C}$ with linear growth. Then we have*

$$c^2(\mu) \leq C\,\|\mu\| + C \iint_0^\infty \left| \frac{\mu(B(x,r))}{r} - \frac{\mu(B(x,2r))}{2r} \right|^2 \frac{dr}{r}\,d\mu(x).$$

To obtain the preceding result we will construct a suitable corona type decomposition of μ by using the following variant of the Main Lemma 5.1:

Lemma 17.4. *Let μ be a compactly supported Radon measure on $\mathbb{C}$. Let $0 < \varepsilon < 1/00$. Suppose that δ and η are small enough positive constants (depending only on ε). Let $R \in \mathcal{D}^{db}$ be a doubling cell such that*

$$\mu\big(\delta^{-1}B_R \setminus \mathsf{G}(R,\delta,\eta)\big) \leq \eta\,\mu(R).$$

Then there exists an AD-regular curve Γ_R (with the AD-regularity constant bounded above by some absolute constant) and a family of pairwise disjoint cells $\mathsf{Stop}(R) \subset \mathcal{D}(R)\setminus\{R\}$ such that, denoting by $\mathsf{Tree}(R)$ the subfamily of the cells from $\mathcal{D}(R)$ which are not strictly contained in any cell from $\mathsf{Stop}(R)$, the following holds:

(a) *μ-almost all $R \setminus \bigcup_{Q\in\mathsf{Stop}(R)} Q$ is contained in Γ_R, and moreover $\mu|_{R\setminus\bigcup_{Q\in\mathsf{Stop}(R)} Q}$ is absolutely continuous with respect to $\mathcal{H}^1|_{\Gamma_R}$.*

(b) *There exists an absolute constant c such that every $Q \in \mathsf{Stop}(R)$ satisfies $cB_Q \cap \Gamma_R \neq \varnothing$.*

(c) *For all $Q \in \mathsf{Tree}(R)$, $\Theta(1.1B_Q) \leq A\,\Theta_\mu(1.1B_R)$, where $A \geq 100$ is some absolute constant.*

(d) *The cells from $\mathsf{Stop}(R)$ satisfy*

$$\sum_{Q\in\mathsf{Stop}(R)} \Theta_\mu(1.1B_Q)^2\,\mu(Q) \leq \varepsilon\,\Theta_\mu(B_R)^2\,\mu(R) + c(\varepsilon) \sum_{Q\in\mathsf{Tree}(R)} \int_{\delta^{-1}B_Q}\int_{\delta\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2\,\frac{dr}{r}\,d\mu(x).$$

Recall that given a cell $Q \in \mathcal{D}$, we denoted by $\mathsf{G}(Q,\delta,\eta)$ the set of points $x \in \mathbb{C}$ such that

$$\int_{\delta\,\ell(Q)}^{\delta^{-1}\ell(Q)} \Delta_\mu(x,r)^2\,\frac{dr}{r} \leq \eta\,\Theta_\mu(2B_Q)^2.$$

Basically, Lemma 17.4 corresponds to the Main Lemma 5.1 in the particular case when $F = \operatorname{supp}\mu$. Further, in (b) we stated the fact that every $Q \in \mathsf{Stop}(R)$ satisfies $cB_Q \cap \Gamma_R \neq \varnothing$, which comes for free from the construction of the curve Γ_R in Section 10, recalling that given $Q \in \mathsf{Stop}(R)$, if c is big enough, then the ball cB_Q contains some cell $Q' \in \mathsf{Good}$ which in turn contains some cell from the family $\{Q_i\}_{i\in I}$. Moreover, unlike in the Main Lemma 5.1, above we do not ask $\ell(R) \leq \delta$. Indeed, this assumption was present in the Main Lemma only because we cared about the truncated square function

$$\left(\int_0^1 \left|\frac{\mu(B(x,r))}{r} - \frac{\mu(B(x,2r))}{2r}\right|^2 \frac{dr}{r}\right)^{1/2}.$$

Let $R_0 \in \mathcal{D}$ be a cell which contains $\operatorname{supp}\mu$ with $\ell(R_0) \approx \operatorname{diam}(\operatorname{supp}\mu)$. Consider the family of cells Top constructed in Subsection 5.2 (with $F = \operatorname{supp}\mu$ and $\mathcal{B}_1 = \mathcal{B}_2 = \varnothing$ now). Recall that this is a family of doubling cells (i.e., $\mathsf{Top} \subset \mathcal{D}^{db}$) contained in R_0 and that $R_0 \in \mathsf{Top}$.

Given a cell $Q \in \mathsf{Top}$, we let $\mathsf{End}(Q)$ be the subfamily of the cells $P \in \mathsf{Top}$ satisfying

- $P \subsetneq Q$,
- P is maximal, in the sense that there does not exist another cell $P' \in \mathsf{Top}$ such that $P \subset P' \subsetneq Q$.

In fact, it turns out that $\mathsf{End}(Q)$ coincides with the family $\mathcal{MD}(Q)$ from Subsection 5.2.

Also, we denote by $\mathsf{Tr}(Q)$ (the tree associated with Q) the family of cells $\mathcal{D}$ which are contained in Q and are not contained in any cell from $\mathsf{End}(Q)$. The set of good points for Q is

$$G(Q) := Q \setminus \bigcup_{P \in \mathsf{End}(Q)} P.$$

Further, given two cells $Q, R \in \mathcal{D}$ with $Q \subset R$, we set

$$\delta_\mu(Q,R) := \int_{2B_R \setminus Q} \frac{1}{|y - z_Q|}\, d\mu(y),$$

where z_Q stands for the center of Q.

We have:

Lemma 17.5 (The corona decomposition)**.** *Let μ be a compactly supported measure on $\mathbb{C}$. The family $\mathsf{Top} \subset \mathcal{D}^{db}$ constructed above satisfies the following. For each cell $Q \in \mathsf{Top}$ there exists an AD regular curve Γ_Q (with the AD-regularity constant uniformly bounded above by some absolute constant) such that:*

(a) *μ almost all $G(Q)$ is contained in Γ_Q.*

(b) *For each $P \in \mathsf{End}(Q)$ there exists some cell $\widetilde{P}$ containing P such that $\delta_\mu(P, \widetilde{P}) \leq C\Theta_\mu(Q)$ and $B_{\widetilde{P}} \cap \Gamma_Q \neq \varnothing$.*

(c) *If $P \in \mathsf{Tr}(Q)$, then $\Theta_\mu(1.1B_P) \leq C\,\Theta_\mu(B_Q)$.*

Further, the following packing condition holds:

(17.4)

$$\sum_{Q \in \mathsf{Top}} \Theta_\mu(B_Q)^2 \mu(Q) \leq C\,\Theta_\mu(B_{R_0})^2\,\mu(R_0) + C \iint_0^\infty \left| \frac{\mu(B(x,r))}{r} - \frac{\mu(B(x,2r))}{2r} \right|^2 \frac{dr}{r}\, d\mu(x).$$

The preceding lemma follows immediately from Lemmas 17.4 and 5.4. Let us remark that the property (b) in Lemma 17.5 is a consequence of the property (b) of Lemma 5.1, the construction of the family $\mathcal{MD}(Q) = \mathsf{End}(Q)$, and Lemma 4.4.

The corona decomposition of Lemma 17.5 is a variant of the one in [To1, Main Lemma 3.1]. In the latter reference, the corona decomposition is stated in terms of the usual dyadic squares of $\mathbb{C}$ instead of the dyadic cells of David-Mattila, and the left hand side of (17.4) is estimated in terms of the curvature of μ, instead of the $L^2(\mu)$ norm of the square integral $T\mu$.

We have now the following.

Lemma 17.6. *Let μ be a compactly supported measure on $\mathbb{C}$ such that $\mu(B(x,r)) \leq c_0\, r$ for all $x \in \mathbb{C}$, $r > 0$. Suppose that there exists a family $\mathsf{Top} \subset \mathcal{D}^{db}$ such that Top contains a cell R_0 such that $\operatorname{supp}\mu \subset R_0$, and so that for each cell $Q \in \mathsf{Top}$ there exists an AD regular curve Γ_Q (with the AD-regularity constant uniformly bounded by some absolute constant) such that the properties (a), (b) and (c) of Lemma 17.5 hold (with the set $G(Q)$ and the families $\mathsf{End}(Q)$, $\mathsf{Tr}(Q)$ defined in terms of the family Top as above). Then,*

$$c^2(\mu) \leq c \sum_{Q \in \mathsf{Top}} \Theta_\mu(B_Q)^2 \mu(Q). \tag{17.5}$$

The proof of this lemma is very similar to the one of Main Lemma 8.1 of [To1], where this is proved to hold for bilipschitz images of the corona decomposition of [To1, Main Lemma 3.1]. We will skip the details.

Clearly, Theorem 17.3 follows from Lemmas 17.5 and 17.6. Indeed, by (17.5) and (17.4) we have

$$c^2(\mu) \leq c \sum_{Q\in\mathsf{Top}} \Theta_\mu(B_Q)^2\mu(Q) \leq C\,\|\mu\| + C \iint_0^\infty \left|\frac{\mu(B(x,r))}{r} - \frac{\mu(B(x,2r))}{2r}\right|^2 \frac{dr}{r}\,d\mu(x).$$

18. Some Calderón-Zygmund theory for T_μ

Before proving that the boundedness of $\mathcal{C}_\mu$ in $L^2(\mu)$ implies the $L^2(\mu)$ boundedness of T_μ, we need to show that some typical results from Calderón-Zygmund theory also hold for the operator T_μ. Since the kernel of T_μ is not smooth, the results available in the literature (of which I am aware) are not suitable for T_μ.

For more generality, we consider the n-dimensional version of T_μ:

$$T_\mu^n f(x) = \left(\int_0^\infty \left|\frac{(f\mu)(B(x,r))}{r^n} - \frac{(f\mu)(B(x,2r))}{(2r)^n}\right|^2 \frac{dr}{r}\right)^{1/2}.$$

We have:

Proposition 18.1. *Let μ be an n-AD-regular measure in $\mathbb{R}^d$. If T_μ^n is bounded in $L^2(\mu)$, then T_μ^n is also bounded in $L^p(\mu)$ for $1 < p < \infty$.*

Proof. In [TT, Theorem 5.1] it is shown that the boundedness of T_μ^n in $L^2(\mu)$ implies the boundedness from the space of measures $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$. Then by interpolation T_μ^n is bounded in $L^p(\mu)$ for $1 < p \leq 2$. To get $L^p(\mu)$ the boundedness for $2 < p < \infty$, by interpolation again, it is enough to show that T_μ is bounded from $L^\infty(\mu)$ to $BMO(\mu)$. The arguments to prove this are rather standard.

Consider $f \in L^\infty(\mu)$ and let Q be some cell of the dyadic lattice $\mathcal{D}$ associated with μ. We have to show that, for some constant c_Q,

$$\frac{1}{\mu(Q)} \int_Q |T_\mu^n f - c_Q|\,d\mu \leq c\,\|f\|_{L^\infty(\mu)}.$$

Set $f_1 = f\,\chi_{4B_Q}$ and $f_2 = f - f_1$. Since T_μ^n is sublinear and positive we have $|T_\mu^n(f_1+f_2)(x) - T_\mu^n f_2(x)| \leq T_\mu^n f_1(x)$. Thus,

$$|T_\mu^n f(x) - c_Q| \leq |T_\mu^n(f_1+f_2)(x) - T_\mu^n f_2(x)| + |T_\mu^n f_2(x) - c_Q| \leq T_\mu^n f_1(x) + |T_\mu^n f_2(x) - c_Q|.$$

Hence,

$$\int_Q |T_\mu^n f - c_Q|\,d\mu \leq \int_Q T_\mu^n f_1\,d\mu + \int_Q |T_\mu^n f_2(x) - c_Q|\,d\mu. \tag{18.1}$$

The first term on the right hand side is estimated by using Cauchy-Schwarz inequality and the $L^2(\mu)$ boundedness of T_μ:

$$\int_Q T_\mu^n f_1\,d\mu \leq \|T_\mu^n f_1\|_{L^2(\mu)}\,\mu(Q)^{1/2} \leq c\,\|f_1\|_{L^2(\mu)}\,\mu(Q)^{1/2} \leq c\,\|f\|_{L^\infty(\mu)}\,\mu(Q).$$

To deal with the last integral on the right hand side of (18.1) we choose $c_Q = T^n_\mu f_2(z_Q)$, where z_Q stands for the center of Q. To show that

$$\int_Q |T^n_\mu f_2(x) - c_Q|\, d\mu \leq c\, \|f\|_{L^\infty(\mu)}\, \mu(Q)$$

and finish the proof of the proposition it is enough to show that

$$|T^n_\mu f_2(x) - T^n_\mu f_2(z_Q)| \leq c\, \|f\|_{L^\infty(\mu)} \quad \text{for } x \in Q. \tag{18.2}$$

To this end, write

(18.3)

$$\begin{aligned}
|T^n_\mu f_2(x) - T^n_\mu f_2(z_Q)| &= \left| \left(\int_0^\infty \Delta_{f_2\,\mu}(x,r)^2\, \frac{dr}{r} \right)^{1/2} - \left(\int_0^\infty \Delta_{f_2\,\mu}(z_Q,r)^2\, \frac{dr}{r} \right)^{1/2} \right| \\
&\leq \left(\int_0^\infty |\Delta_{f_2\,\mu}(x,r) - \Delta_{f_2\,\mu}(z_Q,r)|^2\, \frac{dr}{r} \right)^{1/2} \\
&\lesssim \left(\int_0^\infty \big| (|f_2|\,\mu)(A(z_Q, r - r(B_Q), r + r(B_Q)) \big|^2\, \frac{dr}{r^{2n+1}} \right)^{1/2} \\
&\quad + \left(\int_0^\infty \big| (|f_2|\,\mu)(A(z_Q, 2r - r(B_Q), 2r + r(B_Q)) \big|^2\, \frac{dr}{r^{2n+1}} \right)^{1/2} \\
&\leq \|f\|_{L^\infty(\mu)} \left(\int_{r \geq 3r(B_Q)} \big| \mu(A(z_Q, r - r(B_Q), r + r(B_Q)) \big|^2\, \frac{dr}{r^{2n+1}} \right)^{1/2}.
\end{aligned}$$

To estimate the last integral, first we take into account that $\mu(A(z_Q, r - r(B_Q), r + r(B_Q)) \lesssim |r + r(B_Q)|^n \lesssim r^n$ for $r \geq 3r(B_Q)$ and $x \in Q$ and then we use Fubini:

(18.4)

$$\begin{aligned}
\int_{r \geq 3r(B_Q)} \big| \mu(A(z_Q, r - r(B_Q), r + r(B_Q)) \big|^2 \frac{dr}{r^{2n+1}} &\lesssim \int_{r \geq 3r(B_Q)} \mu(A(z_Q, r - r(B_Q), r + r(B_Q))) \frac{dr}{r^{n+1}} \\
&\lesssim \int_{|x - z_Q| \geq 2r(B_Q)} \int_{|x-z_Q| - r(B_Q)}^{|x-z_Q| + r(B_Q)} \frac{dr}{r^{n+1}}\, d\mu(x).
\end{aligned}$$

In the last double integral, for x and r in the domain of integration we have $r \approx |x - z_Q|$, and so we get

$$\begin{aligned}
\int_{|x - z_Q| \geq 2r(B_Q)} \int_{|x-z_Q| - r(B_Q)}^{|x-z_Q| + r(B_Q)} \frac{dr}{r^{n+1}}\, d\mu(x) &\approx \int_{|x - z_Q| \geq 2r(B_Q)} \frac{1}{|x - z_Q|^{n+1}} \int_{|x-z_Q| - r(B_Q)}^{|x-z_Q| + r(B_Q)} dr\, d\mu(x) \\
&\approx \int_{|x - z_Q| \geq 2r(B_Q)} \frac{r(B_Q)}{|x - z_Q|^{n+1}}\, d\mu(x) \lesssim 1.
\end{aligned}$$

Gathering (18.3), (18.4), and the last inequality, (18.1) follows and we are done. □

In the next proposition we intend to prove a Cotlar type inequality involving the operators

$$T^n_{\mu,\ell} f(x) = \left(\int_{r > \ell} \Delta_{f\mu}(x,r)^2\, \frac{dr}{r} \right)^{1/2},$$

$$M_{\mu,\ell}f(x) = \sup_{r>\ell} \frac{1}{\mu(B(x,2r))} \int_{B(x,r)} |f|\,d\mu \quad \text{and,} \quad M^n_{\mu,\ell}f(x) = \sup_{r>\ell} \frac{1}{r^n} \int_{B(x,r)} |f|\,d\mu,$$

where ℓ is some non-negative constant which may depend on x.

Proposition 18.2. *Let μ be a doubling measure in $\mathbb{R}^d$. That is*

$$\mu(B(y,2r)) \leq c_{db}\,\mu(B(x,r)) \quad \text{for all } y \in \operatorname{supp}\mu \text{ and all } r > 0.$$

Let $f \in L^p(\mu)$, for some $1 \leq p < \infty$. For any $x \in \mathbb{R}^d$ and any $\ell > 0$ we have

$$T^n_{\mu,\ell}f(x) \lesssim_{c_{db}} M_{\mu,\ell}(T_{\mu,\ell}f)(x) + M^n_{\mu,\ell}f(x). \tag{18.5}$$

Proof. Take $x \in \mathbb{R}^d$ and let $t = \max(\ell, \operatorname{dist}(x, \operatorname{supp}\mu))$. It is straightforward to check that

$$T^n_{\mu,\ell}f(x) \leq T^n_{\mu,5t}f(x) + c\,M^n_{\mu,\ell}f(x). \tag{18.6}$$

We claim now that for all $y \in B(x,2t)$,

$$|T^n_{\mu,5t}f(x) - T^n_{\mu,5t}f(y)| \leq c\,M^n_{\mu,\ell}f(x). \tag{18.7}$$

To see that (18.5) follows from the preceding claim, just take the mean over the ball $B(x,2t)$ of the inequality (18.7) to get

$$\begin{aligned} T^n_{\mu,5t}f(x) &\leq \frac{1}{\mu(B(x,2t))} \int_{B(x,2t)} T^n_{\mu,5t}f(y)\,d\mu(y) + c\,M^n_{\mu,\ell}f(x) \\ &\lesssim \frac{1}{\mu(B(x,4t))} \int_{B(x,2t)} T^n_{\mu,5t}f(y)\,d\mu(y) + c\,M^n_{\mu,\ell}f(x), \end{aligned}$$

where we took into account that $\mu(B(x,2t)) \approx \mu(B(x,4t))$, since there exists $x' \in \operatorname{supp}\mu$ such that $|x-x'| = t$ and μ is doubling. Together with (18.6) and the fact that $T^n_{\mu,5t}f(y) \leq T^n_{\mu,\ell}f(y)$, this yields the inequality (18.5).

We turn our attention to (18.7). Some of the estimates will be similar to the ones in the previous proposition in connection with the boundedness from $L^\infty(\mu)$ to $BMO(\mu)$. We write

$$\begin{aligned} |T^n_{\mu,5t}f(x) - T^n_{\mu,5t}f(y)| &\leq \left(\int_{r>5t} \big|\Delta_{f\,\mu}(x,r) - \Delta_{f\,\mu}(y,r)\big|^2 \frac{dr}{r} \right)^{1/2} \\ &\leq \left(\int_{r>5t} |\Delta_{f\,\mu}(x,r) - \Delta_{f\,\mu}(y,r)|^2 \frac{dr}{r} \right)^{1/2} \\ &\lesssim \left(\int_{r>5t} \big|(|f|\,\mu)(A(x,r-2t,r+2t)\big|^2 \frac{dr}{r^{2n+1}} \right)^{1/2} \\ &\quad + \left(\int_{r>5t} \big|(|f|\,\mu)(A(x,2r-2t,2r+2t)\big|^2 \frac{dr}{r^{2n+1}} \right)^{1/2} \\ &\lesssim \left(\int_{r>5t} \big|(|f|\,\mu)(A(x,r-2t,r+2t)\big|^2 \frac{dr}{r^{2n+1}} \right)^{1/2}. \end{aligned} \tag{18.8}$$

To estimate the last integral we use the fact that for x and r in the domain of integration

$$\frac{(|f|\,\mu)(A(x,r-2t,r+2t)}{r^n} \leq \frac{1}{r^n} \int_{B(x,2r)} |f|\,d\mu \leq c\,M^n_{\mu,\ell}f(x)$$

and Fubini:

$$\begin{aligned}\int_{r\geq 5t}\big|(|f|\,\mu)(A(x,r-2t,r+2t)\big|^2\,\frac{dr}{r^{2n+1}} &\leq M^n_{\mu,\ell}f(x)\int_{r>5t}(|f|\,\mu)(A(x,r-2t,r+2t)\,\frac{dr}{r^{n+1}}\\ &\leq M^n_{\mu,\ell}f(x)\int_{|x-y|\geq 3t}|f(y)|\int_{|x-y|-2t}^{|x-y|+2t}\frac{dr}{r^{n+1}}\,d\mu(y)\\ &\lesssim M^n_{\mu,\ell}f(x)\int_{|x-y|>3t}\frac{|f(y)|}{|x-y|^{n+1}}\int_{|x-y|-2t}^{|x-y|+2t}dr\,d\mu(y)\\ &\lesssim M^n_{\mu,\ell}f(x)\int_{|x-y|>3t}\frac{t\,|f(y)|}{|x-y|^{n+1}}\,d\mu(y)\;\lesssim\;M^n_{\mu,\ell}f(x)^2.\end{aligned}$$

Plugging this estimate into (18.8) yields the claim (18.7). □

Proposition 18.3. *Let Γ be an AD-regular curve in $\mathbb{R}^d$, and for a given $a>0$ set $\sigma=a\,\mathcal{H}^1|_\Gamma$. Let μ be a measure in $\mathbb{R}^d$ such that $\mu(B(x,r))\leq a\,r$ for all $x\in\operatorname{supp}\mu$ and all $r\geq\ell(x)$, for some given function $\ell:\operatorname{supp}\mu\to[0,\infty)$. Then $T_{\sigma,\ell}:L^p(\sigma)\to L^p(\mu)$ is bounded for $1<p<\infty$ with norm not exceeding $c_{18}\,a$, with c_{18} depending only on p and the AD-regularity constant of Γ.*

Proof. Since $T_{\mathcal{H}^1|_\Gamma}$ is bounded in $L^p(\mathcal{H}^1|_\Gamma)$, we deduce that T_σ is bounded in $L^p(\sigma)$ with norm not exceeding $c_{19}\,a$, with c_{19} depending only on p and the AD-regularity constant of Γ. Abusing notation, for $x\in\operatorname{supp}\mu$ we set:

$$T_{\sigma,\ell}f(x):=T_{\sigma,\ell(x)}f(x),\quad M_{\sigma,\ell}f(x)=M_{\sigma,\ell(x)}f(x),\quad M^1_{\sigma,\ell}f(x):=M^1_{\sigma,\ell(x)}f(x).$$

By (18.5) we have

$$T_{\sigma,\ell}f(x)\lesssim_{c_{db}}M_{\sigma,\ell}(T_{\sigma,\ell}f)(x)+M^1_{\sigma,\ell}f(x)\qquad\text{for all } x\in\operatorname{supp}\mu. \tag{18.9}$$

Note that the doubling constant c_{db} of σ depends on the AD-regularity constant of $\mathcal{H}^1|_\Gamma$ but not on a.

Bt (18.9), to prove the proposition it is enough to show that $M_{\sigma,\ell}$ and $M^1_{\sigma,\ell}$ are bounded from $L^p(\sigma)$ to $L^p(\mu)$ with

$$\|M_{\sigma,\ell}\|_{L^p(\sigma)\to L^p(\mu)}\leq c\quad\text{and}\quad\|M^1_{\sigma,\ell}\|_{L^p(\sigma)\to L^p(\mu)}\leq c\,a.$$

The arguments to show this are very standard. For completeness, we will show the details.

Concerning $M_{\sigma,\ell}$, it is clear that it is bounded from $L^\infty(\sigma)$ to $L^\infty(\mu)$. Also, it is bounded from $L^1(\mu)$ to $L^{1,\infty}(\sigma)$. Indeed, given $\lambda>0$ and $f\in L^1(\sigma)$, denote

$$\Omega_\lambda=\{x:M_{\sigma,\ell}f(x)>\lambda\}.$$

For each $x\in\Omega_\lambda\cap\operatorname{supp}\mu$, consider a ball $B(x,r_x)$ with $r_x\geq\ell(x)$ and $B(x,r_x)\cap\operatorname{supp}\sigma\neq\varnothing$ such that

$$\frac{1}{\sigma(B(x,2r_x))}\int_{B(x,r_x)}|f|\,d\sigma>\lambda.$$

Consider a Besicovitch covering of $\Omega_\lambda \cap \operatorname{supp}\mu$ with balls $B(x_i, r_{x_i})$ with finite overlap, with $x_i \in \Omega_\lambda \cap \operatorname{supp}\mu$. Then we have

$$\begin{aligned}\mu(\Omega_\lambda) &\leq \sum_i \mu(B(x_i, r_{x_i})) \leq a \sum_i r_{x_i} \\ &\leq c \sum_i \sigma(B(x_i, 2r_{x_i})) \leq c \sum_i \int_{B(x_i, r_{x_i})} |f|\, d\sigma \leq c\, \|f\|_{L^1(\sigma)}.\end{aligned}$$

Above we took into account that $a\, r_{x_i} \lesssim \sigma(B(x_i, 2r_{x_i}))$, which follows from the fact that $B(x_i, r_{x_i}) \cap \operatorname{supp}\sigma \neq \varnothing$. So $M_{\sigma,\ell}$ is bounded from $L^1(\sigma)$ to $L^{1,\infty}(\mu)$, and by interpolation it is bounded from $L^p(\sigma)$ to $L^p(\mu)$.

On the other hand, regarding $M^1_{\sigma,\ell}$, note that if $x \in \operatorname{supp}\mu$ and $B(x,r) \cap \operatorname{supp}\sigma \neq \varnothing$, with $r \geq \ell(x)$, then

$$\frac{1}{r} \int_{B(x,r)} |f|\, d\sigma \leq \frac{c\, a}{\sigma(B(x, 2r))} \int_{B(x,r)} |f|\, d\sigma \leq c\, a\, M_{\sigma,\ell} f(x).$$

Taking the supremum over the radii $r \geq \ell(x)$ such that $B(x,r) \cap \operatorname{supp}\sigma \neq \varnothing$, we infer that $M^1_{\sigma,\ell} f(x) \leq c\, a\, M_{\sigma,\ell} f(x)$, and thus $M^1_{\sigma,\ell}$ is bounded from $L^p(\sigma)$ to $L^p(\mu)$ with its norm not exceeding $c(p)\, a$. □

19. Proof of Theorem 1.3: boundedness of the Cauchy transform implies boundedness of T_μ

In this section we will show that if μ has linear growth and the Cauchy transform $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$, then

$$\|T_\mu \chi_Q\|_{L^2(\mu|_Q)} \leq c\, \mu(Q)^{1/2}$$

for every square $Q \subset \mathbb{C}$. Because of the connection between the Cauchy kernel and curvature, the preceding result is an immediate corollary of the following.

Theorem 19.1. *Let μ be a finite Radon measure on $\mathbb{C}$ with linear growth. Then we have*

$$\iint_0^\infty \left| \frac{\mu(B(x,r))}{r} - \frac{\mu(B(x,2r))}{2r} \right|^2 \frac{dr}{r}\, d\mu(x) \leq C\, \|\mu\| + C\, c^2(\mu).$$

To prove this theorem we will use the corona decomposition of [To1]. To state the precise result we need, first we will introduce some terminology which is very similar to the one of the preceding subsection. The most relevant difference is that it involves the usual dyadic lattice $\mathcal{D}(\mathbb{C})$, instead of the David-Mattila lattice $\mathcal{D}$.

Let μ be a finite Radon measure, and assume that there exists a dyadic square $R_0 \in \mathcal{D}(\mathbb{C})$ such that $\operatorname{supp}\mu \subset R_0$ with $\ell(R_0) \leq 10\, \operatorname{diam}(\operatorname{supp}(\mu))$, say. Let $\mathsf{Top}_* \subset \mathcal{D}(\mathbb{C})$ be a family of dyadic squares contained in R_0, with $R_0 \in \mathsf{Top}_*$.

Given $Q \in \mathsf{Top}_*$, we denote by $\mathsf{End}_*(Q)$ the subfamily of the squares $P \in \mathsf{Top}_*$ satisfying

- $P \subsetneq Q$,
- P is maximal, in the sense that there does not exist another square $P' \in \mathsf{Top}_*$ such that $P \subset P' \subsetneq Q$.

Also, we denote by $\mathsf{Tr}_*(Q)$ the family of squares $\mathcal{D}(\mathbb{C})$ which are contained in Q and are not contained in any square from $\mathsf{End}_*(Q)$. We set

$$G_*(Q) := Q \cap \operatorname{supp}(\mu) \setminus \bigcup_{P \in \mathsf{End}_*(Q)} P.$$

Given a square $Q \subset \mathbb{C}$, we denote

$$\Theta_\mu(Q) = \frac{\mu(Q)}{\ell(Q)},$$

and given two squares $Q \subset R$, we set

$$\delta_{*,\mu}(Q,R) := \int_{2R\setminus Q} \frac{1}{|y - z_Q|}\, d\mu(y),$$

where z_Q stands for the center of Q.

We have:

Lemma 19.2 (The dyadic corona decomposition of [To1])**.** *Let μ be a Radon measure on $\mathbb{C}$ with linear growth and finite curvature $c^2(\mu)$. Suppose that there exists a dyadic square $R_0 \in \mathcal{D}(\mathbb{C})$ such that $\operatorname{supp}\mu \subset R_0$ with $\ell(R_0) \leq 10\operatorname{diam}(\operatorname{supp}(\mu))$. Then there exists a family Top_* as above which satisfies the following. For each square $Q \in \mathsf{Top}_*$ there exists an AD-regular curve Γ_Q (with the AD-regularity constant uniformly bounded by some absolute constant) such that:*

(a) *μ almost all $G_*(Q)$ is contained in Γ_Q.*

(b) *For each $P \in \mathsf{End}_*(Q)$ there exists some square $\widetilde{P} \in \mathcal{D}(\mathbb{C})$ containing P, concentric with P, such that $\delta_{*,\mu}(P, \widetilde{P}) \leq C\Theta_\mu(7Q)$ and $\frac{1}{2}\widetilde{P} \cap \Gamma_Q \neq \varnothing$.*

(c) *If $P \in \mathsf{Tr}_*(Q)$, then $\Theta_\mu(7P) \leq C\,\Theta_\mu(7Q)$.*

Further, the following packing condition holds:

$$\sum_{Q \in \mathsf{Top}_*} \Theta_\mu(7Q)^2 \mu(Q) \leq C\,\|\mu\| + C\,c^2(\mu). \tag{19.1}$$

Let us remark that the squares from the family Top_* may be non-doubling.

The preceding lemma is not stated explicitly in [To1]. However it follows immediately from the Main Lemma 3.1 of [To1], just by splitting the so called 4-dyadic squares in [To1, Lemma 3.1] into dyadic squares. Further, the family Top_* above is the same as the family $\mathsf{Top}_{\rm dy}$ from [To1, Section 8.2].

Quite likely, by arguments analogous to the ones used to prove Lemma 3.1 of [To1] (or the variant stated in Lemma 19.2 of the present paper), one can prove an analogous result in terms of cells from the dyadic lattice of David and Mattila. This would read exactly as Lemma 17.5, but one should replace the inequality (17.4) by the following:

$$\sum_{Q \in \mathsf{Top}} \Theta_\mu(B_Q)^2 \mu(Q) \leq C\,\Theta_\mu(B_{R_0})^2\,\mu(R_0) + C\,c^2(\mu).$$

Perhaps this would simplify some of the technical difficulties arising from the lack of a well adapted dyadic lattice to the measure μ in [To1]. However, proving this would take us too long and so this is out of the reach of the present paper.

To prove Theorem 19.1, we split $T\mu(x)$ as follows. Given $Q \in \mathcal{D}(\mathbb{C})$, we denote

$$T_Q\mu(x)^2 = \chi_Q(x) \int_{\ell(Q)/2}^{\ell(Q)} \Delta_\mu(x,r)^2\, \frac{dr}{r}.$$

Then we have

$$(19.2)\qquad T\mu(x)^2 = \sum_{Q\in\mathcal{D}(\mathbb{C})} T_Q\mu(x)^2 = \sum_{R\in\mathsf{Top}_*}\sum_{Q\in\mathsf{Tr}_*(R)} T_Q\mu(x)^2 + \sum_{Q\in\mathcal{D}(\mathbb{C}):Q\not\subset R_0} T_Q\mu(x)^2.$$

The last sum is easy to estimate:

Lemma 19.3. *We have*

$$\sum_{Q\in\mathcal{D}(\mathbb{C}):Q\not\subset R_0} \|T_Q\mu\|^2_{L^2(\mu)} \leq c\,\Theta_\mu(R_0)^2\,\|\mu\|.$$

Proof. Since $\operatorname{supp}\mu\subset R_0$, we have

$$\begin{aligned}\sum_{Q\in\mathcal{D}(\mathbb{C}):Q\not\subset R_0} \|T_Q\mu\|^2_{L^2(\mu)} &= \sum_{Q\in\mathcal{D}(\mathbb{C}):Q\supset R_0} \|T_Q\mu\|^2_{L^2(\mu)}\\ &= \int_{R_0}\int_{\ell(R_0)}^\infty \Delta_\mu(x,r)^2\,\frac{dr}{r}\\ &\lesssim \mu(R_0)^2\int_{R_0}\int_{\ell(R_0)}^\infty \frac{dr}{r^3}\approx \Theta_\mu(R_0)^2\,\mu(R_0).\end{aligned}$$

□

To deal with the first term on the right hand side of (19.2) we need a couple of auxiliary results from [To1]. The first one is the following.

Lemma 19.4. *Let Top_* be as in Lemma 19.2. For each $R\in\mathsf{Top}_*$ there exists a family of dyadic squares $\mathsf{Reg}_*(R)$ which satisfies the following properties:*

(a) *The squares from $\mathsf{Reg}_*(R)$ are contained in Q and are pairwise disjoint.*
(b) *Every square from $\mathsf{Reg}_*(R)$ is contained in some square from $\mathsf{End}_*(R)$.*
(c) *If $P,Q\in\mathsf{Reg}_*(R)$ and $2P\cap 2Q\neq\varnothing$, then $\ell(Q)/2\leq\ell(P)\leq 2\ell(Q)$.*
(d) *If $Q\in\mathsf{Reg}_*(R)$ and $x\in Q$, $r\geq\ell(Q)$, then $\mu(B(x,r)\cap 4R)\leq C\Theta_\mu(7R)\,r$.*
(e) *For each $Q\in\mathsf{Reg}_*(R)$, there exists some square $\widetilde{Q}$, concentric with Q, which contains Q, such that $\delta_{*,\mu}(Q,\widetilde{Q})\leq C\Theta_\mu(7R)$ and $\frac12\widetilde{Q}\cap\Gamma_R\neq\varnothing$.*

This result is proved in Lemmas 8.2 and 8.3 of [To1]. For the reader's convenience, let us say that this follows by a regularization procedure analogous to the one used in the present paper to construct the families Reg and NReg.

Next lemma shows how, in a sense, the measure μ can be approximated on a tree $\mathsf{Tr}_*(R)$ by another measure supported on Γ_R which is absolutely continuous with respect to length. This is proved in Lemma 8.4 of [To1].

Lemma 19.5. *For $R\in\mathsf{Top}_*$, denote $\mathsf{Reg}_*(R)=:\{P_i\}_{i\geq1}$. For each i, let $\widetilde{P}_i\in\mathcal{D}(\mathbb{C})$ be a square containing P_i such that $\delta_{*,\mu}(P_i,\widetilde{P}_i)\leq C\Theta_\sigma(7R)$ and $\frac12\widetilde{P}_i\cap\Gamma_R\neq\varnothing$ (as in (e) of Lemma 19.4). For each $i\geq1$ there exists some function $g_i\geq0$ supported on $\Gamma_R\cap\widetilde{P}_i$ such that*

$$(19.3)\qquad \int_{\Gamma_R} g_i\,d\mathcal{H}^1 = \mu(P_i),$$

$$(19.4)\qquad \sum_i g_i \lesssim \Theta_\mu(7R),$$

and

$$\|g_i\|_\infty \, \ell(\widetilde{P}_i) \lesssim \mu(P_i). \tag{19.5}$$

Recalling the splitting in (19.2), to prove Theorem 19.1, it suffices to show that for every $R \in \mathsf{Top}_*$

$$\sum_{Q \in \mathsf{Tr}_*(R)} \int T_Q\mu^2 \, d\mu \leq c\, \Theta_\mu(7R)^2 \, \mu(R),$$

because of the packing condition (19.1). To this end, denote

$$S_R\mu(x)^2 = \chi_R(x) \int_{\ell(x)/2}^{\ell(R)} \Delta_\mu(x,r)^2 \, \frac{dr}{r},$$

where $\ell(x) = \ell(Q)$ if $x \in Q \in \mathsf{Reg}_*$ and $\ell(x) = 0$ if $x \notin \bigcup_{Q \in \mathsf{Reg}_*} Q$. By (b) of Lemma 19.4,

$$\sum_{Q \in \mathsf{Tr}_*(R)} T_Q\mu(x)^2 \leq S_R\mu(x)^2.$$

Thus the proof of Theorem 19.1 will be concluded after proving the next result.

Lemma 19.6. *For every $R \in \mathsf{Top}_*$, we have*

$$\|S_R\mu\|_{L^2(\mu)}^2 \leq c\, \Theta_\mu(7R)^2 \, \mu(R).$$

Proof. Consider the measure $\sigma = \Theta_\mu(7R)\, \mathcal{H}^1|_{\Gamma_R}$, and take the functions g_i, $i \geq 0$, from Lemma 19.5. Set

$$\beta_i = \mu|_{P_i} - g_i \, \mathcal{H}^1|_{\Gamma_R},$$

and denote $h_i = \Theta_\mu(7R)^{-1} g_i$, so that $g_i \, \mathcal{H}^1|_{\Gamma_R} = h_i \, \sigma$. Denote also $h = \sum_i h_i$ and notice that

$$\mu = \sum_i \beta_i + h\, \sigma.$$

As S_R is subadditive, we have

$$S_R\mu \leq S_R(h\,\sigma) + \sum_i S_R\beta_i. \tag{19.6}$$

Since $\mu(B(x,r) \cap R) \lesssim \Theta_\mu(7R)\, r$ for all $x \in R \cap \operatorname{supp}\mu$ and all $r \geq \ell(x)$, by Proposition 18.3 $T_{\sigma,\ell} : L^2(\sigma) \to L^2(\mu|_R)$ is bounded with norm not exceeding $c\,\Theta_\mu(7R)$. So we get

$$\|S_R(h\,\sigma)\|_{L^2(\mu)}^2 \leq \|T_{\sigma,\ell} h\|_{L^2(\mu|_R)}^2 \leq c\, \Theta_\mu(7R)^2 \, \|h\|_{L^2(\sigma)}^2.$$

To estimate $\|h\|_{L^2(\sigma)}^2$, write

$$\|h\|_{L^2(\sigma)}^2 \leq \|h\|_{L^\infty(\sigma)} \, \|h\|_{L^1(\sigma)},$$

and recall that

$$\|h\|_{L^\infty(\sigma)} = \Theta_\mu(7R)^{-1} \|g\|_{L^\infty(\sigma)} \lesssim 1$$

and

$$\|h\|_{L^1(\sigma)} = \|g\|_{L^1(\mathcal{H}^1|_{\Gamma_R})} \leq \mu(R).$$

Hence we obtain

$$\|S_R(h\,\sigma)\|_{L^2(\mu)}^2 \lesssim \Theta_\mu(7R)^2 \, \mu(R). \tag{19.7}$$

Now we will estimate the term $\sum_i S_R\beta_i$ from (19.6). We split $S_R\beta_i(x)$ as follows:

$$\begin{aligned}
(19.8)\quad S_R\beta_i(x) &\leq \left(\chi_R(x)\int_{\ell(x)}^{4\ell(\widetilde P_i)} \Delta_{\beta_i}(x,r)^2\,\frac{dr}{r}\right)^{1/2} + \left(\chi_R(x)\int_{4\ell(\widetilde P_i)}^{\ell(R)} \Delta_{\beta_i}(x,r)^2\,\frac{dr}{r}\right)^{1/2}\\
&\leq \left(\chi_R(x)\int_{\ell(x)}^{\ell(P_i)/8} \Delta_{\mu|_{P_i}}(x,r)^2\,\frac{dr}{r}\right)^{1/2} + \left(\chi_R(x)\int_{\ell(P_i)/8}^{4\ell(\widetilde P_i)} \Delta_{\mu|_{P_i}}(x,r)^2\,\frac{dr}{r}\right)^{1/2}\\
&\quad + \left(\chi_R(x)\int_{\ell(x)}^{4\ell(\widetilde P_i)} \Delta_{h_i\,\sigma}(x,r)^2\,\frac{dr}{r}\right)^{1/2} + \left(\chi_R(x)\int_{4\ell(\widetilde P_i)}^{\ell(R)} \Delta_{\beta_i}(x,r)^2\,\frac{dr}{r}\right)^{1/2}\\
&= A_i(x)+B_i(x)+C_i(x)+D_i(x).
\end{aligned}$$

To deal with $A_i(x)$, note if $x\notin 2P_i$, then $A_i(x)$ vanishes. Recall now that, by Lemma 19.4, if $x\in 2P_i$, then $\ell(x)\approx\ell(P_i)$. So we obtain

$$A_i(x)\leq \chi_{2P_i}(x)\left(\int_{c\ell(P_i)}^{\ell(P_i)/8} \Delta_{\mu|_{P_i}}(x,r)^2\,\frac{dr}{r}\right)^{1/2} \lesssim \chi_{2P_i}(x)\,\frac{\mu(P_i)}{\ell(P_i)}.$$

Let us turn our attention to the term $B_i(x)$ from (19.8). Notice that $B_i(x)=0$ if $x\notin 20\widetilde P_i$, and moreover $\Delta_{\mu|_{P_i}}(x,r)=0$ if $2r<\operatorname{dist}(x,P_i)$. Hence, in the domain of integration of $B_i(x)$ we can assume both that $r\geq \ell(P_i)/8$ and that $r\geq\frac12\operatorname{dist}(x,P_i)$, which imply that

$$r\geq \frac12\left(\frac18\,\ell(P_i)+\frac12\operatorname{dist}(x,P_i)\right)\approx |x-z_{P_i}|+\ell(P_i).$$

So we have

$$\begin{aligned}
B_i(x)^2 &\leq \chi_{20\widetilde P_i}(x)\int_{c(|x-z_{P_i}|+\ell(P_i))}^{4\ell(\widetilde P_i)} \Delta_{\mu|_{P_i}}(x,r)^2\,\frac{dr}{r}\\
&\lesssim \chi_{20\widetilde P_i}(x)\,\mu(P_i)^2\int_{c(|x-z_{P_i}|+\ell(P_i))}^{4\ell(\widetilde P_i)} \frac{dr}{r^3} \lesssim \chi_{20\widetilde P_i}(x)\,\frac{\mu(P_i)^2}{\big(|x-z_{P_i}|+\ell(P_i)\big)^2}.
\end{aligned}$$

Thus,

$$B_i(x)\lesssim \chi_{20\widetilde P_i}(x)\,\frac{\mu(P_i)}{|x-z_{P_i}|+\ell(P_i)}.$$

Concerning $C_i(x)$, again it is easy to check that $C_i(x)=0$ if $x\notin 20\widetilde P_i$. So we have

$$C_i(x)\leq \chi_{20\widetilde P_i}T_{\sigma,\ell}h_i(x).$$

Next we consider the term $D_i(x)$ from (19.8). Since $\int d\beta_i=0$, it turns out that $\Delta_{\beta_i}(x,r)=0$ unless

$$\big(\partial B(x,r)\cup\partial B(x,2r)\big)\cap\operatorname{supp}\beta_i\neq\varnothing.$$

If this condition holds, we write $r\in I(i,x)$. This condition, together with the fact that $r\geq 4\,\ell(\widetilde P_i)$ in the domain of integration of the integral that defines $C_i(x)$, implies that $r\approx|x-z_{P_i}|\approx |x-z_{P_i}|+\ell(\widetilde P_i)$. Therefore,

$$\int_{4\ell(\widetilde P_i)}^{\ell(R)} \Delta_{\beta_i}(x,r)^2\,\frac{dr}{r}\lesssim \frac{\|\beta_i\|^2}{\big(|x-z_{P_i}|+\ell(\widetilde P_i)\big)^3}\int_{r\in I(i,x)} dr\leq \frac{\mu(P_i)^2\,\ell(\widetilde P_i)}{\big(|x-z_{P_i}|+\ell(\widetilde P_i)\big)^3}.$$

Hence,

$$D_i(x) \lesssim \frac{\mu(P_i)\,\ell(\widetilde{P}_i)^{1/2}}{\big(|x - z_{P_i}| + \ell(\widetilde{P}_i)\big)^{3/2}}.$$

Gathering the estimates we have obtained for $A_i(x)$, $B_i(x)$, $C_i(x)$, and $D_i(x)$, we get

$$\begin{aligned} S_R\beta_i(x) &\lesssim \chi_{2P_i}(x)\,\frac{\mu(P_i)}{\ell(P_i)} + \chi_{20\widetilde{P}_i}(x)\,\frac{\mu(P_i)}{|x - z_{P_i}| + \ell(P_i)} + \frac{\mu(P_i)\,\ell(\widetilde{P}_i)^{1/2}}{\big(|x - z_{P_i}| + \ell(\widetilde{P}_i)\big)^{3/2}} + \chi_{20\widetilde{P}_i} T_\sigma h_i(x) \\ &\lesssim \left[\chi_{20\widetilde{P}_i}(x)\,\frac{\mu(P_i)}{|x - z_{P_i}| + \ell(P_i)} + \frac{\mu(P_i)\,\ell(\widetilde{P}_i)^{1/2}}{\big(|x - z_{P_i}| + \ell(\widetilde{P}_i)\big)^{3/2}}\right] + \chi_{20\widetilde{P}_i} T_\sigma h_i(x) \\ &=: E_i(x) + \chi_{20\widetilde{P}_i} T_\sigma h_i(x). \end{aligned}$$

We will estimate the $L^2(\mu|_R)$ norm of $\sum_i S_R\beta_i$ by duality. To this end, consider a non-negative function $f \in L^2(\mu|_R)$ and write

$$\int_R f \sum_i S_R\beta_i\, d\mu \lesssim \sum_i \int f\, E_i\, d\mu + \sum_i \int_{20\widetilde{P}_i} f\, T_{\sigma,\ell} h_i\, d\mu =: \textcircled{1} + \textcircled{2}. \tag{19.9}$$

First we deal with $\textcircled{1}$. We consider the centered maximal Hardy-Littlewood operator

$$M^c_{\mu|_R} f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r) \cap R)} \int_{B(x,r)\cap R} |f|\, d\mu.$$

It is easy well known that $M^c_{\mu|_R}$ is bounded in $L^p(\mu|_R)$, $1 < p \leq \infty$, and of weak type $(1,1)$ with respect to $\mu|_R$.

For each i we have

$$\int f\, E_i\, d\mu = \int_{20\widetilde{P}_i} \frac{\mu(P_i)}{|x - z_{P_i}| + \ell(P_i)}\, f(x)\, d\mu(x) + \int \frac{\mu(P_i)\,\ell(\widetilde{P}_i)^{1/2}}{\big(|x - z_{P_i}| + \ell(\widetilde{P}_i)\big)^{3/2}}\, f(x)\, d\mu(x).$$

We claim now that the following holds:

$$\int_{20\widetilde{P}_i} \frac{1}{|x - z_{P_i}| + \ell(P_i)}\, f(x)\, d\mu(x) \lesssim \Theta_\mu(7R) \inf_{y\in P_i} M^c_{\mu|_R} f(y) \tag{19.10}$$

and

$$\int \frac{\ell(\widetilde{P}_i)^{1/2}}{\big(|x - z_{P_i}| + \ell(\widetilde{P}_i)\big)^{3/2}}\, f(x)\, d\mu(x) \lesssim \Theta_\mu(7R) \inf_{y\in P_i} M^c_{\mu|_R} f(y). \tag{19.11}$$

Assuming these estimates for the moment, we deduce

$$\int f\, E_i\, d\mu \lesssim \Theta_\mu(7R) \inf_{y\in P_i} M^c_{\mu|_R} f(y)\, \mu(P_i) \leq \Theta_\mu(7R) \int_{P_i} M^c_{\mu|_R} f(y)\, d\mu(y),$$

and then, since the squares P_i are pairwise disjoint and contained in R,

$$\begin{aligned} \textcircled{1} &\lesssim \Theta_\mu(7R) \sum_i \int_{P_i} M^c_{\mu|_R} f(y)\, d\mu(y) \leq \Theta_\mu(7R) \int_R M^c_{\mu|_R} f\, d\mu \\ &\leq \Theta_\mu(7R)\, \|M^c_{\mu|_R} f\|_{L^2(\mu|_R)}\, \mu(R)^{1/2} \leq \Theta_\mu(7R)\, \|f\|_{L^2(\mu)}\, \mu(R)^{1/2}. \end{aligned}$$

Next we estimate the term ② from (19.9). By Hölder's inequality and the $L^p(\mu)$ boundedness of $T_{\sigma,\ell}$ from $L^p(\sigma)$ to $L^p(\mu|_R)$ with norm not exceeding $c\,\Theta_\mu(7R)$, we get

$$\textcircled{2} \leq \sum_i \left(\int_{20\widetilde{P}_i\cap R} |T_{\sigma,\ell} h_i|^4\,d\mu\right)^{1/4} \left(\int_{20\widetilde{P}_i} |f|^{4/3}\,d\mu\right)^{3/4}$$
$$\lesssim \Theta_\mu(7R)\sum_i \|h_i\|_{L^4(\sigma)} \left(\int_{20\widetilde{P}_i\cap R} |f|^{4/3}\,d\mu\right)^{3/4}.$$

Consider the following centered maximal operator

$$M^{c,4/3}_{\mu|_R} f(x) = \sup_{r>0}\left(\frac{1}{\mu(B(x,r)\cap R)}\int_{B(x,r)\cap R} |f|^{4/3}\,d\mu\right)^{3/4}.$$

This is bounded in $L^p(\mu|_R)$ for $4/3 < p \leq \infty$ and of weak type $(4/3, 4/3)$ with respect to $\mu|_R$. Notice that for all $y \in P_i$

$$\left(\int_{20\widetilde{P}_i\cap R} |f|^{4/3}\,d\mu\right)^{3/4} \leq \left(\int_{B(y,\ell(40\widetilde{P}_i))\cap R} |f|^{4/3}\,d\mu\right)^{3/4}$$
$$\leq \mu(B(y,\ell(40\widetilde{P}_i))\cap R)^{3/4}\, M^{c,4/3}_{\mu|_R} f(y)$$
$$\lesssim \Theta_\mu(7R)^{3/4}\,\ell(\widetilde{P}_i)^{3/4}\, M^{c,4/3}_{\mu|_R} f(y).$$

Therefore,

$$\textcircled{2} \lesssim \Theta_\mu(7R)^{1+3/4}\sum_i \|h_i\|_{L^4(\sigma)}\,\ell(\widetilde{P}_i)^{3/4} \inf_{y\in P_i} M^{c,4/3}_{\mu|_R} f(y). \tag{19.12}$$

Recalling that $h_i = \Theta_\mu(7R)^{-1}\,g_i$ and that $\|g_i\|_\infty\,\ell(\widetilde{P}_i) \lesssim \mu(P_i)$, we get

$$\|h_i\|_{L^4(\sigma)}\,\ell(\widetilde{P}_i)^{3/4} \leq \|h_i\|_{L^\infty(\sigma)}\,\sigma(\widetilde{P}_i)^{1/4}\,\ell(\widetilde{P}_i)^{3/4}$$
$$\lesssim \Theta_\mu(7R)^{-1+1/4}\|g_i\|_{L^\infty(\sigma)}\,\ell(\widetilde{P}_i) \lesssim \Theta_\mu(7R)^{-3/4}\,\mu(P_i).$$

Plugging this estimate into (19.12) and using the $L^2(\mu)$ boundedness of $M^{c,4/3}_{\mu|_R}$, we obtain

$$\textcircled{2} \lesssim \Theta_\mu(7R)\sum_i \mu(P_i) \inf_{y\in P_i} M^{c,4/3}_{\mu|_R} f(y) \leq \Theta_\mu(7R)\int_R M^{c,4/3}_{\mu|_R} f(y)\,d\mu(y)$$
$$\leq \Theta_\mu(7R)\,\|M^{c,4/3}_{\mu|_R} f\|_{L^2(\mu)}\,\mu(R)^{1/2} \lesssim \Theta_\mu(7R)\,\|f\|_{L^2(\mu)}\,\mu(R)^{1/2}.$$

Gathering the estimates obtained for ① and ②, we get

$$\int_R f\sum_i S_R\beta_i\,d\mu \lesssim \Theta_\mu(7R)\,\|f\|_{L^2(\mu)}\,\mu(R)^{1/2}$$

for any non-negative function $f \in L^2(\mu)$, which implies that

$$\Big\|\sum_i S_R\beta_i\Big\|_{L^2(\mu)} \lesssim \Theta_\mu(7R)\,\mu(R)^{1/2},$$

as wished.

To conclude the proof of the lemma it just remains to prove the claims (19.10) and (19.11). We carry out this task in the following lemma. □

Lemma 19.7. *Let $f \in L^2(\mu|_R)$ be non-negative. We have*

$$\int_{20\widetilde{P}_i} \frac{1}{|x - z_{P_i}| + \ell(P_i)} f(x)\, d\mu(x) \lesssim \Theta_\mu(7R) \inf_{y \in P_i} M^c_{\mu|_R} f(y) \tag{19.13}$$

and

$$\int \frac{\ell(\widetilde{P}_i)^{1/2}}{\big(|x - z_{P_i}| + \ell(\widetilde{P}_i)\big)^{3/2}} f(x)\, d\mu(x) \lesssim \Theta_\mu(7R) \inf_{y \in P_i} M^c_{\mu|_R} f(y). \tag{19.14}$$

Proof. First we deal with the inequality (19.13). Given a non-negative function $f \in L^2(\mu|_R)$ and $y \in P_i$, we set

$$\begin{aligned}\int_{20\widetilde{P}_i} \frac{1}{|x - z_{P_i}| + \ell(P_i)} f(x)\, d\mu(x) &\lesssim \frac{1}{\ell(P_i)} \int_{B(y,2\ell(P_i))} f(x)\, d\mu|_R(x) \\ &\quad + \int_{\frac{1}{2}\ell(P_i) \le |x - z_{P_i}| \le 40\ell(\widetilde{P}_i)} \frac{1}{|x - z_{P_i}|} f(x)\, d\mu|_R(x) \\ &= I_1 + I_2.\end{aligned}$$

Concerning I_1, we have

$$I_1 \lesssim \frac{\mu(B(y, 2\ell(P_i)) \cap R)}{\ell(P_i)} \frac{1}{\mu(B(y, 2\ell(P_i)) \cap R)} \int_{B(y,2\ell(P_i))\cap R} f\, d\mu \lesssim \Theta_\mu(7R)\, M^c_{\mu|_R} f(y).$$

To deal with I_2 we apply Fubini:

$$\begin{aligned} I_2 &= \int_{\frac{1}{2}\ell(P_i) \le |x - z_{P_i}| \le 40\ell(\widetilde{P}_i)} f(x) \int_{r > |x - z_{P_i}|} \frac{1}{r^2}\, dr\, d\mu|_R(x) \\ &= \int_{\frac{1}{2}\ell(P_i)}^{\infty} \frac{1}{r^2} \int_{|x - z_{P_i}| < \min(r, 40\ell(\widetilde{P}_i))} f(x)\, d\mu|_R(x)\, dr \\ &\le \int_{\frac{1}{2}\ell(P_i)}^{\infty} \frac{1}{r^2} \int_{B(y, \min(2r, 80\ell(\widetilde{P}_i)))} f\, d\mu|_R\, dr \\ &\le M^c_{\mu|_R} f(y) \int_{\frac{1}{2}\ell(P_i)}^{\infty} \frac{\mu\big(B(y, \min(2r, 80\ell(\widetilde{P}_i))) \cap R\big)}{r^2}\, dr. \end{aligned}$$

Since $B(y, \min(2r, 80\ell(\widetilde{P}_i))) \subset B(z_{P_i}, \min(4r, 160\ell(\widetilde{P}_i)))$, we get

$$I_2 \le M^c_{\mu|_R} f(y) \int_{\frac{1}{2}\ell(P_i)}^{\infty} \frac{\mu\big(B(z_{P_i}, \min(4r, 160\ell(\widetilde{P}_i))) \cap R\big)}{r^2}\, dr.$$

Notice now that, by Fubini, the integral above equals

$$\begin{aligned}\int_{\frac{1}{2}\ell(P_i)}^{\infty} \frac{1}{r^2} \int_{|x - z_{P_i}| < \min(4r, 160\ell(\widetilde{P}_i))} d\mu|_R(x)\, dr &= \int_{\frac{1}{2}\ell(P_i) \le |x - z_{P_i}| \le 160\ell(\widetilde{P}_i)} \int_{r > \frac{1}{4}|x - z_{P_i}|} \frac{1}{r^2}\, dr\, d\mu|_R(x) \\ &= \int_{\frac{1}{2}\ell(P_i) \le |x - z_{P_i}| \le 160\ell(\widetilde{P}_i)} \frac{4}{|x - z_{P_i}|}\, d\mu|_R(x).\end{aligned}$$

From the condition $\delta_{*,\mu}(P_i, \widetilde{P}_i) \lesssim \Theta_\mu(7R)$, it follows easily that the last integral above is bounded by $c\,\Theta_\mu(7R)$. Thus,

$$I_2 \lesssim \Theta_\mu(7R)\, M^c_{\mu|_R} f(y),$$

and (19.13) follows.

The arguments for (19.14) are quite similar. Consider a non-negative function $f \in L^2(\mu|_R)$ and $y \in P_i$, and write

$$\begin{aligned}\int \frac{\ell(\widetilde{P}_i)^{1/2}}{\big(|x - z_{P_i}| + \ell(\widetilde{P}_i)\big)^{3/2}} f(x)\, d\mu(x) &\lesssim \frac{1}{\ell(\widetilde{P}_i)} \int_{B(y,2\ell(\widetilde{P}_i))} f(x)\, d\mu|_R(x) \\ &\quad + \int_{|x - z_{P_i}| \geq \frac{1}{2}\ell(\widetilde{P}_i)^{3/2}} \frac{\ell(\widetilde{P}_i)^{1/2}}{|x - z_{P_i}|^{3/2}} f(x)\, d\mu|_R(x) \\ &= J_1 + J_2.\end{aligned}$$

Arguing as in the case of I_1, we get

$$J_1 \lesssim \Theta_\mu(7R)\, M^c_{\mu|_R} f(y).$$

To deal with J_2 we apply Fubini again:

$$\begin{aligned}J_2 &= c \int_{|x - z_{P_i}| \geq \frac{1}{2}\ell(\widetilde{P}_i)} f(x) \int_{r > |x - z_{P_i}|} \frac{\ell(\widetilde{P}_i)^{1/2}}{r^{5/2}}\, dr\, d\mu|_R(x) \\ &= \int_{\frac{1}{2}\ell(\widetilde{P}_i)}^\infty \frac{\ell(\widetilde{P}_i)^{1/2}}{r^{5/2}} \int_{|x - z_{P_i}| < r} f(x)\, d\mu|_R(x)\, dr \\ &\leq \int_{\frac{1}{2}\ell(\widetilde{P}_i)}^\infty \frac{\ell(\widetilde{P}_i)^{1/2}}{r^{5/2}} \int_{B(y,2r)} f\, d\mu|_R\, dr \\ &\leq M^c_{\mu|_R} f(y) \int_{\frac{1}{2}\ell(\widetilde{P}_i)}^\infty \frac{\ell(\widetilde{P}_i)^{1/2}\, \mu\big(B(y,2r) \cap R\big)}{r^{5/2}}\, dr.\end{aligned}$$

Since

$$\frac{\mu\big(B(y,2r) \cap R\big)}{r} \lesssim \Theta_\mu(7R) \quad \text{for } r \geq \tfrac{1}{2}\ell(P_i),$$

we obtain

$$J_2 \lesssim \Theta_\mu(7R)\, M^c_{\mu|_R} f(y) \int_{\frac{1}{2}\ell(\widetilde{P}_i)}^\infty \frac{\ell(\widetilde{P}_i)^{1/2}}{r^{3/2}}\, dr \lesssim \Theta_\mu(7R)\, M^c_{\mu|_R} f(y),$$

and so (19.14) is proved. □

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