What did Gauss read in the *Appendix* ?

Agustí Reventós[a], *Carlos J. Rodríguez*[b]

[a]*Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, Barcelona, Spain*
[b]*Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, Barcelona, Spain and Universidad del Valle, Colombia*

Abstract

Lambert says, in a clear analogy with spherical geometry, that in an imaginary sphere the sum of the angles of a triangle would be less than π. There are historical reasons for believing Gauss could have used this analogy as a method of discovering non-Euclidean geometry.

In this paper we analyze the reading Gauss made of Bolyai's *Appendix* assuming that his investigations into the foundations of geometry were aimed at finding, among the surfaces in $\mathbb{R}^3$, Lambert's hypothetical *imaginary sphere*.

Lambert sagte, in einer deutlichen Analogie mit der Kugelgeometrie, dass in einer imaginären Kugel die Summe der Winkel eines Dreiecks kleiner als π sein würde. Es gibt historische Gründe, um zu glauben, dass Gauss diese Analogie als Methode in der Entdeckung der nichteuklidischen Geometrie nützen haben könnte.

In diese Artikel, wir analysieren die Vorlesung, dass Gauss über den Bolyai Appendix gemacht hat, mit den Annahme, dass seinen Forschungen über die Grundlagen der Geometrie orientiert wurden an, zwischen die Fläschen in R^3, die hypotetische imaginären Kugelfläche zu finden.

Key words: Non-Euclidean Geometry, Imaginary Sphere, Horocycle, Horosphere, Constant Curvature, Gauss, Lambert, Bolyai
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Contents

1. The classical problem

In definition *XXIII* of the *Elements*[1], EUCLID defines *straight parallel lines* as those *straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*

The *Euclidean Theory of Parallels* is based on the fifth postulate, that says: *That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.*

The *classical problem* of the Euclidean Theory of Parallels consists of demonstrating that this postulate is a consequence of the other postulates of the *Elements*.

In the 1st century B.C. POSIDONIUS had already attempted to solve this problem, confusing straight parallel lines with equidistant straight lines (see [Bon55], page 2).

Nevertheless, the problem was resolved negatively at the end of 19th century, two thousand years later. The definitive proof of this independence is attributed to BELTRAMI, in 1868[2]. He represented the *new plane* by the points inside a circle, its *new lines* by chords, and parallel lines by chords meeting at a point on the circumference of the circle. In this way he obtains a geometry that satisfies all of Euclid's postulates except the fifth (see [Bel68] and the footnote on page 11). This geometry is called *non-Euclidean geometry.*

In the two thousands years that separate POSIDONIUS from BELTRAMI, many mathematicians had thought they had resolved the problem of the Euclidean Theory of Parallels positively.

One of the most important works during this time was the *Euclides ab omni naevo vindicatus*, of 1733, by SACCHERI, [Sac20]. Using Saccheri's quadrilateral— a quadrilateral in which two opposite sides are equal and are perpendicular to the base— he obtains results from the Euclid's postulates, first without using the fifth postulate and later using the negation of it, expecting to find a contradiction and hence prove that the fifth postulate is, in fact, a theorem. He proves, for example, the remarkable fact that this negation implies the existence of asymptotic straight lines. The only error that he commits is to consider as ordinary points of the *new plane* points that are not on it; but the true reason was that some results disgust him, because they were against his Euclidean intuition.

[1] See, for instance, [Euc56], pages 154-155.

[2] As GRAY has remarked in [Gra04], BELTRAMI was not aware of this, and it was BONOLA reading BELTRAMI who in fact noticed it. See [Bon55], page 177 and page 234 and also [Rod09].

In a similar way the work of LAMBERT *Theorie der Parallellinien* is developed ([Lam86]), but without arriving at any satisfactory conclusion. In fact he says: *I should almost conclude that the third hypothesis holds on some imaginary sphere*[3]. This idea, the existence of a sphere of imaginary radius, was the most important tool for the discovery of non-Euclidean geometry. The *Analogy* consists of the formal substitution of R by the imaginary number Ri in all formulas that appear in the study of the geometry of a sphere of radius R. Recall that $\sin ix = i \sinh x$, and $\cos ix = \cosh x$ (see [RR05] and [Rod06]).

However the difficult acceptance of complex numbers during the 18th and the first part of the 19th centuries led to insufficient discussion of the *Analogy*. GAUSS deserves great recognition, because in 1830 he had boldness to defend the complex numbers as the numbers that describe the plane, the basic example of a doubly extended quantity, in the same way that real numbers describe the line, the basic example of a simply extended quantity.

In the famous letter to FARKAS BOLYAI of 6–3–1832, on non-Euclidean geometry, GAUSS suggested Farkas should study complex numbers[4], thus relating non-Euclidean geometry and complex numbers. This is further evidence that GAUSS was using the *Analogy* (see letter 8 page 13).

Many articles have been written about this history, but we believe that the strong relation between classical and differential geometry and the key role played by the *imaginary sphere* in the discovery of non-Euclidean geometry have not been sufficiently emphasized. These are the main reasons that impelled us to write the present paper.

2. Gauss's isolation

In 1794 Legendre published his *Eléments de géométrie*. There, and in posterior editions, he gave

several proofs of the fifth postulate. See [Leg94], [Leg33], and [Bon55], pages 55-60. Independently of whether these proofs were or were not correct, it is clear that LEGENDRE was convinced not only of the certainty of this postulate, but also that *he had finally removed the serious difficulties surrounding the foundations of geometry*, see [Bon55], page 60. We believe that, because of the great influence of LEGENDRE, mainly on French mathematicians,[5] the problem

[3] *Ich sollte daraus fast den Schlufs machen, die dritte Hypothese komme bey einer imaginären Kugelfläche vor.* In 1980 BORIS L. LAPTEV stated that Lambert also arrived at a contradiction. See [Ros88], page 101. See also [Rod06].

[4] Farkas followed Gauss's advice, see [Kis99].

[5] Lutzen, mentioning KARIN REICH, says that it was principally due to LIOUVILLE that Gauss's ideas on differential geometry became known in France: *To be sure,* SOPHIE GERMAIN *had read Gauss's Disquisitiones generales circa superficies curvas [1828], but during the following 15 years Lame's theories of systems of orthogonal surfaces dominated the French scene, and Gauss's work was forgotten. In 1843, in a paper in Liouville's Journal on this subject,* BERTRAND *admitted that "After having written this memoir, I have learned about a memoir by Mr.* GAUSS *entitled Disquisitiones generales..."[Bertrand 1843]. The following year,* BONNET *also referred to* GAUSS. *It is not impossible that* LIOUVILLE *himself had called*

of the Euclidean Theory of Parallels was not sufficiently considered by the great mathematical schools of these times.

GAUSS believed in the importance of this problem, but he felt alone among the great mathematicians of that period.

He did have a group of supporting friends: BESSEL, FARKAS BOLYAI, GERLING, OLBERS, and SCHUMACHER; he must thank them for news of the important works of SCHWEIKART, TAURINUS, and JÁNOS BOLYAI. But all of them were outsiders or amateur mathematicians. LOBACHEVSKY, the other important person in this story, was a professor of mathematics at the peripheral University of Kazan, whose ideas on the Euclidean Theory of Parallels were held up to ridicule by his russian colleagues. He was rector of Kazan University and achieved fame as a reformer of education.

3. The three ds^2 of Bolyai's *Appendix*

As is well known, in 1831 GAUSS, talking about the Euclidean Theory of Parallels and more particularly about one equivalent formulation of the fifth postulate, said to SCHUMAKER (letter of 05–17–1831)[6]: *In the last few weeks I have begun to put down a few of my own meditations, which are already to some extent nearly 40 years old. These I have never put in writing, so that I have been compelled 3 or 4 times to go over the whole matter afresh in my head. I did not wish it to perish with me.*[7]

Nevertheless, some months later, in February 1832, GAUSS read Bolyai's *Appendix*, and he decided not write on the subject anymore. In a letter to GERLING (14–02–1832) he said: *In addition I note that in recent days I received a small work from Hungary on non-Euclidean geometry in which I find all of my ideas and results developed with great elegance, although in a concentrated form that is difficult for one to follow who is not familiar with the subject. The author is a very young Austrian officer, the son of a friend of my youth with whom I had often discussed the subject in 1798, although my ideas at that time were much less developed and mature than those obtained by this young man through his own reflections. I consider this young geometer, v. Bolyai, to be a genius of the first class.*[8]

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the attention of these two young talents to the Disquisitones, *and it is certain that when the interest in Gauss's ideas spread in France after 1847 it was due to Liouville.* See [Lüt90]. This happened twenty years after the publication of *Disquitiones*!

[6] Gauss's letters on non-Euclidean geometry are commented, using the *Analogy*, in [RR05]. See also [Rev04].

[7] *Von meinen eigenen Meditationen, die zum Theil schon gegen 40 Jahr alt sind, wovon ich aber nie etwas aufgeschrieben habe, und daher manches 3 oder 4 mal von neuem auszusinnen genöthigt gewesen bin, habe ich vor einigen Wochen doch einiges aufzuschreiben angefangen. Ich wünschte doch, dass es nicht mit mir unterginge.*

[8] *Noch bemerke ich, dass ich dieser Tage eine Schrift aus Ungarn über die Nicht-Euklidische Geometrie erhalten habe, worin ich alle meine eigenen Ideen und Resultate wiederfinde, mit grosser Eleganz entwicklet, obwohl in einer für jemand, dem die Sache fremd ist, wegen der Concentrirung etwas schwer zu folgenden Form. Der Verfasser ist ein sehr*

Would GAUSS have said this if he had thought that the work of BOLYAI was a mere formal manipulation of concepts, along the lines of TAURINUS,[9] without any content?

Would GAUSS have stopped writing his notes if he had not considered that the problem was completely solved?

Moreover, in the above letter to FARKAS BOLYAI (6–03–1832), after the words *Now some remarks about the work of your son*[10], he said: *Praising him would mean praising myself [...] In truth I am astonished [...] And I am very glad that it is precisely my old friend's son who so wonderfully outmatched me.*[11]

It is in this letter that GAUSS suggests the name *Parasphere* for the surface called only F by J. BOLYAI and *Horosphere* by LOBACHEVSKY. He says: *For instance, the surface and the line your son calls F and L might be named parasphere and paracycle, respectively: they are, in essence the sphere and circle of infinite radii. One might call hypercycle the collection of all points at equal distance from a straight line with which they lie in the same plane; similarly for hypersphere*[12]. See [Kár87].

BOLYAI introduces the surface F, cited in GAUSS's letter above, in section §11 of the *Appendix*.

The first ds^2

In later sections, concretely in §24, BOLYAI proves that the relation between the length z of the paracycle (horocycle) cd, the length y of the paracycle ab and the length x of the geodesic ac (see Fig. 1) is given by

$$z = ye^{-x/R},$$

where R is the constant denoted i by Bolyai, and it represents the radius of the imaginary sphere.

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junger ósterreichischer Officier, Sohn eines Jugendfreundes von mir, mit dem ich 1798 mich oft über die Sache unterhalten hatte, wiewohl damals meine Ideen noch viel weiter von der Ausbildung und Reife entfernt waren, die sie durch das eigene Nachdenken dieses jungen Mannes erhalten haben. Ich halte diesen jungen Geometer v. BOLYAI *für ein Genie erster Grösse.*

[9] TAURINUS developed formally non-Euclidean geometry using the *imaginary sphere*. The results were correct, but it should be proved first that this *imaginary sphere* really exists.

[10] *Jetzt einiges über die Arbeit Deines Sohnes.*

[11] *sie loben hiesse mich selbst loben [...] In der That bin ich dadurch auf das Ausserste überrascht [...] Und höchst erfreulich ist es mir, dass gerade der Sohn meines alten Freundes es ist, der mir auf eine so merkwürdige Art zuvorgekommen ist.*

[12] *So könnte z. B. die Fläche, die Dein Sohn F nennt, eine Parasphäre, die Linie L ein Paracykel genannt werden: es ist im Grunde Kugelfläche, oder Kreislinie von unendlichem Radius. Hypercykel könnte der Complexus aller Punkte heissen, die von einer Geraden, mit der sie in Einer Ebene liegen, gleiche Distanz haben; eben so Hypersphäre.*

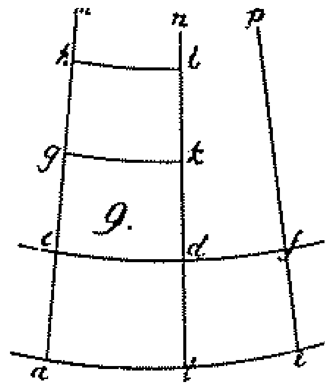

Figure 1.
Corresponds to figure 9 *of the* Appendix. *See page 11.*

From this it is easy to see that

$$ds^2 = dx^2 + e^{-2x/R}\, dy^2. \tag{1}$$

We affirm that this computation, that we give next, could have been done by Gauss. We also emphasize that this expression is obtained *without trigonometry* and without resorting to three dimensions.

Moreover, it seems impossible to look at Bolyai's figure above without seeing a system of local coordinates.

In fact, it is clear that the length element, in the sense used by Gauss, can be written in x, y coordinates as

$$ds^2 = dx^2 + f^2(x)dy^2$$

for a certain function $f(x)$, since

- this coordinate system is orthogonal[13] (so the term $dx\,dy$ does not appear),
- the lines $y =$ constant are geodesics parametrized by the arc length (so the coefficient of dx is 1), and
- it is invariant under translation in the y direction (so $f(x, y) = f(x)$).

To find $f(x)$ one takes the curve $\gamma(t) = (x, t)$, for a constant value of x, with $0 \le t \le y$. The length L of γ is given by

$$L = \int_0^y |\gamma'(t)|dt = \int_0^y f(x)dt = yf(x)$$

But, since $L = ye^{-x/R}$, we have $f(x) = e^{-x/R}$.

[13] The paracycles are orthogonal to the family of parallel straight lines.

The second ds^2

In Section §30, he gives the length of a circle in function of its radius r. This relation is

$$L(r) = 2\pi\, R \sinh \frac{r}{R}.$$

But similar calculations as above imply[14] that the metric in cyclic coordinates (r, θ) is given by

$$ds^2 = dr^2 + R^2 \sinh^2 \frac{r}{R}\, d\theta^2. \tag{2}$$

In fact, it is clear that

$$ds^2 = dr^2 + f^2(r) d\theta^2$$

for a certain function $f(r)$, since this coordinate system is orthogonal[15] (so the term $dr\, d\theta$ does not appear), $\theta =$ constant are geodesics parametrized by the arc length (so the coefficient of dr is 1), and it is invariant under rotation (so $f(r, \theta) = f(r)$).

To find $f(r)$ one takes the curve $\gamma(t) = (r, t)$, for a constant value of r, with $a \le t \le b$. The length L of $\gamma(t)$ is given by

$$L = \int_a^b |\gamma'(t)| dt = \int_a^b f(r) dt = (b - a) f(r)$$

But, since $L(r) = 2\pi\, R \sinh \frac{r}{R}$, the length of γ (a portion of the circle) is

$$L = (b - a) R \sinh \frac{r}{R}.$$

Hence $f(r) = R \sinh \dfrac{r}{R}$, and the metric of the Bolyai plane in cyclic coordinates is the metric of the *imaginary sphere*!

Note that the metric of the sphere in cyclic coordinates is given by $ds^2 = dr^2 + R^2 \sin^2 \dfrac{r}{R}\, d\theta^2$. Applying here the *Analogy* we obtain expression (2).

Did Gauss see this in Section §30 of the *Appendix*?

The third ds^2

In Section §32 of the *Appendix* a metric appears explicitly! Bolyai says:

II. Demonstrari potest, esse $\dfrac{dz^2}{dy^2 + bh^2} \sim 1$;

Figure 2.

[14] This computation does not appear in the *Appendix*; but it is something that would be easy for Gauss to do.

[15] Gauss's lemma, proved in [Gau28].

That is,

$$\textit{It can be proved, that} \quad \frac{dz^2}{dy^2 + bh^2} \sim 1;$$

which is equivalent, using the computation of bh given in the *Appendix*, to

$$\frac{ds^2}{dy^2 + \cosh^2 \frac{y}{R}\, dx^2} = 1,$$

that is

$$ds^2 = dy^2 + \cosh^2 \frac{y}{R}\, dx^2, \tag{3}$$

which is the expression of the metric in hypercyclic coordinates.

In fact, expression (3) is almost evident to any person (GAUSS, for instance) that knows the local theory of surfaces well.[16]

Concretely, it is clear that

$$ds^2 = dy^2 + f^2(y)dx^2$$

for a certain function $f(y)$, independent of x, since, by Gauss's lemma, this coordinate system is orthogonal (so the term $dx\,dy$ does not appear), the lines $x =$constant are geodesics (so the coefficient of dy is 1), and it is invariant under translation in the x direction (so $f(x, y) = f(y)$).

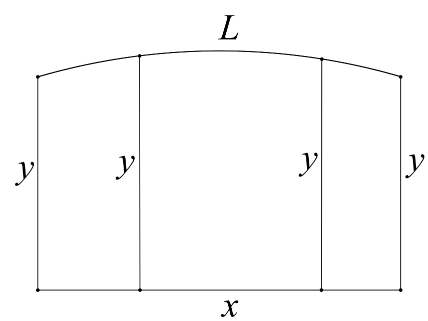

Figure 3.

To find $f(y)$ we take the equidistant curve $\gamma(t) = (t, y)$, for a constant value of y, with $a \leq t \leq b$. The length of $\gamma(t)$ is

$$L = \int_a^b |\gamma'(t)|dt = \int_a^b f(y)dt = f(y)(b - a)$$

[16]Unfortunately J. BOLYAI never knew GAUSS's work on the theory of surfaces: KÁRTESZI in [Kár87], page 32 says: *"Even of Gauss's results only a small proportion was known to him; for example, he has not heard of the investigations of* GAUSS *in surface theory contained in the work* Disquisitiones generales circa superficies curvas *through his life"*.

But in Section §27 of the *Appendix* Bolyai gives the relation between the length of the equidistant in function of the length of the base in the mixed quadrilateral of the above figure. This relation is

$$L = x \cosh \frac{y}{R}.$$

Hence $f(y) = \cosh \dfrac{y}{R}$, as we wanted to demonstrate.

Note that the metric of the sphere in hypercyclic coordinates is given by $ds^2 = dy^2 + \cos^2 \dfrac{y}{R}\, dx^2$. Applying here the *Analogy* we obtain expression (3).

Moreover, the curvature formula

$$k = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial^2 r},$$

known by Gauss since his first version of the *Disquisitiones generales circa superficies curvas* in 1825, could be applied to the expressions (1), (2) and (3), with $G = e^{-2x/R}$, $G = R^2 \sinh^2 \frac{y}{R}$ and $G = \cosh^2 \frac{y}{R}$ respectively, to prove that Bolyai's plane is represented by a surface of constant negative curvature $-1/R^2$.

Gauss may have seen that Bolyai's metrics (2) and (3) could be directly obtained by *Analogy* from the corresponding cyclic and hypercyclic metrics on the sphere. However, the paracyclic metric (1) can not appear by *Analogy*. The concept of paracycle is characteristic of hyperbolic geometry. However, Gauss's manuscripts on the Theory of Parallels of 1831 may be the beginning of a synthetic approach to finding this paracyclic metric, see [Gau27], *Zur Theorie der Parallellinien*, Bd. VIII, pages 202-209.

We make explicit this analytical change of coordinates in appendix A, page 16.

Did Gauss see that the hypercyclic coordinates on the *new plane* were global, unlike on the sphere, where they are not? In particular, that the *imaginary sphere* can be covered with only one chart?

Did Gauss see the proof of the consistency in the *Appendix*? The letters to Gerling and Farkas Bolyai that we have mentioned on page 4 lead us to conjecture that the answer is yes. But, did Gauss have the problem of the consistency clear enough?

Gauss and Riemann, who could have done the computations that we do in this article, did not realize that the problem of the consistency was solved, because the question that they set themselves: *Which surface of* $\mathbb{R}^3$ *has one of these metrics?* was incorrect. Gauss was the founder of the intrinsic geometry of surfaces, but all the *length elements* (metrics) used by Gauss came from the Euclidean metric of $\mathbb{R}^3$.

This epistemological mistake is quite understandable: they were discovering a new world and, like all discoverers, passed by something very important without seeing it.

It seems that Beltrami also made the same mistake, see note number 2 on page 2.

If one assumes that GAUSS used the *Analogy* to find the ds^2 of the imaginary sphere, it is easy to explain all results of the new geometry that GAUSS showed he knew in his letters. It also explains why he did not enclose proofs: the use of imaginary numbers was not sufficiently accepted and especially if they were used by analogy.

But, after reading the *Appendix*, GAUSS saw all these results deduced axiomatically and without any reference to imaginary numbers.

4. The drawings of the *Appendix*

Although we agree with Gray's comments on the *Appendix*, see [Gra04], pages 123-127, we would like to make some further remarks. We hope that these remarks contribute to enrich the recognition of Bolyai's work made by GRAY.

First of all, the coordinates used by BOLYAI are the hypercyclic coordinates (the lines x =constant are geodesic, while y =constant are not geodesic but equidistant).

It is in this sense that *Gray* uses the expression 'usual system of Cartesian (x, y) coordinates'.

Some of Gray's expressions can be considered as moderate criticism of Bolyai's work, for instance:

- *Without as much as a hint in the direction just outlined, Bolyai supposed that his readers would recognise these arguments [...]*
- *but it requires an interpretation that Bolyai was unwilling to provide [...]*
- *Bolyai escaped the pedagogic problem, not for the first or only time in the* Appendix *by saying: "It can be demonstrated"[...].*

All of these can be very well understood if we accept the hypothesis that the *Appendix* was written for GAUSS. Or at least, having in mind that the first and most important reader should be GAUSS. In fact, the *Appendix* was sent to Gauss in 1831 and the *Tentamen* was published in 1832.

JÁNOS BOLYAI had sent a first version of his work to his former professor Herr JOHANN WALTER VON ECKWEHR in 1826[17], and *on the prompting of his father* he had translated it from German into Latin, to be published in *Tentamen*, see [Bon55], page XXVIII of HALSTED's introduction. Given the friendship between GAUSS and FARKAS, it is logical to assume that FARKAS had already decided to send this latin version to GAUSS.

HALSTED also remarks that JÁNOS contributed with 104 florins and 50 kreuzers for the printing of the *Appendix* (about three and a half florins per page, so he must have been careful with the number of pages!) Perhaps for this reason he did not make explicit some details and it explains Gauss's comment in

[17]This manuscript has not been found.

his letter to Gerling (see page 4): [the results of the *Appendix* are developed] *in a concentrated form that is difficult for one to follow who is not familiar with the subject.*

The 23 drawings of the *Appendix*, with the legend *Tabula Appendicis* in the top right, that we reproduce from [Bol02],

should not be interpreted as

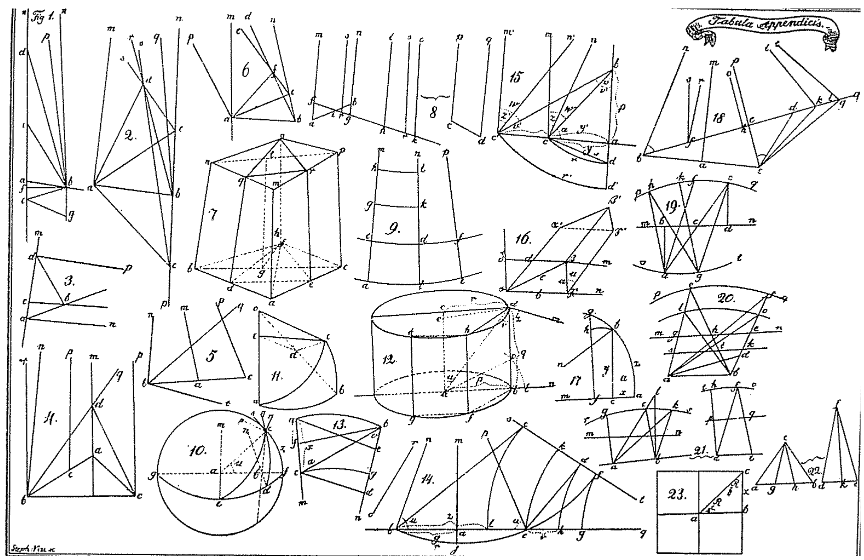

Figure 4.

drawings in the Euclidean plane, as might be infered from Gray's remark: *He drew a picture of a curve ABC in the familiar Cartesian plane with x- and y-axes and outlined an interpretation of it as a picture of non-Euclidean geometry drawn in a Euclidean plane.*

These figures play the same role as the figures that appear in the majority of versions of the Euclid's *Elements*: They are only guides for the proofs. In fact, Bolyai does not use the Euclidean plane at all. Note that Gauss few notes on the subject uses similar drawings.

Nevertheless, a valid objection to Bolyai's drawings is that he represents non-Euclidean segments in the same way that we usually represent the Euclidean ones. This problem was skillfully solved by Battaglini, see [Rod09], and was the basis for the proof of the consistency given by Beltrami, using a model where *non-Euclidean segments were represented by the Euclidean ones!*

As Gray says, it is a pity that Bolyai did not find the hyperbolic half-plane model: *With a bit of extra work, he could have shown that the entire picture of non-Euclidean two-dimensional geometry could appear in the right half-plane (the region defined by* $x > 0$*), and that in his new space straight lines were curves of a certain appearance [...].*

But in order to prove the consistency it is not necessary to have this specific model of hyperbolic geometry. It suffices to have, as Bolyai has, a "plane" with an appropriate metric. But this presupposes the idea of abstract Riemannian manifold. This was the great contribution of Riemann many years later. But we defend that this idea, though not in Bolyai's mind, would be clear to Gauss when he read the *Appendix* and decided not to write any more on the subject (see section 6 below).

Finally we completely agree with Gray when he says: *But the fact that* Bolyai *got as close as he did to formulating the elements of his new geometry in terms of the calculus is striking testimony to his insight, and seems not to have been appreciated sufficiently in his day or since.*

5. The modern solution

At the time of Gauss the consistency of Euclidean geometry was accepted without discussion. For this reason, the negation of the fifth postulate demanded a proof the consistency of this new geometry. This is what we term the *modern problem*: the problem of the independence of the fifth postulate. Moreover we must consider that the foundation of real numbers was still not rigorously achieved. They were only an instrument of calculus.

There are some letters written or received by Gauss, talking about *astral* geometry or *antieuclidean* geometry, from which we can deduce that Gauss was convinced of the consistency of this new geometry.

We mention[18]:

1. Gauss to Olbers. Goettingen, 28 April 1817. *I am coming ever more to the conviction that the necessity of our geometry cannot be proved at least not by human comprehension nor for human comprehension. Perhaps in another life we will come to other views on the nature of space which are currently unobtainable for us. Until then one must not put Geometry into the same rank as Arithmetic, which stands a priori, but rather in the same rank as, say, Mechanics*[19].
2. Schweikart's Note to Gauss. Marburg, December 1818. *There is a two-fold geometry, a geometry in the narrow sense, the Euclidean; and an astral study of magnitudes*[20].
3. Gauss to Gerling. Marburg, 16 March 1819. *The note of Herr Professor Schweikart gave me an incredible amount of pleasure, [...] because although I can imagine quite well that the Euclidean geometry is not correct*[21].

[18] See [Gau27], Vol 8, Grundlagen der Geometrie, Nachträge zu Band IV, pages 177-220.

[19] *Ich komme immer mehr zu der Überzeugung, dass die Nothwendigkeit unserer Geometrie nicht bewiesen werden kann, wenigstens nicht vom menschlichen Verstande noch für den menschlichen Verstand. Vielleicht kommen wir in einem andern Leben zu andern Einsichten in das Wesen das Raums, die uns jetzt unerreichbar sind. Bis dahin müsste man die Geometrie nicht mit der Arithmetik, die rein a priori steht, sondern etwa mit der Mechanik in gleichen Rang setzen.*

[20] *Es gibt eine zweifache Geometrie, eine Geometrie im engern Sinn, die Euklidische; und eine astralische Grössenlehre*

[21] *Die Notiz von Hrn. Prof. Schweikart hat mir ungemein viel Vergnügen gemacht, [...] denn obgleich ich mir recht gut die Unrichtigkeit der Euklidischen Geometrie denken kann*

4. Gauss to Taurinus. Goettingen, 8 November 1824. *The assumption that the sum of the three angles is smaller than* 180° *leads to a geometry that is quite different from ours (Euclidean), which is consistent, and which I have developed quite satisfactorily to the point that I can resolve every question in it with the exception of the determination of a constant which does not present itself a priori.* [...] *All of my efforts to find a contradiction, an inconsistency in this non-Euclidean geometry have been fruitless*[22].
5. Gauss to Bessel. Goettingen, 27 January 1829. *[...] my conviction that we cannot completely establish geometry a priori has become stronger*[23].
6. Bessel to Gauss. Koenigsberg, 10 February 1829. *[...] our geometry is incomplete and needs a correction which is hypothetical and which disappears if the sum of the angles of a triangle* = 180°[24].
7. Gauss to Bessel. Goettingen, 9 April 1830. *My innermost conviction is that the study of space is a priori completely different than the study of magnitudes; our knowledge of the former is missing that complete conviction of necessity (thus of absolute truth) that is characteristic of the latter*[25].
8. Gauss to Bolyai (senior)[26]. Goettingen, 6 March 1832. *Precisely the impossibility of deciding a priori between* Σ *and* S *gives the clearest proof that Kant was not justified in asserting that space is just the form of our perception. Another equally strong reason is in a brief essay in the Scholarly Notices of Goettingen* 1831, *article* 64, *page* 625. *Perhaps it will not be a disappointment if you try to procure that volume of the G.G.A. (which may be accomplished through any bookseller in Vienna or Buda*[27]*), as you also find there, developed in a few pages, the essence of my views concerning imaginary quantities*[28].

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[22] *Die Annahme, dass die Summe der* 3 *Winkel kleiner sei als* 180°, *führt auf eine eigene, von der unsrigen (Euklidischen) ganz verschieden Geometrie, die in sich selbst durchaus consequent ist, und die ich für mich selbst ganz befriedigend ausgebildet habe, so dass ich jede Aufgabe in derseklben auflösen kann mit Ausnahme der Bestimmung einer Constante, die sich a priori nicht ausmitteln lässt. [...] Alle meine Bemühungen, einen Widerspruch, eine Inconsequenz in dieser Nicht-Euklidischen Geometrie zu finden, sind fruchtlos gewesen.*

[23] *und meine Überzeugung, dass wir die Geometrie nicht vollst¨ndig a priori begründen können, ist, wo möglich, noch fester geworden.*

[24] *dass unsere Geometrie unvollständig ist, und eine Correction erhalten sollte, welche hypothetisch ist und, wenn die Summe der Winkel des ebenen Dreiecks* = 180°.

[25] *Nach meiner innigsten Überzeugung hat die Raumlehre in unserm Wissen a priori eine ganz andere Stellung, wie die reine Grössenlehre; es geht unserer Kenntniss von jener durchaus diejenige vollständige Überzeugung von ihrer Nothwendigkeit (also auch von ihrer absoluten Wahrheit) ab, die der letztern eigen ist.*

[26] The same letter is discussed on pages 3 and 5. See also page 14.

[27] The german name for Buda is Ofen. Budapest became a single city with the unification in 1873, of Buda and Óbuda (Old Buda) together with Pest.

[28] *Gerade in der Unmöglichkeit, zwischen* Σ *und* S *a priori zu entscheiden, liegt der klarste Beweis, dass Kant Unrecht hatte zu behaupten, der Raum sei nur Form unserer Anschauung. Einen andern ebenso starken Grund habe ich in einem kleinen Aufsatze angedeutet, der in den Göttingischen Gelehrten Anzeigen* 1831 *steht Stück* 64, *pag.* 625. *Vielleicht wird es Dich nicht gereuen, wenn Du Dich bemühest Dir diesen Band der G.G.A. zu verschaffen (was*

The arguments put forward by GAUSS in these letters for the belief in the consistency of non-Euclidean geometry were of inductive and physical type. Inductive: no matter how much he had looked for an inconsistency with the hypothesis of the acute angle, he had not found it. Physical: Although the Euclidean geometry was a very good candidate for the geometry of the physical space, an *antieuclidean* geometry with small negative curvature could also be the answer.

Had GAUSS the concept of mathematical model? Certainly not, but we believe that he had the idea that a surface in the space of three dimensions, with constant negative curvature and without singularities (MINDING's *pseudosphere* had singularities), would be a proof of the possibility of a new plane. We completely agree on this point with [BBI01], page 158.

GAUSS says to F. BOLYAI, in the above-mentioned letter of 1832, that he had obtained the same results as his son in similar ways: *because the whole contents of the writing, the path that your son has taken and the results to which it leads, are almost perfectly in agreement with my own meditations, some going back* $30-35$ *years*[29]. Nevertheless in his letter to SCHUMACHER of 1846 he says that LOBACHEVSKY had obtained the same results but in a different way: *Lobachevsky carried out the task in a masterly fashion and in a truly geometric spirit*,[30] (see [RR05], page 106). We believe that GAUSS insinuates that his discoveries were made using the length element ds^2 of the *imaginary sphere*, that he obtained by *Analogy* (whereas J. BOLYAI had deduced one of them explicitly, and the other two implicitly); but, as he could not show a *complete* surface[31] in the space of three dimensions with this length element, he did not publish anything; the synthetic rewritting of Theory of Parallels, that GAUSS began in 1831, was widely surpassed by the *Appendix*, a complete and magistral synthetic deduction of a hyperbolic arc length.

Perhaps GAUSS thought that the *imaginary sphere* could emerge using the geometrical interpretation of complex numbers[32]; that explains the suggestion made to FARKAS BOLYAI at the end of the letter. JÁNOS indeed read GAUSS's paper[33], and developed independently a conception of complex numbers that applied to Number Theory. As far as we know, J. BOLYAI did not relate the new

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jeder Buchhändler in Wien oder Ofen leicht bewirken kann), da darin unter andern auch die Quintessenz meiner Ansicht von den imaginären Grössen auf ein Paar Seiten dargelegt ist.

[29] *denn der ganze Inhalt der Schrift, der Weg, den Dein Sohn eingeschlagen hat, und die Resultate, zu denen er geführt ist, kommen fast durchgehends mit meinen eigenen, zum Teile schon seit 30–35 Jahren angestellten Meditationen überein.*

[30] *und zwar von* LOBATSCHEWSKY *auf eine meisterhafte Art in ächt geometrischem Geiste.* GAUSS refers to the german version [Lob55] that does not use differential calculus.

[31] A surface without singularities, where the straight lines are infinite.

[32] A proof that the *Analogy* was its source of inspiration. GAUSS, and a small number of friends with whom he spoke about the subject, knew LAMBERT's work very well; see BESSEL's letter of 1829 mentioned above. In section B we sketch an elementary proof of consistency using complex numbers.

[33] Kiss comments in [Kis99] that GAUSS does not give the correct reference, because the subject was completely developed in another of Gauss's works.

geometrical conception of complex numbers with the problem of consistency of the new geometry.

It is also possible that GAUSS made the same suggestion to RIEMANN; but RIEMANN was by this time occupied with other mathematical and physical problems, that lead him to the discovery of Riemann surfaces (the first example of a topological manifold of dimension two that it is not a surface of a three dimensional space: the first example of an abstract manifold!), and to a conception of physical space as a perfectly elastic and massless medium formed by an elastic fluid, affected by the energy-momentum of the physical fields within it. KLEIN compared RIEMANN with FARADAY who had described the electromagnetic field with the idea of *lines of force*. With RIEMANN geometry became a physical geometry. This idea will be developed in [Rod09].

6. The mistake

GAUSS made one mistake, possibly the biggest of his life: *to look for an imaginary sphere in* $\mathbb{R}^3$.

In fact, there exists no *imaginary sphere* in the usual sense of a surface in $\mathbb{R}^3$ of constant negative curvature[34]. Therefore, the search for an *imaginary sphere* was an endless struggle. It is possible that in 1831 he was aware that this way was a dead end, and decided to take the deductive point of view; but it was too late: JÁNOS BOLYAI had already followed this path in the *Appendix*.

The impossibility of finding a complete surface in $\mathbb{R}^3$ of constant negative curvature could have made GAUSS doubt his belief in the consistency of non-Euclidean Geometry, and could be the main reason by which GAUSS did not make any effort to publicize the *Appendix*.

The *Appendix* proves that analytical geometry of the new plane is very similar to that of the Euclidean plane; indeed, the problem of the consistency is almost the same in both geometries. Nobody has proved that Euclidean geometry is consistent. Because of this strong relation between the sphere and the *imaginary sphere*, the new geometry has the same status as the spherical one, in the sense that this geometry can be considered either as a section of the spacial Euclidean geometry or as the geometry of a curved surface of the space.

It would be interesting to answer the following question: Why did GAUSS only look at surfaces in three dimensional space?

The answer is surprising. Numbers and geometry were on different levels. The identification of $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ as geometrical objects was still not clearly made. One must wait for DEDEKIND for the foundation of real numbers; and BETTI, who learned probably from RIEMANN the importance of thinking mathematics conceptually, for giving the definitive step in the geometrization of $\mathbb{R}^n$.

[34] HILBERT proved in 1910 that there exists no complete regular surface of constant negative curvature immersed in $\mathbb{R}^3$. In 1955 KUIPER proved that such a surface does exist if we change "regular" for $\mathcal{C}^1$. See [Kui55].

Gauss, and other contemporary mathematicians, do not identify the set of pairs of real numbers as done today.

It is precisely with Riemann, as was observed by Ferreirós, that the idea of a conceptual mathematics comes up, a mathematics that studies manifolds and their mappings (see [Fer00], pages 93–95, and [Fer07]). Riemann made this giant's step because he needed to extend the geometric intuition to other areas of mathematics different from geometry. But, at the same time, he also found this way useful for thinking about geometry without any spacial intuition (see [Fer00], page 94); Riemann coincides on this point with Lambert and his *analytical program*, introduced with the hope of solving the *classical problem* of the Euclidean Theory of Parallels (see [RR05], page 16). This program was completed by Hilbert in his fundamental work on foundations of geometry of 1899 ([Hil99]), using set theory introduced by Cantor. As Hilbert said[35]: *No one shall expel us from the paradise that Cantor has created for us.*

Appendix

A. Changes of coordinates

In the hyperbolic plane, apart from the polar or *cyclic* coordinates and the cartesian or *hypercyclic* coordinates, there are also the *paracyclic* or *horocyclic* coordinates in which one of the distances is measured on paracycles.

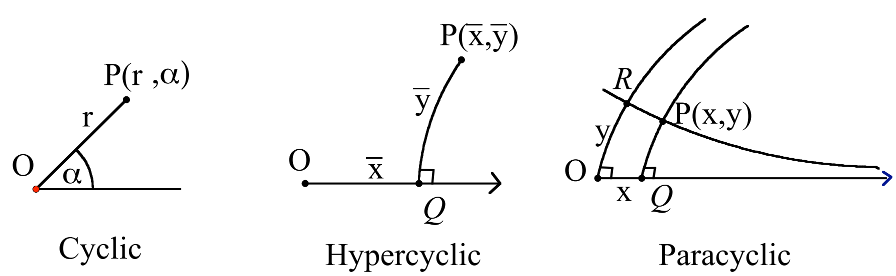

Figure 5.

Cyclic (r, α). Here r is the distance between the point P and the origin O; and α is the angle between the geodesic PO and a given geodesic through O. Observe that $r = constant$ is a hyperbolic circle.

Hypercyclic $(\bar{x}, \bar{y})$. Here $\bar{x}$ is the distance between the origin O and the point Q, intersection of the line through P orthogonal to a given line through O; and $\bar{y}$ is the distance between the point P and Q. Observe that $\bar{y} = constant$ is a hypercycle (equidistant).

[35] Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können.

Paracyclic (x, y). Here x is the distance between the origin O and the point Q, intersection with a given line through O of the horocycle through P and axis this line; and y is the length of the horocycle OR, where R is the intersection of the axe through P with the horocycle of this family through O. Observe that $x = constant$ is a paracycle (horocycle).

Recall that three points of the Hyperbolic plane determine a straight line, a circle, a hypercycle or a paracycle. The assumption that three points not on a line determine a circle is equivalent to the fifth postulate. In fact, this was the mistake made by Farkas Bolyai in his proof of this postulate.

Hypercyclic-Cyclic

The change of coordinates cyclic-hypercyclic is immediate applying trigonometry to a right triangle of catheti $\bar{x}, \bar{y}$ and hypotenuse r (see [RR05], page 120)

$$\begin{aligned} \cosh\frac{r}{R} &= \cosh\frac{\bar{x}}{R}\cosh\frac{\bar{y}}{R} \\ \sinh\frac{\bar{y}}{R} &= \sinh\frac{r}{R}\sin\theta, \end{aligned}$$

From this system we can write: $x = x(r, \theta)$, $y = y(r, \theta)$.

In particular,

$$d\bar{y}^2 + \cosh^2\frac{\bar{y}}{R}\, d\bar{x}^2 = dr^2 + R^2\sinh^2\frac{r}{R}\, d\theta^2.$$

Hypercyclic-Paracyclic

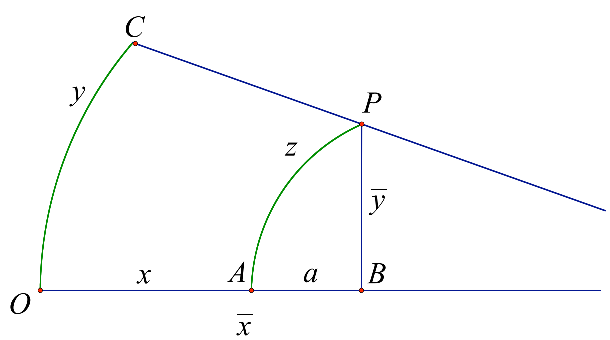

Figure 6.

Let us assume that the point P has hypercyclic coordinates $(\bar{x}, \bar{y})$, and paracyclic coordinates (x, y). In the picture, CO and PA are arcs of horocycles orthogonal to the parallel geodesics CP, OA. The hypercyclic coordinates are given by $\bar{x} = OB$, $\bar{y} = PB$; and the paracyclic coordinates are given by $x = OA$, $y = CO$.

The relation between the length z of the horocycle PA and the length $\bar{y}$ of the geodesic PB is

$$z = ye^{-x}. \tag{4}$$

Also

$$z = \sinh \bar{y}. \tag{5}$$

And

$$e^a = \cosh \bar{y}, \tag{6}$$

where $a = AB$. We remark that equations (4), (5) and (6) are given directly in the *Appendix*! (Equation (4) in §24 and equations (5) and (6) in §32). Bolyai writes $z = i \cot CBN$, that in our notation is $z = \cot \Pi(\bar{y})$, (we are assuming curvature $= -1$, i.e. $i = 1$), but it is easy to see that $\cot \Pi(\bar{y}) = \sinh \bar{y}$, and thus we have equation (5).

From these equations we can explicit the change of coordinates

$$\begin{aligned} \bar{x} &= x + \frac{1}{2} \ln(1 + y^2 e^{-2x}) \\ \bar{y} &= \ln(ye^{-x} + \sqrt{y^2 e^{-2x} + 1}). \end{aligned}$$

In particular,

$$d\bar{y}^2 + \cosh^2 \bar{y}\, d\bar{x}^2 = dx^2 + e^{-2x}\, dy^2.$$

B. A wasted opportunity

The stereographic projection between the sphere S_R of radius R and the plane that contains the equator is given by

$$\begin{aligned} p &= \frac{Rx}{R - z} \\ q &= \frac{Ry}{R - z} \end{aligned}$$

with $x^2 + y^2 + z^2 = R^2$.

Equivalently, the image of the point $(x, y, z) \in S_R$ is the complex number $w = p + iq$.

Let us *translate* the geometry of S_R to the extended complex plane $\mathbb{C}$ via this stereographic projection. First we note that the equator is given by

$$w\bar{w} = R^2.$$

Moreover, if w, w' are the images under the stereographic projection of antipodal points, then

$$w' = -\frac{R^2}{\bar{w}}. \tag{7}$$

Since stereographic projection takes circles to circles, the image of a meridian is a circle in the complex plane. Hence, if $P, Q \in \mathbb{C}$, the *straight line* PQ is the circle determined by the three points $P, Q, -P^*$, where P^* is the inverse point of P with respect to the circle $w\bar{w} = R^2$.

The *angles* of this geometry on $\mathbb{C}$ are the angles in S_R. Also *congruent* relations can be derived in this way. It is the geometry of the sphere considered in $\mathbb{C} \cup \{\infty\}$.

If we apply now the *Analogy* changing formally R by Ri in (7), we obtain

$$w' = \frac{R^2}{\bar{w}}.$$

Which are the *straight lines* of this new geometry? If $P, Q \in \mathbb{C}$, the *new straight line* PQ is the circle determined by the three points P, Q, P^*. It is well known that this circle is orthogonal to the circle $w\bar{w} = R^2$. Thus, the new straight lines are circles orthogonal to the boundary of the disc of radius R.

Remark that we have had to exclude the case $P = P^*$ because the three points must be different. But the set of points P with $P = P^*$ is the boundary of the disk. Thus this boundary does not belong to the new geometry.

Thus we have the open disc and its complement, which are 'equal' through inversion. If we consider the open disc with the *straight lines* defined above, and we consider that *movements* are generated by inversions, we have the classical Poincaré disc. That is, we have a model of non-Euclidean geometry and the problem of the consistency is solved. In fact, an inconsistency in non Euclidean geometry would be translated into an inconsistency in inversion geometry, and hence into an inconsistency in Euclidean geometry. The non Euclidean geometry is thus as consistent as Euclidean geometry.

We think that this construction of the Poincaré disc using stereographic projection and *Analogy*, that we have just done, could be done easily by MONGE or his school in the *École Polytechnique*, thirty years before the *Appendix*. This school had as leitmotif the translation of geometric properties using geometric transformations. In particular stereographic projections of the quadrics over the plane.

But, as we have mentioned before, we think that due LEGENDRE's influence they were not interested in the *classical problem* of the Euclidean Theory of Parallels. Moreover, LAGRANGE's analytical point of view spread rapidly all over the world and the synthetical approach was left buried until POINCARÉ found it again.

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Figure captions

Fig 1. Corresponds to figure 9 of the *Appendix*. See page 11.

Fig 2. The metric of the *Appendix*.

Fig 3. Length of a equidistant.

Fig 4. All figures in the *Appendix*.

Fig 5. Three coordinate systems.

Fig 6. Relation between hypercyclic and paracyclic coordinates.