ALGORITHM FOR DETERMINING THE GLOBAL GEOMETRIC CONFIGURATIONS OF SINGULARITIES OF QUADRATIC DIFFERENTIAL SYSTEMS WITH THREE DISTINCT REAL SIMPLE FINITE SINGULARITIES

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Abstract. In this work we consider the problem of classifying all configurations of singularities, both finite and infinite of quadratic differential systems, with respect to the *geometric equivalence relation* defined in [2]. This relation is finer than the *topological equivalence relation* which does not distinguish between a focus and a node or between a strong and a weak focus or between foci (or saddles) of different orders. Such distinctions are however important in the production of limit cycles close to the foci (or loops) in perturbations of the systems. The notion of *geometric equivalence relation* of configurations of singularities allows us to incorporate all these important geometric features which can be expressed in purely algebraic terms. This equivalence relation is also finer than the *qualitative equivalence relation* introduced in [18]. The *geometric classification* of all configurations of singularities, finite and infinite, of quadratic systems was initiated in [3] where the classification was done for systems with total multiplicity m_f of finite singularities less than or equal to one. That work was continued in [4] where the geometric classification was done for the case $m_f = 2$. In this article we go one step further and obtain the *geometric classification* of singularities, finite and infinite, for the subclass of quadratic differential systems possessing three distinct real finite singularities. We obtain 147 *geometrically distinct* configurations of singularities for this family. We also give here the global bifurcation diagram of configurations of singularities, both finite and infinite, with respect to the *geometric equivalence relation*, for this class of systems. The bifurcation diagram is done in the 12-dimensional space of parameters and it is expressed in terms of polynomial invariants, fact which gives us an algorithm for determining the geometric configuration of singularities for any quadratic system.

1. Introduction and statement of main results

We consider here differential systems of the form

$$\frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1}$$

where $p,\, q \in \mathbb{R}[x,y]$, i.e. $p,\ q$ are polynomials in $x,\ y$ over $\mathbb{R}$. We call *degree* of a system (1) the integer $m = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1) with $m = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

The study of the class **QS** has proved to be quite a challenge since hard problems formulated more than a century ago, are still open for this class. It is expected that we have a finite number of phase portraits in **QS**. We have phase portraits for several subclasses of **QS** but the complete list of phase portraits of this class is not known and attempting to topologically classify these systems, which occur rather often in applications, is a very complex task. This is partly due to the elusive nature of limit cycles and partly to the rather large number of parameters involved. This family of systems depends on twelve parameters but due to the group action of real affine transformations and time homotheties, the class ultimately depends on five parameters

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which is still a rather large number of parameters. For the moment only subclasses depending on at most three parameters were studied globally, including global bifurcation diagrams (for example [1]). On the other hand we can restrict the study of the whole quadratic class by focusing on specific global features of the systems in this family. We may thus focus on the global study of singularities and their bifurcation diagram. The singularities are of two kinds: finite and infinite. The infinite singularities are obtained by compactifying the differential systems on the sphere or on the Poincaré disk as defined in Section 2 (see also [15]).

The global study of quadratic vector fields in the neighborhood of infinity was initiated by Coll in [14] where he characterizes all possible phase portraits in a neighborhood of infinity in terms of the coefficients of the normal forms. Later on Nikolaev and Vulpe in [21] classified topologically the configurations of singularities at infinity in terms of invariant polynomials. Schlomiuk and Vulpe used geometrical concepts defined in [26], and also introduced some new geometrical concepts in [27] in order to simplify the invariant polynomials and to render more transparent the classification. To reduce the number of phase portraits in half, in both cases the *topological equivalence relation* was taken to mean the existence of a homeomorphism of the phase plane carrying orbits to orbits and *preserving or reversing* the orientation. In [5] the authors classified topologically (adding also the distinction between nodes and foci) the whole quadratic class, according to configurations of their finite singularities.

In the topological classification no distinction was made among the various types of foci or saddles, strong or weak of various orders. However these distinctions of an algebraic nature are very important in the study of perturbations of systems possessing such singularities. Indeed, the maximum number of limit cycles which can be produced close to the weak foci in perturbations depends on the orders of the foci.

The distinction among weak saddles is also important because for example when a loop is formed using two separatrices of one weak saddle, the maximum number of limit cycles that can be obtained close to the loop in perturbations is the order of weak saddle.

There are also three kinds of simple nodes as we can see in Figure 1 below where the local phase portraits around the singularities are given.

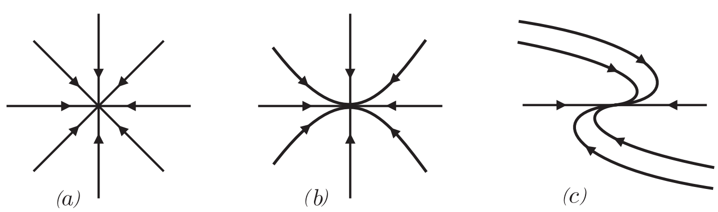


Figure 1. Different types of nodes.

In the three phase portraits of Figure 1 the corresponding three singularities are stable nodes. These portraits are topologically equivalent but the solution curves do not arrive at the nodes in the same way. In the first case, any two distinct non-trivial phase curves arrive at the node with distinct slopes. Such a node is called a star node. In the second picture all non-trivial solution curves excepting two of them arrive at the node with the same slope but the two exception curves arrive at the node with different slopes. This is the generic node with two directions. In the third phase portrait all phase curves arrive at the node with the same slope. The three kinds of nodes are distinguished algebraically. By this we mean that the linearization matrices at these nodes and their eigenvalues, distinguish the nodes. More precisely we have the two direction nodes (b) which have distinct eigenvalues, the one direction nodes (c) with identical eigenvalues but linearization is not diagonal and star nodes (a) with identical eigenvalues and linearization is diagonal. (see [2], [3], [4]).

We recall that the first and the third types of nodes could produce foci in perturbations and the first type of nodes is also involved in the existence of invariant straight lines of differential systems. For example it can easily be shown that if a quadratic differential system has two finite star nodes then necessarily the system possesses invariant straight lines of total multiplicity 6.

Furthermore, a generic node at infinity may or may not have the two exceptional curves lying on the line at infinity. This leads to two different situations for the phase portraits. For this reason we split the generic nodes at infinity in two types as indicated in Section 4.

The geometric equivalence relation for finite or infinite singularities, introduced in [2] and used in [3] and [4], takes into account such distinctions. This equivalence relation is finer than the *qualitative equivalence relation* introduced by Jiang and Llibre in [18] because it distinguishes among the foci (or saddles) of different orders and among the various types of nodes. This equivalence relation also induces a finer distinction among the more complicated degenerate singularities.

To distinguish among the foci (or saddles) of various orders we use the algebraic concept of Poincaré-Lyapunov constants. We call strong focus (or strong saddle) a focus (or a saddle) with non–zero trace of the linearization matrix at this point. Such a focus (or saddle) will be considered to have the order zero and will be denoted by f or by $f^{(0)}$ (respectively s or $s^{(0)}$). A focus (or saddle) with trace zero is called a weak focus (weak saddle). In quadratic systems these could be of orders 1, 2 or 3 [10]. For details on Poincaré-Lyapunov constants and weak foci of various orders we refer to [25], [19]. In [30] necessary and sufficient conditions for a quadratic system to have weak foci (saddles) of orders i, i=1,2,3 are given in invariant form.

Algebraic information may not be significant for the local (topological) phase portrait around a singularity. For example, topologically there is no distinction between a focus and a node or between a weak and a strong focus. However, as indicated before, algebraic information plays a fundamental role in the study of perturbations of systems possessing such singularities.

Although we now know that in trying to understand these systems, there is a limit to the power of algebraic methods, these methods have not been used far enough. For example the global classification of configurations of singularities, finite and infinite, using the *geometric equivalence relation* (see further below), can be done by using only algebraic methods. To obtain this global *geometric classification* by purely algebraic tools is a long term goal since we expect to finally obtain over 800 distinct geometric configurations of singularities. The first step in this direction was done in [2] where the global classification of singularities at infinity of *the whole class* **QS**, was done according to the *geometric equivalence relation* of configurations of infinite singularities by using only algebraic methods. This work needs to be extended to also incorporate all the finite singularities. We initiated the work in this direction in [3] where this classification was done for all global configurations of singularities which have a total finite multiplicity $m_f \leq 1$ and we continued the classification for $m_f = 2$ in [4].

In the present article our goal is to go one step further in the *geometric classification* of global configurations of singularities by studying here the case of three real distinct finite singularities.

We recall here below the notion of *geometric configuration of singularities* defined in [4] for both finite and infinite singularities. We distinguish two cases:

1) If we have a finite number of infinite singular points and a finite number of finite singularities we call *geometric configuration of singularities*, finite and infinite, the set of all these singularities each endowed with its own multiplicity together with the local phase portraits around real singularities endowed with additional geometric structure involving the concepts of tangent, order and blow–up equivalences defined in Section 4 of [2] and using the notations described here in Section 4.

2) If the line at infinity $Z = 0$ is filled up with singularities, in each one of the charts at infinity $X \neq 0$ and $Y \neq 0$, the corresponding system in the Poincaré compactification (see Section 2) is degenerate and we need to do a rescaling of an appropriate degree of the system, so that the degeneracy be removed. The resulting systems have only a finite number of singularities on the line $Z = 0$. In this case we call *geometric configuration of singularities*, finite and infinite, the set of all points at infinity (they are all singularities)

in which we single out the singularities at infinity of the "reduced" system, taken together with their local phase portraits and we also take the local phase portraits of finite singularities each endowed with additional geometric structure to be described in Section 4.

Main Theorem. (*A*) *We consider here all geometric configurations of singularities, finite and infinite, of quadratic vector fields with three distinct real finite singularities. These configurations are classified in* DIAGRAM 1 *according to the geometric equivalence relation. The total number of geometrically distinct configurations of singularities, finite and infinite, is 147.*

(*B*) *Necessary and sufficient conditions for each one of the 147 different geometric equivalence classes can be assembled from these diagrams in terms of 21 polynomial invariants with respect to the action of the affine group and time rescaling, given in Section 5.*

(*C*) *The* DIAGRAM 1 *actually contains the global bifurcation diagram in the 12-dimensional space of parameters, of the global geometric configurations of singularities, finite and infinite, of this family of quadratic differential systems and provides an algorithm for deciding for any given system which is its respective configuration.*

We use the concepts and notations introduced in [2] and [3] which we describe in Sections 3 and 4. We denote by $f^{(i)}$ ($s^{(i)}$) the weak foci (weak saddles) of order i and by c and $\$$ the centers and integrable saddles. For more notations see Section 4.

We introduce two affine invariants: TOW=the total order, i.e. the sum of the orders of the weak finite singularities, saddles or foci; MOW=the maximum of the orders of the weak finite singularities.

Corollary. *We sum up below relevant data regarding the configurations of singularities which follow from the bifurcation diagram* DIAGRAM 1 *(for notations see Section 4.*

Regarding the singularities at infinity:

- *We have no configurations with a star node at infinity.*
- *We have no configurations with an intricate singular point at infinity.*
- *A nilpotent point could have either four or two sectors in which case the point is an elliptic saddle described as* $\widehat{\binom{1}{2}}\, H-E$.
- *We have 46 configurations with a (unique) nilpotent singular point at infinity, 39 of multiplicity* $\widehat{\binom{1}{2}}$, *7 of multiplicity* $\widehat{\binom{1}{3}}$.
- *We have 75 configurations with a single semi-elemental saddle-node* $\overline{\binom{1}{1}}SN$ *out of which we have 29 cases with* $\overline{\binom{1}{1}}SN$, ©, ©, *29 cases with* $\overline{\binom{1}{1}}SN$, S *and* N^∞, *and 17 cases with* $\overline{\binom{1}{1}}SN$, N^f, N^f.
- *There are 24 cases with two semi-elemental saddle-nodes* $\overline{\binom{1}{1}}SN$, $\overline{\binom{0}{2}}SN$.
- *We have only two configurations with the line at infinity filled up with singularities:* s, n, n; $[\infty; \emptyset]$ *and* $\$, n, n$; $[\infty; \emptyset]$.

Regarding the finite singularities and their corresponding singularities at infinity:

- *We have 17 configurations with 2 foci which necessarily appear with a saddle and we have* $MOW \leq 2$ *and* $TOW \leq 2$. *For* $MOW \leq 1$, *apart from a saddle-node* $\overline{\binom{1}{1}}SN$ *at infinity, in all configurations we have either* ©, © *or* S, N^∞ *or just* $\overline{\binom{0}{2}}SN$. *If* $MOW = 2 = TOW$ *then at infinity, apart from a saddle-node* $\overline{\binom{1}{1}}SN$, *we just have* ©, ©.
- *We have 53 configurations with two nodes, the third point necessarily being a saddle* $s^{(i)}$ *with order* $i = 0, 1, 2, 3$ *or an integrable saddle. All possible combination of nodes* n *and* n^d *occur. The possible configurations at infinity are: 1) one saddle-node* $\overline{\binom{1}{1}}SN$ *and two simple singular points either* ©, © *or* S, N^∞*; 2) two saddle-nodes* $\overline{\binom{1}{1}}SN$, $\overline{\binom{0}{2}}SN$*; 3) a nilpotent point of multiplicity three and either an* S *or an* N^∞, *and 4) a single singular point which is nilpotent of multiplicity 4.*

- *We have 11 configurations with 2 saddles and a focus. Any combination with $TOW \leq 2$ occurs in some configuration. If in a combination we have $TOW = 3$ then necessarily we have s, s, and $f^{(3)}$. We have 8 configurations with $\overline{\binom{1}{1}}SN$ and N^f, N^f, and 3 with a nilpotent singular point at infinity and an N^f and these only occur in the case $TOW \leq 1$.*
- *We have 13 configurations with 2 saddles and a node. In such a case $TOW \leq 2$. We have 9 configurations for which at infinity we have $\overline{\binom{1}{1}}SN$ and N^f, N^f, and 4 configurations for which at infinity we have a nilpotent point and an N^f.*
- *We have only 2 configurations with a center which are: s, s, c and $\$, \$, c$, in this last case the system is Hamiltonian. In both cases at infinity we have $\widehat{\binom{1}{2}}\ \widehat{P}_\lambda E\widehat{P}_\lambda - H$, N^f.*
- *We have 51 configurations with a saddle, a focus and a node. In this case $TOW \leq 3$ and all combinations (regarding the focus and the saddle) with $TOW \leq 2$ are possible. There is only one possibility with a focus $f^{(3)}$ in which case at infinity we have $\overline{\binom{1}{1}}SN$, S, N^∞, but three possibilities with a saddle $s^{(3)}$ in which case apart from the saddle-node $\overline{\binom{1}{1}}SN$, at infinity we have either ©, ©, or S, N^∞, or just $\overline{\binom{0}{2}}SN$.*
- *We have only three configurations with a finite star node, i.e., s, n, n^*; $\overline{\binom{1}{1}}SN$, S, N^∞ and s, n^d, n^*; $\overline{\binom{1}{1}}SN$, S, N^∞, and s, s, n^*; $\overline{\binom{1}{1}}SN$, N^f, N^f.*

The invariants and comitants of differential equations used for proving our main results are obtained following the theory of algebraic invariants of polynomial differential systems, developed by Sibirsky and his disciples (see for instance [29, 32, 23, 7, 13]).

Remark 1. *We note that the geometric equivalence relation for configurations is much finer than the topological equivalence. Indeed, for example the topological equivalence does not distinguish between the following three configurations which are geometrically non-equivalent: 1) n, f; $\overline{\binom{1}{1}}SN$, ©, ©, 2) n, $f^{(1)}$; $\overline{\binom{1}{1}}SN$, ©, ©, and 3) n^d, $f^{(1)}$; SN, ©, © where n and n^d mean singularities which are nodes, capital letters indicate points at infinity, © in case of a complex point and SN a saddle–node at infinity.*

2. Compactifications associated to planar polynomial differential systems

2.1. **Compactification on the sphere and on the Poincaré disk.** Planar polynomial differential systems (1) can be compactified on the 2–dimensional sphere. For this we consider the affine plane of coordinates (x, y) as being the plane $Z = 1$ in $\mathbb{R}^3$ with the origin located at $(0, 0, 1)$, the x–axis parallel with the X–axis in $\mathbb{R}^3$, and the y–axis parallel to the Y–axis. We use central projection to project this plane on the sphere as follows: for each point $(x, y, 1)$ we consider the line joining the origin with $(x, y, 1)$. This line intersects the sphere in two points $P_1 = (X, Y, Z)$ and $P_2 = (-X, -Y, -Z)$ where $(X, Y, Z) = (1/\sqrt{x^2 + y^2 + 1})(x, y, 1)$. The applications $(x, y) \mapsto P_1$ and $(x, y) \mapsto P_2$ are bianalytic and associate to a vector field on the plane (x, y) an analytic vector field Ψ on the upper hemisphere and also an analytic vector field Ψ' on the lower hemisphere. A theorem stated by Poincaré and proved in [16] says that there exists an analytic vector field Θ on the whole sphere which simultaneously extends the vector fields on the two hemispheres. We call *Poincaré compactification on the sphere* of the planar polynomial system, the restriction $\bar{\Psi}$ of the vector field Θ to the upper hemisphere completed by the equator. For more details we refer to [15].

The vertical projection of $\bar{\Psi}$ on the plane $Z = 0$ indices a vector field Φ on the unit disk, called the *Poincaré compactification on the disk* of the polynomial differential system. This vector field is diffeomorphic to $\bar{\Psi}$. By a *singular point at infinity* of a planar polynomial vector field we mean a singular point of the vector field $\bar{\Psi}$ which is located on the equator of the sphere, respectively a singular point of the vector field Φ located on the boundary circle of the Poincaré disk.

2.2. **Compactification on the projective plane.** To a polynomial system (1) we can associate a differential equation $\omega_1 = q(x, y)dx - p(x, y)dy = 0$. Since the differential system (1) is with real coefficients, we may associate to it a foliation with singularities on the real, respectively complex, projective plane as indicated

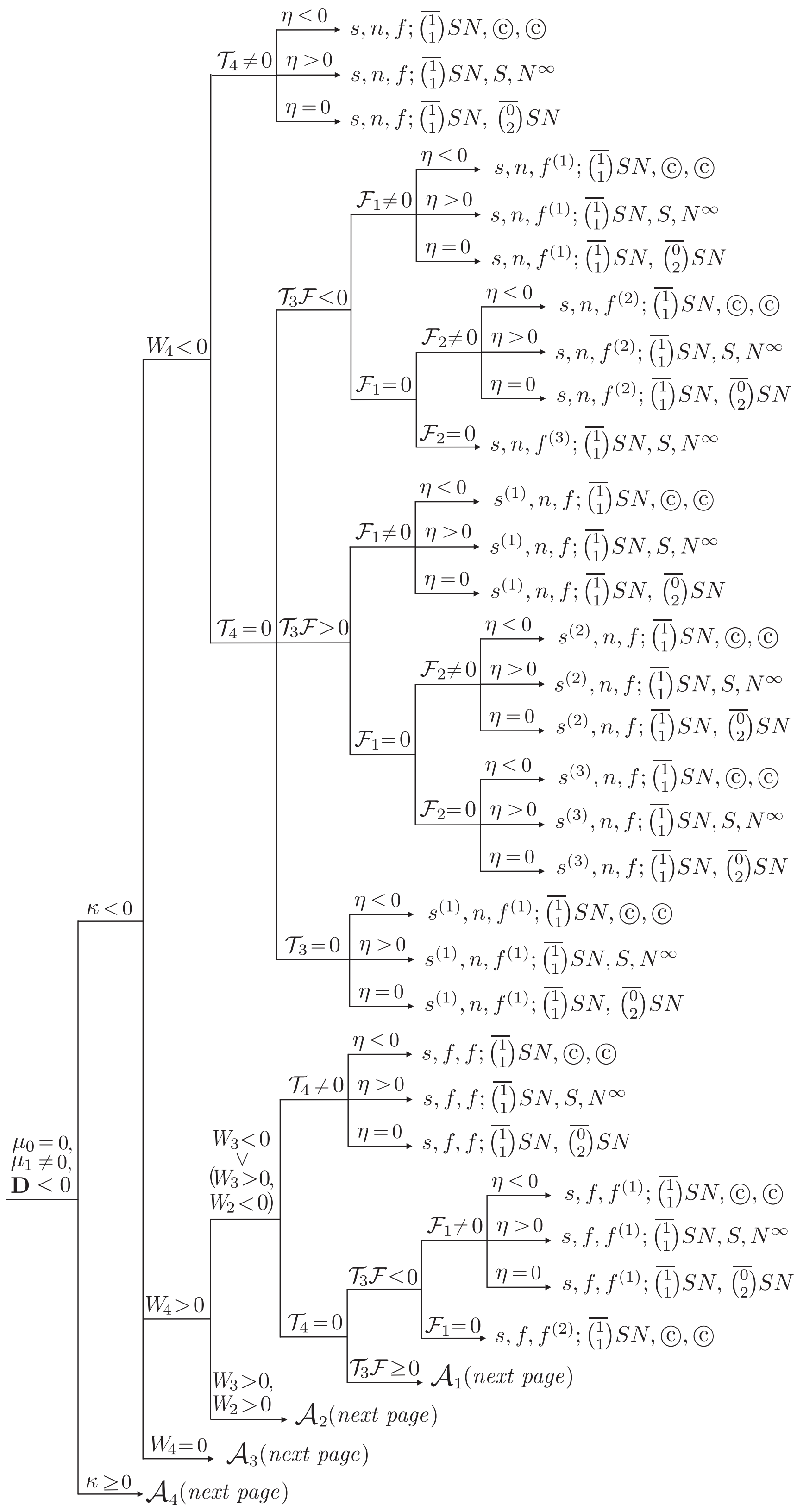


DIAGRAM 1. **Global configurations: the case $\mu_0 = 0$, $\mu_1 \neq 0$, $D < 0$.**

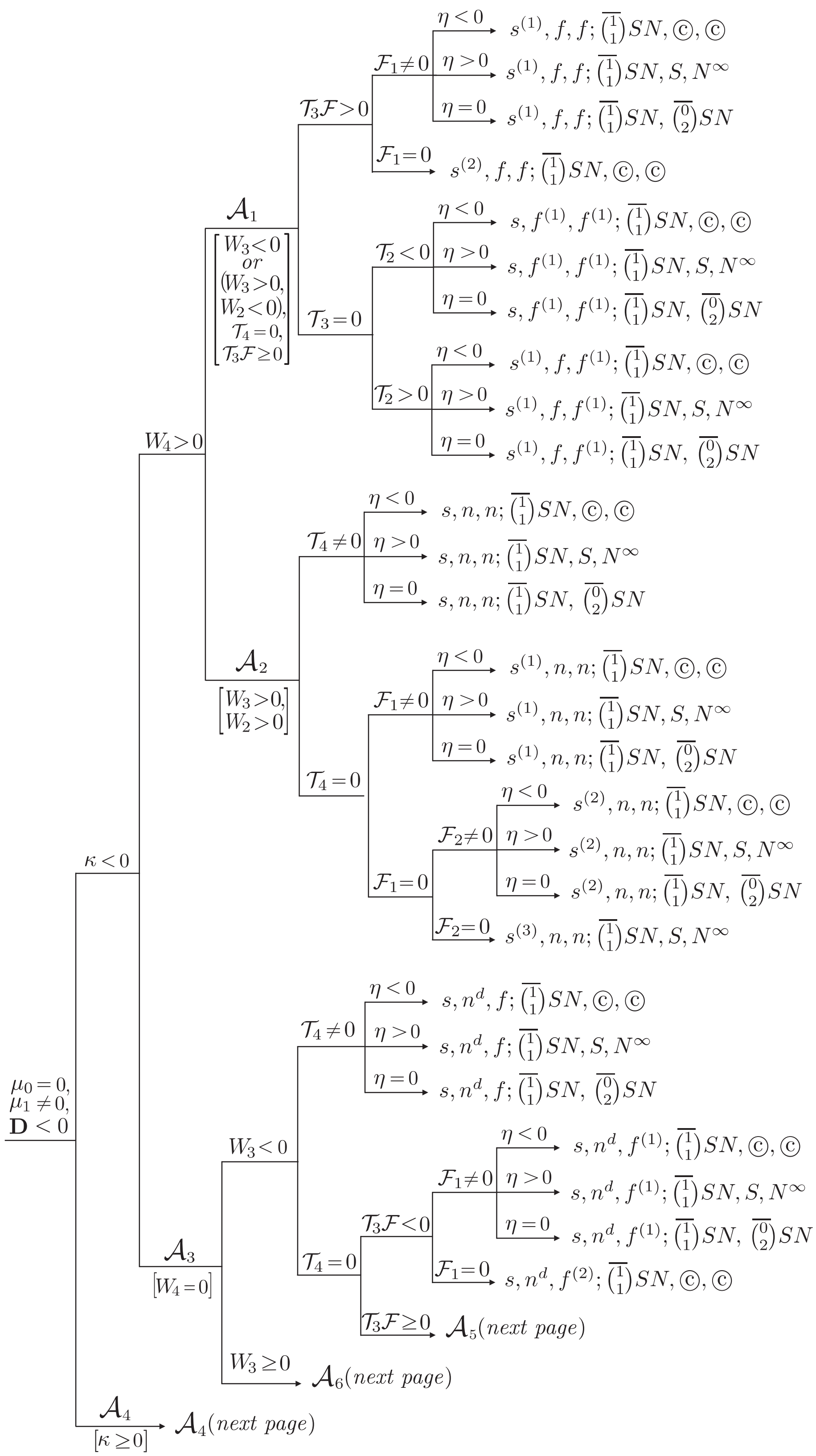


DIAGRAM 1 (*continued*). **Global configurations: the case $\boldsymbol{\mu_0=0}$, $\boldsymbol{\mu_1\neq 0}$, $\boldsymbol{D<0}$.**

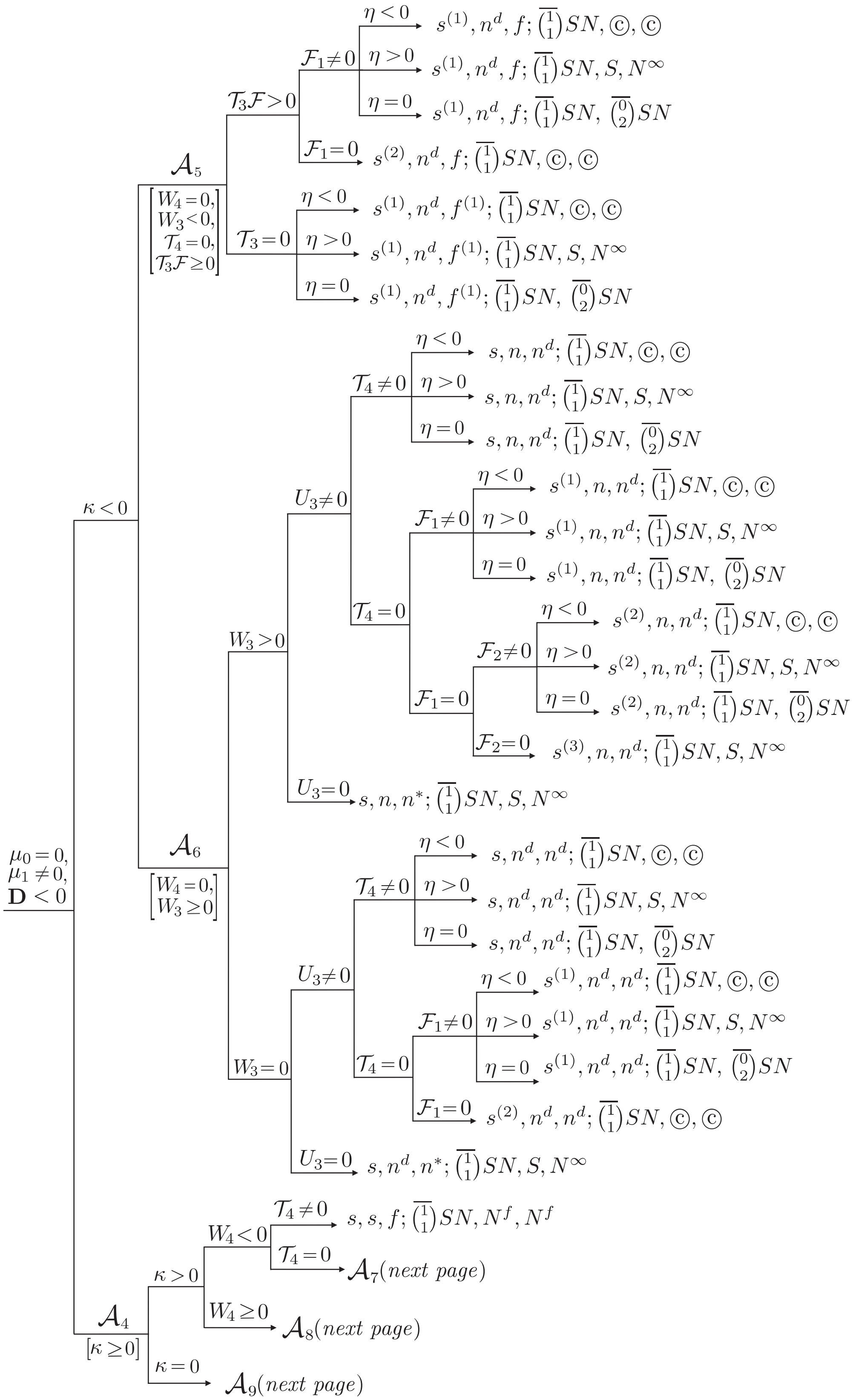


DIAGRAM 1 (*continued*). **Global configurations: the case $\mu_0 = 0$, $\mu_1 \neq 0$, $D < 0$.**

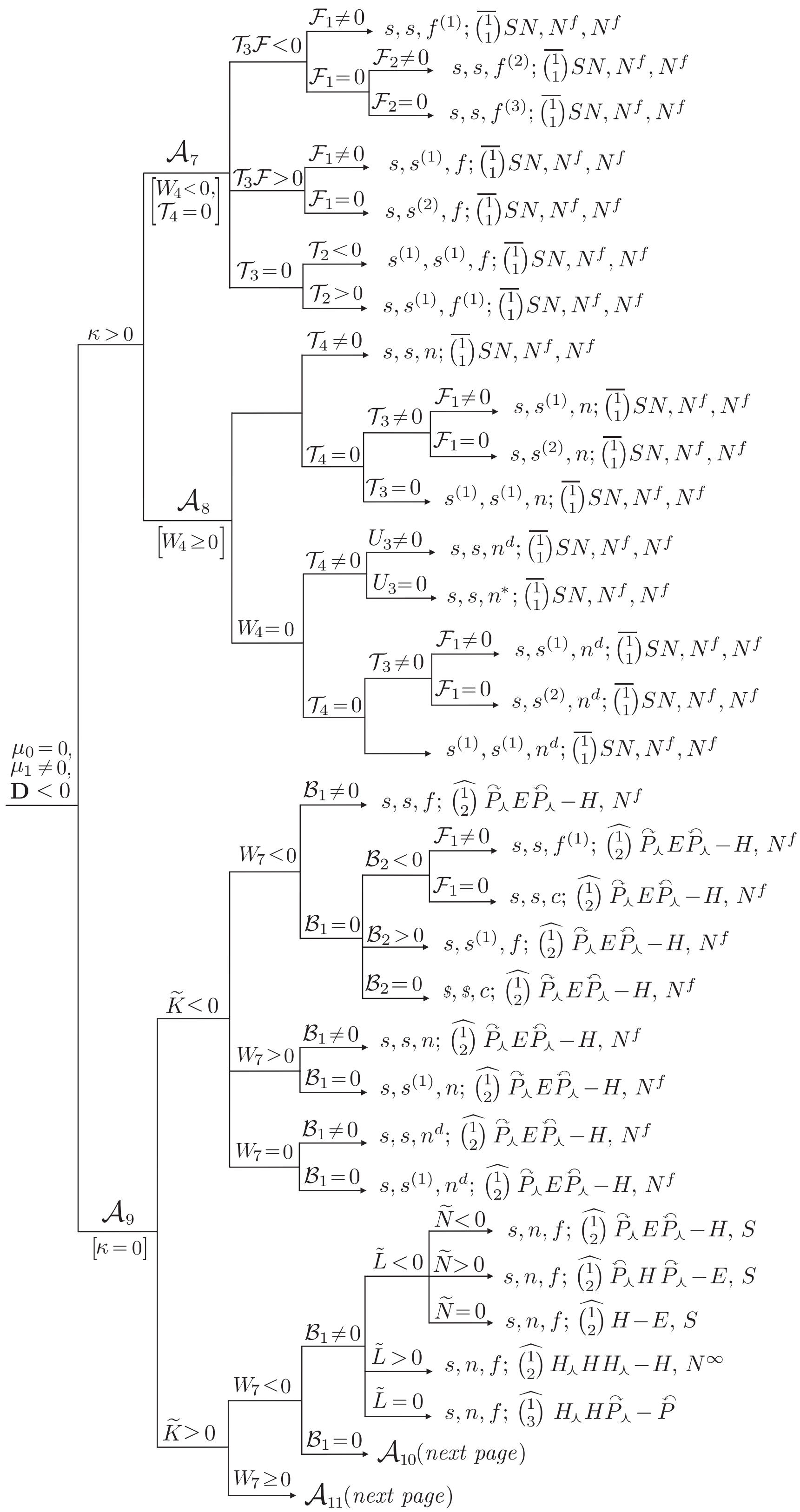


Diagram 1 (*continued*). **Global configurations: the case $\mu_0=0$, $\mu_1\neq0$, $D<0$.**

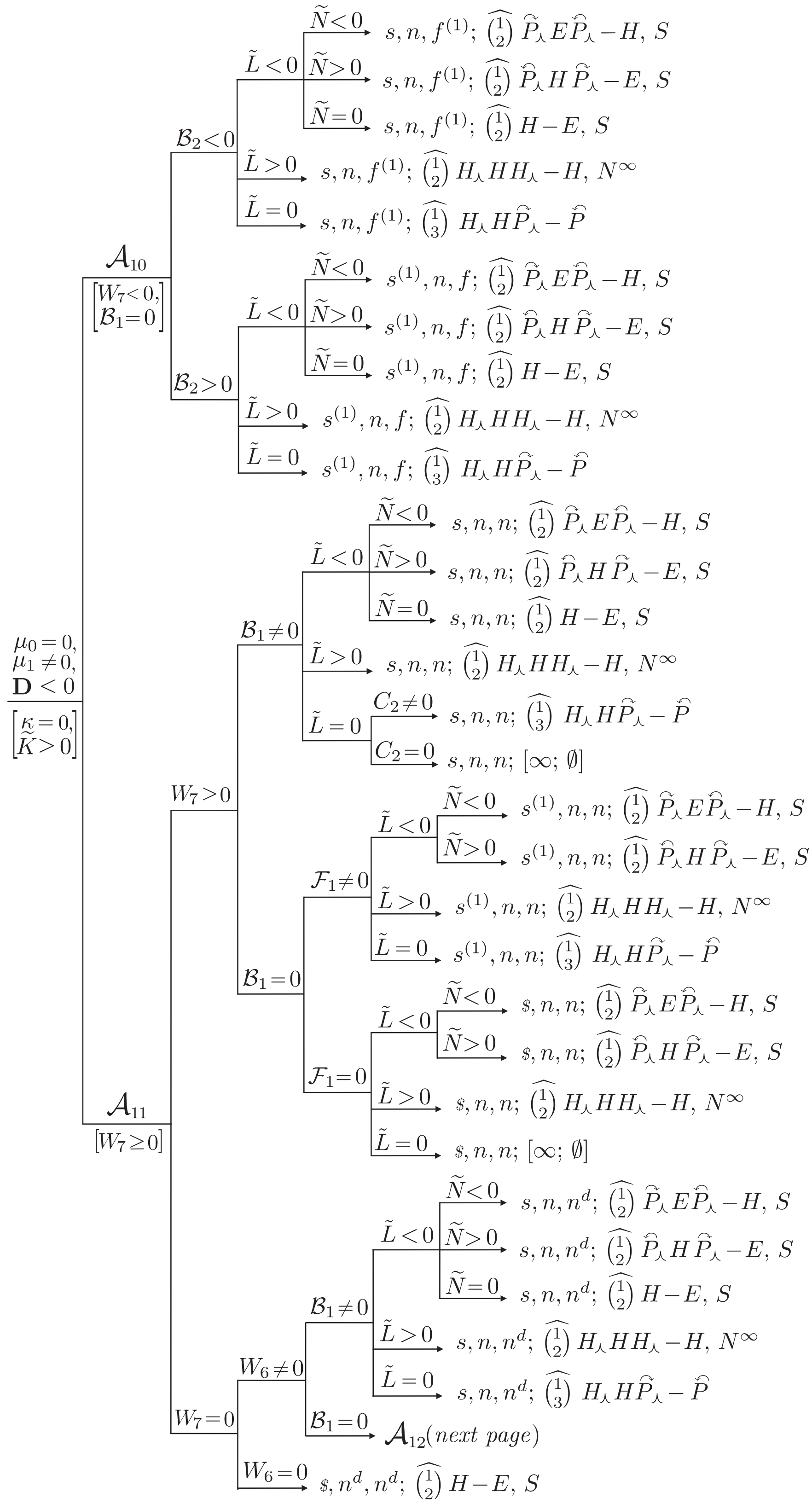


DIAGRAM 1 (*continued*). **Global configurations: the case $\mu_0=0$, $\mu_1\neq 0$, $D<0$.**

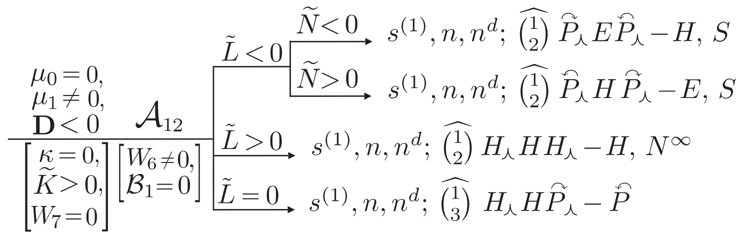


DIAGRAM 1 (*continued*). **Global configurations: the case $\mu_0 = 0$, $\mu_1 \neq 0$, $D < 0$.**

below. The equation $\omega_1 = 0$ defines a foliation with singularities on the real or complex plane depending if we consider the equation as being defined over the real or complex affine plane. It is known that we can compactify these foliations with singularities on the real respectively complex projective plane. In the study of real planar polynomial vector fields, their associated complex vector fields and their singularities play an important role. In particular such a vector field could have complex, non-real singularities, by this meaning singularities of the associated complex vector field. We briefly recall below how these foliations with singularities are defined.

The application $\Upsilon : \mathbb{K}^2 \longrightarrow P_2(\mathbb{K})$ defined by $(x,y) \mapsto [x:y:1]$ is an injection of the plane $\mathbb{K}^2$ over the field $\mathbb{K}$ into the projective plane $P_2(\mathbb{K})$ whose image is the set of $[X:Y:Z]$ with $Z \neq 0$. If $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ this application is an analytic injection. If $Z \neq 0$ then $(\Upsilon)^{-1}([X:Y:Z]) = (x,y)$ where $(x,y) = (X/Z, Y/Z)$. We obtain a map $i : \mathbb{K}^3 \setminus \{Z=0\} \longrightarrow \mathbb{K}^2$ defined by $[X:Y:Z] \mapsto (X/Z, Y/Z)$.

Considering that $dx = d(X/Z) = (ZdX - XdZ)/Z^2$ and $dy = (ZdY - YdZ)/Z^2$, the pull-back of the form ω_1 via the map i yields the form $i_*(\omega_1) = q(X/Z, Y/Z)(ZdX - XdZ)/Z^2 - p(X/Z, Y/Z)(ZdY - YdZ)/Z^2$ which has poles on $Z = 0$. Then the form $\omega = Z^{m+2} i_*(\omega_1)$ on $K^3 \setminus \{Z=0\}$, K being $\mathbb{R}$ or $\mathbb{C}$ and m being the degree of systems (1) yields the equation $\omega = 0$:

$$A(X,Y,Z)dX + B(X,Y,Z)dY + C(X,Y,Z)dZ = 0$$

on $K^3 \setminus \{Z=0\}$ where A, B, C are homogeneous polynomials over K with $A(X,Y,Z) = ZQ(X,Y,Z)$, $Q(X,Y,Z) = Z^m q(X/Z, Y/Z)$, $B(X,Y,Z) = ZP(X,Y,Z)$, $P(X,Y,Z) = Z^m p(X/Z, Y/Z)$ and $C(X,Y,Z) = YP(X,Y,Z) - XQ(X,Y,Z)$.

The equation $AdX + BdY + CdZ = 0$ defines a foliation F with singularities on the projective plane over K with K either $\mathbb{R}$ or $\mathbb{C}$, which is the *compactification on the projective plane* of the foliation with singularities defined by $\omega_1 = 0$ on the real (respectively complex) affine plane. The *points at infinity* of the foliation defined by $\omega_1 = 0$ on the affine plane are the points $[X:Y:0]$ and the line $Z = 0$ is called the *line at infinity* of this foliation.

The singular points of the foliation F are the solutions of the three equations $A = 0$, $B = 0$, $C = 0$. In view of the definitions of A, B, C it is clear that the singular points at infinity are the points of intersection of $Z = 0$ with $C = 0$.

2.3. **Assembling multiplicities of singularities in divisors of the line at infinity and in zero-cycles of the plane.** There are two types of multiplicities for a singular point p at infinity: one expresses the maximum number m of finite singularities which can split from p, in small perturbations of the system and the other expresses the maximum number m' of infinite singularities which can split from p, in small perturbations of the system. We shall use a column $(m, m')^t$ to indicate this situation.

We are interested in the global picture which includes *all* singularities, finite and infinite. Therefore we need to assemble the data for individual singularities in a convenient, precise way. We use for the notion of *cycle* on an algebraic variety as indicated in [22] and which was used in [19] as well as in [27].

We briefly recall here the definition of cycle. Let V be an irreducible algebraic variety over a field $\mathbb{K}$. A *cycle* of dimension r or $r-cycle$ on V is a formal sum $\sum_W n_W W$, where W is a subvariety of V of dimension

r which is not contained in the singular locus of V, $n_W \in \mathbb{Z}$, and only a finite number of the coefficients n_W are non-zero. The *degree* $\deg(J)$ of a cycle J is defined by $\sum_W n_W$. An $(n-1)$-cycle is called a *divisor* on V. These notions were used for classification purposes of planar quadratic differential systems in [22, 19, 27].

To a system (1) with isolated singularities at infinity we can associate two divisors on the line at infinity $Z = 0$ of the complex projective plane: $D_S(P, Q; Z) = \sum_w I_w(P, Q)w$ and $D_S(C, Z) = \sum_w I_w(C, Z)w$ where $w \in \{Z = 0\}$ and where by $I_w(F, G)$ we mean the intersection multiplicity at w of the curves $F(X, Y, Z) = 0$ and $G(X, Y, Z) = 0$, with F and G homogeneous polynomials in X, Y, Z over $\mathbb{C}$. For more details see [19].

Following [27] we assemble the above two divisors on the line at infinity into just one but with values in the ring $\mathbb{Z}^2$:

$$D_S = \sum_{\omega \in \{Z=0\}} \begin{pmatrix} I_w(P, Q) \\ I_w(C, Z) \end{pmatrix} w.$$

This divisor encodes the total number of singularities at infinity of a system (1) as well as the two kinds of multiplicities which each singularity has. The meaning of these two kinds of multiplicities are described in the definition of the two divisors $D_S(P, Q; Z)$ and $D_S(C, Z)$ on the line at infinity (see [19], [1]).

For systems (1) with isolated finite singularities we also consider the zero-cycle $D_S(p, q) = \sum_{\omega \in \mathbb{R}^2} I_w(p, q)w$ assembling its multiplicities of all its finite singularities.

3. Some geometrical concepts

Firstly we recall some terminology.

We call *elemental* a singular point with its both eigenvalues not zero.

We call *semi–elemental* a singular point with exactly one of its eigenvalues equal to zero.

We call *nilpotent* a singular point with both its eigenvalues zero but with its Jacobian matrix at this point not identically zero.

We call *intricate* a singular point with its Jacobian matrix identically zero.

The *intricate* singularities are usually called in the literature *linearly zero*. We use here the term *intricate* to indicate the rather complicated behavior of phase curves around such a singularity.

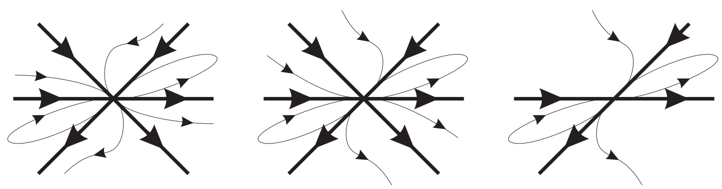
In this section we use the same concepts we considered in [2] and [3] such as *orbit γ tangent to a semi–line L at p*, *well defined angle at p*, *characteristic orbit at a singular point p*, *characteristic angle at a singular point*, *characteristic direction at p*. Since these are basic concepts for the notion of *geometric equivalence relation* we recall here these notions as well as a few others.

We assume that we have an isolated singularity p. Suppose that in a neighborhood U of p there is no other singularity. Consider an orbit γ in U defined by a solution $\Gamma(t) = (x(t), y(t))$ such that $\lim_{t\to+\infty} \Gamma(t) = p$ (or $\lim_{t\to-\infty} \Gamma(t) = p$). For a fixed t consider the unit vector $C(t) = (\overrightarrow{\Gamma(t) - p})/\|\overrightarrow{\Gamma(t) - p}\|$. Let L be a semi–line ending at p. We shall say that *the orbit γ is tangent to a semi–line L at p* if $\lim_{t\to+\infty} C(t)$ (or $\lim_{t\to-\infty} C(t)$) exists and L contains this limit point on the unit circle centered at p. In this case we call *well defined angle of Γ at p* the angle between the positive x–axis and the semi–line L measured in the counterclockwise sense. We may also say that *the solution curve $\Gamma(t)$ tends to p with a well defined angle*. A *characteristic orbit at a singular point p* is the orbit of a solution curve $\Gamma(t)$ which tends to p with a well defined angle. We call *characteristic angle at the singular point p* a well defined angle of a solution curve $\Gamma(t)$. The line through p extending the semi-line L is called a *characteristic direction*.

If a singular point has an infinite number of characteristic directions, we will call it a *star–like* point.

It is known that the neighborhood of any isolated singular point of a polynomial vector field, which is not a focus or a center, is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [15]). It is also known that any degenerate singular point can be desingularized by means of a finite number of changes of variables, called blow–up's, into elementary singular points (for more details see the Section on blow–up in [2] or [15]).

Consider the three singular points given in Figure 2. All three are topologically equivalent and their neighborhoods can be described as having two elliptic sectors and two parabolic ones. But we can easily detect some geometric features which distinguish them. For example (a) and (b) have three characteristic directions and (c) has only two. Moreover in (a) the solution curves of the parabolic sectors are tangent to only one characteristic direction and in (b) they are tangent to two characteristic directions. All these properties can be determined algebraically.



(a) (b) (c)

Figure 2. Some topologically equivalent singular points.

The usual definition of a sector is of topological nature and it is local with respect to a neighborhood around the singular point. We work with a new notion, namely of *geometric local sector*, introduced in [2] which distinguishes the phase portraits of Figure 2. As we shall later see this notion is characterized in algebraic terms.

We begin with the elemental singular points having characteristic directions. These are either two-directions nodes, one-direction nodes, star nodes or saddles. The first three cases are distinguished algebraically using their eigenvalues (see Figure 1). In the case of saddles the notion or geometric local sector coincides with usual notion of topological sector.

We consider now the semi–elemental singular points. These could be saddles, nodes or saddle–nodes. Each saddle has four separatrices and four hyperbolic sectors. Here again we call geometric local sector any one of these hyperbolic sectors and we call *borsec* (contraction of border with sector) any one of the four separatrices.

A semi–elemental node has two characteristic directions generating four half lines. For each one of these half lines there exists at least one orbit tangent to that half line and we pick an orbit tangent to that half line. Removing these four orbits together with the singular point, we are left with four sectors which we call *geometric local sectors* and we call *borsecs* these four orbits.

Consider now a semi–elemental saddle–node. Such a singular point has three separatrices and three topological sectors, two hyperbolic ones and one parabolic sector. Such a singular point has four characteristic half lines and one of them separates the parabolic sector in two. By removing an orbit tangent to a half line for each half lines as well as the singular point we obtain four sectors which we call geometric local sectors. We call borsecs these four orbits.

We now proceed to extend the notion of geometric local sector and of borsec for nilpotent and intricate singular points.

The introduction of the concept of borsec in the general case will play a role in distinguishing a semi–elemental saddle–node from an intricate saddle–node such as the one indicate in Figure 3. In the elemental saddle–node all orbits inside the parabolic sector are tangent to the same half–line but in the saddle–node of Figure 3 the orbits in the parabolic sector are not all tangent to the same half–line. The orbits in this parabolic sector are of three kinds: the ones tangent to separatrix (a), the ones tangent to separatrix (c) and a single orbit which is tangent to other half–line of the characteristic direction defined by separatrix (b). In this case this last orbit is called the borsec. The other three borsecs are separatrices as in the case of the semi–elemental node.

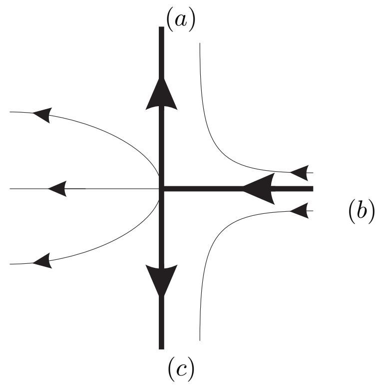


FIGURE 3. Local phase portrait of a non semi–elemental saddle–node.

To extend the notion of geometric local sector and of borsec for nilpotent and intricate singular points we start by introducing some terminology.

Let δ be the border of a sufficiently small open disc D centered at point p so that δ intersects all the elliptic, parabolic and hyperbolic sectors of a nilpotent or intricate singular point p.

Consider a solution $\Gamma: (a,b) \to \mathbb{R}^2$ where (a,b) is its maximal interval of definition and let γ be the orbit of Γ, i.e. $\gamma = \{\Gamma(t) \,|\, t \in (a,b)\}$. We call *half orbit* of γ at a singular point p a subset $\gamma' \subseteq \gamma$ such that there exists $t_1 \in (a,b)$ for which we have either $\gamma' = \{\Gamma(t) \,|\, t \in (a,t_1)\}$ in which case we have $a = -\infty$, $\lim_{t\to-\infty} \Gamma(t) = p$, $\Gamma(t_1) \in \delta$ and $\Gamma(t) \in D$ for $t \in (-\infty, t_1)$, or $\gamma' = \{\Gamma(t) \,|\, t \in (t_1, b)\}$, $b = +\infty$, $\lim_{t\to+\infty} \Gamma(t) = p$, $\Gamma(t_1) \in \delta$ and $\Gamma(t) \in D$ for $t \in (t_1, \infty)$.

We note that in the case of elliptic sectors there may exist orbits which are divided exactly in two half orbits.

Let $\Omega_p = \{\gamma' \,|\, \gamma' \text{ is a half orbit at } p\}$.

We shall define a relation of equivalence on Ω_p by using the complete desingularization of the singular point p in case this point is nilpotent or intricate. There are two ways to desingularize such a singular point: by passing to polar coordinates or by using rational changes of coordinates. The first has the inconvenience of using trigonometrical functions, and this becomes a serious problem when a chain of blow–ups are needed in order to complete the desingularization of the degenerate point. The second uses rational changes of coordinates, convenient for our polynomial systems. In such a case two blow–ups in different directions are needed and information from both must be glued together to obtain the desired portrait.

Here for desingularization we use the second possibility, namely with rational changes of coordinates at each stage of the process. Two rational changes are needed, one for each direction of the blow–up. If at a stage the coordinates are (x,y) and we do a blow–up in y-direction, the change $(x,y) \to (x,zx)$. This change is a diffeomorphism when $x \neq 0$. This diffeomorphism transfers our vector field on a subset of the algebraic surface $y = zx$. The point $p = (0,0)$ is then replaced by the straight line $x = 0$ and $y = 0$ in the 3-dimensional space of coordinates x, y, z and which is also located on the ruled algebraic surface $y = xz$ which is the z-axis. This line is called *blow–up line* for this blow–up and it is also present as the line $x = 0$ in the (x,z) plane.

Analogously we can do a blow-up in the x-direction using the change $(x,y) \to (zy,y)$ which is a diffeomorphism for $y \neq 0$.

The two directional blow–ups can be simplified in just one 1–direction blow–up if we make sure that the direction in which we do a blow–up is not a characteristic direction, so as to be sure that we are not going to lose information doing the blow–up in the chosen direction. This can be easily solved by a simple linear change of coordinates of the type $(x, y) \to (x + ky, y)$ where k is a constant (usually 1). It seems natural to call this linear change a *k–twist* as the y–axis gets twisted with some angle depending on k. It is obvious that the phase portrait of the degenerate point which is studied cannot depend on the set of k's used in the desingularization process.

Since the complete desingularization of a nilpotent or an intricate singular point in general needs more than one blow–up, we have as many blow–up lines as we have blow–ups. As indicated above a blow–up line may be transformed by means of linear changes and through other blow–up's in other straight lines. We will call such straight lines *blow–up lines of higher order* .

We now introduce an equivalent relation on Ω_p. We say that two half orbits γ_1', $\gamma_2' \in \Omega_p$ are equivalent if and only if i) for both γ_1' and γ_2' we have $\lim_{t\to-\infty} \Gamma_1(t) = p = \lim_{t\to-\infty} \Gamma_2(t)$ or $\lim_{t\to+\infty} \Gamma_1(t) = p = \lim_{t\to+\infty} \Gamma_2(t)$, and ii) after the complete desingularization, these orbits lifted to the final stage are tangent to the same half–line at the same singular point, or end as orbits of a star node on the same half–plane defined by the blown–up line, and iii) both orbits must remain in the same half–plane in all the successive blow–up's.

We recall that after a complete desingularization all singular points are elemental or semi–elemental. We now single out two types of equivalence classes:

(a) Suppose that an equivalence class $\mathcal{C} \in \Omega_p/\sim$ is such that its half orbits lifted to the last stage in the desingularization process lead to orbits which possess the following properties: i) they belong to an elemental two–directions node or to a semi–elemental saddle–node, and ii) they are all tangent to the same half–line which lies on the blow–up line.

(b) Suppose that an equivalence class $\mathcal{C} \in \Omega_p/\sim$ is such that i) its half orbits lifted to the final stage of the desingularization process, are tangent to a blow–up line of higher order, and ii) its lifted orbits blown–down to the previous stage of the desingularization, form a part of an elliptic sector.

Let $\Omega_p'/\sim$ be the set of all equivalence classes which are of type (a) or (b). Then consider the complement $\mathcal{B}_p = (\Omega_p/\sim) - (\Omega_p'/\sim)$ and consider a set of representatives of $\mathcal{B}_p$. We call *borsec* anyone of these representatives.

Note that the definition of borsec is independent of the choice of the disc D with boundary δ if D is sufficiently small.

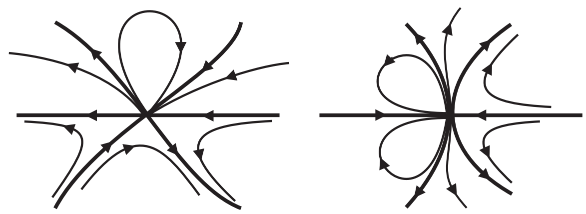
We call *geometric local sector* of a singular point p with respect to a neighborhood V, a region in V delimited by two consecutive borsecs.

For more details and a complete example of the desingularization of an intricate singular point see [4].

Generically a *geometric local sector* is defined by two borsecs arriving at the singular point with two different well defined angles and which are consecutive. If this sector is parabolic, then the solutions can arrive at the singular point with one of the two characteristic angles, and this is a geometrical information that can be revealed with the blow–up.

There is also the possibility that two borsecs defining a geometric local sector tend to the singular point with the same well defined angle. Such a sector will be called a *cusp–like sector* which can either be hyperbolic, elliptic or parabolic denoted by $H_{\curlywedge}$, $E_{\curlywedge}$ and $P_{\curlywedge}$ respectively.

In the case of parabolic sectors we want to include the information about how the orbits arrive at the singular points namely tangent to one or to the other borsec. We distinguish the two cases by writing $\overset{\curvearrowleft}{P}$ if they arrive tangent to the borsec limiting the previous sector in clockwise sense or $\overset{\curvearrowright}{P}$ if they arrive tangent to the borsec limiting the next sector. In the case of a cusp–like parabolic sector, all orbits must arrive with only one well determined angle, but the distinction between $\overset{\curvearrowleft}{P}$ and $\overset{\curvearrowright}{P}$ is still valid because it occurs at some stage of the desingularization and this can be algebraically determined. Thus complicate intricate singular points like the two we see in FIGURE 4 may be described as $\widehat{P}E\widehat{P}HHH$ (case (a)) and $E\widehat{P}_{\curlywedge}HH\widehat{P}_{\curlywedge}E$ (case (b)), respectively.



(a) *(b)*

FIGURE 4. Two phase portraits of degenerate singular points.

A star–like point can either be a node or something much more complicated with elliptic and hyperbolic sectors included. In case there are hyperbolic sectors, they must be cusp–like. Elliptic sectors can either be cusp–like or star–like. We call *special characteristic angle* for a polynomial differential system any well defined angle of a star-like point, in which either none or more than one solution curve tends to p within this well defined angle. We will call *special characteristic direction* any line such that at least one of the two angles defining it, is a special characteristic angle.

4. NOTATIONS FOR SINGULARITIES OF POLYNOMIAL DIFFERENTIAL SYSTEMS

In this work we limit ourselves to the class of quadratic systems with three real singularities in the finite plane. In [2] we introduced convenient notations which we also used in [3] and [4] some of which we also need here. Because these notations are essential for understanding the bifurcation diagram, we indicate below the notations needed for this article.

We describe the finite and infinite singularities, denoting the first ones with small letters and the second with capital letters. When describing in a sequence both finite and infinite singular points, we will always place first the finite ones and later the infinite ones, separating them by a semicolon‘;’.

Elemental points: We use the letters ‘s’,‘S’ for “saddles”; $\$$ for “integrable saddles”; ‘n’, ‘N’ for “nodes”; ‘f’ for “foci”; ‘c’ for “centers” and ⓒ (respectively Ⓒ) for complex finite (respectively infinite) singularities. We distinguish the finite nodes as follows:

- ‘n’ for a node with two distinct eigenvalues (generic node);
- ‘n^d’ (a one–direction node) for a node with two identical eigenvalues whose Jacobian matrix is not diagonal;
- ‘n^*’ (a star–node) for a node with two identical eigenvalues whose Jacobian matrix is diagonal.

In the case of an elemental infinite generic node, we want to distinguish whether the eigenvalue associated to the eigenvector directed towards the affine plane is, in absolute value, greater or lower than the eigenvalue associated to the eigenvector tangent to the line at infinity. This is relevant because this determines if all the orbits except one on the Poincaré disk arrive at infinity tangent to the line at infinity or transversal to this line. We will denote them as ‘N^∞’ and ‘N^f’ respectively.

Finite elemental foci and saddles are classified as strong or weak foci, respectively strong or weak saddles. When the trace of the Jacobian matrix evaluated at those singular points is not zero, we call them strong saddles and strong foci and we maintain the standard notations ‘s’ and ‘f’. But when the trace is zero, except for centers, and saddles of infinite order (i.e. with all their Poincaré-Lyapounov constants equal to zero), it is known that the foci and saddles, in the quadratic case, may have up to 3 orders. We denote them by ‘$s^{(i)}$’ and ‘$f^{(i)}$’ where $i = 1, 2, 3$ is the order. In addition we have the centers which we denote by ‘c’ and saddles of infinite order (integrable saddles) which we denote by ‘$\$$’.

Foci and centers cannot appear as singular points at infinity and hence there is no need to introduce their order in this case. In case of saddles, we can have weak saddles at infinity but the maximum order of weak singularities in cubic systems is not yet known. For this reason, a complete study of weak saddles at infinity cannot be done at this stage. Due to this, in [2], [3], [4] and here we chose not even to distinguish between a saddle and a weak saddle at infinity.

All non–elemental singular points are multiple points, in the sense that there are perturbations which have at least two elemental singular points as close as we wish to the multiple point. For finite singular points we denote with a subindex their multiplicity as in '$\overline{s}_{(5)}$' or in '$\widehat{es}_{(3)}$' (the notation '$^{-}$' indicates that the saddle is semi–elemental and '$\widehat{es}_{(3)}$' indicates that the singular point is nilpotent, in this case a triple *elliptic saddle* (i.e. it has two sectors, one elliptic and one hyperbolic)). In order to describe the various kinds of multiplicity for infinite singular points we use the concepts and notations introduced in [27]. Thus we denote by '$\binom{a}{b}...$' the maximum number a (respectively b) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point. For example '$\overline{\binom{1}{1}}SN$' means a saddle–node at infinity produced by the collision of one finite singularity with an infinite one; '$\overline{\binom{0}{3}}S$' means a saddle produced by the collision of 3 infinite singularities. The meaning of the notation '$^{-}$' in the general case will be described in the next paragraph.

Semi–elemental points: They can either be nodes, saddles or saddle–nodes, finite or infinite (see [15]). We denote the semi–elemental ones always with an overline, for example '$\overline{sn}$', '$\overline{s}$' and '$\overline{n}$' with the corresponding multiplicity. In the case of infinite points we put '$^{-}$' on top of the parenthesis with multiplicities.

Moreover, in cases which will be explained later on (see the paragraph reserved for intricate points), an infinite saddle–node may be denoted by '$\overline{\binom{1}{1}}NS$' instead of '$\overline{\binom{1}{1}}SN$'. Semi–elemental nodes could never be 'n^d' or 'n^*' since their eigenvalues are always different. In case of an infinite semi–elemental node, the type of collision determines whether the point is denoted by 'N^f' or by 'N^∞' where '$\overline{\binom{2}{1}}N$' is an 'N^f' and '$\overline{\binom{0}{3}}N$' is an 'N^∞'.

Nilpotent points: They can either be saddles, nodes, saddle–nodes, elliptic saddles, cusps, foci or centers (see [15]). The first four of these could be at infinity. We denote the nilpotent singular points with a hat '$\widehat{\ }$' as in $\widehat{es}_{(3)}$ for a finite nilpotent elliptic saddle of multiplicity 3 and $\widehat{cp}_{(2)}$ for a finite nilpotent cusp point of multiplicity 2. In the case of nilpotent infinite points, we will put the '$\widehat{\ }$' on top of the parenthesis with multiplicity, for example $\widehat{\binom{1}{2}}PEP-H$ (the meaning of $PEP-H$ will be explained in the next paragraph). The relative position of the sectors of an infinite nilpotent point, with respect to the line at infinity, can produce topologically different phase portraits. This forces us to use a notation for these points similar to the notation which we will use for the intricate points.

Intricate points: It is known that the neighborhood of any singular point of a polynomial vector field (except for foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [15]). Then, a reasonable way to describe intricate and nilpotent points is to use a sequence formed by the types of their sectors. The description we give is the one which appears in the clockwise direction (starting anywhere) once the blow–down of the desingularization is done. Thus in *non-degenerate* quadratic systems (that is, the components of the system are coprime), we have just seven possibilities for finite intricate singular points of multiplicity four (see [5]) which are the following ones: $phpphp_{(4)}$; $phph_{(4)}$; $hh_{(4)}$; $hhhhhh_{(4)}$; $peppep_{(4)}$; $pepe_{(4)}$; $ee_{(4)}$.

The small letters used here indicate that we have finite singularities and subindex (4) indicates the multiplicity 4 of the singularities.

For infinite intricate and nilpotent singular points, we insert a dash (hyphen) between the sectors to split those which appear on one side or the other of the equator of the sphere. In this way we will distinguish between $\binom{2}{2}PHP-PHP$ and $\binom{2}{2}PPH-PPH$.

Whenever we have an infinite nilpotent or intricate singular point, we will always start with a sector bordering the infinity (to avoid using two dashes).

For the description of the topological phase portraits around the isolated singular points the information described above is sufficient. However we are interested in additional geometrical features such as the number of characteristic directions which figure in the final global picture of the desingularization. In order to add this information we need to introduce more notation. If two borsecs (the limiting orbits of a sector) arrive at the singular point with the same direction, then the sector will be denoted by $H_{\curlywedge}$, $E_{\curlywedge}$ or $P_{\curlywedge}$. The index in this notation refers to the cusp–like form of limiting trajectories of the sectors. Moreover, in the case of parabolic sectors we want to make precise whether the orbits arrive tangent to one borsec or to the other. We distinguish the two cases by $\overset{\curvearrowleft}{P}$ if they arrive tangent to the borsec limiting the previous sector in clockwise sense or $\overset{\curvearrowright}{P}$ if they arrive tangent to the borsec limiting the next sector. A parabolic sector will be P^* when all orbits arrive with all possible slopes between the two consecutive borsecs. In the case of a cusp–like parabolic sector, all orbits must arrive with only one direction, but the distinction between $\overset{\curvearrowleft}{P}$ and $\overset{\curvearrowright}{P}$ is still valid if we consider the different desingularizations we obtain from them. Thus, complicated intricate singular points like the two we see in FIGURE 4 may be described as $\binom{4}{2}\ \overset{\curvearrowright}{P}E\overset{\curvearrowleft}{P}-HHH$ (case (a)) and $\binom{4}{3}\ E\overset{\curvearrowleft}{P}_{\curlywedge}H-H\overset{\curvearrowright}{P}_{\curlywedge}E$ (case (b)), respectively.

Finally there is also the possibility that we have an infinite number of infinite singular points.

Line at infinity filled up with singularities: It is known that any such system has in a sufficiently small neighborhood of infinity one of 6 topological distinct phase portraits (see [28]). The way to determine these phase portraits is by studying the reduced systems on the infinite local charts after removing the degeneracy of the systems within these charts. In case a singular point still remains on the line at infinity we study such a point. In [28] the tangential behavior of the solution curves was not considered in the case of a node. If after the removal of the degeneracy in the local charts at infinity a node remains, this could either be of the type N^d, N and $N^\star$ (this last case does not occur in quadratic systems as it was shown in [2]). Since no eigenvector of such a node N (for quadratic systems) will have the direction of the line at infinity we do not need to distinguish N^f and N^∞ (see [2]). Other types of singular points at infinity of quadratic systems, after removal of the degeneracy, can be saddles, foci, centers, semi–elemental saddle–nodes or nilpotent elliptic saddles. We also have the possibility of no singularities after the removal of the degeneracy. To convey the way these singularities were obtained as well as their nature, we use the notation $[\infty;\emptyset]$, $[\infty;N]$, $[\infty;N^d]$, $[\infty;S]$, $[\infty;C]$, $[\infty;\overline{\binom{1}{0}}SN]$ or $[\infty;\widehat{\binom{3}{0}}ES]$.

5. Invariant polynomials and preliminary results

Consider real quadratic systems of the form

$$\begin{aligned}\frac{dx}{dt}&=p_0+p_1(x,y)+\,p_2(x,y)\equiv P(x,y),\\ \frac{dy}{dt}&=q_0+q_1(x,y)+\,q_2(x,y)\equiv Q(x,y),\end{aligned}\tag{2}$$

with homogeneous polynomials p_i and q_i $(i=0,1,2)$ of degree i in x,y written as

$$\begin{aligned}p_0&=a_{00},\quad p_1(x,y)=a_{10}x+a_{01}y,\quad p_2(x,y)=a_{20}x^2+2a_{11}xy+a_{02}y^2,\\ q_0&=b_{00},\quad q_1(x,y)=b_{10}x+b_{01}y,\quad q_2(x,y)=b_{20}x^2+2b_{11}xy+b_{02}y^2.\end{aligned}$$

Let $\tilde{a}=(a_{00},a_{10},a_{01},a_{20},a_{11},a_{02},b_{00},b_{10},b_{01},b_{20},\ b_{11},b_{02})$ be the 12-tuple of the coefficients of systems (2) and denote $\mathbb{R}[\tilde{a},x,y]=\mathbb{R}[a_{00},\ldots,b_{02},x,y]$.

5.1. **Affine invariant polynomials associated to infinite singularities.** It is known that on the set **QS** of all quadratic differential systems (2) acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane (cf. [27]). For every subgroup $G\subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **QS**. We can identify the set **QS** of systems (2) with a subset of $\mathbb{R}^{12}$ via the map $\mathbf{QS}\longrightarrow \mathbb{R}^{12}$ which associates to each system (2) the 12–tuple $\tilde{a}=(a_{00},\ldots,b_{02})$ of its coefficients.

For the definitions of a GL–comitant and invariant as well as for the definitions of a T–comitant and a CT–comitant we refer the reader to the paper [27] (see also [29]). Here we shall only construct the necessary T–comitants and CT–comitants associated to configurations of infinite singularities (including multiplicities) of quadratic systems (2). All polynomials constructed here are GL–comitants. But some are also affine invariants or even affine comitants.

Consider the polynomial $\Phi_{\alpha,\beta}=\alpha P^*+\beta Q^*\in\mathbb{R}[\tilde{a},X,Y,Z,\alpha,\beta]$ where $P^*=Z^2P(X/Z,Y/Z)$, $Q^*=Z^2Q(X/Z,Y/Z)$, $P,\,Q\in\mathbb{R}[\tilde{a},x,y]$ and $\max(\deg_{(x,y)}P,\deg_{(x,y)}Q)=2$. Then

$$\Phi_{\alpha,\beta}=s_{11}(\tilde{a},\alpha,\beta)X^2+2s_{12}(\tilde{a},\alpha,\beta)XY+s_{22}(\tilde{a},\alpha,\beta)Y^2+2s_{13}(\tilde{a},\alpha,\beta)XZ+2s_{23}(\tilde{a},\alpha,\beta)YZ+s_{33}(\tilde{a},\alpha,\beta)Z^2$$

and we denote $\widetilde{D}(\tilde{a},x,y)=4\det\left|\left|s_{ij}(\tilde{a},y,-x)\right|\right|_{i,j\in\{1,2,3\}}$ and $\widetilde{H}(\tilde{a},x,y)=4\det\left|\left|s_{ij}(\tilde{a},y,-x)\right|\right|_{i,j\in\{1,2\}}$.

We consider the polynomials

$$C_i(\tilde{a},x,y)=yp_i(\tilde{a},x,y)-xq_i(\tilde{a},x,y),\quad D_i(\tilde{a},x,y)=\frac{\partial}{\partial x}p_i(\tilde{a},x,y)+\frac{\partial}{\partial y}q_i(\tilde{a},x,y), \tag{3}$$

in $\mathbb{R}[\tilde{a},x,y]$ for $i=0,1,2$ and $i=1,2$ respectively. The polynomials C_2 and D_2 are trivial T-comitants (trivial because they only depend on the coefficients of the quadratic terms). Using the so–called *transvectant of order* k (see [17], [20]) of two polynomials $f,\,g\in\mathbb{R}[\tilde{a},x,y]$

$$(f,g)^{(k)}=\sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\;\frac{\partial^k g}{\partial x^h\partial y^{k-h}},$$

we construct the following GL—comitants of degree two with respect to the coefficients of the initial system

$$\begin{array}{lll} T_1(\tilde{a},x,y)=(C_0,C_1)^{(1)}, & T_2(\tilde{a},x,y)=(C_0,C_2)^{(1)}, & T_3(\tilde{a})=(C_0,D_2)^{(1)},\\ T_4(\tilde{a})=(C_1,C_1)^{(2)}, & T_5(\tilde{a},x,y)=(C_1,C_2)^{(1)}, & T_6(\tilde{a},x,y)=(C_1,C_2)^{(2)},\\ T_7(\tilde{a},x,y)=(C_1,D_2)^{(1)}, & T_8(\tilde{a},x,y)=(C_2,C_2)^{(2)}, & T_9(\tilde{a},x,y)=(C_2,D_2)^{(1)}. \end{array} \tag{4}$$

Using these GL—comitants as well as the polynomials (3) we construct the additional invariant polynomials (see also [27])

$$\begin{aligned} \widetilde{M}(\tilde{a},x,y)=&(C_2,C_2)^{(2)}\equiv 2\mathrm{Hess}\,\big(C_2(\tilde{a},x,y)\big);\\ \eta(\tilde{a})=&(\widetilde{M},\widetilde{M})^{(2)}/384\equiv \mathrm{Discrim}\,\big(C_2(\tilde{a},x,y)\big);\\ \widetilde{K}(\tilde{a},x,y)=&\mathrm{Jacob}\,\big(p_2(\tilde{a},x,y),q_2(\tilde{a},x,y)\big);\\ \kappa(\tilde{a})=&(\widetilde{M},\widetilde{K})^{(2)}/4;\\ \widetilde{N}(\tilde{a},x,y)=&\widetilde{K}(\tilde{a},x,y)+\widetilde{H}(\tilde{a},x,y). \end{aligned}$$

The geometrical meaning of the invariant polynomials C_2, $\widetilde{M}$ and η is revealed in the next lemma (see [27]).

Lemma 1. *The form of the divisor $D_S(C,Z)$ for systems (2) is determined by the corresponding conditions indicated in Table 1, where we write $w_1^c+w_2^c+w_3$ if two of the points, i.e. w_1^c,w_2^c, are complex but not real. Moreover, for each form of the divisor $D_S(C,Z)$ given in Table 1 the quadratic systems (2) can be brought via a linear transformation to one of the following canonical systems $(\mathbf{S}_I)-(\mathbf{S}_V)$ corresponding to their behavior at infinity.*

Table 1

Case	Form of $D_S(C,Z)$	Necessary and sufficient conditions on the comitants
I	$w_1+w_2+w_3$	$\eta>0$
II	$w_1^c+w_2^c+w_3$	$\eta<0$
III	$2w_1+w_2$	$\eta=0,\quad \widetilde{M}\neq 0$
IV	$3w$	$\widetilde{M}=0,\quad C_2\neq 0$
V	$D_S(C,Z)$ undefined	$C_2=0$

$$\begin{aligned}\dot{x} &= a+cx+dy+gx^2+(h-1)xy,\\ \dot{y} &= b+ex+fy+(g-1)xy+hy^2;\end{aligned}\qquad (\mathbf{S}_I)$$

$$\begin{aligned}\dot{x} &= a+cx+dy+gx^2+(h+1)xy,\\ \dot{y} &= b+ex+fy-x^2+gxy+hy^2;\end{aligned}\qquad (\mathbf{S}_{II})$$

$$\begin{aligned}\dot{x} &= a+cx+dy+gx^2+hxy,\\ \dot{y} &= b+ex+fy+(g-1)xy+hy^2;\end{aligned}\qquad (\mathbf{S}_{III})$$

$$\begin{aligned}\dot{x} &= a+cx+dy+gx^2+hxy,\\ \dot{y} &= b+ex+fy-x^2+gxy+hy^2,\end{aligned}\qquad (\mathbf{S}_{IV})$$

$$\begin{aligned}\dot{x} &= a+cx+dy+x^2,\\ \dot{y} &= b+ex+fy+xy.\end{aligned}\qquad (\mathbf{S}_{V})$$

5.2. **Affine invariant polynomials associated to finite singularities.** Consider the differential operator $\mathcal{L}=x\cdot\mathbf{L}_2-y\cdot\mathbf{L}_1$ constructed in [9] and acting on $\mathbb{R}[\tilde{a},x,y]$, where

$$\begin{aligned}\mathbf{L}_1 = &\ 2a_{00}\tfrac{\partial}{\partial a_{10}}+a_{10}\tfrac{\partial}{\partial a_{20}}+\tfrac{1}{2}a_{01}\tfrac{\partial}{\partial a_{11}}+2b_{00}\tfrac{\partial}{\partial b_{10}}+b_{10}\tfrac{\partial}{\partial b_{20}}+\tfrac{1}{2}b_{01}\tfrac{\partial}{\partial b_{11}},\\ \mathbf{L}_2 = &\ 2a_{00}\tfrac{\partial}{\partial a_{01}}+a_{01}\tfrac{\partial}{\partial a_{02}}+\tfrac{1}{2}a_{10}\tfrac{\partial}{\partial a_{11}}+2b_{00}\tfrac{\partial}{\partial b_{01}}+b_{01}\tfrac{\partial}{\partial b_{02}}+\tfrac{1}{2}b_{10}\tfrac{\partial}{\partial b_{11}}.\end{aligned}$$

Using this operator and the affine invariant $\mu_0=\operatorname{Res}_x\big(p_2(\tilde{a},x,y),q_2(\tilde{a},x,y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a},x,y)=\frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ \ i=1,..,4,$$

where $\mathcal{L}^{(i)}(\mu_0)=\mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$ and $\mathcal{L}^{(0)}(\mu_0)=\mu_0$.

These polynomials are in fact comitants of systems (2) with respect to the group $GL(2,\mathbb{R})$ (see [9]). Their geometrical meaning is revealed in Lemmas 2 and 3 below.

Lemma 2. ([8]) *The total multiplicity of all finite singularities of a given quadratic system* (2) *with coefficients* $\tilde{\boldsymbol{a}}$ *equals* k *if and only if for every* $i\in\{0,1,\dots,k-1\}$ *we have* $\mu_i(\tilde{\boldsymbol{a}},x,y)=0$ *in the ring* $\mathbb{R}[x,y]$ *and* $\mu_k(\tilde{\boldsymbol{a}},x,y)\neq 0$. *Moreover a given system* (2) *with coefficients* $\tilde{\boldsymbol{a}}$ *is degenerate (i.e.* $\gcd(P,Q)\neq$ *constant) if and only if* $\mu_i(\tilde{\boldsymbol{a}},x,y)=0$ *in* $\mathbb{R}[x,y]$ *for every* $i=0,1,2,3,4$.

Lemma 3. ([9]) *The point* $M_0(0,0)$ *is a singular point of multiplicity* k $(1\leq k\leq 4)$ *for a given quadratic system* (2) *with coefficients* $\tilde{\boldsymbol{a}}$ *if and only if for every* $i\in\{0,1,\dots,k-1\}$ *we have* $\mu_{4-i}(\tilde{\boldsymbol{a}},x,y)=0$ *in* $\mathbb{R}[x,y]$ *and* $\mu_{4-k}(\tilde{\boldsymbol{a}},x,y)\neq 0$.

We denote

$$\sigma(\tilde{a},x,y)=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=\sigma_0(\tilde{a})+\sigma_1(\tilde{a},x,y)\ (\equiv D_1(\tilde{a})+D_2(\tilde{a},x,y)),$$

and observe that the polynomial $\sigma(\tilde{a},x,y)$ is an affine comitant of systems (2). It is know that if (x_i,y_i) is a singular point of a system (2) then for the trace ρ_i of its respective linear matrix we have $\rho_i=\sigma(x_i,y_i)$.

Applying the differential operators $\mathcal{L}$ and $(*,*)^{(k)}$ (i.e. transvectant of index k) we define the following polynomial function which governs the values of the traces for finite singularities of systems (2).

Definition 1 ([30])**.** We call *trace polynomial* $\mathfrak{T}(w)$ over the ring $\mathbb{R}[\tilde{a}]$ the polynomial defined as follows

$$\mathfrak{T}(w)=\sum_{i=0}^{4}\frac{1}{(i!)^2}\left(\sigma_1^i,\ \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0)\right)^{(i)}\ w^{4-i}=\sum_{i=0}^{4}\mathcal{G}_i(\tilde{a})\,w^{4-i},\tag{5}$$

where the coefficients $\mathcal{G}_i(\tilde{a})=\dfrac{1}{(i!)^2}(\sigma_1^i,\ \mu_i)^{(i)}\in\mathbb{R}[\tilde{a}],\quad i=0,1,2,3,4\ \big(\mathcal{G}_0(\tilde{a})\equiv\mu_0(\tilde{a})\big)$ are GL–invariants.

Using the polynomial $\mathfrak{T}(w)$ we could construct the following four affine invariants $\mathcal{T}_4$, $\mathcal{T}_3$, $\mathcal{T}_2$, $\mathcal{T}_1$, which are responsible for the weak singularities:

$$\mathcal{T}_{4-i}(\tilde{a})=\frac{1}{i!}\,\frac{d^i\mathfrak{T}}{dw^i}\Big|_{w=\sigma_0}\in\mathbb{R}[\tilde{a}],\quad i=0,1,2,3\quad\big(\mathcal{T}_4\equiv\mathfrak{T}(\sigma_0)\big).$$

The geometric meaning of these invariants is revealed by Theorem 1 below (see [30] for its proof).

In order to be able to calculate directly for every canonical system the values of invariant polynomials, we define here a family of T–comitants (see [27] for detailed definitions) expressed through C_i $(i=0,1,2)$ and D_j $(j=1,2)$:

$$\begin{aligned}
\hat{A}&=\left(C_1,T_8-2T_9+D_2^2\right)^{(2)}/144,\\
\widehat{D}&=\Big[2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6-(C_1,T_5)^{(1)}+6D_1(C_1D_2-T_5)-9D_1^2C_2\Big]/36,\\
\widehat{E}&=\Big[D_1(2T_9-T_8)-3\,(C_1,T_9)^{(1)}-D_2(3T_7+D_1D_2)\Big]/72,\\
\widehat{F}&=\big[6D_1^2(D_2^2-4T_9)+4D_1D_2(T_6+6T_7)+48C_0\,(D_2,T_9)^{(1)}-9D_2^2T_4+288D_1\widehat{E}\\
&\quad-24\left(C_2,\widehat{D}\right)^{(2)}+120\left(D_2,\widehat{D}\right)^{(1)}-36C_1\,(D_2,T_7)^{(1)}+8D_1\,(D_2,T_5)^{(1)}\Big]/144,
\end{aligned}$$

$$\begin{aligned}
\widehat{B}=\Big\{&16D_1\,(D_2,T_8)^{(1)}\,(3C_1D_1-2C_0D_2+4T_2)+32C_0\,(D_2,T_9)^{(1)}\,(3D_1D_2-5T_6+9T_7)\\
&+2\,(D_2,T_9)^{(1)}\left(27C_1T_4-18C_1D_1^2-32D_1T_2+32\,(C_0,T_5)^{(1)}\right)\\
&+6\,(D_2,T_7)^{(1)}\,[8C_0(T_8-12T_9)-12C_1(D_1D_2+T_7)+D_1(26C_2D_1+32T_5)+C_2(9T_4+96T_3)]\\
&+6\,(D_2,T_6)^{(1)}\,[32C_0T_9-C_1(12T_7+52D_1D_2)-32C_2D_1^2]+48D_2\,(D_2,T_1)^{(1)}\left(2D_2^2-T_8\right)\\
&-32D_1T_8\,(D_2,T_2)^{(1)}+9D_2^2T_4\,(T_6-2T_7)-16D_1\,(C_2,T_8)^{(1)}\left(D_1^2+4T_3\right)\\
&+12D_1\,(C_1,T_8)^{(2)}\,(C_1D_2-2C_2D_1)+6D_1D_2T_4\left(T_8-7D_2^2-42T_9\right)\\
&+12D_1\,(C_1,T_8)^{(1)}\,(T_7+2D_1D_2)+96D_2^2\left[D_1\,(C_1,T_6)^{(1)}+D_2\,(C_0,T_6)^{(1)}\right]\\
&-16D_1D_2T_3\left(2D_2^2+3T_8\right)-4D_1^3D_2\left(D_2^2+3T_8+6T_9\right)+6D_1^2D_2^2\,(7T_6+2T_7)\\
&-252D_1D_2T_4T_9\big\}/(2^83^3),\\
\widehat{K}&=(T_8+4T_9+4D_2^2)/72\quad\equiv\widetilde{K}/4,\\
\widehat{H}&=(8T_9-T_8+2D_2^2)/72\equiv-\widetilde{H}/4.
\end{aligned}$$

These polynomials in addition to (3) and (4) will serve as bricks in constructing affine invariant polynomials for systems (2).

The following 42 affine invariants $A_1, \ldots, A_{42}$ form the minimal polynomial basis of affine invariants up to degree 12. This fact was proved in [11] by constructing $A_1, \ldots, A_{42}$ using the above bricks.

$$
\begin{aligned}
A_1 &= \hat{A}, \\
A_2 &= (C_2, \widehat{D})^{(3)}/12, \\
A_3 &= \big[C_2, D_2)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/48, \\
A_4 &= (\widehat{H}, \widehat{H})^{(2)}, \\
A_5 &= (\widehat{H}, \widehat{K})^{(2)}/2, \\
A_6 &= (\widehat{E}, \widehat{H})^{(2)}/2, \\
A_7 &= \big[C_2, \widehat{E})^{(2)}, D_2\big)^{(1)}/8, \\
A_8 &= \big[\widehat{D}, \widehat{H})^{(2)}, D_2\big)^{(1)}/8, \\
A_9 &= \big[\widehat{D}, D_2)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/48, \\
A_{10} &= \big[\widehat{D}, \widehat{K})^{(2)}, D_2\big)^{(1)}/8, \\
A_{11} &= (\widehat{F}, \widehat{K})^{(2)}/4, \\
A_{12} &= (\widehat{F}, \widehat{H})^{(2)}/4, \\
A_{13} &= \big[C_2, \widehat{H})^{(1)}, \widehat{H}\big)^{(2)}, D_2\big)^{(1)}/24, \\
A_{14} &= (\widehat{B}, C_2)^{(3)}/36, \\
A_{15} &= (\widehat{E}, \widehat{F})^{(2)}/4, \\
A_{16} &= \big[\widehat{E}, D_2)^{(1)}, C_2\big)^{(1)}, \widehat{K}\big)^{(2)}/16, \\
A_{17} &= \big[\widehat{D}, \widehat{D})^{(2)}, D_2\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{18} &= \big[\widehat{D}, \widehat{F})^{(2)}, D_2\big)^{(1)}/16, \\
A_{19} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{H}\big)^{(2)}/16, \\
A_{20} &= \big[C_2, \widehat{D})^{(2)}, \widehat{F}\big)^{(2)}/16, \\
A_{21} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{K}\big)^{(2)}/16, \\
A_{22} &= \tfrac{1}{1152}\big[C_2, \widehat{D})^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)} D_2\big)^{(1)}, \\
A_{23} &= \big[\widehat{F}, \widehat{H})^{(1)}, \widehat{K}\big)^{(2)}/8, \\
A_{24} &= \big[C_2, \widehat{D})^{(2)}, \widehat{K}\big)^{(1)}, \widehat{H}\big)^{(2)}/32, \\
A_{25} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{E}\big)^{(2)}/16, \\
A_{26} &= (\widehat{B}, \widehat{D})^{(3)}/36, \\
A_{27} &= \big[\widehat{B}, D_2)^{(1)}, \widehat{H}\big)^{(2)}/24, \\
A_{28} &= \big[C_2, \widehat{K})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{E}\big)^{(2)}/16, \\
A_{29} &= \big[\widehat{D}, \widehat{F})^{(1)}, \widehat{D}\big)^{(3)}/96, \\
A_{30} &= \big[C_2, \widehat{D})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{D}\big)^{(3)}/288, \\
A_{31} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{K}\big)^{(1)}, \widehat{H}\big)^{(2)}/64, \\
A_{32} &= \big[\widehat{D}, \widehat{D})^{(2)}, D_2\big)^{(1)}, \widehat{H}\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{33} &= \big[\widehat{D}, D_2)^{(1)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/128, \\
A_{34} &= \big[\widehat{D}, \widehat{D})^{(2)}, D_2\big)^{(1)}, \widehat{K}\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{35} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{E}\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/128, \\
A_{36} &= \big[\widehat{D}, \widehat{E})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{H}\big)^{(2)}/16, \\
A_{37} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{D}\big)^{(3)}/576, \\
A_{38} &= \big[C_2, \widehat{D})^{(2)}, \widehat{D}\big)^{(2)}, \widehat{D}\big)^{(1)}, \widehat{H}\big)^{(2)}/64, \\
A_{39} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{F}\big)^{(1)}, \widehat{H}\big)^{(2)}/64, \\
A_{40} &= \big[\widehat{D}, \widehat{D})^{(2)}, \widehat{F}\big)^{(1)}, \widehat{K}\big)^{(2)}/64, \\
A_{41} &= \big[C_2, \widehat{D})^{(2)}, \widehat{D}\big)^{(2)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{42} &= \big[\widehat{D}, \widehat{F})^{(2)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}/16.
\end{aligned}
$$

In the above list, the bracket "[" is used in order to avoid placing the otherwise necessary up to five parenthesizes "(".

Using the elements of the minimal polynomial basis given above we construct the affine invariants

$$
\begin{aligned}
\mathcal{F}_1(\tilde{a}) =& A_2, \\
\mathcal{F}_2(\tilde{a}) =& -2A_1^2A_3 + 2A_5(5A_8 + 3A_9) + A_3(A_8 - 3A_{10} + 3A_{11} + A_{12}) - \\
& - A_4(10A_8 - 3A_9 + 5A_{10} + 5A_{11} + 5A_{12}), \\
\mathcal{F}_3(\tilde{a}) =& -10A_1^2A_3 + 2A_5(A_8 - A_9) - A_4(2A_8 + A_9 + A_{10} + A_{11} + A_{12}) + \\
& + A_3(5A_8 + A_{10} - A_{11} + 5A_{12}), \\
\mathcal{F}_4(\tilde{a}) =& 20A_1^2A_2 - A_2(7A_8 - 4A_9 + A_{10} + A_{11} + 7A_{12}) + A_1(6A_{14} - 22A_{15}) - 4A_{33} + 4A_{34}, \\
\mathcal{F}(\tilde{a}) =& A_7, \\
\mathcal{B}(\tilde{a}) =& -(3A_8 + 2A_9 + A_{10} + A_{11} + A_{12}), \\
\mathcal{H}(\tilde{a}) =& -(A_4 + 2A_5),
\end{aligned}
\tag{6}
$$

as well as the GL-comitants

$$
\begin{aligned}
\mathcal{B}_1(\tilde a) =& \Big\{ \big(T_7, D_2\big)^{(1)} \big[12 D_1 T_3 + 2D_1^3 + 9 D_1 T_4 + 36 \big(T_1, D_2\big)^{(1)}\big] \\
& - 2 D_1 \big(T_6, D_2\big)^{(1)} \big[D_1^2 + 12 T_3\big] + D_1^2 \big[D_1 \big(T_8, C_1\big)^{(2)} + 6 \big(\big(T_6, C_1\big)^{(1)}, D_2\big)^{(1)}\big]\Big\}/144, \\
\mathcal{B}_2(\tilde a) =& \Big\{ \big(T_7, D_2\big)^{(1)} \big[8 T_3 \big(T_6, D_2\big)^{(1)} - D_1^2 \big(T_8, C_1\big)^{(2)} - 4 D_1 \big(\big(T_6, C_1\big)^{(1)}, D_2\big)^{(1)}\big] + \\
& + \Big[\big(T_7, D_2\big)^{(1)}\Big]^2 (8T_3 - 3T_4 + 2D_1^2)\Big\}/384, \\
\mathcal{B}_3(\tilde a, x, y) =& - D_1^2 (4D_2^2 + T_8 + 4T_9) + 3 D_1 D_2 (T_6 + 4T_7) - 24 T_3 (D_2^2 - T_9), \\
\mathcal{B}_4(\tilde a, x, y) =& \ D_1 (T_5 + 2 D_2 C_1) - 3 C_2 (D_1^2 + 2T_3), \\
\mathcal{G}(\tilde a, x, y) =& \widetilde{M} + 32 \widetilde{H}, \\
\mathbf{D}(\tilde a) =& - ((\widehat{D}, \widehat{D})^{(2)}, \widehat{D})^{(1)}, \widehat{D})^{(3)}/576, \\
\mathbf{P}(\tilde a, x, y) =& 12 \mu_0 \mu_4 - 3 \mu_1 \mu_3 + \mu_2^2, \\
\mathbf{R}(\tilde a, x, y) =& \mu_1^2 - 8 \mu_0 \mu_2, \\
\mathbf{S}(\tilde a, x, y) =& \mathbf{R}^2 - 16 \mu_0^2 \mathbf{P}, \\
\mathbf{U}(\tilde a, x, y) =& \mu_3^2 - 4 \mu_2 \mu_4, \\
\mathbf{T}(\tilde a, x, y) =& 18 \mu_0^2 (3\mu_3^2 - 8 \mu_2 \mu_4) + 2 \mu_0 (2 \mu_2^3 - 9 \mu_1 \mu_2 \mu_3 + 27 \mu_1^2 \mu_4) - \mathbf{PR}.
\end{aligned} \tag{7}
$$

We note that the invariant polynomials $\mathcal{T}_i$ (i=1,2,3,4), σ, (6) and (7) are responsible for weak singularities of the family of quadratic systems as it is shown in the next theorem proved in [30].

Theorem 1. *Consider a non-degenerate quadratic system* (2).

(*a*) *If* $\mathcal{T}_4 \neq 0$ *then this system has no weak singularity.*

(*b*) *If* $\mathcal{T}_4 = 0$ *and* $\mathcal{T}_3 \neq 0$ *then the system has exactly one weak singularity. Moreover this singularity is either a weak focus (respectively a weak saddle) of the below indicated order, or a center (respectively an integrable saddle) if and only if* $\mathcal{T}_3 \mathcal{F} < 0$ *(respectively* $\mathcal{T}_3 \mathcal{F} > 0$*) and one of the following conditions holds*

(b_1)	$f^{(1)}$	*(respectively* $s^{(1)}$*)*	$\Leftrightarrow$	$\mathcal{F}_1 \neq 0$;
(b_2)	$f^{(2)}$	*(respectively* $s^{(2)}$*)*	$\Leftrightarrow$	$\mathcal{F}_1 = 0,\ \mathcal{F}_2 \neq 0$;
(b_3)	$f^{(3)}$	*(respectively* $s^{(3)}$*)*	$\Leftrightarrow$	$\mathcal{F}_1 = \mathcal{F}_2 = 0,\ \mathcal{F}_3 \mathcal{F}_4 \neq 0$;
(b_4)	c	*(respectively* $\$$*)*	$\Leftrightarrow$	$\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 \mathcal{F}_4 = 0$.

(*c*) *If* $\mathcal{T}_4 = \mathcal{T}_3 = 0$ *and* $\mathcal{T}_2 \neq 0$*, then the system could possess two and only two weak singularities and none of them is of order* 2 *or* 3*. Moreover this system possesses two weak singularities, which are of the types indicated below, if and only if* $\mathcal{F} = 0$ *and one of the following conditions holds*

(c_1)	$s^{(1)}$,	$s^{(1)}$	$\Leftrightarrow$	$\mathcal{F}_1 \neq 0,\ \mathcal{T}_2 < 0,\ \mathcal{B} \leq 0,\ \mathcal{H} > 0$;
(c_2)	$s^{(1)}$,	$f^{(1)}$	$\Leftrightarrow$	$\mathcal{F}_1 \neq 0,\ \mathcal{T}_2 > 0,\ \mathcal{B} < 0$;
(c_3)	$f^{(1)}$,	$f^{(1)}$	$\Leftrightarrow$	$\mathcal{F}_1 \neq 0,\ \mathcal{T}_2 < 0,\ \mathcal{B} < 0,\ \mathcal{H} < 0$;
(c_4)	$\$$,	$\$$	$\Leftrightarrow$	$\mathcal{F}_1 = 0,\ \mathcal{T}_2 < 0,\ \mathcal{B} < 0,\ \mathcal{H} > 0$;
(c_5)	$\$$,	c	$\Leftrightarrow$	$\mathcal{F}_1 = 0,\ \mathcal{T}_2 > 0,\ \mathcal{B} < 0$;
(c_6)	c,	c	$\Leftrightarrow$	$\mathcal{F}_1 = 0,\ \mathcal{T}_2 < 0,\ \mathcal{B} < 0,\ \mathcal{H} < 0$.

(*d*) *If* $\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = 0$ *and* $\mathcal{T}_1 \neq 0$*, then the system could possess one and only one weak singularity (which is of order 1). Moreover this system has one weak singularity of the type indicated below if and only if* $\mathcal{F} = 0$ *and one of the following conditions holds*

(d_1)	$s^{(1)}$	$\Leftrightarrow$	$\mathcal{F}_1 \neq 0,\ \mathcal{B} < 0,\ \mathcal{H} > 0$;
(d_2)	$f^{(1)}$	$\Leftrightarrow$	$\mathcal{F}_1 \neq 0,\ \mathcal{B} < 0,\ \mathcal{H} < 0$.

(*e*) *If* $\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = \mathcal{T}_1 = 0$ *and* $\sigma(a,x,y) \neq 0$, *then the system could possess one and only only one weak singularity. Moreover this system has one weak singularity, which is of the type indicated below, if and only if one of the following conditions holds*

$$
\begin{array}{lll}
(e_1)\ s^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \mathcal{H} = \mathcal{B}_1 = 0,\ \mathcal{B}_2 > 0;\\
(e_2)\ f^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \mathcal{H} = \mathcal{B}_1 = 0,\ \mathcal{B}_2 < 0;\\
(e_3)\quad \$ & \Leftrightarrow & \begin{cases} [\alpha]\ \ \mathcal{F}_1 = 0,\ \mathcal{F} = 0,\ \mathcal{B} < 0,\ \mathcal{H} > 0,\ \textit{or}\\ [\beta]\ \ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B}_1 = 0,\ \mathcal{B}_2 > 0,\ \textit{or}\\ [\gamma]\ \ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = \mu_0 = 0,\ K(\mu_2^2 + \mu_3^2) \neq 0,\ \textit{or}\\ [\delta]\ \ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = K = 0,\ \mu_2\mathcal{G} \neq 0,\ \textit{or}\\ [\varepsilon]\ \ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = \mathcal{B}_4 = K = \mu_2 = 0,\ \mu_3 \neq 0; \end{cases}\\
(e_4)\quad c & \Leftrightarrow & \begin{cases} [\alpha]\ \ \mathcal{F}_1 = 0,\ \mathcal{F} = 0,\ \mathcal{B} < 0,\ \mathcal{H} < 0,\ \textit{or}\\ [\beta]\ \ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B}_1 = 0,\ \mathcal{B}_2 < 0. \end{cases}
\end{array}
$$

(*f*) *If* $\sigma(a,x,y) = 0$, *then the system is Hamiltonian and it possesses* i *(with* $1 \leq i \leq 4$*) weak singular points of the types indicated below if and only if one of the following conditions holds*

$$
\begin{array}{llll}
(f_1) & \$, \$, \$, c & \Leftrightarrow & \mu_0 < 0,\ \mathbf{D} < 0,\ \mathbf{R} > 0,\ \mathbf{S} > 0;\\
(f_2) & \$, \$,\ c,\ c & \Leftrightarrow & \mu_0 > 0,\ \mathbf{D} < 0,\ \mathbf{R} > 0,\ \mathbf{S} > 0;\\
(f_3) & \$, \$,\ c & \Leftrightarrow & \mu_0 = 0,\ \mathbf{D} < 0,\ \mathbf{R} \neq 0;\\
(f_4) & \$, \$ & \Leftrightarrow & \begin{cases} [\alpha]\ \ \mu_0 < 0,\ \mathbf{D} > 0,\ \textit{or}\\ [\beta]\ \ \mu_0 < 0,\ \mathbf{D} = 0,\ \mathbf{T} < 0,\ \textit{or}\\ [\gamma]\ \ \mu_0 = \mathbf{R} = 0,\ \mathbf{P} \neq 0,\ \mathbf{U} > 0,\ \widetilde{K} \neq 0; \end{cases}\\
(f_5) & \$,\ c & \Leftrightarrow & \begin{cases} [\alpha]\ \ \mu_0 > 0,\ \mathbf{D} > 0,\ \textit{or}\\ [\beta]\ \ \mu_0 > 0,\ \mathbf{D} = 0,\ \mathbf{T} < 0,\ \textit{or}\\ [\gamma]\ \ \mu_0 = \mathbf{R} = 0,\ \mathbf{P} \neq 0,\ \mathbf{U} > 0,\ \widetilde{K} = 0; \end{cases}\\
(f_6) & \$ & \Leftrightarrow & \begin{cases} [\alpha]\ \ \mu_0 < 0,\ \mathbf{D} = \mathbf{T} = \mathbf{P} = 0,\ \mathbf{R} \neq 0,\ \textit{or}\\ [\beta]\ \ \mu_0 = 0,\ \mathbf{D} > 0,\ \mathbf{R} \neq 0,\ \textit{or}\\ [\gamma]\ \ \mu_0 = 0,\ \mathbf{D} = 0,\ \mathbf{PR} \neq 0,\ \textit{or}\\ [\delta]\ \ \mu_0 = \mathbf{R} = \mathbf{P} = 0,\ \mathbf{U} \neq 0; \end{cases}\\
(f_7) & c & \Leftrightarrow & \mu_0 > 0,\ \mathbf{D} = \mathbf{T} = \mathbf{P} = 0,\ \mathbf{R} \neq 0.
\end{array}
$$

Now we also need the invariant polynomials which are responsible for the types of the finite singularities. These were constructed in [5]. Here we need only the following ones (we keep the notations from [5]):

$$
\begin{aligned}
W_2(\tilde{a}) =& \big[144A_1A_2A_6 - 23A_2^2A_4 - 72A_3(A_{19} + A_{21}) + 72A_5(A_{17} + 4A_{19}) + A_4(14A_{20} - 50A_{17}\\
& - 32A_{18} - 75A_{19} + 93A_{21}) - 288A_6A_{15}\big]/9,\\
W_3(\tilde{a}) =& \big[9A_1^2(36A_{18} - 19A_2^2 + 134A_{17} + 165A_{19}) + 3A_{11}(42A_{18} - 102A_{17} + 195A_{19})\\
& + 2A_2^2(A_{10} + 3A_{11}) + 102A_3(3A_{30} - 14A_{29}) - 63A_6(17A_{25} + 30A_{26}) + 3A_{10}(14A_{18}\\
& - 118A_{17} + 153A_{19} + 120A_{21}) + 6A_7(329A_{25} - 108A_{26}) + 3A_8(164A_{18} + 153A_{19} - 442A_{17})\\
& + 9A_{12}(2A_{20} - 160A_{17} - 2A_{18} - 59A_{19}) + 3A_1(77A_2A_{14} + 235A_2A_{15} - 54A_{36})\\
& + 18A_{21}(21A_9 - 5A_{11}) + 302A_2A_{34} - 366A_{14}^2 - 12A_{15}(71A_{14} + 80A_{15})\big]/9,
\end{aligned}
$$

$$\begin{aligned}
W_4(\tilde a) =& \big[1512A_1^2(A_{30}-2A_{29})-648A_{15}A_{26}+72A_1A_2(49A_{25}+39A_{26})+6A_2^2(23A_{21}-1093A_{19})\\
&-87A_2^4+4A_2^2(61A_{17}+52A_{18}+11A_{20})-6A_{37}(352A_3+939A_4-1578A_5)\\
&-36A_8(396A_{29}+265A_{30})+72A_{29}(17A_{12}-38A_9-109A_{11})\\
&+12A_{30}(76A_9-189A_{10}-273A_{11}-651A_{12})-648A_{14}(23A_{25}+5A_{26})\\
&-24A_{18}(3A_{20}+31A_{17})+36A_{19}(63A_{20}+478A_{21})+18A_{21}(2A_{20}+137A_{21})\\
&-4A_{17}(158A_{17}+30A_{20}+87A_{21})-18A_{19}(238A_{17}+669A_{19})\big]/81,\\
W_6(\tilde a) =& 2A_1^2(2A_{29}-A_{30})-A_{26}(A_1A_2+2A_{14}),\\
W_7(\tilde a) =& 12A_{26}(A_{26}-2A_{25})+(2A_{29}-A_{30})(A_2^2-20A_{17}-12A_{18}+6A_{19}+6A_{21})\\
&+48A_{37}(A_1^2-A_8-A_{12}),\\
G_9(\tilde a) =& (A_4+2A_5)/4.
\end{aligned}$$

Finally we need the invariant polynomials which are responsible for the existence of one (or two) star node(s) arbitrarily located on the phase plane of a system (2). We have the following lemma (see [33]).

Lemma 4. *A quadratic system* (2) *possesses one star node if and only if one of the following sets of conditions holds*

$$(i)\quad U_1\neq 0,\ U_2\neq 0,\ U_3=Y_1=0;$$
$$(ii)\quad U_1=U_4=U_5=U_6=0,\quad Y_2\neq 0;$$

and it possesses two star nodes if and only if

$$(iii)\quad U_1=U_4=U_5=0,\quad U_6\neq 0,\quad Y_2>0,$$

where

$$\begin{aligned}
&U_1(\tilde a,x,y)=\widetilde N,\quad U_2(\tilde a,x,y)=(C_1,\widetilde H-\widetilde K)^{(1)}-2D_1\widetilde N,\\
&U_3(\tilde a,x,y)=3\widetilde D(D_2^2-16\widetilde K)+C_2\big[(C_2,\widetilde D)^{(2)}-5(D_2,\widetilde D)^{(1)}+6\,\widetilde F\big],\\
&U_4(\tilde a,x,y)=2T_5+C_1D_2,\quad U_5(\tilde a,x,y)=3C_1D_1+4T_2-2C_0D_1,\\
&U_6(\tilde a,x,y)=\widetilde H,\quad Y_1(\tilde a)=A_1,\quad Y_2(\tilde a,x,y)=2D_1^2+8T_3-T_4.
\end{aligned}$$

We base our work here on results obtained in [2] and [5].

6. The proof of the Main Theorem

According to [30] for a quadratic system to have three real finite simple singularities the conditions $\mu_0=0$, $\mu_1\neq 0$ and $\mathbf{D}<0$ must be satisfied. Such systems have finite singularities of total multiplicity 3 and therefore by [2] the following lemma is valid.

Lemma 5. *The geometric configurations of singularities at infinity of the family of quadratic systems possessing finite singularities of total multiplicity 3 (i.e. $\mu_0=0$, $\mu_1\neq 0$) are classified in* Diagram 2 *according to the geometric equivalence relation. Necessary and sufficient conditions for each one of the 11 different equivalence classes can be assembled from these diagrams in terms of* 9 *invariant polynomials with respect to the action of the affine group and time rescaling, given in Section 5.*

Consider quadratic systems (2) with real coefficients and variables x and y. Assume that these systems possess three real elemental finite singularities. Then via an affine transformation we may bring two of the singularities at the points $M_1(0,0)$ and $M_2(0,1)$ and hence we may assume the equations as being:

$$\dot x=cx+dy+gx^2+2hxy-dy^2,\quad \dot y=ex+fy+lx^2+2mxy-fy^2 \tag{8}$$

with $d^2+f^2\neq 0$ (otherwise we get degenerate systems). For these systems the condition $\mu_0=0$ and $\mu_1\neq 0$ must be fulfilled (i.e. only one real finite singular point has coalesced with an infinite singularity).

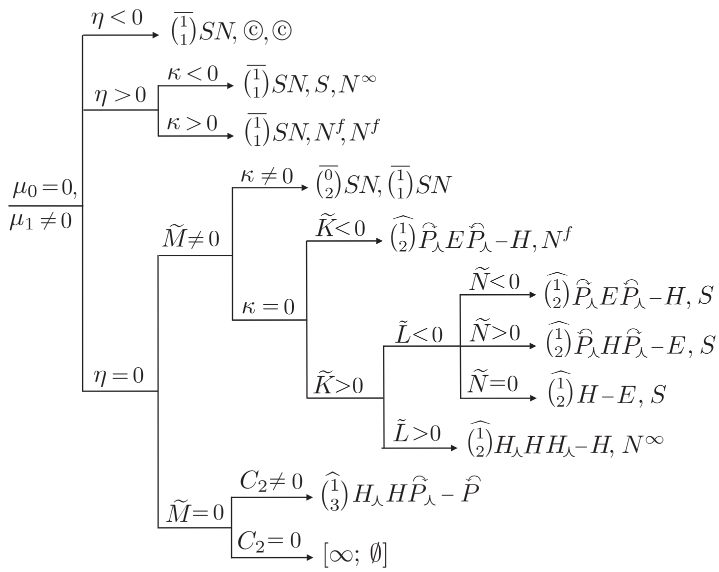


Diagram 2. **Configurations of infinite singularities: the case $\mu_0 = 0$, $\mu_1 \neq 0$.**

On the other hand the invariant polynomial μ_0 is the discriminant of the form $\widetilde{K}(\tilde{a}, x, y)$. So in this case the homogeneous quadratic polynomial $\widetilde{K}$ has the form: $\widetilde{K} = (ux + vy)^2$, where $u, v \in \mathbb{R}[\tilde{a}]$. Moreover the linear form $ux + vy$ is a common factor of the quadratic parts of systems (8) and, hence, the fourth finite singularity has coalesced with the infinite singular point $N[-v, u, 0]$. We observe that the condition $v \neq 0$ has to be satisfied, otherwise the common factor of the quadratic parts of systems (8) will be x and this implies $d = f = 0$ and we get degenerate systems. So $v \neq 0$ and via the transformation $x_1 = x$ and $y_1 = ux/v + y$, which keeps the singular points $M_1(0, 0)$ and $M_2(0, 1)$, we get $\widetilde{K}(\tilde{a}, x, y) = v^2y^2$. This means that the common factor of the homogeneous quadratic parts of systems (8) will be y and therefore the transformation we applied above, implies the conditions $g = l = 0$.

Thus we arrive at the family of systems

$$\dot{x} = cx + dy + 2hxy - dy^2, \quad \dot{y} = ex + fy + 2mxy - fy^2 \tag{9}$$

possessing three distinct finite singularities $M_1(0, 0)$, $M_2(0, 1)$ and $M_3\left(\dfrac{\alpha(\alpha-2\gamma)}{4\beta\gamma}, \dfrac{\alpha}{2\gamma}\right)$, where $\alpha = cf - de$, $\beta = cm - eh$ and $\gamma = dm - fh$. For these systems calculations yield

$$\mu_0 = 0, \quad \mu_1 = 4\beta\gamma y, \quad \kappa = -128m^2\gamma.$$

As for the above systems we have $\mu_1 \neq 0$ (i.e. $\gamma \neq 0$) we observe that the condition $\kappa = 0$ is equivalent to $m = 0$.

Since by Lemma 5 the sign of the invariant polynomial κ is relevant, in what follows we shall consider three cases: $\kappa < 0$, $\kappa > 0$ and $\kappa = 0$.

If $\kappa \neq 0$ we have $m \neq 0$ and we may assume $m = 1$ due to a time rescaling. So we get the systems

$$\dot{x} = cx + dy + 2hxy - dy^2, \quad \dot{y} = ex + fy + 2xy - fy^2, \tag{10}$$

and for the singular points M_i we have the following values for the traces ρ_i, for the determinants Δ_i, for the discriminants τ_i and for the linearization matrix $\mathcal{M}_1$ and $\mathcal{M}_2$:

$$(11)\qquad \begin{gathered}\mathcal{M}_1=\begin{pmatrix} c & d\\ e & f\end{pmatrix},\quad \mathcal{M}_2=\begin{pmatrix} c+2h & -d\\ e+2 & -f\end{pmatrix},\\ \rho_1=c+f,\quad \Delta_1=\alpha;\qquad \rho_2=c-f+2h,\quad \Delta_2=2\gamma-\alpha;\\ \rho_3=\frac{1}{2\beta\gamma}\big[d^2e(2+e)-f^2(c^2-2ceh-2eh^2)+2d(c^2-2ceh-2efh-e^2fh+e^2h^2)\big],\\ \Delta_3=\frac{\alpha(\alpha-2\gamma)}{2\gamma};\quad \tau_i=\rho_i^2-4\Delta_i,\quad i=1,2,3.\end{gathered}$$

Then for systems above we calculate

$$(12)\qquad \begin{gathered}\kappa=-128\gamma,\quad \widetilde{K}=4\gamma y^2,\quad \mu_1=4\beta\gamma y,\quad \eta=4\big[(f-2h)^2-8\gamma\big],\\ G_9=\gamma,\quad \widetilde{M}=-32x^2+16(f+2h)xy-8\big[(f+2h)^2-6d\big]y^2,\\ \mathcal{T}_4=8\beta\gamma\rho_1\rho_2\rho_3,\quad \mathcal{T}_3=8\beta\gamma\big[\rho_1(\rho_2+\rho_3)+\rho_2\rho_3\big],\quad \mathcal{T}_2=8\beta\gamma(\rho_1+\rho_2+\rho_3),\\ W_4=64\beta^2\gamma^2\tau_1\tau_2\tau_3,\quad W_2=64\beta^2\gamma^2(\tau_1+\tau_2+\tau_3),\\ W_3=64\beta^2\gamma^2\big[\tau_1\tau_2+\tau_1\tau_3+\tau_2\tau_3\big].\end{gathered}$$

First of all we prove the following lemma.

Lemma 6. *A system* (10) *possesses a star node if and only if the conditions* $W_4=U_3=0$ *hold. Moreover in this case the star node is unique and the conditions* $W_3\geq 0$ *and* $\mathcal{T}_4\neq 0$ *hold. In addition if* $W_3=0$ *then* $\eta>0$.

Proof: Assuming that a system (10) possesses a star node, without loss of generality we can consider that such a singular point is $M_1(0,0)$. In this case considering (11) we must have $d=e=0$ and $f=c$ and this implies $W_4=U_3=0$.

Conversely, assume that the conditions $W_4=U_3=0$ hold. Since $\mu_1\neq 0$ (i.e. $\beta\gamma\neq 0$) the relation $W_4=0$ is equivalent to $\tau_1\tau_2\tau_3=0$, i.e. at least one of the discriminants corresponding to finite singularities vanishes. Then without loss of generality we can consider that the singular point with vanishing discriminant is $M_1(0,0)$ and hence the relation $(c-f)^2+4de=0$ holds.

Assume first $e=0$. Then $f=c$ and we have

$$\text{Coefficient}[U_3,x^3y^2]=-72cd,\qquad \mu_1=4c(d-ch)y.$$

As $\mu_1\neq 0$ the condition $U_3=0$ implies $d=0$ (then U_3 vanishes) and we get a star node. We note that in this case we obtain

$$W_3=64c^6(c+h)^2(c+4h)^2\geq 0,\quad \eta=4(c+2h)^2,\quad \mathcal{T}_4=-16c^5h,\quad \mu_1=-4c^2hy.$$

It is clear that the condition $\mu_1\neq 0$ gives $\mathcal{T}_4\neq 0$ and that the condition $W_3=0$ implies $\eta>0$.

Suppose now $e\neq 0$. In this case we obtain $d=-(c-f)^2/(4e)$ and we calculate Coefficient$[U_3,x^5]=-12ef(e+2)$ and since $e\neq 0$ we consider two cases: $e+2=0$ and $e+2\neq 0$.

1) If $e=-2$, then calculations yield

$$\text{Coefficient}[U_3,x^4y]=-24(c+2h)(c+f+2h),\qquad \mu_1=(c+2h)(c^2-2cf+f^2-8fh)y/2.$$

Due to $\mu_1\neq 0$ the condition $U_3=0$ implies $f=-(c+2h)$ and we obtain Coefficient$[[U_3,x^3y^2]=36(c+h)^2(c+2h)=0$. Since $c+2h\neq 0$ we get $h=-c$. In this case we have $U_3=0$ and at the same time considering the matrix $\mathcal{M}_2$ from (11) we observe that the singular point $M_2(0,1)$ is a star node. We remark that in this case $M_1(0,0)$ is a node n^d and the condition $W_3=0$ is fulfilled. On the other hand we have $\eta=4c^2>0$ and $\mathcal{T}_4=16c^6\neq 0$ due to $\mu_1=-4c^3y\neq 0$.

2) Suppose now $e+2\neq 0$. Then $f=0$ and we calculate $\mu_1=-c^2(c-eh)y/e$ and

$$\text{Coefficient}[U_3,x^4y]=-3(10c^2+c^2e-16ceh+8e^2h^2)\equiv -3\Phi_1,$$
$$\text{Coefficient}[U_3,x^3y^2]=6(3c^3+3c^3e+5c^2eh-c^2e^2h-8ce^2h^2+4e^3h^3)/e\equiv 6\Phi_2/e.$$

Since $U_3=0$ we necessarily have $\text{Res}_{\,h}(\Phi_1,\Phi_2)=144c^6e^6(2+e)(4+e)^2=0$ and as $e(2+e)\neq 0$ and $\mu_1\neq 0$ (then $c\neq 0$) we get $e=-4$. Thus we obtain

$$U_3=3(3c+8h)y(8x+cy)\big[-256(c+8h)x^3+32(7c^2+16ch+64h^2)x^2y$$
$$-8c(c+4h)(c+8h)xy^2-c^2(11c^2-8ch-64h^2)y^3\big]/1024,$$

and it is easy to determine that the condition $U_3=0$ is equivalent to $3c+8h=0$. Hence $h=-3c/8$ and we obtain that in this case the singular point $M_3(-c/4,2)$ is a star node (besides the node n^d at $M_1(0,0)$) and the condition $W_3=0$ holds. We note that in this case $\eta=c^2/4>0$ and $\mathcal{T}_4=c^6/16\neq 0$ due to $\mu_1=-c^3y/8\neq 0$.

To prove the uniqueness of the star node it is enough to apply Lemma 4. For systems (10) we calculate

$$U_1=-4x^2+8hxy+4(d-fh-h^2)y^2\neq 0,$$

and by Lemma 4 (see statement (iii)) these systems could not have two star nodes.

Since all the cases are examined Lemma 6 is proved. ∎

Lemma 7. *Assume that a quadratic system possesses exactly 3 real finite simple singular points, one of which being a weak saddle of order 3. Then the condition $\kappa<0$ is satisfied. Moreover the other singular points could not be both foci and if both are nodes then at infinity there are three distinct real singularities (i.e. $\eta>0$).*

Proof: Instead of using systems (9) and forcing the existence of the weak saddle of order three, it is more convenient to start with a normal form that already assures such property. We take then the normal form

$$\dot{x}=x+ax^2+bxy+cy^2,\quad \dot{y}=-y-kx^2-lxy-my^2, \tag{13}$$

introduced by Cai Sui Lin [12] for systems having a weak saddle of order 1 and used in [6]. We note that in [6] the study was split in 3 different subcases $b=l=1$; $b=1,l=0$ and $b=l=0$.

In the case $b=l=0$ we obtain $\mathcal{F}_1=\mathcal{F}_2=\mathcal{F}_3=0$ and by Theorem 1 the weak saddle is integrable and it is not the case that we need.

In the case $b=1$ and $l=0$ we have $\mathcal{F}_1=-2a$ and by Theorem 1 in order to have a weak saddle of order 3 the condition $\mathcal{F}_1=0$ is necessary and then we get $a=0$. Thus we calculate

$$\mathcal{F}_2=-k^2(2+m)(2m-1)^4/2,\quad \mathcal{F}_3=-k^2m(2m-1)^4/2,$$

and according to Theorem 1 in order to have a weak saddle of the third order the conditions $\mathcal{F}_2=0$ and $\mathcal{F}_3\neq 0$ must hold. Hence we obtain $m=-2$ and then we calculate $\mu_0=k(c^2k-2)$. In order to have three finite singularities we need the condition $\mu_0=0$ and as $k\neq 0$ clearly the condition $c\neq 0$ holds. Therefore we have $k=2/c^2$ and in this case we obtain $W_4=-124400/c^4<0$ and $\kappa=-800/c^2<0$ and according to [5] we have exactly one focus.

Finally we consider the case $b=l=1$. Then we have $\mathcal{F}_1=-2(a-2m)$ and the condition $\mathcal{F}_1=0$ gives $a=2m$. In this case calculations yield

$$\mathcal{F}_2=-(8c-k)^2(2+m)(-1+2m)^4/2,$$
$$\mathcal{F}_3=-(8c-k)^2m(-1+2m)^4/2,$$

and considering Theorem 1 the conditions $\mathcal{F}_2=0$ and $\mathcal{F}_3\neq 0$ imply $m=-2$. Herein we calculate $\mu_0=48-16c-2k-18ck+c^2k^2$ and the condition $\mu_0=0$ is necessary to have exactly three real finite singularities. We observe that in this case the condition $\text{Discrim}\,[\mu_0,c]=4(8+k)^2(1+2k)\geq 0$ must hold and hence we either have $k=-8$ or $2k+1=u^2\geq 0$.

We observe that $k+8\neq 0$, otherwise for $k=-8$ we get a contradiction: $\mu_0=64(1+c)^2=0$ and $\mathcal{F}_3=40000(c+1)^2\neq 0$. Therefore $k=(u^2-1)/2$ and then we obtain

$$\mu_0=\big[c(1+u)^2-2(7-u)\big]\big[c(1-u)^2-2(7+u)\big],$$

and we need to impose $\mu_0=0$. If $u=\pm 1$ then $k=0$ and $c=3$. Then we get a system with configuration $s^{(3)},n,f$ and at infinity we have three distinct real singularities, i.e. $\eta>0$.

Assuming $u\neq\pm1$ we get $c=2(7\pm u)/(1\mp u)^2$. Then we calculate

$$\begin{aligned}
\mathbf{D}=-\frac{1296(5\mp u)^2(13\pm 2u+u^2)}{(1\mp u)^2},\quad G_9=\frac{25(3\pm u)^2(15+u^2)}{16(1\mp u)^2},\\
\kappa=-\frac{200(3\pm u)^2(15+u^2)}{(1\mp u)^2},\quad \mathcal{F}_3=\frac{25(5\mp u)^2(3\pm u)^2(15+u^2)^2}{4(1\mp u)^4}\neq 0,\\
W_4=-\frac{25(3\pm u)^2}{(1\mp u)^4}\Phi_4^{\pm}(u),\quad W_2=-\frac{75(3\pm u)^2}{2(1\mp u)^4}\Phi_2^{\pm}(u),\\
W_3=-\frac{625(3\pm u)^2(15\pm u^2)}{4(1\mp u)^4}\Phi_3^{\pm}(u),\quad \eta=-\frac{25(3\pm u)^2(5\pm 10u+u^2)}{(1\mp u)^2},
\end{aligned}\tag{14}$$

where

$$\begin{aligned}
\Phi_4^{\pm}(u)&=u^2(393945+46913u^2+311u^4)\pm 2u(217125+93750u^2+2981u^4)+204975,\\
\Phi_3^{\pm}(u)&=u^2(776+23u^2)\pm 2u(1227+101u^2)+2433),\\
\Phi_2^{\pm}(u)&=u^2(5u^4-1117u^2-7605)\pm 2u(16875-150u^2+11u^4)-4275.
\end{aligned}$$

Since $\Phi_i^-(u)=\Phi_i^+(-u)$, $i=2,3,4$ we shall consider only the solutions of $\Phi_i^+(u)=0$, $i=2,3,4$. We obtain that the polynomial $\Phi_4^+(u)$ has only two real solutions $u_1\approx-5.06133$ and $u_2\approx-1.63968$ and it is positive in the open interval (u_1,u_2). It is not too difficult to detect that the polynomials W_3, W_2 and η have no real roots inside the interval $[u_1,u_2]$ and all of them are positive.

On the other hand by (14) the condition $\mathcal{F}_3\neq 0$ implies $\kappa<0$ and $G_9\neq 0$. Therefore, since the condition $W_4\geq 0$ implies $W_3>0$, $W_2>0$ and $\eta>0$ (i.e. at infinity there are 3 distinct real singularities), according to [5] we conclude that Lemma 7 is proved. ∎

Remark 2. *We observe that for systems* (10) *the invariant polynomial* $\widetilde{M}\neq 0$ *and in addition the condition* $\kappa\neq 0$ *implies* $\widetilde{K}G_9\neq 0$. *Moreover we have* $\operatorname{sign}(\widetilde{K})=-\operatorname{sign}(\kappa)$.

Now we continue to study the cases $\kappa<0$, $\kappa>0$ and $\kappa=0$.

6.1. **The subcase $\kappa<0$.** This implies $\widetilde{K}>0$ and according to [5] (see Table 1, lines 115 - 122) systems (10) possess one saddle and two anti-saddles. Moreover the types of the anti-saddles are governed by the invariant polynomials W_i, $i=4,3,2$ (because in the case we consider here we have $G_9\neq 0$).

6.1.1. **The possibility $W_4<0$.** By [5] besides the saddle we have a node (which is generic) and a focus.

6.1.1.1. **Assume first $\mathcal{T}_4\neq 0$.** Then by Theorem 1 the focus is strong and considering the condition $\widetilde{M}\neq 0$ (see Remark 2) by Lemma 5 we arrive at the following three configurations of singularities:

- $s,n,f;\overline{\binom{1}{1}}SN,ⓒ,ⓒ$: *Example* $\Rightarrow$ $(c=1,\ h=3,\ e=-1,\ d=10,\ f=1)$ (if $\eta<0$);
- $s,n,f;\overline{\binom{1}{1}}SN,S,N^{\infty}$: *Example* $\Rightarrow$ $(c=1,\ h=3,\ e=-1,\ d=4,\ f=1/5)$ (if $\eta>0$);
- $s,n,f;;\overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow$ $(c=1,\ h=3,\ e=-1,\ d=961/200,\ f=1/5)$ (if $\eta=0$).

6.1.1.2. **Suppose now $\mathcal{T}_4=0$.** According to Theorem 1 we have either a weak focus, or a weak saddle depending on the sign of the product $\mathcal{T}_3\mathcal{F}$ (in the case $\mathcal{T}_3\neq 0$).

6.1.1.2.1. **The case $\mathcal{T}_3\mathcal{F} < 0$.** Then we have a weak focus, the order of which is governed by the invariant polynomials $\mathcal{F}_i \neq 0$ $(i = 1, 2, 3, 4)$ (see Theorem 1).

1) *The subcase* $\mathcal{F}_1 \neq 0$. In this case considering Lemma 5 and the condition $\kappa < 0$ we get the three global configurations of singularities:

- $s, n, f^{(1)}; \overline{\binom{1}{1}}SN, \textcircled{c}, \textcircled{c}$: *Example* $\Rightarrow (c = 1,\ h = 3,\ e = -1,\ d = 5,\ f = -1)$ (if $\eta < 0$);
- $s, n, f^{(1)}; \overline{\binom{1}{1}}SN, S, N^{\infty}$: *Example* $\Rightarrow (c = 1,\ h = 3,\ e = -1,\ d = 2,\ f = -1)$ (if $\eta > 0$);
- $s, n, f^{(1)}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = 1,\ h = 3,\ e = -1,\ d = 25/8,\ f = -1)$ (if $\eta = 0$).

2) *The subcase* $\mathcal{F}_1 = 0$. Then the weak focus is of order at least two and this depends on $\mathcal{F}_2$.

a) *The possibility* $\mathcal{F}_2 \neq 0$. In this case the weak focus has the order two and we arrive at the following three global configurations of singularities (to construct the examples below we use the normal form

$$\dot{x} = -y + (b - v)x^2/3 + 2axy + y^2, \quad \dot{y} = x + ax^2 + (b + 2v)xy/3 - ay^2) : \tag{15}$$

- $s, n, f^{(2)}; \overline{\binom{1}{1}}SN, \textcircled{c}, \textcircled{c}$: *Example* $\Rightarrow (b = -23/7,\ v = -6/7,\ a = \sqrt{1338 - 4\sqrt{2559}}/42)$ (if $\eta < 0$);
- $s, n, f^{(2)}; \overline{\binom{1}{1}}SN, S, N^{\infty}$: *Example* $\Rightarrow (b = -163/49,\ v = -6/7,\ a = \sqrt{66534 - 26\sqrt{127374}}/294)$ (if $\eta > 0$);
- $s, n, f^{(2)}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (b = -162/49,\ v = -6/7,\ a = 4\sqrt{2}/7)$ (if $\eta = 0$).

b) *The possibility* $\mathcal{F}_2 = 0$. According to [31] we could not have a center besides a node and a saddle. So we have a focus of order three. On the other hand in the case we consider, according to [19] we could have only three real distinct infinite singularities (i.e. $\eta > 0$). Hence we have one configuration (see above systems):

- $s, n, f^{(3)}; \overline{\binom{1}{1}}SN, S, N^{\infty}$: *Example* $\Rightarrow (b = -15/4,\ v = 3/2,\ a = \sqrt{(31 - 3\sqrt{57})/2}/8)$.

6.1.1.2.2. **The case $\mathcal{T}_3\mathcal{F} > 0$.** Then we have a weak saddle and we consider two subcases: $\mathcal{F}_1 \neq 0$ and $\mathcal{F}_1 = 0$.

1) *The subcase* $\mathcal{F}_1 \neq 0$. Then by Theorem 1 the weak saddle is of order one and considering Lemma 5 we get the three global configurations of singularities:

- $s^{(1)}, n, f; \overline{\binom{1}{1}}SN, \textcircled{c}, \textcircled{c}$: *Example* $\Rightarrow (c = 1,\ h = 3,\ e = -1,\ d = 5,\ f = -9/5)$ (if $\eta < 0$);
- $s^{(1)}, n, f; \overline{\binom{1}{1}}SN, S, N^{\infty}$: *Example* $\Rightarrow (c = 1,\ h = 3,\ e = -1,\ d = 2,\ f = 2(3 - 8\sqrt{6})/25)$ (if $\eta > 0$);
- $s^{(1)}, n, f; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = 1,\ h = 3,\ e = -1,\ d = 2(28\sqrt[3]{4} - 16\sqrt[3]{2} - 23),$ $f = 2(4\sqrt[3]{4} - 8\sqrt[3]{2} + 3))$ (if $\eta = 0$).

2) *The subcase* $\mathcal{F}_1 = 0$. By Theorem 1 the weak saddle is of order at least two and this order depends on $\mathcal{F}_2$.

a) *The possibility* $\mathcal{F}_2 \neq 0$. We have a weak saddle of order two and we arrive at the next three global configurations of singularities, for which to construct examples we use the canonical systems:

$$\dot{x} = x + ax^2 + bxy + cy^2, \quad \dot{y} = -y - kx^2 - lxy - my^2. \tag{16}$$

- $s^{(2)}, n, f; \overline{\binom{1}{1}}SN, \textcircled{c}, \textcircled{c}$: *Example* $\Rightarrow (a = 0,\ b = 1,\ c = 0,\ m = 0,\ k = 5,\ l = 4)$ (if $\eta < 0$);
- $s^{(2)}, n, f^{(2)}; \overline{\binom{1}{1}}SN, S, N^{\infty}$: *Example* $\Rightarrow (a = 0,\ b = 1,\ c = 0,\ m = 0,\ k = 5,\ l = 6)$ (if $\eta > 0$);
- $s^{(2)}, n, f^{(2)}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (a = 0,\ b = 1,\ c = 0,\ m = 0,\ k = 5,\ l = 2\sqrt{5})$ (if $\eta = 0$).

b) *The possibility* $\mathcal{F}_2 = 0$. According to [6] we could not have an integrable saddle. So we have a weak saddle of order three and we get the next three configurations.

- $s^{(3)}, n, f; \overline{\binom{1}{1}}SN, \textcircled{c}, \textcircled{c}$: *Example* $\Rightarrow (a = 2/5,\ b = 1,\ c = (27\sqrt{201} - 227)/250,\ m = -2,\ k = 1,\ l = -1/5)$ (if $\eta < 0$);
- $s^{(3)}, n, f^{(2)}; \overline{\binom{1}{1}}SN, S, N^{\infty}$: *Example* $\Rightarrow (a = 4/5,\ b = 1,\ c = (33\sqrt{51} - 233)/125,\ m = -2,\ k = 1,\ l = -2/5)$ (if $\eta > 0$);

- $s^{(3)}, n, f^{(2)}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (a=\sqrt{2},\ b=1,\ c=(5\sqrt{5}-11)/\sqrt{2},\ m=-2,\ k=1/(5\sqrt{5}-11),\ l=-1/\sqrt{2})$ (if $\eta=0$).

6.1.1.2.3. **The case $\mathcal{T}_3=0$.** Since $\mathcal{T}_4=0$ considering (12) we conclude that two traces are zero, but the third one is different from zero (as one point is a node). So according to Theorem 1 (see the statement (c)) we have either a weak focus and a weak saddle both of order one (if $\mathcal{F}_1\neq 0$), or a center and an integrable saddle (if $\mathcal{F}_1\neq 0$). However by [6] we could not have besides a node an integrable saddle and a center. So we get the following three configurations:

- $s^{(1)}, n, f^{(1)}; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c=-1/10,\ h=1/10,\ e=1,\ d=2,\ f=1/10)$ (if $\eta<0$);
- $s^{(1)}, n, f^{(1)}; \overline{\binom{1}{1}}SN, S, N^{\infty}$: *Example* $\Rightarrow (c=-7/5,\ h=7/5,\ e=1,\ d=2,\ f=7/5)$ (if $\eta>0$);
- $s^{(1)}, n, f^{(1)}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c=-4/3,\ h=4/3,\ e=1,\ d=2,\ f=4/3)$ (if $\eta=0$).

6.1.2. **The possibility $W_4>0$.** By [5] (see Table 1, lines 115, 121, 122) besides the saddle we have either two foci or two nodes and this situation if governed by the invariant polynomials W_3 and W_2.

6.1.2.1. **Assume first $W_3<0$ *or* $(W_3>0, W_2<0)$.** In this case we have two foci.

6.1.2.1.1. **The case $\mathcal{T}_4\neq 0$.** Then by Theorem 1 all three singularities are strong and according to Lemma 5 we get the following three geometrically distinct configurations:

- $s, f, f; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c=1,\ h=3,\ e=-1,\ d=7,\ f=-3)$ (if $\eta<0$);
- $s, f, f; \overline{\binom{1}{1}}SN, S, N^{\infty}$: *Example* $\Rightarrow (c=1,\ h=3,\ e=-1,\ d=40,\ f=12)$ (if $\eta>0$);
- $s, f, f; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c=1,\ h=3,\ e=-1,\ d=40,\ f=2(4\sqrt{5}-3))$ (if $\eta=0$).

6.1.2.1.2. **The case $\mathcal{T}_4=0$.** Then at least one singularity is weak.

1) *The subcase $\mathcal{T}_3\mathcal{F}<0$.* In this case by Theorem 1 we have one weak focus, the order of which is governed by the polynomials $\mathcal{F}_i,\ i=1,2,3,4$.

a) *The possibility $\mathcal{F}_1\neq 0$.* The focus is of order one and we arrive at the next three configurations:

- $s, f, f^{(1)}; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c=1,\ h=1,\ e=-1,\ d=5/4,\ f=-1)$ (if $\eta<0$);
- $s, f, f^{(1)}; \overline{\binom{1}{1}}SN, S, N^{\infty}$: *Example* $\Rightarrow (c=1,\ h=-2,\ e=-5/11,\ d=11/4,\ f=-1)$ (if $\eta>0$);
- $s, f, f^{(1)}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c=1,\ h=-5/4,\ e=-40/49,\ d=49/32,\ f=-1)$ (if $\eta=0$).

b) *The possibility $\mathcal{F}_1=0$.* In this case the weak focus is of order 2 or 3 because by [31] a quadratic system could not have a center besides a focus and a saddle. On the other hand according to [19] systems with three finite real singularities could not have a weak focus of order three and a strong focus. Moreover it is proved in [1] that we could have the singularities $s, f, f^{(2)}$ only in the case $\eta<0$. So we get the configuration (see systems (15)):

- $s, f, f^{(2)}; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (b=-3\sqrt[3]{4},\ v=-3,\ a=1)$.

2) *The subcase $\mathcal{T}_3\mathcal{F}>0$.* By Theorem 1 in this case we have a weak saddle.

a) *The possibility $\mathcal{F}_1\neq 0$.* The saddle is of order one and we arrive at the next three configurations:

- $s^{(1)}, f, f; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c=-11/10,\ h=1/10,\ e=-42/47,\ d=47/200,\ f=11/10)$ (if $\eta<0$);
- $s^{(1)}, f, f; \overline{\binom{1}{1}}SN, S, N^{\infty}$: *Example* $\Rightarrow (c=-21/20,\ h=7/5,\ e=-82/145,\ d=29/160,\ f=21/20)$ (if $\eta>0$);
- $s^{(1)}, f, f; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c=-(1+\sqrt{226})/15,\ h=1/10,\ e=-8(1+\sqrt{226})/(121+4\sqrt{226}),\ d=(121+4\sqrt{226})/900,\ f=(1+\sqrt{226})/15)$ (if $\eta=0$).

b) *The possibility $\mathcal{F}_1=0$.* In this case the weak saddle is of order 2 or 3. However according to Lemma 7 we could not have two foci besides a weak saddle of order three. Moreover, examining the graphics of the functions $W_i = 0$ $(i = 4, 3, 2)$, $\mathcal{T}_4 = 0$, $\mathcal{F}_1 = 0$ and $\eta = 0$, it can be proved that in the case under consideration we could not have $\eta \geq 0$. So we get the next configuration:

- $s^{(2)}, f, f; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c = -2,\ h = 1,\ e = -6/5,\ d = 5/2,\ f = 2)$.

3) *The subcase* $\mathcal{T}_3 = 0$. We note that in this case the systems could not be Hamiltonian, otherwise we get 2 centers and an integrable saddle, which is impossible (see [6]). On the other hand according to Theorem 1 in the case of non Hamiltonian systems we could have only two weak singularities and this implies $\mathcal{T}_2 \neq 0$. So we could apply the statement (c) of Theorem 1 which says that we have two weak foci if $\mathcal{T}_2 < 0$ and a weak focus and a weak saddle if $\mathcal{T}_2 > 0$.

a) *The possibility* $\mathcal{T}_2 < 0$. By Theorem 1 we have two weak foci (each one of order one) if $\mathcal{F}_1 \neq 0$ and we have two centers if $\mathcal{F}_1 = 0$. However according to [6] systems (10) could not have as finite singularities c, c, s. Thus in the case we consider we arrive at the following three configurations:

- $s, f^{(1)}, f^{(1)}; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c = -1,\ h = 1,\ e = -4/7,\ d = 7/2,\ f = 1)$ (if $\eta < 0$);
- $s, f^{(1)}, f^{(1)}; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = -1,\ h = 1,\ e = -306/313,\ d = 313/288,\ f = 1)$ (if $\eta > 0$);
- $s, f^{(1)}, f^{(1)}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = -1,\ h = 1,\ e = -17/18,\ d = 9/8,\ f = 1)$ (if $\eta = 0$).

b) *The possibility* $\mathcal{T}_2 > 0$. Then by the statement (c) of Theorem 1 we have either a weak focus and a weak saddle (each of order one) if $\mathcal{F}_1 \neq 0$, or we have a center and an integrable saddle if $\mathcal{F}_1 = 0$. However according to [6] in the second case we could not have a focus. So we get the configurations:

- $s^{(1)}, f, f^{(1)}; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c = -1,\ h = 1,\ e = -3/4,\ d = 5/4,\ f = 1)$ (if $\eta < 0$);
- $s^{(1)}, f, f^{(1)}; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = -1,\ h = 1,\ e = -6/7,\ d = 35/32,\ f = 1)$ (if $\eta > 0$);
- $s^{(1)}, f, f^{(1)}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = -1,\ h = 1,\ e = -5/6,\ d = 9/8,\ f = 1)$ (if $\eta = 0$).

6.1.2.2. **Suppose now** $W_3 > 0, W_2 > 0$**.** By [5] (see Table 1, row 115) besides a saddle we have two nodes which are generic due to $W_4 \neq 0$.

6.1.2.2.1. **The case** $\mathcal{T}_4 \neq 0$**.** Then the saddle is strong and we get the following three geometrically distinct configurations:

- $s, n, n; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c = 1,\ h = 1,\ e = 1/100,\ d = 3,\ f = 2)$ (if $\eta < 0$);
- $s, n, n; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = 1,\ h = 1,\ e = 1/3,\ d = 6,\ f = 5)$ (if $\eta > 0$);
- $s, n, n; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = 1,\ h = 1,\ e = 1/2,\ d = 3 + 2\sqrt{2},\ f = 2 + 2\sqrt{2})$ (if $\eta = 0$).

6.1.2.2.2. **The case** $\mathcal{T}_4 = 0$**.** In this case we have $\mathcal{T}_3 \neq 0$ because systems (10) possess two nodes besides a saddle. Then we have a weak saddle the order of which is governed by the invariant polynomials $\mathcal{F}_i$, $i = 1, 2, 3, 4$.

1) *The subcase* $\mathcal{F}_1 \neq 0$. By Theorem 1 the weak saddle is of order one and we arrive at the next three configurations:

- $s^{(1)}, n, n; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c = 1,\ h = 2,\ e = 1,\ d = 49/40,\ f = -1)$ (if $\eta < 0$);
- $s^{(1)}, n, n; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = 1,\ h = 2,\ e = 1,\ d = 1/8,\ f = -1)$ (if $\eta > 0$);
- $s^{(1)}, n, n; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = 1,\ h = 2,\ e = 1,\ d = 9/8,\ f = -1)$ (if $\eta = 0$).

2) *The subcase* $\mathcal{F}_1 = 0$. Then the order of the weak saddle is greater than or equal to 2.

a) *The possibility* $\mathcal{F}_2 \neq 0$. In this case we have a weak saddle of order two and this leads to the next three configurations (see systems(16)):

- $s^{(2)}, n, n; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (a = 1,\ b = -9/10,\ c = -27/100,\ m = 1,\ k = (127\sqrt{21} - 581)/54,\ l = -9/10)$ (if $\eta < 0$);

• $s^{(2)}, n, n; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow a = 1,\ b = -9/10,\ c = -1/4,\ m = 1,\ k = (9\sqrt{181} - 121)/10,\ l = -9/10)$ (if $\eta > 0$);

• $s^{(2)}, n, n; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow a = 1,\ b = -9/10,\ c = -(87\sqrt{29} + 419)/3380,\ m = 1,\ k = 13(-5089747 - 317811\sqrt{29} + 261\sqrt{58(7312891 + 821733\sqrt{29})}\ )/(10(419 + 87\sqrt{29})^2),\ l = -9/10)$ (if $\eta = 0$).

b) *The subcase* $\mathcal{F}_2 = 0$. Since $\mathcal{T}_3 \neq 0$, according to [6] we could not have an integrable saddle with two nodes. On the other hand by Lemma 7 in this case we could only have one configuration with a weak saddle of order three (here we use systems (16)):

• $s^{(3)}, n, n; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (a = -2,\ b = 1,\ c = (5 - \sqrt{17})/2,\ k = 2,\ l = 1,\ m = -2)$.

6.1.3. **The possibility** $W_4 = 0$**.** Considering (12) we obtain $\tau_1\tau_2\tau_3 = 0$, i.e. at least one discriminant vanishes and this leads to a node with coinciding eigenvalues.

6.1.3.1. **Assume first** $W_3 < 0$**.** In this case by [5] (see Table 1, row 119) besides the node, systems (10) possess a focus and a saddle. Since $W_3 < 0$, according to Lemma 6 systems (10) could not have a star node and therefore we have a node n^d.

6.1.3.1.1. **The case** $\mathcal{T}_4 \neq 0$**.** Then by Theorem 1 we do not have weak singularities and this leads to the following three configurations:

• $s, n^d, f; \overline{\binom{1}{1}}SN, \copyright, \copyright$: *Example* $\Rightarrow (c = 2,\ h = -1,\ e = -1/4,\ d = 1,\ f = 1)$ (if $\eta < 0$);

• $s, n^d, f; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = 2,\ h = -2,\ e = -1/4,\ d = 1,\ f = 1)$ (if $\eta > 0$);

• $s, n^d, f; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = 2,\ h = -(1 + 2\sqrt{2})/2,\ e = -1/4,\ d = 1,\ f = 1)$ (if $\eta = 0$).

6.1.3.1.2. **The subcase** $\mathcal{T}_4 = 0$**.** We have at least one weak singularity.

a) *The possibility* $\mathcal{T}_3\mathcal{F} < 0$. By Theorem 1 in this case we have one weak focus, the order of which is governed by the polynomials $\mathcal{F}_i$, $i = 1, 2, 3, 4$.

α) *The case* $\mathcal{F}_1 \neq 0$. The weak focus is of order one and we arrive at the next three configurations:

• $s, n^d, f^{(1)}; \overline{\binom{1}{1}}SN, \copyright, \copyright$: *Example* $\Rightarrow (c = -3/2,\ h = 5/4,\ e = -25/32,\ d = 2,\ f = 1)$ (if $\eta < 0$);

• $s, n^d, f^{(1)}; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = -9/4,\ h = 13/8,\ e = -169/128,\ d = 2,\ f = 1)$ (if $\eta > 0$);

• $s, n^d, f^{(1)}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = -2,\ h = 3/2,\ e = -9/8,\ d = 2,\ f = 1)$ (if $\eta = 0$).

β) *The case* $\mathcal{F}_1 = 0$. Then by Theorem 1 the weak focus is of order at least 2. However according to [19] and [6] systems with three finite real singularities could neither have the configuration $s, n^d, f^{(3)}$, nor s, n^d, c. On the other hand by [1] in the case of the weak focus of order two, only the case $\eta < 0$ is realizable. So we get the configuration (the example below is based on systems (15)):

• $s, n^d, f^{(2)}; \overline{\binom{1}{1}}SN, \copyright, \copyright$: *Example* $\Rightarrow (a = 1,\ (b, v) \in \{\mu^{-1}(0), W_4^{-1}(0)) \approx (-4.1394, -1.4439))$.

b) *The possibility* $\mathcal{T}_3\mathcal{F} > 0$. In this case we have one weak saddle.

α) *The case* $\mathcal{F}_1 \neq 0$. By Theorem 1 the weak saddle is of order one and we obtain three configurations:

• $s^{(1)}, n^d, f; \overline{\binom{1}{1}}SN, \copyright, \copyright$: *Example* $\Rightarrow (c = -27/20,\ h = 1,\ e = -58/45,\ d = 45/32,\ f = 27/20)$ (if $\eta < 0$);

• $s^{(1)}, n^d, f; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = -13/10,\ h = 1,\ e = -84/67,\ d = 67/50,\ f = 13/10)$ (if $\eta > 0$);

• $s^{(1)}, n^d, f; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = -(10 + \sqrt{103})/15,\ h = 1,\ e = -2(17 + 2\sqrt{103})(23 + 2\sqrt{103})/(1703 + 80\sqrt{103}),\ d = (1703 + 80\sqrt{103})/1800,\ f = (10 + \sqrt{103})/15)$ (if $\eta = 0$).

β) *The case* $\mathcal{F}_1 = 0$. The order of the weak saddle is ≥ 2. On the other hand by [6] in the case under examination we could not have the configuration $\$, n^d, f$. It was shown earlier (see the proof of Lemma 7) that in the case of a weak saddle of order three the condition $W_4 = 0$ implies $W_3 > 0$ and $W_2 > 0$. By [5] this means that we could not have the configuration $s^{(3)}, n^d, f$. Moreover examining the graphics of the functions $W_i = 0$ $(i = 4, 3, 2)$, $\mathcal{T}_4 = 0$, $\mathcal{F}_1 = 0$ and $\eta = 0$ it can be proved from these graphics that in the case under consideration we could not have $\eta \geq 0$. So considering Lemma 5 we get the configuration:

- $s^{(2)}, n^d, f; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c = -2,\ h = 1,\ e = -14/9,\ d = 9/4,\ f = 2)$.

c) *The possibility* $\mathcal{T}_3 = 0$. In this case considering (12) two traces vanish and we have two weak singularities: a focus and a saddle. On the other hand by [6] we could not have the configuration $\$, n^d, c$. So according to Theorem 1 both weak singularities are of order one and we get three configurations:

- $s^{(1)}, n^d, f^{(1)}; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c = -1,\ h = 9/5,\ e = -7/9,\ d = 729/275,\ f = 1)$ (if $\eta < 0$);
- $s^{(1)}, n^d, f^{(1)}; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = -1,\ h = 89/50,\ e = -139/178,\ d = 704969/271250,\ f = 1)$ (if $\eta > 0$);
- $s^{(1)}, n^d, f^{(1)}; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = -1,\ h = (3 + \sqrt{17})/4,\ e = -(7 + \sqrt{17})/(6 + 2\sqrt{17}),\ d = (45 + 11\sqrt{17})/(10 + 6\sqrt{17}),\ f = 1)$ (if $\eta = 0$).

6.1.3.2. **Suppose now** $W_3 > 0$**.** In this case by [5] (see Table 1, row 116) systems (10) possess one saddle and two nodes (one generic and another one with coinciding eigenvalues).

6.1.3.2.1. **The case** $U_3 \neq 0$**.** Then by Lemma 6 we have a node n^d.

a) *The subcase* $\mathcal{T}_4 \neq 0$. By Theorem 1 the saddle is strong and we obtain three configurations:

- $s, n, n^d; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c = 33/8,\ h = 1,\ e = -2,\ d = 625/512,\ f = 1)$ (if $\eta < 0$);
- $s, n, n^d; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = 31/8,\ h = 1,\ e = -2,\ d = 529/512,\ f = 1)$ (if $\eta > 0$);
- $s, n, n^d; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = 4,\ h = 1,\ e = -2,\ d = 9/8,\ f = 1)$ (if $\eta = 0$).

b) *The subcase* $\mathcal{T}_4 = 0$. Then we have a weak saddle.

α) *The possibility* $\mathcal{F}_1 \neq 0$. By Theorem 1 the weak saddle is of order one and we arrive at the following three configurations

- $s^{(1)}, n, n^d; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c = -2/5,\ h = 7/10,\ e = -2441/3600,\ d = 1764/2441,\ f = 1)$ (if $\eta < 0$);
- $s^{(1)}, n, n^d; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = -2/5,\ h = 7/10,\ e = -1243/1800,\ d = 882/1243,\ f = 1)$ (if $\eta > 0$);
- $s^{(1)}, n, n^d; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = -2/5,\ h = 7/10,\ e = -49/72,\ d = 18/25,\ f = 1)$ (if $\eta = 0$).

β) *The possibility* $\mathcal{F}_1 = 0$.

β.**1**) *The case* $\mathcal{F}_2 \neq 0$. According to Theorem 1 we have a weak saddle of order two and this leads to the following three configurations:

- $s^{(2)}, n, n^d; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (c = 1 - 2\xi,\ h = \xi,\ e = -11,\ d = \xi^2/11,\ f = 1)$ (if $\eta < 0$) (where $\xi = \mathcal{F}_1^{-1}(0) \approx -0.8549$);
- $s^{(2)}, n, n^d; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = 1 - 2\xi,\ h = \xi,\ e = -12,\ d = \xi^2/12,\ f = 1)$ (if $\eta > 0$) (where $\xi = \mathcal{F}_1^{-1}(0) \approx -0.8557$);
- $s^{(2)}, n, n^d; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = 1 - 2\xi,\ h = \xi,\ d = -\xi^2/e,\ f = 1)$ (if $\eta = 0$) (where $(e, \xi) \in \{\mathcal{F}_1^{-1}(0), \eta^{-1}(0)\} \approx (-11.5890,\ -0.8553)$).

β.**2**) *The case* $\mathcal{F}_2 = 0$. By [6] in this case we could not have an integrable saddle. On the other hand by Lemma 7 systems (10) have finite singularities $s^{(3)}, n, n^d$ only in the case $\eta > 0$. So we get the configuration:

- $s^{(3)}, n, n^d; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = 1 - 2\xi,\ h = \xi,\ d = -\xi^2/e,\ f = 1)$ (where $(e, \xi) \in \{\mathcal{F}_1^{-1}(0), \mathcal{F}_2^{-1}(0)\} \approx (-4.7882,\ -2.9346)$);

6.1.3.2.2. **The case $U_3 = 0$.** Then by Lemma 6 we have a star node. Moreover from the proof of Lemma 6 it follows that in the case $W_3 \neq 0$ the condition $W_4 = U_3 = 0$ yields $d = e = 0$ and $f = c$. In this case we calculate $\eta = 4(c+2h)^2$ and $\mathbf{D} = -192c^8(c+2h)^2$ and therefore the condition $\mathbf{D} < 0$ implies $\eta > 0$. Since by Lemma 6 the condition $\mathcal{T}_4 \neq 0$ holds (i.e. the saddle is strong) we could only have one configuration:

- $s, n, n^*; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = 1,\ h = -2,\ e = 0,\ d = 0,\ f = 1)$

6.1.3.3. **Suppose now $W_3 = 0$.** In this case by [5] (see Table 1, row 116) systems (10) possess one saddle and two nodes each one with coinciding eigenvalues.

6.1.3.3.1. **The case $U_3 \neq 0$.** Then by Lemma 6 we have two nodes n^d.

a) *The subcase* $\mathcal{T}_4 \neq 0$. According to Theorem 1 the saddle is strong and we obtain the following three configurations:

- $s, n^d, n^d; \overline{\binom{1}{1}}SN, \copyright, \copyright$: *Example* $\Rightarrow (c = 1,\ h = -2,\ e = 3/(\sqrt{6}-4),\ d = (4-\sqrt{6})/2,\ f = 1+\sqrt{6})$ (if $\eta < 0$);
- $s, n^d, n^d; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = 1,\ h = -2,\ e = (5-2\sqrt{6})/(3\sqrt{6}-8),\ d = (8-3\sqrt{6})/2,\ f = \sqrt{6}-1)$ (if $\eta > 0$);
- $s, n^d, n^d; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = 1,\ h = -2,\ e = (7-2\sqrt{6})/(4\sqrt{6}-11),\ d = (11-4\sqrt{6})/4,\ f = \sqrt{6})$ (if $\eta = 0$).

b) *The subcase* $\mathcal{T}_4 = 0$. In this case we have a weak saddle.

α) *The possibility* $\mathcal{F}_1 \neq 0$. By Theorem 1 the weak saddle is of order one and we arrive at three configurations:

- $s^{(1)}, n^d, n^d; \overline{\binom{1}{1}}SN, \copyright, \copyright$: *Example* $\Rightarrow (c = 1,\ h = -(8+5f^2)/20,\ e = -200(f-1)^2/(104-160f+180f^2-100f^3+25f^4),\ d = (104-160f+180f^2-100f^3+25f^4)/800$ (if $\eta < 0$) (where $f = \mathcal{T}_4^{-1}(0) \approx -3.2840$);
- $s^{(1)}, n^d, n^d; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c = 1,\ h = -2,\ e = 0,\ d = 1/2,\ f = 1)$ (if $\eta > 0$);
- $s^{(1)}, n^d, n^d; \overline{\binom{1}{1}}SN,\ \overline{\binom{0}{2}}SN$: *Example* $\Rightarrow (c = 1,\ h = -(2+f^2)/4,\ e = -8(f-1)^2/(2-2f+f^2)^2,\ d = (2-2f+f^2)^2/32),\ f = (\sqrt{2(5+\sqrt{41})}-1-\sqrt{41})/4$ (if $\eta = 0$).

β) *The possibility* $\mathcal{F}_1 = 0$. In this case the weak saddle is of the order ≥ 2. According to Lemma 7 in the case of the existence of a weak saddle of order three the condition $W_4 = 0$ implies $W_3 > 0$, i.e. we could not have $s^{(3)}, n^d, n^d$. So we could only obtain the configuration $s^{(2)}, n^d, n^d$, and we claim that in this case the condition $\eta < 0$ is satisfied. Indeed, to prove this we consider the 6-parameter family of systems (9) and we force the conditions on the two discriminants

$$\tau_1 = (c-f)^2 + 4de = 0, \quad \tau_2 = (c+f+2h)^2 - 4d(e+2m) = 0,$$

in order to have two nodes with coinciding eigenvalues, and namely the singularities $M_1(0,0)$ and $M_2(0,1)$. Moreover we also need the condition $\rho_3 = 0$ in order to force the singular point M_3 to be a weak singularity.

First of all we observe that due to $\kappa \neq 0$ the condition $d \neq 0$ must hold. Indeed, assume $d = 0$. Then the relations $\tau_1 = \tau_2 = 0$ yield $f = c$, $h = -c$ and therefore we obtain $\rho_3 = -cm/(2(e+m)) = 0$ which contradicts $\kappa = -128c^2m^2 \neq 0$.

Thus $d \neq 0$ and solving the equations $\tau_1 = 0$ and $\tau_2 = 0$ with respect to the parameters e and m we obtain

$$e = -(c-f)^2/(4d), \quad m = (c^2+f^2+2ch+2fh+2h^2)/(4d).$$

Then $W_4 = W_3 = 0$ and now it is necessary to force the conditions $\rho_3 = 0$ (i.e. $\mathcal{T}_4 = 0$) and $\mathcal{F}_1 = 0$. We calculate

$$\rho_3 = \frac{(c+f)(c-f+2h)}{8(c+h)(c^2+f^2+2ch)\big[(c+h)^2+(f-h^2)\big]}\, Z_1(c,f,h),$$
$$\mathcal{F}_1 = \frac{1}{8d}\big[(f-h)^2(f+h)^3 - (c+h)^2(7c^2f+2f^3+3c^2h+14cfh-2f^2h$$
$$+6ch^2+fh^2+h^3)\big] \equiv -\frac{1}{8d}\, Z_2(c,f,h),$$

where

$$Z_1(c,f,h) = (f-h)^2(f+h)^2 + (c+h)^2(7c^2+8f^2+14ch-4fh-h^2).$$

On the other hand we have

$$\mathbf{D} = -\frac{3}{64d^2}(c+f)^4(c+h)^2(c-f+2h)^4(c^2+f^2+2ch)^2,$$
$$\eta = -\frac{1}{4d^2}(2c^2+f^2+4ch)\big[(c+h)^2+(f+h)^2\big]^2.$$

Therefore due to $\mathbf{D} < 0$, the condition $\rho_3 = 0$ is equivalent to $Z_1(c,f,h) = 0$.

Thus it remains to determine the sign of the polynomial η when $Z_1(c,f,h) = Z_2(c,f,h) = 0$. Clearly if $h = 0$ then $\eta < 0$.

Assume $h \neq 0$. Then we may consider $h = 1$ due to a time rescaling in systems (9). We observe that in this case we have $\eta = 0$ if and only if $2c^2+f^2+4c = 0$. Moreover $\operatorname{sign}(\eta) = -\operatorname{sign}(2c^2+f^2+4c)$. Examining the common real solutions of the polynomials $Z_1(c,f,1)$ and $Z_2(c,f,1)$ we detect, that each such solution (c_0, f_0) satisfying the condition $\mathbf{D}(c_0,f_0) \neq 0$ lies outsides the ellipse $2c^2+f^2+4c = 0$, where $\eta < 0$. This proves our claim.

Thus in the case of two nodes n^d and of a weak saddle of order two we arrive at the configuration:

- $s^{(2)}, n^d, n^d; \overline{\binom{1}{1}}SN, ©, ©$: *Example* $\Rightarrow (h=1,\ e=-(c-f)^2/[(c+1)^2+(f+1)^2],\ d=[(c+1)^2+(f+1)^2]/4,)$, where $(c,f) \in \{\mathcal{F}_1^{-1}(0), \mathcal{T}_4^{-1}(0)\} \approx (-1.9245,\ 0.7926)$.

6.1.3.3.2. **The case** $U_3 = 0$**.** According to Lemma 6 the conditions $W_4 = W_3 = U_3 = 0$ imply $\mathcal{T}_4 = 16c^6 \neq 0$ and $\eta > 0$. So the saddle is strong and at infinity we have three real singularities. This leads to the configuration:

- $s, n^d, n^*; \overline{\binom{1}{1}}SN, S, N^\infty$: *Example* $\Rightarrow (c=1,\ h=-1,\ e=0,\ d=0,\ f=1)$.

6.2. **The subcase** $\kappa > 0$**.** According to Remark 2 the condition $\kappa > 0$ implies $\widetilde{K} < 0$ and by [5] (see Table 1, lines 110 - 114) systems (10) possess two saddles and one anti-saddle. On the other hand considering (12) the condition $\kappa > 0$ implies $\gamma < 0$ and then $\eta > 0$. By Lemma 5 this means that at infinity we only have the configuration of singularities: $\overline{\binom{1}{1}}SN, N^f, N^f$.

We observe that due to the condition $G_9 \neq 0$ according to Table 1 [5] the type of the anti-saddle is governed by the invariant polynomials W_i, $i = 4, 3, 2$.

6.2.1. **The possibility** $W_4 < 0$**.** The anti-saddle is a focus. If $\mathcal{T}_4 \neq 0$ then we do not have weak singularities and hence we get only one configuration:

- : $s, s, f; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c=1,\ h=2,\ e=1,\ d=-1/3,\ f=2)$

Assume now $\mathcal{T}_4 = 0$, i.e. we have at least one weak singularity. We consider two cases: $\mathcal{T}_3 \neq 0$ and $\mathcal{T}_3 = 0$.

6.2.1.1. **The case** $\mathcal{T}_3 \neq 0$**.** Then by Theorem 1 systems (10) have exactly one weak singularity. If $\mathcal{F}_1 \neq 0$ or $\mathcal{F}_1 = 0$ and $\mathcal{F}_2 \neq 0$ considering Theorem 1 we conclude that there are 4 possible configurations and they indeed are all realizable:

- $s, s, f^{(1)}; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c=0,\ h=1,\ e=1,\ d=-1,\ f=0)$ (if $\mathcal{T}_3\mathcal{F} < 0$, $\mathcal{F}_1 \neq 0$);
- $s, s^{(1)}, f; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c=-1,\ h=2,\ e=2,\ d=0,\ f=1)$ (if $\mathcal{T}_3\mathcal{F} > 0$, $\mathcal{F}_1 \neq 0$);

- $s, s, f^{(2)}; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = -1/2,\ h = 1,\ e = 2,\ d = -1/3,\ f = 1/2)$ (if $\mathcal{T}_3\mathcal{F} < 0,\ \mathcal{F}_1 = 0,\ \mathcal{F}_2 \neq 0$);
- $s, s^{(2)}, f; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = -1,\ h = 6/5,\ e = 1,\ d = 3/5,\ f = 1)$ (if $\mathcal{T}_3\mathcal{F} > 0,\ \mathcal{F}_1 = 0,\ \mathcal{F}_2 \neq 0$).

Suppose now $\mathcal{F}_1 = \mathcal{F}_2 = 0$, i.e. we have a single weak singularity of the order ≥ 2. By [6] in the case under consideration we could neither have a center, nor could we have an integrable saddle. Moreover as $\kappa > 0$, by Lemma 7 in this case a weak saddle of order three is also impossible. Therefore considering the only possible configuration here for infinite singularities we arrive at the following global configuration:

- $s, s, f^{(3)}; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = -14/11,\ h = 1,\ e = -7/4,\ d = 6496/5203,\ f = 14/11)$.

6.2.1.2. **The case $\mathcal{T}_3 = 0$.** Then at least two singularities are weak and since $\sigma = c + f + 2x - 2(f - h)y \neq 0$ (i.e. systems (10) could not be Hamiltonian) we conclude that we could only have two weak singularities. This implies the condition $\mathcal{T}_2 \neq 0$. Moreover as it was mentioned above we could neither have a center, nor could we have an integrable saddle and in this case by Theorem 1 (see statement (c)) the condition $\mathcal{F}_1 \neq 0$ is satisfied. So we obtain the following two configurations of singularities:

- $s^{(1)}, s^{(1)}, f; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = 2,\ h = -2,\ e = 1,\ d = 1,\ f = -2)$ (if $\mathcal{T}_2 < 0$);
- $s, s^{(1)}, f^{(1)}; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = -2,\ h = 2,\ e = -5,\ d = 1,\ f = 2)$ (if $\mathcal{T}_2 > 0$).

6.2.2. *The possibility $W_4 > 0$.* In this case besides two saddles, systems (10) possess a node, which is generic. By [6] in the case under consideration we could neither have a center, nor could we have an integrable saddle. Moreover by Lemma 7 it is impossible to have a weak saddle of order three together with a saddle and a node. This leads to the next four configurations:

- $s, s, n; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = 1,\ h = 2,\ e = 1,\ d = 0,\ f = 2)$ (if $\mathcal{T}_4 \neq 0$);
- $s, s^{(1)}, n; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = -1,\ h = 2,\ e = -3/4,\ d = 0,\ f = 1)$ (if $\mathcal{T}_4 = 0,\ \mathcal{T}_3 \neq 0,\ \mathcal{F}_1 \neq 0$);
- $s, s^{(2)}, n; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = -1,\ h = 621/500,\ e = 1,\ d = 129/250,\ f = 1)$ (if $\mathcal{T}_4 = 0,\ \mathcal{T}_3 \neq 0,\ \mathcal{F}_1 = 0$);
- $s^{(1)}, s^{(1)}, n; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = 2,\ h = -2,\ e = -4/5,\ d = 1,\ f = -2)$ (if $\mathcal{T}_4 = \mathcal{T}_3 = 0$).

6.2.3. *The possibility $W_4 = 0$.* Then the anti-saddle is a node with coinciding eigenvalues. Moreover by Lemma 6 this node is a star node if and only if $U_3 = 0$, and in this case the condition $\mathcal{T}_4 \neq 0$ holds. It was mentioned above that we could not have a weak saddle of order three besides a saddle and besides a node of any type. So we get the following five configurations of singularities:

- $s, s, n^d; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = 2,\ h = 2,\ e = -1/4,\ d = 1,\ f = 1)$ (if $\mathcal{T}_4 \neq 0,\ U_3 \neq 0$);
- $s, s, n^*; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = 1,\ h = 1,\ e = 0,\ d = 0,\ f = 1)$ (if $\mathcal{T}_4 \neq 0,\ U_3 = 0$);
- $s, s^{(1)}, n^d; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = -7/2,\ h = 9/4,\ e = -81/32,\ d = 2,\ f = 1)$ (if $\mathcal{T}_4 = 0,\ \mathcal{T}_3 \neq 0,\ \mathcal{F}_1 \neq 0$);
- $s, s^{(2)}, n^d; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = -3/4,\ h = 1,\ e = -2(3+\sqrt{39})/(23+\sqrt{39}),\ d = (23+\sqrt{39})/40,\ f = 3/4)$ (if $\mathcal{T}_4 = 0,\ \mathcal{T}_3 \neq 0,\ \mathcal{F}_1 = 0$);
- $s^{(1)}, s^{(1)}, n^d; \overline{\binom{1}{1}}SN, N^f, N^f \Rightarrow (c = -1,\ h = 4/5,\ e = -9/8,\ d = 128/175,\ f = 1)$ (if $\mathcal{T}_4 = \mathcal{T}_3 = 0$).

6.3. **The subcase $\kappa = 0$.** In this case for systems (9) we get $m = 0$ and we have $\mu_1 = 4efh^2y \neq 0$. Therefore we may assume $e = h = 1$ due to the rescaling $(x, y, t) \mapsto (hx/e, y, t/h)$ and this leads to the 3-parameter family of systems

$$\dot{x} = cx + dy + 2xy - dy^2, \quad \dot{y} = x + fy - fy^2. \tag{17}$$

These systems possess the singular points

$$M_1(0,0), \quad M_2(0,1), \quad M_3\Big(\frac{d-cf)(d-cf-2f)}{4f}, \frac{d-cf}{2f}\Big),$$

for which we have

$$\begin{aligned} &\rho_1 = c+f, \quad \Delta_1 = cf-d; \qquad \rho_2 = c-f+2,\\ &\Delta_2 = d-2f-cf; \quad \rho_3 = f(c+1)+d(1-f)/f,\\ &\Delta_3 = (cf-d)(d-2f-cf)/(2f); \quad \tau_i = \rho_i^2-4\Delta_i, \quad i=1,2,3. \end{aligned} \tag{18}$$

Then for the systems above we calculate

$$\begin{aligned} &\kappa=\eta=\mu_0=0, \quad G_9=0, \quad \mathcal{T}_i=0=W_i, \quad i=1,2,3,4,\\ &\mu_1=4fy\neq 0, \quad \widetilde{K}=-4fy^2, \quad \widetilde{N}=-4(1+f)fy^2, \quad \widetilde{M}=-8(2+f)^2y^2,\\ &\mathcal{B}_1=2f\rho_1\rho_2\rho_3, \quad \mathcal{B}_2=2f(f-1)^2\big[\rho_1\rho_2+\rho_1\rho_3+\rho_2\rho_3\big], \quad \tilde{L}=8f(2+f)y^2,\\ &W_7=12f^2\tau_1\tau_2\tau_3, \quad W_6=4f^2\big[\tau_1\tau_2+\tau_1\tau_3+\tau_2\tau_3\big], \quad W_5=1444f^2(\tau_1+\tau_2+\tau_3). \end{aligned} \tag{19}$$

We shall examine step by step the global configurations of singularities of systems (17).

We observe that the condition $\mu_1\neq 0$ implies $\widetilde{K}\neq 0$ and $\operatorname{sign}(\widetilde{K})=-\operatorname{sign}(f)$.

Remark 3. *For systems* (17) *the condition* $\tilde{L}=0$ *implies* $\widetilde{M}=0$*. Moreover if* $\tilde{L}\widetilde{N}\neq 0$ *then we have* $\operatorname{sign}(\tilde{L})=\operatorname{sign}(f(2+f))$ *and* $\operatorname{sign}(\widetilde{N})=-\operatorname{sign}(1+f)$.

6.3.1. **The possibility $\widetilde{K}<0$.** According to [5], systems (17) have two saddles and one anti-saddle.

6.3.1.1. **The case $W_7<0$.** Then the anti-saddle is a focus.

6.3.1.1.1. **The subcase $\mathcal{B}_1\neq 0$.** We have $\rho_1\rho_2\rho_3\neq 0$, i.e. all three singularities are strong. On the other hand we observe that the condition $\widetilde{K}<0$ implies $f>0$ and then $\widetilde{M}\neq 0$. So considering Lemma 5 we obtain the configuration:

- $s, s, f;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ N^f$: *Example* $\Rightarrow (c=-1,\ d=-2,\ f=3)$.

6.3.1.1.2. **The subcase $\mathcal{B}_1=0$.** Then $\rho_1\rho_2\rho_3=0$ and at least one of the traces is zero. Therefore without loss of generality we may assume $\rho_1=c+f=0$, i.e. $c=-f$ and the calculations yield

$$\begin{aligned} &\mathcal{B}_1=\mathcal{H}=0, \quad \mathcal{B}_2=4(f-1)^4(d+f^2), \quad \sigma=2(1-f)y,\\ &\mathbf{D}=-192(d+f^2)^2(d-2f+f^2)^2, \quad \mathcal{F}_1=2(1-f)(d+2f+f^2)\\ &W_7=192(1-d)(d+f^2)^2(d-3f^2+df^2+f^4), \quad C_2=(2+f)xy^2-dy^3. \end{aligned} \tag{20}$$

1) *The possibility* $\mathcal{B}_2\neq 0$. This implies $\sigma\neq 0$. Moreover since $W_7<0$ the condition $\mathcal{B}_2>0$ implies $\mathcal{F}_1\neq 0$, otherwise setting $d=-f(f+2)$ (i.e. $\mathcal{F}_1=0$) we obtain $W_7=-1536f^3(1+f)^4<0$, and therefore $\mathcal{B}_2=-8(f-1)^4f<0$.

Thus according to Theorem 1 (see statements (e_1), (e_2) and (e_4), $[\beta]$) and considering Lemma 5 we get the following three configurations:

- $s, s, f^{(1)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ N^f$: *Example* $\Rightarrow (c=-5/4,\ d=2,\ f=3/4)$ (if $\mathcal{B}_2<0$, $\mathcal{F}_1\neq 0$);
- $s, s, c;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ N^f$: *Example* $\Rightarrow (c=-2,\ d=-8,\ f=2)$ (if $\mathcal{B}_2<0$, $\mathcal{F}_1=0$);
- $s, s^{(1)}, f;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ N^f$: *Example* $\Rightarrow (c=-1/2,\ d=2,\ f=3/2)$ (if $\mathcal{B}_2>0$).

2) *The possibility* $\mathcal{B}_2=0$. In this case $(f-1)(d+f^2)=0$ and due to $W_7\neq 0$ we get $f=1$, i.e. $\sigma=0$. So we arrive at the following configuration:

- $\$, \$, c;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ N^f$: *Example* $\Rightarrow (c=-1,\ d=2,\ f=1)$;

6.3.1.2. **The case $W_7>0$.** Then the anti-saddle is a node which is generic.

6.3.1.2.1. **The subcase $\mathcal{B}_1\neq 0$.** In this case both saddle are strong and as $\widetilde{K}<0$, considering Lemma 5 we get the following configuration:

- $s, s, n;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ N^f$: *Example* $\Rightarrow (c=-1,\ d=-2,\ f=9/4)$.

6.3.1.2.2. **The subcase $\mathcal{B}_1 = 0$.** We set again $\rho_1 = 0$ which implies $c = -f$. We claim that the condition $W_7 > 0$ and $\widetilde{K} < 0$ imply $\mathcal{B}_2 > 0$ and $\mathcal{F}_1 \neq 0$. We first show that we could not have $\mathcal{F}_1 = 0$ and neither could we have $\mathcal{B}_2 = 0$. Supposing the contrary by (20) we obtain either

$$(f-1)(d+f^2) = 0 \quad \text{or} \quad (1-f)(d+2f+f^2) = 0.$$

Since the condition $W_7 > 0$ implies $d + f^2 \neq 0$ we must have either $f = 1$ or $d = -f(f+2)$, and we obtain either $W_7 = -384(d^2-1)^2$, or $W_7 = -1536f^3(1+f)^4$. The condition $\widetilde{K} = -4fy^2 < 0$ gives $f > 0$, and therefore in both cases we obtain the relation $W_7 < 0$ which contradicts the assumption.

Assume now $\mathcal{B}_2 < 0$. Then $d + f^2 < 0$ and we can set $d = -f^2 - u^2$ (with $u \neq 0$), and this leads to the relation $W_7 = -192u^4(1+f^2+u^2)(4f^2+u^2+f^2u^2) < 0$. So our claim is proved.

Thus $\mathcal{B}_2 > 0$ and $\mathcal{F}_1 \neq 0$ and considering Theorem 1 and Lemma 5 we get the configuration:

- $s, s^{(1)}, n;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} E \widehat{P}_{\lambda} - H,\ N^f$: *Example* $\Rightarrow (c = -3/2,\ d = 7/13,\ f = 3/2)$.

6.3.1.3. **The case $W_7 = 0$.** Then the anti-saddle is a node with the coinciding eigenvalues. For systems (17) we have $U_1 = -4(1+f)y^2 \neq 0$ and $Y_1 = 1 + f \neq 0$ due to $\widetilde{K} < 0$ (i.e. $f > 0$), and by Lemma 4 we could not have a star node.

6.3.1.3.1. **The subcase $\mathcal{B}_1 \neq 0$.** In this case both saddles are strong and considering Lemma 5 we obtain the configuration:

- $s, s, n^d;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} E \widehat{P}_{\lambda} - H,\ N^f$: *Example* $\Rightarrow (c = 1,\ d = -1/4,\ f = 2)$.

6.3.1.3.2. **The subcase $\mathcal{B}_1 = 0$.** Then one saddle is weak and without loss of generality we may assume that it is located at the origin and the node n^d is at the point $M_1(0,1)$. Then $\rho_1 = 0$ gives $c = -f$, whereas $\tau_2 = 4(1-d) = 0$ implies $d = 1$. In this case calculations yield

$$\mathcal{B}_1 = \mathcal{H} = 0,\ \ \mathcal{B}_2 = 4(f-1)^4(1+f^2),\ \ \sigma = 2(1-f)y,$$
$$W_7 = 0,\ \ \mathcal{F}_1 = 2(1-f)(f+1)^2,\ \ \mathbf{D} = -192(f-1)^4(1+f^2)^2.$$

Since for the family of systems (17) we have $\mathbf{D} < 0$ this implies $\mathcal{F}_1\mathcal{B}_2\sigma \neq 0$. Then by Theorem 1 (see statement (e_1)) the weak saddle is of order one. Therefore we get the configuration:

- $s, s^{(1)}, n^d;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} E \widehat{P}_{\lambda} - H,\ N^f$: *Example* $\Rightarrow (c = -2,\ d = 1,\ f = 2)$.

6.3.2. **The possibility $\widetilde{K} > 0$.** According to [5] systems (17) possess one saddle and two anti-saddles, the types of which are governed by the invariant polynomial W_7.

6.3.2.1. **The case $W_7 < 0$.** Then besides the saddle we have a node (which is generic) and a focus. We observe that the condition $W_7 < 0$ implies $C_2 \neq 0$. Indeed, otherwise setting $C_2 = 0$ (i.e. $f = -2$ and $d = 0$) we get $W_7 = 192c^2(c+2)^2 \geq 0$.

6.3.2.1.1. **The subcase $\mathcal{B}_1 \neq 0$.** Then we could not have weak singularities and considering the condition $C_2 \neq 0$ and Remark 3 and Lemma 5 we arrive at the next five configurations:

- $s, n, f;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} E \widehat{P}_{\lambda} - H,\ S$: *Example* $\Rightarrow (c = 5/4,\ d = 1,\ f = -1/2)$ (if $\tilde{L} < 0,\ \widetilde{N} < 0$);
- $s, n, f;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} H \widehat{P}_{\lambda} - E,\ S$: *Example* $\Rightarrow (c = 9/4,\ d = 1,\ f = -3/2)$ (if $\tilde{L} < 0,\ \widetilde{N} > 0$);
- $s, n, f;\ \widehat{\binom{1}{2}}\ H - E,\ S$: *Example* $\Rightarrow (c = 2,\ d = 3,\ f = -1)$ (if $\tilde{L} < 0,\ \widetilde{N} = 0$);
- $s, n, f;\ \widehat{\binom{1}{2}}\ H_{\lambda} H H_{\lambda} - H,\ N^{\infty}$: *Example* $\Rightarrow (c = 4,\ d = -5,\ f = -3)$ (if $\tilde{L} > 0$);
- $s, n, f;\ \widehat{\binom{1}{3}}\ H_{\lambda} H \widehat{P}_{\lambda} - \widehat{P}$: *Example* $\Rightarrow (c = -5,\ d = 1,\ f = -2)$ (if $\tilde{L} = 0$).

6.3.2.1.2. **The subcase $\mathcal{B}_1 = 0$.** Then $\rho_1\rho_2\rho_3 = 0$ and at least one of the traces (which corresponds to the saddle or to the focus) is zero. Without loss of generality we may assume $\rho_1 = c + f = 0$, i.e. $c = -f$ and we arrive at the relations given in (20). Therefore we deduce that the conditions $\widetilde{K} > 0$ (i.e. $f < 0$) and $W_7 < 0$ imply $\mathcal{B}_2\sigma \neq 0$. Moreover we have $\mathcal{F}_1 \neq 0$ otherwise setting $d = -f(f+2)$ we obtain $W_7 = -1536f^3(1+f)^4 \geq 0$ due to $f < 0$.

Thus by Lemma 5 considering the condition $C_2 \neq 0$ we get in the case $\mathcal{B}_2 < 0$ the configurations:

- $s, n, f^{(1)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} E \widehat{P}_{\lambda} - H,\ S:$ *Example* $\Rightarrow (c = 1/2,\ d = -1,\ f = -1/2)$ (if $\tilde{L} < 0,\ \widetilde{N} < 0$);
- $s, n, f^{(1)};\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} H \widehat{P}_{\lambda} - E,\ S:$ *Example* $\Rightarrow (c = 3/2,\ d = -3,\ f = -3/2)$ (if $\tilde{L} < 0,\ \widetilde{N} > 0$);
- $s, n, f^{(1)};\ \widehat{\binom{1}{2}}\ H - E,\ S:$ *Example* $\Rightarrow (c = 1,\ d = -2,\ f = -1)$ (if $\tilde{L} < 0,\ \widetilde{N} = 0$);
- $s, n, f^{(1)};\ \widehat{\binom{1}{2}}\ H_{\lambda} H H_{\lambda} - H,\ N^{\infty}:$ *Example* $\Rightarrow (c = 3,\ d = -10,\ f = -3)$ (if $\tilde{L} > 0$);
- $s, n, f^{(1)};\ \widehat{\binom{1}{3}}\ H_{\lambda} H \widehat{P}_{\lambda} - \widehat{P}:$ *Example* $\Rightarrow (c = 2,\ d = -5,\ f = -2)$ (if $\tilde{L} = 0$).

In the case $\mathcal{B}_2 > 0$ we obtain the configurations:

- $s^{(1)}, n, f;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} E \widehat{P}_{\lambda} - H,\ S:$ *Example* $\Rightarrow (c = 1/2,\ d = 1/4,\ f = -1/2)$ (if $\tilde{L} < 0,\ \widetilde{N} < 0$);
- $s^{(1)}, n, f;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} H \widehat{P}_{\lambda} - E,\ S:$ *Example* $\Rightarrow (c = 3/2,\ d = 1/4,\ f = -3/2)$ (if $\tilde{L} < 0,\ \widetilde{N} > 0$);
- $s^{(1)}, n, f;\ \widehat{\binom{1}{2}}\ H - E,\ S:$ *Example* $\Rightarrow (c = 1,\ d = 0,\ f = -1)$ (if $\tilde{L} < 0,\ \widetilde{N} = 0$);
- $s^{(1)}, n, f;\ \widehat{\binom{1}{2}}\ H_{\lambda} H H_{\lambda} - H,\ N^{\infty}:$ *Example* $\Rightarrow (c = 3,\ d = -8,\ f = -3)$ (if $\tilde{L} > 0$);
- $s^{(1)}, n, f;\ \widehat{\binom{1}{3}}\ H_{\lambda} H \widehat{P}_{\lambda} - \widehat{P}:$ *Example* $\Rightarrow (c = 2,\ d = -3,\ f = -2)$ (if $\tilde{L} = 0$).

6.3.2.2. **The case** $W_7 > 0$**.** Then beside the saddle we have two generic nodes.

6.3.2.2.1. **The subcase** $\mathcal{B}_1 \neq 0$**.** Then the saddle is strong and considering Remark 3 and Lemma 5 we arrive at the next six configurations:

- $s, n, n;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} E \widehat{P}_{\lambda} - H,\ S:$ *Example* $\Rightarrow (c = 1,\ d = 1,\ f = -1/2)$ (if $\tilde{L} < 0,\ \widetilde{N} < 0$);
- $s, n, n;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} H \widehat{P}_{\lambda} - E,\ S:$ *Example* $\Rightarrow (c = 2,\ d = 1,\ f = -3/2)$ (if $\tilde{L} < 0,\ \widetilde{N} > 0$);
- $s, n, n;\ \widehat{\binom{1}{2}}\ H - E,\ S:$ *Example* $\Rightarrow (c = 2,\ d = 2,\ f = -1)$ (if $\tilde{L} < 0,\ \widetilde{N} = 0$);
- $s, n, n;\ \widehat{\binom{1}{2}}\ H_{\lambda} H H_{\lambda} - H,\ N^{\infty}:$ *Example* $\Rightarrow (c = 1,\ d = -7/2,\ f = -3)$ (if $\tilde{L} > 0$);
- $s, n, n;\ \widehat{\binom{1}{3}}\ H_{\lambda} H \widehat{P}_{\lambda} - \widehat{P}:$ *Example* $\Rightarrow (c = 1,\ d = -7/2,\ f = -2)$ (if $\tilde{L} = 0,\ C_2 \neq 0$).
- $s, n, n;\ [\infty; \emptyset]:$ *Example* $\Rightarrow (c = 3,\ d = 0,\ f = -2)$ (if $\tilde{L} = 0,\ C_2 = 0$).

6.3.2.2.2. **The subcase** $\mathcal{B}_1 = 0$**.** Then the saddle is weak and we may assume that it is located at the origin, i.e. $c = -f$ and we get the relations (20). We observe that due to $f < 0$ and $\mathbf{D} < 0$ we obtain $\mathcal{B}_2 \neq 0$. Moreover the condition $\mathcal{B}_2 > 0$ has to be satisfied, otherwise we have $d + f^2 < 0$, and setting $d = -f^2 - u^2$ (where $u \neq 0$) we obtain $W_7 = -192u^4(1 + f^2 + u^2)(4f^2 + u^2 + f^2u^2) < 0$.

On the other hand we observe that in the case under examination we have $\widetilde{N} \neq 0$, otherwise the condition $\widetilde{N} = 0$ yields $f = -1$, and then we arrive at the contradiction: $W_7 = -384(d^2 - 1)^2 \leq 0$.

1) *The possibility* $\mathcal{F}_1 \neq 0$. Then $C_2 \neq 0$ because the condition $C_2 = 0$ implies $d = f + 2 = 0$ and then $\mathcal{F}_1 = 0$. So considering Remark 3 and Lemma 5 we arrive at the next four configurations:

- $s^{(1)}, n, n;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} E \widehat{P}_{\lambda} - H,\ S:$ *Example* $\Rightarrow (c = 1/2,\ d = 4/5,\ f = -1/2)$ (if $\tilde{L} < 0,\ \widetilde{N} < 0$);
- $s^{(1)}, n, n;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\lambda} H \widehat{P}_{\lambda} - E,\ S:$ *Example* $\Rightarrow (c = 3/2,\ d = 4/5,\ f = -3/2)$ (if $\tilde{L} < 0,\ \widetilde{N} > 0$);
- $s^{(1)}, n, n;\ \widehat{\binom{1}{2}}\ H_{\lambda} H H_{\lambda} - H,\ N^{\infty}:$ *Example* $\Rightarrow (c = 3,\ d = 1/2,\ f = -3)$ (if $\tilde{L} > 0$);
- $s^{(1)}, n, n;\ \widehat{\binom{1}{3}}\ H_{\lambda} H \widehat{P}_{\lambda} - \widehat{P}:$ *Example* $\Rightarrow (c = 2,\ d = 1/2,\ f = -2)$ (if $\tilde{L} = 0$).

2) *The possibility* $\mathcal{F}_1 = 0$. Then by Theorem 1 (see statement $(e_3), \beta$) we have an integrable saddle. Considering (20) we have $d = -f(f + 2)$ and calculations yield

$$C_2 = (f + 2)y^2(x - fy), \quad \tilde{L} = 8f(f + 2)y^2.$$

Since $f \neq 0$ the condition $\tilde{L} = 0$ is equivalent to $C_2 = 0$ and considering Remark 3 and Lemma 5 we arrive at the next four configurations:

- $\$, n, n;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ S:$ *Example* $\Rightarrow (c = 1/2,\ d = 3/4,\ f = -1/2)$ (if $\tilde{L} < 0,\ \widetilde{N} < 0$);
- $\$, n, n;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} H \widehat{P}_{\curlywedge} - E,\ S:$ *Example* $\Rightarrow (c = 3/2,\ d = 3/4,\ f = -3/2)$ (if $\tilde{L} < 0,\ \widetilde{N} > 0$);
- $\$, n, n;\ \widehat{\binom{1}{2}}\ H_{\curlywedge} H H_{\curlywedge} - H,\ N^{\infty}:$ *Example* $\Rightarrow (c = 3,\ d = -3,\ f = -3)$ (if $\tilde{L} > 0$);
- $\$, n, n;\ [\infty; \emptyset]:$ *Example* $\Rightarrow (c = 2,\ d = 0,\ f = -2)$ (if $\tilde{L} = 0$).

6.3.2.3. **The case $W_7 = 0$.** Then besides the saddle we have two nodes and at least one of them has coinciding eigenvalues. Without loss of generality we may assume that this point is located at the origin and therefore the condition $\tau_1 = (c - f)^2 + 4d = 0$ gives $d = -(c - f)^2/4$. Then calculations yield

$$U_1 = -4(1 + f)y^2,\quad Y_1 = 1 + f,\quad U_6 = -4y^2,\quad \tilde{L} = 8f(f + 2)y^2,$$
$$\mathbf{D} = -3/4(c + f)^4(c^2 + 8f + 2cf + f^2)^2,\quad C_2 = (2 + f)xy^2 + 1/4(c - f)^2y^3.$$

and considering Lemma 4 we conclude that systems (17) could not possess a star node. Moreover $C_2 \neq 0$ otherwise the conditions $c = f = -2$ implies $\mathbf{D} = 0$.

6.3.2.3.1. **The subcase $W_6 \neq 0$.** Considering (19) we conclude that only one discriminant vanishes, i.e. in this case we could not have two nodes n^d.

1) *Assume first* $\mathcal{B}_1 \neq 0$. Then the saddle is strong and considering Remark 3 and Lemma 5 we arrive at the next five configurations:

- $s, n, n^d;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ S:$ *Example* $\Rightarrow (c = 1,\ d = -9/16,\ f = -1/2)$ (if $\tilde{L} < 0,\ \widetilde{N} < 0$);
- $s, n, n^d;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} H \widehat{P}_{\curlywedge} - E,\ S:$ *Example* $\Rightarrow (c = 1,\ d = -25/16,\ f = -3/2)$ (if $\tilde{L} < 0,\ \widetilde{N} > 0$);
- $s, n, n^d;\ \widehat{\binom{1}{2}}\ H - E,\ S:$ *Example* $\Rightarrow (c = 2,\ d = -9/4,\ f = -1)$ (if $\tilde{L} < 0,\ \widetilde{N} = 0$);
- $s, n, n^d;\ \widehat{\binom{1}{2}}\ H_{\curlywedge} H H_{\curlywedge} - H,\ N^{\infty}:$ *Example* $\Rightarrow (c = 1,\ d = -4,\ f = -3)$ (if $\tilde{L} > 0$);
- $s, n, n^d;\ \widehat{\binom{1}{3}}\ H_{\curlywedge} H \widehat{P}_{\curlywedge} - \widehat{P}:$ *Example* $\Rightarrow (c = 1,\ d = -9/4,\ f = -2)$ (if $\tilde{L} = 0$).

2) *Suppose now* $\mathcal{B}_1 = 0$. In this case we have $\rho_1\rho_2\rho_3 = 0$ and $\rho_1 \neq 0$ (due to $\tau_1 = 0$). So without loss of generality we may assume $\rho_2 = 2 + c - f = 0$ and this gives $c = f - 2$. Then we have

$$\mathcal{B}_1 = \mathcal{H} = 0,\quad \mathcal{B}_2 = 4(f - 1)^4(1 + f^2),\quad \sigma = 2(1 - f)(y - 1),\quad \widetilde{N} = -4(f + 1)y^2,$$
$$W_7 = 0,\quad \mathcal{F}_1 = 2(1 - f)(f + 1)^2,\quad \mathbf{D} = -192(f - 1)^4(1 + f^2)^2,$$

and hence, the condition $\mathbf{D} < 0$ implies $\mathcal{B}_2 > 0$. We note that in this case $\mathcal{F}_1 \neq 0$ and $\widetilde{N} \neq 0$ otherwise we get $f = -1$, $c = -3$ and this implies $W_6 = 0$. Therefore by Theorem 1 (see statement (e_1)) we have a weak saddle of first order. Since $C_2\widetilde{N} \neq 0$, considering Remark 3 and Lemma 5 we arrive at the next four configurations:

- $s^{(1)}, n, n^d;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} E \widehat{P}_{\curlywedge} - H,\ S:$ *Example* $\Rightarrow (c = -5/2,\ d = -1,\ f = -1/2)$ (if $\tilde{L} < 0,\ \widetilde{N} < 0$);
- $s^{(1)}, n, n^d;\ \widehat{\binom{1}{2}}\ \widehat{P}_{\curlywedge} H \widehat{P}_{\curlywedge} - E,\ S:$ *Example* $\Rightarrow (c = -7/2,\ d = -1,\ f = -3/2)$ (if $\tilde{L} < 0,\ \widetilde{N} > 0$);
- $s^{(1)}, n, n;\ \widehat{\binom{1}{2}}\ H_{\curlywedge} H H_{\curlywedge} - H,\ N^{\infty}:$ *Example* $\Rightarrow (c = -5,\ d = -1,\ f = -3)$ (if $\tilde{L} > 0$);
- $s^{(1)}, n, n^d;\ \widehat{\binom{1}{3}}\ H_{\curlywedge} H \widehat{P}_{\curlywedge} - \widehat{P}:$ *Example* $\Rightarrow (c = -4,\ d = -1,\ f = -2)$ (if $\tilde{L} = 0$).

6.3.2.3.2. **The subcase $W_6 = 0$.** Then besides the node n^d located at the origin we have another node n^d and we can assume that this is the singularity $M_2(0, 1)$. Then we have $\tau_2 = 2\big[(c + 1)^2 + (f + 1)^2\big] = 0$ which implies $c = f = -1$. In this case we calculate

$$\mathcal{B}_1 = \mathcal{H} = \mathcal{F}_1 = 0,\quad \mathcal{B}_2 = 32 > 0,\quad \sigma = 2(2y - 1) \neq 0,$$
$$W_7 = W_6 = 0,\quad \widetilde{K} = 4y^2 > 0,\quad \tilde{L} = -8y^2 < 0,\quad \widetilde{N} = 0.$$

So by Theorem 1 (see statement $(e_3), \beta$) the saddle is integrable and considering Lemma 5 we get the configuration

- $\$, n^d, n^d;\ \widehat{\binom{1}{2}}\ H - E,\ S:$ *Example* $\Rightarrow (c = -1,\ d = 0,\ f = -1)$.

As all the cases are examined, we have constructed all 147 possible configurations for the family of quadratic systems with exact three real distinct finite singularities.

References

[1] J. C. Artés, J. Llibre and D. Schlomiuk, *The geometry of quadratic differential systems with a weak focus of second order.* International J. of Bifurcation and Chaos **16** (2006), 3127–3194.

[2] J. C. Artés, J. Llibre and D. Schlomiuk, N. Vulpe, *From topological to geometric equivalence in the classification of singularities at infinity for quadratic vector fields.* Rocky Mountain J. of Math., **43** (2013), No. 6.

[3] J. C. Artés, J. Llibre and D. Schlomiuk, N. Vulpe, *Configurations of singularities for quadratic differential systems with total finite multiplicity lower than 2.* Bul. Acad. Stiinte Repub. Mold. Mat., **71** (2013), 72–124.

[4] J. C. Artés, J. Llibre and D. Schlomiuk, N. Vulpe, *Algorithm for determining the global geometric configurations of singularities of total finite multiplicity 2 for quadratic differential systems.* Preprint, UAB Barcelona, 2013.

[5] J. C. Artés, J. Llibre and N. I. Vulpe, *Singular points of quadratic systems: A complete classification in the coefficient space* $\mathbb{R}^{12}$. International J. of Bifurcation and Chaos **18** (2008), 313–362.

[6] Joan C. Artés, Jaume Llibre, Nicolae Vulpe, *Complete geometric invariant study of two classes of quadratic systems.* Electron. J. Differential Equations, **2012** (2012) No. 9, 1–35.

[7] V. Baltag, *Algebraic equations with invariant coefficients in qualitative study of the polynomial homogeneous differential systems.* Bul. Acad. Stiinte Repub. Mold. Mat., **2**(2003), 31–46

[8] V. A. Baltag and N. I. Vulpe, *Affine-invariant conditions for determining the number and multiplicity of singular points of quadratic differential systems.* Izv. Akad. Nauk Respub. Moldova Mat., **1** (1993), 39–48.

[9] V. A. Baltag and N. I. Vulpe, *Total multiplicity of all finite critical points of the polynomial differential system.* Planar nonlinear dynamical systems (Delft, 1995), Differ. Equ. Dyn. Syst., **5** (1997), 455–471.

[10] N. N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type.* Mat. Sb. (N.S) **30** (7)2 (1952), 181-196 (Russian), Transl. Amer. Math. Soc. Ser. (1) **100** (1954), 397-413.

[11] D. Bularas, Iu. Calin. L. Timochouk and N. Vulpe, *T–comitants of quadratic systems: A study via the translation invariants.* Delft University of Technology, Faculty of Technical Mathematics and Informatics, Report no. 96-90, 1996; (URL: `ftp://ftp.its.tudelft.nl/publications/tech- reports/1996/DUT-TWI-96- 90.ps.gz`).

[12] Cai, Sui Lin, *The weak saddle and separatrix cycle of a quadratic system.* Acta Math. Sinica, **30**(1987), No.4, 553–559 (Chinese).

[13] Iu. Calin, *On rational bases of* $GL(2,\mathbb{R})$*-comitants of planar polynomial systems of differential equations.* Bul. Acad. Stiinte Repub. Mold. Mat., **2** (2003), 69–86.

[14] B. Coll, *Qualitative study of some classes of vector fields in the plane.* Ph. D., Universitat Autónoma de Barcelona, (1987), 5–34.

[15] F. Dumortier, J. Llibre and J. C. Artés, *Qualitative Theory of Planar Differential Systems.* Universitext, Springer–Verlag, New York–Berlin, *ISBN: 3–540–32893–9*, 2008.

[16] E. A. González Velasco, *Generic properties of polynomial vector fields at infinity.* Trans. Amer.Math. Soc., **143** (1969), 201–222.

[17] J. H. Grace and A. Young, *The algebra of invariants.* Stechert, New York, 1941.

[18] Q. Jiang and J. Llibre, *Qualitative classification of singular points.* Qual. Theory Dyn. Syst., **6** (2005), 87–167.

[19] J. Llibre and D. Schlomiuk, *Geometry of quadratic differential systems with a weak focus of third order.* Canad. J. of Math., **56** (2004), 310–343.

[20] P. J. Olver, *Classical Invariant Theory.* London Math. Soc. Student Texts **44**, Cambridge University Press, 1999.

[21] I. Nikolaev and N. Vulpe, *Topological classification of quadratic systems at infinity.* J. London Math. Soc., **2** (1997), 473–488.

[22] J. Pal and D. Schlomiuk, *Summing up the dynamics of quadratic Hamiltonian systems with a center.* Canad. J. Math., **56** (1997), 583–599.

[23] M. N. Popa, *Applications of algebraic methods to differential systems.* Romania, Piteşti Univers., The Flower Power Edit., 2004.

[24] D. Schlomiuk, *Algebraic particular integrals, integrability and the problem of the center.* Trans. Amer. Math. Soc., **338** (1993), 799–841.

[25] D. Schlomiuk *Algebraic and Geometric Aspects of the Theory of Polynomial vector fields*, in Bifurcations and periodic Orbits of Vector Fields, Dana Schlomiuk Editor, NATO ASI Series, Series C: Vol.408, Kluwer Academic Publishers, (1993), 429–467.

[26] D. Schlomiuk and J. Pal, *On the geometry in the neighborhood of infinity of quadratic differential phase portraits with a weak focus.* Qual. Theory Dyn. Syst., **2** (2001), 1–43.

[27] D. Schlomiuk and N. I. Vulpe, *Geometry of quadratic differential systems in the neighborhood of infinity*. J. Differential Equations, **215** (2005), 357–400.

[28] D. Schlomiuk and N. I. Vulpe, *The full study of planar quadratic differential systems possessing a line of singularities at infinity*. J. Dynam. Differential Equations, **20** (2008), 737–775.

[29] K. S. Sibirskii, *Introduction to the algebraic theory of invariants of differential equations*. Translated from the Russian. Nonlinear Science: Theory and Applications. Manchester University Press, Manchester, 1988.

[30] N. Vulpe, *Characterization of the finite weak singularities of quadratic systems via invariant theory*. Nonlinear Analysis, **74** (2011), No. 4, 6553–6582.

[31] N. I. Vulpe, *Affine–invariant conditions for the topological discrimination of quadratic systems with a center*. Differential Equations, **19** (1983), 273–280.

[32] N. I. Vulpe, *Polynomial bases of comitants of differential systems and their applications in qualitative theory* (in Russian). "Ştiinţa", Kishinev, 1986.

[33] N. Vulpe and M. Lupan, *Quadratic systems with dicritical points*. Bul. Acad. Stiinte Repub. Mold. Mat., **3** (1994), 52–60.

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