A fully discrete approximation of the one-dimensional stochastic wave equation

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Abstract A fully discrete approximation of one-dimensional nonlinear stochastic wave equations driven by multiplicative noise is presented. A standard finite difference approximation is used in space and a stochastic trigonometric method for the temporal approximation. This explicit time integrator allows for error bounds in $L^p(\Omega)$, uniformly in time and space, in such a way that the time discretisation do not suffer from any kind of CFL condition. Moreover, uniform almost sure convergence of the numerical solution is also proved. Numerical experiments are presented and confirm the theoretical results.

Keywords Nonlinear stochastic wave equation · Multiplicative noise · Strong convergence · Finite differences · Stochastic trigonometric methods.

Mathematics Subject Classification (2000) 65C20 · 65C30 · 60H10 · 60H15 · 60H35

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1 Introduction

We consider the numerical discretisation of the one-dimensional nonlinear stochastic wave equation

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial^2 x}(t,x)+f\big(t,x,u(t,x)\big)+\sigma\big(t,x,u(t,x)\big)\frac{\partial^2 W}{\partial x \partial t}(t,x) && \text{in } [0,T]\times[0,1],\\ &u(t,0)=u(t,1)=0 && \text{for } t\in[0,T],\\ &u(0,\cdot)=u_0,\ \frac{\partial u}{\partial t}(0,\cdot)=v_0 && \text{in } (0,1), \end{aligned} \tag{1.1}$$

where $T>0$ is a fixed time horizon and W is a Brownian sheet on $[0,T]\times[0,1]$ defined on some probability space $(\Omega,\mathscr{F},\mathscr{P})$. Precise conditions on the functions f and σ and on the initial values u_0 and v_0 are given below. For the numerical discretisation of (1.1), we first discretise in space by a standard finite difference scheme (as in [24]) and then in time by a stochastic trigonometric method (see e.g. [2,5,4]).

While much efforts have been devoted to the numerical discretisation of stochastic parabolic problems (see e.g. [11,12,25,23,32,22,17,21,20,18,30]), our paper offers one of the few attempts to the numerical discretisation of stochastic nonlinear hyperbolic problems. In fact, as far as strong approximations for stochastic wave equations with multiplicative noise is concerned, references [24,27] used finite difference discretisations in space and both in time and space, respectively. We also point out that weak approximations, in the probabilistic sense, have also been studied in [15] and more recently in the preprint [29]. On the other hand, in the case of linear problems with additive noise, the paper [19] used a finite element discretisation, while in [4] a stochastic trigonometric method has been applied for the time discretisation of such problems. More recently, we point out that the preprint [31] presents a full discretisation of the semilinear wave equation with additive noise: a spectral Galerkin approximation is used in space and an adapted stochastic trigonometric method, using linear functionals of the noise as in [16], is employed in time. Eventually, time discretisation of nonlinear stochastic wave equations by stochastic trigonometric methods, without the use of filter functions, is analysed in the preprint [29]. Note that all these latter references deal with $L^p([0,1])$ convergence in the space variable, whereas we are concerned with space-time uniform convergence in $L^{2p}(\Omega)$.

The author of [27] noted that the spatial convergence rate of the scheme proposed in [24] was unexpectedly slow and that it would be interesting to know whether time-discretisations of this method would converge faster. In the present paper, we will answer positively to this question and, moreover, show that our numerical scheme for the time discretisation of (1.1) does not suffer a stepsize restriction due to the CFL condition, as does the numerical integrator proposed in [27]. In the latter reference, this condition thus forces the numerical scheme to use (at most) the same step sizes in time and in space.

In order to discretise efficiently the problem (1.1) in time, one is often interested in using explicit methods with large step sizes (see for example [9] for deterministic problems). A standard approach in the deterministic case is the leap-frog scheme, but unfortunately one has a step-size restriction due to stability issues (as seen above).

Much efficient numerical integrators for the time discretisation of deterministic wave equations are the trigonometric methods considered in [3,9] and more recently [8], for example. Observe, that these explicit numerical methods were firstly designed for an efficient discretisation of highly oscillatory problems (see [13, Chapter XIII] and references therein). In [2,5,4], an extension of the trigonometric methods to stochastic problems is presented and analysed. This is the numerical method that will be used for the time discretisation of (1.1) in the present publication.

Throughout the paper we will assume that the functions f and σ satisfy the following conditions:

$$\sup_{t\in[0,T]}\Big(|f(t,x,z)-f(t,y,v)|+|\sigma(t,x,z)-\sigma(t,y,v)|\Big)\leq C\Big(|x-y|+|z-v|\Big) \quad (1.2)$$

and

$$\sup_{(t,x)\in[0,T]\times[0,1]}\Big(|f(t,x,z)|+|\sigma(t,x,z)|\Big)\leq C\Big(1+|z|\Big), \quad (1.3)$$

for every $x,y\in[0,1]$ and $z,v\in\mathbb{R}$. On the other hand, let us introduce the spaces where the initial data u_0 and v_0 will be assumed to take their values. Namely, for any $\alpha\in\mathbb{R}$, we denote by $H^\alpha([0,1])$ the subspace of the fractional Sobolev space of order α formed by functions $g:[0,1]\to\mathbb{R}$ such that

$$\|g\|_\alpha := \left(\sum_{j=1}^\infty (1+j^2)^\alpha \langle g,\varphi_j\rangle^2_{L^2([0,1])}\right)^{1/2} < +\infty,$$

where $\varphi_j(x):=\sqrt{2}\sin(j\pi x)$, $j\geq 1$, and we note that $(\varphi_j)_{j\geq 1}$ forms a complete orthonormal system of $L^2([0,1])$. Moreover, we assume the obvious compatibility condition $u_0(0)=v_0(0)=0$.

As far as the rigorous formulation of our equation (1.1), we will use the random field approach set up by Walsh in [28]. That is, if we let $(\mathscr{F}_t)_{t\geq 0}$ be the filtration generated by the Brownian sheet W, a (mild) solution to equation (1.1) will be an $\mathscr{F}_t$-adapted process $\{u(t,x),\,(t,x)\in[0,T]\times[0,1]\}$ satisfying

$$\begin{aligned} u(t,x) &= \int_0^1 G(t,x,y)v_0(y)\,\mathrm{d}y + \frac{\partial}{\partial t}\Big(\int_0^1 G(t,x,y)u_0(y)\,\mathrm{d}y\Big) \\ &\quad + \int_0^t\int_0^1 G(t-s,x,y)f(s,y,u(s,y))\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \int_0^t\int_0^1 G(t-s,x,y)\sigma(s,y,u(s,y))\,W(\mathrm{d}s,\mathrm{d}y), \end{aligned} \quad (1.4)$$

where $G=G(t,x,y)$ is the Green function of the wave equation with homogeneous Dirichlet boundary conditions. The following expansion will be very useful in the sequel [24]:

$$G(t,x,y)=\sum_{j=1}^\infty \frac{\sin(j\pi t)}{j\pi}\varphi_j(x)\varphi_j(y).$$

Existence and uniqueness of solution to our stochastic partial differential equation (1.1) under the above assumptions can be obtained using standard arguments (see e.g. [28,1]). Additionally, assuming that $u_0 \in H^\alpha([0,1])$ and $v_0 \in H^\beta([0,1])$ for some $\alpha > 1/2$ and $\beta > -1/2$, one has almost surely Hölder continuity of the sample paths of the solution of order δ, for all $\delta \in (0,\delta_0)$, where $\delta_0 = \frac{1}{2} \wedge \left(\alpha - \frac{1}{2}\right) \wedge \left(\beta + \frac{1}{2}\right)$ (see [24, Prop. 2]).

The present paper is organised as follows. In Section 2, we will recall the spatial discretisation method used in [24] and prove an auxiliary result. Section 3 will be devoted to set up the time discretisation method for our stochastic wave equation and define a suitable space-time continuous interpolation process associated to it. The main convergence result of the paper will be stated and proved in Section 4. Finally, numerical experiments are presented in Section 5.

2 A finite difference approximation of the nonlinear stochastic wave equation

In this section, we will recall how in [24] the problem (1.1) has been discretised in space using a standard finite difference scheme and state the main result on strong convergence of the spatial discretisation contained therein (cf. [24, Thm. 1]). Using some arguments contained in the latter paper, we will also deduce a straightforward result which will be needed in the sequel (see Lemma 2.1 below).

Let an integer $M \geq 1$ and the partition $x_m = m/M$, for $m = 1,\dots,M-1$, of the unit interval $(0,1)$ with equidistant (spatial) mesh size $\Delta x = 1/M$. Then, the spatial semi-discretisation of (1.1) is defined as the solution of the following system of stochastic differential equations:

$$\begin{aligned} \mathrm{d}u_m^M(t) &= v_m^M(t)\,\mathrm{d}t \\ \mathrm{d}v_m^M(t) &= M^2 \sum_{\ell=1}^{M-1} d_{m\ell} u_\ell^M(t)\,\mathrm{d}t + f(t,x_m,u_m^M(t))\,\mathrm{d}t \\ &\quad + \sqrt{M}\sigma(t,x_m,u_m^M(t))\,\mathrm{d}W_m^M(t), \end{aligned} \tag{2.1}$$

for $m = 1,\dots,M-1$, where $u_m^M(t) := u^M(t,x_m)$ and $v_m^M(t) := v^M(t,x_m)$. Here, $W^M(t) = (W_1^M(t),\dots,W_{M-1}^M(t))$ is an $(M-1)$-dimensional standard Brownian motion with $W_m^M(t) := \sqrt{M}(W(t,x_{m+1}) - W(t,x_m))$. The $d_{m\ell}$ are the entries of the tri-diagonal $(M-1)\times(M-1)$ matrix

$$D = \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 \end{pmatrix}.$$

Defining the vector $w^M(t) := (u^M(t), v^M(t))^T \in \mathbb{R}^{2(M-1)}$, one can rewrite the above system of stiff stochastic differential equations as

$$\mathrm{d}w^M(t) = Aw^M(t)\,\mathrm{d}t + F(w^M(t))\,\mathrm{d}t + \Sigma(w^M(t)) \begin{pmatrix} 0 \\ \mathrm{d}W^M(t) \end{pmatrix}, \tag{2.2}$$

where $F(w^M(t)) = (0, f(t,x_1,u_1^M(t)),\dots,f(t,x_{M-1},u_{M-1}^M(t))^T \in \mathbb{R}^{2(M-1)}$,

$$A = \begin{pmatrix} 0 & I \\ M^2 D & 0 \end{pmatrix} \quad \text{and} \quad \Sigma(w^M(t)) = \sqrt{M} \begin{pmatrix} 0 & 0 \\ 0 & B_\sigma(w^M(t)) \end{pmatrix}$$

with a diagonal matrix $B_\sigma(w^M(t)) \in \mathbb{R}^{(M-1)\times(M-1)}$ of entries $\sigma(t,x_m,u_m^M(t))$ for $m = 1,\dots,M-1$.

By Itô's formula, one easily proves that the solution of (2.2) satisfies the following mild equation:

$$w^M(t) = \mathrm{e}^{tA} w^M(0) + \int_0^t \mathrm{e}^{(t-s)A} F(w^M(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{(t-s)A} \Sigma(w^M(s)) \begin{pmatrix} 0 \\ \mathrm{d}W^M(s) \end{pmatrix}. \tag{2.3}$$

For $x \in [0,1]$, a continuous version of the above approximation can be obtained by linear interpolation:

$$u^M(t,x) := u^M(t,x_m) + (Mx-m)\big(u^M(t,x_{m+1}) - u^M(t,x_m)\big),$$

if $x \in [x_m, x_{m+1})$. This sequence of processes, $\{u^M(t,x)\}_{M\geq 1}$, approximates the solution of our stochastic wave equation (1.1) and can be shown to satisfy the following evolution equation (see [24] for details):

$$\begin{aligned} u^M(t,x) = &\int_0^1 G^M(t,x,y) v_0(\kappa_M(y))\,\mathrm{d}y \\ &+ \frac{\partial}{\partial t}\Big(\int_0^1 G^M(t,x,y) u_0(\kappa_M(y))\,\mathrm{d}y\Big) \\ &+ \int_0^t \int_0^1 G^M(t-s,x,y) f(s,\kappa_M(y),u^M(s,\kappa_M(y)))\,\mathrm{d}y\,\mathrm{d}s \\ &+ \int_0^t \int_0^1 G^M(t-s,x,y) \sigma(s,\kappa_M(y),u^M(s,\kappa_M(y)))\,W(\mathrm{d}s,\mathrm{d}y), \end{aligned}$$

for $x \in (0,1)$ and $t \in (0,T]$. Here, we use the notation $\kappa_M(y) = [My]/M$ and the discrete Green function

$$G^M(t,x,y) = \sum_{j=1}^{M-1} \frac{\sin\Big(j\pi t\sqrt{c_j^M}\Big)}{j\pi\sqrt{c_j^M}} \varphi_j^M(x) \varphi_j(\kappa_M(y)), \tag{2.4}$$

with $\frac{4}{\pi^2} \leq c_j^M := \frac{\sin^2\left(\frac{j\pi}{2M}\right)}{\left(\frac{j\pi}{2M}\right)^2} \leq 1$ and

$$\varphi_j^M(x) = \varphi_j(x_m) + (Mx-m)\big(\varphi_j(x_{m+1}) - \varphi_j(x_m)\big)$$

for $x \in (x_m, x_{m+1})$, where we recall that $\varphi_j(x) = \sqrt{2}\sin(j\pi x)$ for $j = 1,\dots,M-1$. As pointed out in [24, Eq. (20)], the function G^M verifies that

$$\sup_{M\geq 1} \sup_{(t,x)\in[0,T]\times[0,1]} \int_0^1 |G^M(t,x,y)|^2\,\mathrm{d}y < +\infty. \tag{2.5}$$

Moreover, [24, Prop. 3] asserts that, for all $p \geq 1$,

$$\sup_{M\geq 1} \sup_{(t,x)\in[0,T]\times[0,1]} \mathbb{E}[|u^M(t,x)|^p] < +\infty. \tag{2.6}$$

Before stating the main convergence result of [24], let us prove the following simple lemma, which will be used in the proof of our main result, Theorem 4.1.

Lemma 2.1 *There is a positive constant C independent of M such that, for all $0 < s < t$ and all $x \in (0,1)$, it holds*

$$\int_0^1 |G^M(s,x,y) - G^M(t,x,y)|^2 \,\mathrm{d}y \leq C\,(t-s).$$

Proof It follows with similar arguments as those in the proof of [24, Prop. 2] (see the analysis of the term $D_{11}(s,t,x)$ therein). Namely, the very definition of G^M and the fact that

$$\int_0^1 \varphi_j(\kappa_M(y))\varphi_k(\kappa_M(y))\,\mathrm{d}y = \delta_{\{j=k\}},$$

with the Kronecker delta function $\delta_{\{j=k\}}$, implies that

$$\int_0^1 |G^M(s,x,y) - G^M(t,x,y)|^2 \,\mathrm{d}y \leq C \sum_{j=1}^{M-1} \frac{\left(\sin\left(j\pi s\sqrt{c_j^M}\right) - \sin\left(j\pi t\sqrt{c_j^M}\right)\right)^2}{j^2\pi^2 c_j^M}.$$

Then, since $c_j^M \in [\frac{4}{\pi^2}, 1]$, we have

$$\int_0^1 |G^M(s,x,y) - G^M(t,x,y)|^2 \,\mathrm{d}y \leq C \sum_{j=1}^{\infty} \frac{1}{j^2} \min\left(1, j^2(t-s)^2\right),$$

where the constant C does not depend on M. It can be seen that the last series is bounded by $(t-s)$, which concludes the proof. □

The following result establishes the convergence of the above semi-discrete solution $u^M(t,x)$ to the exact solution $u(t,x)$ of our stochastic wave equation (1.1) (cf. [24, Thm. 1]).

Theorem 2.1 *Suppose that $u_0 \in H^\alpha([0,1])$ with $\alpha > 3/2$ and $v_0 \in H^\beta([0,1])$ with $\beta > 1/2$. Assume that the functions f and σ satisfy the Lipschitz condition* (1.2) *and the linear growth condition* (1.3).

Let $p \geq 1$. Then, there exists a positive constant C independent of M such that

$$\sup_{(t,x)\in[0,1]\times[0,T]} \left(\mathbb{E}\left[\left|u^M(t,x) - u(t,x)\right|^{2p}\right]\right)^{1/(2p)} \leq C(\Delta x)^{\rho-\varepsilon}$$

for all $\varepsilon > 0$ with $\rho = 1/3 \wedge (\alpha - 3/2) \wedge (\beta - 1/2)$. Moreover, $u^M(t,x)$ converges almost surely to $u(t,x)$ as $\Delta x = 1/M$ tends to zero, uniformly with respect to $(t,x) \in [0,T]\times[0,1]$.

3 Time discretisation by a stochastic trigonometric method

This section is devoted to present the time discretisation method that will be applied to the semi-discrete problem (2.1) (or (2.2)). As explained in the Introduction, our method corresponds to a particular case of the so-called trigonometric schemes for second order differential equations and, on the other hand, if we focus on the mild evolution equation (2.3), it can be seen as an explicit Euler-Maruyama scheme for this formulation of the problem.

For ease of exposition and for the rest of the presentation, we will now assume that the functions f and σ only depend on the variable u. All forthcoming results can be easily extended to the general setting.

Let $\Delta t = T/N$ denote the step size of our numerical time integrator and $t_n = n\Delta t$, for $n = 0,1,\ldots,N$, denote the discrete times. Looking at the mild solution (2.3) of our problem (2.2) on the interval $[t_n,t_{n+1}]$, and discretising the integrals (by freezing the integrands at the left-end point of the interval), one can iteratively define the following (explicit) stochastic trigonometric scheme. We note that, for the sake of simplicity, we will omit the explicit dependence on M in the vectors $\mathscr{W}^n$, U^n, V^n and ΔW^n defined below.

$$\begin{aligned}\mathscr{W}^0 &:= w^M(0),\\ \mathscr{W}^{n+1} &:= \mathrm{e}^{\Delta tA}\mathscr{W}^n + \Delta t\,\mathrm{e}^{\Delta tA}F(\mathscr{W}^n) + \mathrm{e}^{\Delta tA}\Sigma(\mathscr{W}^n)\begin{pmatrix}0\\ \Delta W^n\end{pmatrix}, \qquad n \geq 0.\end{aligned} \tag{3.1}$$

Here, $\mathscr{W}^n$ is a vector in $\mathbb{R}^{2(M-1)}$ which can be written as $\mathscr{W}^n =: (U^n,V^n)^T$, where each component defines a $(M-1)$-dimensional vector. The terms $\Delta W^n := W^M(t_{n+1}) - W^M(t_n)$ denote the $M-1$-dimensional Wiener increments. Computing explicitly the C_0-semigroup $\mathrm{e}^{\Delta tA}$, one obtains that the above scheme can be equivalently written as

$$\begin{aligned}\begin{pmatrix}U^{n+1}\\ V^{n+1}\end{pmatrix} &= \begin{pmatrix}\cos(\Delta t\Theta_M) & \Theta_M^{-1}\sin(\Delta t\Theta_M)\\ -\Theta_M\sin(\Delta t\Theta_M) & \cos(\Delta t\Theta_M)\end{pmatrix}\begin{pmatrix}U^n\\ V^n\end{pmatrix}\\ &+ \begin{pmatrix}\Delta t^2\,\mathrm{sinc}(\Delta t\Theta_M)f(U^n)\\ \Delta t\cos(\Delta t\Theta_M)f(U^n)\end{pmatrix}\\ &+ \begin{pmatrix}\Theta_M^{-1}\sin(\Delta t\Theta_M)\sqrt{M}B_\sigma(U^n)\Delta W^n\\ \cos(\Delta t\Theta_M)\sqrt{M}B_\sigma(U^n)\Delta W^n\end{pmatrix},\end{aligned} \tag{3.2}$$

where $\Theta_M = \sqrt{-M^2D}$. The components of the vector U^n (resp. V^n) will be denoted by U_m^n (resp. V_m^n). We also note that the $(M-1)\times(M-1)$ matrix $B_\sigma(U^n)$ is defined analogously as the corresponding one in Section 2, namely it is diagonal with entries $\sigma(U_m^n)$, $m = 1,\ldots,M-1$. We will sometimes use the notation $t\,\mathrm{sinc}(t\Theta_M)$ for $\Theta_M^{-1}\sin(t\Theta_M)$, which is defined for arbitrary matrices Θ_M. We thus obtain a numerical approximation $U^n \approx u^M(t_n)$ (resp. $V^n \approx v^M(t_n)$), of the exact solution (resp. derivative of the solution), of our finite difference problem (2.1) at the discrete times $t_n = n\Delta t$.

The time integrator (3.2) can be seen as simple representative of stochastic trigonometric methods with simple choices of filter functions (see e.g. [2,5,4]). Observe that, the purpose of these filter functions is to attenuate numerical resonances (see e.g. [13, Chapter XIII] for the deterministic setting and [2] for the stochastic one). Furthermore, we remark that the choice of the filter functions may also have a substantial influence on the long-time properties of the method (see e.g. [13, Chapter XIII] for the deterministic case). We will not deal with these issues in the present paper.

Remark 3.1 We note that an effective numerical computation of the matrix functions present in the integrators (3.2) can be done using (rational) Krylov subspace approximations (see for example [10] and references therein).

The above formulation (3.2) of the numerical method will be used for practical computations in Section 5. For the theoretical parts presented below, we will make use of the discrete Green function G^M introduced in the previous section in order to write the numerical method (3.2) in mild form. Namely, performing explicit computations of the matrices $\cos(\Delta t\Theta_M)$ and $\sin(\Delta t\Theta_M)$ in equation (3.2) above, one obtains that the mth component of the vector U^{n+1} is given by

$$
\begin{aligned}
&U_m^{n+1} \\
&= \frac{1}{M}\sum_{l=1}^{M-1}\sum_{j=1}^{M-1}\frac{\sin\left(j\pi\Delta t\sqrt{c_j^M}\right)}{j\pi\sqrt{c_j^M}}\varphi_j(x_m)\varphi_j(x_l)\,V_l^n \\
&\quad+\frac{1}{M}\sum_{l=1}^{M-1}\sum_{j=1}^{M-1}\cos\left(j\pi\Delta t\sqrt{c_j^M}\right)\varphi_j(x_m)\varphi_j(x_l)\,U_l^n \\
&\quad+\Delta t\,\frac{1}{M}\sum_{l=1}^{M-1}\sum_{j=1}^{M-1}\frac{\sin\left(j\pi\Delta t\sqrt{c_j^M}\right)}{j\pi\sqrt{c_j^M}}\varphi_j(x_m)\varphi_j(x_l)\,f(U_l^n) \\
&\quad+\frac{1}{\sqrt{M}}\sum_{l=1}^{M-1}\sum_{j=1}^{M-1}\frac{\sin\left(j\pi\Delta t\sqrt{c_j^M}\right)}{j\pi\sqrt{c_j^M}}\varphi_j(x_m)\varphi_j(x_l)\,\sigma(U_l^n)\big(W_l^M(t_{n+1})-W_l^M(t_n)\big),
\end{aligned}
$$

for $m\in\{1,\ldots,M-1\}$, where we recall that $\varphi_j(x)=\sqrt{2}\sin(j\pi x)$, $c_j^M=\frac{\sin^2\left(\frac{j\pi}{2M}\right)}{\left(\frac{j\pi}{2M}\right)^2}$,

$$
W_l^M(t_n)=\sqrt{M}\big(W(t_n,x_{l+1})-W(t_n,x_l)\big),
$$

and V_l^n is the lth component of the vector V^n defined in (3.2). Then, owing to the definition of the discretised Green function (2.4), we can infer that, for all $n=0,\ldots,N$ and $m=1,\ldots,M-1$,

$$
\begin{aligned}
U_m^{n+1} &= \int_0^1 G^M(t_{n+1}-t_n,x_m,y)V^n_{M\kappa_M(y)}\,\mathrm{d}y+\int_0^1\frac{\partial G^M}{\partial t}(t_{n+1}-t_n,x_m,y)U^n_{M\kappa_M(y)}\,\mathrm{d}y \\
&\quad+\int_{t_n}^{t_{n+1}}\int_0^1 G^M(t_{n+1}-t_n,x_m,y)f(U^n_{M\kappa_M(y)})\,\mathrm{d}y\,\mathrm{d}s \\
&\quad+\int_{t_n}^{t_{n+1}}\int_0^1 G^M(t_{n+1}-t_n,x_m,y)\sigma(U^n_{M\kappa_M(y)})\,W(\mathrm{d}s,\mathrm{d}y). \qquad (3.3)
\end{aligned}
$$

In order to exhibit a more convenient mild form for U_m^{n+1}, we should iterate the above expression with respect to n. However, it is much easier to iterate the unified expression (3.1), and this procedure yields, for all $n \in \{0,\dots,N-1\}$,

$$\mathscr{W}^{n+1} = \mathrm{e}^{(n+1)\Delta t A}\mathscr{W}^0 + \Delta t \sum_{r=0}^{n} \mathrm{e}^{(n+1-r)\Delta t A} F\left(\mathscr{W}^r\right) + \sum_{r=0}^{n} \mathrm{e}^{(n+1-r)\Delta t A} \Sigma\left(\mathscr{W}^r\right) \begin{pmatrix} 0 \\ \Delta W^r \end{pmatrix}.$$

Writing the first component of $\mathscr{W}^{n+1}$, that is U^{n+1}, componentwise, we obtain that

$$\begin{aligned} U_m^{n+1} &= \int_0^1 G^M(t_{n+1},x_m,y) v_0(\kappa_M(y))\,\mathrm{d}y \\ &+ \int_0^1 \frac{\partial G^M}{\partial t}(t_{n+1},x_m,y) u_0(\kappa_M(y))\,\mathrm{d}y \\ &+ \sum_{r=0}^{n} \int_{t_r}^{t_{r+1}} \int_0^1 G^M(t_{n+1}-t_r,x_m,y) f(U^r_{M\kappa_M(y)})\,\mathrm{d}y\,\mathrm{d}s \\ &+ \sum_{r=0}^{n} \int_{t_r}^{t_{r+1}} \int_0^1 G^M(t_{n+1}-t_r,x_m,y) \sigma(U^r_{M\kappa_M(y)})\,W(\mathrm{d}s,\mathrm{d}y). \end{aligned} \tag{3.4}$$

At this point, we introduce a continuous version of our time discretisation scheme, as follows. For any $(t,x) \in [0,T]\times[0,1]$, we define

$$\begin{aligned} u^{M,N}(t,x) := &\int_0^1 G^M(t,x,y) v_0(\kappa_M(y))\,\mathrm{d}y \\ &+ \int_0^1 \frac{\partial G^M}{\partial t}(t,x,y) u_0(\kappa_M(y))\,\mathrm{d}y \\ &+ \int_0^t \int_0^1 G^M(t-\kappa_N^T(s),x,y) f\left(U^{\kappa_N^T(s)/\Delta t}_{M\kappa_M(y)}\right)\mathrm{d}y\,\mathrm{d}s \\ &+ \int_0^t \int_0^1 G^M(t-\kappa_N^T(s),x,y) \sigma\left(U^{\kappa_N^T(s)/\Delta t}_{M\kappa_M(y)}\right) W(\mathrm{d}s,\mathrm{d}y), \end{aligned}$$

where we have used the notation $\kappa_N^T(s) := T\kappa_N(s/T)$. First, let us observe that, for all $n = 0,\dots,N$ and $m = 0,\dots,M-1$, we have $u^{M,N}(t_n,x_m) = U_m^n$. Indeed, this can be deduced from (3.4) because, for instance, we clearly have that

$$\begin{aligned} &\int_0^{t_n} \int_0^1 G^M(t_n-\kappa_N^T(s),x_m,y) f\left(U^{\kappa_N^T(s)/\Delta t}_{M\kappa_M(y)}\right)\mathrm{d}y\,\mathrm{d}s \\ &\quad = \sum_{r=0}^{n-1} \int_{t_r}^{t_{r+1}} \int_0^1 G^M(t_n-t_r,x_m,y) f(U^r_{M\kappa_M(y)})\,\mathrm{d}y\,\mathrm{d}s. \end{aligned}$$

In particular, the process $\{u^{M,N}(t,x),\,(t,x)\in[0,T]\times[0,1]\}$ satisfies the following integral equation:

$$\begin{aligned}
u^{M,N}(t,x) := &\int_0^1 G^M(t,x,y) v_0(\kappa_M(y))\,\mathrm{d}y \\
&+\int_0^1 \frac{\partial G^M}{\partial t}(t,x,y) u_0(\kappa_M(y))\,\mathrm{d}y \\
&+\int_0^t\int_0^1 G^M(t-\kappa_N^T(s),x,y) f\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big)\,\mathrm{d}y\,\mathrm{d}s \\
&+\int_0^t\int_0^1 G^M(t-\kappa_N^T(s),x,y) \sigma\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big)\,W(\mathrm{d}s,\mathrm{d}y).
\end{aligned} \tag{3.5}$$

As we will deduce from the forthcoming Proposition 4.2, the random field $\{u^{M,N}(t,x),\,(t,x)\in[0,T]\times[0,1]\}$ admits a modification with Hölder-continuous paths. The main task to be done in the remaining of the paper will be to compare the random fields $u^{M,N}(t,x)$ and $u^M(t,x)$ in $L^{2p}(\Omega)$. This will be the main part of Theorem 4.1 below.

4 Strong convergence of the stochastic trigonometric methods

This section is devoted to state and prove the main result of the paper. Namely, we will derive $L^{2p}(\Omega)$-error estimates for the stochastic trigonometric method (3.3) applied to the solution (1.4) of our problem (1.1). After stating the main results to be addressed (Theorems 4.1 and 4.2 below), in Subsection 4.1 we will consider two preliminary results, while the proof of Theorem 4.2 will be developed in Subsection 4.2.

Recall that $\{u(t,x),\,(t,x)\in[0,T]\times[0,1]\}$ denotes the solution to our stochastic wave equation (1.1), $\{u^M(t,x),\,(t,x)\in[0,T]\times[0,1]\}$ is the numerical approximation of (1.1) by the finite difference scheme with mesh size $\Delta x=1/M$, and $\{u^{M,N}(t,x),\,(t,x)\in[0,T]\times[0,1]\}$ refers to the numerical solution given by the explicit stochastic trigonometric method (3.2) with a time step size $\Delta t=T/N$ on the interval $[0,T]$. The main result reads as follows.

Theorem 4.1 *Suppose that $u_0\in H^\alpha([0,1])$ with $\alpha>3/2$ and $v_0\in H^\beta([0,1])$ with $\beta>1/2$. Assume that the functions f and σ satisfy the Lipschitz condition* (1.2) *and the linear growth condition* (1.3).

Let $p\geq 1$. Then, the following estimate of the error for the full discretisation holds:

$$\sup_{(t,x)\in[0,T]\times[0,1]}\Big(\mathbb{E}\big[\big|u^{M,N}(t,x)-u(t,x)\big|^{2p}\big]\Big)^{\frac{1}{2p}}\leq C_1\,(\Delta x)^{\rho-\varepsilon}+C_2\,(\Delta t)^{\tau},$$

with $\rho=\frac13\wedge(\alpha-\frac32)\wedge(\beta-\frac12)$ and $\tau=\frac12\wedge(\alpha-\frac12)\wedge(\beta+\frac12)$, for all small enough $\varepsilon>0$. The constants C_1 and C_2 are positive and do not depend neither on M nor on N.

Moreover, $u^{M,N}(t,x)$ converges to $u(t,x)$ $\mathbb{P}$-a.s., as M and N tend to infinity, uniformly with respect to $(t,x)\in[0,T]\times[0,1]$.

As an immediate consequence of the above result, we observe that, in the case where the initial data u_0 and v_0 vanish, one obtains that

$$\sup_{(t,x)\in[0,T]\times[0,1]}\left(\mathbb{E}\left[\left|u^{M,N}(t,x)-u(t,x)\right|^{2p}\right]\right)^{\frac{1}{2p}}\leq C_1\,(\Delta x)^{\frac{1}{3}-\varepsilon}+C_2\,(\Delta t)^{\frac{1}{2}},$$

for all small enough $\varepsilon>0$.

Remark 4.1 As already pointed out in the Introduction, the space and time steps Δx and Δt do not need to satisfy any kind of CFL condition, which turns out to be optimal as far as the numerical implementation of the method is concerned. In Section 5, we will perform several numerical experiments illustrating the above theoretical result.

The proof of Theorem 4.1 will immediately follow from the spatial convergence result of Theorem 2.1 and the following one.

Theorem 4.2 *Under the assumptions of Theorem 4.1, we have the following error estimate for the stochastic trigonometric method* (3.3) *applied to* (2.1)*:*

$$\sup_{(t,x)\in[0,T]\times[0,1]}\left(\mathbb{E}\left[\left|u^{M,N}(t,x)-u^{M}(t,x)\right|^{2p}\right]\right)^{\frac{1}{2p}}\leq C\,(\Delta t)^{\tau},$$

with $\tau=\frac{1}{2}\wedge(\alpha-\frac{1}{2})\wedge(\beta+\frac{1}{2})$, *where the constant* C *is non-negative and do not depend neither on* M *nor on* N. *In particular, in the case where the initial data vanish, the above error estimate will be simply of order* $(\Delta t)^{\frac{1}{2}}$.

Moreover, $u^{M,N}(t,x)$ *converges to* $u^{M}(t,x)$ $\mathbb{P}$*-a.s., as* N *tends to infinity, uniformly with respect to* $(t,x)\in[0,T]\times[0,1]$ *and* $M\in\mathbb{N}$.

4.1 Preliminary results

In order to proceed with the proof of Theorem 4.2, we will need two auxiliary results which will be addressed in this Subsection.

Proposition 4.1 *Under the assumptions of Theorem 4.1, we have, for all* $p\geq 1$,

$$\sup_{M,N\geq 1}\;\sup_{(t,x)\in[0,T]\times[0,1]}\mathbb{E}\left[\left|u^{M,N}(t,x)\right|^{2p}\right]<+\infty.$$

Proof By the proof of [24, Prop. 3], the first two terms in (3.5) can be estimated by

$$\sup_{M,N\geq 1}\;\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^1 G^M(t,x,y)v_0(\kappa_M(y))\,\mathrm{d}y\right|^{2p}\leq C_1$$

$$\sup_{M,N\geq 1}\;\sup_{(t,x)\in[0,T]\times[0,1]}\left|\int_0^1 \frac{\partial G^M}{\partial t}(t,x,y)u_0(\kappa_M(y))\,\mathrm{d}y\right|^{2p}\leq C_2,$$

where the constants C_1 and C_2, as well as the forthcoming $C_3,\dots,C_6$, are generic constants which do not depend on M nor N nor on the mesh parameters Δt and Δx.

For the term containing the stochastic integral, an application of Burkholder-Davis-Gundy's inequality, Hölder's inequality with respect to the finite measure $|G^M(t-\kappa_N^T(s),x,y)|^2\,\mathrm{d}s\,\mathrm{d}y$, property (2.5), and assumption (1.3) for the function σ, yield

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t\int_0^1 G^M(t-\kappa_N^T(s),x,y)\sigma\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big)\,W(\mathrm{d}s,\mathrm{d}y)\right|^{2p}\right]\\
&\leq C\,\mathbb{E}\left[\left(\int_0^t\int_0^1 |G^M(t-\kappa_N^T(s),x,y)|^2|\sigma\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big)|^2\,\mathrm{d}y\,\mathrm{d}s\right)^p\right]\\
&\leq C\int_0^t\int_0^1 |G^M(t-\kappa_N^T(s),x,y)|^2\mathbb{E}\left[|\sigma\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big)|^{2p}\right]\mathrm{d}y\,\mathrm{d}s\\
&\leq C_3+C_4\int_0^t \sup_{(r,x)\in[0,s]\times[0,1]}\mathbb{E}\left[|u^{M,N}(r,x)|^{2p}\right]\mathrm{d}s.
\end{aligned}$$

In order to estimate the remaining term in (3.5), we use Hölder's inequality, Hölder's inequality with respect to the finite measure $|G^M(t-\kappa_N^T(s),x,y)|^2\,\mathrm{d}s\,\mathrm{d}y$, property (2.5), and assumption (1.3) for the function f to get

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t\int_0^1 G^M(t-\kappa_N^T(s),x,y)f\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\,\mathrm{d}y\,\mathrm{d}s\right|^{2p}\right]\\
&\leq C\,\mathbb{E}\left[\left(\int_0^t\int_0^1 |G^M(t-\kappa_N^T(s),x,y)|^2|f\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big)|^2\,\mathrm{d}y\,\mathrm{d}s\right)^p\right]\\
&\leq C\left(\int_0^t\int_0^1 |G^M(t-\kappa_N^T(s),x,y)|^2\,\mathrm{d}y\,\mathrm{d}s\right)^{p-1}\\
&\quad\times\mathbb{E}\left[\int_0^t\int_0^1 |G^M(t-\kappa_N^T(s),x,y)|^2|f\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big)|^{2p}\,\mathrm{d}y\,\mathrm{d}s\right]\\
&\leq C_5+C_6\int_0^t \sup_{(r,x)\in[0,s]\times[0,1]}\mathbb{E}\left[|u^{M,N}(r,x)|^{2p}\right]\mathrm{d}s.
\end{aligned}$$

Collecting all the above estimates, we arrive at

$$\sup_{x\in[0,1]}\mathbb{E}\left[\left|u^{M,N}(t,x)\right|^{2p}\right]\leq \tilde{C}_1+\tilde{C}_2\int_0^t \sup_{(r,x)\in[0,s]\times[0,1]}\mathbb{E}\left[|u^{M,N}(r,x)|^{2p}\right]\mathrm{d}s$$

and an application of Gronwall's lemma concludes the proof. □

Proposition 4.2 *Set*

$$w^{M,N}(t,x):=u^{M,N}(t,x)-\int_0^1 G^M(t,x,y)v_0(\kappa_M(y))\,\mathrm{d}y-\int_0^1\frac{\partial G^M}{\partial t}(t,x,y)u_0(\kappa_M(y))\,\mathrm{d}y.$$

Then, there is a positive constant C which does not depend neither on M nor on N such that, for all $s,t\in[0,T]$ and $x,y\in[0,1]$, it holds

$$\mathbb{E}\left[\left|w^{M,N}(t,x)-w^{M,N}(s,y)\right|^{2p}\right]\leq C\big\{|t-s|^p+|x-y|^p\big\}.$$

This implies that the random field $w^{M,N}$ has a version with jointly δ-Hölder continuous paths, for any $\delta\in(0,\frac{1}{2})$.

Proof Making use of the above Proposition 4.1, one can follow exactly the same lines as part of the proof of [24, Prop. 2], where the same type of estimate has been obtained for the moments of $u(t,x)-u(s,y)$. Indeed, the fact that the discretising functions κ_M and κ_N are involved in the expression of $w^{M,N}$ does not alter the main steps to follow. We leave the details to the reader. □

4.2 Proof of Theorem 4.2

To start with, observe that the difference between the continuous version of the numerical solution given by the stochastic trigonometric method (3.5) and the solution of the finite difference discretisation $u^M(t,x)$ of the stochastic wave equation reads

$$\begin{aligned}u^{M,N}(t,x)-u^M(t,x) = &\int_0^t\int_0^1 \Big\{G^M(t-\kappa_N^T(s),x,y)f\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big)\\ &\quad - G^M(t-s,x,y)f\big(u^M(s,\kappa_M(y))\big)\Big\}\,\mathrm{d}y\,\mathrm{d}s\\ &+\int_0^t\int_0^1 \Big\{G^M(t-\kappa_N^T(s),x,y)\sigma\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big)\\ &\quad - G^M(t-s,x,y)\sigma\big(u^M(s,\kappa_M(y))\big)\Big\}\,W(\mathrm{d}s,\mathrm{d}y).\end{aligned}$$

These differences can be decomposed as the sum of the following six terms:

$$\begin{aligned}D_1 &:= \int_0^t\int_0^1 G^M(t-\kappa_N^T(s),x,y)\\ &\qquad \times\big\{f\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big)-f\big(u^M(\kappa_N^T(s),\kappa_M(y))\big)\big\}\,\mathrm{d}y\,\mathrm{d}s,\\ D_2 &:= \int_0^t\int_0^1 \big\{G^M(t-\kappa_N^T(s),x,y)-G^M(t-s,x,y)\big\}\\ &\qquad \times f\big(u^M(\kappa_N^T(s),\kappa_M(y))\big)\,\mathrm{d}y\,\mathrm{d}s,\\ D_3 &:= \int_0^t\int_0^1 G^M(t-s,x,y)\big\{f\big(u^M(\kappa_N^T(s),\kappa_M(y))\big)-f\big(u^M(s,\kappa_M(y))\big)\big\}\,\mathrm{d}y\,\mathrm{d}s,\\ D_4 &:= \int_0^t\int_0^1 G^M(t-\kappa_N^T(s),x,y)\\ &\qquad \times\big\{\sigma\big(u^{M,N}(\kappa_N^T(s),\kappa_M(y))\big)-\sigma\big(u^M(\kappa_N^T(s),\kappa_M(y))\big)\big\}\,W(\mathrm{d}s,\mathrm{d}y),\\ D_5 &:= \int_0^t\int_0^1 \big\{G^M(t-\kappa_N^T(s),x,y)-G^M(t-s,x,y)\big\}\\ &\qquad \times\sigma\big(u^M(\kappa_N^T(s),\kappa_M(y))\big)\,W(\mathrm{d}s,\mathrm{d}y),\\ D_6 &:= \int_0^t\int_0^1 G^M(t-s,x,y)\\ &\qquad \times\big\{\sigma\big(u^M(\kappa_N^T(s),\kappa_M(y))\big)-\sigma\big(u^M(s,\kappa_M(y))\big)\big\}\,W(\mathrm{d}s,\mathrm{d}y).\end{aligned}$$

Let us proceed with the estimation of the above terms. To start with, using Hölder's inequality and the Lipschitz condition of the function f (1.2), we arrive at

$$\mathbb{E}\big[\big|D_1\big|^{2p}\big] \leq C\mathbb{E}\Big[\Big|\int_0^t \int_0^1 G^M(t-\kappa_N^T(s),x,y)^2 \times \big|u^{M,N}(\kappa_N^T(s),\kappa_M(y)) - u^M(\kappa_N^T(s),\kappa_M(y))\big|^2 \,\mathrm{d}y\,\mathrm{d}s\Big|^p\Big],$$

where, here and in the following, we recall that the constant C is a generic constant which does not depend on M nor on the mesh parameters Δt and Δx. We next apply Hölder's inequality with respect to the measure

$$G^M(t-\kappa_N^T(s),x,y)^2\,\mathrm{d}y\,\mathrm{d}s \quad \text{on} \quad [0,t]\times[0,1].$$

Hence,

$$\mathbb{E}\big[\big|D_1\big|^{2p}\big] \leq C\int_0^t \int_0^1 G^M(t-\kappa_N^T(s),x,y)^2\,\mathrm{d}y \sup_{x\in[0,1]} \mathbb{E}\big[|u^{M,N}(\kappa_N^T(s),x) - u^M(\kappa_N^T(s),x)|^{2p}\big]\,\mathrm{d}s.$$

Using (2.5), we finally obtain

$$\mathbb{E}\big[\big|D_1\big|^{2p}\big] \leq C\int_0^t \sup_{x\in[0,1]} \mathbb{E}\big[|u^{M,N}(\kappa_N^T(s),x) - u^M(\kappa_N^T(s),x)|^{2p}\big]\,\mathrm{d}s. \tag{4.1}$$

In a similar fashion, but using Hölder's inequality with respect to the measure

$$|G^M(t-\kappa_N^T(s),x,y) - G^M(t-s,x,y)|^2\,\mathrm{d}y\,\mathrm{d}s \quad \text{on} \quad [0,t]\times[0,1],$$

one obtains

$$\mathbb{E}\big[\big|D_2\big|^{2p}\big] \leq C\Big(\int_0^t \int_0^1 |G^M(t-\kappa_N^T(s),x,y) - G^M(t-s,x,y)|^2\,\mathrm{d}y\,\mathrm{d}s\Big)^{p-1} \times \mathbb{E}\Big[\int_0^t \int_0^1 |G^M(t-\kappa_N^T(s),x,y) - G^M(t-s,x,y)|^2 \times |f(u^M(\kappa_N^T(s),\kappa_M(y)))|^{2p}\,\mathrm{d}y\,\mathrm{d}s\Big].$$

Using the properties (2.6) and (1.3) and invoking Lemma 2.1, we get that

$$\mathbb{E}\big[\big|D_2\big|^{2p}\big] \leq C(\Delta t)^p. \tag{4.2}$$

For the last term D_3, using similar techniques as above, we arrive at

$$\mathbb{E}\big[\big|D_3\big|^{2p}\big] \leq C\int_0^t \sup_{x\in[0,1]} \mathbb{E}\big[|u^M(\kappa_N^T(s),x) - u^M(s,x)|^{2p}\big]\,\mathrm{d}s.$$

The regularity properties of the process $u^M(t,x)$ given in [24, Lem. 2] permits to show that

$$\mathbb{E}\big[\big|D_3\big|^{2p}\big] \leq C(\Delta t)^{2p\tau}, \tag{4.3}$$

where we recall that $\tau = \frac{1}{2} \wedge (\alpha - \frac{1}{2}) \wedge (\beta + \frac{1}{2})$, and α and β come from the regularity assumptions on the initial data.

Next, owing at Burkholder-Davies-Gundy's inequality, the Lipschitz condition on the function σ (1.2), Hölder's inequality with respect to the measure $G^M(t - \kappa_N^T(s), x, y)^2 \, \mathrm{d}y \, \mathrm{d}s$, and using property (2.5), we arrive at

$$\begin{aligned}
\mathbb{E}\big[\big|D_4\big|^{2p}\big] &\le C\,\mathbb{E}\Big[\Big(\int_0^t \int_0^1 G^M(t-\kappa_N^T(s),x,y)^2 \\
&\qquad \times |\sigma(u^{M,N}(\kappa_N^T(s),\kappa_M(y)) - \sigma(u^M(\kappa_N^T(s),\kappa_M(y)))|^2 \,\mathrm{d}y\,\mathrm{d}s\Big)^p\Big] \\
&\le C \int_0^t \int_0^1 G^M(t-\kappa_N^T(s),x,y)^2 \\
&\qquad \times \sup_{x\in[0,1]} \mathbb{E}\big[|u^{M,N}(\kappa_N^T(s),x) - u^M(\kappa_N^T(s),x)|^{2p}\big]\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}$$

Taking into account again property (2.5), it follows

$$\mathbb{E}\big[\big|D_4\big|^{2p}\big] \le C \int_0^t \sup_{(r,x)\in[0,s]\times[0,1]} \mathbb{E}\big[|u^{M,N}(r,x) - u^M(r,x)|^{2p}\big]\,\mathrm{d}s. \tag{4.4}$$

For the term D_5, applying Burkholder-Davis-Gundy's and Hölder's inequalities (the latter with respect to $\big|G^M(t-\kappa_N^T(s),x,y) - G^M(t-s,x,y)\big|^2 \,\mathrm{d}y\,\mathrm{d}s$), and similar arguments as before, we can infer that

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^t \int_0^1 \big\{G^M(t-\kappa_N^T(s),x,y) - G^M(t-s,x,y)\big\} \\
&\qquad \times \sigma(u^M(\kappa_N^T(s),\kappa_M(y)))\,W(\mathrm{d}s,\mathrm{d}y)\Big|^{2p}\Big] \\
&\le C \left(\int_0^t\int_0^1 \big|G^M(t-\kappa_N^T(s),x,y) - G^M(t-s,x,y)\big|^2 \,\mathrm{d}y\,\mathrm{d}s\right)^{p-1} \\
&\qquad \times \int_0^t \int_0^1 \big|G^M(t-\kappa_N^T(s),x,y) - G^M(t-s,x,y)\big|^2 \,\mathrm{d}y \\
&\qquad \times \sup_{(r,x)\in[0,T]\times[0,1]} \mathbb{E}[|1+|u^M(r,x)||^{2p}]\,\mathrm{d}s.
\end{aligned}$$

Taking into account estimate (2.6) and the result of Lemma 2.1, we obtain

$$\mathbb{E}\big[\big|D_5\big|^{2p}\big] \le C\,(\Delta t)^p. \tag{4.5}$$

Let us now deal with the term D_6. By Burkholder-Davis-Gundy's and Hölder's inequalities, followed by the Lipschitz condition on σ and result (2.5), one obtains that

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^t\int_0^1 G^M(t-s,x,y)\big\{\sigma(u^M(\kappa_N^T(s),\kappa_M(y))) - \sigma(u^M(s,\kappa_M(y)))\big\}\,W(\mathrm{d}s,\mathrm{d}y)\Big|^{2p}\Big] \\
&\le C \int_0^t \int_0^1 G^M(t-s,x,y)^2 \,\mathrm{d}y \sup_{x\in[0,1]} \mathbb{E}[|u^M(\kappa_N^T(s),x) - u^M(s,x)|^{2p}]\,\mathrm{d}s.
\end{aligned}$$

The regularity properties of the process $u^M(t,x)$ given in [24, Lem. 2] (see also [24, Prop. 2] for the initial values) and estimate (2.5) finally give us

$$\mathbb{E}\big[\big|D_6\big|^{2p}\big] \leq C\,(\Delta t)^{2p\tau}. \tag{4.6}$$

Putting together estimates (4.1)-(4.4), (4.5) and (4.6), we arrive at

$$\sup_{(t,x)\in[0,T]\times[0,1]} \mathbb{E}\big[\big|u^{M,N}(t,x)-u^M(t,x)\big|^{2p}\big] \leq C_1\,(\Delta t)^{2p\tau} + C_2\,(\Delta t)^p + C_3 \int_0^t \sup_{(r,x)\in[0,s]\times[0,1]} \mathbb{E}\big[\big|u^{M,N}(r,x)-u^M(r,x)\big|^{2p}\big]\,\mathrm{d}s,$$

for some positive constants C_1, C_2 and C_3 independent of N and M. An application of Gronwall's lemma let us conclude the proof of the first part of Theorem 4.2.

In order to prove the assertion about the almost sure convergence of the numerical solution, we first use [24, Thm. 1] which asserts that $u^M(t,x)$ converges to $u(t,x)$ $\mathbb{P}$-a.s. uniformly in (t,x). It thus suffices to show that $u^{M,N}(t,x)$ converges to $u^M(t,x)$ $\mathbb{P}$-a.s., as N tends to infinity, uniformly with respect to $(t,x)\in[0,T]\times[0,1]$ and $M\in\mathbb{N}$. Note that it suffices to prove such almost surely convergence for $w^{M,N}$ and w^M, where the former has been defined in Proposition 4.2 and the latter is given by

$$w^M(t,x) := u^M(t,x) - \int_0^1 G^M(t,x,y)v_0(\kappa_M(y))\,\mathrm{d}y - \int_0^1 \frac{\partial G^M}{\partial t}(t,x,y)u_0(\kappa_M(y))\,\mathrm{d}y.$$

Similarly as in the proof of [24, Thm. 1], we observe that

$$\big|w^{M,N}(t,x)-w^M(t,x)\big|^{2p} \leq C(A_1+A_2+A_3),$$

where C denotes a positive constant and

$$A_1 = \sum_{n=0}^N \sum_{i=0}^N \left| w^{M,N}\Big(t_n,\frac{i}{N}\Big) - w^M\Big(t_n,\frac{i}{N}\Big)\right|^{2p},$$

$$A_2 = \sup_{n,i=0,\ldots,N}\ \sup_{|x-\frac{i}{N}|\leq\frac{1}{N}}\ \sup_{|t-t_n|\leq\Delta t} \left| w^{M,N}(t,x) - w^{M,N}\Big(t_n,\frac{i}{N}\Big)\right|^{2p},$$

$$A_3 = \sup_{n,i=0,\ldots,N}\ \sup_{|x-\frac{i}{N}|\leq\frac{1}{N}}\ \sup_{|t-t_n|\leq\Delta t} \left| w^{M}(t,x) - w^{M}\Big(t_n,\frac{i}{N}\Big)\right|^{2p}.$$

By the first part of the proof, we can infer that

$$\mathbb{E}[A_1] \leq C\left(\frac{1}{N}\right)^{2p\tau-2}.$$

On the other hand, by Proposition 4.1, the paths of $w^{M,N}$ are δ-Hölder continuous jointly in time and space, for all $\delta\in(0,\frac{1}{2})$. Moreover, by [24, Lem. 2], the process w^M also has the same path regularity. Thus, we obtain that

$$\mathbb{E}[A_2+A_3] \leq C\left(\frac{1}{N}\right)^{2p\delta}.$$

For the sake of clarity in the notation, let us assume that the initial data are sufficiently regular so that τ becomes equal to $\frac{1}{2}$. In this case, we have proved that, for all $\delta \in (0,\frac{1}{2})$ and $p \geq 1$,

$$\mathbb{E}\left[\sup_{M\geq 1}\sup_{(t,x)\in[0,T]\times[0,1]}\left|w^{M,N}(t,x)-w^{M}(t,x)\right|^{2p}\right] \leq C\Big(\frac{1}{N}\Big)^{2p\delta},$$

where the constant C does not depend on M neither on N. At this point, Chebyshev's inequality yields

$$\mathbb{P}\left\{\sup_{M\geq 1}\sup_{(t,x)\in[0,T]\times[0,1]}\left|w^{M,N}(t,x)-w^{M}(t,x)\right|^{2p} > \Big(\frac{1}{N}\Big)^{2}\right\} \leq C\Big(\frac{1}{N}\Big)^{2p\delta-2}.$$

Hence, the Borel-Cantelli lemma implies that, for sufficiently large p,

$$\sup_{M\geq 1}\sup_{(t,x)\in[0,T]\times[0,1]}\left|w^{M,N}(t,x)-w^{M}(t,x)\right|^{2p} \leq \frac{1}{N^2} \quad \mathbb{P}\text{-a.s.}$$

which concludes the proof of the theorem. □

5 Numerical experiments

Let us first consider the one-dimensional hyperbolic Anderson model [6,7]

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial^2 x}(t,x) + u(t,x)\frac{\partial^2 W}{\partial x \partial t}(t,x), && (t,x)\in\ (0,1)\times(0,1),\\
&u(t,0) = u(t,1) = 0, && t\in(0,1),\\
&u(0,x) = \sin(2\pi x),\ \frac{\partial u}{\partial t}(0,x) = \sin(3\pi x), && x\in(0,1).
\end{aligned}$$

This linear stochastic partial differential equation with multiplicative noise is now discretised in space by a finite difference method with mesh Δx (Section 2). This leads to a system of stiff stochastic differential equations of the form (2.1). This last problem is then discretised in time by a stochastic trigonometric method using a step size Δt (Section 3).

Figure 5.1 confirms the results on the spatial discretisation of our numerical discretisation as stated in Theorem 2.1. The spatial mean-square errors at time $T_{\text{end}} = 1$

$$\sup_{x\in[0,1]}\sqrt{\mathbb{E}\left[\,|u^{M,N}(T_{\text{end}},x)-u(T_{\text{end}},x)|^2\,\right]}$$

are displayed for various values of the parameter $\Delta x = 1/M$. The expected convergence rate $\mathscr{O}(\Delta x^{1/3})$ is observed. Here, since no exact solution is available, we therefore simulate the exact solution $u(t,x)$ with the numerical one using very small step sizes, i. e., $\Delta t_{\text{exact}} = 2^{-9}$ and $\Delta x_{\text{exact}} = 2^{-9}$. The expected values are approximated by computing averages over $M_s = 1000$ samples. We have checked that, in all numerical experiments that we present, the Monte-Carlo errors are small enough.

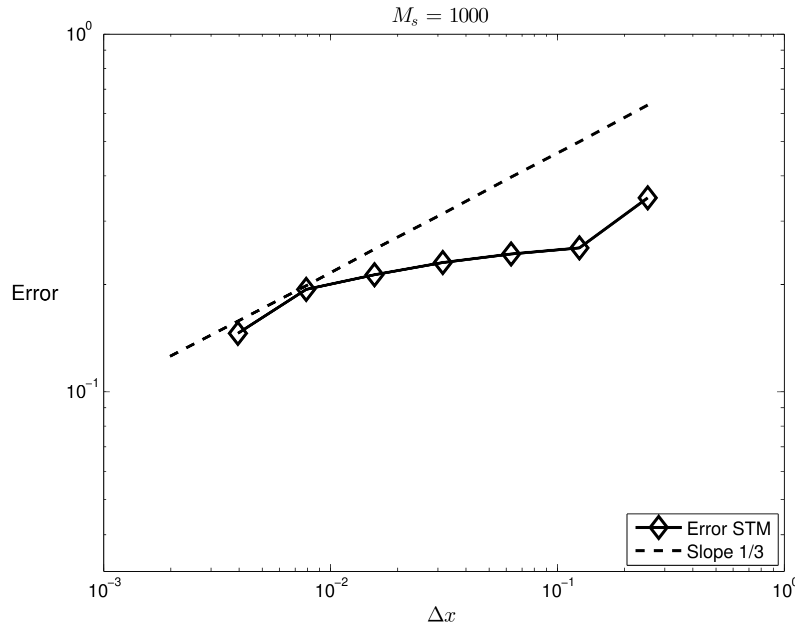

Fig. 5.1 Anderson model: Spatial rate of convergence of order $\Delta x^{1/3}$. The reference line has slope $1/3$ (dashed line).

We are now interested in the time-discretisation of the above stochastic partial differential equation. In Figure 5.2 one can observe the rate of convergence $\mathscr{O}(\Delta t^{1/2})$ of the mean-square errors in time, as stated by Theorem 4.2. Again, the exact solution is approximated by the stochastic trigonometric method with a very small step size $\Delta t_{\text{exact}} = 2^{-9}$ and uses $\Delta x_{\text{exact}} = 2^{-9}$ for the spatial discretisation. $M_s = 1000$ samples are used for the approximation of the expected values. For sake of comparison, we also display the errors of two different time integrators applied to (2.2) (see for example [14] or [26]). These numerical schemes are: the semi-implicit Euler-Maruyama scheme

$$\mathscr{W}^{n+1} = \mathscr{W}^n + \Delta t A \mathscr{W}^{n+1} + \Delta t F(\mathscr{W}^n) + \Sigma(\mathscr{W}^n) \begin{bmatrix} 0 \\ \Delta W^n \end{bmatrix}$$

and the semi-implicit Crank-Nicolson-Maruyama scheme

$$\mathscr{W}^{n+1} = \mathscr{W}^n + \frac{\Delta t}{2} A(\mathscr{W}^{n+1} + \mathscr{W}^n) + \Delta t F(\mathscr{W}^n) + \Sigma(\mathscr{W}^n) \begin{bmatrix} 0 \\ \Delta W^n \end{bmatrix}.$$

Note that no convergence results for nonlinear hyperbolic problems are known for these numerical integrators.

We next consider a version of the stochastic sine-Gordon equation with multiplicative noise

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial^2 x}(t,x) - \sin(u(t,x)) - \sin(u(t,x))\frac{\partial^2 W}{\partial x \partial t}(t,x), && (t,x) \in (0,1)\times(0,1),\\
&u(t,0) = u(t,1) = 0, && t \in (0,1),\\
&u(0,x) = \sin(2\pi x),\ \frac{\partial u}{\partial t}(0,x) = \sin(3\pi x), && x \in (0,1).
\end{aligned}$$

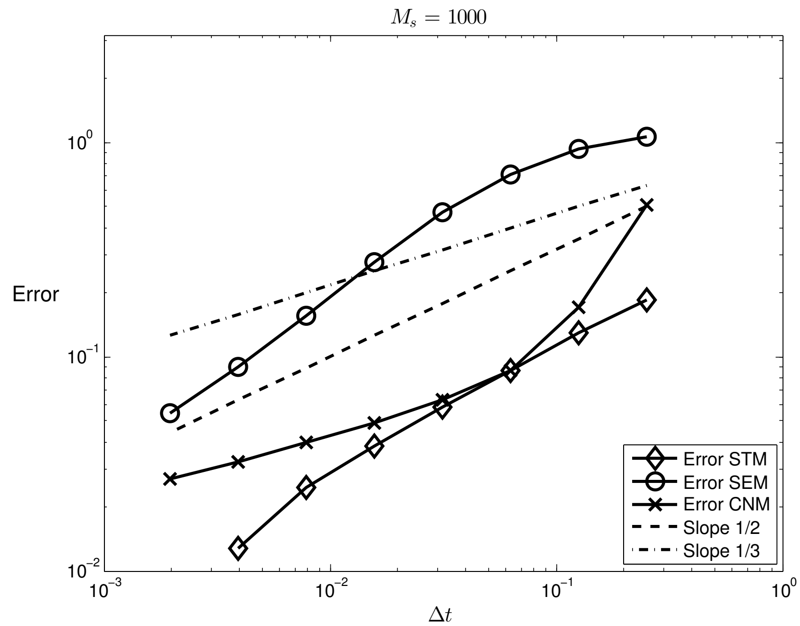

Fig. 5.2 Anderson model: Temporal rates of convergence for the stochastic trigonometric method (STM), the Euler-Maruyama scheme (SEM) and the Crank-Nicolson-Maruyama scheme (CNM). The reference lines have slopes $1/2$ and $1/3$ (dashed and dashdotted lines).

As in the first example, we discretise this nonlinear stochastic partial differential equation by a finite difference method with mesh Δx (in space) and the stochastic trigonometric method using a step size Δt (in time).

Figure 5.3 displays the spatial mean-square errors at time $T_{\text{end}} = 1$ and a convergence rate $\mathscr{O}(\Delta x^{1/3})$ is observed. Again, we simulate the exact solution with the numerical one using very small step sizes, i. e., $\Delta t_{\text{exact}} = 2^{-9}$ and $\Delta x_{\text{exact}} = 2^{-9}$. The expected values are approximated by computing averages over $M_s = 1000$ samples.

In Figure 5.4 one can observe the rate of convergence in time $\mathscr{O}(\Delta t^{1/2})$ for the stochastic trigonometric method as stated by Theorem 4.2. One can also observe a faster convergence for this scheme than for the two other semi-implicit numerical methods. Here again, the exact solution is approximated by the stochastic trigonometric method with very small step sizes $\Delta t_{\text{exact}} = 2^{-9}$ and uses $\Delta x_{\text{exact}} = 2^{-9}$ for the spatial discretisation. $M_s = 1000$ samples are used for the approximation of the expected values.

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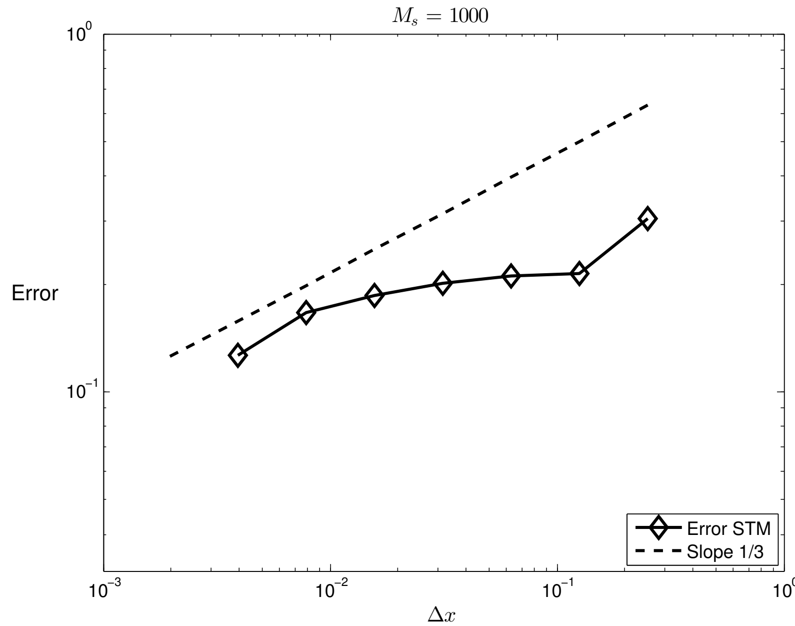

Fig. 5.3 Sine-Gordon equation: Spatial rate of convergence of order $\Delta x^{1/3}$. The reference line has slope $1/3$ (dotted line).

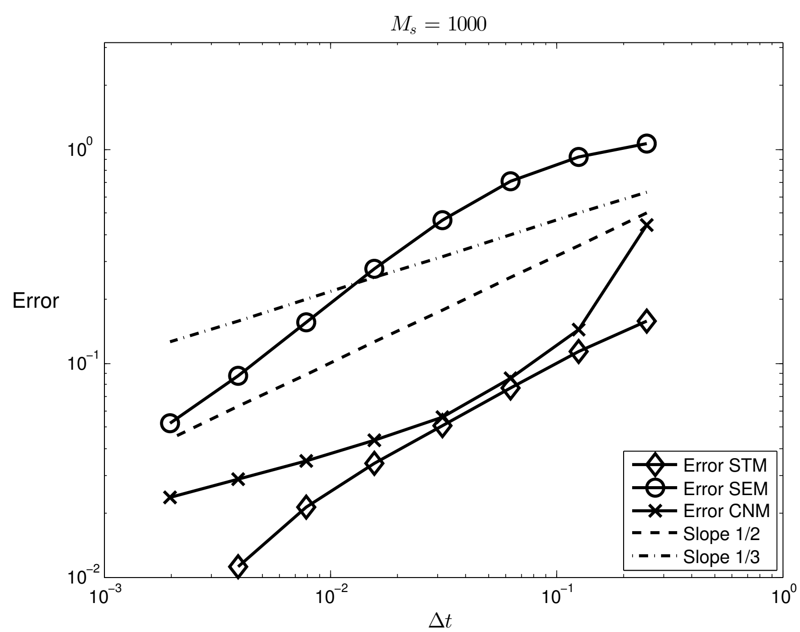

Fig. 5.4 Sine-Gordon equation: Temporal rates of convergence for the stochastic trigonometric method (STM), the Euler-Maruyama scheme (SEM) and the Crank-Nicolson-Maruyama scheme (CNM). The reference lines have slopes $1/2$ and $1/3$ (dashed and dashdotted lines).

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