ON SOME NEW EXAMPLES OF p-LOCAL COMPACT GROUPS

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Abstract. In this paper we present new examples of p-local compact groups for all $p \geq 3$. These examples do not correspond neither to p-compact groups nor compact Lie groups, and split in two families for $p \geq 5$. One of the families consists of simple p-local compact groups and there is a relation of inclusion with the other one. More precisely, for each $p \geq 5$, the simple example is a normal fusion subsystem of index 2 in the other one. As we have not found this type of inclusion in the literature for p-local compact groups, we provide an appendix with the classification of saturated fusion subsystems of finite index prime to p.

1. Introduction

Given a finite group G and a prime number p dividing the order of G, we can consider S, a Sylow p-subgroup of G, and construct a category $\mathcal{F}$ which models the action of G on the subgroups of S. Considering this, L. Puig replaced the existence of a group G by axioms in [23]. These axioms were used by C. Broto, R. Levi and B. Oliver to define the concept of p-local finite group in [6]: a p-local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$, where S is a finite p group, $\mathcal{F}$ a category with objects all the subgroups of S and morphisms fulfilling some technical conditions, and $\mathcal{L}$ is a category with objects some subgroups of S and satisfying also some technical conditions. If we have a finite group G and S a Sylow p-subgroup of G, we can construct a p-local finite group $(S, \mathcal{F}_S(G), \mathcal{L}_S(G))$, and it satisfies that $|\mathcal{L}_S(G)|^\wedge_p \simeq BG^\wedge_p$. It was shown in [6] that not all the p-local finite groups were of these type, defining the concept of exotic p-local finite group.

The definition of p-local finite group was generalized by C. Broto, R. Levi and B. Oliver in [7] considering S in a larger class of p groups. In the new definition, a p-local compact group is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a discrete p-toral group, and $\mathcal{F}$ and $\mathcal{L}$ are two categories with the same axioms as in the finite case with an extra condition. In [7], the authors also prove that these are generalizations of compact Lie groups and p-compact groups, in the sense that given a compact Lie group G (respectively a p-compact group X), we can find a p-local compact group $(S, \mathcal{F}, \mathcal{L})$ such that $|\mathcal{L}|^\wedge_p \simeq BG^\wedge_p$ (respectively $|\mathcal{L}|^\wedge_p \simeq BX$).

Here we present some examples of p-local compact groups which do not correspond neither to compact Lie groups, nor to p-compact groups. These can be considered as the limit of families of exotic p-local finite groups. The starting point are the examples by A. Díaz, A. Ruiz and A. Viruel in [10] for $p = 3$. The generalization to bigger primes p is considered first at the level of p-groups: the Sylow 3-groups considered in [10] were 3-groups of maximal class classified by Blackburn

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in [3] which fit in a family defined for every odd prime p with the same parameters. The second step needs to deal with the automorphism group of the maximal torus, which for $p = 3$ is the general linear group of rank 2 over $\mathbb{F}_3$ and for bigger primes is the product of the symmetric group of p letters and a cyclic group (see Proposition 3.7 for more details). For $p \geq 5$, these examples of exotic p-local finite groups were already known as Example 9.3 in [6]. We enlarge this family with a simple saturated fusion subsystem of index 2 for each prime $p \geq 5$. In both cases, $p = 3$ and $p > 3$, we can consider the limit and produce structures of p-local compact groups.

To proceed with the main result of this paper we need some notation which will be used in all this paper.

Notation 1.1. For n a natural number, we denote by C_n and $\mathbb{Z}/n$ the cyclic group of order n in multiplicative and additive notation respectively, while we use A_n and Σ_n for the alternating and symmetric groups on a set of order n respectively. The number p will be an odd prime and all the p-groups considered here will be either finite p-groups or discrete p-toral groups of the form:

$$1 \longrightarrow T_{p,k} \longrightarrow S_{p,k} \longrightarrow C_p \longrightarrow 1$$

where $T_{p,k} \cong (C_{p^k})^{p-1}$ (allowing $k = \infty$ and writing T_p in this case). This means that the order of $S_{p,k}$ is $p^{k(p-1)+1}$ when $k < \infty$. The generators of $T_{p,k}$ will be denoted either by $\langle s_1, \dots s_{p-1}\rangle$ or by $\langle v_1, v_2, \dots, v_{p-1}\rangle$, depending on whether we consider the action in Equation (6) or (8) (which we will show to be equivalent). All these extensions will be split and we use s for an element of order p which projects nontrivially to C_p. The center of $S_{p,k}$ is cyclic of order p and generated by an element that we shall call ζ. We use the notation $V \stackrel{\text{def}}{=} \langle \zeta, s\rangle$, a rank 2 elementary abelian p-group.

Normal subsystems of saturated fusion systems where introduced in [1] in the finite case, although the very same definition applies to saturated fusion systems over discrete p-toral groups. This way, a p-local compact group $(S, \mathcal{F}, \mathcal{L})$ of positive rank is said to be *simple* if the only normal subsystems of $\mathcal{F}$ are all finite. We give an explicit definition of these concepts in Definition 2.3 below.

With the notation above, we have the following:

Theorem 1.2. *There are p-local compact groups $(S_p, \mathcal{F}_p, \mathcal{L}_p)$ for each prime number $p \geq 3$, and $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$ for $p \geq 5$, with the fusion characterized by the following data:*

$\mathcal{F}$	$\mathrm{Out}_{\mathcal{F}}(S_p)$	$\mathrm{Out}_{\mathcal{F}}(T_p)$	$\mathrm{Out}_{\mathcal{F}}(V)$	*prime*
$\mathcal{F}_3$	$C_2 \times C_2$	$\mathrm{GL}_2(\mathbb{F}_3)$	$\mathrm{GL}_2(\mathbb{F}_3)$	$p = 3$
$\mathcal{F}_p$	$C_{p-1} \times C_{(p-1)/2}$	$A_p \times C_{p-1}$	$\mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{(p-1)/2}$	$p \geq 5$
$\widetilde{\mathcal{F}}_p$	$C_{p-1} \times C_{p-1}$	$\Sigma_p \times C_{p-1}$	$\mathrm{GL}_2(\mathbb{F}_p)$	

None of these examples can be realized either by a compact Lie group, or by a p-compact group. Moreover, the members of the family $\{(S_p, \mathcal{F}_p, \mathcal{L}_p)\}_{p\geq 3}$ are simple p-local compact groups, $(S_p, \mathcal{F}_p, \mathcal{L}_p)$ is the only proper normal subsystem of

$(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$ *for* $p \geq 5$*, and all the fundamental groups of the corresponding classifying spaces are trivial.*

Proof. A more detailed description of these saturated fusion systems can be found in Section 4.1. The saturation of $(S_p, \mathcal{F}_p)$ and $(S_p, \widetilde{\mathcal{F}}_p)$ is proved in Theorem 4.3 and the exoticness results are proved in Theorems 4.8 and 4.10. The simplicity and normality conditions, together with the property of the fundamental groups, are proved in Proposition 4.5. □

Although these are very specific examples, we have proved some general results which apply to other cases and which are interesting by themselves:

- A saturation criterion for p-local compact groups (Proposition 2.11).
- If $(S, \mathcal{F}, \mathcal{L})$ is a p-local compact group corresponding to a p-compact group and $P \leq S$ is a fully $\mathcal{F}$-centralized subgroup, then the centralizer of P in $(S, \mathcal{F}, \mathcal{L})$ as a p-local compact group is the same as the centralizer of P in $|\mathcal{L}|^\wedge_p$ as a p-compact group (Proposition 4.6).
- An appendix with the classification of saturated fusion subsystems of index prime to p in a p-local compact group. This result is a generalization of the corresponding result in the finite case by C. Broto, N. Castellana, J. Grodal, R. Levi and B. Oliver [4] and the classification can be described in the same parameters as in [4]. However, the tools to reach the classification must be adapted carefully to the infinite case. Moreover, these results complement those proved previously by the first author in [16], where the saturated fusion subsystems of p-power index of a p-local compact group are classified.

Remark 1.3. The [10] examples also include the case where the $\mathcal{F}$-essential subgroups are extraspecial groups instead of elementary abelian (in fact these extraspecial groups are the normalizers of V in Notation 1.1). These examples also generalize to $p \geq 5$ in a similar way that we present here, but they do not produce exotic p-local compact groups in the sense of Section 4. It can be checked that for $p = 3$ we get the fusion system of BDI_2 [9, Proposition 10.2] and for bigger primes we get the fusion systems of $U(p)$ and $PU(p)$ respectively (an explicit description of the automorphism groups of essential subgroups of $U(n)$ is found in [21], from where we deduce our previous assertion).

Remark 1.4. The authors are aware that D. Craven, B. Oliver and J. Semeraro are writing the classification of simple p-local finite groups over p-groups with an abelian subgroup of index p. The finite p-groups $S_{p,k}$ have this property, so this classification should cover the examples of Section 3. We are just interested in these concrete examples and we do not claim that the finite examples in this paper have to be considered a classification of the p-local finite groups with Sylow p-subgroup isomorphic to $S_{p,k}$.

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2. On p-local compact groups

In this section we recall and generalize some concepts about p-local compact groups that we will use throughout this paper. For a more exhaustive treatment of this topic, the reader is referred to [6], [7] and [2]. Let p be a prime to remain fixed for the rest of this section.

2.1. **Normal fusion subsystems.** We start by reviewing the concept of a normal subsystem of a fusion system. Let S be a discrete p-toral group, and let $\mathcal{F}$ be a saturated fusion system over S. Let also P be a subgroup of S. Recall that:

- P is *strongly closed in* $\mathcal{F}$ if we have $\varphi(g) \in P$ for all $g \in P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle g \rangle, S)$.
- P is *normal in* $\mathcal{F}$ if P is normal in S and each morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ in $\mathcal{F}$ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, PR)$ such that $\overline{\varphi}(P) = P$. The maximal normal p-subgroup of a $\mathcal{F}$ is denoted by $O_p(\mathcal{F})$.

The following definition was introduced by Aschbacher [1, Section 6] for finite fusion systems, although it applies to fusion systems over discrete p-toral groups without modification.

Definition 2.1. Let S be a discrete p-toral group, let $\mathcal{F}$ be a saturated fusion system over S, and let $(S', \mathcal{F}') \subseteq (S, \mathcal{F})$ be a subsystem. Then, $\mathcal{F}'$ *is normal in* $\mathcal{F}$ if the following conditions are satisfied.

(N1) S' is strongly closed in $\mathcal{F}$.

(N2) For each $P \leq Q \leq S$ and each $\gamma \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$, the map that sends each $f \in \mathrm{Hom}_{\mathcal{F}'}(P, Q)$ to $\gamma \circ f \circ \gamma^{-1}$ defines a bijection between the sets $\mathrm{Hom}_{\mathcal{F}'}(P, Q)$ and $\mathrm{Hom}_{\mathcal{F}'}(\gamma(P), \gamma(Q))$.

(N3) $\mathcal{F}'$ is a saturated fusion system over S'.

(N4) Each $f \in \mathrm{Aut}_{\mathcal{F}'}(S')$ extends to some $\widetilde{f} \in \mathrm{Aut}_{\mathcal{F}}(S' C_S(S'))$ such that

$$[\widetilde{f}, C_S(S')] = \{\widetilde{f}(g) \cdot g^{-1} \mid g \in C_S(S')\} \leq Z(S').$$

Lemma 2.2. *Let G be a compact Lie group and $S \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup. Let $H \trianglelefteq G$ a normal closed subgroup and write $R = S \cap H$. Then $R \in \mathrm{Syl}_p(H)$ and the saturated fusion system $\mathcal{F}_R(H)$ is normal in $\mathcal{F}_S(G)$.*

Proof. We have to show that $R \in \mathrm{Syl}_p(H)$ and that $(R, \mathcal{F}_R(H)) \subseteq (S, \mathcal{F}_S(G))$ satisfies the properties of Definition 2.1.

As we have to deal with Sylow p-subgroups of compact Lie groups we need the notations introduced in [7, Section 9]: if G is a compact Lie group, $\overline{\mathrm{Syl}}_p(G)$ is the set of all maximal p-toral subgroups while $\mathrm{Syl}_p(G)$ is the set of all discrete p-toral subgroups $P \leq G$ such that $\overline{P} \in \overline{\mathrm{Syl}}_p(G)$, where $\overline{P}$ is the topological closure of P in G. It is shown in [7] that, for any compact Lie group G, the elements in $\mathrm{Syl}_p(G)$ are Sylow p-subgroups of G.

First note that, since H is a closed subgroup of G, it is itself a compact Lie group, so it has Sylow p-subgroups, and we can choose some $P \in \mathrm{Syl}_p(H)$. The group P is a discrete p-toral subgroup of G, so we know that there exists $g \in G$ such that $gPg^{-1} \leq S$, since $S \in \mathrm{Syl}_p(G)$. Then, we have that $gPg^{-1} \leq R$, because H is a normal subgroup of G.

Now, since R is a discrete p-toral subgroup and $P \in \mathrm{Syl}_p(H)$, there exists $h \in H$ such that $hRh^{-1} \leq P$. Hence, we have the chain of inclusions

$$hRh^{-1} \leq P \leq g^{-1}Rg.$$

This implies $gh \in N_G(R)$, and then,

$$h^{-1}Ph \leq h^{-1}g^{-1}Rgh \leq R.$$

We conclude that $P \leq hRh^{-1}$, so in fact $R = h^{-1}Ph$. To finish the proof we will see that $hPh^{-1} \in \mathrm{Syl}_p(H)$ for all $h \in H$. That is, we must show that $\overline{hPh^{-1}} \in \overline{\mathrm{Syl}}_p(H)$ and $hPh^{-1} \in \mathrm{Syl}_p(\overline{hPh^{-1}})$ for all $h \in H$.

We have that $(\overline{hPh^{-1}})_0 = (h\overline{P}h^{-1})_0 = h\overline{P}_0h^{-1}$ and, since $\overline{P}_0$ its a maximal torus of H by hypothesis, so it is $h\overline{P}_0h^{-1}$. Now, $\overline{P}/\overline{P}_0$ is isomorphic to $h\overline{P}h^{-1}/h\overline{P}_0h^{-1}$ and $N_H(\overline{P})/\overline{P}_0$ is isomorphic to $N_H(h\overline{P}h^{-1})/h\overline{P}_0h^{-1}$. Then, if $\overline{P}/\overline{P}_0 \in \mathrm{Syl}_p(N_H(\overline{P})/\overline{P}_0)$, we have that

$$h\overline{P}h^{-1}/h\overline{P}_0h^{-1} \in \mathrm{Syl}_p(N_H(h\overline{P}h^{-1})/h\overline{P}_0h^{-1}).$$

This proves that $\overline{hPh^{-1}} \in \overline{\mathrm{Syl}}_p(H)$.

Now, $hPh^{-1}(\overline{hPh^{-1}})_0 = hPh^{-1}h\overline{P}_0h^{-1} = hP\overline{P}_0h^{-1} = h\overline{P}h^{-1}$, where in the last equality we use that $P \in \mathrm{Syl}_p(\overline{P})$. Finally, let $hxh^{-1} \in h\overline{P}_0h^{-1}$ an element of p-power order. Then, $x \in \overline{P}_0$ has p-power order, so we have, by hypothesis, that $x \in P$, hence $hxh^{-1} \in hPh^{-1}$. This proves that $hPh^{-1} \in \mathrm{Syl}_p(\overline{hPh^{-1}})$.

In conclusion, we have that $R \in \mathrm{Syl}_p(H)$. It remains to prove that $(R, \mathcal{F}_R(H)) \subseteq (S, \mathcal{F}_S(G))$ satisfies the properties of Definition 2.1.

(N1) R is strongly closed in $\mathcal{F}_S(G)$.

Let $a \in R = H \cap S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}_S(G)}(\langle a \rangle, S)$. Then, we have that $\varphi = c_g$ for some $g \in G$ and, since H is normal in G, we have that $c_g(a) \in H$. Moreover, $c_g \in \mathrm{Hom}_{\mathcal{F}_S(G)}(\langle a \rangle, S)$, so we also have $c_g(a) \in S$. Hence $c_g(a) \in H \cap S = R$.

(N2) For each $P \leq Q \leq S$ and each $\gamma \in \mathrm{Hom}_{\mathcal{F}_S(G)}(Q, S)$, the map that sends each $f \in \mathrm{Hom}_{\mathcal{F}_R(H)}(P, Q)$ to $\gamma \circ f \circ \gamma^{-1}$ defines a bijection between the sets $\mathrm{Hom}_{\mathcal{F}_R(H)}(P, Q)$ and $\mathrm{Hom}_{\mathcal{F}_R(H)}(\gamma(P), \gamma(Q))$.

Fix $\gamma \in \mathrm{Hom}_{\mathcal{F}_S(G)}(Q, S)$, we have that $\gamma = c_g$ for some $g \in G$. For every $f \in \mathrm{Hom}_{\mathcal{F}_R(H)}(P, Q)$ we have that $f = c_h$ for some $h \in H$, so $\gamma \circ f \circ \gamma^{-1} = c_{ghg^{-1}}$ for $ghg^{-1} \in H$, since H is normal in G. Moreover, $(ghg^{-1})\gamma(P)(ghg^{-1})^{-1} \leq \gamma(Q)$, so we can define a map

$$\begin{array}{cccc} \gamma^*\colon & \mathrm{Hom}_{\mathcal{F}_R(H)}(P, Q) & \longrightarrow & \mathrm{Hom}_{\mathcal{F}_R(H)}(\gamma(P), \gamma(Q)) \\ & f & \mapsto & \gamma \circ f \circ \gamma^{-1} \end{array}$$

To proof the injectivity of γ^* let $f, f' \in \mathrm{Hom}_{\mathcal{F}_R(H)}(P, Q)$ such that $\gamma^*(f) = \gamma^*(f')$. Note then that $f = c_h$ and $f' = c_{h'}$ for some $h, h' \in H$ and $\gamma^*(f) = c_{ghg^{-1}}$ while $\gamma^*(f') = c_{gh'g^{-1}}$. It is clear that $c_h = c_{h'}$, and hence $f = f'$.

For the surjectivity of γ^*, let $f \in \mathrm{Hom}_{\mathcal{F}_R(H)}(\gamma(P), \gamma(Q))$. We know that $f = c_h$ for some $h \in H$, and since $H \trianglelefteq G$, we can take $c_{g^{-1}hg} \in \mathrm{Hom}_{\mathcal{F}_S(G)}(Q, S)$ so we have that $\gamma^*(c_{g^{-1}hg}) = f$.

(N3) $\mathcal{F}_R(H)$ is a saturated fusion system over H.

Since H is a closed subgroup of G, it is itself a compact Lie group. Moreover, $R \in \mathrm{Syl}_p(H)$, so $\mathcal{F}_R(H)$ is a saturated fusion system over H by [7, Lemma 9.5].

(N4) Each $f \in \mathrm{Aut}_{\mathcal{F}_R(H)}(R)$ extends to some $\widetilde{f} \in \mathrm{Aut}_{\mathcal{F}}(RC_S(R))$ such that

$$[\widetilde{f}, C_S(R)] = \{\widetilde{f}(g) \cdot g^{-1} \mid g \in C_S(R)\} \leq Z(R).$$

Consider first $\overline{S}$, the topological closure of S. We will proof first that $[N_H(R), C_{\overline{S}}(R)] \leq C_H(R)$. Indeed, let $g \in N_H(R)$ and $x \in C_{\overline{S}}(R)$, then $gxg^{-1}x^{-1} \in H$ since H normal in G implies $xg^{-1}x^{-1} \in H$. Moreover, $gxg^{-1}x^{-1} \in C_G(R)$, hence $gxg^{-1}x^{-1} \in H \cap C_G(R) = C_H(R)$.

Now we can see that $C_H(R)C_{\overline{S}}(R)$ is a normal subgroup of $N_H(R)C_{\overline{S}}(R)$. Since R is strongly closed in $\mathcal{F}_S(G)$, it is also fully centralized in $\mathcal{F}_S(G)$, so we have $C_S(R) \in \mathrm{Syl}_p(C_G(R))$ by [7, Lemma 9.5]. Then, we have that $C_S(R) \in \mathrm{Syl}_p(C_H(R)C_{\overline{S}}(R))$ and we can apply the Frattini argument to obtain

$$N_H(R)C_{\overline{S}}(R) = C_H(R)C_{\overline{S}}(R)N_{N_H(R)C_{\overline{S}}(R)}(C_S(R)).$$

Finally, let $f \in \mathrm{Aut}_{\mathcal{F}_R(H)}(R)$, then $f = c_g$ with $g \in N_H(R)$, by the previous decomposition, we can write $g = xh$ for some $x \in C_H(R)C_{\overline{S}}(R)$ and $h \in N_{N_H(R)C_{\overline{S}}(R)}(C_S(R))$. Then, take $\widetilde{f} = c_h$. It is clear that $\widetilde{f} \in \mathrm{Aut}_{\mathcal{F}}(RC_S(R))$ and it is an extension of f. Moreover, if $g \in C_S(R)$, $\widetilde{f}(g) \cdot g^{-1} = hgh^{-1}g^{-1}$ is in H because $H \trianglelefteq G$ and is in $C_S(R)$ because h normalizes it, then, $\widetilde{f}(g) \cdot g^{-1} \in H \cap C_S(R) = C_R(R) = Z(R)$. □

Next we define simplicity for saturated fusion systems and p-local compact groups.

Definition 2.3. Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Then, $\mathcal{F}$ is *simple* if it satisfies one of the following conditions.

(i) $\mathrm{rk}(\mathcal{F}) = 0$ and $\mathcal{F}$ has no proper normal subsystems.
(ii) $\mathrm{rk}(\mathcal{F}) \geq 1$ and every proper normal subsystem of $\mathcal{F}$ is finite.

The p-local compact group $\mathcal{G}$ is *simple* if $\mathcal{F}$ is simple.

Note that if $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ is a simple p-local compact group of positive rank $r \geq 1$, then the fusion system $\mathcal{F}$ may still contain finite normal subsystems. The reader may compare this situation with the definitions of finite simple group and simple compact Lie group, where a similar phenomenon occurs.

In some parts of this paper we deal with subsystems of $\mathcal{F}$ of *p-power index* and of *index prime to p*. Such subsystems can be detected via the computation of certain subgroups. The detection techniques in the finite case where developed in [4]. In the compact case, the tools for the detection of subsystems of p-power index where developed in [16, Appendix B]. Regarding the detection of subsystems of index prime to p, the techniques where first extended from finite to compact in [14] (unpublished), and for the sake of completion they are also included at the end of this paper as an appendix.

For a (possibly infinite) group G, let $O^p(G)$ be the intersection of all the normal subgroups of G that have finite p-power index. Similarly, let $O^{p'}(G)$ be the intersection of all the normal subgroups of G that have finite index prime to p.

Definition 2.4. The *hyperfocal subgroup of* $\mathcal{F}$ is the subgroup

$$O^p_{\mathcal{F}}(S) = \langle T, \{g \cdot \varphi(g)^{-1} \mid g \in Q \leq S, \, \varphi \in O^p(\mathrm{Aut}_{\mathcal{F}}(Q))\}\rangle \leq S.$$

Given a saturated subsystem $(S', \mathcal{F}') \subseteq (S, \mathcal{F})$, we say that:

- $\mathcal{F}'$ has *p-power index in* $\mathcal{F}$ if $S' \geq O^p_{\mathcal{F}}(S)$, and $\mathrm{Aut}_{\mathcal{F}'}(P) \geq O^p(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \leq S'$.

- $\mathcal{F}'$ has *index prime to* p if $S' = S$, and $\mathrm{Aut}_{\mathcal{F}'}(P) \geq O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \leq S'$.

The following result states the correspondence between subgroups of S that contain the hyperfocal subgroup, and subsystems of $\mathcal{F}$ of p-power index. It corresponds to [16, Theorem B.12], together with [16, Corollary B.13].

Theorem 2.5. *For each $R \leq S$ such that $O^p_{\mathcal{F}}(S) \leq R$, there exists a unique subsystem $(R, \mathcal{F}_R) \subseteq (S, \mathcal{F})$ of p-power index. In particular, $\mathcal{F}$ contains a minimal subsystem of p-power index, denoted by $O^p(\mathcal{F})$, which is normal in $\mathcal{F}$.*

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group and consider the group defined as (see Equation (9) in the Appendix A for more details):

$$\Gamma_{p'}(\mathcal{F}) \stackrel{\text{def}}{=} \pi_1(|\mathcal{L}|)/O^{p'}(\pi_1(|\mathcal{L}|)).$$

Then, Theorem A.10, together with Corollary A.12, give us the following result.

Theorem 2.6. *For each $H \leq \Gamma_{p'}(\mathcal{F})$ there exists a unique subsystem $(S, \mathcal{F}_H) \subseteq (S, \mathcal{F})$ of index prime to p. In particular, $\mathcal{F}$ contains a minimal subsystem of index prime to p, denoted by $O^{p'}(\mathcal{F})$, which is normal in $\mathcal{F}$.*

Definition 2.7. The saturated fusion system $(S, \mathcal{F})$ is *reduced* if $O_p(\mathcal{F}) = 1$ and $O^p(\mathcal{F}) = O^{p'}(\mathcal{F}) = \mathcal{F}$.

Remark 2.8. Let $(S, \mathcal{F})$ be a saturated fusion system. As pointed out in [22], if $\mathcal{F}$ is reduced and contains no proper strongly $\mathcal{F}$-closed subgroups, then $\mathcal{F}$ is simple. However, the converse is only true when $\mathcal{F}$ is a finite fusion system: if $\mathcal{F}$ is infinite and simple, we can still have $O_p(\mathcal{F}) > 1$. An example of that occurs with the fusion system associated to the compact Lie group S^3 at the prime 2, which contains a central (in particular normal) subgroup isomorphic to $\mathbb{Z}/2$. See [16, Example 3.8] for further details.

2.2. **A saturation criterion.** In this subsection we present a generalization of [20, Proposition 1.1] for infinite fusion systems (see [9, Proposition 4.4] for a topological analogue of the result in [20]). Before we prove the main result of this subsection, Proposition 2.11, we need two technical results.

Lemma 2.9. *Let π be a finite p-group, G a locally finite p-group and $\rho\colon \pi \to \mathrm{Aut}(G)$ an action of π on G. Then, G has a fixed point of order p.*

Proof. Since π is finite, we can write $\pi = \{g_1, \ldots, g_n\}$. Then, choose an element $x \in G$ and consider the subgroup $H = \langle \rho(g_1)(x), \ldots, \rho(g_n)(x)\rangle$. Since H is finitely generated and G is locally finite, we have that H is finite. Observe that, by construction, the action of π restricts to the subgroup H:

$$\begin{aligned}\rho(g_j)[\rho(g_{k_1})(x)\cdots\rho(g_{k_l})(x)] &= \rho(g_j)[\rho(g_{k_1})(x)]\cdots\rho(g_j)[\rho(g_{k_l})(x)]\\ &= \rho(g_jg_{k_1})(x)\cdots\rho(g_jg_{k_l})(x)\\ &= \rho(g_{m_1})(x)\cdots\rho(g_{m_l})(x) \in H\end{aligned}$$

It follows that the action of π on H has nontrivial fixed points, and hence so does the action of π on G. $\square$

Lemma 2.10. *Let S be a discrete p-toral group and $P \leq S$ any subgroup. Then, there exists an element $x \in Z(P)$ of order p which is fixed by all $\varphi \in \mathrm{Aut}_S(P)$.*

Proof. Set $\pi = \mathrm{Out}_S(P)$ and $\rho\colon \pi \to \mathrm{Aut}(Z(P))$ the action given by choosing a representative and restricting to $Z(P)$, which is well defined because $Z(P)$ is characteristic in P and the inner automorphisms of P restrict to the identity in $Z(P)$.

Since $Z(P) \leq P \leq S$ is a discrete p-toral group, $Z(P)$ is locally finite. By the lemma above, there exists an element $x \in Z(P)$ of order p fixed by all morphisms in $\mathrm{Out}_S(P)$, because $\mathrm{Out}_S(P)$ is finite. Finally, since every $\psi \in \mathrm{Inn}(P)$ restricts to the identity in $Z(P)$, we have that the element x is fixed by all $\varphi \in \mathrm{Aut}_S(P)$. □

Proposition 2.11. *Let $(S, \mathcal{F})$ be a fusion system over a discrete p-toral group. Then, $\mathcal{F}$ is saturated if and only if it satisfies axiom (III) of saturated fusion systems and there exists a set $\mathfrak{X}$ of elements of order p in S such that the following conditions hold:*

(i) *each $x \in S$ of order p is $\mathcal{F}$-conjugate to some $y \in \mathfrak{X}$;*
(ii) *if x, y are $\mathcal{F}$-conjugate and $y \in \mathfrak{X}$, then there is some morphism*
$$\rho \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$$
such that $\rho(x) = y$ and
(iii) *for each $x \in \mathfrak{X}$, $C_{\mathcal{F}}(x)$ is a saturated fusion system over $C_S(x)$.*

Proof. First, if $\mathcal{F}$ is saturated, the set $\mathfrak{X}$ of all elements $x \in S$ of order p such that $\langle x \rangle$ is fully $\mathcal{F}$-centralized satisfies the conditions in the statement. Suppose then that $\mathcal{F}$ satisfies axiom (III) and that such a set $\mathfrak{X}$ exists. Define the sets
$$\begin{aligned} U = \{(P, x) \mid P \leq S, x \in Z(P)^{\Gamma} \text{ of order } p,\ &\text{where } \Gamma \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P)) \\ &\text{is such that } \mathrm{Aut}_S(P) \leq \Gamma\} \text{ and} \\ U_0 = \{(P, x) \in U \mid x \in \mathfrak{X}\}. \end{aligned}$$
Note that, by Lemma 2.10, for each $1 \neq P \leq S$ finite, there is some $x \in P$ such that $(P, x) \in U$.

We first check the following claim.

(A) If $(P, x) \in U_0$ and P is fully centralized in $C_{\mathcal{F}}(x)$, then P is fully centralized in $\mathcal{F}$.

Assume otherwise and let $P' \in P^{\mathcal{F}}$ be fully $\mathcal{F}$-centralized and $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, P')$. Let also $x' = \varphi(x) \in Z(P')$. By property (ii) of $\mathfrak{X}$, there is $\rho \in \mathrm{Hom}_{\mathcal{F}}(C_S(x'), C_S(x))$ such that $\rho(x') = x$. Set then $P'' = \rho(P')$. In particular, $\rho \circ \varphi \in \mathrm{Iso}_{C_{\mathcal{F}}(x)}(P, P'')$ and P'' is $C_{\mathcal{F}}(x)$-conjugate to P. Also, since $\langle x' \rangle \leq P$, we have $C_S(P') \leq C_S(x')$ and then ρ sends $C_S(P')$ injectively into $C_S(P'')$, obtaining that
$$|C_S(P)| < |C_S(P')| \leq C_S(P'').$$
But we have the equalities $C_S(P) = C_{C_S(x)}(P)$ and $C_S(P'') = C_{C_S(x)}(P'')$, which contradict the assumption that P is fully $C_{\mathcal{F}}(x)$-centralized, which proves (A).

Note that, by definition, $N_S(P) \leq C_S(x)$ for all $(P, x) \in U$, and hence
$$\mathrm{Aut}_{C_S(x)}(P) = \mathrm{Aut}_S(P).$$
Also, if $(P, x) \in U$ and $\Gamma \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ is as in the definition of U, then $\Gamma \leq \mathrm{Aut}_{C_{\mathcal{F}}(x)}(P)$. In particular:

(B) For all $(P, x) \in U$,
$$\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P)) \Longleftrightarrow \mathrm{Aut}_{C_S(x)}(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{C_{\mathcal{F}}(x)}(P)).$$

We are ready to check that $\mathcal{F}$ satisfies axioms (I) and (II) of saturated fusion systems. Since $\mathcal{F}$ satisfies axiom (III) by assumption, this will prove the saturation of $\mathcal{F}$.

Axiom (I). For all $P \leq S$ which is fully $\mathcal{F}$-normalized, P is fully $\mathcal{F}$-centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.

Fix $P \leq S$ a fully $\mathcal{F}$-normalized subgroup. By definition, $|N_S(P)| \geq |N_S(P')|$ for all $P' \in P^{\mathcal{F}}$. Choose $x \in Z(P)$ such that $(P, x) \in U$ and let $\Gamma \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ be such that $\mathrm{Aut}_S(P) \leq \Gamma$ and such that $x \in Z(P)^{\Gamma}$. Then, by properties (i) and (ii) of the set $\mathfrak{X}$, there is some $y \in \mathfrak{X}$ and $\rho \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$ such that $\rho(x) = y$. Set $P' = \rho(P)$ and $\Gamma' = \rho \circ \Gamma \circ \rho^{-1} \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P'))$.

Note that, since P is assumed to be fully $\mathcal{F}$-normalized and $N_S(P) \leq C_S(x)$, it follows that $\rho(N_S(P)) = N_S(P')$ and hence P' is also fully $C_{\mathcal{F}}(y)$-normalized. Also, since $\mathrm{Aut}_S(P) \leq \Gamma$, then $\mathrm{Aut}_S(P') \leq \Gamma'$, and clearly $y \in Z(P')^{\Gamma'}$. That is, $(P', y) \in U_0$.

Now, by property (iii) of $\mathfrak{X}$, the fusion system $C_{\mathcal{F}}(y)$ is saturated. Then, P' is fully centralized in $C_{\mathcal{F}}(y)$ and $\mathrm{Aut}_{C_S(y)}(P') \in \mathrm{Syl}_p(\mathrm{Aut}_{C_{\mathcal{F}}(y)}(P'))$, since P' is fully normalized in $C_{\mathcal{F}}(y)$. The following holds then:

(i) By (A) above, P' is fully centralized in $\mathcal{F}$, since it is fully centralized in $C_{\mathcal{F}}(y)$.
(ii) By (B) above, $\mathrm{Aut}_S(P') \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P'))$.

Since $\rho(N_S(P)) = N_S(P')$, it follows that the same holds for P, and axiom (I) is proved.

Axiom (II). Let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ be such that $P' = \varphi(P)$ is fully centralized in $\mathcal{F}$. Then, there exists $\widetilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_{\varphi}, S)$ such that $\varphi = \widetilde{\varphi}|_P$, and where

$$N_{\varphi} = \{g \in N_S(P) \mid \varphi \circ c_g \circ \varphi^{-1} \in \mathrm{Aut}_S(P')\}.$$

Choose $x' \in Z(P')$ of order p and which is fixed under the action of $\mathrm{Aut}_S(P')$, which exists by Lemma 2.10. Let $x = \varphi^{-1}(x') \in Z(P)$. For all $g \in N_{\varphi}$, $\varphi \circ c_g \circ \varphi^{-1}$ fixes x', and thus $c_g(x) = x$. Hence,

(C) $x \in Z(N_f)$, and $N_f \leq C_S(x)$, $N_S(P') \leq C_S(x')$.

Let $y \in \mathfrak{X}$ be $\mathcal{F}$-conjugate to x and x'. By property (ii) of $\mathfrak{X}$ there exist $\rho \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$ and $\rho' \in \mathrm{Hom}_{\mathcal{F}}(C_S(x'), C_S(y))$ such that $\rho(x) = y = \rho'(x')$. Set also $Q = \rho(P)$ and $Q' = \rho'(P')$. Since P' is fully $\mathcal{F}$-centralized and $C_S(P') \leq C_S(x')$, it follows that

(D) $\rho'(C_{C_S(x')}(P')) = \rho'(C_S(P')) = C_S(Q') = C(Q')$,

and then Q' is fully centralized in $C_{\mathcal{F}}(y)$.

Set $\omega = \rho' \circ f \circ \rho^{-1} \in \mathrm{Iso}_{\mathcal{F}}(Q, Q')$. By construction, $\omega(y) = y$, and thus $\omega \in \mathrm{Iso}_{C_{\mathcal{F}}(y)}(Q, Q')$, and we can apply axiom (II) of saturated fusion systems on ω as a morphism in $C_{\mathcal{F}}(y)$ (which is a saturated fusion system by property (iii) of $\mathfrak{X}$): ω extends to some $\widetilde{\omega} \in \mathrm{Hom}_{C_{\mathcal{F}}(y)}(N_{\omega}, C_S(y))$, where

$$N_{\omega} = \{g \in N_{C_S(y)}(Q) \mid \omega \circ c_g \circ \omega^{-1} \in \mathrm{Aut}_{C_S(y)}(Q')\}.$$

Note that, by (C), for all $g \in N_f \leq C_S(x)$,

$$\begin{aligned} c_{\widetilde{\omega}(\rho(g))} &= \omega \circ c_{\rho(g)} \circ \omega^{-1} \\ &= (\omega \circ \rho) \circ c_g \circ (\omega \circ \rho)^{-1} \\ &= (\rho' \circ \varphi) \circ c_g \circ (\rho' \circ \varphi)^{-1} \\ &= c_{\rho'(h)} \in \mathrm{Aut}_{C_S(y)}(Q') \end{aligned}$$

for some $h \in N_S(P')$ such that $\varphi \circ c_g \circ \varphi^{-1} = c_h$. In particular, $\rho(N_\varphi) \leq N_\omega$, and, by (D), $\widetilde{\omega}(\rho(N_\varphi)) \leq \rho'(N_{C_S(x)}(P'))$. We can then define

$$\widetilde{\varphi} = (\rho')^{-1} \circ (\widetilde{\omega} \circ \rho)|_{N_\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S),$$

which clearly satisfies axiom (II) above. This finishes the proof. □

3. Some families of exotic p-local finite groups

In this section we will deal with the finite case. As we will see, examples for $p = 3$ and the non-simple examples for $p \geq 5$ were already known as exotic examples of p-local finite groups. Here, we construct a family of simple examples for $p \geq 5$ which can be seen either as generalization of the $p = 3$ case adapting the Weyl group of the maximal torus, or as a subsystem of index 2 of the previously known examples for $p \geq 5$.

3.1. A family of exotic 3-local finite groups.

We are interested in the p-local finite groups denoted in [10, Table 6] as $\mathcal{F}(3^{2k+1}, 3)$. These saturated fusion systems are over 3-groups which can be expressed as an split extension (see Notation 1.1):

$$1 \longrightarrow T_{3,k} \longrightarrow S_{3,k} \longrightarrow C_3 \longrightarrow 1$$

with a fixed (non-trivial) action of C_3 on T_{3^k}. Consider the action of $\mathrm{GL}_2(\mathbb{F}_3)$ on $T_{3,k}$ as explained in [10], ζ a generator of the center of $S_{3,k}$, s an element (of order 3) not in $T_{3,k}$ and $V \overset{\text{def}}{=} \langle \zeta, s \rangle$ an elementary abelian 3-group of rank 2.

Theorem 3.1 ([10]). *The following outer automorphisms groups characterize simple saturated fusion systems* $(S, \mathcal{F}) \overset{def}{=} (S_{3,k}, \mathcal{F}_{3,k})$*:*

- $\mathrm{Out}_{\mathcal{F}}(S) \cong C_2 \times C_2$,
- $\mathrm{Out}_{\mathcal{F}}(T_{3,k}) \cong \mathrm{GL}_2(\mathbb{F}_3)$ *and*
- $\mathrm{Out}_{\mathcal{F}}(V)) \cong \mathrm{GL}_2(\mathbb{F}_3)$.

All other morphisms are given by composition and restriction of these and inner automorphisms in S*.*

Proof. The existence of these 3-local finite groups is done in [10], and we have just to see that are simple: if $\mathcal{F}$ is not simple, there would be a smaller saturated fusion system over a strongly closed subgroup or of index prime to p. In [10] it is proved that there is no any strongly closed subgroup in $\mathcal{F}$. Then, if there is any saturated fusion subsystem of index prime to p in $\mathcal{F}$ it would have the same Sylow 3-subgroup, so it would appear in the classification of [10, Table 6], and this does not happen. □

3.2. Some p-groups of maximal class.

One natural generalization of the Sylow 3-subgroups listed in the previous subsection can be viewed in the paper of N. Blackburn [3, Page 88]. We recall here the presentation given there.

Fix and odd prime p and define $S_{p,k}$ as the group of order $p^{(p-1)k+1}$ with parameters $\alpha = \beta = \gamma = \delta = 0$ in [3]. This means that $S_{p,k}$ can be given by the following presentation: $\{s, s_1, s_2, \ldots, s_{(p-1)k}\}$ is a generating set, with relations

$$[s, s_{i-1}] = s_i \text{ for } i = 2, \ldots, (p-1)k \tag{1}$$

$$[s_1, s_i] = 1 \text{ for } i = 2, \ldots, (p-1)k \tag{2}$$

$$s^p = 1 \tag{3}$$

$$s_i^{\binom{p}{1}} s_{i+1}^{\binom{p}{2}} \cdots s_{i+p-1}^{\binom{p}{p}} = 1 \text{ for } i = 1, \ldots, (p-1)k \tag{4}$$

In the last equation, we are assuming $s_j = 1$ for $j > (p-1)k$.

Consider $\gamma(S_{p,k}) \stackrel{\text{def}}{=} \langle s_i \rangle_{1 \le i \le (p-1)k}$ and the lower central series

$$\gamma_2(S_{p,k}) = [S_{p,k}, S_{p,k}],\ \gamma_i(S_{p,k}) = [\gamma_{i-1}(S_{p,k}), S_{p,k}].$$

The following proposition gives two properties of $S_{p,k}$ which will allow us to see it as an extension of a finite torus by an element of order p:

Proposition 3.2. *Consider $S_{p,k}$ the group defined above.*

(a) *The subgroup $\gamma(S_{p,k})$ is isomorphic to $(C_{p^k})^{p-1}$ with generators $s_1, \ldots, s_{p-1}$.*
(b) *There are p $S_{p,k}$-conjugacy classes of subgroups of order p not contained in $\gamma(S_{p,k})$. We can consider the subgroups $\langle s s_1^i \rangle$, for $i \in \{0, 1, \ldots, p-1\}$, as representatives of each class.*

Proof. Let us see first that the center of $\gamma \stackrel{\text{def}}{=} \gamma(S_{p,k})$, that we denote by $Z(\gamma)$, is all of γ. From Equation (2) we obtain that $s_1 \in Z(\gamma)$. Conjugation by s induces an automorphism of γ, so $c_s(Z(\gamma)) = Z(\gamma)$. From Equation (1), $c_s(s_1) = s_1 s_2 \in Z(\gamma)$, which implies that $s_2 \in Z(\gamma)$. As $c_s(s_i) = s_i s_{i+1}$ we can iterate this argument and $Z(\gamma) = \gamma$. So γ is abelian.

Use now Equation (4) for $i = (p-1)k$, obtaining that $s^p_{(p-1)k} = 1$. Using the same relation inductively for $i = (p-1)k - j$, $1 \le j \le p-2$ we get $s^p_{(p-1)k-j} = 1$ for $0 \le j < p-1$. So we obtain that $\langle s_m \rangle_{(p-1)(k-1)+1 \le m \le (p-1)k} \cong (C_p)^{p-1}$. We can iterate this process, getting that $\langle s_m \rangle_{(p-1)(k-l)+1 \le m \le (p-1)(k-l+1)} \cong (C_{p^l})^{p-1}$, getting the first assertion.

For the second item, remark that using Equation (1) we get that the element s is conjugated to $s s_2^{i_2} s_3^{i_3} \cdots s_{p-1}^{i_{p-1}}$ for all $0 \le i_j \le p^k - 1$. Use now Equation (4) for $i = 1$ to see that conjugating s by powers of s_{p-1} we obtain all possible elements of the type $s s_1^{p i_1}$ modulo $\gamma_2(S_{p,k})$. The result follows from the fact that these are all possible conjugations of elements of order p which are not in $\gamma(S_{p,k})$ $\square$

This proposition tells us that $S_{p,k}$ fits in a split extension:

$$1 \to (C_{p^k})^{p-1} \to S_{p,k} \to C_p \to 1 \tag{5}$$

and the action of $\langle s \rangle \cong C_p$ over $(C_{p^k})^{p-1}$ is given by the matrix (using additive notation, so with coefficients in $\mathbb{Z}/p^k$, on the generators $\{s_1, \ldots, s_{p-1}\}$):

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -\binom{p}{1} \\ 1 & 1 & 0 & \cdots & 0 & -\binom{p}{2} \\ 0 & 1 & 1 & \cdots & 0 & -\binom{p}{3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\binom{p}{p-2} \\ 0 & 0 & 0 & \cdots & 1 & 1-\binom{p}{p-1} \end{pmatrix}. \tag{6}$$

Consider $(\mathbb{Z}/p^k)^p$ generated by $e_1, \ldots, e_p$ and the action of Σ_p by permutation of the elements of the basis. This action leaves invariant the submodule T, isomorphic to $(\mathbb{Z}/p^k)^{p-1}$, generated by the basis $\langle v_1, \ldots, v_{p-1}\rangle$, where $v_1 \overset{\text{def}}{=} e_1 - e_2$, $v_2 \overset{\text{def}}{=} e_2 - e_3$, $\ldots$, $v_{p-1} \overset{\text{def}}{=} e_{p-1} - e_p$, so we get an action of Σ_p on $(\mathbb{Z}/p^k)^{p-1}$, and allows us to construct a split extension:

$$1 \longrightarrow (C_{p^k})^{p-1} \longrightarrow (C_{p^k})^{p-1} \rtimes \Sigma_p \longrightarrow \Sigma_p \longrightarrow 1 \tag{7}$$

In the basis $\{v_1, v_2, \ldots, v_{p-1}\}$ the permutation $(1, 2, \ldots, p)$ corresponds to the matrix:

$$B = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \tag{8}$$

Lemma 3.3. *The split extensions of the type:*

$$1 \to (C_{p^k})^{p-1} \to S \to C_p \to 1\,,$$

where the action of C_p on $(C_{p^k})^{p-1}$ are the induced by matrices A and B from Equations (6) and (8) are equivalent.

Proof. Consider the change of basis given by the formula $v_j = \sum_{m=0}^{j-1} \binom{j-1}{m} s_{m+1}$, for $j = 1, \ldots, p-1$. It is an exercise to check that this change of basis transforms A on B. □

With all these computations we have obtained the following inclusion of extensions:

$$\begin{tikzcd} 1 \arrow[r] & (C_{p^k})^{p-1} \arrow[r] \arrow[d, equal] & S_{p,k} \arrow[r] \arrow[d] & C_p \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & (C_{p^k})^{p-1} \arrow[r] & (C_{p^k})^{p-1} \rtimes \Sigma_p \arrow[r] & \Sigma_p \arrow[r] & 1 \end{tikzcd}$$

Where each piece of the first row is a p-Sylow subgroup of the corresponding position of the second row.

Remark 3.4. Here we can see that there is a difference between $p = 3$ and $p > 3$: while the definition and properties of $S_{p,k}$ are the same for any odd prime p, for $p = 3$, the inclusion $C_3 \subset \Sigma_3$ is a normal subgroup, so the action of Σ_3 over $T_{3,k}$ will not produce a saturated fusion system $\mathcal{F}$ with $T_{3,k}$ an $\mathcal{F}$-essential subgroup. For $p > 3$, $C_p \subset \Sigma_p$ is not a normal subgroup and this will become a tool to construct saturated fusion systems with $T_{p,k}$ an $\mathcal{F}$-essential subgroup.

We will enumerate some properties of $S_{p,k}$. To make it clear we consider the elements of $S_{p,k}$ written uniquely as $v_1^{i_1} \cdots v_{p-1}^{i_{p-1}} s^i$ with $0 \le i_j \le p^k - 1$ and $0 \le i \le p-1$. Let also $T_{p,k}$ denote the subgroup of elements of the form $v_1^{i_1} \cdots v_{p-1}^{i_{p-1}}$, which we call the maximal torus.

Lemma 3.5. *Let $v_1^{i_1} \cdots v_{p-1}^{i_{p-1}} \in T_{p,k} \subset S_{p,k}$. Then, the following holds*

(a) $s \cdot v_1^{i_1} \cdots v_{p-1}^{i_{p-1}} \cdot s^{-1} = v_1^{-i_{p-1}} v_2^{i_1 - i_{p-1}} v_3^{i_2 - i_{p-1}} \cdots v_{p-1}^{i_{p-2} - i_{p-1}}$ *and*

(b) $v_1^{i_1}\cdots v_{p-1}^{i_{p-1}}\cdot s\cdot(v_1^{i_1}\cdots v_{p-1}^{i_{p-1}})^{-1}=v_1^{i_{p-1}+i_1}v_2^{i_{p-1}+i_2-i_1}\cdots v_{p-1}^{i_{p-1}+i_{p-1}-i_{p-2}}s$.

(c) $v_1^{i_1}\cdots v_{p-1}^{i_{p-1}}s$ *and* $v_1^{j_1}\cdots v_{p-1}^{j_{p-1}}s$ *are* $S_{p,k}$*-conjugate if and only if* $\sum_{l=1}^{p-1} i_l \equiv \sum_{l=1}^{p-1} j_l \pmod p$.

(d) *The center of* $S_{p,k}$ *is generated by* $\zeta=(v_1^1v_2^2\cdots v_{p-1}^{p-1})^{p^{k-1}}$.

(e) *There are* p *conjugacy classes of subgroups of order* p *not contained in* $T_{p,k}$*, represented by the elements* v_1^is *for* $0\le i\le p-1$.

Proof. The proof of (a) and (b) are direct computation using the action of s on $v_k^{j_k}$, and statement (e) is the same as Proposition 3.2 (b), just changing the generators.

We prove (c) applying (a) and (b): consider the set $X\overset{\text{def}}{=}\{vs\}_{v\in T_{p,k}}$. This set has $p^{k(p-1)}$ elements. By (a) and (b) there is an action of $S_{p,k}$ on X by conjugation. This action keeps the congruence modulo p of the sum of the exponents, so there are at least p $S_{p,k}$-conjugacy classes of elements in X. Given an element $vs\in X$, the centralizer in $S_{p,k}$, $C_{S_{p,k}}(sv)$, is the elementary abelian group of order p^2 generated by $\langle sv,\zeta\rangle$, so the orbits of the action have $p^{k(p-1)-1}$ elements. So, there are exactly p $S_{p,k}$-conjugacy classes of elements in X and the congruence modulo p of the sum of the exponents determine if two of them are $S_{p,k}$-conjugated.

Finally we prove (d). To compute a generator of the center, consider the action of Σ_p in the basis $\{e_1,\ldots,e_p\}$. It is easy to see that the elements of the form $\lambda(e_1+\cdots+e_p)$ are invariant under the action of Σ_p (in particular, by the action of C_p). In particular, if λ is a multiple of p^{k-1}, then the corresponding element belongs to $T_{p,k}$. Statement (e) follows by passing to the basis $\{v_1,\ldots,v_{p-1}\}$ and using multiplicative notation. □

3.3. **Two families of exotic p-local finite groups for $p>3$.** We now describe some generalizations to $p>3$ of the examples constructed in Subsection 3.1 for $p=3$. We start reducing the possible outer automorphism group over S of any saturated fusion system $(S_{p,k},\mathcal{F})$.

Proposition 3.6. *If* $(S_{p,k},\mathcal{F})$ *is a saturated fusion system, then* $\mathrm{Out}_{\mathcal{F}}(S_{p,k})\le C_{p-1}\times C_{p-1}$.

Proof. As $\mathcal{F}$ is saturated, $\mathrm{Out}_{\mathcal{F}}(S_{p,k})$ must be a p'-subgroup of $\mathrm{Out}(S_{p,k})$.

Consider now the Frattini subgroup $\Phi(S_{p,k})=\langle s_2,s_3,\ldots s_{(p-1)k}\rangle$. The kernel of the map

$$\rho\colon\ \mathrm{Out}(S_{p,k})\to\mathrm{Out}(S_{p,k}/\Phi(S_{p,k}))$$

is a p-group, so $\rho(\mathrm{Out}_{\mathcal{F}}(S_{p,k}))$ is isomorphic to a subgroup of $\mathrm{Out}(S_{p,k}/\Phi(S_{p,k}))$. Use now that $S_{p,k}/\Phi(S_{p,k})$ is a rank two elementary abelian group, so we can consider as an $\mathbb{F}_p$ vector space with basis $\{\overline{s},\overline{s_1}\}$ (representatives of s and s_1 in $S_{p,k}/\Phi(S_{p,k})$). As γ is a characteristic subgroup, in this basis, $\rho(\mathrm{Out}_{\mathcal{F}}(S_{p,k}))$ is included in lower triangular matrices of $\mathrm{GL}_2(p)$. Now use again that $p\nmid\#\,\mathrm{Out}_{\mathcal{F}}(S_{p,k})$ to get the result. □

As we want to work with concrete examples, consider ϕ and ψ, elements in $\mathrm{Aut}(S_{p,k})$ of order $p-1$ defined as:

- The normalizer of $\langle s\rangle$ in Σ_p is isomorphic to $C_p\rtimes C_{p-1}$. Consider $\phi\in\Sigma_p$ an element of order $p-1$ normalizing $\langle s\rangle$ and consider the action over $S_{p,k}=T_{p,k}\rtimes\langle s\rangle$ as the induced by conjugation as a subgroup of $T_{p,k}\rtimes\Sigma_p$. This action sends $s\mapsto s^{\lambda}$, λ a generator of $\mathbb{F}_p^{\times}$, while $\phi(\zeta)=\zeta$ (ζ the generator of the center from Lemma 3.5 (d)).

- Consider $\mu \in \mathrm{Aut}(C_{p^k})(\cong (\mathbb{Z}/p^k\mathbb{Z})^\times)$ an element of order $p-1$, and define ψ as the element in $\mathrm{Aut}(S_{p,k})$ which restricts to $\mu \times \overset{p-1}{\cdots} \times \mu$ in the maximal torus and the identity on $\langle s \rangle$. To get easier computations, consider μ such that reduces to λ modulo p. With this definition, $\psi(\zeta) = \zeta^\lambda$.

It can be checked that $\langle \phi, \psi \rangle \cong C_{p-1} \times C_{p-1}$.

Proposition 3.7. *Consider $S_{p,k}$, $T_{p,k}$ and V as in Notation 1.1 and ϕ and ψ from the previous paragraph.*

(a) *There are exotic saturated fusion systems $(S, \widetilde{\mathcal{F}}) \overset{def}{=} (S_{p,k}, \widetilde{\mathcal{F}}_{p,k})$ characterized by the morphisms:*
  - $\mathrm{Out}_{\widetilde{\mathcal{F}}}(S) = \langle \phi, \psi \rangle \cong C_{p-1} \times C_{p-1}$,
  - $\mathrm{Out}_{\widetilde{\mathcal{F}}}(T_{p,k}) = \Sigma_p \times \langle \psi \rangle \cong \Sigma_p \times C_{p-1}$ *and*
  - $\mathrm{Out}_{\widetilde{\mathcal{F}}}(V) = \mathrm{GL}_2(\mathbb{F}_p) \cong \mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{p-1}$.

  All other morphisms are given by composition and restriction of these and inner automorphisms in S.

(b) *$(S, \widetilde{\mathcal{F}})$ contains an exotic simple saturated fusion subsystem $(S, \mathcal{F})$ of index 2, generated by the morphisms*
  - $\mathrm{Out}_{\mathcal{F}}(S) = \langle \phi^2, \psi\phi^{-1} \rangle \cong C_{\frac{p-1}{2}} \times C_{p-1}$,
  - $\mathrm{Out}_{\mathcal{F}}(T_{p,k}) = A_p \times \langle \psi\phi^{-1} \rangle \cong A_p \times C_{p-1}$ *and*
  - $\mathrm{Out}_{\mathcal{F}}(V) = \mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{\frac{p-1}{2}} < \mathrm{GL}_2(\mathbb{F}_p)$.

  and the conjugation by the elements in S.

Proof. To prove (a), the existence of the saturated fusion systems $(S, \widetilde{\mathcal{F}})$ can be found in [6, Example 9.3]. In the same result the authors also prove that these examples are exotic.

To get (b), we can proceed classifying all the saturated fusions subsystems of $(S, \widetilde{\mathcal{F}})$ of index prime to p as in [4, Section 5.1] or [2, Part I.7]. To do this, we need to compute $\mathcal{E}_0$, the fusion system generated by $O^{p'}(\mathrm{Aut}_{\widetilde{\mathcal{F}}}(P))$ for all $P \in \widetilde{\mathcal{F}}^c$:

- For $P = S$, $O^{p'}(\mathrm{Aut}_{\widetilde{\mathcal{F}}}(S)) = \mathrm{Inn}(S)$.
- For $P = T_{p,k}$, $O^{p'}(\mathrm{Aut}_{\widetilde{\mathcal{F}}}(T_{p,k})) \cong A_p$: the elements of order p in Σ_p for odd prime p generate the alternating group A_p.
- For $P = V$, $O^{p'}(\mathrm{Aut}_{\widetilde{\mathcal{F}}}(V)) \cong \mathrm{SL}_2(\mathbb{F}_p)$: for odd prime p the elements of order p in $\mathrm{GL}_2(\mathbb{F}_p)$ generate $\mathrm{SL}_2(\mathbb{F}_p)$.

Now we have to detect the elements in $\mathrm{Out}_{\widetilde{\mathcal{F}}}(S)$ which restrict to morphisms in $\mathrm{Out}_{\mathcal{E}_0}(S)$:

- ϕ is an odd permutation, so the restriction to $T_{p,k}$ does not give an element of $O^{p'}(\mathrm{Aut}_{\widetilde{\mathcal{F}}}(T_{p,k}))$, and it does not restrict to an automorphism of determinant one in V, so $\phi \notin \mathrm{Out}_{\mathcal{E}_0}(S)$.
- ϕ^2 is an even permutation, so it restricts to an element in $O^{p'}(\mathrm{Aut}_{\widetilde{\mathcal{F}}}(T_{p,k}))$.
- $\psi^i\phi^{-i}$ restricts to an automorphism of determinant one in V.

These computations show us that $\mathrm{Out}_{\widetilde{\mathcal{F}}}(S)/\mathrm{Out}^0_{\widetilde{\mathcal{F}}}(S) \cong \mathbb{Z}/2$, so there is just one proper saturated fusion subsystem of index prime to p, and it is of index 2. Now, if we add the necessary morphisms to $\mathcal{E}_0$ to get a saturated fusion system (or we remove the restrictions of ϕ in $\widetilde{\mathcal{F}}$) we get the structure of $\mathcal{F}$. By [2, Theorem I.7.7], we have proved that $O^{p'}(\widetilde{\mathcal{F}}) = \mathcal{F}$.

Let us see now that $\mathcal{F}$ is simple: by Remark 2.8 we have to check that:

(i) $O_p(\mathcal{F}) = 1$: there is not any non-trivial proper strongly closed subgroup in $\mathcal{F}$, so there is not any non-trivial proper normal subgroup in $\mathcal{F}$.

(ii) $O^p(\mathcal{F}) = \mathcal{F}$: by [22, Proposition 1.3 (d)], we need to show that $\mathfrak{foc}(\mathcal{F}) = S$, where $\mathfrak{foc}(\mathcal{F})$ is the *focal subgroup of* $\mathcal{F}$, defined by

$$\mathfrak{foc}(\mathcal{F}) = \langle g \cdot \varphi(g)^{-1} \,|\, g \in S,\, \varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle g \rangle, S) \rangle$$

(remark that this definition applies only to finite saturated fusion systems). First, note that there are elements $\varphi, \varphi' \in \mathrm{Aut}_{\mathcal{F}}(V_0)$ such that $\varphi(s) = s\zeta$ and also $\varphi'(\zeta) = s\zeta$, and hence we get $V_0 \subset \mathfrak{foc}(\mathcal{F})$. The action of C_{p-1} on the maximal torus T includes φ such that $\varphi(v) = v^{-1}$ for all $v \in T$, so we have all elements $\langle v_1^2, \dots, v_{p-1}^2 \rangle \subset \mathfrak{foc}(\mathcal{F})$. Taking now the expression of ζ, we get that $v_1 v_3 \cdots v_{p-2} \in \mathfrak{foc}(\mathcal{F})$. Conjugating this element by s we get $v_2 v_4 \cdots v_{p-1} \in \mathfrak{foc}(\mathcal{F})$. Consider now φ the conjugation by an element of order p in A_p on v_{p-1}. This tells us that $v_{p-1}\varphi(v_{p-1}^{-1}) = v_1 v_2 \dots v_{p-2} v_{p-1}^2 \in \mathfrak{foc}(\mathcal{F})$. So $v_1 v_2 \dots v_{p-2} \in \mathfrak{foc}(\mathcal{F})$ and we get that $v_{p-1} \in \mathfrak{foc}(\mathcal{F})$. Conjugating $v_1 v_2 \dots v_{p-2}$ by s and using that $v_{p-1} \in \mathfrak{foc}(\mathcal{F})$ we get $v_1 \in \mathfrak{foc}(\mathcal{F})$, getting that $S = \langle s, v_1 \rangle \subset \mathfrak{foc}(\mathcal{F})$.

(iii) $O^{p'}(\mathcal{F}) = \mathcal{F}$: as $\widetilde{\mathcal{F}}^c = \mathcal{F}^c$, the computations in the first part of this proof show us that $O^{p'}(\mathcal{F}) = O^{p'}(\widetilde{\mathcal{F}}) = \mathcal{F}$.

Finally, $(S, \mathcal{F})$ and $(S, \widetilde{\mathcal{F}})$ are exotic because these examples do not appear in the list of [6, Proposition 9.5], where there are all the finite groups which realize simple saturated fusion systems over p-groups of this type. □

4. On p-local compact groups

In this section we will introduce the p-local compact version of the finite examples described in Section 3 and we will prove their exoticness. The definition of exotic p-local finite groups is very precise: there is no finite group realizing the saturated fusion system. When trying to generalize this definition to p-local compact groups, it seems natural to replace the finiteness condition by compactness conditions. From the point of view of local homotopy theory, the most developed frameworks are the compact Lie groups and p-compact groups. Thus, more precisely, in this section we prove that these examples do not correspond neither to compact Lie groups, nor to p-compact groups.

Remark 4.1. As we can see in [17], there always exists some infinite group which realizes a given saturated fusion system over a discrete p-toral group, so we must restrict our attention to the two cases mentioned above.

Remark 4.2. There are several ways of producing examples of *non-simple* p-local compact groups which are not p-compact groups: take an extension of a torus by a non p-nilpotent finite group. By Lemma 4.9, this will produce a p-local compact group which do not correspond to a p-compact group.

4.1. **Families of p-local compact groups.**

In this subsection we describe the properties which characterize the p-local compact groups we are interested in. Consider the extension $T_{p,k} \to S_{p,k} \to C_p$ described in Equation (5). We can construct a discrete p-toral group by taking the monomorphims $I_{p,k}\colon S_{p,k} \to S_{p,k+1}$, defined by $I_{p,k}(s) = s$ and $I_{p,k}(v_i) = v_i^p$, which are compatible with an obvious choice of sections of the extensions. Thus, the discrete p-toral group

$$S_p = \bigcup_{k\geq 2} S_{p,k}$$

fits in a split extension $T_p \to S_p \to C_p$, where $T_p = (\mathbb{Z}/p^\infty)^{p-1}$. Observe that the center of S_p is the subgroup $Z(S_p) = \langle \zeta \rangle \cong \mathbb{Z}/p$ and define $V = \langle s, \zeta \rangle \cong \mathbb{Z}/p \times \mathbb{Z}/p$.

Theorem 4.3. *Consider $(S_p, \mathcal{F}_p)$ for each prime number $p \geq 3$, and $(S_p, \widetilde{\mathcal{F}}_p)$ for $p \geq 5$, the fusion systems characterized by the information.*

$\mathcal{F}$	$\mathrm{Out}_{\mathcal{F}}(S_p)$	$\mathrm{Out}_{\mathcal{F}}(T_p)$	$\mathrm{Out}_{\mathcal{F}}(V)$	*prime*
$\mathcal{F}_3$	$C_2 \times C_2$	$\mathrm{GL}_2(\mathbb{F}_3)$	$\mathrm{GL}_2(\mathbb{F}_3)$	$p = 3$
$\mathcal{F}_p$	$C_{p-1} \times C_{(p-1)/2}$	$A_p \times C_{p-1}$	$\mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{(p-1)/2}$	$p \geq 5$
$\widetilde{\mathcal{F}}_p$	$C_{p-1} \times C_{p-1}$	$\Sigma_p \times C_{p-1}$	$\mathrm{GL}_2(\mathbb{F}_p)$	

Then, $\mathcal{F}_p$ (for $p \geq 3$) and $\widetilde{\mathcal{F}}_p$ (for $p \geq 5$) are saturated fusion systems over S_p and there exist p-local compact groups $(S_p, \mathcal{F}_p, \mathcal{L}_p)$ (for $p \geq 3$) and $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$ (for $p \geq 5$).

Before applying Proposition 2.11 to prove this statement, we need the following easy calculation.

Lemma 4.4. *The elements of order p not contained in T_p belong to a single $\mathcal{F}_p$-conjugacy class. Moreover, the elements of this class are $\mathcal{F}_p$-conjugated to the maximal torus.*

Proof. Let $v \in T_p$ and vs^i, with $i \neq 0$, an element not contained in T_p. An easy computation shows that $\langle vs^i \rangle = \langle ws \rangle$ for some $w \in T_p$. Since $T_p = (\mathbb{Z}/p^\infty)^{p-1}$, we can write $w = w_1^{pi_1} \cdots w_{p-1}^{pi_{p-1}}$, for some $w_1, \ldots, w_{p-1}$ in $\mathbb{Z}/p^k \subset \mathbb{Z}/p^\infty$, for k big enough. Then, by Lemma 3.5 (c), $w_1^{pi_1} \cdots w_{p-1}^{pi_{p-1}} s$ and $(1, s)$ are conjugate in $S_{p,k}$, and also in S_p. So we have vs^i conjugated to s^j for some $j \neq 0$. Finally, as $\mathrm{Out}_{\mathcal{F}_3}(V) = \mathrm{GL}_2(\mathbb{F}_3)$ and $\mathrm{Out}_{\mathcal{F}_p}(V) = \mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{\frac{p-1}{2}}$ for $p > 3$, we have that s is conjugated to s^j for all $j \neq 0$.

Then, every element not in the maximal torus is conjugated to s, and using again that $\mathrm{Out}_{\mathcal{F}_3}(V) = \mathrm{GL}_2(\mathbb{F}_3)$ and $\mathrm{Out}_{\mathcal{F}_p}(V) = \mathrm{SL}_2(\mathbb{F}_p) \rtimes C_{\frac{p-1}{2}}$ for $p > 3$, we obtain that s is conjugated to $\zeta \in T_p$. $\square$

Proof of Theorem 4.3. Fix and odd prime p and let $\mathcal{F}$ be $\mathcal{F}_3$, $\mathcal{F}_p$ or $\widetilde{\mathcal{F}}_p$ for $p \geq 5$ as in the statement. We want to apply Proposition 2.11 to $\mathcal{F}$.

Note that, by construction, $\mathcal{F}$ satisfies axiom (III) of saturated fusion systems, and now we have to prove the existence of a set $\mathfrak{X}$ as in Proposition 2.11. Let $\mathfrak{X} = \{v \in T_p \subset S_p \mid v \text{ has order } p\}$.

Condition (i) follows from Lemma 4.4: every element of order p not in the maximal torus is $\mathcal{F}_p$-conjugated to $\zeta \in T_p$.

We proceed now checking condition (ii). Observe that, for $v \in T_p$, $C_S(v) = T_p$ if $v \notin \langle \zeta \rangle$ and $C_S(v) = S_p$ if $v \in \langle \zeta \rangle$. Let first $v \in T_p$ be and element of order p not in $\langle \zeta \rangle$.

If v is $\mathcal{F}_p$-conjugated to other $v' \in T_p$, then, by construction, there is an automorphism $\rho \in \mathrm{Aut}_{\mathcal{F}}(T_p)$ such that $\rho(v) = v'$. If $v \in \langle \zeta \rangle$, then $v = \zeta^\lambda$, and ζ^λ is conjugated to ζ^μ for all $\lambda, \mu \neq 0$ by an $\mathcal{F}$-automorphism of S_p.

Finally, let vs^i be an element of order p, by Lemma 4.4 and Lemma 3.5 (c), we know that vs^i is S_p-conjugated to s^j for some $j \neq 0$, and hence it is enough to prove that there is some $\rho \in \mathrm{Hom}_{\mathcal{F}}(C_{S_p}(s^j), C_{S_p}(\zeta^\lambda))$ such that $\rho(s^j) = \zeta^\lambda$ for some $\lambda \neq 0$. Recall that $C_{S_p}(s^j) = V$ and $C_{S_p}(\zeta^\lambda)) = S_p$, by construction, there is an automorphism $\rho \in \mathrm{Aut}_{\mathcal{F}_p}(V)$ sending s^j to ζ^λ. Then, composing with inclusion in S_p, we obtain

$$\rho \in \mathrm{Hom}_{\mathcal{F}}(C_{S_p}(s^j), C_{S_p}(\zeta^\lambda))$$

and thus $\mathfrak{X}$ satisfies condition (ii).

We are left to check condition (iii): for each $v \in \mathfrak{X}$, the fusion system $C_{\mathcal{F}}(v)$ is saturated. By definition, $C_{\mathcal{F}}(v) \subseteq \mathcal{F}$ is the fusion subsystem over $C_{S_p}(v)$ whose morphisms are those morphisms in $\mathcal{F}$ which fix the element v. A careful inspection of the generating morphisms of $\mathcal{F}$ for $\mathcal{F}$ being $\mathcal{F}_3$, $\mathcal{F}_p$ or $\widetilde{\mathcal{F}}_p$ ($p \geq 5$) shows that $C_{\mathcal{F}}(v) = \mathcal{F}_{C_{S_p}(v)}(T_p \rtimes L)$, where $L \leq W$ is the subgroup fixing the element v in W, with W being $\mathrm{GL}_2(\mathbb{F}_3)$, A_p or Σ_p ($p \geq 5$). In either case, $C_{\mathcal{F}}(v)$ is saturated.

It follows by Proposition 2.11 that $\mathcal{F}$ is saturated. The existence and uniqueness of a linking systems for every saturated fusion system is proved by R. Levi and A. Libman in [19]. $\square$

We finish this section with some properties of these examples:

Proposition 4.5. *Let $(S_p, \mathcal{F}_p, \mathcal{L}_p)$ for $p \geq 3$ and $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$ for $p \geq 5$ be the p-local compact groups described in Theorem 4.3. Then, the following holds.*

(a) *$(S_p, \mathcal{F}_p, \mathcal{L}_p)$ is simple for $p \geq 3$.*

(b) *$(S_p, \mathcal{F}_p, \mathcal{L}_p)$ has index 2 in $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$. Moreover, $(S_p, \mathcal{F}_p, \mathcal{L}_p)$ is the only proper normal subsystem in $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$ for $p \geq 5$.*

(c) *$\pi_1(|\mathcal{L}|^\wedge_p) = \{e\}$ for $\mathcal{L}$ being $\mathcal{L}_p$ for $p \geq 3$ or $\widetilde{\mathcal{L}}_p$ for $p \geq 5$.*

Proof. To prove (a) we have to check that every proper normal subsystem of $(S_p, \mathcal{F}_p)$ is finite. As there is not any proper $\mathcal{F}_p$-strongly closed subgroup in S_p, we only have to check that any normal subsystem over S_p must be all $\mathcal{F}_p$. Let $(S_p, \mathcal{F}'_p) \subseteq (S_p, \mathcal{F}_p)$ be a normal subsystem, then by condition (N2) in Definition 2.1, as we have to consider fusion subsystems over S_p, all the elements of order p in $\mathrm{Out}_{\mathcal{F}_p}(P)$ must be in $\mathrm{Out}_{\mathcal{F}'_p}(P)$. This implies that:

- for $p = 3$, $\mathrm{Out}_{\mathcal{F}'_3}(T) \geq \mathrm{SL}_2(\mathbb{F}_3)$ and $\mathrm{Out}_{\mathcal{F}'_3}(V) \geq \mathrm{SL}_2(\mathbb{F}_3)$;
- for $p > 3$, $\mathrm{Out}_{\mathcal{F}'_p}(T) \geq A_p$ and $\mathrm{Out}_{\mathcal{F}'_3}(V) \geq \mathrm{SL}_2(\mathbb{F}_p)$.

We have now to apply condition (N3) and force $\mathcal{F}'_p$ to be saturated, and, as the computations are different, we distinguish $p = 3$ from $p \geq 5$.

- Consider first $p = 3$: $-\mathrm{Id} \in \mathrm{SL}_2(\mathbb{F}_3) \leq \mathrm{Out}_{\mathcal{F}'_3}(V)$ must extend to $N_{S_3}(V)$ (see [7, Definition 2.2 (II)]), which is strictly larger than V. Applying Alperin's fusion theorem [7, Theorem 3.6], this morphism must extend to an automorphism $\varphi \in \mathrm{Aut}_{\mathcal{F}'}(S_3)$. The restriction of φ to the maximal torus T_3 will give an element in $\mathrm{Out}_{\mathcal{F}'}(T_3)$ with determinant -1, which implies that $\mathrm{Aut}_{\mathcal{F}'_3}(T_3) = \mathrm{Aut}_{\mathcal{F}_3}(T_3)$. This argument can be applied now extending $-\mathrm{Id} \in \mathrm{SL}_2(\mathbb{F}_3) \leq \mathrm{Aut}_{\mathcal{F}'_3}(T_3)$ and restricting to $\mathrm{Aut}_{\mathcal{F}'_3}(V)$, obtaining that $\mathrm{Aut}_{\mathcal{F}'_3}(V) = \mathrm{Aut}_{\mathcal{F}_3}(V)$.

- Consider now the case $p \geq 5$: the part of the proof concerning the extension of elements in $\mathrm{SL}_2(\mathbb{F}_p) \leq \mathrm{Aut}_{\mathcal{F}'_p}(V)$ is the same, replacing $-\mathrm{Id}$ by diagonal matrices of the type $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, for $\lambda \in \mathbb{F}_p^\times$. The extension and restriction argument implies that $\mathrm{Aut}_{\mathcal{F}'_p}(T_p) \geq A_p \times C_{p-1} \cong \mathrm{Aut}_{\mathcal{F}_p}(T_p)$. Finally, consider φ an element of order $(p-1)/2$ in $N_{A_p}(\langle s \rangle)$ and the induced action on T_p. This action must extend to $N_{S_p}(T_p)$ and the restriction to V gives matrices of determinant a square in $\mathbb{F}_p^\times$, obtaining $\mathrm{Aut}_{\mathcal{F}'_p}(T_p) = \mathrm{Aut}_{\mathcal{F}_p}(T_p)$.

Now the result follows as we have seen that $\mathcal{F}'_p$ must contain all the generators of $\mathcal{F}_p$.

To see (b), we have to proceed similarly, but all the computations have been already done: any normal saturated fusion subsystem of $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$ must be over S_p (there is not any proper strongly $\widetilde{\mathcal{F}}_p$-closed subgroup) and must contain all the automorphims of order p. So, applying the computations in the poof of (a), we see that $(S_p, \mathcal{F}_p)$ is $O^{p'}(\widetilde{\mathcal{F}}_p)$ (see Remark A.11), so by Corollary A.12, a normal subsystem of index prime to p of $(S_p, \widetilde{\mathcal{F}}_p)$. This computations also show that $\Gamma_{p'}(\widetilde{\mathcal{F}}_p) \cong \mathbb{Z}/2$, where $\Gamma_{p'}(\widetilde{\mathcal{F}}_p)$ is the group considered in Theorem A.10. This tells that all the possible saturated fusion subsystems of index prime to p are in bijective correspondence with the subgroups of $\mathbb{Z}/2$, which implies that there is only $(S_p, \mathcal{F}_p, \mathcal{L}_p)$ and the total.

Part (c) for $\mathcal{L} = \mathcal{L}_p$ follows immediately from [16, Proposition 3.3], since $\mathcal{L}_p$ is simple by part (a). Since this argument does not apply to $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$, we include a variation of the proof of [16, Proposition 3.3] that works for both $(S_p, \mathcal{F}_p, \mathcal{L}_p)$ and $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$.

Let $(S_p, \mathcal{F}, \mathcal{L})$ denote either $(S_p, \mathcal{F}_p, \mathcal{L}_p)$ or $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$, and also suppose that $\pi_1(|\mathcal{L}|_p^\wedge) \neq \{e\}$. By the Hyperfocal Subgroup Theorem [16, Theorem B.5], we have

$$\pi_1(|\mathcal{L}|_p^\wedge) = S_p / O^p_{\mathcal{F}}(S_p),$$

where $O^p_{\mathcal{F}}(S_p)$ is the hyperfocal subgroup defined in Definition 2.4. By [16, B.12-B.13], it follows that $O^p_{\mathcal{F}}(S_p) \lneqq S_p$ is a proper strongly $\mathcal{F}$-closed subgroup, which is impossible. $\square$

4.2. **On the exoticness of p-local compact groups.** In this subsection we prove that there does not exist any compact Lie group or p-compact group realizing any of the fusions $(S_p, \mathcal{F}_p)$ and $(S_p, \widetilde{\mathcal{F}}_p)$ described in Theorem 4.3.

We start proving that there does not exist any p-compact group with this fusion. In particular, as a connected compact Lie group corresponds to a p-compact group, this also shows that there does not exist any connected compact Lie group realizing any of these fusions.

Let us fix first the usual definitions and notations when working with p-compact groups (we refer to W. Dwyer and C. Wilkerson papers [12] and [13] for more details): a p-compact group is a triple (X, BX, e) where X is a space such that $H^*(X; \mathbb{F}_p)$ is finite, BX a pointed p-complete space and $e \colon X \to \Omega(BX)$ is a homotopy equivalence. We refer to X as a p-compact group and BX and e are assumed. If X and Y are p-compact groups, a homomorphism $f \colon X \to Y$ is a pointed map $Bf \colon BX \to BY$. Two homomorphisms $f, f' \colon X \to Y$ are conjugate if Bf and Bf' are freely homotopic.

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group, and let $P \leq S$ be a fully $\mathcal{F}$-centralized subgroup. In this situation, there is a well-defined notion of centralizer p-local compact group of P, denoted by $(C_S(P), C_{\mathcal{F}}(P), C_{\mathcal{L}}(P))$, see [8, Section 2] for the explicit definition and properties. The following proposition describes the relation between algebraic centralizers and mapping spaces for p-local compact groups associated to p-compact groups.

Proposition 4.6. *Let $(S, \mathcal{F}, \mathcal{L})$ a p-local compact group such that $|\mathcal{L}|^\wedge_p \simeq BX$, where (X, BX, e) is a p-compact group. Then:*

(a) *X is connected if and only if all the elements in S are $\mathcal{F}$-conjugate to elements in the maximal torus.*
(b) *Let P be a fully $\mathcal{F}$-centralized subgroup of S. Let also $(C_S(P), C_{\mathcal{F}}(P), C_{\mathcal{L}}(P))$ be the centralizer p-local compact group of P, and let $Bf|_{BP}\colon BP \to |\mathcal{L}|^\wedge_p$ be the composition of the inclusion $BP \to BS$ with the map $Bf\colon BS \to |\mathcal{L}|^\wedge_p$. Then $\mathrm{Map}(BP, |\mathcal{L}|^\wedge_p)_{Bf|_{BP}} \simeq |C_{\mathcal{L}}(P)|^\wedge_p$.*

Proof. We prove first (a): by [7, Proposition 10.1] we can consider $f\colon S \to X$ a maximal discrete p-toral subgroup. Consider $T \subset S$ the inclusion of the maximal torus in S. Define $i\colon T \to X$ the composition of the inclusion and f. Recall that the fusion system over S corresponding to X, which is denoted by $\mathcal{F}_{S,f}(X)$ is defined as:

$$\mathrm{Mor}_{\mathcal{F}_{S,f}(X)}(P, Q) \stackrel{\text{def}}{=} \{\varphi \in \mathrm{Hom}(P, Q) \mid Bf|_{BQ} \circ B\varphi \simeq Bf|_{BP}\}.$$

Assume first that X is connected and let $x \in S$. The composition of the inclusion of $\langle x \rangle$ in S and f gives a monomorphism $g\colon \mathbb{Z}/p^n \to X$, where $p^n = |\langle x \rangle|$. By [13, Proposition 3.11], as X is connected, any morphism $f\colon \mathbb{Z}/p^n \to X$ extends to $\overline{g}\colon \mathbb{Z}/p^\infty \to X$. Applying now [12, Proposition 8.11] we get that there is $h\colon \mathbb{Z}/p^\infty \to T$ such that $i \cdot h$ is conjugate to $\overline{g}$. The restriction of h to $\mathbb{Z}/p^n$ gives a morphism $\varphi \in \mathrm{Mor}_{\mathcal{F}_{S,f}(X)}(\langle x \rangle, T)$.
If X is not connected, also by [13, Proposition 3.11], there exists $g\colon \mathbb{Z}/p^n \to X$ which does not extend to $\mathbb{Z}/p^\infty$. By the maximality of S, this map factors through $\widetilde{g}\colon \mathbb{Z}/p^n \to S$. Consider $x \stackrel{\text{def}}{=} \widetilde{g}(1) \in S$. This element cannot be conjugated to the maximal torus, otherwise, we would be able to extend $\widetilde{g}$ to a map from $\mathbb{Z}/p^\infty$, providing an extension of g.

To prove (b) consider first $(C_S(P), C_{\mathcal{F}}(P), C_{\mathcal{L}}(P))$, the p-local compact group defined as the centralizer of P in $(S, \mathcal{F}, \mathcal{L})$. This p-local compact group exists by [7] because P is fully $\mathcal{F}$-centralized.

Consider also $\mathrm{Map}(BP, |\mathcal{L}|^\wedge_p)_{Bf|_{BP}}$ as a p-compact group (here, to simplify the notation, we consider $Bf\colon BS \to |\mathcal{L}|^\wedge_p$ directly, using that there is a homotopy equivalence $BX \simeq |\mathcal{L}|^\wedge_p$), which will have a p-local compact group structure which we denote as $(S', \mathcal{F}', \mathcal{L}')$. This means that $|\mathcal{L}'|^\wedge_p \simeq \mathrm{Map}(BP, |\mathcal{L}|^\wedge_p)_{Bf|_{BP}}$.

By [7, Proposition 10.4 (c)], if P is fully centralized $C_S(P)$ is a Sylow p-Subgroup of $\mathrm{Map}(BP, |\mathcal{L}|^\wedge_p)_{Bf|_{BP}}$, so we can consider $S' = C_S(P)$.

So it remains to prove that for all Q, Q' subgroups of $C_S(P)$, the morphism $\varphi\colon Q \to Q'$ belongs to $\mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q, Q')$ if and only if φ belongs to $\mathrm{Hom}_{\mathcal{F}'}(Q, Q')$:

A group morphism $\varphi\colon Q \to Q'$ belongs to $\mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q, Q')$ if there exists $\widetilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, PQ)$ such that $\widetilde{\varphi}|_Q = \varphi$ and $\widetilde{\varphi}|_P = \mathrm{Id}_P$. But, by definition of $\mathcal{F}$, as a fusion system corresponding to a p-compact group X, this is equivalent to $Bf|_{BPQ} \simeq Bf|_{BPQ'} \circ B\widetilde{\varphi}$ (see diagram $(**)$ below).

To understand what it means for φ to belong to $\mathcal{F}'$, we must deal with the inclusion of the Sylow p-subgroups $BC_S(P)$ in $\mathrm{Map}(BP, |\mathcal{L}|^\wedge_p)_{Bf|_{BP}}$, which we denote by Bg. Now $\varphi \in \mathcal{F}'$ if and only if $Bg|_{BQ} \simeq Bg|_{BQ'} \circ B\varphi$. And, considering adjoin maps, this is equivalent to verify that the composition $Bf|_{BPQ} \circ B\mu$ and $Bf_{BPQ'} \circ B\mu' \circ \mathrm{Id} \times B\varphi$ are homotopy equivalent, where $\mu\colon P \times Q \to PQ$ and $\mu'\colon P \times Q' \to PQ'$ are the corresponding multiplications (see diagram $(***)$ below).

So, we have the following three diagrams:

$$\begin{tikzcd}
BP \times BQ \arrow[r, "\mathrm{Id} \times B\varphi"] \arrow[d, "B\mu"'] \arrow[dr, phantom, "(*)"] & BP \times BQ' \arrow[d, "B\mu'"] \\
BPQ \arrow[r, "B\widetilde{\varphi}"'] & BPQ'
\end{tikzcd}
\qquad
\begin{tikzcd}
BPQ \arrow[rr, "B\widetilde{\varphi}"] \arrow[dr, "Bf|_{BPQ}"'] & (**) & BPQ' \arrow[dl, "Bf|_{BPQ'}"] \\
& |\mathcal{L}|^\wedge_p &
\end{tikzcd}$$

$$\begin{tikzcd}
BP \times BQ \arrow[rr, "\mathrm{Id} \times B\varphi"] \arrow[d, "B\mu"'] & & BP \times BQ' \arrow[d, "B\mu'"] \\
BPQ \arrow[dr, "Bf|_{BPQ}"'] & (***) & BPQ' \arrow[dl, "Bf|_{BPQ'}"] \\
& |\mathcal{L}|^\wedge_p &
\end{tikzcd}$$

Where:

- For any $\varphi \in \mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q, Q')$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}'}(Q, Q')$, we can construct the commutative diagram $(*)$. In fact, in both cases, it can be done at the level of groups.
- Once we have the commutative diagram $(*)$, φ belongs to $C_{\mathcal{F}}(P)$ if and only if diagram $(**)$ is homotopy commutative. In this case, we have that $(***)$ is also homotopy commutative and we get that $\varphi \in \mathcal{F}'$.
- If φ is a morphism in $\mathcal{F}'$, then diagram $(***)$ is homotopy commutative. We want to see that this implies that $(**)$ is also homotopy commutative. For that, consider K to be the kernel of μ. The map from $BK \to |\mathcal{L}|^\wedge_p$ which is the composition $Bf|_{BPQ} \circ B\mu|_{BK}$ is a central map in $|\mathcal{L}|^\wedge_p$, so $\mathrm{Map}(BK, |\mathcal{L}|^\wedge_p)_{Bf|_{BPQ} \circ B\mu|_{BK}} \simeq |\mathcal{L}|^\wedge_p$ (here we are using that $|\mathcal{L}|^\wedge_p$ is the classifying space of a p-compact group). So, we can apply Zabrodsky Lemma as stated in [11, Proposition 3.5] and we get that $B\mu$ induce a equivalence $\mathrm{Map}(BPQ, |\mathcal{L}|^\wedge_p) \to \mathrm{Map}(BP \times BQ, |\mathcal{L}|^\wedge_p)_{[Bf|_{BPQ} \circ B\mu]}$. The class $[Bf|_{BPQ}] \in \pi_0(\mathrm{Map}(BPQ, |\mathcal{L}|^\wedge_p))$ corresponds to $[Bf|_{BPQ} \circ B\mu] \in \pi_0(\mathrm{Map}(BP \times BQ, |\mathcal{L}|^\wedge_p))$, and the class $[Bf|_{BPQ'} \circ B\widetilde{\varphi}]$ corresponds to $[Bf|_{BPQ'} \circ B\widetilde{\varphi} \circ B\mu]$. Using diagrams $(*)$ and $(***)$, we have that the following classes in $\pi_0(\mathrm{Map}(BP \times BQ, |BK|^\wedge_p))$ are the same:

$$[Bf|_{BPQ'} \circ B\widetilde{\varphi} \circ B\mu] = [Bf|_{BPQ'} \circ B\mu' \circ \mathrm{Id} \times B\varphi] = [Bf|_{BPQ} \circ B\mu].$$

  This implies that $(**)$ is also homotopy commutative and $\varphi \in C_{\mathcal{F}}(P)$.

This finishes the proof. □

Remark 4.7. In [15, Theorem D] the first author proves a more general version of Proposition 4.6 (b). The proof we give above is independent from [15].

Theorem 4.8. *There does not exist any p-compact group X such that $X \simeq |\mathcal{L}|_p^\wedge$ for $\mathcal{L}$ any of the examples in Theorem 4.3.*

Before the proof of the theorem we need a lemma which follows from a result by K. Ishiguro [18, Proposition 3.1]

Lemma 4.9. *Let p be a prime number and H a finite non p-nilpotent group acting on a torus T. Then, there does not exist any p-compact group realizing the fusion system of $T \rtimes H$ over the prime p.*

Proof. Consider the compact Lie group $G \stackrel{\text{def}}{=} T \rtimes H$. By [7, Theorem 9.10] there is a p-local compact group $(S, \mathcal{F}, \mathcal{L})$ with $\mathcal{F}$ the fusion system of G over a Sylow p-subgroup S and $|\mathcal{L}|_p^\wedge \simeq BG_p^\wedge$. Assume there is a p-compact group X realizing also the p-local compact group $(S, \mathcal{F}, \mathcal{L})$. Then, by [7, Theorem 10.7], $|\mathcal{L}|_p^\wedge \simeq BX$, hence $BG_p^\wedge \simeq BX$. In this case, by [18, Proposition 3.1], the group of components of G must be a p-nilpotent group, in contradiction with the hypothesis in H. □

Proof of Theorem 4.8. For short, in this proof we denote by $(S_p, \mathcal{F}, \mathcal{L})$ any of the examples in Theorem 4.3. Assume that there exists some p-compact group X such that $BX \simeq |\mathcal{L}|_p^\wedge$ and we will get a contradiction.

Let Z be the centre of S_p, which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. By definition, it is a fully centralized subgroup, so we can construct the centralizer of Z in $(S_p, \mathcal{F}, \mathcal{L})$, which is again a p-local compact group that we denote by $(S_p, C_{\mathcal{F}}(Z), C_{\mathcal{L}}(Z))$. We can compute $C_{\mathcal{F}}(Z)$ and we get that it corresponds to the saturated fusion system of the group $T_3 \rtimes \Sigma_3$ for $p = 3$ and $\mathcal{F} = \mathcal{F}_3$, $T_p \rtimes A_p$ for $p \geq 5$ and $\mathcal{F} = \mathcal{F}_p$, and $T_p \rtimes \Sigma_p$ for $\mathcal{F} = \widetilde{\mathcal{F}}_p$ and $p \geq 5$. As neither Σ_p for $p \geq 3$, nor A_p for $p \geq 5$ are p-nilpotent, it follows by Lemma 4.9 that none of these is the fusion system of a p-compact group.

If we denote by $C_X(Z)$ the centralizer in X of the composition of maps $Z \hookrightarrow S_p \to X$ we have that $C_X(Z)$ is again a p-compact group by [12]. But by Proposition 4.6 (b), $C_X(Z) \simeq |C_{\mathcal{L}}(Z)|_p^\wedge$, so $(S_p, C_{\mathcal{F}}(Z), C_{\mathcal{L}}(Z))$ is the fusion system of a p-compact group, getting a contradiction with the previous paragraph. □

Until now we have proved that the p-local compact groups described in Theorem 4.3 cannot be realized by p-compact groups. This result includes the impossibility of these p-local compact groups to be realized by compact Lie groups whose group of components is a p-group. In order to prove that the p-local compact groups of Theorem 4.3 are not realized by any compact Lie group, it remains to eliminate the case of compact Lie groups whose group of components is not a p-group.

Theorem 4.10. *There does not exist any compact Lie group realizing the p-local compact groups of Theorem 4.3.*

Proof. Assume there is a compact Lie group G such that $\mathcal{F}_p \cong \mathcal{F}_S(G)$ for $S \in \mathrm{Syl}_p(G)$, and let $G_0 \trianglelefteq G$ be the connected component of the identity in G. By Lemma 2.2, $\mathcal{F}_S(G_0) \trianglelefteq \mathcal{F}_S(G)$, but, since $\mathcal{F}_p$ is a simple saturated fusion system by Proposition 4.5(a), we must have $\mathcal{F}_S(G_0) \cong \mathcal{F}_S(G)$. This is impossible since a connected compact Lie group gives rise to a p-compact group, and the fusion system is not realized by any p-compact group by Theorem 4.8.

Assume now that there is a compact Lie group $\widetilde{G}$ such that $\widetilde{\mathcal{F}}_p \cong \mathcal{F}_S(\widetilde{G})$ for $S \in \mathrm{Syl}_p(\widetilde{G})$, and let $\widetilde{G}_0 \trianglelefteq \widetilde{G}$ be the connected component of the identity in $\widetilde{G}$.

Again, by Lemma 2.2, $\mathcal{F}_S(\widetilde{G}_0) \trianglelefteq \mathcal{F}_S(\widetilde{G})$. We know, by Proposition 4.5(b), that $(S_p, \mathcal{F}_p, \mathcal{L}_p)$ is the only proper normal subsystem of $(S_p, \widetilde{\mathcal{F}}_p, \widetilde{\mathcal{L}}_p)$, so in this case we must have $\mathcal{F}_S(\widetilde{G}_0) \cong \mathcal{F}_p$ or $\mathcal{F}_S(\widetilde{G}_0) \cong \widetilde{\mathcal{F}}_p$, but we have proved in Theorem 4.8 that there is no p-compact group realizing any of these two fusion systems, hence $\widetilde{G}$ cannot exist. □

Appendix A. Fusion subsystems of index prime to p in p-local compact groups

In this section we generalize to the compact case the results in [4] about detection of subsystems of index prime to p of a given fusion system (see Definition 2.4). We also show that the *minimal* subsystem of index prime to p is always a normal subsystem. For the sake of completion we also reproduce here some definitions and results from [16, Appendix B] that we will need, but omitting those proofs that already appeared there.

We use the following notation from [4]. Given a discrete p-toral group S, a *restrictive category* over S is a category $\mathcal{F}^*$ such that $\mathrm{Ob}(\mathcal{F}^*)$ is the set of all subgroups of S, morphisms in $\mathcal{F}^*$ are group monomorphisms, and satisfying the following two properties:

(i) for each $P' \leq P \leq S$ and $Q' \leq Q \leq S$, and for each $f \in \mathrm{Hom}_{\mathcal{F}^*}(P,Q)$ such that $f(P') \leq f(Q')$, the restriction $f|_{P'}$ is a morphism in $\mathrm{Hom}_{\mathcal{F}^*}(P',Q')$ and
(ii) for each $P \leq S$, $\mathrm{Aut}_{\mathcal{F}}(P)$ is an Artinian locally finite group.

For a (discrete) group Γ, let $\mathfrak{Sub}(\Gamma)$ denote the set of nonempty subsets of Γ. Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. A subgroup $P \leq S$ is $\mathcal{F}$*-quasicentric* if, for all $Q \in P^{\mathcal{F}}$, the centralizer fusion system $C_{\mathcal{F}}(Q)$ is the fusion system of $C_S(Q)$. Given a subset $\mathcal{H} \subseteq \mathrm{Ob}(\mathcal{F})$, we denote by $\mathcal{F}_{\mathcal{H}} \subseteq \mathcal{F}$ the full subcategory with object set $\mathcal{H}$. For simplicity, $\mathcal{F}^q \subseteq \mathcal{F}$ denotes the full subcategory with object set the family of all $\mathcal{F}$-quasicentric subgroups of S.

The main tool to detect subsystems of a given fusion system are the so-called fusion mapping triples, defined below, which were already generalized from [4] to the context of p-local compact groups in [16, Appendix B].

Definition A.1. Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let $\mathcal{H} \subseteq \mathrm{Ob}(\mathcal{F})$ be a subset of $\mathcal{F}$-quasicentric subgroups which is closed by conjugation in $\mathcal{F}$. A *fusion mapping triple* for $\mathcal{F}_{\mathcal{H}}$ consists of a triple (Γ, θ, Θ), where Γ is a discrete group, $\theta\colon S \to \Gamma$ is a homomorphism, and

$$\Theta\colon \mathrm{Mor}(\mathcal{F}_{\mathcal{H}}) \longrightarrow \mathfrak{Sub}(\Gamma)$$

is a map satisfying the following properties for all subgroups $P, Q, R \in \mathcal{H}$:

(i) for all $P \xrightarrow{f} Q \xrightarrow{f'} R$ in $\mathcal{F}$ and all $x \in \Theta(f')$, $\Theta(f' \circ f) = x \cdot \Theta(f)$;
(ii) if P is fully $\mathcal{F}$-centralized, then $\Theta(\mathrm{Id}_P) = \theta(C_S(P))$;
(iii) if $f = c_g \in \mathrm{Hom}_S(P,Q)$, then $\theta(g) \in \Theta(f)$;
(iv) for all $f \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, all $x \in \Theta(f)$ and all $g \in P$, $x \cdot \theta(g) \cdot x^{-1} = \theta(f(g))$;
(v) $\Theta(\mathrm{Id}_P)$ is a subgroup of Γ, and Θ restricts to a homomorphism

$$\Theta_P\colon \mathrm{Aut}_{\mathcal{F}}(P) \longrightarrow N_{\Gamma}(\Theta(\mathrm{Id}_P))/\Theta(\mathrm{Id}_P).$$

In particular $\Theta_P(f) = \Theta(f)$ (as a coset of $\Theta(\mathrm{Id}_P)$) for all $f \in \mathrm{Aut}_{\mathcal{F}}(P)$;

(vi) for all $P \xrightarrow{f} Q \xrightarrow{f'} R$ in $\mathcal{F}_{\mathcal{H}}$ and all $x \in \Theta(f)$, $\Theta(f' \circ f) \subseteq \Theta(f') \cdot x$, with equality if $f(P) = Q$. In particular $\Theta(f'|_P) \supseteq \Theta(f')$ if $P \leq Q$ and

(vii) if $S \in \mathcal{H}$, then for all $P \xrightarrow{f} Q$ in $\mathcal{F}$, all $\gamma \in \mathrm{Aut}_{\mathcal{F}}(S)$ and all $x \in \Theta(\gamma)$, there is an equality $\Theta(\gamma \circ f \circ \gamma^{-1}) = x \cdot \Theta(f) \cdot x^{-1}$.

When constructing fusion mapping triples we may have to deal with infinitely many conjugacy classes of subgroups of S. The *bullet functor* $(-)^{\bullet}$ defined in [7, Section 3] is the tool to reduce to situations involving only finitely many $\mathcal{F}$-conjugacy classes. Given a full subcategory (or a fusion subsystem) $\mathcal{F}_0 \subseteq \mathcal{F}$, we denote by $\mathcal{F}_0^{\bullet} \subseteq \mathcal{F}_0$ the full subcategory whose objects are the subgroups $P \in \mathrm{Ob}(\mathcal{F}_0)$ such that $P = P^{\bullet}$.

Lemma A.2. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S.*

(i) *Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be a full subcategory, closed by $\mathcal{F}$-conjugacy and over-groups, and let (Γ, θ, Θ) be a fusion mapping triple for $\mathcal{F}_0$. Then $(\Gamma, \theta, \Theta^{\bullet})$ is a fusion mapping triple for $\mathcal{F}_0^{\bullet}$, with $\Theta^{\bullet} = \Theta \circ \mathrm{incl}$.*

(ii) *Let $\mathcal{F}_0^{\bullet} \subseteq \mathcal{F}^{\bullet}$ be a full subcategory which is closed under $\mathcal{F}$-conjugacy and over-groups (in $\mathcal{F}^{\bullet}$), and let $\mathcal{F}_0 \subseteq \mathcal{F}$ be the greatest subcategory such that $(P)^{\bullet} \in \mathcal{F}_0^{\bullet}$ for all $P \in \mathrm{Ob}(\mathcal{F}_0)$, and such that*

$$\{f^{\bullet} \mid f \in \mathrm{Mor}(\mathcal{F}_0)\} = \mathrm{Mor}(\mathcal{F}_0^{\bullet}).$$

Let also (Γ, θ, Θ) be a fusion mapping triple on $\mathcal{F}^{\bullet}$. Then, $(\Gamma, \theta, \Theta^{\circ})$ is a fusion mapping triple for $\mathcal{F}_0$, with $\Theta^{\circ} = \Theta \circ (-)^{\bullet}$.

Proof. This is an immediate consequence of the properties of $(-)^{\bullet}$. □

The following result is the key to inductively construct fusion mapping triples for a given fusion system.

Lemma A.3. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, $\mathcal{H}_0 \subseteq \mathrm{Ob}(\mathcal{F}^{\bullet q})$ be a subset closed by $\mathcal{F}$-conjugacy and over-groups (in $\mathcal{F}^{\bullet}$) and $\mathcal{P}$ be an $\mathcal{F}$-conjugacy class in $\mathcal{F}^{\bullet q}$ maximal among those not contained in $\mathcal{H}_0$. Set $\mathcal{H} = \mathcal{H}_0 \bigcup \mathcal{P}$ and let $\mathcal{F}_{\mathcal{H}_0} \subseteq \mathcal{F}_{\mathcal{H}} \subseteq \mathcal{F}^{\bullet q}$ be the corresponding full subcategories. Fix a group Γ and a homomorphism $\theta \colon S \to \Gamma$, and let*

$$\Theta \colon \mathrm{Mor}(\mathcal{F}_{\mathcal{H}_0}) \longrightarrow \mathfrak{Sub}(\Gamma)$$

be such that (Γ, θ, Θ) is a fusion mapping triple for $\mathcal{F}_{\mathcal{H}_0}$.

For $P \in \mathcal{P}$, a fully $\mathcal{F}$-normalized subgroup, fix a homomorphism

$$\Theta_P \colon \mathrm{Aut}_{\mathcal{F}}(P) \longrightarrow N_{\Gamma}(\theta(C_S(P)))/\theta(C_S(P))$$

such that the following conditions hold:

(a) *$x \cdot \theta(f) \cdot x^{-1} = \theta(f(g))$ for all $g \in P$, $f \in \mathrm{Aut}_{\mathcal{F}}(P)$ and $x \in \Theta_P(f)$ and*

(b) *$\Theta_P(f) \supseteq \Theta(f')$ for all $P \lneqq Q \leq S$ such that $P \lhd Q$ and Q is fully $\mathcal{F}$-normalized, and for all $f \in \mathrm{Aut}_{\mathcal{F}}(P)$ and $f' \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $f = f'|_P$.*

Then, there exists a unique extension of Θ to a fusion mapping triple $(\Gamma, \theta, \widetilde{\Theta})$ on $\mathcal{F}_{\mathcal{H}}$ such that $\widetilde{\Theta}(f) = \Theta_P(f)$ for all $f \in \mathrm{Aut}_{\mathcal{F}}(P)$.

Proof. This is [16, Lemma B.9] with minor modifications to restrict to $\mathcal{F}^{\bullet q}$. □

Lemma A.4. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let (Γ, θ, Θ) be a fusion mapping triple on $\mathcal{F}^c$. Then there is a unique extension*

$$\widetilde{\Theta} \colon \mathrm{Mor}(\mathcal{F}^q) \longrightarrow \mathfrak{Sub}(\Gamma)$$

of Θ such that $(\Gamma, \theta, \widetilde{\Theta})$ is a fusion mapping triple on $\mathcal{F}^q$.

Proof. By Lemma A.2, we can restrict the fusion mapping triple (Γ, θ, Θ) to a fusion mapping triple $(\Gamma, \theta, \Theta^\bullet)$ on $\mathcal{F}^{\bullet c}$. To prove the above statement we now extend $(\Gamma, \theta, \Theta^\bullet)$ to a fusion mapping triple for $\mathcal{F}^{\bullet q}$, which will in turn extend to a fusion mapping triple for $\mathcal{F}^q$ by Lemma A.2 again. Since there is no place for confusion we denote $\Theta^\bullet$ simply by Θ.

Let then $\mathcal{H}_0 \subseteq \mathrm{Ob}(\mathcal{F}^{\bullet q})$ be a set closed under $\mathcal{F}$-conjugacy and over-groups (in $\mathcal{F}^{\bullet q}$), and such that it contains $\mathrm{Ob}(\mathcal{F}^{\bullet c})$, and let $\mathcal{P}$ be a conjugacy class in $\mathcal{F}^{\bullet q}$, maximal among those not in $\mathcal{H}_0$. We want to extend Θ to $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{P}$.

Let $P \in \mathcal{P}$ be fully $\mathcal{F}$-normalized. For each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$, there is an extension $\beta \in \mathrm{Aut}_{\mathcal{F}}(R)$, where $R = P{\cdot}C_S(P)$, which in turn induces a unique $\beta^\bullet \in \mathrm{Aut}_{\mathcal{F}}(R^\bullet)$. Furthermore, by [7, Proposition 2.7] both R and $R^\bullet$ are $\mathcal{F}$-centric (because P is fully $\mathcal{F}$-normalized), and in particular $R^\bullet \in \mathcal{H}_0$. Thus we can define a map

$$\Theta_P\colon\ \mathrm{Aut}_{\mathcal{F}}(P) \longrightarrow \mathfrak{Sub}(N_\Gamma(\theta(C_S(P))))$$

by the formula $\Theta_P(\alpha) = \Theta(\beta^\bullet) \cdot \theta(C_S(P))$. By properties (i) and (ii) of fusion mapping triples, $\Theta(\beta^\bullet)$ is a left coset of $\theta(C_S(R))$ (because $Z(R) = Z(R^\bullet)$ by [7, Lemma 3.2 (d)]), and by (iv) it is also a right coset (where the left and right coset representatives can be chosen to be the same). Hence $\Theta_P(\alpha)$ is a left and right coset of $\theta(C_S(P))$.

If $\beta' \in \mathrm{Aut}_{\mathcal{F}}(P)$ is any other extension of α, then by [5, Lemma 3.8], which applies in this situation without restriction, there is some $g \in C_S(P)$ such that $\beta' = c_g \circ \beta$, and then $\Theta((\beta')^\bullet) = \Theta(c_g\,\beta^\bullet) = \theta(g)\Theta(\beta^\bullet)$, and

$$\begin{aligned}\Theta((\beta')^\bullet) \cdot \theta(C_S(P)) &= \theta(g) \cdot \Theta(\beta^\bullet) \cdot \theta(C_S(P)) = \\ &= \Theta(\beta^\bullet) \cdot \theta(\beta^\bullet(g)) \cdot \theta(C_S(P)) = \Theta(\beta^\bullet) \cdot \theta(C_S(P))\end{aligned}$$

and so the definition of $\Theta_P(\alpha)$ is independent of the choice of the extension of α. This shows that Θ_P is well defined.

Note also that Θ_P respects compositions and, since $\Theta_P(\alpha) = x \cdot \theta(C_S(P)) = \theta(C_S(P)) \cdot x$ for some $x \in \Gamma$, we conclude that $x \in N_\Gamma(\theta(C_S(P))$. Thus Θ_P induces a homomorphism

$$\Theta_P\colon\ \mathrm{Aut}_{\mathcal{F}}(P) \longrightarrow N_\Gamma(\theta(C_S(P)))/\theta(C_S(P)).$$

We can now apply Lemma A.3 to extend Θ to $\mathcal{H}$.

If $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ and $x \in \Theta_P(\alpha)$, then $x = y \cdot \theta(h)$ for some $h \in C_S(P)$ and $y \in \Theta(\beta^\bullet)$, where $\beta^\bullet$ is some extension of α to $R = P \cdot C_S(P)$. Hence, for any $g \in P$,

$$x\theta(g)x^{-1} = y\theta(hgh^{-1})y^{-1} = y\theta(g)y^{-1} = \theta(\beta^\bullet(g)) = \theta(\alpha(g)).$$

This shows that condition (i) in Lemma A.3 holds.

Assume now that $P \lneqq Q \leq N_S(P)$, and let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$, $\beta \in \mathrm{Aut}_{\mathcal{F}}(Q)$ be such that $\alpha = \beta|_P$. Then, in the notation of axiom (II) for saturated fusion systems, $Q \cdot C_S(P) \leq N_\alpha$, and hence α extends to some other $\gamma \in \mathrm{Aut}_{\mathcal{F}}(Q \cdot C_S(P))$, and

$$\Theta_P(\alpha) = \Theta(\gamma^\bullet) \cdot \theta(C_S(P))$$

by definition of Θ_P. By [5, Lemma 3.8], $\gamma|_Q = c_g \circ \beta$ for some $g \in C_S(P)$, and hence by definition of fusion mapping triple, $\Theta(\gamma^\bullet) = \Theta(c_g \circ \beta^\bullet) = \theta(g) \cdot \Theta(\beta^\bullet)$, and

$$\begin{aligned}\Theta_P(\alpha) &= \theta(g) \cdot \Theta(\beta^\bullet) \cdot \theta(C_S(P)) = \\ &= \Theta(\beta^\bullet) \cdot \theta(\beta^\bullet(g)) \cdot \theta(C_S(P)) = \Theta(\beta^\bullet) \cdot \theta(C_S(P)).\end{aligned}$$

In particular, $\Theta_P(\alpha) \supseteq \Theta(\beta^\bullet)$, and condition (ii) in Lemma A.3 also holds. $\square$

Let $\mathcal{F}$ be a saturated fusion system over S, and let (Γ, θ, Θ) be a fusion mapping triple for $\mathcal{F}_{\mathcal{H}}$, for some $\mathcal{H}$. For a subgroup $H \leq \Gamma$, let $\mathcal{F}_H^* \subseteq \mathcal{F}$ be the smallest restrictive subcategory which contains all $f \in \mathrm{Mor}(\mathcal{F}^q)$ such that $\Theta(f) \cap H \neq \emptyset$. Let also $\mathcal{F}_H \subseteq \mathcal{F}_H^*$ be the full subcategory whose objects are the subgroups of $\theta^{-1}(H)$.

The following result is a modification of [16, Proposition B.8] for groups of order prime to p (the statement in [16] dealed with p-groups). However the same proof applies here, so we omit it.

Proposition A.5. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S, and let (Γ, θ, Θ) be a fusion mapping triple on $\mathcal{F}^q$, where Γ is a finite group of order prime to p. Then the following holds for all $H \leq \Gamma$.*

(i) *$\mathcal{F}_H$ is a saturated fusion system over $S_H = \theta^{-1}(H)$.*
(ii) *A subgroup $P \leq S_H$ is $\mathcal{F}_H$-quasicentric if and only if it is $\mathcal{F}$-quasicentric.*

When Γ is a group of order prime to p, there is only one morphism possible from a discrete p-toral group S to Γ, the trivial one. The existence of a fusion mapping triple in this case is equivalent to the existence of a functor $\widehat{\Theta}\colon \mathcal{F}^q \to \mathcal{B}(\Gamma)$ such that $\Theta(f) = \{\widehat{\Theta}(f)\}$ for each $f \in \mathrm{Mor}(\mathcal{F}^q)$. This equivalent approach will be useful later on when constructing fusion mapping triples. The following generalization of [4, Proposition 2.6] will be needed later.

Proposition A.6. *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Then the induced map*

$$\pi_1(|\mathcal{L}|) \longrightarrow \pi_1(|\mathcal{F}^c|)$$

is surjective, and its kernel is generated by elements of p-power order.

Proof. The proof for the finite case applies verbatim in this situation. $\square$

Let G and H be (possibly infinite) groups. If $f\colon G \to H$ is an epimorphism with $\mathrm{Ker}(f) \leq O^{p'}(G)$, then there is an induced isomorphism $G/O^{p'}(G) \cong H/O^{p'}(H)$. In particular, given a p-local compact group $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$, Proposition A.6 implies an isomorphism

$$\Gamma_{p'}(\mathcal{F}) \stackrel{\text{def}}{=} \pi_1(|\mathcal{L}|)/O^{p'}(\pi_1(|\mathcal{L}|)) \cong \pi_1(|\mathcal{F}^c|)/O^{p'}(\pi_1(|\mathcal{F}^c|)). \tag{9}$$

We will show that the natural functor

$$\varepsilon\colon \mathcal{F}^c \longrightarrow \mathcal{B}(\Gamma_{p'}(\mathcal{F}))$$

induces a bijective correspondence between subgroups of $\Gamma_{p'}(\mathcal{F})$ and fusion subsystems of $\mathcal{F}$ of index prime to p. First we need to introduce some notation.

Definition A.7. Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S.

(i) $O^{p'}_*(\mathcal{F}) \subseteq \mathcal{F}$ is the smallest fusion subsystem over S (not necessarily saturated) whose morphism set contains $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \leq S$.
(ii) $\mathrm{Out}^0_{\mathcal{F}}(S) \leq \mathrm{Out}_{\mathcal{F}}(S)$ is the subgroup generated by the elements $[f] \in \mathrm{Out}_{\mathcal{F}}(S)$ such that $f|_P \in \mathrm{Mor}_{O^{p'}_*(\mathcal{F})}(P, S)$ for some $P \in \mathrm{Ob}(\mathcal{F}^c)$ and $f \in \mathrm{Aut}_{\mathcal{F}}(S)$ representing $[f]$.

Lemma A.8. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then*

(i) *$O_*^{p'}(\mathcal{F})$ is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$: for all $f \in \mathrm{Mor}(O_*^{p'}(\mathcal{F}))$ and all $\gamma \in \mathrm{Aut}_{\mathcal{F}}(S)$, we have*
$$\gamma \circ f \circ \gamma^{-1} \in \mathrm{Mor}(O_*^{p'}(\mathcal{F}));$$
(ii) *the fusion system $\mathcal{F}$ is generated by $O_*^{p'}(\mathcal{F})$ together with $\mathrm{Aut}_{\mathcal{F}}(S)$ and*
(iii) *$\mathrm{Out}^0_{\mathcal{F}}(S)$ is a normal subgroup of $\mathrm{Out}_{\mathcal{F}}(S)$.*

Proof. Parts (i) and (ii) follow from [4, Lemma 3.4], since the same proof applies here without modification (all the properties required have the necessary counterpart for fusion systems over discrete p-toral groups). Part (iii) follows then from part (i). □

Proposition A.9. *There is a unique functor $\widehat{\theta}\colon \mathcal{F}^c \to \mathcal{B}(\mathrm{Out}_{\mathcal{F}}(S)/\mathrm{Out}^0_{\mathcal{F}}(S))$ with the following properties:*

(i) *$\widehat{\theta}(f) = [f]$ for all $f \in \mathrm{Aut}_{\mathcal{F}}(S)$.*
(ii) *$\widehat{\theta}(f) = [\mathrm{Id}]$ if $f \in \mathrm{Mor}(O_*^{p'}(\mathcal{F})^c)$. In particular, $\widehat{\theta}$ sends inclusion morphisms to the identity.*

Furthermore, there is an isomorphism $\overline{\theta}\colon \Gamma_{p'}(\mathcal{F}) \xrightarrow{\cong} \mathrm{Out}_{\mathcal{F}}(S)/\mathrm{Out}^0_{\mathcal{F}}(S)$ such that $\widehat{\theta} = \mathcal{B}(\overline{\theta}) \circ \varepsilon$.

Proof. By Lemma A.8 (ii), there exist $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $f' \in \mathrm{Hom}_{O_*^{p'}(\mathcal{F})^c}(\alpha(P), Q)$ such that $f = f' \circ \alpha|_P$. Thus, if we have two such decompositions $f = f'_1 \circ (\alpha_1)|_P = f'_2 \circ (\alpha_2)|_P$, then (after factoring out inclusions) we have
$$(\alpha_2 \circ \alpha_1^{-1})|_P = (f'_2)^{-1} \circ f_1 \in \mathrm{Iso}_{O_*^{p'}(\mathcal{F})^c}(\alpha_1(P), \alpha_2(P)),$$
which implies that $\alpha_2 \circ \alpha_1^{-1} \in \mathrm{Out}^0_{\mathcal{F}}(S)$, and we can define
$$\widehat{\theta}(f) = [\alpha_1] = [\alpha_2] \in \mathrm{Out}_{\mathcal{F}}(S)/\mathrm{Out}^0_{\mathcal{F}}(S).$$
This also proves that $\widehat{\theta}$ is well defined on morphisms and maps all objects in $\mathcal{F}^c$ to the unique object of $\mathcal{B}(\mathrm{Out}_{\mathcal{F}}(S)/\mathrm{Out}^0_{\mathcal{F}}(S))$. By Lemma A.8 (ii) again, this functor preserves compositions, and thus is well defined. Furthermore, it satisfies conditions (i) and (ii) above by construction. The uniqueness of $\widehat{\theta}$ is clear.

Let us prove then the last part of the statement. Since $\mathrm{Out}_{\mathcal{F}}(S)/\mathrm{Out}^0_{\mathcal{F}}(S)$ is a finite p'-group, the morphism $\pi_1(|\widehat{\theta}|)$ factors through a homomorphism
$$\overline{\theta}\colon \pi_1(|\mathcal{F}^c|)/O^{p'}(\pi_1(|\mathcal{F}^c|)) \longrightarrow \mathrm{Out}_{\mathcal{F}}(S)/\mathrm{Out}^0_{\mathcal{F}}(S),$$
and the inclusion of $B\,\mathrm{Aut}_{\mathcal{F}}(S)$ into $|\mathcal{F}^c|$ (as a subcomplex with a single vertex S) induces then a homomorphism
$$\tau\colon\ \mathrm{Out}_{\mathcal{F}}(S) \longrightarrow \pi_1(|\mathcal{F}^c|)/O^{p'}(\pi_1(|\mathcal{F}^c|)).$$
Furthermore, τ is an epimorphism since $\mathcal{F}$ is generated by $O_*^{p'}(\mathcal{F})$ and $\mathrm{Aut}_{\mathcal{F}}(S)$, by Lemma A.8 (ii), and because every automorphism on $O_*^{p'}(\mathcal{F})$ is a composite of restrictions of automorphisms of p-power order.

By (i), the composite $\overline{\theta} \circ \tau$ is the projection of $\mathrm{Out}_{\mathcal{F}}(S)$ onto the quotient $\mathrm{Out}_{\mathcal{F}}(S)/\mathrm{Out}^0_{\mathcal{F}}(S)$, and $\mathrm{Out}^0_{\mathcal{F}}(S) \leq \mathrm{Ker}(\tau)$ by definition of $\mathrm{Out}^0_{\mathcal{F}}(S)$. Thus $\overline{\theta}$ is an isomorphism. □

Theorem A.10. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then, there is a bijective correspondence between*

$$\{H \leq \Gamma_{p'}(\mathcal{F}) = \operatorname{Out}_{\mathcal{F}}(S)/\operatorname{Out}^0_{\mathcal{F}}(S)\}$$

and the set of saturated fusion subsystems $\mathcal{F}_H \subseteq \mathcal{F}$ of index prime to p. The correspondence is given by associating to H the fusion system generated by $\big(\widehat{\theta}\big)^{-1}(\mathcal{B}(H))$, where $\widehat{\theta}$ is the functor in Proposition A.9.

Proof. Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be a saturated subsystem of index prime to p. That is, $\mathcal{F}_0$ is a saturated subsystem over S which contains $O^{p'}_*(\mathcal{F})$. Then $\operatorname{Out}^0_{\mathcal{F}}(S) \lhd \operatorname{Out}_{\mathcal{F}_0}(S)$, and we can set

$$H = \operatorname{Out}_{\mathcal{F}_0}(S)/\operatorname{Out}^0_{\mathcal{F}}(S) \leq \Gamma_{p'}(\mathcal{F}).$$

We have to show that this provides the bijection in the statement. We first show that a morphism $f \in \operatorname{Mor}(\mathcal{F}^c)$ is in $\mathcal{F}_0$ if and only if $\widehat{\theta}(f) \in H$, which in turn implies that

$$\mathcal{F}_0 = \big(\widehat{\theta}\big)^{-1}(\mathcal{B}(H)).$$

Clearly it is enough to prove this for isomorphisms in $\mathcal{F}^c$.

Let $P, Q \leq S$ be $\mathcal{F}$-centric, $\mathcal{F}$-conjugate subgroups, and fix $f \in \operatorname{Iso}_{\mathcal{F}}(P,Q)$. By Lemma A.8 we can write $f = f' \circ \alpha|_P$, where $\alpha \in \operatorname{Aut}_{\mathcal{F}}(S)$ and $f' \in \operatorname{Iso}_{O^{p'}_*(\mathcal{F})}(P,Q)$. Then f is in $\mathcal{F}_0$ if and only if $\alpha|_P$ is in $\mathcal{F}_0$. Also, by definition of $\widehat{\theta}$ (and also h), $\widehat{\theta}(f) \in H$ if and only if $\alpha \in \operatorname{Aut}_{\mathcal{F}_0}(S)$. Thus we have to show that $\alpha|_P \in \operatorname{Mor}(\mathcal{F}_0)$ if and only if $\alpha \in \operatorname{Aut}_{\mathcal{F}_0}(S)$.

The case when $\alpha \in \operatorname{Aut}_{\mathcal{F}_0}(S)$ is clear, so let us prove the converse. Note that $\alpha(P)$ is $\mathcal{F}_0$-centric, and hence fully $\mathcal{F}_0$-centralized. Since $\alpha|_P$ extends to an (abstract) automorphism of S, axiom (II) implies that it extends to some $\alpha_1 \in \operatorname{Hom}_{\mathcal{F}_0}(N_S(P), S)$. By [7, Proposition 2.8],

$$\alpha_1 = (\alpha|_{N_S(P)}) \circ c_g$$

for some $g \in Z(P)$, and hence $\alpha|_{N_S(P)} \in \operatorname{Hom}_{\mathcal{F}_0}(N_S(P), S)$. Furthermore, $P \lneqq N_S(P)$ since $P \lneqq S$ by hypothesis. Applying this process repeatedly it follows that $\alpha \in \operatorname{Aut}_{\mathcal{F}_0}(S)$.

Now, fix a subgroup $H \leq \Gamma_{p'}(\mathcal{F})$, and let $\mathcal{F}_H$ be the smallest fusion system over S which contains $\big(\widehat{\theta}\big)^{-1}(\mathcal{B}(H))$. We show then that $\mathcal{F}_H$ is a saturated fusion subsystem of $\mathcal{F}$ of index prime to p. Let $P, Q \leq S$ be $\mathcal{F}$-centric subgroups, and note that $\operatorname{Hom}_{\mathcal{F}_H}(P,Q)$ is the set of all morphisms $f \in \operatorname{Hom}_{\mathcal{F}}(P,Q)$ such that $\widehat{\theta}(f) \in H$. Thus, in particular $\mathcal{F}_H$ contains $O^{p'}_*(\mathcal{F})$ because all morphisms in $O^{p'}_*(\mathcal{F})$ are sent by $\widehat{\theta}$ to the identity.

Define then a map $\Theta\colon \operatorname{Mor}(\mathcal{F}^c) \to \mathfrak{Sub}(\Gamma_{p'}(\mathcal{F}))$ by setting $\Theta(f) = \{\widehat{\theta}(f)\}$, that is, each image is a singleton. Let also $\theta \in \operatorname{Hom}(S, \Gamma_{p'}(\mathcal{F}))$ be the trivial homomorphism. Then it follows that $(\Gamma_{p'}(\mathcal{F}), \theta, \Theta)$ is a fusion mapping triple of $\mathcal{F}^c$ which, by Lemma A.4 extends to a unique fusion mapping triple of $\mathcal{F}^q$. Thus $\mathcal{F}_H$ is saturated by Proposition A.5.

By Alperin's Fusion Theorem, [7, Theorem 3.6], $\mathcal{F}_H$ is the unique saturated fusion subsystem of $\mathcal{F}$ with the property that a morphism $f \in \operatorname{Hom}_{\mathcal{F}}(P,Q)$ between $\mathcal{F}$-centric subgroups of S lies in $\mathcal{F}_H$ if and only if $\widehat{\theta}(f) \in H$. This shows that the correspondence is bijective. $\square$

Remark A.11. Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Then there is a minimal fusion subsystem $O^{p'}(\mathcal{F}) \subseteq \mathcal{F}$ of index prime to p, corresponding to the trivial subgroup $\{1\} \leq \Gamma_{p'}(\mathcal{F})$. By [19, Theorem B], $O^{p'}(\mathcal{F})$ has a unique associated centric linking system $O^{p'}(\mathcal{L})$ (up to isomorphism), and thus there is a p-local compact group $O^{p'}(\mathcal{G}) = (S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$.

Corollary A.12. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then $O^{p'}(\mathcal{F})$ is a normal subsystem of $\mathcal{F}$.*

Proof. Since $O^{p'}(\mathcal{F})$ is a saturated fusion subsystem over S, conditions (N1) and (N3) in Definition 2.1 follow immediately. Also condition (N4) is immediate since $S = S \cdot C_S(S)$. Finally, condition (N2) is a consequence of the following. The fusion mapping triple (Γ, θ, Θ) associated to $O^{p'}(\mathcal{F})$ corresponds to a functor $\widehat{\Theta}\colon \mathcal{F}^q \to \mathcal{B}(\Gamma)$ which sends the morphisms in $O^{p'}(\mathcal{F})$ to the trivial automorphism. □

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