Tensor products and regularity properties of Cuntz semigroups

Ramon Antoine

Francesc Perera

Hannes Thiel

R. Antoine, Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
E-mail address: `ramon@mat.uab.cat`

F. Perera, Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain
E-mail address: `perera@mat.uab.cat`

H. Thiel, Mathematisches Institut, Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany
E-mail address: `hannes.thiel@uni-muenster.de`

Prepublicació Núm 17, octubre 2014.
Departament de Matemàtiques.
http://www.uab.cat/matematiques

UAB Universitat Autònoma de Barcelona

2010 *Mathematics Subject Classification.* Primary 06B35, 06F05, 15A69, 46L05. Secondary 06B30, 06F25, 13J25, 16W80, 16Y60, 18B35, 18D20, 19K14, 46L06, 46M15, 54F05.

Key words and phrases. Cuntz semigroup, tensor product, continuous posets, C^*-algebra

Abstract. The Cuntz semigroup of a C^*-algebra is an important invariant in the structure and classification theory of C^*-algebras. It captures more information than K-theory but is often more delicate to handle. We systematically study the lattice and category theoretic aspects of Cuntz semigroups.

Given a C^*-algebra A, its (concrete) Cuntz semigroup $\mathrm{Cu}(A)$ is an object in the category Cu of (abstract) Cuntz semigroups, as introduced by Coward, Elliott and Ivanescu in [**CEI08**]. To clarify the distinction between concrete and abstract Cuntz semigroups, we will call the latter Cu-semigroups.

We establish the existence of tensor products in the category Cu and study the basic properties of this construction. We show that Cu is a symmetric, monoidal category and relate $\mathrm{Cu}(A \otimes B)$ with $\mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B)$ for certain classes of C^*-algebras.

As a main tool for our approach we introduce the category W of precompleted Cuntz semigroups. We show that Cu is a full, reflective subcategory of W. One can then easily deduce properties of Cu from respective properties of W, e.g. the existence of tensor products and inductive limits. The advantage is that constructions in W are much easier since the objects are purely algebraic.

For every (local) C^*-algebra A, the classical Cuntz semigroup $W(A)$ together with a natural auxiliary relation is an object of W. This defines a functor from C^*-algebras to W which preserves inductive limits. We deduce that the assignment $A \mapsto \mathrm{Cu}(A)$ defines a functor from C^*-algebras to Cu which preserves inductive limits. This generalizes a result from [**CEI08**].

We also develop a theory of Cu-semirings and their semimodules. The Cuntz semigroup of a strongly self-absorbing C^*-algebra has a natural product giving it the structure of a Cu-semiring. For C^*-algebras, it is an important regularity property to tensorially absorb a strongly self-absorbing C^*-algebra. Accordingly, it is of particular interest to analyse the tensor products of Cu-semigroups with the Cu-semiring of a strongly self-absorbing C^*-algebra. This leads us to define 'solid' Cu-semirings (adopting the terminology from solid rings), as those Cu-semirings S for which the product induces an isomorphism between $S \otimes_{\mathrm{Cu}} S$ and S. This can be considered as an analog of being strongly self-absorbing for Cu-semirings. As it turns out, if a strongly self-absorbing C^*-algebra satisfies the UCT, then its Cu-semiring is solid.

If R is a solid Cu-semiring, then a Cu-semigroup S is a semimodule over R if and only if $R \otimes_{\mathrm{Cu}} S$ is isomorphic to S. Thus, analogous to the case for C^*-algebras, we can think of semimodules over R as Cu-semigroups that tensorially absorb R. We give explicit characterizations when a Cu-semigroup is such a semimodule for the cases that R is the Cu-semiring of a strongly self-absorbing C^*-algebra satisfying the UCT. For instance, we show that a Cu-semigroup S tensorially absorbs the Cu-semiring of the Jiang-Su algebra if and only if S is almost unperforated and almost divisible, thus establishing a semigroup version of the Toms-Winter conjecture.

In the course of our investigations, we often incorporate two additional axioms that do not appear in the initial description of the category Cu: these are the 'almost algebraic order axiom', introduced by Rørdam and Winter in [**RW10**], and the 'almost Riesz decomposition axiom', introduced by Robert in [**Rob13a**]. We identify a stronger version of the almost algebraic order axiom and establish that it is satisfied by all Cuntz semigroups of C^*-algebras. The advantage of our formulation is that, in this stronger form, the axiom passes to inductive limits in Cu.

Contents

CHAPTER 1

Introduction

1.1. Background

This paper is concerned with a number of regularity properties of Cuntz semigroups, which are invariants naturally associated to C^*-algebras. To put our results into perspective, we first review the C^*-motivation behind our work, as well as the importance of these semigroups in the context of the Elliott classification program.

1.1.1. The Elliott classification program. The Cuntz semigroup $\mathrm{W}(A)$ of a C^*-algebra A is an important invariant in the structure theory of C^*-algebras, particularly in connection with the classification program of simple, nuclear C^*-algebras initiated by George Elliott. In itself, nuclearity is a finite-dimensional approximation property that includes a large number of our stock-in-trade C^*-algebras.

The original Elliott Conjecture asserts that simple, separable, unital, nuclear C^*-algebras can be classified by an invariant of a K-theoretic nature. In the stably finite case, the invariant used is

$$\mathrm{Ell}(A) = ((K_0(A), K_0(A)^+, [1_A]), K_1(A), T(A), r_A)\,,$$

consisting of the ordered K_0-group, the topological K_1-group, the trace simplex $T(A)$ and the pairing between traces and projections.

Conjecture 1.1.1 (Elliott's Classification Conjecture). For C^*-algebras A and B as above, we have $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ if and only if $A \cong B$.

The Elliott program has had tremendous success in the classification of wide classes of algebras (see, e.g. [**Rør92**] and [**ET08**]). However, the first counterexamples to the conjecture as stated above appeared in the work of Rørdam ([**Rør03**]) and Toms ([**Tom05**]). Both examples allowed to repair the conjecture by adding a minimal amount of information to the invariant (in this case, the real rank). Soon after that though, Toms produced in [**Tom08**] two simple AH-algebras that agreed not only on the Elliott invariant, but also on a whole collection of topological invariants (among them the real and stable rank).

The distinguishing factor for the said algebras is precisely the Cuntz semigroup. This is an object that was introduced by Cuntz in [**Cun78**] as equivalence classes of positive elements in matrices over a C^*-algebra A, in very much the same way the semigroup $V(A)$ (as a precursor of $K_0(A)$) is constructed via Murray-von Neumann equivalence classes of projections in matrices over A.

Briefly, if A is a C^*-algebra and a and b are positive elements in A, set $a \precsim b$ if and only if $a = \lim_n x_n b x_n^*$ for some sequence $(x_n)_n$ in A. This can be extended to the directed union $M_\infty(A)$ of matrices over A and, setting $a \sim b$ if both $a \precsim b$ and $b \precsim a$ occur, one obtains a partially ordered set $W(A) = M_\infty(A)_+/{\sim}$. This set can be endowed with an addition by setting $[a] + [b] = [\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)]$, so that it becomes a partially ordered semigroup. It is worth noting that the relation $\precsim$, when restricted to projections, becomes the usual Murray-von Neumann subequivalence.

One of the key features of the Cuntz semigroup is its ordering, which is in general not algebraic. As a matter of fact, it is one order property – almost unperforation – that is used to distinguish the algebras mentioned above.

It is important to mention that the Cuntz semigroup for purely infinite, simple C^*-algebras is degenerate (it is equal to $\{0, \infty\}$) and hence its use is only of relevance in the stably finite case.

Many of the classes of algebras considered in the classification program admit an inductive limit decomposition, and hence it is desirable that any addition to the original Elliott invariant behaves well with respect to inductive limits. This is not the case of the Cuntz semigroup, when considered as an invariant from the category of C^*-algebras to the category of semigroups. This shortcoming can be remedied by passing to stable algebras and considering as a target category a suitable category Cu of ordered semigroups (see below). This was carried out by Coward, Elliott and Ivanescu in [**CEI08**], where they defined $\mathrm{Cu}(A)$ using Hilbert modules (and showed it is naturally isomorphic to $W(A \otimes K)$). In this way, the assignment $A \mapsto \mathrm{Cu}(A)$ defines a sequentially continuous functor.

To this date, there is no counterexample to the conjecture of whether the Elliott invariant, together with the Cuntz semigroup, constitutes a complete invariant for the class of unital, simple, separable, nuclear C^*-algebras. It is therefore natural to ask what is the largest possible class for which the Elliott Conjecture can be proved to hold.

It is important to point out that the Cuntz semigroup alone has become a useful tool in the classification of certain classes of nonsimple algebras. A remarkable instance of this situation is found in the work of Robert, [**Rob12**], where the Cuntz semigroup is used to classify, up to approximate unitary equivalence, *-homomorphisms out of an inductive limit of 1-dimensional noncommutative CW-complexes with trivial K_1-groups into a stable rank one algebra. As a consequence, Robert classifies all (not necessarily simple) inductive limits of 1-dimensional NCCW-complexes with trivial K_1-groups using the Cuntz semigroup.

1.1.2. Strongly self-absorbing C^*-algebras. $\mathcal{Z}$-stability. Toms and Winter, [**TW07**, Definition 1.3], termed a C^*-algebra D strongly self-absorbing if $D \neq \mathbb{C}$ and if there is an isomorphism $\varphi\colon D \to D \otimes D$ that is approximately unitarily equivalent to the inclusion in the first factor (or, as it turns out, in the second). Such C^*-algebras are automatically simple, nuclear (Effros-Rosenberg), and are either purely infinite or stably finite with a unique trace (Kirchberg). The only known examples of strongly self-absorbing C^*-algebras are: The Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, every UHF-algebra of infinite type, the tensor products $\mathcal{O}_\infty \otimes U$ where U is a UHF-algebra of infinite type, and the Jiang-Su algebra $\mathcal{Z}$. All these algebras satisfy the Universal Coefficient Theorem (UCT). The Elliott classification program predicts that they are in fact the only strongly self-absorbing C^*-algebras satisfying the UCT. It remains an important open problem to determine whether there is a strongly self-absorbing C^*-algebra outside the UCT class, as this would provide a nuclear, non-UCT C^*-algebra. If D is strongly self-absorbing, a C^*-algebra A is called D-stable provided that $A \cong A \otimes D$. Winter showed that all strongly self-absorbing C^*-algebras are $\mathcal{Z}$-stable (see [**Win11**] and also [**DR09**]). It follows from this result that $\mathcal{Z}$ becomes an initial object in the category of strongly self-absorbing C^*-algebras.

The Jiang-Su algebra $\mathcal{Z}$ has the same Elliott invariant as the complex numbers, and it has become prominent in the classification program. In fact, tensoring a C^*-algebra with $\mathcal{Z}$ is inert at the level of K-Theory and traces (although it may change the order of the K_0-group, except under some additional assumptions). It is thus reasonable to expect that classification can be achieved within the class of simple, separable, unital, nuclear, $\mathcal{Z}$-stable algebras. In this way, $\mathcal{Z}$-stability postulates itself as a regularity property for C^*-algebras.

All classes of simple, nuclear C^*-algebras for which the Elliott Conjecture has been verified consist of $\mathcal{Z}$-stable C^*-algebras, see [**TW08**]. One may therefore wonder what role the Cuntz semigroup plays in these results, if any. As proved in [**ADPS14**] (see also [**Tik11**], [**BPT08**]), for the class of unital, simple, separable, nuclear and $\mathcal{Z}$-stable C^*-algebras, the Elliott invariant and the Cuntz semigroup of any such algebra tensored with the circle determine one another in a functorial way. Thus, $\mathrm{Ell}(\_)$ is a classifying functor if and only if so is $\mathrm{Cu}(C(\mathrm{T},\_))$.

1.1.3. The regularity Conjecture. This conjecture, which is also known as the Toms-Winter conjecture, see [**TW09**, Remarks 3.5] and [**Win12**, Conjecture 0.1], links three seemingly unrelated regularity properties that a simple, separable, nonelementary C^*-algebra A may enjoy. The first of these properties is that A has finite nuclear dimension. We will not define nuclear dimension here. Instead let us just say that it is a strengthening of the definition of nuclearity that uses completely positive order-zero maps (i.e. completely positive maps that preserve orthogonality of elements).

The second regularity property is $\mathcal{Z}$-stability, and the third one is strict comparison of positive elements, which may be roughly stated by saying that comparison of positive elements (modulo Cuntz subequivalence) is determined by the states on the Cuntz semigroup. This is equivalent to saying that the Cuntz semigroup is almost unperforated.

Conjecture 1.1.2 (Toms-Winter). Let A be a simple, separable, nonelementary C^*-algebra. Then, the following conditions are equivalent:

(i) The C^*-algebra A has finite nuclear dimension.
(ii) The C^*-algebra A is $\mathcal{Z}$-stable.
(iii) The Cuntz semigroup $\mathrm{W}(A)$ is almost unperforated.

Rørdam showed that (ii) implies (iii) (see [**Rør04**]). It is known that (iii) implies (ii) if $T(A)$ has finite dimensional extreme boundary (see [**KR12**], [**TWW12**], [**Sat12**]). Sato, White and Winter ([**SWW14**]) showed that (ii) implies (i) in the case that A has a unique tracial case. Very recently, Bosa, Brown, Sato, Tikuisis, White and Winter proved that $\mathcal{Z}$-stability implies that the nuclear dimension is at most one in the case that the trace simplex is a Bauer simplex. If A has no tracial states then it is purely infinite and the conjecture had been confirmed in that case by work of Kirchberg, Phillips and Rørdam. It follows that the Toms-Winter conjecture is verified whenever the C^*-algebra has at most one tracial state.

Winter proved in [**Win12**] that (i) implies (ii). The Jiang-Su algebra $\mathcal{Z}$ stands out as the C^*-analogue of the hyperfinite II_1-factor. Following this analogy, a $\mathcal{Z}$-stable, simple C^*-algebra can be considered as the analog of a McDuff factor, which, by definition, tensorially absorbs the hyperfinite II_1-factor. We may therefore view Winter's result as the analogue of Connes' theorem that, for II_1-factors, injectivity implies McDuff.

Conjecture 1.1.3 (Elliott). If A is a simple, separable C^*-algebra satisfying the UCT and with finite nuclear dimension, then A admits an inductive limit decomposition whose blocks are either Cuntz algebras over the circle, or subhomogeneus algebras whose primitive ideal spaces have dimension at most 2.

Therefore, if classification and the regularity conjecture hold, the Elliott conjecture above would also hold and we would get deep insight into the structure of simple, nuclear C^*-algebras.

1.2. The categories W and Cu

As we have mentioned in 1.1.1, Coward, Elliott and Ivanescu introduced a category of ordered semigroups Cu such that $\mathrm{Cu}(A)$ is an object in Cu for every C^*-algebra A. The four axioms defining this category capture the continuous nature of the Cuntz semigroup. The first axiom asks for every increasing sequence to admit an order-theoretic supremum, while in the second axiom it is required that every element can be reached as a supremum of a sequence (s_n) such that $s_n \ll s_{n+1}$ for each n. (Here, $a \ll b$ means that whenever $b \leq \sup_n c_n$, for an increasing sequence $(c_n)_n$, then $a \leq c_k$ for some k.) Given a C^*-algebra A, a positive element a in A and $\varepsilon > 0$, one always has that $[(a-\varepsilon)_+] \ll [a]$ in $\mathrm{Cu}(A)$. A projection p in A always satisfies $[p] \ll [p]$. The elements s satisfying that $s \ll s$ play an important role and are termed *compact*. We may think with advantage that they are equivalence classes of projections. The third and fourth axioms express compatibility between order, addition, suprema, and the relation $\ll$.

It is natural to ask, then, what continuity properties are reflected already in $W(A)$ and how $\mathrm{Cu}(A)$ is obtained out of them. Attempts in this direction may be found in [**ABP11**].

We introduce here a new category of semigroups W parallel to the category Cu and show that $W(A)$ is an object of this category. One of the key ingredients here is that the objects in W are semigroups equipped with an additional relation, sufficiently compatible with addition, referred to as an auxiliary relation ([**GHK$^+$03**]). We show that $W(A)$ can be endowed with such a relation; this is also the case with $\mathrm{Cu}(A)$, as was already noted in [**CEI08**], and where one takes $\ll$ as an auxiliary relation. Another ingredient in our approach consists of considering the larger category C^*_{loc} of local C^*-algebras. Essentially, these are pre-C^*-algebras that admit functional calculus on finite sets of positive elements.

We then prove:

Theorem. *The following conditions hold true:*

(i) *The category* W *admits arbitrary inductive limits and the assignment $A \mapsto W(A)$ defines a continuous functor from the category C^*_{loc} of local C^*-algebras to the category* W.

(ii) *The category* Cu *is a full, reflective subcategory of* W. *Therefore,* Cu *also admits arbitrary inductive limits.*

(iii) *There is a diagram, that commutes up to natural isomorphisms:*

$$\begin{tikzcd} C^*_{\mathrm{loc}} \arrow[r, "\mathrm{W}"] \arrow[d, bend right, "\gamma"'] & \mathrm{W} \arrow[d, bend left, "\gamma"] \\ C^* \arrow[u, phantom, "\cup"] \arrow[r, "\mathrm{Cu}"] & \mathrm{Cu} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

where $\gamma\colon \mathrm{W} \to \mathrm{Cu}$ is the reflection functor and $\gamma\colon C^*_{\mathrm{loc}} \to C^*$ is the completion functor that assigns to a local C^*-algebra its completion (which is a C^*-algebra).

In particular, the assignment $A \mapsto \mathrm{Cu}(A)$ is also a continuous functor from the category of C^*-algebras to the category Cu.

Notice that condition (i) above sets up the right framework for the functor W to be continuous, by enlarging the source category to C^*_{loc} and identifying the range category W. Condition (iii) generalizes [**CEI08**, Theorem 2] from sequential to arbitrary inductive limits.

A key concept in the proof is that of a Cu-completion of a semigroup S in the category W. This may be thought of as a pair (T, α), where $T \in \mathrm{Cu}$ and $\alpha\colon S \to T$ is a morphism that, suitably interpreted, is an embedding with dense image.

1.2.1. The range problem. Additional axioms. It is an important problem to determine which semigroups in the category Cu come as Cuntz semigroups of C^*-algebras. For example, we know that, for any finite dimensional, compact Hausdorff space, the semigroup $\mathrm{Lsc}(X, \mathbb{N}\cup\{\infty\})$ of lower semicontinuous functions is an object of Cu ([**APS11**]), but if fails to be the Cuntz semigroup of a C^*-algebra whenever the dimension of X is larger than 2 ([**Rob13b**]).

There are two additional axioms that the Cuntz semigroup of any C^*-algebra satisfies, and which are not derived from the original set of axioms used to define the category Cu. The first of such axioms was established by Rørdam and Winter ([**RW10**]) and indicates how far the partial order in $\mathrm{Cu}(A)$ is from being algebraic. It is usually referred to as the *almost algebraic order* axiom. Given three elements a', a and b, the axiom says that:

$$a' \ll a \leq b \implies \text{ there is } c \text{ such that } a' + c \leq b \leq a + c\,.$$

It is worth pointing out that, if $a \ll a$, then the above implies that whenever $a \leq b$, there is an element c with $a + c = b$. Thus, this axiom is a generalization of the fact that the order among Cuntz classes of projections is algebraic.

The second axiom was established by Robert ([**Rob13a**]) and is a condition of a Riesz decomposition type, usually referred to as the *almost Riesz decomposition* axiom. Given elements a', a, b and c, the axiom reads as follows:

$$a' \ll a \leq b + c \implies \text{ there are } b' \leq b, a \quad c' \leq c, a \text{ with } a' \leq b' + c'\,.$$

In Definition 4.1, we introduce a strengthening of the almost algebraic order axiom, and we prove that it is satisfied by the Cuntz semigroup $\mathrm{Cu}(A)$ of any C^*-algebra A. It is equivalent to the original formulation if the semigroup is weakly cancellative (i.e., if elements cancel from inequalities with $\ll$). With this new formulation, the axiom passes to inductive limits.

We also introduce corresponding versions of these axioms for the category W and show that they are satisfied by $W(A)$, for any local C^*-algebra A. The Cu-completion process, as described above, relates exactly each one of the W-axioms with its Cu-counterpart. All the axioms considered pass to inductive limits.

Although it may be premature to recast the category Cu by adding the axioms of almost algebraic order and almost Riesz decomposition (as new axioms may emerge in the near future), it is quite pertinent to add them to our basket of assumptions in many results of the paper.

1.2.2. Softness and pure noncompactness. While compact elements may be thought of as 'projections', the class of purely noncompact elements can be placed at the other end of the scale, that is, as far as possible from projections. This may be phrased by saying that the element in question only becomes compact in a quotient when it is zero or properly infinite. It was shown by Elliott, Robert, and Santiago that the purely noncompact elements in $\mathrm{Cu}(A)$ are, in the almost unperforated case, the ones that can be compared by means of traces [**ERS11**].

It is natural to seek for a result of this nature in the framework of the category Cu alone. For this, given a semigroup $S \in \mathrm{Cu}$, we need to consider the set $F(S)$ of functionals on S, that is, extended states on S that respect suprema of increasing sequences. Note that, in this way, any element $s \in S$ then can be represented as a linear, lower semicontinuous, $[0,\infty]$-valued function $\hat{s}$ on $F(S)$ by evaluation. It is to be noted that $F(\mathrm{Cu}(A))$ is homeomorphic to the trace simplex of non-normalized traces on A (when A is exact), as shown in [**ERS11**, Theorem 4.4]. Indeed, given a trace τ, its corresponding functional d_τ maps $[a]$ to $\lim \tau(a^{1/n})$.

The key notion in the abstract setting of Cu-semigroups is that of a soft element. By definition, $a \in S$ is soft if any $a' \ll a$ satisfies $(n+1)a' \leq na$ for some n. As it turns out, the subset S_{soft} of soft elements is a submonoid of S. If S is furthermore simple and stably finite, then S_{soft} is also a Cu-semigroup. Our definition of softness is inspired by [**GH82**].

For almost unperforated Cu-semigroups, soft elements are the ones whose comparison theory is completely determined by functionals. Namely, if a, $b \in S$ with a soft, then $a \leq b$ precisely when $\hat{a} \leq \hat{b}$. This includes [**ERS11**, Theorem 6.6]. It is worth mentioning that, in the presence of the almost algebraic order axiom, softness is equivalent to a suitable weakening of pure noncompactness. The concept of softness is, however, easier to state and to use.

1.2.3. Algebraic semigroups. A particularly interesting class of Cu-semigroups is that of algebraic semigroups. These are Cu-semigroups where the compact elements are dense, and they are modelled after C^*-algebras of real rank zero, where the structure of projections determines a great deal of the structure of the algebra.

We show that this is also the case at the semigroup level. Of particular significance is the fact that axioms of interest have a translation into properties of the compact elements, namely:

THEOREM. *Let S be an algebraic* Cu*-semigroup, and let S_c be the submonoid of compact elements. Then:*

(i) The Cu*-semigroup S satisfies the axiom of almost algebraic order if and only if S_c is algebraically ordered.*
(ii) The Cu*-semigroup S is weakly cancellative if and only if S_c is a cancellative semigroup.*
(iii) If S_c has Riesz decomposition, then S satisfies the axiom of almost Riesz decomposition. Conversely, if S satisfies the axioms of almost algebraic order and almost Riesz decomposition and is weakly cancellative, then S_c has Riesz decomposition.

1.2.4. Near unperforation. The notion of near unperforation allows us to analyse almost unperforation from a different perspective. A positively ordered monoid S is *nearly unperforated* if $a \leq b$ whenever $2a \leq 2b$ and $3a \leq 3b$. (This is not our original definition, but a useful restatement.) Nearly unperforated semigroups are always almost unperforated. In the simple case, a converse is available:

THEOREM. *Let S be a simple, stably finite* Cu*-semigroup that satisfies the almost algebraic order axiom. Then S is nearly unperforated precisely when it is almost unperforated and weakly cancellative.*

This result has as a corollary that, if A is a $\mathcal{Z}$-stable C^*-algebra, $\mathrm{Cu}(A)$ is nearly unperforated in a variety of situations. For example, if A is simple or has real rank zero and stable rank one, or also if A has no K_1-obstructions. It then remains an interesting open problem to decide whether $\mathrm{Cu}(A)$ is always nearly unperforated for any $\mathcal{Z}$-stable C^*-algebra A. We conjecture this is always the case, and prove it in a number of instances.

1.3. Tensor products

Tensor products with the Jiang-Su algebra or, more generally, by a strongly self-absorbing C^*-algebra, are of particular relevance in connection with the current status of the classification program.

The tensor product construction at the level of (ordered) semigroups has a long tradition (for completeness, we have included a review of the necessary definitions and results in Appendix B). It is therefore a very natural question to ask how $\mathrm{Cu}(A \otimes B)$ and $\mathrm{Cu}(A) \otimes \mathrm{Cu}(B)$ are related. The first step towards a solution to this question resides in carrying out a construction of the tensor product within the category Cu, so as to 'equip' the usual semigroup tensor product with the necessary continuity structure.

Our approach has a categorical flavor, and at the same time allows for computations of examples. A central notion is that of a bimorphism $\varphi\colon S\times T\to R$, that is, a biadditive map that is required to satisfy certain additional conditions depending on the category where the objects S, T and R live. Thus, for example, if we focus on the category Cu, we speak of Cu-bimorphisms and we shall be asking that φ is continuous in each variable (i.e. preserves suprema) and is jointly preserving the relation $\ll$, that is, $s'\ll s$ and $t'\ll t$ imply $\varphi(s',t')\ll\varphi(s,t)$. One then asks a tensor product in Cu of S and T to be a pair (Q,φ), where Q is an object in Cu and $\varphi\colon S\times T\to Q$ is a Cu-bimorphism with certain universal properties.

We can also regard the tensor product as an object that represents the bimorphism bifunctor $\mathrm{BiCu}(S\times T,\_)$. We use $\mathrm{Cu}(\_,\_)$ and $\mathrm{W}(\_,\_)$ below to denote the corresponding morphism sets, which are naturally positively ordered semigroups. We prove:

THEOREM. *Let S and T be Cu-semigroups. There is a Cu-semigroup $S\otimes_{\mathrm{Cu}}T$ and a Cu-bimorphism $\varphi\colon S\times T\to S\otimes_{\mathrm{Cu}}T$ such that the pair $(S\otimes_{\mathrm{Cu}}T,\,\varphi)$ represents the bimorphism functor $\mathrm{BiCu}(S\times T,\_)$ that takes values in the category of positively ordered semigroups. Thus, for every Cu-semigroup R, φ induces a positively ordered semigroup isomorphism of the following (bi)morphisms sets:*

$$\mathrm{Cu}(S\otimes_{\mathrm{Cu}}T,R)\to\mathrm{BiCu}(S\times T,R)\,.$$

In outline, the construction of the object $S\otimes_{\mathrm{Cu}}T$ in the Theorem above uses the reflector functor $\gamma\colon\mathrm{W}\to\mathrm{Cu}$ as described in 1.2, and so the tensor product in Cu comes as a completion of the corresponding object in W. In fact, recalling that Cu is a reflective subcategory of W, we have:

THEOREM. *Let S and T be semigroups in the category W. There is then a W-semigroup $S\otimes_{\mathrm{W}}T$ and a W-bimorphism that induces a commutative diagram where every row and column are semigroup isomorphisms:*

$$\begin{tikzcd}
\mathrm{W}(S\otimes_{\mathrm{W}}T,R) \arrow[r, "\cong"] & \mathrm{BiW}(S\times T,R) \\
\mathrm{Cu}(\gamma(S\otimes_{\mathrm{W}}T),R) \arrow[u, "\cong"'] \arrow[r, "\cong"] & \mathrm{BiCu}(\gamma(S)\times\gamma(T),R). \arrow[u, "\cong"]
\end{tikzcd}$$

In particular, we can identify $\gamma(S)\otimes_{\mathrm{Cu}}\gamma(T)$ with $\gamma(S\otimes_{\mathrm{W}}T)$.

Applied to C^*-algebras, the results above yield:

THEOREM. *The following conditions hold true:*

(i) Let A and B be local C^*-algebras. Then

$$\mathrm{Cu}(A)\otimes_{\mathrm{Cu}}\mathrm{Cu}(B)=\gamma(W(A)\otimes_{\mathrm{W}}W(B))\,.$$

(ii) Let D be a strongly self-absorbing C^*-algebra of real rank zero that satisfies the UCT. Then

$$\mathrm{Cu}(A\otimes D)\cong\mathrm{Cu}(A)\otimes_{\mathrm{Cu}}\mathrm{Cu}(D)\,,$$

for any C^*-algebra A.

1.4. Multiplicative structure of Cu-semigroups. Solid Cu-semirings

As noted in 1.1.2, the class of D-stable C^*-algebras, where D is strongly self-absorbing, is relevant for the theory, and thus a description of their Cuntz semigroup is of particular interest. Towards this end, we identify which semigroups should play the role of strongly self absorbing C^*-algebras. If D is such an algebra, then the isomorphism $D \otimes D \cong D$ induces a Cu-bimorphism $\mathrm{Cu}(D) \times \mathrm{Cu}(D) \to \mathrm{Cu}(D)$, which in turn can be used to equip $\mathrm{Cu}(D)$ with a unital semiring structure (see Section B.4 for the definition of a semiring). This is also compatible with the continuous properties of $\mathrm{Cu}(D)$ and leads us to introduce the notion of a Cu-semiring. In the case of the Jiang-Su algebra $\mathcal{Z}$, its Cuntz semigroup may be identified with $Z := \mathbb{N} \sqcup (0, \infty]$, where the product of two elements is the obvious one in either $\mathbb{N}$ or $(0, \infty]$ and mixed terms multiply into $(0, \infty]$.

In a similar vein, if A is a D-stable C^*-algebra, there is natural Cu-bimorphism $\mathrm{Cu}(D) \times \mathrm{Cu}(A) \to \mathrm{Cu}(A)$ which is moreover compatible with the multiplicative structure of $\mathrm{Cu}(D)$. This leads us to define the notion of a Cu-semimodule S over a Cu-semiring R. We refer to this situation by saying that S has an R-multiplication.

Of particular importance is the structure of Cu-semimodules over semirings that come from strongly self-absorbing algebras, or from the Jacelon-Razak algebra $\mathcal{R}$, whose Cuntz semigroup is $[0, \infty]$ (see [**Jac13**] and also [**Rob13a**]). As Robert points out for $\mathrm{Cu}(\mathcal{R})$ (see [**Rob13a**]), having a $\mathrm{Cu}(\mathcal{R})$-multiplication is in fact a property of the semigroup rather than an additional structure. Denote by R_q the Cuntz semigroup of a UHF-algebra of infinite type (and supernatural number q). We then prove the following:

THEOREM. *Let S be a* Cu*-semigroup. Then:*

(i) The Cu*-semigroup S has Z-multiplication if and only if S is almost divisible and almost unperforated.*
(ii) The Cu*-semigroup S has R_q-multiplication if and only if S is p-divisible and p-unperforated whenever p is an integer that divides q.*
(iii) The Cu*-semigroup S has $[0, \infty]$-multiplication if and only if S is unperforated, divisible and every element of S is soft.*
(iv) The Cu*-semigroup S has $\{0, \infty\}$-multiplication if and only if $2x = x$ for every $x \in S$.*

Condition (i) above allows us to prove a semigroup version of the Toms-Winter conjecture:

THEOREM. *Let S be a* Cu*-semigroup. Then, the following conditions are equivalent:*

(i) We have $S \cong S \otimes_{\mathrm{Cu}} Z$.
(ii) The Cu*-semigroup S is almost unperforated and almost divisible.*

A key ingredient in the above Theorem is the fact that $Z \otimes_{\mathrm{Cu}} Z \cong Z$, where the isomorphism is induced by the natural product. This naturally poses the question of which is the right notion for a 'strongly self-absorbing Cu-semigroup'. We adopt here the terminology of a solid ring, as introduced in [**BK72**], and call a unital Cu-semiring R *solid* if the multiplication induces an isomorphism $R \otimes_{\mathrm{Cu}} R \cong R$. Every such semiring is automatically simple and, in the stably finite case, has a unique normalized functional. We then prove:

THEOREM. *Let D be a strongly self-absorbing C^*-algebra satisfying the UCT. Then* $\mathrm{Cu}(D)$ *is a solid* Cu*-semiring, and so* $\mathrm{Cu}(D) \otimes_{\mathrm{Cu}} \mathrm{Cu}(D) \cong \mathrm{Cu}(D)$.

As solid Cu-semirings have good structural properties, it is natural to analyse the tensor product of a Cu-semigroup with one of these semirings. This process may be termed a *regularization*, as the final object enjoys regularity properties (e.g. it absorbs the Cu-semiring Z tensorially). We explore two such constructions, closely related to C^*-algebras: the rationalization and the realification of a semigroup.

The *rationalization* of a Cu-semigroup S is, by definition, its tensor product with a semigroup of the form R_q, where R_q is, as mentioned above, the Cuntz semigroup of a UHF-algebra of infinite type, so that $q = (p_i)$ is a supernatural number of infinite type. The tensor product $S \otimes_{\mathrm{Cu}} R_q$ can be realized as the inductive limit S_q constructed as $S \xrightarrow{\cdot p_1} S \xrightarrow{\cdot p_2} S \xrightarrow{\cdot p_3} \ldots$.

Given a Cu-semigroup S, Robert introduced in [**Rob13a**] the *realification* of S, which is a Cu-semigroup denoted by S_R. Robert indicates that his construction may be thought of as the tensor product with $[0, \infty]$. This semigroup is, by definition, the subsemigroup of lower semicontinuous, linear, $[0, \infty]$-valued functions defined on $F(S)$ that can be obtained as pointwise suprema of functions of the type $\frac{1}{n}\hat{s}$, where $s \in S$. Robert obtains in [**Rob13a**, Theorem 3.2.1] a more abstract characterization of S_R. We make the connection of S_R with the tensor product construction precise and we show that we indeed have $S_R \cong S \otimes_{\mathrm{Cu}} [0, \infty]$. It then follows from our results and [**Rob13a**, Theorem 5.1.2] that $\mathrm{Cu}(A \otimes \mathcal{R}) \cong \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(\mathcal{R})$ for any C^*-algebra A.

Using the classification theory for solid rings (see [**BK72**], [**BS77**]) we prove a classification theorem for solid Cu-semirings. In essence, this result states that if R is a solid Cu-semiring then it is either $\{0, 1, \ldots, k, \infty\}$, for some k, $[0, \infty]$, Z, or is constructed out of a solid ring $T \ncong \mathbb{Z}$ with non-torsion unit.

Now denote by $Q = \mathbb{Q}_+ \sqcup (0, \infty]$ the Cuntz semigroup of the universal UHF-algebra. As a consequence of our classification theorem, we obtain that Z and Q can be (uniquely) characterized as initial and final objects in the category of nonelementary, solid Cu-semirings satisfying the almost algebraic order axiom. Likewise, $[0, \infty]$ is the unique solid Cu-semiring that contains no nonzero compact elements. This is an exact parallell of Winter's result that strongly self-absorbing C^*-algebras are $\mathcal{Z}$-stable and in fact, our methods allow us to recover this result.

It would be interesting to know whether the Cuntz semigroup of any strongly self-absorbing C^*-algebra D is always solid. This is clearly the case if D is purely infinite simple. As D is $\mathcal{Z}$-stable by Winter's result, [**Win11**], and monotracial in the stably finite case, it follows that $\mathrm{Cu}(D)$ may be identified with $V(D) \sqcup (0, \infty]$. With our classification theorem of solid Cu-semirings at hand, this would shed light on whether there could exist a non-UCT strongly self-absorbing example.

Acknowledgments

We thank Joan Bosa, Joachim Cuntz, George Elliott, Martin Engbers, Henning Petzka, Leonel Robert, Luis Santiago, Aaron Tikuisis, Fred Wehrung, Stuart White and Wilhelm Winter for valuable comments. We are grateful to Ken Goodearl for several discussions on tensor products of semigroups. We also thank Snigdhayan Mahanta for pointing out the notion of a solid ring, and Eusebio Gardella for carefully reading the first chapters of a draft of this paper.

This work was initiated at the Centre de Recerca Matemàtica (Bellaterra) during the program 'The Cuntz Semigroup and the Classification of C^*-algebras' in 2011. The authors would like to thank the CRM for financial support and working conditions provided. Part of this research was conducted while the third named author was visiting the Universitat Autònoma de Barcelona (UAB) in 2012 and 2013, and while he was attending the Thematic Program on Abstract Harmonic Analysis, Banach and Operator Algebras at the Fields Institute in January-June 2014. He would like to thank both Mathematics departments for their kind hospitality.

The two first named authors were partially supported by DGI MICIIN (grant No. MTM2011-28992-C02-01), and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya. The third named author was partially supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92), and by the Deutsche Forschungsgemeinschaft (SFB 878).

CHAPTER 2

Pre-completed Cuntz semigroups

In the first part of this chapter, we introduce the categories PreW and W of (abstract) pre-completed Cuntz semigroups and we develop their general theory. An object of PreW or W is a positively ordered monoid (see Paragraph B.2.1 for the definition) equipped with an auxiliary relation such that certain axioms are satisfied; see Paragraph 2.1.1 and Definition 2.1.2. We show that W is a full, reflective subcategory of PreW, see Proposition 2.1.5, and that both categories have inductive limits; see Theorem 2.1.8 and Corollary 2.1.9.

In the second part, we associate to every local C^*-algebra A its *pre-completed Cuntz semigroup* $W(A)$, which naturally belongs to the category W. It is given as the original definition of the Cuntz semigroup (equivalence classes of positive elements in matrices over A), together with a natural auxiliary relation; see Proposition 2.2.5. We show that the assignment $A \mapsto W(A)$ extends to a continuous functor from local C^*-algebras to the category W; see Corollary 2.1.9. This is inspired by [**CEI08**], where the analogous results are shown for the completed Cuntz semigroup; see Chapter 3.

2.1. The categories PreW and W

We refer to Section B.2 for the basic theory of positively ordered monoids.

2.1.1 (Axioms for the category W). Let S be a positively ordered monoid. Following [**GHK$^+$03**, Definition I-1.11, p.57], an *auxiliary relation* on S is a binary relation $\prec$ such that the following conditions hold:

(i) We have that $a \prec b$ implies $a \leq b$, for any $a, b \in S$.
(ii) We have that $a \leq b \prec c \leq d$ implies $a \prec d$, for any $a, b, c, d \in S$.
(iii) We have that $0 \prec a$, for any $a \in S$.

Let S be a positively ordered monoid and fix an auxiliary relation $\prec$ on S. We say that S is *countably-based* if there exists a countable subset $B \subset S$ such that for any two elements a', a in S satisfying $a' \prec a$, there exists $b \in B$ such that $a' \leq b \prec a$. A subset B with these properties is called a *basis* for S; cf. [**GHK$^+$03**, Proposition III.4.2, p. 241].

A particularly interesting auxiliary relation is the following: Given elements a and b in a positively ordered monoid S, we say that a is *compactly contained in* b (or a is *way-below* b), denoted $a \ll b$, if whenever $(b_n)_{n\in\mathbb{N}}$ is an increasing sequence in S for which the supremum $\sup_n b_n$ exists, then $b \leq \sup_n b_n$ implies that there is k such that $a \leq b_k$. If $a \in S$ satisfies $a \ll a$, we say that a is *compact*, and we shall denote the set of compact elements by S_c.

Note that the compact containment relation is usually defined by considering suprema of arbitrary upwards directed sets; c.f. [**GHK$^+$03**, Definition I-1.1, p.49]. The definition given here is a sequential version. In Remarks 3.1.3 we will see that both notions agree under a suitable separability assumption.

We will use the following axioms to define the objects in the categories PreW and W. Given an element $a \in S$, we use the notation $a^\prec := \{x \in S \mid x \prec a\}$ for the set of predecessors of a.

(W1) For each $a \in S$, there exists a sequence $(a_k)_k \subset a^\prec$ such that $a_k \prec a_{k+1}$ for each k and such that for any $b \in a^\prec$ there exists an index k such that $b \prec a_k$.
(W2) For each $a \in S$, we have $a = \sup a^\prec$.
(W3) If $a', a, b', b \in S$ satisfy $a' \prec a$ and $b' \prec b$, then $a' + b' \prec a + b$.
(W4) If $a, b, c \in S$ satisfy $a \prec b+c$, then there exist $b', c' \in S$ such that $a \prec b'+c'$, $b' \prec b$ and $c' \prec c$.

Axiom (W1) means that for each $a \in S$, the set $a^\prec$ is upward directed and contains a cofinal increasing sequence with respect to $\prec$. Given $a, b \in S$, axiom (W3) means that the set

$$a^\prec + b^\prec = \{a' + b' \mid a' \prec a, b' \prec b\},$$

is contained in $(a+b)^\prec$. Moreover, axiom (W4) means that $a^\prec + b^\prec$ is cofinal in $(a+b)^\prec$. We also remark that axioms (W3) and (W4) mean precisely that the addition map

$$+\colon S \times S \to S, \quad (a,b) \mapsto a+b, \quad (a,b \in S)$$

is a W-bimorphism, as defined in Definition 6.2.3.

Definition 2.1.2. A PreW-*semigroup* is a pair $(S, \prec)$, where S is a positively ordered monoid and $\prec$ is a fixed auxiliary relation on S satisfying axioms (W1), (W3) and (W4) from Paragraph 2.1.1. If $(S, \prec)$ also satisfies axiom (W2), then it is called a W-*semigroup*. We often drop the reference to the auxiliary relation and simply write S for a (Pre)W-semigroup.

Given two PreW-semigroups S and T, a *generalized* W-*morphism* $f\colon S \to T$ is a PoM-morphism that is continuous in the following sense:

(M) For every $a \in S$ and $b \in T$ with $b \prec f(a)$, there exists $a' \in S$ such that $a' \prec a$ and $b \leq f(a')$.

We denote the collection of all such maps by $\mathrm{W}[S,T]$. A W-morphism is a generalized W-morphism that preserves the auxiliary relation and we denote the set of all such maps by $\mathrm{W}(S,T)$.

We let PreW be the category that has as objects all PreW-semigroups, and whose morphisms are the W-morphisms. We let W be the full subcategory of PreW whose objects are W-semigroups. Note that we call the morphisms in both categories W-morphisms.

Remarks 2.1.3. (1) The order of the axioms (W1)-(W4) has been chosen so that it roughly correspond to the axioms (O1)-(O4) for Cu-semigroups, see Paragraph 3.1.1. Indeed, one should think of the W-axioms as a version of the O-axioms formulated in such a way that the semigroup is not required to have suprema of increasing sequences.

(2) Let S be a semigroup with an auxiliary relation $\prec$ such that (W1), (W3) and (W4) are satisfied. It is easy to check that S satisfies (W2) if and only if for every $a, b \in S$ we have that $a^\prec \subset b^\prec$ implies $a \leq b$. Note that the converse of this statement is always true, that is, if $a \leq b$ then $a^\prec \subset b^\prec$. This means that in the presence of (W2), the partial order may be derived from the auxiliary relation. For a Cu-semigroup, the converse is also true, since then the auxiliary relation is the compact containment relation which is defined in terms of the partial order.

(3) In a W-semigroup, the relation $\ll$ is stronger than $\prec$. The class of W-semigroups where $\prec$ is equal to $\ll$ was studied in [**ABP11**].

(4) Let $f\colon S \to T$ be a PoM-morphism between W-semigroups. If f is continuous, then for each $a \in S$ we have

$$f(a) = \sup f(a^\prec).$$

Indeed, by (W2) in T, we have that $f(a) = \sup f(a)^{\prec}$. Continuity of the map f implies that $f(a^{\prec})$ is cofinal in $f(a)^{\prec}$ in the sense that for every $t' \in f(a)^{\prec}$ there exists $a' \in a^{\prec}$ with $t' \leq f(s')$. However, let us remark that we do not assume that f preserves the auxiliary relation, so that we do not necessarily have $f(a^{\prec}) \subset f(a)^{\prec}$. Nevertheless, it follows from this version of cofinality that the suprema of the sets $f(a^{\prec})$ and $f(a)^{\prec}$ agree.

The converse of this statement in the context of Cu-semigroups is given in Lemma 3.1.4.

2.1.4. We will now show that the category W is a reflective subcategory of PreW. This means that the inclusion functor $\mathrm{W} \to \mathrm{PreW}$ admits a left adjoint. We use the idea of a universal completion as described in [**KL09**, § 2]. Thus, given a PreW-semigroup, we will construct its W-completion.

Let $(S, \prec)$ be a PreW-semigroup. In order to enforce (W2), we consider the binary relation $\preceq$ on S given by $a \preceq b$ if and only if $a^{\prec} \subset b^{\prec}$, for any $a, b \in S$. It is clear that $\preceq$ is a pre-order on S. By symmetrizing $\preceq$, we obtain an equivalence relation $\sim$ on S such that for any $a, b \in S$ we have

$$a \sim b \quad \text{if and only if} \quad a \preceq b \preceq a \quad \text{if and only if} \quad a^{\prec} = b^{\prec}.$$

We let $\mu(S) = S/\sim$ denote the set of equivalence classes, and we denote the class of an element $a \in S$ by $[a]$. By construction, the pre-order $\preceq$ induces a partial order on $\mu(S)$ by setting $[a] \leq [b]$ if and only if $a^{\prec} \subset b^{\prec}$, for any $a, b \in S$. It is easy to check that the addition on S induces an addition on $\mu(S)$ and that this endows $\mu(S)$ with the structure of a positively ordered monoid.

We define an auxiliary relation on $\mu(S)$ by setting $[a] \prec [b]$ if and only if $a \in b^{\prec}$, for any $a, b \in S$. With this structure, $\mu(S)$ becomes a W-semigroup. Moreover, the map

$$\beta\colon S \to \mu(S), \quad a \mapsto [a], \quad (a \in S)$$

is easily seen to be a W-morphism. The only detail that needs some verification is that $\mu(S)$ satisfies (W2), and for this we use the second observation in Remarks 2.1.3. Thus assume that $a, b \in S$ satisfy $[a]^{\prec} \subset [b]^{\prec}$. By definition, this means that $[a'] \prec [b]$ for any $a' \in S$ satisfying $[a'] \prec [a]$. This in turn mean that $a' \prec b$ whenever $a' \in S$ satisfies $a' \prec a$. Therefore $a^{\prec} \subset b^{\prec}$, which means $[a] \leq [b]$, as desired.

The map $\beta\colon S \to \mu(S)$ has the following universal property: For every W-semigroup R and for every W-morphism $f\colon S \to R$, there is a unique W-morphism $\tilde{f}\colon \mu(S) \to R$ such that $f = \tilde{f} \circ \beta$. This means precisely that $\beta\colon S \to \mu(S)$ is the W-completion of S. By [**KL09**, § 2], this induces a reflection functor $\mu\colon \mathrm{PreW} \to \mathrm{W}$.

Proposition 2.1.5. *The category* W *is a full, reflective subcategory of* PreW.

2.1.6 (Inductive limits in PoM). Let $(S_i, \varphi_{i,j})$ be an inductive system in PoM, indexed over a directed set Ω. We define an equivalence relation $\sim$ on the disjoint union $\bigsqcup_{i \in \Omega} S_i$, by setting for any $a \in S_i$ and any $b \in S_j$:

$$a \sim b \quad \text{if and only if there exists } k \geq i, j \text{ such that } \varphi_{i,k}(a) = \varphi_{j,k}(b).$$

The set of equivalence classes is denoted by PoM-$\varinjlim S_i$. We denote the equivalence class of an element $a \in S_i$ by $[a]$. Given $a \in S_i$ and $b \in S_j$, we define

$$[a] + [b] = [\varphi_{i,k}(a) + \varphi_{j,k}(b)], \quad \text{for any } k \geq i, j.$$

It is clear that this is a well-defined addition on PoM-$\varinjlim S_i$. We also define a partial order by setting for $a \in S_i$ and $b \in S_j$:

$$[a] \leq [b] \quad \text{if and only if there exists } k \geq i, j \text{ such that } \varphi_{i,k}(a) \leq \varphi_{j,k}(b).$$

This gives PoM-$\varinjlim S_i$ the structure of a positively ordered monoid, and it is well-known that this is the inductive limit in the category PoM. The PoM-morphism from S_i to the inductive limit is denoted by $\varphi_{i,\infty}$.

Definition 2.1.7 (Auxiliary relation on PoM-$\varinjlim S_i$). Let $(S_i, \varphi_{i,j})$ be an inductive system in PreW. We define a relation $\prec$ on the inductive limit PoM-$\varinjlim S_i$ of the underlying positively ordered monoids as follows: For $a \in S_i$ and $b \in S_j$, we set

$$[a] \prec [b] \quad \text{if and only if there exists } k \geq i, j \text{ such that } \varphi_{i,k}(a) \prec \varphi_{j,k}(b).$$

Theorem 2.1.8. *The category* PreW *has inductive limits. More precisely, let* $(S_i, \varphi_{i,j})$ *be an inductive system in* PreW. *Let* $S =$ PoM-$\varinjlim S_i$ *be the inductive limit of the underlying positively ordered monoids, together with* PoM-*morphisms* $\varphi_{i,\infty} \colon S_i \to S$. *The relation* $\prec$ *on* S *as defined in Definition 2.1.7 is an auxiliary relation and* $(S, \prec)$ *is a* PreW-*semigroup, denoted by* PreW-$\varinjlim S_i$. *Moreover, the maps* $\varphi_{i,\infty}$ *are* W-*morphisms and* PreW-$\varinjlim S_i$ *is the inductive limit of the system* $(S_i, \varphi_{i,j})$ *in* PreW.

Proof. Let us first show that $\prec$ is an auxiliary relation. Let $[a], [b], [c]$ and $[d]$ be elements in S. It is easy to see that $0 \prec [a]$, and that $[a] \prec [b]$ implies $[a] \leq [b]$. Now suppose that $[a] \leq [b] \prec [c] \leq [d]$. We have to show that $[a] \prec [d]$. We may assume that there is an index i such that all four elements are represented in S_i. Since $[a] \leq [b]$, there is $j \geq i$ such that $\varphi_{i,j}(a) \leq \varphi_{i,j}(b)$. Similarly, there are $k \geq i$ and $l \geq i$ such that $\varphi_{i,k}(b) \prec \varphi_{i,k}(c)$ and $\varphi_{i,l}(c) \leq \varphi_{i,l}(d)$. Let $n \geq j, k, l$. Using that the connecting maps preserve the order and the auxiliary relation, we obtain

$$\varphi_{i,n}(a) \leq \varphi_{i,n}(b) \prec \varphi_{j,n}(c) \leq \varphi_{i,n}(d),$$

which implies that $[a] \prec [d]$.

Next, we show that $(S \prec)$ is a PreW-semigroup. In order to verify (W1), let $a \in S_i$ for some i. Since S_i satisfies (W1), there is a cofinal $\prec$-increasing sequence $(a_k)_{k \in \mathbb{N}}$ in S_i. Then $[a_k] \prec [a]$ for all k. Further, if $[b] \prec [a]$ for some j and $b \in S_j$, then there is $n \geq i, j$ with $\varphi_{j,n}(b) \prec \varphi_{i,n}(a)$. Using that $\varphi_{i,n}$ is continuous, there is k such that $\varphi_{j,n}(b) \prec \varphi_{i,n}(a_k)$. Thus $[b] \prec [a_k]$. This shows that $[a]^{\prec}$ is upward directed and contains a cofinal $\prec$-increasing sequence.

It is routine to check (W3). In order to show that $(S, \prec)$ satisfies (W4), suppose that $[c] \prec [a] + [b]$ for some elements $[a], [b]$ and $[c]$ in S. We may assume that there is an index i such all three elements are represented in S_i. Then there is $j \geq i$ such that $\varphi_{i,j}(c) \prec \varphi_{i,j}(a) + \varphi_{i,j}(b)$. Since S_j satisfies (W4), there are elements $d, e \in S_j$ such that

$$\varphi_{i,j}(c) \leq d + e, \quad d \prec \varphi_{i,j}(a), \quad e \prec \varphi_{i,j}(b).$$

Using that $\varphi_{i,j}$ is continuous, there are elements $a', b' \in S_i$ such that

$$a' \prec a, \quad b' \prec b, \quad d \leq \varphi_{i,j}(a'), \quad e \leq \varphi_{i,j}(b').$$

Then $[a'] \prec [a]$, $[b'] \prec [b]$ and $[c] \leq [a'] + [b']$, which shows that the elements $[a']$ and $[b']$ have the desired properties to verify (W4) in S.

The natural maps $\varphi_{i,\infty} \colon S_i \to S$ clearly preserve the auxiliary relation. We need to check that they are also continuous. So let $a \in S_i$ and $s \in S$ satisfy $s \prec [a]$. Then there is j and $b \in S_j$ such that $s = [b]$. It follows that there is $k \geq i, j$ such that $\varphi_{j,k}(b) \prec \varphi_{i,k}(a)$ in S_k. Since $\varphi_{i,k}$ is continuous, there is $a' \in S_i$ such that $a' \prec a$ and $\varphi_{j,k}(b) \leq \varphi_{i,k}(a')$. It follows that $s \leq \varphi_{i,\infty}(a')$, which shows that a' has the desired properties.

Finally, we show that S is the inductive limit in the category PreW. So let T be a PreW-semigroup and let $\lambda_i \colon S_i \to T$ be are W-morphisms such that $\lambda_j \circ \varphi_{i,j} = \lambda_i$

whenever $i \leq j$. Since S is the limit in the category PoM, there is a unique PoM-morphism $\alpha\colon S \to T$ such that $\alpha \circ \varphi_{i,\infty} = \lambda_i$ for each i. It is left to the reader to verify that α is a W-morphism. $\square$

Given an inductive system $(S_i, \varphi_{i,j})$ in W, the inductive limit PreW-$\varinjlim S_i$ from Theorem 2.1.8 need not be a W-semigroup. However, since W is a reflective subcategory of PreW, it follows from general category theory that W has inductive limits and that the reflection functor $\mu\colon \mathrm{PreW} \to \mathrm{W}$ is continuous; see e.g. [**Bor94**, Proposition 3.2.2, p.106].

Corollary 2.1.9. *The category* W *has inductive limits.*

More precisely, let $(S_i, \varphi_{i,j})$ be an inductive system in W*. Then the inductive limit in* W *is the* W*-completion of the inductive limit in* PreW*:*

$$\mathrm{W}\text{-}\varinjlim S_i = \mu(\mathrm{PreW}\text{-}\varinjlim S_i).$$

2.2. The pre-completed Cuntz semigroup of a C*-algebra

2.2.1 (Local C^*-algebras). A *pre-C^*-algebra* A is a *-algebra over the complex numbers together with a C^*-norm $\|\_\|$, that is, $\|a^*a\| = \|a\|^2$ for all $a \in A$. It is known that such a norm is automatically submultiplicative and that the involution becomes isometric; see [**Pal01**, Theorem 9.5.14, p.956]. Every pre-C^*-algebra A naturally embeds as a dense sub-*-algebra in its completion $\overline{A}$, which is a C^*-algebra.

In this paper, we will say that A is a *local C^*-algebra* if there is a family of complete, *-invariant subalgebras $A_i \subset A$ such that for any i_1, i_2 there is i_3 such that $A_{i_1} \cup A_{i_2} \subset A_{i_3}$ and such that $A = \bigcup_i A_i$. Note that each A_i is a C^*-algebra. Viewing a pre-C^*-algebra A inside its completion $\overline{A}$, it is a local C^*-algebra if and only if for any finite subset $F \subset A$, the C^*-algebra $C^*(F)$ generated inside $\overline{A}$ is contained in A. The main point is that local C^*-algebras are closed under continuous functional calculus.

We say that A is *separable* if it contains a countable dense subset (equivalently, $\overline{A}$ is separable). If A is a local C^*-algebra, then so is every matrix algebra $M_k(A)$, and there is a natural dense embedding $\mathrm{M}_k(A) \subset M_k(\bar{A})$. The C^*-algebras $M_k(\bar{A})$ sit (as upper left corners) inside the stabilization $\bar{A} \otimes \mathrm{K}$, and we may consider the union

$$M_\infty(A) := \bigcup_k M_k(A) \subset \bar{A} \otimes \mathrm{K}.$$

This is a dense embedding, hence $\overline{M_\infty(A)} = \bar{A} \otimes \mathrm{K}$, and one sees that $M_\infty(A)$ is again a local C^*-algebra.

A *-homomorphism between local C^*-algebras is automatically continuous and even norm-decreasing. We let C^*_{loc} be the category whose objects are local C^*-algebras, and whose morphisms are *-homomorphisms.

We remark that there are other definitions of a local C^*-algebra in the literature, in particular in [**Bla98**, 3.1], [**Mey99**], [**CMR07**] and [**BH82**, Definition I.1.1(a)]. Some of these definitions seem to be more general than the one given here. It is conceivable that the theory of pre-completed Cuntz semigroups can be carried out in this more general framework, but we will not pursue this here.

2.2.2 (Cuntz comparison in a local C^*-algebra). Let A be a local C^*-algebra and let $x, y \in A_+$ be positive elements. We say that x is *Cuntz sub-equivalent* to y, in symbols $x \precsim y$, if there exists a sequence $(z_n)_n$ in A such that $x = \lim_n z_n^* y z_n$. We say x is *Cuntz equivalent* to y, in symbols $x \sim y$, if $x \precsim y$ and $y \precsim x$. These relations were introduced in [**Cun78**].

Rørdam's fundamental results on Cuntz comparison, [**Rør92**, Proposition 2.4] (see also [**APT11**, Proposition 2.17]), remain valid in local C^*-algebras, that is, for any $x, y \in A_+$ the following conditions are equivalent:

(1) We have $x \precsim y$.
(2) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $(x-\varepsilon)_+ \precsim (y-\delta)_+$.
(3) For every $\varepsilon > 0$ there exist $\delta > 0$ and $r \in A$ such that $(x-\varepsilon)_+ = r(y-\delta)_+ r^*$.

2.2.3. Given a local C^*-algebra A, the (original) definition of the Cuntz semigroup of A is

$$W(A) = M_\infty(A)_+/{\sim},$$

the set of Cuntz equivalence classes of positive elements in matrix algebras over A. The equivalence class of an element $x \in M_\infty(A)_+$ is denoted by $[x]$. Given $x, y \in M_\infty(A)_+$, we set $[x] \leq [y]$ if $x \precsim y$, and we define $[x] + [y] = [\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)]$. This defines a partial order and a well-defined abelian addition on $W(A)$. The zero element in $W(A)$ is given by the class of the zero element $0 \in A$. This equips $W(A)$ with the structure of a positively ordered monoid. Next, we will endow $W(A)$ with an auxiliary relation, making it a W-semigroup.

Definition 2.2.4. Let A be a local C^*-algebra. We define a relation $\prec$ on the positively ordered monoid $W(A) = M_\infty(A)_+/\sim$ as follows: For $a, b \in M_\infty(A)_+$, we set

$$[a] \prec [b] \quad \text{ if and only if there exists } \varepsilon > 0 \text{ such that } [a] \leq [(b-\varepsilon)_+].$$

We call $W(A) = (W(A), \prec)$ the *pre-completed Cuntz semigroup* of A.

Proposition 2.2.5. *Let A be a local C^*-algebra. Then the relation $\prec$ defined in Definition 2.2.4 is an auxiliary relation and $(W(A), \prec)$ is a* W*-semigroup. If A is separable, then $W(A)$ is countably-based.*

Proof. By abusing notation, let us define a relation $\prec$ on positive elements in $M_\infty(A)$ by setting $a \prec b$ if there exists $\varepsilon > 0$ such that $a \precsim (b-\varepsilon)_+$. Rørdam's results on Cuntz comparison show that $a \precsim b$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $(a-\varepsilon)_+ \precsim (b-\delta)_+$; see Paragraph 2.2.2 and [**Rør92**, Proposition 2.4], Thus, given any three positive elements a, b and c, we see that $a \prec b \precsim c$ implies $a \prec c$.

It follows that $\prec$ is a well-defined auxiliary relation on $W(A)$. The axioms (W1)-(W4) are now straightforward to check. Finally, if A is separable, it follows that $W(A)$ countably-based; the argument can be found in the proof of [**APS11**, Lemma 1.3]; see also [**Rob13a**, Proposition 5.1.1]. □

Remarks 2.2.6. (1) Usually, by $W(A)$ we denote the pre-completed Cuntz semigroup of A, considered as a W-semigroup which is understood to be equipped with an auxiliary relation. It should be clear from the context when by $W(A)$ we only mean the underlying positively ordered monoid.

(2) It is not known whether the auxiliary relation on $W(A)$ can be deduced from its structure as a positively ordered monoid, but it seems unlikely that this is the case without assuming certain regularity properties on the C^*-algebra. Thus, we consider the auxiliary relation on $W(A)$ as an additional structure, not just as a property. This is in contrast to Cu-semigroups, where the auxiliary relation (the way-below relation) is defined in terms of the order structure, see Chapter 3.

The case when the auxiliary relation in $W(A)$ is just the way-below relation was studied in [**ABP11**]. The class of such semigroups was denoted by PreCu. See also Remarks 2.1.3(3).

We want to extend the assignment $A \mapsto W(A)$ to a functor from the category of local C^*-algebras to the category W. Thus, we need to show that a *-homomorphism $\varphi\colon A \to B$ induces a W-morphism between the respective W-semigroups. We will first prove a related result for c.p.c. order-zero maps. As shown in [**WZ09**, Corollary 4.5], a c.p.c. order-zero map between C^*-algebras induces a PoM-morphism between the respective W-semigroups. In the result below we establish that this map is also continuous, hence a generalized W-morphism.

Note that a c.p.c. order-zero map $\varphi\colon A \to B$ naturally extends to a c.p.c. order-zero map between the local C^*-algebras $M_\infty(A)$ and $M_\infty(B)$. By abuse of notation, we will denote this extension also by φ.

PROPOSITION 2.2.7 (cf. Winter, Zacharias, [**WZ09**, Corollary 4.5]). *Let A and B be local C^*-algebras. Then every c.p.c. order-zero map $\varphi\colon A \to B$ naturally induces a generalized* W*-morphism*

$$W(\varphi)\colon W(A) \to W(B), \quad [x] \mapsto [\varphi(x)]. \quad (x \in M_\infty(A)_+)$$

If φ is a *-homomorphism, then $W(\varphi)$ also preserves the auxiliary relation and thus is a W*-morphism.*

PROOF. We denote by the map $W(\varphi)$ by f. It follows from [**WZ09**, Corollary 4.5] that f is a well-defined PoM-morphism, that is, f preserves addition, order and the zero element.

To check that f is continuous, let $b \in W(B)$ and $a \in W(A)$ satisfy $a \prec f(b)$. We need to show that there exists $a' \in W(A)$ such that $a' \prec a$ and $b \leq f(a')$. To that end, let $x \in M_\infty(A)_+$ such that $a = [x]$. Then, by definition of $\prec$ on $W(B)$, there exists $\varepsilon > 0$ such that $b \leq [(\varphi(x) - \varepsilon 1_{\widetilde{B}})_+]$. We have

$$\varphi(x - \varepsilon 1_{\widetilde{A}}) = \varphi(x) - \varepsilon\varphi(1_{\widetilde{A}}) \geq \varphi(x) - \varepsilon 1_{\widetilde{B}}.$$

Then, using that $\varphi(x - \varepsilon 1_{\widetilde{A}})$ commutes with $\varphi(x) - \varepsilon 1_{\widetilde{B}}$ for the first step, and that $\varphi(y)_+ = \varphi(y_+)$ for every self-adjoint y at the second step, we deduce that

$$(\varphi(x) - \varepsilon 1_{\widetilde{B}})_+ \leq \varphi(x - \varepsilon 1_{\widetilde{A}})_+ = \varphi((x - \varepsilon 1_{\widetilde{A}})_+).$$

For $a' := [(x - \varepsilon 1_{\widetilde{A}})_+]$ we have $a' \prec a$ in $W(A)$ and $b \leq f(a')$, as desired.

If φ is a *-homomorphism, then $\varphi((x-\varepsilon)_+) = (\varphi(x)-\varepsilon)_+$ for each $x \in M_\infty(A)_+$, which implies that $f = W(\varphi)$ preserves the auxiliary relation. $\square$

It follows that W is a functor from the category C^*_{loc} of local C^*-algebras with *-homomorphisms to the category W.

2.2.8 (Inductive limits in C^*_{loc}). Let $(A_i, \varphi_{i,j})$ be an inductive system in the category C^*_{loc}, indexed over the directed set I. To construct the inductive limit in C^*_{loc}, we first consider the algebraic inductive limit A_{alg} with the pre-norm defined by

$$\|x\| = \inf\left\{\|\varphi_{i,j}(x)\| \ \mid \ j \in J, j \geq i\right\},$$

for $x \in A_i$. The set $N = \{x \in A_{\mathrm{alg}} \mid \|x\| = 0\}$ is a two-sided *-ideal. Set

$$C^*_{\mathrm{loc}}\text{-}\varinjlim A_i := A_{\mathrm{alg}}/N,$$

which is a local C^*-algebra satisfying the universal properties of an inductive limit.

Observe that each for each $i, j \in I$ satisfying $i \leq j$, the map $\varphi_{i,j}$ induces a natural *-homomorphism $M_\infty(A_i) \to M_\infty(A_j)$, which we denote by $\tilde{\varphi}_{i,j}$. The limit of the inductive system $(M_\infty(A_i), \tilde{\varphi}_{ij})$ is naturally isomorphic to $M_\infty(C^*_{\mathrm{loc}}\text{-}\varinjlim A_i)$.

THEOREM 2.2.9. *The functor $W\colon C^*_{\mathrm{loc}} \to$ W is continuous.*

Proof. Let $(A_i, \varphi_{i,j})$ be an inductive system in C^*_{loc}. We let A be the algebraic inductive limit, and we let N denote the *-ideal defined in Paragraph 2.2.8 so that the inductive limit in C^*_{loc} is given by A/N with *-homomorphisms $\varphi_{i,\infty}\colon A_i \to A/N$. As explained in Paragraph 2.2.8, we may replace each A_i by $M_\infty(A_i)$. Then every Cuntz class in $W(A_i)$ is realized by a positive element in A_i. It follows that also every class in $W(A/N)$ is realized by a positive element in A/N, that is, $W(A/N) = (A/N)_+/{\sim}$.

The following diagram shows the algebras and maps to be constructed.

$$\begin{tikzcd}
A_i \arrow[r, "\varphi_{i,j}"] & A_j \arrow[r] & \ldots \arrow[r] & A \arrow[r, two heads] & A/N
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & W(A/N) \\
W(A_i) \arrow[r, "\psi_{i,j}"] & W(A_j) \arrow[r] \arrow[urrr, "W(\varphi_{j,\infty})"] \arrow[drr, "\psi_{j,\infty}"'] & \ldots \arrow[r] & \operatorname{PreW-\varinjlim} W(A_i) \arrow[r, two heads] \arrow[ur, "\omega"] \arrow[d, equal] & \operatorname{W-\varinjlim} W(A_i) \arrow[d, equal] \\
 & & & S \arrow[r, two heads, "\beta"'] & \mu(S)
\end{tikzcd}$$

For $i \leq j$, set $\psi_{i,j} := W(\varphi_{i,j})$ and consider the induced W-inductive system $(W(A_i), \psi_{i,j})$. Let S be the inductive limit in PreW; see Corollary 2.1.9. Denote the W-morphisms into the limit by $\psi_{i,\infty}\colon W(A_i) \to S$. The maps $W(\varphi_{i,\infty})\colon W(A_i) \to W(A/N)$ induce a W-morphism $\omega\colon S \to W(A/N)$. The limit in the category W is given as the W-completion of the limit in PreW; see Corollary 2.1.9. Thus, to show that $W(A/N)$ is the inductive limit of the $W(A_i)$ in the category W, we need to prove that ω has the universal property of the W-completion.

Since every positive element in A/N is the image of a positive element in some A_i, we conclude that ω is surjective. It remains to show that ω has the following property: Given $s, t \in S$ with $\omega(s) \leq \omega(t)$, it follows that $s^{\prec} \subset t^{\prec}$ in S.

So let $s, t \in S$ satisfy $\omega(s) \leq \omega(t)$ and let s' be an element in S with $s' \prec s$. Then there is i and elements $a', a \in W(A_i)$ such that $a' \prec a$ in $W(A_i)$ and $s' = \psi_{i,\infty}(a')$ and $s = \psi_{i,\infty}(a)$. We may assume that t is also realized by an element in $W(A_i)$ (by passing to a larger index, if necessary). This means there is $b \in W(A_i)$ with $t = \psi_{i,\infty}(b)$.

Let x and y be positive elements in A_i such that $a = [x]$ and $b = [y]$. By definition of the relation $\prec$ in $W(A_i)$, there is $\varepsilon > 0$ such that $a' \leq [(x-\varepsilon)_+]$. Note that $\omega(s)$ and $\omega(t)$ are the Cuntz classes of $\varphi_{i,\infty}(x)$ and $\varphi_{i,\infty}(y)$ in A/N, respectively. By assumption, $\varphi_{i,\infty}(x) \precsim \varphi_{i,\infty}(y)$. Using Rørdam's lemma, see Paragraph 2.2.2, there is $\delta > 0$ and $r \in A/N$ such that

$$\left(\varphi_{i,\infty}(x) - \frac{\varepsilon}{2}\right)_+ = r\left(\varphi_{i,\infty}(y) - \delta\right)_+ r^*.$$

Then there is j and $\bar{r} \in A_j$ such that $r = \varphi_{j,\infty}(\bar{r})$. We may assume $j \geq i$. It follows that

$$\varphi_{i,\infty}\left((x - \frac{\varepsilon}{2})_+\right) = \varphi_{j,\infty}\left(\bar{r}\varphi_{i,j}((y-\delta)_+)\bar{r}^*\right).$$

Using the description of the limit in C^*_{loc}, see Paragraph 2.2.8, this implies that there exists $k \geq j$ such that

$$\left\|\varphi_{i,k}\left((x - \frac{\varepsilon}{2})_+\right) - \varphi_{j,k}\left(\bar{r}\varphi_{i,j}((y-\delta)_+)\bar{r}^*\right)\right\| \leq \frac{\varepsilon}{2}.$$

Using [**Rør92**, Lemma 2.2] at the second step, we get that

$$\begin{aligned}(\varphi_{i,k}(x)-\varepsilon)_+ &= \left(\varphi_{i,k}\left((x-\frac{\varepsilon}{2})_+\right)-\frac{\varepsilon}{2}\right)_+ \\ &\precsim \varphi_{j,k}\left(\bar{r}\varphi_{i,j}((y-\delta)_+)\bar{r}^*\right) \\ &\precsim \left(\varphi_{j,k}(y)-\delta\right)_+ .\end{aligned}$$

It follows that $\psi_{i,\infty}([(x-\varepsilon)_+]) \leq \psi_{i,\infty}([(y-\delta)_+])$. Using this at the third step, we obtain that

$$s' = \psi_{i,\infty}(a') \leq \psi_{i,\infty}([(x-\varepsilon)_+]) \leq \psi_{i,\infty}([(y-\delta)_+]) \prec \psi_{i,\infty}([y]) = t.$$

Hence, $s' \prec t$ as desired. $\square$

CHAPTER 3

Completed Cuntz semigroups

In the first part of this chapter, we recall the definition of the category Cu of (abstract) completed Cuntz semigroups, as introduced in [**CEI08**]. We show that Cu is a full, reflective subcategory of PreW, see Theorem 3.1.10. The reflection of a PreW-semigroup S in Cu is called its Cu-*completion*. Since PreW has inductive limits, the same holds for Cu. This generalizes [**CEI08**, Theorem 2] and it provides a new description of inductive limits in Cu; see Corollary 3.1.11.

In the second part, we consider the functor $\mathrm{Cu}\colon C^* \to \mathrm{Cu}$, as introduced in [**CEI08**]. It associates to a C^*-algebra A the set of Cuntz equivalence classes of positive elements in the stabilization of A, that is, $\mathrm{Cu}(A) = (A \otimes \mathrm{K})_+/{\sim}$. It turns out that $\mathrm{Cu}(A)$ is an object in Cu and we call it the *completed Cuntz semigroup* of A. The main result of this chapter, Theorem 3.2.8, states that for every C^*-algebra A, its completed Cuntz semigroup $\mathrm{Cu}(A)$ is naturally isomorphic to the Cu-completion of its pre-completed Cuntz semigroup $W(A)$. Moreover, all involved functors are continuous.

3.1. The category Cu

3.1.1 (Axioms for the category Cu). Given a positively ordered monoid S, the following axioms were introduced in [**CEI08**], see also [**Rob13a**]. Recall the definition of the compact containment relation $\ll$ from Paragraph 2.1.1.

(O1) Every increasing sequence $(a_n)_{n\in\mathbb{N}}$ in S has a supremum $\sup_n a_n \in S$.
(O2) Every element $a \in S$ is the supremum of a sequence $(a_n)_n$ such that $a_n \ll a_{n+1}$ for all n.
(O3) If $a', a, b', b \in S$ satisfy $a' \ll a$ and $b' \ll b$, then $a' + b' \ll a + b$.
(O4) If $(a_n)_n$ and $(b_n)_n$ are increasing sequences in S, then $\sup_n(a_n + b_n) = \sup_n a_n + \sup_n b_n$.

A sequence as in (O2) is called *rapidly increasing*.

Definition 3.1.2 (The category Cu; c.f. Coward-Elliott-Ivanescu, [**CEI08**]). A Cu-*semigroup* is a positively ordered monoid that satisfies axioms (O1)-(O4) from Paragraph 3.1.1. Given two Cu-semigroups S and T, a Cu-*morphism* $f\colon S \to T$ is a PoM-morphism that preserves compact containment and suprema of increasing sequences. We denote the collection of such maps by $\mathrm{Cu}(S,T)$. We let Cu be the category whose objects are Cu-semigroups and whose morphisms are Cu-morphisms.

A *generalized* Cu-*morphism* between two Cu-semigroups is a Cu-morphism that does not necessarily preserve compact containment, i.e., a PoM-morphism that preserves suprema of increasing sequences. We denote the set of generalized Cu-morphisms by $\mathrm{Cu}[S,T]$.

Remarks 3.1.3. (1) In lattice theory, a partially ordered set M is called a *directed complete partially ordered set*, often abbreviated to **dcpo**, if each upward directed set in M has a supremum; see [**GHK$^+$03**, Definition 0.2.1, p. 9]. If the existence of suprema is only required for increasing sequences, then M is called an ω-**dcpo**.

A **dcpo** M is called *continuous* if each element a in M is the supremum of the elements compactly contained in a; see [**GHK$^+$03**, Definition I.1.6, p. 54]. Recall from Paragraph 2.1.1 that we use a sequential version of compact containment. An ω-**dcpo** is called *ω-continuous* if every element a is the supremum of a sequence $(a_k)_k$ where a_k is sequentially compactly contained in a_{k+1} for each k.

Thus, axioms (O1) and (O2) mean exactly that the positively ordered monoid in question is a ω-continuous ω-**dcpo**.

(2) Let S be a positively ordered monoid, considered with the derived auxiliary relation $\ll$. Recall from Paragraph 2.1.1 that S is called *countably-based* if there exists a countable subset $B \subset S$ such that, whenever $a', a \in S$ satisfy $a' \ll a$, there exists $b \in B$ with $a' \leq b \ll a$. If S satisfies (O1) and (O2) from Paragraph 3.1.1, then this is equivalent to the condition that every $a \in S$ is the supremum of a rapidly increasing sequence $(a_k)_k$ with $a_k \in B$ for each k. It is easy to check that a countably-based positively ordered monoid satisfying (O1) and (O2) is a continuous **dcpo**.

(4) A PoM-morphism between Cu-semigroups preserves suprema of increasing sequences if and only if it is sequentially continuous for the so-called Scott-topology; see [**GHK$^+$03**, Definition II-2.2, p. 158].

(3) Axioms (O3) and (O4) mean precisely that the addition map $+\colon S \times S \to S$ is a Cu-bimorphism, as defined in Definition 6.3.1.

Given a Cu-semigroup S, it is easily checked that the pair $(S, \ll)$ is a W-semigroup. The next result implies that under this identification, the notions of (generalized) Cu-morphisms and (generalized) W-morphisms agree.

LEMMA 3.1.4. *Let S and T be two* Cu*-semigroups, and let $f\colon S \to T$ be a* PoM*-morphism. Then the following are equivalent:*

(1) The map $f\colon S \to T$ preserves suprema of increasing sequences.
(2) The map $f\colon S \to T$ is continuous in the sense of Definition 2.1.2.
(3) We have $f(a) = \sup f(a^{\ll})$ for each $a \in S$.

PROOF. For the implication '(1)$\Rightarrow$(2)', let $a \in S$ and $b \in T$ satisfy $b \ll f(a)$. Using (O2) for S, there exists a rapidly increasing sequence $(a_k)_k$ in S with $a = \sup_k a_k$. Since $f(a) = \sup_k f(a_k)$ by assumption, it follows that there is k such that $b \leq f(a_k)$. Then a_k has the desired properties.

The implication '(2)$\Rightarrow$(3)' was shown in Remarks 2.1.3(4).

Finally, to show the implication '(3)$\Rightarrow$(1)', let $(a_k)_k$ be an increasing sequence in S. Set $a = \sup_k a_k$. We clearly have $f(a) \geq \sup_k f(a_k)$. For the converse inequality, we choose a rapidly increasing sequence $(c_n)_n$ in S satisfying $a = \sup_n c_n$. It follows from the assumption that $f(a) = \sup_n f(c_n)$. Now, given any $b \in T$ satisfying $b \ll f(a)$, there exists an index n such that $b \leq f(c_n)$. Since $c_n \ll a = \sup_k a_k$, there is an index k such that $c_n \leq a_k$. Then

$$b \leq f(c_n) \leq f(a_k) \leq \sup_k f(a_k).$$

Thus, for every b satisfying $b \ll f(a)$ we have shown that $b \leq \sup_k f(a_k)$. It follows that $f(a) \leq \sup_k f(a_k)$, as desired. $\square$

3.1.5 (Cu is a full subcategory of W). Consider the functor Cu $\to$ W that maps a Cu-semigroup S to the W-semigroup $(S, \ll)$, and that sends a Cu-morphism $f\colon S \to T$ to the same map f, considered as a W-morphism. It follows from Lemma 3.1.4 that this functor is fully faithful. This means that we may consider Cu as a full subcategory of W, and we will therefore usually not distinguish between a Cu-semigroup S and the associated W-semigroup $(S, \ll)$.

We will now show that Cu is a reflective subcategory of W, using again the idea of a universal completion as described in [**KL09**, § 2]. We first show that every PreW-semigroup S can be suitably completed to a Cu-semigroup $\gamma(S)$; see Proposition 3.1.6. We then show that this has the desired universal properties; see Theorem 3.1.8.

The proof of the following result is inspired by the so-called round ideal completion, that associates to a partially ordered set with an auxiliary relation a continuous **dcpo**, see [**Law97**, Theorem 2.4]. The construction given here is a sequential version that also takes the additive structure into account.

PROPOSITION 3.1.6. *Let $(S,\prec)$ be a* PreW*-semigroup. Then there exist a* Cu*-semigroup $\gamma(S)$ and a* W*-morphism $\alpha\colon S\to\gamma(S)$ satisfying the following conditions:*

(1) The map α is an 'embedding' in the sense that $a'\prec a$ whenever $\alpha(a')\ll\alpha(a)$, for any $a',a\in S$.
(2) The map α has 'dense image' in the sense that for every $b',b\in\gamma(S)$ with $b'\ll b$ there exists $a\in S$ such that $b'\leq\alpha(a)\leq b$.

In particular, if S is countably-based, then so is $\gamma(S)$.

PROOF. To construct $\gamma(S)$, first consider the set $\overline{S}$ of $\prec$-increasing sequences in S. We write such sequences as $\mathbf{a}=(a_k)_k=(a_1,a_2,\ldots)$. For $\mathbf{a}$ and $\mathbf{b}$ in $\overline{S}$, we define their sum as $\mathbf{a}+\mathbf{b}=(a_k+b_k)_k$. We define a binary relation $\subset$ on $\overline{S}$ by setting for any $\mathbf{a},\mathbf{b}\in\overline{S}$:

$$\mathbf{a}\subset\mathbf{b}\quad\text{if and only if for every } k\in\mathbb{N} \text{ there exists } n\in\mathbb{N} \text{ such that } a_k\prec b_n.$$

It is easy to check that $\subset$ is a preorder on $\overline{S}$. We obtain an equivalence relation by setting for any $\mathbf{a},\mathbf{b}\in\overline{S}$:

$$\mathbf{a}\sim\mathbf{b}\quad\text{if and only if}\quad\mathbf{a}\subset\mathbf{b}\text{ and }\mathbf{b}\subset\mathbf{a}.$$

We denote the set of equivalence classes by

$$\gamma(S)=\overline{S}/\!\sim.$$

For an element $\mathbf{a}\in\overline{S}$, we denote its class in $\gamma(S)$ by $[\mathbf{a}]$. The relation $\subset$ induces a partial order $\leq$ on $\gamma(S)$ by setting $[\mathbf{a}]\leq[\mathbf{b}]$ if and only if $\mathbf{a}\subset\mathbf{b}$, for any $\mathbf{a},\mathbf{b}\in\overline{S}$.

Since $0\prec 0$ in S, the sequence $\mathbf{0}=(0,0,\ldots)$ is an element of $\overline{S}$. We denote its class in $\gamma(S)$ by 0. For each $\mathbf{a}\in\overline{S}$, we have that $\mathbf{0}\subset\mathbf{a}$, and therefore $0\leq[\mathbf{a}]$. It follows from axiom (W3) for S that $\mathbf{a}\subset\mathbf{b}$ implies $\mathbf{a}+\mathbf{c}\subset\mathbf{b}+\mathbf{c}$, for any $\mathbf{a},\mathbf{b},\mathbf{c}\in\overline{S}$. Thus, the addition on $\overline{S}$ induces an addition on $\gamma(S)$. Together with the partial order and the zero element, this gives $\gamma(S)$ the structure of a positively ordered monoid.

We define a binary relation $\Subset$ on $\overline{S}$ by setting for any $\mathbf{a},\mathbf{b}\in\overline{S}$:

$$\mathbf{a}\Subset\mathbf{b}\quad\text{if and only if there exists } n\in\mathbb{N} \text{ such that } a_k\prec b_n \text{ for all } k\in\mathbb{N}.$$

It is easy to check that $\Subset$ is an auxiliary relation on $\overline{S}$. This induces an auxiliary relation $\prec$ on $\gamma(S)$ by setting $[\mathbf{a}]\prec[\mathbf{b}]$ if and only if $\mathbf{a}\Subset\mathbf{b}$, for any $\mathbf{a},\mathbf{b}\in\overline{S}$.

We will now check that $\gamma(S)$ satisfies axioms (O1)-(O4). To show (O1), let an increasing sequence $\mathbf{a}^{(1)}\subset\mathbf{a}^{(2)}\subset\ldots$ in $\overline{S}$ be given. We employ a standard diagonalization argument, which is also used to show existence of suprema in the inductive limit construction in Cu, see e.g. [**APT11**, Theorem 4.34]. Write $\mathbf{a}^{(k)}=(a_1^{(k)},a_2^{(k)},\ldots)$ for each k. We inductively find indices n_k such that

$$a^{(i)}_{n_i+j}\prec a^{(k)}_{n_k}\quad\text{for all}\quad i,j\text{ with }i+j\leq k.$$

We start with $n_1 = 0$. Since $\mathbf{a}^{(1)} \subset \mathbf{a}^{(2)}$, there exists n_2 such that $a^{(1)}_{n_1+1} = a^{(1)}_1 \prec a^{(2)}_{n_2}$. Now assume n_i has been constructed for $i \leq k$. Since $\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(k)} \subset \mathbf{a}^{(k+1)}$, there exists n_{k+1} such that

$$a^{(1)}_{n_1+k}, a^{(2)}_{n_2+k-1}, \ldots, a^{(k-1)}_{n_{k-1}+2}, a^{(k)}_{n_k+1} \prec a^{(k+1)}_{n_{k+1}}$$

This completes the inductive step. After reindexing the sequences, we may assume that $n_i = i$ and therefore $a^{(i)}_{i+j} \prec a^{(i+j)}_{i+j}$ for all $i, j \geq 1$. Set $b_j = a^{(j)}_j$ for each j, and set $\mathbf{b} = (b_1, b_2, \ldots) \in \overline{S}$. It is straightforward to check that $\mathbf{b}$ is the supremum of the sequence $(\mathbf{a}^{(k)})_k$ in $\overline{S}$. It follows that

$$[\mathbf{b}] = \sup_k [\mathbf{a}^{(k)}],$$

in $\gamma(S)$, which verifies (O1). It is left to the reader to prove axiom (O4) for $\gamma(S)$.

Next, we show that $\Subset$ induces the compact containment relation $\ll$ on $\gamma(S)$. To that end, we first show that the analog of (O2) holds for $\Subset$, that is, for every $\mathbf{a} = (a_n)_n$ in $\overline{S}$ there exist elements $\mathbf{a}^{(k)} \in \overline{S}$ for $k \in \mathbb{N}$ such that

$$\mathbf{a} = \sup_k \mathbf{a}^{(k)}, \quad \text{and} \quad \mathbf{a}^{(1)} \Subset \mathbf{a}^{(2)} \Subset \mathbf{a}^{(3)} \Subset \ldots.$$

By (W1) for S, for each k there is a sequence $a_{[k,1]} \prec a_{[k,2]} \prec \ldots$ with $a_k \prec a_{[k,1]}$ and $a_{[k,i]} \prec a_{k+1}$ for all i. Set

$$\mathbf{a}^{(k)} = (a_1, a_2, \ldots, a_k, a_{[k,1]}, a_{[k,2]}, \ldots).$$

It is straightforward to check that this sequence has the desired properties.

Now let $\mathbf{a}, \mathbf{b} \in \overline{S}$ satisfy $[\mathbf{a}] \ll [\mathbf{b}]$. As explained in the previous paragraph, we can choose a sequence of elements $\mathbf{b}^{(k)} \in \overline{S}$ such that $\mathbf{b} = \sup_k \mathbf{b}^{(k)}$ and such that $b^{(k)}_k = b_k$ and $b^{(k)}_i \prec b_{k+1}$ for each $k, i \in \mathbb{N}$. Then

$$[\mathbf{a}] \ll [\mathbf{b}] = \sup_k [\mathbf{b}^{(k)}],$$

which implies that there exists an index k such that $[\mathbf{a}] \leq [\mathbf{b}^{(k)}]$. This mean that $\mathbf{a} \subset \mathbf{b}^{(k)}$, and it follows easily that $\mathbf{a} \Subset \mathbf{b}$.

Conversely, let $\mathbf{a}, \mathbf{b} \in \overline{S}$ satisfy $\mathbf{a} \Subset \mathbf{b}$. We want to show that $[\mathbf{a}] \ll [\mathbf{b}]$ in $\gamma(S)$. So let $(\mathbf{c}^{(k)})_k$ be an increasing sequence in $\overline{S}$ satisfying $[\mathbf{b}] \leq \sup_k [\mathbf{c}^{(k)}]$. Set $\mathbf{c} = \sup_k \mathbf{c}^{(k)}$ in $\overline{S}$. Then

$$\mathbf{b} \subset \mathbf{c} = \sup_k \mathbf{c}^{(k)}.$$

After reindexing, we may assume $\mathbf{c} = (c^{(k)}_k)_k$. Since $\mathbf{a} \Subset \mathbf{b}$, there is an index n such that $a_i \prec b_n$ for all i. Since $\mathbf{b} \subset \mathbf{c}$, there is some m such that $b_n \prec c_m = c^{(m)}_m$. It follows that $\mathbf{a} \subset \mathbf{c}^{(m)}$ and therefore $[\mathbf{a}] \leq [\mathbf{c}^{(m)}]$. Thus, for any two elements $\mathbf{a}, \mathbf{b} \in \overline{S}$ we have shown that

$$\mathbf{a} \Subset \mathbf{b} \text{ in } \overline{S} \quad \text{if and only if} \quad [\mathbf{a}] \ll [\mathbf{b}] \text{ in } \gamma(S).$$

It easily follows that (O2) holds in $\gamma(S)$.

To verify (O3) for $\gamma(S)$, we first show the analog for $\overline{S}$. So let $\mathbf{a}', \mathbf{a}, \mathbf{b}', \mathbf{b} \in \overline{S}$ satisfy $\mathbf{a}' \Subset \mathbf{a}$ and $\mathbf{b}' \Subset \mathbf{b}$. Then there are two indices m and n such that $a'_k \prec a_m$ for all k and such that $b'_k \prec b_n$ for all k. Set $d = \max\{m, n\}$. Using (W3) for S, it follows that

$$a'_k + b'_k \prec a_n + b_n,$$

for all k. This shows that $\mathbf{a}' + \mathbf{b}' \Subset \mathbf{a} + \mathbf{b}$. It easily follows that $\gamma(S)$ satisfies (O3). This completes the proof that $\gamma(S)$ is a Cu-semigroup.

We define the map $\alpha\colon S \to \gamma(S)$ as follows: Given $a \in S$, there exists by (W1) a sequence $a_1 \prec a_2 \prec \ldots$ that is cofinal in $a^{\prec}$. Set $\alpha(a) = [(a_1, a_2, \ldots)]$. It is straightforward to check that $\alpha(a)$ does not depend on the choice of the cofinal sequence in $a^{\prec}$ and that α is a W-morphism satisfying conditions (i) and (ii) of the statement. $\square$

DEFINITION 3.1.7. Let S be a PreW-semigroup. A Cu-*completion* of S is a Cu-semigroup T together with a W-morphism $\alpha\colon S\to T$ satisfying the following universal property: For every Cu-semigroup R and for every W-morphism $f\colon S\to R$, there exists a unique Cu-morphism $\tilde{f}\colon T\to R$ such that $f=\tilde{f}\circ\alpha$.

Any two Cu-completions of a W-semigroup are isomorphic in the following sense: If $\alpha_i\colon S\to T_i$ are two Cu-completions, then there is a unique isomorphism $\varphi\colon T_1\to T_2$ such that $\alpha_2=\varphi\circ\alpha_1$.

THEOREM 3.1.8. *Let S be a* PreW*-semigroup, let T be a* Cu*-semigroup and let $\alpha\colon S\to T$ be a* W*-morphism. Then the following are equivalent:*

(1) The map α satisfies the conditions of Proposition 3.1.6, namely:
  - *(i) The map α is an 'embedding' in the sense that $a'\prec a$ whenever $\alpha(a')\ll\alpha(a)$, for any $a',a\in S$.*
  - *(ii) The map α has 'dense image' in the sense that for every $b',b\in T$ with $b'\ll b$ there exists $a\in S$ such that $b'\leq\alpha(a)\leq b$.*

(2) The map α is a Cu*-completion of S.*

(3) For every Cu*-semigroup R and every generalized* W*-morphism $f\colon S\to R$, there exists a generalized* Cu*-morphism $\tilde{f}\colon T\to R$ such that $f=\tilde{f}\circ\alpha$. Moreover, f is a* W*-morphism if and only if $\tilde{f}$ is a* Cu*-morphism. Moreover, if $g_1,g_2\colon T\to R$ are generalized* Cu*-morphisms such that $g_1\circ\alpha\leq g_2\circ\alpha$, then $g_1\leq g_2$. (We consider the pointwise ordering among morphisms.)*

PROOF. The implication '(3)⇒(2)' is clear. To show the implication '(1)⇒(3)', let R be a Cu-semigroup, and let $f\colon S\to R$ be a generalized W-morphism. It follows from conditions (i) and (ii) that for every $t\in T$, the set $\{a\in S \mid \alpha(a)\ll t\}$ is $\prec$-upwards directed and contains a cofinal sequence. Thus, we may define a map $\tilde{f}\colon T\to R$ by

$$\tilde{f}(t):=\sup\{f(s) \mid \alpha(a)\ll t\}.$$

Let us show that $\tilde{f}$ is a generalized Cu-morphism. Let $t_1,t_2\in T$. If $t_1\leq t_2$, then it follows from condition (ii) of an auxiliary relation (see Paragraph 2.1.1) that

$$\{a\in S \mid \alpha(a)\ll t_1\}\subset\{a\in S \mid \alpha(a)\ll t_2\},$$

and therefore $\tilde{f}(t_1)\leq\tilde{f}(t_2)$.

Similarly, it follows from axiom (W3) for S that there is an inclusion

$$\{a\in S \mid \alpha(a)\ll t_1\}+\{a\in S \mid \alpha(a)\ll t_2\}\subset\{a\in S \mid \alpha(a)\ll t_1+t_2\},$$

which is moreover cofinal by (W4). It follows that $\tilde{f}(t_1+t_2)=\tilde{f}(t_1)+\tilde{f}(t_2)$. It is easy to check that $\tilde{f}$ is continuous in the sense of Definition 2.1.2. By Lemma 3.1.4, this implies that $\tilde{f}$ preserves suprema of increasing sequences. Thus, $\tilde{f}$ is a generalized Cu-morphism.

Let us check that $f=\tilde{f}\circ\alpha$. Let $a\in S$ be given. Using at the second step that α preserves the auxiliary relations and satisfies condition (i), and at the third step that f is continuous, we obtain:

$$\tilde{f}\circ\alpha(a)=\sup\{f(a') \mid \alpha(a')\ll\alpha(a)\}=\sup\{f(a') \mid a'\prec a\}=f(a).$$

We claim that if f is additionally assumed to preserve the auxiliary relation, then so does $\tilde{f}$. To see this, let $t'\ll t$ in T. Choose $x\in T$ such that $t'\ll x\ll t$. By condition (ii) for α, there is $a\in S$ such that $x\leq\alpha(a)\leq t$. Then $t'\ll\alpha(a)$, and since α is continuous, there is $a' \in S$ such that $a' \prec a$ and $t' \leq \alpha(a')$. We obtain that

$$\tilde{f}(t') \leq \tilde{f}(\alpha(a')) = f(a') \ll f(a) = \tilde{f}(\alpha(a)) \leq \tilde{f}(t).$$

Finally, assume $g_1, g_2 \colon T \to R$ are two generalized Cu-morphisms such that $g_1 \circ \alpha \leq g_2 \circ \alpha$. Given $t \in T$, let $(a_k)_k$ be an increasing sequence in S such that $t = \sup_k \alpha(a_k)$. Then

$$g_1(t) = g_1(\sup_k \alpha(a_k)) = \sup_k (g_1 \circ \alpha)(a_k) \leq \sup_k (g_2 \circ \alpha)(a_k) = g_2(t),$$

which shows that $g_1 \leq g_2$.

Let us show the implication '(2)⇒(1)'. By Proposition 3.1.6, there exists a Cu-semigroup $\gamma(S)$ and a W-morphism $\tilde{\alpha} \colon S \to \gamma(S)$ satisfying (1). We have seen that (1) implies (2). Thus, $\tilde{\alpha}$ is a Cu-completion satisfying (1). Since every two Cu-completions of S are isomorphic, it follows that every Cu-completion satisfies (1), as desired. □

Remarks 3.1.9. (1) By Proposition 3.1.6 and Theorem 3.1.8, for every PreW-semigroup S there exists a Cu-completion $\alpha \colon S \to \gamma(S)$. Given a Cu-semigroup R, assigning to a (generalized) Cu-morphism $f \colon \gamma(S) \to R$ the (generalized) W-morphism $f \circ \alpha$ is an isomorphism of the following morphism sets:

$$\begin{array}{ccc} \mathrm{W}[S,R] & \xleftarrow{\cong} & \mathrm{Cu}[\gamma(S),R] \\ \cup & & \cup \\ \mathrm{W}(S,R) & \xleftarrow{\cong} & \mathrm{Cu}(\gamma(S),R). \end{array}$$

(2) Given a PreW-semigroup S, let $\alpha \colon S \to \gamma(S)$ be its Cu-completion, as constructed in Proposition 3.1.6. As remarked in Remarks 2.1.3(2), S satisfies (W2) if and only if $a^\prec \subset b^\prec$ implies $a \leq b$. It follows that α is an order-embedding if and only if S is a W-semigroup.

(3) More generally, let $\alpha \colon S \to T$ be a W-morphism from a W-semigroup S to a Cu-semigroup T. Then α is a Cu-completion if and only if α is an order-embedding that has 'dense image' in the sense of condition (ii) from Theorem 3.1.8.

Necessity follows from (2) above. For the converse, assume α is an order-embedding. Let $a', a \in S$ such that $\alpha(a') \ll \alpha(a)$. Since α is continuous, there exists $x \in S$ with $x \prec a$ and $\alpha(a') \leq \alpha(x)$. Since α is an order-embedding, $a' \leq x$ and then $a' \prec a$, as desired.

By Paragraph 3.1.5, the category Cu is a full subcategory of PreW. Moreover, for every PreW-semigroup S, there exists a Cu-completion. As described in [**KL09**, § 2], this induces a reflection functor $\gamma \colon \mathrm{PreW} \to \mathrm{Cu}$ (see also Remarks 3.1.9(1)).

Theorem 3.1.10. *The category* Cu *is a full, reflective subcategory of* PreW.

As noticed before Corollary 2.1.9, it follows from general category theory that the category Cu has inductive limits and that the reflection functor $\gamma \colon \mathrm{PreW} \to \mathrm{Cu}$ is continuous. Thus, we obtain the following generalization of [**CEI08**, Theorem 2].

Corollary 3.1.11. *The category* Cu *has inductive limits.*

More precisely, let $(S_i, \varphi_{i,j})$ *be an inductive system in* Cu. *Then the inductive limit in* Cu *is the* Cu*-completion of the inductive limit in* PreW*:*

$$\mathrm{Cu}\text{-}\varinjlim S_i = \gamma(\mathrm{PreW}\text{-}\varinjlim S_i).$$

Remark 3.1.12. The reflection functors $\mu \colon \mathrm{PreW} \to \mathrm{W}$ from Paragraph 2.1.4 and $\gamma \colon \mathrm{PreW} \to \mathrm{Cu}$ from Theorem 3.1.10 commute in the sense that the composed functor $\gamma \circ \mu \colon \mathrm{PreW} \to \mathrm{Cu}$ is naturally isomorphic to γ. The situation is shown in the following diagram:

$$\begin{tikzcd} \mathrm{PreW} \arrow[r, "\mu"] \arrow[rr, bend right, "\gamma"'] & \mathrm{W} \arrow[r, "\gamma"] & \mathrm{Cu} \end{tikzcd}$$

More precisely, starting with a PreW-semigroup S, let us first consider the universal W-morphism $\beta_S \colon S \to \mu(S)$ from S to its W-completion. There is a universal W-morphism $\alpha_{\mu(S)} \colon \mu(S) \to \gamma(\mu(S))$ to the Cu-completion of $\mu(S)$. The composition $\alpha_{\mu(S)} \circ \beta_S \colon S \to \gamma(\mu(S))$ is a Cu-completion of S. Since any two Cu-completions are isomorphic, there is an isomorphism $\gamma(S) \cong \gamma(\mu(S))$ which intertwines α_S and $\alpha_{\mu(S)} \circ \beta_S$.

3.2. The completed Cuntz semigroup of a C*-algebra

3.2.1. Let A be C^*-algebra. In [**CEI08**], a new definition of the Cuntz semigroup is introduced as $\mathrm{Cu}(A) = (A \otimes \mathrm{K})_+/\!\sim$, the set of Cuntz equivalence classes of positive elements in the stabilization of A. (See Paragraph 2.2.2 for the definition of Cuntz equivalence.) The relation of Cuntz subequivalence induces a partial order on $\mathrm{Cu}(A)$. Using an isomorphism $M_2(\mathrm{K}) \cong \mathrm{K}$ we get an isomorphism $\psi \colon M_2(A \otimes \mathrm{K}) \to A \otimes \mathrm{K}$, which is used to obtain a well-defined addition $[x] + [y] = [\psi(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix})]$. With this structure, $\mathrm{Cu}(A)$ becomes a positively ordered monoid.

Definition 3.2.2. Let A be a C^*-algebra. We call

$$\mathrm{Cu}(A) = (A \otimes \mathrm{K})_+/\!\sim$$

the *completed Cuntz semigroup* of A.

Proposition 3.2.3 ([**CEI08**, Theorem 1]). *Let A be a C^*-algebra. Then the positively ordered monoid* $\mathrm{Cu}(A)$ *satisfies (O1)-(O4) from Paragraph 3.1.1 and is therefore a* Cu-*semigroup. If A is separable, then* $\mathrm{Cu}(A)$ *is countably-based.*

Remarks 3.2.4. (1) We will show in Theorem 3.2.8 that $\mathrm{Cu}(A)$ is isomorphic to the Cu-completion of $\mathrm{W}(A)$. This is why we call $\mathrm{Cu}(A)$ the *completed* Cuntz semigroup, and $W(A)$ the *pre-completed* Cuntz semigroup of A.

(2) Another way of looking at $\mathrm{Cu}(A)$ is to identify it with $W(A \otimes \mathrm{K})$. In fact, the *-homomorphism $A \otimes \mathrm{K} \to M_\infty(A \otimes \mathrm{K})$ given by embedding an element in the upper-left corner induces a bijection of Cuntz equivalence classes, respecting the given order and addition. Thus, as a positively ordered monoid, $\mathrm{Cu}(A)$ is nothing but $W(A \otimes \mathrm{K})$. The auxiliary relation $\prec$ on $W(A \otimes \mathrm{K})$ as defined in Definition 2.2.4 is precisely the compact containment relation, which is deduced from the order structure.

Indeed, it was shown in [**CEI08**] that for every $b \in (A \otimes \mathrm{K})_+$ and $\varepsilon > 0$ we have that $[(b-\varepsilon)_+] \ll [b]$ in $W(A \otimes \mathrm{K})$. It follows in particular that $[p] \ll [p]$ for any projection p, so projections are a natural source of compact elements in $\mathrm{Cu}(A)$ (sometimes the only source, see [**BC09**]). Given $a, b \in (A \otimes \mathrm{K})_+$, we have by definition that $[a] \prec [b]$ if and only if $[a] \leq [(b-\varepsilon)_+]$ for some $\epsilon > 0$. It follows that

$$[a] \prec [b] \text{ in } \mathrm{W}(A \otimes \mathrm{K}) \quad \text{ if and only if } \quad [a] \ll [b] \text{ in } \mathrm{Cu}(A) = W(A \otimes \mathrm{K}).$$

3.2.5. Let $\varphi \colon A \to B$ be a *-homomorphism (respectively, a c.p.c. order-zero map). Then φ naturally extends to a *-homomorphism (respectively, a c.p.c. order-zero map) between the stabilizations, which we denote by $\bar{\varphi} \colon A \otimes \mathrm{K} \to B \otimes \mathrm{K}$. By Proposition 2.2.7, this induces a (generalized) W-morphism $W(\bar{\varphi}) \colon W(A \otimes \mathrm{K}) \to$

$W(B\otimes \mathrm{K})$. Using the identification $\mathrm{Cu}(A) = W(A\otimes \mathrm{K})$, see Remarks 3.2.4(2), and Lemma 3.1.4, the map $W(\bar{\varphi})$ corresponds to a (generalized) Cu-morphism

$$\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B).$$

One obtains a functor $\mathrm{Cu}\colon C^* \to \mathrm{Cu}$; see [**CEI08**, Theorem 2].

We have seen that c.p.c. order-zero maps between C^*-algebras naturally induce generalized Cu-morphisms between their respective completed Cuntz semigroups. Another source of generalized Cu-morphisms are lower-semicontinuous 2-quasitraces. For each C^*-algebra A, these are in natural one-to-one correspondence with the generalized Cu-morphisms from $\mathrm{Cu}(A)$ to the extended positive real line, $[0,\infty]$, also called the functionals on $\mathrm{Cu}(A)$, see Paragraph 5.2.1. We refer the reader to [**BK04**, Section 2.9] and [**ERS11**, Section 4] for details.

3.2.6. Let us clarify the connection between the functors

$$\mathrm{W}\colon C^*_{\mathrm{loc}} \to \mathrm{W} \quad \text{ and } \quad \mathrm{Cu}\colon C^* \to \mathrm{Cu}.$$

The category C^* of C^*-algebras with *-homomorphisms is a full, reflective subcategory of C^*_{loc}. Indeed, assigning to a local C^*-algebra A its completion $\overline{A}$ extends to a functor $\gamma\colon C^*_{\mathrm{loc}} \to C^*$ which is left adjoint to the inclusion of C^* in C^*_{loc}.

On the other hand, we have the functor $\gamma\colon \mathrm{W} \to \mathrm{Cu}$ from Theorem 3.1.10. In Theorem 3.2.8, we will show that the functors W and Cu are intertwined by these completion functors.

Lemma 3.2.7. *Let A be a local C^*-algebra, considered as a dense subalgebra of its completion $\overline{A}$. Then:*

(1) For $x,y\in A_+$, we have $x\precsim y$ in A if and only if $x\precsim y$ in $\overline{A}$.
(2) For every $x\in \overline{A}_+$ and $\varepsilon>0$, there exists $y\in A$ such that $(x-\varepsilon)_+\precsim y\precsim x$ in $\overline{A}$.

Proof. (1): The forward implication is obvious. So assume $x\precsim y$ in $\overline{A}$. Then $x=\lim_k z_k^*yz_k$ for some sequence $(z_k)_k$ in $\overline{A}$. Since A is dense in $\overline{A}$, we may approximate each z_k arbitrarily well by elements from A. A diagonalization argument shows that $x\precsim y$ in A.

(2): Let x and ε be as in the statement. Since A is dense in $\overline{A}$, there exists $z\in A$ with $\|x-z\|<\frac{\varepsilon}{2}$. By [**Rør92**, Proposition 2.2], $(z-\frac{\varepsilon}{2})_+\precsim x$ in $\overline{A}$. Similarly, since $\|x-(z-\frac{\varepsilon}{2})_+\|<\varepsilon$, we obtain that $(x-\varepsilon)_+\precsim(z-\frac{\varepsilon}{2})_+$. Thus, the element $y:=(z-\frac{\varepsilon}{2})_+$ has the desired properties. □

Theorem 3.2.8. *The compositions $\gamma\circ\mathrm{W}$ and $\mathrm{Cu}\circ\gamma$ are naturally isomorphic as functors $C^*_{\mathrm{loc}}\to\mathrm{Cu}$. This means that the following diagram commutes (up to natural isomorphism):*

$$\begin{tikzcd} C^*_{\mathrm{loc}} \arrow[r, "\mathrm{W}"] \arrow[d, bend right, "\gamma"'] & \mathrm{W} \arrow[d, bend left, "\gamma"] \\ C^* \arrow[r, "\mathrm{Cu}"] \arrow[u, phantom, "\cup"] & \mathrm{Cu} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

In particular, if A is a C^*-algebra, then its completed Cuntz semigroup $\mathrm{Cu}(A)$ is naturally isomorphic to the Cu-completion $\gamma(W(A))$ of its pre-completed Cuntz semigroup $W(A)$.

Proof. Let A be a local C^*-algebra. Set $B:=M_\infty(A)$, which is again a local C^*-algebra. Let $\iota\colon B\to\overline{B}$ be the natural inclusion map into the completion. Note that there is a natural isomorphism $\overline{B}\cong\gamma(A)\otimes\mathrm{K}$.

We have $W(A)=B_+/{\sim}$ and $\mathrm{Cu}(\gamma(A))=\overline{B}_+/{\sim}$, and the *-homomorphism ι induces a W-morphism $W(\iota)\colon W(A)\to\mathrm{Cu}(\gamma(A))$. It is enough to show that $W(\iota)$ is the Cu-completion of $W(A)$. By Remarks 3.1.9(2), since $\mathrm{W}(A)$ is a W-semigroup, it is enough to check that $W(\iota)$ is an order-embedding satisfying condition (ii) of Theorem 3.1.8. But this follows directly from Lemma 3.2.7. $\square$

By Theorem 2.2.9, the functor $\mathrm{W}\colon C^*_{\mathrm{loc}} \to \mathrm{W}$ is continuous. Since $\gamma\colon \mathrm{W} \to \mathrm{Cu}$ is also continuous, we obtain from Theorem 3.2.8 the following generalization of [**CEI08**, Theorem 2].

Corollary 3.2.9. *The functor* $\mathrm{Cu}\colon C^* \to \mathrm{Cu}$ *is continuous.*

More precisely, given an inductive system of C^*-algebras A_i, there are natural isomorphisms:

$$\mathrm{Cu}(C^*\text{-}\varinjlim A_i) \quad\cong\quad \gamma(\mathrm{W}\text{-}\varinjlim \mathrm{W}(A_i)) \quad\cong\quad \mathrm{Cu}\text{-}\varinjlim \mathrm{Cu}(A_i).$$

CHAPTER 4

Additional axioms

In this chapter, we consider additional axioms for (pre)completed Cuntz semigroups. For Cu-semigroups, these are denoted by (O5) and (O6), and they are satisfied by all completed Cuntz semigroups of C^*-algebras; see Proposition 4.7. We work with a slightly stronger version of (O5) than the one that has previously appeared in the literature. The advantage is that the new (O5) passes to inductive limits in Cu; see Theorem 4.5.

We also introduce axioms (W5) and (W6) for pre-completed Cuntz semigroups, which are the exact counterparts of (O5) and (O6). Indeed, a PreW-semigroup satisfies (W5) if and only if its Cu-completion satisfies (O5), and analogously for (W6) and (O6); see Theorem 4.4.

Let A be a C^*-algebra. The axiomatic description of $\mathrm{Cu}(A)$ as an object in Cu had a positive impact in the study of the Cuntz semigroup as an invariant. For instance, the structure as a ω-continuous ω-**dcpo** provides $\mathrm{Cu}(A)$ with nice topological properties.

However, the category Cu of (abstract) Cuntz semigroups is still far bigger than the subcategory of concrete Cuntz semigroups that are isomorphic to $\mathrm{Cu}(A)$ for some C^*-algebra A. For example, it is shown in [**Rob13b**, Theorem 1.3] that the semigroup $\mathrm{Lsc}(S^2, \overline{\mathbb{N}})$ of lower-semicontinuous functions from the sphere to $\overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}$ is not the Cuntz semigroup of any C^*-algebra. In order to get a better understanding of the class of concrete Cuntz semigroups, it has been useful to determine additional axioms satisfied by Cuntz semigroups of C^*-algebras.

Definition 4.1. Let S be a Cu-semigroup.

(O5) We say that S has *almost algebraic order*, or that S satisfies (O5), if for every $a', a, b', b, c \in S$ that satisfy
$$a + b \leq c, \quad a' \ll a, \quad b' \ll b,$$
there exists $x \in S$ such that
$$a' + x \leq c \leq a + x, \quad b' \leq x.$$

(O6) We say that S has *almost Riesz decomposition*, or that S satisfies (O6), if for every $a', a, b, c \in S$ that satisfy
$$a' \ll a \leq b + c,$$
there exist elements e and f in S such that
$$a' \leq e + f, \quad e \leq a, b, \quad f \leq a, c.$$

(C) We say that S has *weak cancellation*, or that S is *weakly cancellative*, if for every $a, b, x \in S$ we have that $a + x \ll b + x$ implies that $a \ll b$.

Remarks 4.2. (1) The axiom (O5) of almost algebraic order was first considered in [**RW10**, Lemma 7.1]. It also appeared in [**ORT11**, Corollary 4.16] and [**Rob13a**, 2.1]. Note, however, that the version of (O5) given here is slightly stronger than the original versions in the literature. Nevertheless, (O5) is satisfied by every Cuntz semigroup coming from a C^*-algebra, see Proposition 4.7, and the proof is essentially the same as the original one in [**RW10**, Lemma 7.1], with additional care in the choice of the complement.

The new (O5) has the advantage that it passes to inductive limits in the category Cu; see Theorem 4.4. This seems unlikely for the original (O5), although we have no example where the original (O5) does not pass to an inductive limit. In Remarks 4.9, we show that for weakly cancellative Cu-semigroups, the new (O5) is equivalent to the original version of the axiom.

(2) Axiom (O6) was introduced in [**Rob13a**, §4]. It was shown to hold for completed Cuntz semigroups of C^*-algebras in [**Rob13a**, Proposition 5.1.1].

(3) The axiom of weak cancellation was introduced in [**RS10**, after Lemma 1]. The definition given there is equivalent to Definition 4.1. It is also equivalent to the property that $a+c \leq b+c'$ for $c' \ll c$ implies $a \leq b$, which was shown to hold in completed Cuntz semigroups of C^*-algebras with stable rank one; see [**RW10**, Theorem 4.3].

The spirit of W-semigroups is that the order relation $\leq$ is derived from the auxiliary relation $\prec$. It is therefore natural to formulate versions of axioms (O5) and (O6) only in terms of the auxiliary relation, without using the partial order. That the axioms (W5) and (W6) of Definition 4.3 are the 'correct' analogs is justified by Theorem 4.4. We also formulate the axiom of weak cancellation for PreW-semigroups, simply by replacing the compact containment relation by an arbitrary auxiliary relation. It should cause no confusion that we call this axiom 'weak cancellation' as well.

DEFINITION 4.3. Let $(S, \prec)$ be a PreW-semigroup. We define the axioms (W5), (W6) and weak cancellation for S as follows:

(W5) We say that S satisfies (W5) if for every $a', a, b', b, c, \tilde{c} \in S$ that satisfy

$$a+b \prec c, \quad a' \prec a, \quad b' \prec b, \quad c \prec \tilde{c},$$

there exist elements x' and x in S such that:

$$a'+x \prec \tilde{c}, \quad c \prec a+x', \quad b' \prec x' \prec x.$$

(W6) We say that S satisfies (W6) if for every $a', a, b, c \in S$ that satisfy

$$a' \prec a \prec b+c,$$

there exist elements e and f in S such that

$$a' \prec e+f, \quad e \prec a, b, \quad f \prec a, c.$$

(C) We say that M satisfies *weak cancellation*, or that M is *weakly cancellative*, if for every $a, b, x \in S$ we have that $a+x \prec b+x$ implies that $a \prec b$.

THEOREM 4.4. *Let S be a* PreW*-semigroup and let $\gamma(S)$ be its* Cu*-completion. Then:*

(1) The semigroup S satisfies (W5) if and only if $\gamma(S)$ satisfies (O5).
(2) The semigroup S satisfies (W6) if and only if $\gamma(S)$ satisfies (O6).
(3) The semigroup S is weakly cancellative if and only if $\gamma(S)$ is.

PROOF. Given an element $s \in S$, we will denote its image in $\gamma(S)$ by $\bar{s}$.

First, let us show that weak cancellation passes from S to its Cu-completion. Let $a, b, c \in \gamma(S)$ satisfy $a+c \ll b+c$. Using that α has dense image in the sense of Theorem 3.1.8, we can find elements $s, t \in S$ such that $\bar{s} \ll b$, $\bar{t} \ll c$ and $a+c \ll \bar{s}+\bar{t}$. Moreover, there is an increasing sequence $(r_n)_n$ in S such that $a = \sup_n \bar{r}_n$. Then $\bar{r}_n + \bar{t} \ll \bar{s}+\bar{t}$ for each n. Since α is an embedding in the sense of Theorem 3.1.8, we obtain the same inequality for the pre-images in S. Since S is weakly cancellative, we have $r_n \prec s$ for each n. Then

$$a = \sup_n \bar{r}_n \leq \bar{s} \ll b,$$

which verifies that $\gamma(S)$ is weakly cancellative.

The converse follows directly from the properties of α. Using similar methods one proves (2) and the implication that S satisfies (W5) whenever $\gamma(S)$ satisfies (O5). Details are left to the reader.

The other implication of (1) is, however, not straightforward. So assume that S satisfies (W5). We have to show that $\gamma(S)$ satisfies (O5). Let $a', a, b', b, c \in \gamma(S)$ satisfy

$$a + b \leq c, \quad a' \ll a, \quad b' \ll b.$$

Choose a rapidly decreasing sequence $(a_n)_n$ in S such that $a' \ll \bar{a}_{n+1} \ll \bar{a}_n \ll a$ for all n. Choose s' and s in S such that $b' \ll \bar{s}' \ll \bar{s} \ll b$. Finally, choose a rapidly increasing sequence $(c_n)_n$ in S such that $c = \sup_n \bar{c}_n$. We can moreover assume that $\bar{a}_1 + \bar{s} \ll \bar{c}_1$, and so $a_1 + s \prec c_1$.

We will inductively define elements x_n and x'_n in S satisfying:

$$(R_n) \qquad x'_{n-1} \prec x'_n \prec x_n, \quad a_{n+1} + x_n \prec c_{n+1}, \quad c_n \prec a_n + x'_n.$$

To make sense of (R_1), we set $x_0 := s'$. Then:

$$a_1 + s \prec c_1, \quad a_2 \prec a_1, \quad x_0 \prec s, \quad c_1 \prec c_2.$$

By (W5), there exist x'_1 and x_1 in S fulfilling (R_1).

For the inductive step, assume x'_n and x_n have been constructed satisfying (R_n). Applying (W5) to $a_{n+1} + x_n \prec c_{n+1}$ and $a_{n+2} \prec a_{n+1}$, $x'_n \prec x_n$ and $c_{n+1} \prec c_{n+2}$, we can find x'_{n+1} and x_{n+1} fulfilling (R_{n+1}).

We obtain a rapidly increasing sequence $(x'_n)_n$ in S. Using the existence of suprema in $\gamma(S)$, we may set $x := \sup_n \bar{x}'_n$. For each n, we have:

$$a' + \bar{x}'_n \leq \bar{a}_{n+1} + \bar{x}'_n \leq \bar{a}_{n+1} + \bar{x}_n \leq \bar{c}_{n+1}, \quad \bar{c}_n \leq \bar{a}_n + \bar{x}'_n \leq a + \bar{x}_n.$$

Therefore:

$$a' + x = \sup_n (a' + \bar{x}'_n) \leq \sup_n \bar{c}_{n+1} = c = \sup_n \bar{c}_n \leq \sup_n (a + \bar{x}'_n) = a + x.$$

Moreover, $x \geq \bar{x}'_1 \geq b'$, which shows that x has the desired properties. ☐

Theorem 4.5. *Let (S_i) be an inductive system in* PreW. *If each S_i satisfies (W5) (resp. (W6), or weak cancellation), then so does the inductive limit in* PreW.

Similarly, axiom (O5) (resp. (O6), weak cancellation) passes to inductive limits in Cu.

Proof. Let us verify that weak cancellation passes to inductive limits in PreW. Let a, b, c be elements in $S :=$ PreW-$\varinjlim S_i$ satisfying $a+c \prec b+c$. Then here are an index i and elements $x, y, z \in S_i$ such that $a = [x], b = [y]$ and $c = [z]$. By definition of $\prec$ on S, see Definition 2.1.7, there is $j \geq i$ such that $\varphi_{i,j}(x+z) \prec \varphi_{i,j}(y+z)$ in S_j. Using that S_j is weakly cancellative, we deduce that $\varphi_{i,j}(x) \prec \varphi_{i,j}(y)$ in S_j. It follows that $a \prec b$ in S, as desired.

It is shown analogously that the other axioms pass to inductive limits in PreW.

Now, let (S_i) be an inductive system in Cu. Assume that each S_i is weakly cancellative. Considering S_i as a PreW-semigroup that is isomorphic to its own Cu-completion, we obtain from Theorem 4.4 that S_i is weakly cancellative as a PreW-semigroup. It follows that the limit in PreW, $S :=$ PreW-$\varinjlim_i S_i$, is weakly cancellative. By Corollary 3.1.11, the limit in Cu is isomorphic to the Cu-completion of S. Using the 'only if' implication of part (3) of Theorem 4.4, we deduce that Cu-$\varinjlim_i S_i$ is weakly cancellative.

The argument for (O5) and (O6) is completely analogous. ☐

Proposition 4.6. *Let A be a local C^*-algebra. Then $W(A)$ is a* W*-semigroup satisfying (W5) and (W6).*

Proof. By Proposition 2.2.5, $\mathrm{W}(A)$ is a W-semigroup. Its Cu-completion is $\mathrm{Cu}(A)$, which satisfies (O6) by [**Rob13a**, Proposition 5.1.1]. Therefore, (W6) for $\mathrm{W}(A)$ follows from Theorem 4.4.

The proof of (W5) is based on that of [**RW10**, Lemma 7.1]. Let $a', a, b', b, c \in \mathrm{W}(A)$ satisfy $a + b \leq c$, $a' \prec a$ and $b' \prec b$. We will show that there is an element $s \in W(A)$ such that $a' + s \leq c \leq a + s$ and $b' \leq s$. One verifies that this implies that (W5) holds.

Let $\tilde{x}, \tilde{y}, z \in M_\infty(A)_+$ such that $a = [\tilde{x}]$, $b = [\tilde{y}]$ and $c = [z]$. Choose $\varepsilon > 0$ such that $a' \prec [(\tilde{x} - \varepsilon)_+]$ and $b' \prec [(\tilde{y} - \varepsilon)_+]$. Since $\tilde{x} + \tilde{y} \precsim z$, by [**Rør92**, Section 2] and Paragraph 2.2.2, there exist $\delta > 0$ and $r \in A$ such that

$$(\tilde{x} - \varepsilon)_+ + (\tilde{y} - \varepsilon)_+ = (\tilde{x} + \tilde{y} - e)_+ = r^*(z - \delta)_+ r.$$

Set $v = (z - \delta)_+^{1/2} r$, set $x = v(\tilde{x} - \varepsilon)_+^2 v^*$ and set $y = v(\tilde{x} - \varepsilon)_+^2 v^*$.

Then $x \perp y$ and

$$(\tilde{x} - \varepsilon)_+ \sim (\tilde{x} - \varepsilon)_+^2 = (\tilde{x} - \varepsilon)_+^{1/2} v^* v (\tilde{x} - \varepsilon)_+^{1/2} \sim v(\tilde{x} - \varepsilon)_+ v^* = x,$$

and similarly $(\tilde{y} - \varepsilon)_+ \sim y$.

Let B be a sub-C^*-algebra of A that contains x, y and z. We will work inside B from now on. We have shown that there exist $\delta > 0$ and orthogonal positive elements $x, y \in \mathrm{Her}((z - \delta)_+)$ such that $a' \prec [x] \leq a$ and $b' \prec [y] \leq b$.

For $\eta > 0$, let $f_\eta \colon \mathbb{R}^+ \to [0,1]$ be the function that takes value 0 at 0, value 1 on $[\eta, \infty)$ and which is linear on $[0, \eta]$. Set $e = f_\delta(z)$. Then $e \sim z$ and e acts as a unit on x and y. Choose $\beta > 0$ such that $a' \leq [(x - \beta)_+]$. Set $w = e - f_\beta(x)e$, which is positive since e commutes with x and therefore with $f_\beta(x)$. We will show that the element $s = [w]$ has the desired properties.

Observe that $w \in \mathrm{Her}(z)$, that $(x - \beta)_+ \perp w$, and that $x + w$ is strictly positive in $\mathrm{Her}(z)$. Hence,

$$\begin{aligned} a' + [w] \leq [(x - \beta)_+] + [w] = [(x - \beta)_+ \oplus w] \leq [z] = b \\ b = [z] = [x + w] \leq [x] + [w] \leq a + [w]. \end{aligned}$$

Moreover, $x + w = (x + e - f_\beta(x))e \geq \delta' e$ for some $\delta' > 0$. Therefore, since $x \perp y$,

$$y = y^{1/2} e y^{1/2} \leq \frac{1}{\delta'} y^{1/2}(x + w) y^{1/2} = \frac{1}{\delta'} y^{1/2} w y^{1/2} \precsim w,$$

and thus $b' \leq [w]$, as desired. ☐

Proposition 4.7. *Let A be a C^*-algebra. Then* $\mathrm{Cu}(A)$ *is a* Cu*-semigroup satisfying (O5) and (O6).*

Proof. By [**CEI08**, Theorem 1], $\mathrm{Cu}(A)$ is a Cu-semigroup. Axiom (O6) is verified in [**Rob13a**, Proposition 5.1.1]. Our strengthened version of (O5) follows from Proposition 4.6 combined with Theorem 4.4. ☐

Though (O5) holds for all complete Cuntz semigroups coming from C^*-algebras, we note that, under the additional hypothesis of weak cancellation, it is equivalent to the original formulation of the axiom:

(O5') If $a \leq b$ and $a' \ll a$, then there is $x \in S$ such that $a' + x \leq b \leq a + x$.

Lemma 4.8. *Let S be a* Cu*-semigroup. If S satisfies (O5), then it also satisfies (O5'). The converse holds if S is weakly cancellative.*

PROOF. It is clear that (O5) implies (O5') in general. To show the converse, assume that S is a weakly cancellative Cu-semigroup satisfying (O5'). By Theorem 4.4, it is enough to verify (W5). Suppose we are given $a + b \ll c$ and $a' \ll a$, $b' \ll b$ and $c \ll \tilde{c}$. There is $c^\sharp$ with $c \ll c^\sharp \ll \tilde{c}$. Applying (O5') to $a' \ll a \leq c^\sharp$, we obtain x such that $a' + x \leq c^\sharp \leq a + x$. Then $b + a \leq c \ll c^\sharp \leq x + a$, which by weak cancellation implies $b \ll x$. Choose x' such that $x' \ll x$, $b' \ll x'$ and $c \ll a + x'$. The elements x, x' have the desired properties to verify (W5). □

REMARKS 4.9. Let $\mathrm{W}_{5,6}$ be the full subcategory of W consisting of W-semigroups satisfying (W5) and (W6). It follows from Theorem 4.5 that $\mathrm{W}_{5,6}$ is closed under inductive limits in W and therefore has inductive limits itself. By Proposition 4.6, the functor W from Theorem 2.2.9 takes values in $\mathrm{W}_{5,6}$.

Similarly, the full subcategory $\mathrm{Cu}_{5,6}$ of Cu consisting of Cu-semigroups satisfying (O5) and (O6) is closed under inductive limits. By Proposition 4.7, the functor Cu takes values in $\mathrm{Cu}_{5,6}$.

By Theorem 4.4, the reflector $\gamma\colon \mathrm{W} \to \mathrm{Cu}$ maps $\mathrm{W}_{5,6}$ to $\mathrm{Cu}_{5,6}$.

Given a PreW-semigroup $(S, \prec)$, we say that an element $s \in S$ is *full* if, whenever there are $t', t \in S$ satisfying $t' \prec t$, then there is $n \in \mathbb{N}$ such that $t' \leq ns$. We say that an element s *cancels from sums* if $a + s \leq b + s$ implies $a \leq b$ for any a, b.

PROPOSITION 4.10. *Let $(S, \prec)$ be a* PreW-*semigroup satisfying (W5). If S contains a full element e such that $e \prec e$ and e cancels from sums, then S has weak cancellation.*

Similarly, if a Cu-*semigroup satisfying (O5) contains a full compact element that cancels from sums, then it has weak cancellation.*

PROOF. Suppose $a + c \prec b + c$. Using (W1) and (W4), there are $b' \prec b$ and $c' \prec c$ such that $a + c \leq b' + c'$. By (W1), there is c'' with $c' \prec c'' \prec c$. Since e is full, there is n with $c'' \leq ne$. By (W5), applied to $c'' + 0 \prec ne$, $c' \prec c''$, and using that $e \prec e$, there are elements $x' \prec x$ such that $c' + x \prec ne \prec c'' + x'$. Now

$$a + ne \leq a + c'' + x' \leq a + c + x' \prec b' + c' + x \leq b + ne\,,$$

from which it follows that $a \prec b$, as desired.

The analogous result for Cu-semigroups follows from Theorem 4.4. □

CHAPTER 5

Structure of Cu-semigroups

This chapter contains some general results about the structure of Cu-semigroups.

5.1. Ideals and quotients

In this section, we study ideals and quotients of Cu-semigroups. We show that (O5), (O6) and weak cancellation pass to ideals and quotients, see Proposition 5.1.3. Given a Cu-semigroup S, we denote the set of ideals in S by $\mathrm{Lat}(S)$. We show that $\mathrm{Lat}(S)$ has a natural structure as a complete lattice, see Paragraph 5.1.6. The subset of singly-generated ideals forms a sublattice, denoted by $\mathrm{Lat}_{\mathrm{f}}(S)$. We show that $\mathrm{Lat}_{\mathrm{f}}(S)$ is a Cu-semigroup, Proposition 5.1.7. In Proposition 7.2.3, we will see that $\mathrm{Lat}_{\mathrm{f}}(S)$ is naturally isomorphic to the tensor product $S \otimes \{0, \infty\}$.

Then, we consider the case of a concrete Cuntz semigroup $\mathrm{Cu}(A)$ of a C^*-algebra A. We show that there is a natural isomorphism between $\mathrm{Lat}(\mathrm{Cu}(A))$ and the lattice of ideals in A, which we denote by $\mathrm{Lat}(A)$, see Proposition 5.1.10. This isomorphism identifies the Cu-semigroup $\mathrm{Lat}_{\mathrm{f}}(\mathrm{Cu}(A))$ with the subset of $\mathrm{Lat}(A)$ consisting of ideals that contain a full, positive element. In the case that A is separable, every ideal in A is σ-unital and hence contains a positive, full (even strictly positive) element. It follows that in this case, $\mathrm{Lat}(A)$ is a Cu-semigroup, see Corollary 5.1.13.

5.1.1. Let M be a positively ordered monoid. A subset I of M is *order-hereditary* if for every $a, b \in M$ we have that $a \leq b$ and $b \in I$ imply that $a \in I$. An *ideal* (also called *order-ideal*) in M is a subsemigroup which is order-hereditary. Given a Cu-semigroup S, we shall also require that an *ideal* in S is closed under suprema of increasing sequences.

Given an ideal I in a Cu-semigroup S, we define a binary relation $\leq_I$ on S as follows: For elements $a, b \in S$, we set $a \leq_I b$ if and only if there exists $c \in I$ such that $a \leq b + c$. By symmetrizing, we define a relation $\sim_I$ on S: For elements $a, b \in S$, we set $a \sim_I b$ if and only if both conditions $a \leq_I b$ and $b \leq_I a$ are met.

It is easy to see that $\sim_I$ is a congruence relation on S. We denote the set of congruence classes by

$$S/I = S/{\sim_I} \ .$$

The partial order on S induces a partial order on S/I, giving the latter the structure of a positively ordered monoid. Given an element $a \in S$, we denote its congruence class in S/I by a_I. In the next result, we verify that S/I satisfies (O1)-(O4).

Lemma 5.1.2. *Let S be a* Cu*-semigroup, and let I be an ideal in S. Then S/I is a* Cu*-semigroup. Moreover, the map*

$$\pi_I \colon S \to S/I, \quad a \mapsto a_I, \quad (a \in S)$$

is a surjective Cu*-morphism.*

Proof. Let S and I be as in the statement. As explained in Paragraph 5.1.1, we have that S/I is a positively ordered monoid. It is also easy to see that π is a surjective PoM-morphism. The following two claims are easily verified.

Claim 1: Given two elements $x, y \in S/I$, we have that $x \leq y$ if and only if there exist representatives $a, b \in S$ such that $x = a_I$, $y = b_I$ and $a \leq b$.

Claim 2: Given an increasing sequence $(x_k)_k \subset S/I$, there exists an increasing sequence $(a_k)_k \subset S$ such that $x_k = (a_k)_I$ for each k.

To verify (O1) for S/I, let $(x_k)_k \subset S/I$ be an increasing sequence in S/I. By claim 2, we can choose an increasing sequence $(a_k)_k \subset S$ such that for each k, the element x_k is represented by a_k. Since S satisfies (O1), the sequence $(a_k)_k$ has a supremum in S which we denote by $a = \sup_k a_k$. We claim that a_I is the supremum of the sequence $(x_k)_k$ in S/I.

It is clear that $x_k \leq a_I$ for each k. Conversely, let y be an element in S/I such that $x_k \leq y$ for all k. Let $b \in S$ be an element such that $y = b_I$. Then, for each k, there exists an element $c_k \in I$ such that $a_k \leq b + c_k$. Set

$$c = \sum_{k=0}^{\infty} c_k = \sup_n \sum_{k=0}^{n} c_k,$$

which is an element in I. We obtain that

$$a_k \leq b + c_k \leq b + c,$$

for each k. By definition of a, this implies that

$$a = \sup_k a_k \leq b + c.$$

Since $c \in I$, we get that $a \leq_I b$ and therefore $x \leq y$, as desired. It also follows from the above argument that π preserves suprema of increasing sequences.

Next, we show that π preserves the way-below relation. So let $a, b \in S$ satisfy $a \ll b$ in S. To show that $a_I \ll b_I$ in S/I, let $(x_k)_k \subset S/I$ be an increasing sequence satisfying $b_I \leq \sup_k x_k$. By claim 2, we can choose an increasing sequence $(b_k)_k \subset S$ such that $x_k = (b_k)_I$ for each k. Then

$$b_I \leq \sup_k x_k = (\sup_k b_k)_I,$$

whence there exists $c \in I$ such that $b \leq (\sup_k b_k) + c$. Using that S satisfies (O4), we obtain that

$$a \ll b \leq (\sup_k b_k) + c = \sup_k (b_k + c).$$

Therefore, there exists $n \in \mathbb{N}$ such that $a \leq b_n + c$, and hence $a_I \leq (b_n)_I = x_n$, as desired.

To verify (O2) for S/I, let $x \in S/I$. Choose $a \in S$ such that $x = a_I$. Since S satisfies (O2), we can choose a rapidly increasing sequence $(a_k)_k \subset S$ such that $a = \sup_k a_k$. For each k, set $x_k = (a_k)_I$. It follows that $(x_k)_k$ is a rapidly increasing sequence in S/I such that $x = \sup_k x_k$. This finishes the proof of (O2) for S/I. Finally, it is straightforward to verify the axioms (O3) and (O4) for S/I. □

PROPOSITION 5.1.3. *Let S be a* Cu*-semigroup, and let I be an ideal in S. If S satisfies (O5) (resp. (O6), or weak cancellation), the so does the ideal I and the quotient S/I.*

PROOF. It is easy to verify that each of the axioms passes to ideals. To show that (O5) passes to quotients, let S be a Cu-semigroup and let I be an ideal in S. Assume that S satisfies (O5). To verify (O5) for S/I, let $a', a, b', b, c \in S/I$ satisfy

$$a + b \leq c, \quad a' \ll a, \quad b' \ll b.$$

We can choose $s, t, r \in S$ such that

$$a = s_I, \quad b = t_I, \quad c = r_I, \quad s + t \leq r.$$

Since the quotient map is continuous, there exist $s', t' \in S$ such that

$$s' \ll s, \quad t' \ll t, \quad a' \le (s')_I, \quad b' \le (t')_I.$$

Since S satisfies (O5), there exists $x \in S$ such that

$$s' + x \le r \le s + x, \quad t' \le x.$$

Then x_I has the desired properties to verify (O5) for S/I. The proofs that (O6) and weak cancellation pass to quotients can be obtained with the same technique and are left to the reader. □

Remark 5.1.4. It is possible to define the notion of ideals and quotients in the category PreW. We do not pursue this idea.

Problem 5.1.5. Let S be a Cu-semigroup, and let I be an ideal in S. Assume that I and S/I satisfy (O5) (resp. (O6), weak cancellation). Under what assumptions does this imply that S itself satisfies the respective axiom?

5.1.6. Let S be a Cu-semigroup. We denote the set of all ideals in S by $\mathrm{Lat}(S)$. Inclusion of ideals defines a partial order on $\mathrm{Lat}(S)$. Given two ideals I and J, we define their sum as

$$I + J = \{ a \in S \ | \ a \le y + z, \text{ for some } y \in I, z \in J \}.$$

In particular, we have $I + I = I$ for every $I \in \mathrm{Lat}(S)$. Observe also that $I \subseteq J$ if and only if $I + J = J$, whence the ordering is algebraic. This endows $\mathrm{Lat}(S)$ with the structure of an algebraically ordered monoid.

Let $(I_\lambda)_\lambda \subset \mathrm{Lat}(S)$ be a family of ideals. It is easy to check that the intersection $\bigcap_\lambda I_\lambda$ is again an ideal. Clearly, this is the largest ideal contained in each I_λ. Therefore, the family $(I_\lambda)_\lambda$ has an infimum in $\mathrm{Lat}(S)$, given by $\bigwedge_\lambda I_\lambda = \bigcap_\lambda I_\lambda$.

On the other hand, the union $\bigcup_\lambda I_\lambda$ is an order-hereditary submonoid that, however, is in general not closed under suprema of increasing sequences in S. Nevertheless, using that S is itself an ideal in S, the family $(I_\lambda)_\lambda$ has a supremum in $\mathrm{Lat}(S)$ given by

$$\bigvee_\lambda I_\lambda = \bigcap \{ J \in \mathrm{Lat}(S) \ | \ I_\lambda \subset J \text{ for all } \lambda \}.$$

This shows that $\mathrm{Lat}(S)$ is a complete lattice.

Let $M = \bigcup_\lambda I_\lambda$. As mentioned above, M is an order-hereditary submonoid of S. We claim that the supremum of the family $(I_\lambda)_\lambda$ is also given by

$$\bigvee_\lambda I_\lambda = \left\{ \sup_n a_n \ | \ (a_n)_n \subset M \text{ rapidly increasing sequence} \right\}.$$

To see this, let us temporarily denote the right hand side in the equation above by P. Using that S satisfies (O3) and (O4), it follows easily that P is closed under addition. To show that P is order-hereditary, let $a, b \in S$ satisfy $a \le b$ and $b \in P$. By definition of P, there exists an increasing sequence $(b_n)_n \subset M$ such that $b = \sup_n b_n$. Since S satisfies (O2), we can choose a rapidly increasing sequence $(a_k)_k \subset S$ such that $a = \sup_k a_k$. For each k, we have

$$a_k \ll a \le b = \sup_n b_n,$$

whence there exists $n(k) \in \mathbb{N}$ such that $a_k \le b_{n(k)}$. Since M is order-hereditary, this implies that $a_k \in M$, and hence $a \in P$, as desired. Finally, a standard diagonalization argument shows that P is closed under suprema of increasing sequences. Thus, P is an ideal of S that contains I_λ for each λ. Since P is clearly the smallest ideal with this property, we have $P = \bigvee_\lambda I_\lambda$.

It follows that an element $a \in S$ is contained in $\bigvee_\lambda I_\lambda$ if and only if for every $a' \in S$ satisfying $a' \ll a$ we have that a' is contained in $\bigcup_\lambda I_\lambda$.

Given $a \in S$, we denote by $\mathrm{Idl}(a)$ the ideal generated by a, that is:

$$\mathrm{Idl}(a) = \{x \in S \ \mid \ x \leq \infty \cdot a\}.$$

We claim that $\mathrm{Idl}(a') \ll \mathrm{Idl}(a)$ in $\mathrm{Lat}(S)$ for any $a', a \in S$ satisfying $a' \ll a$. To prove the claim, let $(I_k)_{k\in\mathbb{N}} \subset \mathrm{Lat}(S)$ be an increasing sequence with $\mathrm{Idl}(a) \subset \bigvee_k I_k$. Then $a' \ll a \in \bigvee_k I_k$ and therefore $a' \in \bigcup_k I_k$. Thus, there is $n \in \mathbb{N}$ such that $a' \in I_n$. But this implies $\mathrm{Idl}(a') \subset I_n$, which proves the claim.

Let $a \in S$. Since S satisfies (O2), there exists a rapidly increasing sequence $(a_n)_n \subset S$ with $a = \sup_n a_n$. It follows that $\mathrm{Idl}(a)$ is the supremum of the rapidly increasing sequence $(\mathrm{Idl}(a_n))_n$ in $\mathrm{Lat}(S)$.

However, this is no longer true for general ideals in a Cu-semigroup. We define

$$\mathrm{Lat}_\mathrm{f}(S) = \{\mathrm{Idl}(a) \ \mid \ a \in S\} \subset \mathrm{Lat}(S),$$

which is the set of singly-generated ideals in S.

Note that for an ideal I in S, the following are equivalent:

(1) We have $I \in \mathrm{Lat}_\mathrm{f}(S)$, that is, I is generated by a single element.
(2) The ideal I is generated by countably many elements.
(3) The ideal I has a maximal element, denoted by $\bigvee I$, and then

$$I = \mathrm{Idl}\left(\bigvee I\right) = \left\{x \in S \ \mid \ x \leq \bigvee I\right\}.$$

It is clear that (1) implies (2), and that (3) implies (1). To show that (2) implies (3), assume that I is an ideal that is generated by a countable set of elements, say $\{a_0, a_1, a_2, \ldots\} \subset S$. Then the element

$$s = \infty \cdot \sum_{k=0}^{\infty} a_k = \sup_n \sum_{k=0}^{n} n a_k,$$

is contained in I. Since $a_k \leq s$ for each k, it is clear that $I = \mathrm{Idl}(s)$. Since, moreover, $\infty \cdot s = s$, we also have that $s = \bigvee I$.

Proposition 5.1.7. *Let S be a* Cu*-semigroup. Then* $\mathrm{Lat}_\mathrm{f}(S)$ *is a* Cu*-semigroup satisfying (O5). If S satisfies (O6), then so does* $\mathrm{Lat}_\mathrm{f}(S)$*. Moreover, the map*

$$S \to \mathrm{Lat}_\mathrm{f}(S), \quad a \mapsto \mathrm{Idl}(a), \quad (a \in S)$$

is a surjective Cu*-morphism.*

If S is countably-based, then

$$\mathrm{Lat}(S) = \mathrm{Lat}_\mathrm{f}(S).$$

Proof. We denote the map $S \to \mathrm{Lat}_\mathrm{f}(S)$ from the statement by Idl. Let $a, b \in S$. It is easy to see that $\mathrm{Idl}(a+b) = \mathrm{Idl}(a) + \mathrm{Idl}(b)$. Moreover, we have $\mathrm{Idl}(a) \subset \mathrm{Idl}(b)$ if and only if $\infty \cdot a \leq \infty \cdot b$. It follows that $\mathrm{Lat}_\mathrm{f}(S)$ is an algebraically ordered submonoid of $\mathrm{Lat}(S)$. We also get that the map Idl is a PoM-morphism.

To verify (O1) for $\mathrm{Lat}_\mathrm{f}(S)$, let $(I_n)_n$ be an increasing sequence of singly-generated ideals. The supremum $\bigvee_n I_n$ in $\mathrm{Lat}(S)$ is a countably-generated ideal. As observed in Paragraph 5.1.6, this implies that $\bigvee_n I_n \in \mathrm{Lat}_\mathrm{f}(S)$. It follows that $\bigvee_n I_n$ is the supremum of $(I_n)_n$ in $\mathrm{Lat}_\mathrm{f}(S)$, which verifies (O1).

In Paragraph 5.1.6, we have already observed that $\mathrm{Lat}_\mathrm{f}(S)$ satisfies (O2) and that the map Idl preserves the way-below relation. Then, it is easy to check that $\mathrm{Lat}_\mathrm{f}(S)$ satisfies (O3) and (O4), and that Idl is a surjective Cu-morphism. Moreover, since the order on $\mathrm{Lat}_\mathrm{f}(S)$ is algebraic, (O5) holds trivially.

Next, let us show that $\mathrm{Lat}_\mathrm{f}(S)$ satisfies (O6), assuming that S does. So let $I', I, J, K \in \mathrm{Lat}_\mathrm{f}(S)$ satisfy

$$I' \ll I \subset J + K.$$

Choose $a \in S$ such that $I = \mathrm{Idl}(a)$. Since $I' \ll I$, there exists $a' \in S$ such that

$$a' \ll a, \quad I' \subset \mathrm{Idl}(a'), \quad I = \mathrm{Idl}(a).$$

Moreover, since $I \subset J + K$, there exist elements $b, c \in S$ such that

$$a \leq b + c, \quad J = \mathrm{Idl}(b), \quad K = \mathrm{Idl}(c).$$

Using that S satisfies (O6), there exist elements $e, f \in S$ such that

$$a' \leq e + f, \quad e \leq a, b, \quad f \leq a, c.$$

It is now easy to check that the ideals $\mathrm{Idl}(e)$ and $\mathrm{Idl}(f)$ have the desired properties to verify (O6) for $\mathrm{Lat}_{\mathrm{f}}(S)$.

Finally, assume that S is a countably-based Cu-semigroup. Given an ideal I in S, it is straightforward to check that I is generated by countably many elements. As observed in Paragraph 5.1.6, this implies that $I \in \mathrm{Lat}_{\mathrm{f}}(S)$, as desired. $\square$

Remark 5.1.8. In Proposition 7.2.3, we will show that there is a natural isomorphism

$$\mathrm{Lat}_{\mathrm{f}}(S) \cong S \otimes \{0, \infty\}.$$

5.1.9. Let A be a C^*-algebra, and let I be an ideal in A. (By an ideal in a C^*-algebra, we always mean a closed, two-sided ideal.) The inclusion map $\iota\colon I \to A$ induces a Cu-morphism

$$\mathrm{Cu}(\iota)\colon \mathrm{Cu}(I) \to \mathrm{Cu}(A).$$

It is shown in [**Ciu08**, Proposition 3.1.1] that $\mathrm{Cu}(\iota)$ is an order-embedding. We may therefore identify $\mathrm{Cu}(I)$ with a subsemigroup of $\mathrm{Cu}(A)$. (The assumption that the C^*-algebra is separable is not needed in the proof of [**Ciu08**, Proposition 3.1.1].) In fact, the argument is not difficult and we include it for completeness.

First, we show that $\mathrm{Cu}(\iota)$ is an order-embedding. We may assume that A and I are stable. Let $x, y \in I_+$ such that x is Cuntz-subequivalent to y in A. Then, using Rørdam's lemma, see Paragraph 2.2.2, we have that for every $\varepsilon > 0$ there exists $\delta > 0$ and $r \in A$ such that

$$(x - \varepsilon)_+ = r(y - \delta)_+ r^*.$$

Let $f_\delta\colon \mathbb{R} \to [0,1]$ be the function that takes value 0 on $(-\infty, \delta/2)$, that takes value 1 on $[\delta, \infty)$, and that is linear on $[\delta/2, \delta]$. By functional calculus, we obtain that

$$(y - \delta)_+ = f_\delta(y)(y - \delta)_+ f_\delta(y).$$

This implies that

$$(x - \varepsilon)_+ = [r f_\delta(y)](y - \delta)_+ [r f_\delta(y)]^*.$$

Since $f_\delta(y) \in I$ and since I is an ideal, we have that $r f_\delta(y) \in I$. Then, using Rørdam's lemma in the other direction, it follows that x is Cuntz subequivalent to y in I, as desired.

Let us also show that $\mathrm{Cu}(I)$ is an ideal in $\mathrm{Cu}(A)$. First, it is clear that $\mathrm{Cu}(I)$ is a submonoid of $\mathrm{Cu}(A)$. To show that it is an order-hereditary subset, let $a, b \in \mathrm{Cu}(A)$ satisfy $a \leq b$ and $b \in \mathrm{Cu}(I)$. Choose $x \in A_+$ and $y \in I_+$ such that $a = [x]$ and $b = [y]$. By definition, there exists a sequence $(r_k)_k \subset A$ such that $x = \lim_k r_k y r_k^*$. Since I is an ideal, we have that $r_k y r_k^* \in I$ for each k. As I is also closed, we get $x \in I$ and so $a \in \mathrm{Cu}(I)$, as desired.

Finally, we need to show that $\mathrm{Cu}(I)$ is closed under suprema of increasing sequences. So let $(a_k)_k \subset \mathrm{Cu}(I)$ be an increasing sequence with $a = \sup_k a_k \in \mathrm{Cu}(A)$. Choose representatives $x_k \in I_+$ for $k \in \mathbb{N}$ and $x \in A_+$ such that $a = [x]$ and $a_k = [x_k]$ for each k. We need to show that $a \in I$. Let $\varepsilon > 0$. Then

$$[(x - \varepsilon)_+] \ll [x] = a = \sup_k a_k,$$

which implies that there exists $n \in \mathbb{N}$ such that $[(x-\varepsilon)_+] \leq a_k$. We have already observed that this implies that $(x-\varepsilon)_+ \in I$. Since this holds for every $\varepsilon > 0$, we get $x \in I$, and hence $a \in \mathrm{Cu}(I)$, as desired.

We let $\mathrm{Lat}(A)$ denote the collection of ideals of A, equipped with the partial order given by inclusion of ideals. It is well-known that $\mathrm{Lat}(A)$ is a complete lattice. We let $\mathrm{Lat}_{\mathrm{f}}(A)$ denote the subset of ideals in A that contain a full, positive element. We remark that every σ-unital ideal of A belongs to $\mathrm{Lat}_{\mathrm{f}}(A)$, but the converse does not hold. Indeed, in [**BGR77**, Lemma 2.2] an example of a simple C^*-algebra without strictly positive element is given.

It is easy to see that $\mathrm{Lat}_{\mathrm{f}}(A)$ is a sublattice of $\mathrm{Lat}(A)$.

PROPOSITION 5.1.10. *Let A be a C^*-algebra. Then, the map*

$$\mathrm{Lat}(A) \to \mathrm{Lat}(\mathrm{Cu}(A)), \quad I \mapsto \mathrm{Cu}(I), \quad (I \in \mathrm{Lat}(A))$$

is a natural isomorphism of complete lattices.

Moreover, it maps the sublattice $\mathrm{Lat}_{\mathrm{f}}(A)$ of ideals in A that contain a full, positive element onto the sublattice $\mathrm{Lat}_{\mathrm{f}}(\mathrm{Cu}(A))$ of singly-generated ideals in $\mathrm{Cu}(A)$.

PROOF. For the case that A is a separable C^*-algebra, a proof of the statement can be found in [**Ciu08**, Proposition 3.1.2]. Our proof is based on the ideas given by Ciuperca, and we include it for completeness. We may assume that A is stable, so that $\mathrm{Cu}(A) = A_+/{\sim}$. Let us denote the map of the statement by $\varphi\colon \mathrm{Lat}(A) \to \mathrm{Lat}(\mathrm{Cu}(A))$.

Consider the map

$$c\colon A \to \mathrm{Cu}(A), \quad x \mapsto [xx^*], \quad (x \in A)$$

which assigns to an element $x \in A$ the Cuntz class of xx^*. Given an ideal I in A, it is easy to see that

$$\varphi(I) = \{[x] \in \mathrm{Cu}(A) \ \mid \ x \in I_+\} = \{[xx^*] \in \mathrm{Cu}(A) \ \mid \ x \in I\} = c(I).$$

We define a map that will turn out to be the inverse of φ as follows:

$$\begin{aligned} &\psi\colon \mathrm{Lat}(\mathrm{Cu}(A)) \to \mathrm{Lat}(A), \\ &J \mapsto c^{-1}(J) = \{x \in A \ \mid [xx^*] \in J\}. \quad (J \in \mathrm{Lat}(\mathrm{Cu}(A))) \end{aligned}$$

Given an ideal J in $\mathrm{Cu}(A)$, let us check that $\psi(J)$ is an ideal of A.

To show that $\psi(J)$ is closed under addition, let $x, y \in \psi(J)$. We have that

$$(x+y)(x+y)^* \leq (x+y)(x+y)^* + (x-y)(x-y)^* = 2xx^* + 2yy^*,$$

and therefore $[(x+y)(x+y)^*] \leq [xx^*]+[yy^*]$. Since J is an ideal and $[xx^*], [yy^*] \in J$, we get that $[(x+y)(x+y)^*] \in J$ and so $x+y \in \psi(J)$. It is straightforward to check that $\psi(J)$ is closed under scalar multiplication.

To show that $\psi(J)$ is an ideal, let $x \in \psi(J)$ and $y \in A$. We have that

$$(xy)(xy)^* = x(yy^*)x^* \precsim xx^*, \quad (yx)(yx)^* = y(xx^*)y^* \precsim xx^*,$$

which again implies that $xy, yx \in \psi(J)$. It is left to the reader to check that $\psi(J)$ is also closed.

It is clear that both φ and ψ are order-preserving. Next, let us show that these maps are inverses of each other. Given an ideal J in $\mathrm{Cu}(A)$, using that c is a surjective map, it follows easily that

$$\varphi \circ \psi(J) = c(c^{-1}(J)) = J.$$

Conversely, let I be an ideal of A. Then I is clearly a subset of $\psi \circ \varphi(I) = c^{-1}(c(I))$. By definition, if $x \in c^{-1}(c(I))$, then $xx^* \in c(I)$, which means that there exists $y \in I_+$ such that $xx^* \sim y$. We have already seen that this implies $xx^* \in I$ and hence also $x \in I$, as desired.

Finally, let us see that φ maps $\mathrm{Lat_f}(A)$ onto $\mathrm{Lat_f}(\mathrm{Cu}(A))$. In one direction, let $I \in \mathrm{Lat_f}(A)$ and choose a full, positive element $x \in I_+$. Set $a = [x] \in \mathrm{Cu}(I)$. We claim that $\infty \cdot a$ is the largest element of $\mathrm{Cu}(I)$. So let $y \in I_+$, and let $\varepsilon > 0$. Since x is full and y is positive, there exist $K \in \mathbb{N}$ and elements $r_1, \ldots, r_K \in I$ such that

$$\left\| y - \sum_{k=1}^{K} r_k x r_k^* \right\| < \varepsilon.$$

It follows that

$$[(y-\varepsilon)_+] \leq K[x] \leq \infty \cdot a.$$

Since this holds for every $\varepsilon > 0$, we get $[y] \leq \infty \cdot a$, as desired.

Conversely, assume that J is a singly-generated ideal in $\mathrm{Cu}(A)$ and set $I = \psi(J)$. Then, as observed in Paragraph 5.1.6, there exists a largest element in J, which we denote by a. Choose $x \in I_+$ such that $a = [x]$. We claim that x is a full element in I. So let $y \in I$. Since a is the largest element in J, we get that $yy^* \precsim x$. This implies that yy^* and hence y is contained in the ideal generated by x. Hence, x is full in I, as desired. $\square$

Recall that a C^*-algebra A is called *simple* if $\{0\}$ and A are the only ideals of A. Analogously, we define for Cu-semigroups:

Definition 5.1.11. A Cu-semigroup S is called *simple* if $\{0\}$ and S are the only ideals of S.

Corollary 5.1.12. *A C^*-algebra A is simple if and only if its (completed) Cuntz semigroup $\mathrm{Cu}(A)$ is a simple Cu-semigroup.*

Corollary 5.1.13. *Let A be a separable C^*-algebra. Then the ideal lattice $\mathrm{Lat}(A)$ is a Cu-semigroup.*

Remarks 5.1.14. Let A be a separable C^*-algebra.

(1) In Corollary 7.2.13, we will show that there are natural isomorphisms between the following Cu-semigroups:

$$\mathrm{Cu}(A \otimes \mathcal{O}_2) \cong \mathrm{Lat}(A) \cong \mathrm{Lat}(\mathrm{Cu}(A)) \cong \mathrm{Cu}(A) \otimes \{0, \infty\}.$$

(2) The Cu-semigroup $\mathrm{Lat}(A)$ is algebraic (see Section 5.5) if and only if the C^*-algebra A has the ideal property.

Proposition 5.1.15 (c.f. [**CRS10**, Proposition 3.3]). *Let A be a C^*-algebra, and let I be an ideal in A. Then, there is a natural isomorphism*

$$\mathrm{Cu}(A)/\mathrm{Cu}(I) \cong \mathrm{Cu}(A/I).$$

5.1.16 (Elementary semigroups). We call a simple Cu-semigroup S *elementary* if $S \cong \{0\}$ or if S contains a minimal, nonzero element. The typical example is the semigroup of extended natural numbers

$$\overline{\mathbb{N}} = \{0, 1, 2, \ldots, \infty\}.$$

For each $k \in \mathbb{N}$, we define a semigroup

$$E_k = \{0, 1, 2, \ldots, k, \infty\},$$

with the natural order and $a+b$ defined as ∞ if usually one would have $a+b \geq k+1$. For $k = 0$ we obtain $E_0 = \{0, \infty\}$. It is easy to check that these are simple Cu-semigroups satisfying (O5) and (O6), and all elements are compact.

There exist simple, elementary Cu-semigroups satisfying (O5) that are not isomorphic to $\{0\}$, to $\overline{\mathbb{N}}$ or to E_k for some k; see Example 8.1.2. With the assumption of (O6), this is not possible as we will show in Proposition 5.1.18.

Lemma 5.1.17. *Let S be a simple* Cu*-semigroup satisfying (O5) and (O6). Given nonzero elements $a_1, \ldots, a_n \in S$, there exists a nonzero element $x \in S$ such that $x \ll a_k$ for all k.*

Proof. It is enough to prove the case $n = 2$ (and then use induction). So let a and b be nonzero elements in S. We need to find a nonzero element $x \in S$ such that $x \ll a$ and $x \ll b$.

Choose nonzero elements a' and a'' in S such that $a'' \ll a' \ll a$. By simplicity of S, there is $k \in \mathbb{N}$ such that $a' \leq kb$. Considering the situation

$$a'' \ll a' \leq kb = b + b + \ldots + b,$$

we may apply (O6) in S to obtain elements $c_1, \ldots, c_k \in S$ such that

$$a'' \leq c_1 + \cdots + c_k, \quad c_i \leq a', b, \quad i = 1, 2, \ldots, k.$$

Since a'' is nonzero, there has to be an index i_0 with $c_{i_0} \neq 0$. Choose a nonzero element $x \in S$ with $x \ll c_{i_0}$. Then x has the desired properties. □

The following result was observed independently by Engbers, [**Eng14**]. He also noted that one must exclude elementary semigroups to obtain results like Glimm Halving, [**Rob13a**, Proposition 5.2.1]; see Proposition 5.4.1.

Proposition 5.1.18. *Let S be a simple* Cu*-semigroup satisfying (O5) and (O6). Then S is elementary if and only if S is isomorphic to $\{0\}$, to $\overline{\mathbb{N}}$, or to E_k for some $k \in \mathbb{N}$.*

Proof. The 'if' part of the statement is clear. So assume that S is an elementary Cu-semigroup and assume that $S \neq \{0\}$. Then there exists a minimal, nonzero element a in S. By Lemma 5.1.17, the element a is compact. We claim that a is the least nonzero element. Indeed, let $b \in S$ be an arbitrary nonzero element. By Lemma 5.1.17, there exists a nonzero element b' with $b' \leq a, b$. Since a is minimal, we have $b' = a$ and therefore $a \leq b$.

Now, let b be an arbitrary nonzero element in S. Then $a \leq b$ and since a is compact and S satisfies (O5), there exists $x \in S$ such that $a + x = b$. If $x = 0$, we have $b = a$. Otherwise, since a is the least nonzero element, we obtain that $a \leq x$ and so there is $y \in S$ with $a + y = x$ and consequently $2a + y = b$. Continuing in this way, we find that either $b = na$ for some $n \in \mathbb{N}$ or otherwise $na \leq b$ for all $n \in \mathbb{N}$. The latter implies $b = \infty$, whence

$$S = \{\infty\} \cup \{na \mid n \in \mathbb{N}\}.$$

Now, if $na \neq ma$ for any $n, m \in \mathbb{N}$ with $n \neq m$, then we have $S \cong \overline{\mathbb{N}}$. Otherwise, there is $k \in \mathbb{N}$ with $ka = (k+1)a$. For the smallest such k, we have $S \cong E_k$. □

5.2. Functionals

In this section, we study functionals on Cu-semigroups and their connection to the order structure. First, we show that the existence of nontrivial functionals characterizes stable finiteness of simple Cu-semigroups, see Proposition 5.2.5. Then, we study the relation of 'stable domination' of elements in a positively ordered monoid, see Definition 5.2.7.

We recall that comparison by extended states is closely related to stable domination of elements, see Proposition 5.2.8. In the context of Cu-semigroups, we introduce the 'regularization' of a relation, see Definition 5.2.9. The main result of this section is Theorem 5.2.13, where we show that comparison by functional on a Cu-semigroup is closely related to the regularization of the stable domination relation.

5.2.1. Let S be a positively ordered monoid. A *state* on S is a map $f\colon S \to [0,\infty)$ that preserves addition, order and the zero-element. If the value ∞ is allowed, then we call f an *extended state*. Thus, an (extended) state is a PoM-morphism from S to $[0,\infty)$ (resp. from S to $[0,\infty]$).

Assume now that S is a Cu-semigroup. A *functional* on S is a map $\lambda\colon S \to [0,\infty]$ that preserves addition, order, the zero-element, and suprema of increasing sequences. Hence, a functional is a generalized Cu-morphism from S to $[0,\infty]$. The set of functionals on S is denoted by $F(S)$. When equipped with a suitable topology, $F(S)$ becomes a compact Hausdorff space; see [**ERS11**, Theorem 4.8], see also [**Rob13a**]. If S is countably-based, then $F(S)$ is second-countable, hence a compact, metrizable space.

It is clear that by multiplying a functional $\lambda \in F(S)$ with a positive scalar $\theta \in (0,\infty)$, one obtains a functional $\theta \cdot \lambda$. It was shown in the comments before Theorem 4.8 in [**ERS11**] that this can be extended to a jointly continuous scalar multiplication

$$[0,\infty] \times F(S) \to F(S).$$

We equip $F(S)$ with pointwise addition and order, which provides it with the structure of a positively ordered monoid. If S satisfies (O5) (or just the weaker (O5')), then $F(S)$ is algebraically ordered, [**Rob13a**, Proposition 2.2.3].

Given an element $a \in S$, we say that a functional λ is normalized at a provided $\lambda(a) = 1$, and we denote the set of these functionals by $F_a(S)$. If S is simple and $a \in S$ is a compact element, then $F_a(S)$ is a closed, convex subset of $F(S)$.

We denote by $\mathrm{Lsc}(F(S))$ the set of functions $f\colon F(S) \to [0,\infty]$ that are lower-semicontinuous, order-preserving and linear (i.e., f is additive and $f(\theta\lambda) = \theta f(\lambda)$ for $\theta \in [0,\infty]$ and $\lambda \in F(S)$). If $F(S)$ is algebraically ordered (e.g. if S satisfies (O5)), then being order-preserving follows automatically from additivity.

We define a binary relation $\lhd$ on $\mathrm{Lsc}(F(S))$ as follows: For two elements $f, g \in \mathrm{Lsc}(F(S))$, we set $f \lhd g$ if and only if $f \leq (1-\varepsilon)g$ for some $\varepsilon > 0$, and if moreover f is continuous at each $\lambda \in F(S)$ where $g(\lambda) < \infty$; see the paragraph after Remark 3.1.5 in [**Rob13a**]. We let $L(F(S))$ be the subset of $\mathrm{Lsc}(F(S))$ consisting of all $f \in \mathrm{Lsc}(F(S))$ for which there exists a sequence $(f_n)_n \subset \mathrm{Lsc}(F(S))$ satisfying $f = \sup_n f_n$ (the pointwise supremum) and $f_n \lhd f_{n+1}$ for each n.

Any element $a \in S$ induces a function

$$\hat{a}\colon F(S) \to [0,\infty], \quad \hat{a} = (\lambda \mapsto \lambda(a)). \quad (\lambda \in F(S))$$

The assignment $a \mapsto \hat{a}$ defines a map $S \to L(F(S))$ that preserves addition, order and suprema of increasing sequences.

If S is a Cu-semigroup satisfying (O5), then it is shown in [**Rob13a**] that $L(F(S))$ is also a Cu-semigroup satisfying (O5).

5.2.2 (Stable finiteness). Let S be a Cu-semigroup. An element $a \in S$ is *finite* if for every element $b \in S$, we have that $a + b = a$ implies $b = 0$. Equivalently, we have $a < a + b$ for every nonzero element $b \in S$. We call an element *infinite* if it is not finite. An infinite element $a \in S$ is *properly infinite* if $2a = a$. We say that S is *stably finite* if an element $a \in S$ is finite whenever there exists $\tilde{a} \in S$ with $a \ll \tilde{a}$. If S contains a largest element, denoted by ∞, then the latter condition is equivalent to $a \ll \infty$.

In general, a Cu-semigroup does not contain a largest element. There are, however, two important cases when a largest element always exists. First, consider a simple Cu-semigroup S. We may assume that $S \neq \{0\}$. Choose any nonzero element $a \in S$ and consider the increasing sequence $(ka)_{k\in\mathbb{N}}$. By axiom (O1), the supremum of this sequence exists and it is easy to check that it is the largest element of S:

$$\infty = \sup_{k\in\mathbb{N}} ka.$$

In the other case, assume that S is a countably-based Cu-semigroup. Choose a countable set $\{a_0, a_1, a_2, \ldots\}$ in S that is a basis in the sense of Paragraph 2.1.1. For each $n \in \mathbb{N}$, consider the n-th partial sum $\sum_{k=0}^{n} a_k$. It is straightforward to check that the supremum of this increasing sequence of partial sums is the largest element of S:

$$\infty = \sup_n \sum_{k=0}^{n} a_k.$$

Thus, if S is a Cu-semigroup that is simple or countably-based, then S is stably finite if and only if an element $a \in S$ is finite whenever $a \ll \infty$.

Lemma 5.2.3. *Let S be a simple* Cu-*semigroup, and let $a \in S$. Then, the following are equivalent:*

(1) The element a is infinite.
(2) The element a is properly infinite.
(3) We have $a = \infty$.

Moreover, the following are equivalent:

(1) The element a is nonzero, and for every $n \in \mathbb{N}$ the element na is finite.
(2) For every $n, m \in \mathbb{N}$, we have $na = ma$ if and only if $n = m$.
(3) There exists a functional $\lambda \in F(S)$ such that $\lambda(a) = 1$.

Proof. For the first part of the statement, observe that in a general nonzero Cu-semigroup the largest element is properly infinite (if it exists), and that every properly infinite element is infinite. It remains to show that (1) implies (3). So assume that S is a simple Cu-semigroup, and let $a \in S$ be an infinite element. By definition, there exists a nonzero element $b \in S$ such that $a = a + b$. Then $a = a + 2b$, and inductively $a = a + kb$ for every $k \in \mathbb{N}$. Therefore,

$$\infty = \sup_k kb \leq \sup_k (a + kb) = \sup_k a = a \leq \infty,$$

which shows that $a = \infty$, as desired.

To show the second part of the statement, let S be a simple Cu-semigroup, and let $a \in S$. One easily checks that (1) implies (2). Let us show that (3) implies (1). So assume that there exists a functional λ with $\lambda(a) = 1$. This clearly implies that a is nonzero. Given $k \in \mathbb{N}$, we need to show that ka is finite. So let $b \in S$ be an element such that $ka + b = ka$. This implies that $\lambda(b) = 0$. Note that every nonzero functional μ on a simple Cu-semigroup is faithful in the sense that $\mu(x) \neq 0$ for every nonzero element x. Thus, it follows from $\lambda(b) = 0$ that we have $b = 0$, as desired.

Finally, let us show that (2) implies (3). So let $a \in S$ satisfy (2). Then a is nonzero. Consider the following subsets of S:

$$H = \{x \in S \mid x \leq ka, \text{some } k \in \mathbb{N}\}.$$
$$S_0 = \{x \in S \mid x \ll \infty\}.$$

Using that a is nonzero, it is easy to check that $S_0 \subset H$. It is also clear that H is a subsemigroup of S. Endowed with the partial order induced by S, the semigroup H becomes a positively ordered monoid. By construction, a is an order unit of H.

Consider the submonoid of H, generated by a:

$$\langle a \rangle = \{0, a, 2a, 3a, \ldots\}.$$

Using the assumption on a, we may define a function f as follows:

$$f\colon \langle a \rangle \to [0, \infty), \quad na \mapsto n. \quad (n \in \mathbb{N})$$

It is clear that f is a state on $\langle a\rangle$, which by [**BR92**, Corollary 2.7] can be extended to a state $\tilde{f}$ on H. Using that $S_0 \subset H$, we may define a function λ_0 as follows:

$$\lambda_0\colon S \to [0,\infty], \quad \lambda_0(x) = \sup\left\{ \tilde{f}(x') \;\mid\; x' \ll x \right\}. \quad (x \in S)$$

It is straightforward to check that λ_0 is a nonzero functional on S. Then, the functional $\lambda = \frac{1}{\lambda_0(a)}\lambda_0$ has the desired properties to show (3). □

REMARKS 5.2.4. (1) Let S be a simple Cu-semigroup, and let $a \in S$. Then a multiple of a can be infinite even if a itself is finite. This happens for instance in the elementary semigroups E_k from Paragraph 5.1.16.

(2) Let S be a not necessarily simple Cu-semigroup, and let $a \in S$. Considering the statements in the second part of Lemma 5.2.3, it is not known whether (1) and (2) imply (3). For Cuntz semigroups of C^*-algebras, this is related to the global Glimm halving problem; see [**BK04**, Definition 1.2] and [**ER06**, Question 1.2].

For the next result, we call a functional nontrivial if it does not only take the values 0 and ∞.

PROPOSITION 5.2.5. *Let S be a simple* Cu*-semigroup with $S \neq \{0\}$. Then, the following are equivalent:*

(1) The semigroup S is stably finite.
(2) Every compact element in S is finite.
(3) The largest element ∞ is not compact.
(4) There exists a nontrivial functional $\lambda \in F(S)$.

PROOF. In general, every compact element in a stably finite Cu-semigroup is finite. Moreover, the largest element ∞ is never finite. It follows that (1) implies (2), and that (2) implies (3).

Let us show that (1) implies (4). Choose a nonzero element $a \in S$ satisfying $a \ll \infty$. Then $na \ll \infty$ for all $n \in \mathbb{N}$. Since S is stably finite, we obtain that na is finite for every $n \in \mathbb{N}$. By Lemma 5.2.3, there exists a functional $\lambda \in F(S)$ with $\lambda(a) = 1$. This functional is nontrivial, as desired.

Let us show that (4) implies (3). Choose a nontrivial functional $\lambda \in F(S)$. By rescaling if necessary, we may assume that there exists $a \in S$ with $\lambda(a) = 1$. In order to show (3), assume that ∞ is compact. Then, since $\infty = \sup_k(ka)$, there exists $n \in \mathbb{N}$ with $\infty \leq na$, and hence $\infty = na$. This implies that

$$n = \lambda(na) = \lambda(\infty) = \lambda(2\infty) = \lambda(2na) = 2n,$$

which clearly is a contradiction. Hence, ∞ is not compact, which shows (3).

Finally, let us use contraposition to show that (3) implies (1). So assume that S is not stably finite. Then there exists a nonzero, infinite element $a \in S$ satisfying $a \ll \infty$. By Lemma 5.2.3, every infinite element in S is equal to the largest element ∞. It follows that $\infty = a \ll \infty$, and so ∞ is compact. □

REMARK 5.2.6. The equivalence of statements (1) and (4) in Proposition 5.2.5 is well-known, especially for Cuntz semigroups of (simple) C^*-algebras. It is used to show that every unital, simple, stably finite C^*-algebra has a 2-quasitrace. In fact, the correspondence between 2-quasitraces on a C^*-algebra and functionals on its Cuntz semigroup, were one of the original motivations for Cuntz to introduce the semigroups named after him; see [**Cun78**], [**BH82**].

In the next part of this section, we will study the connection between the order-structure of a positively ordered monoid and the set of its functionals. We first recall a notion that has appeared many times in the literature. The notation chosen here follows [**OPR12**, Definition 2.2].

DEFINITION 5.2.7. Let M be a positively ordered monoid, and let $a, b \in M$. We will write $a \propto b$ if there exists $k \in \mathbb{N}$ such that $a \leq kb$.

We say that a is *stably dominated* by b, denoted by $a <_s b$, if there exists $k \in \mathbb{N}$ such that $(k+1)a \leq kb$.

The following result provides useful characterizations of the relation $<_s$. Several versions of this results have appeared in the literature, see e.g. [**OPR12**, Proposition 2.1], and most are based on [**GH76**, Lemma 4.1].

PROPOSITION 5.2.8. *Let M be a positively ordered monoid, and let $a, b \in M$. Then, the following are equivalent:*

(1) We have $a <_s b$, that is, there exists $k \in \mathbb{N}$ such that $(k+1)a \leq kb$.
(2) There exists $k_0 \in \mathbb{N}$ such that $(k+1)a \leq kb$ for all $k \geq k_0$.
(3) Given $n \in \mathbb{N}_+$, there exists $k \in \mathbb{N}$ such that $(k+n)a \leq kb$.
(4) Given $n \in \mathbb{N}_+$, there exists $k_0 \in \mathbb{N}$ such that $(k+n)a \leq kb$ for all $k \geq k_0$.
(5) We have $a \propto b$, and $f(a) < f(b)$ for every extended state on S that is normalized at b.

If b is an order-unit for M, then the above statements are also equivalent to:

(6) We have $f(a) < f(b)$ for every state on S that is normalized at b.

PROOF. It is clear that (4) implies (3) and (2), and that (3) implies (1), and that (2) implies (1). It is also easy to see that (1) implies (3). Indeed, assume that $n \in \mathbb{N}_+$ and that $(k+1)a \leq kb$ for some $k \in \mathbb{N}$. Then $(kn+n)a \leq knb$, as desired.

Let us show that (3) implies (4). So let $n \in \mathbb{N}_+$ be given. By assumption, there exists $d \in \mathbb{N}$ such that $(d+n)a \leq db$. We claim that $k_0 = d(d+1)$ has the desired properties. So let $k \in \mathbb{N}$ satisfy $k \geq k_0$. Then there are $x, y \in \mathbb{N}$ with $k = (d+1)x + y$ and $x \geq d$ and $y \leq d$. Then

$$(k+n)a = [(d+1)x]a + (y+n)a \leq [(d+n)x]a + (d+n)a \leq (dx)b + db \leq kb,$$

as desired.

Finally, the equivalence between (1) and (5) is shown in [**OPR12**, Proposition 2.1]. If b is an order-unit, it is easy to verify that (5) and (6) are equivalent. $\square$

DEFINITION 5.2.9. Let S be a Cu-semigroup, and let $R \subset S \times S$ be a binary relation. The *regularization* of R, denoted by R^*, is the binary relation defined as follows: For any $a, b \in S$, we set aR^*b if and only if $a'Rb$ for every $a' \in S$ satisfying $a' \ll a$.

A relation R is *regular* if it is equal to its own regularization.

EXAMPLE 5.2.10. Let S be a Cu-semigroup.

(1) The usual order-relation $\leq$ on S is regular. Indeed, given $a, b \in S$, it is clear that $a \leq b$ implies that $a \leq^* b$. The converse follows from axiom (O2) for S.

(2) The way-below relation $\ll$ on S is not regular. In fact, it is straightforward to check that the regularization of $\ll$ is nothing but the order-relation $\leq$.

(3) The stable domination relation $<_s$ from Definition 5.2.7 is not regular. However, we will show in Theorem 5.2.13 that the regularization of $<_s$ is closely related to comparison by functionals. In Section 5.3, we will study elements $a \in S$ satisfying $a <_s^* a$. (We call such elements 'soft'.)

(4) The relation $\propto$ is not regular. However, its regularization determines exactly which ideal an element of S generates. More precisely, given $a, b \in S$, we have $a \propto^* b$ if and only if $a \leq \infty \cdot b$, and if and only if $\mathrm{Idl}(a) \subset \mathrm{Idl}(b)$, see Paragraph 5.1.6.

DEFINITION 5.2.11 (Rørdam, [**Rør92**, Section 3]). A positively ordered monoid M is *almost unperforated* if for every $a, b \in M$, we have that $a <_s b$ implies $a \leq b$.

The following result is straightforward to verify.

LEMMA 5.2.12. *Let S be a* Cu*-semigroup. Then S is almost unperforated if for every $a, b \in S$, we have that $a <_s^* b$ implies $a \leq b$.*

THEOREM 5.2.13. *Let S be a* Cu*-semigroup, and let $a, b \in S$. Consider the following statements:*

(1) We have $a <_s b$, that is, there exists $k \in \mathbb{N}$ such that $(k+1)a \leq kb$.
(2) We have $\hat{a} <_s \hat{b}$, that is, there exists $k \in \mathbb{N}$ such that $(k+1)\hat{a} \leq k\hat{b}$.
(3) We have $a \propto^* b$, and $\lambda(a) < \lambda(b)$ for every functional $\lambda \in F(S)$ satisfying $\lambda(b) = 1$.
(4) We have $a <_s^* b$, that is, we have $a' <_s b$ for every $a' \in S$ satisfying $a' \ll a$.
(5) We have $\hat{a} \leq \hat{b}$.

Then, the following implications hold: '(1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5)'.

If a is compact, then (4) implies (1). If the element a satisfies $a <_s^* a$ (such elements will be called 'soft', see Definition 5.3.1), then (5) implies (4). If S is almost unperforated, then (4) implies that $a \leq b$. The different implications are shown in the following diagram:

$$\begin{tikzcd}
a <_s b \arrow[r, Rightarrow] & \hat{a} <_s \hat{b} \arrow[r, Rightarrow] & (3) \arrow[r, Rightarrow] & a <_s^* b \arrow[r, Rightarrow] \arrow[d, dashed, "\substack{S\ \text{is almost}\\ \text{unperforated}}"'] & \hat{a} \leq \hat{b} \arrow[l, dashed, bend right, "a\ \text{soft}"'] \\
& & & a \leq b \arrow[ur, Rightarrow] &
\end{tikzcd}$$

PROOF. It is clear that (1) implies (2), and it is straightforward to check that (4) implies (5). To see that (2) implies (3), assume that $\hat{a} <_s \hat{b}$. This clearly implies that $\lambda(a) < 1$ for every $\lambda \in F_b(S)$. Thus, it remains to show that $a \propto^* b$. Let I be the ideal generated by b, that is, $I = \{x \in S \mid x \leq \infty \cdot b\}$. Consider the following map

$$\lambda_I \colon S \to [0, \infty], \quad \lambda_I(x) = \begin{cases} 0, & \text{if } x \in I \\ \infty, & \text{if } x \notin I \end{cases}. \quad (x \in S)$$

It is easy to check that λ_I is a functional. Since $\lambda_I(b) = 0$ and $\hat{a} <_s \hat{b}$, it follows that $\lambda_I(a) = 0$ and therefore $a \leq \infty \cdot b$, as desired.

Let us show that (3) implies (4). Assume a and b satisfy the statement of (3), and let us show that $a <_s^* b$. Let $a' \in S$ satisfy $a' \ll a$. We want to verify (5) of Proposition 5.2.8 to show that $a' <_s b$. The argument is similar to the one in the proof of [**ERS11**, Proposition 6.2] and [**Rob13a**, Proposition 2.2.6]. Since $a' \ll a$, we get that $a' \propto b$.

Now, let $f \colon S \to [0, \infty]$ be an extended state with $f(b) = 1$. We want to show that $f(a) < f(b)$. Consider the map

$$\tilde{f} \colon S \to [0, \infty], \quad \tilde{f}(x) = \sup \{f(x') \mid x' \ll x\}. \quad (x \in S)$$

It is easy to see that $\tilde{f}$ is a functional on S. In the literature, the functional $\tilde{f}$ is sometimes called the regularization of f. We distinguish two cases:

In the first case, assume that $\tilde{f}(b) = 0$. Since $\hat{a} <_s \hat{b}$, it follows that $\tilde{f}(a) = 0$. Using the definition of $\tilde{f}$ at the first step, we deduce that

$$f(a') \leq \tilde{f}(a) = 0 < 1 = f(b).$$

In the second case, assume that $\tilde{f}(b) > 0$. Since $\hat{a} <_s \hat{b}$, it follows that $\tilde{f}(a) < \tilde{f}(b)$. Using the definition of $\tilde{f}$ at the first and last step, we obtain that

$$f(a') \leq \tilde{f}(a) < \tilde{f}(b) \leq f(b).$$

Thus, in either case, we have $f(a') < f(b)$. Applying Proposition 5.2.8, we get that $a' <_s b$, as desired.

Finally, if a is a compact element, it is clear that (4) implies (1). Moreover, as observed in Lemma 5.2.12, if S is almost unperforation thent $a <_s^* b$ implies $a \leq b$. It remains to show that $\hat{a} \leq \hat{b}$ implies $a <_s^* b$ if a satisfies $a <_s^* a$. Let $x \in S$ satisfy $x \ll a$. Choose y such that $x \ll y \ll a$. By assumption, this implies that $y <_s a$. It follows that

$$\hat{y} <_s \hat{a} \leq \hat{b}.$$

Using that (2) implies (3), we get that $y <_s^* b$. Since $x \ll y$, we obtain that $x <_s b$, as desired. □

The next result describes which information about the order-structure of a Cu-semigroup is recorded by its functionals. It has appeared in [**Rob13a**, Proposition 2.2.6], under the additional assumption that the Cu-semigroup satisfies (O5). However, an inspection of the proof of [**Rob13a**, Proposition 2.2.6] shows that (O5) is not needed.

PROPOSITION 5.2.14. *Let S be a* Cu*-semigroup, and let $a, b \in S$. Then, the following are equivalent:*

(1) We have $\hat{a} \leq \hat{b}$.
(2) For each $n \in \mathbb{N}$, we have $na <_s^* (n+1)b$.
(3) For every $a' \in S$ satisfying $a' \ll a$ and every $\varepsilon > 0$, there exist $k, n \in \mathbb{N}$ such that $(1-\varepsilon) < \frac{k}{n}$ and $ka' \leq nb$.

If S is almost unperforated, then these conditions are also equivalent to:

(4) For each $n \in \mathbb{N}$, we have $na \leq (n+1)b$.

PROOF. Let $a, b \in S$. We first show that (1) implies (2). So assume that $\hat{a} \leq \hat{b}$. This clearly implies that $n\hat{a} <_s (n+1)\hat{b}$ for each $n \in \mathbb{N}$. Then, (2) follows from Theorem 5.2.13.

It is straightforward to check that (2) implies (3), and that (3) implies (1). Finally, statements (4) implies (1) in general. Conversely, it is clear that (2) implies (4) if S is almost unperforated. □

The equivalence of statements (1) and (2) in the next result follows immediately from Proposition 5.2.8 and was first obtained by Rørdam, [**Rør04**, Proposition 3.2] (see also [**Rør92**, Proposition 3.1]). The equivalence with condition (3) follows easily from Theorem 5.2.13 and was first shown in [**ERS11**, Proposition 6.2].

PROPOSITION 5.2.15. *Let S be a positively ordered monoid. Then, the following are equivalent:*

(1) The semigroup S is almost unperforated.
(2) For all $a, b \in S$ we have that $a \leq b$ whenever $a \propto b$ and $f(a) < f(b)$ for every extended state f on S that is normalized at b.

If, moreover, S is a Cu*-semigroup, then these conditions are also equivalent to:*

(3) For all $a, b \in S$ we have that $a \leq b$ whenever $a \propto^* b$ and $\lambda(a) < \lambda(b)$ for every functional *λ on S that is normalized at b.*

For the next result, recall that a 2-quasitrace on a C^*-algebra A is a map

$$\tau\colon (A \otimes \mathrm{K})_+ \to [0, \infty],$$

such that $\tau(0) = 0$, such that $\tau(xx^*) = \tau(x^*x)$ for all $x \in A \otimes \mathrm{K}$, and such that $\tau(x+y) = \tau(x) + \tau(y)$ for all $x, y \in (A \otimes \mathrm{K})_+$ that commute. The set of lower-semicontinuous 2-quasitraces on A is denoted by $\mathrm{QT}_2(A)$, and its structure (e.g. as a lattice and as a noncancellative cone) has been thoroughly studied in [**ERS11**].

Given a 2-quasitrace τ on A, consider the map

$$d_\tau \colon (A \otimes \mathrm{K})_+ \to [0,\infty], \quad d_\tau(x) = \lim_k \tau(x^{1/k}). \quad (x \in (A \otimes \mathrm{K})_+)$$

If x is Cuntz-subequivalent to y, then $d_\tau(x) \leq d_\tau(y)$. It follows that a 2-quasitrace τ on A induces a map

$$d_\tau \colon \mathrm{Cu}(A) \to [0,\infty], \quad d_\tau(x) = \lim_k \tau(x^{1/k}), \quad (x \in (A \otimes \mathrm{K})_+)$$

which is an extended state on $\mathrm{Cu}(A)$. If τ is a lower-semicontinuous, then d_τ is a functional on $\mathrm{Cu}(A)$.

Proposition 5.2.16 ([**ERS11**, Theorem 4.4]). *Let A be a C^*-algebra. Then the map*

$$\mathrm{QT}_2(A) \to F(\mathrm{Cu}(A)), \quad \tau \mapsto d_\tau, \quad (\tau \in \mathrm{QT}_2(A))$$

is a bijection. When $\mathrm{QT}_2(A)$ and $F(\mathrm{Cu}(A)$ are equipped with suitable natural topologies and order-structures, then this map becomes a homeomorphic order-isomorphism.

Corollary 5.2.17. *Let A be a simple, unital C^*-algebra with a unique 2-quasitrace τ that satisfies $\tau(1_A) = 1$. Then the Cuntz semigroup $\mathrm{Cu}(A)$ has a unique functional λ that satisfies $\lambda([1_A]) = 1$.*

5.3. Soft and purely noncompact elements

In this section, we first introduce the notion of 'softness' for elements in a Cu-semigroup, see Definition 5.3.1. This concept is closely related to that of 'pure noncompactness', which was introduced in the Definition before 6.4 in [**ERS11**]. In fact, we will slightly generalize their definition to that of 'weak pure noncompactness', which for elements in a Cu-semigroup S satisfying (O5) is equivalent to softness, see Proposition 5.3.5. In Corollary 5.3.10, we will show that under the addition assumption that S is almost unperforated or residually stably finite, an element $a \in S$ is soft if and only if it is purely noncompact.

The set of soft elements in a Cu-semigroup S forms a submonoid that is closed under suprema of increasing sequences and that is absorbing in a suitable sense, see Theorem 5.3.11. The main result of this section is Theorem 5.3.12, where we show that the order among soft elements in an almost unperforated Cu-semigroup S is determined solely by the functionals of S. This generalizes [**ERS11**, Theorem 6.6], where the analogous result is shown for the comparison of purely noncompact elements in the Cuntz semigroup of a C^*-algebra. We point out that we obtain our result without using (O5), by considering soft elements instead of (weakly) purely noncompact elements, see Remark 5.3.13.

Let M be a positively ordered monoid. An *interval* in M is a subset $I \subset M$ that is upward directed and order-hereditary. An interval I is *soft* if for every $x \in I$ there exist $y \in I$ and $n \in \mathbb{N}$ such that $(n+1)x \leq ny$. This notion was introduced by Goodearl and Handelman; see the Definition before Lemma 7.4 in [**GH82**]. It was also studied in [**Goo96**] and [**Per01**].

Using the relation $<_s$ from Definition 5.2.7, an interval I is soft if and only if for every $x \in I$ there exists $y \in I$ such that $x <_s y$.

Next, we introduce the notion of 'softness' for elements in Cu-semigroups. In Proposition 5.3.3, we will show that it is equivalent to softness of the interval of compactly contained elements.

DEFINITION 5.3.1. Let S be a Cu-semigroup. An element $a \in S$ is *soft* if for every $a' \in S$ satisfying $a' \ll a$ there exists $n \in \mathbb{N}$ such that $(n+1)a' \leq na$.

We denote by S_{soft} the subset of soft elements in S.

REMARK 5.3.2. Let S be a Cu-semigroup, and let $a \in S$.

(1) Consider the set of compactly-contained elements $a^{\ll} = \{x \in S \mid x \ll a\}$. We have that a is soft if and only if for every $x \in a^{\ll}$ we have $x <_s a$. Thus, an element is soft if and only if it stably dominates every compactly-contained element.

(2) Recall that $<_s^*$ denotes the regularization of the stable domination relation $<_s$; see Definitions 5.2.7 and 5.2.9. Then, the element a is soft if and only if $a <_s^* a$. In Theorem 5.2.13, we have seen that for soft elements there is a close connection between the order comparison in the Cu-semigroup and the comparison by functionals. In the case that the Cu-semigroup is almost unperforated, we even have that the functionals record the complete information about comparison between soft elements, see Theorem 5.3.12.

PROPOSITION 5.3.3. *Let S be a* Cu*-semigroup, and let $a \in S$. Then the following are equivalent:*

(1) The element a is soft.
(2) The set of compactly-contained elements, $a^{\ll}$, is a soft interval.
(3) For every $b \in S$ satisfying $b \ll a$, we have that $\hat{b} <_s \hat{a}$.

PROOF. Let $a \in S$. To see that (1) implies (2), assume that a is soft and let x be an element in $a^{\ll}$. Choose an element $\tilde{x} \in S$ such that $x \ll \tilde{x} \ll a$. Since a is soft, there exists $n \in \mathbb{N}$ such that $(n+1)\tilde{x} \leq na$. Then

$$(n+1)x \ll (n+1)\tilde{x} \leq na.$$

It follows that there exists $y \in S$ such that $y \ll a$ and $(n+1)x \leq ny$. Thus, $x <_s y$ and $y \in a^{\ll}$, which shows that $a^{\ll}$ is a soft interval.

It is easy to see that (2) implies (3). So assume that (3) holds and let us show that a is soft. Let $a' \in S$ satisfy $a' \ll a$. Choose an element $b \in S$ such that $a' \ll b \ll a$. By assumption, we have that $\hat{b} <_s \hat{a}$. By Theorem 5.2.13, this implies that $b <_s^* a$. Since $a' \ll b$, we get that $a' <_s a$, as desired. $\square$

For the next definition, recall from Paragraph 5.1.1 that given an ideal I in a Cu-semigroup S, we denote the image of an element $a \in S$ in the quotient S/I by a_I. To notion of 'pure noncompactness' was introduced in the Definition before 6.4 in [**ERS11**]. We will recall their definition and also generalize it to the concept of 'weak pure noncompactness', which is more closely connected to softness; see Proposition 5.3.5.

DEFINITION 5.3.4. Let S be a Cu-semigroup. An element $a \in S$ is *purely noncompact* if for every ideal I in S, we have that $a_I \ll a_I$ implies that $2a_I = a_I$.

An element $a \in S$ is *weakly purely noncompact* if for every ideal I in S, we have that $a_I \ll a_I$ implies that $(k+1)a_I = ka_I$ for some $k \in \mathbb{N}$.

Thus, if a is a (weakly) purely noncompact element, and if I is an ideal such that a_I is compact, then either $a_I = 0$ or (a multiple of) a_I is properly infinite.

The following result clarifies the connection between softness and weak pure noncompactness. In the context of Cuntz semigroups of C^*-algebras, the following result has partially been obtained in [**ERS11**, Proposition 6.4].

PROPOSITION 5.3.5. *Let S be a* Cu*-semigroup, and let $a \in S$. Consider the following statements:*

(1) The element a is soft.
(2) The element a is weakly purely noncompact.

(3) *For every a' and x in S satisfying $a' \ll a \leq a' + x$, there is $k \in \mathbb{N}$ such that $(k+1)a \leq ka' + \infty \cdot x$.*

(4) *For every a' and x in S satisfying $a' \ll a \leq a' + x$, we have that $a' <_s a' + x$.*

Then, the following implications hold: '(1)⇒(2)⇒(3)⇒(4)'.

If S satisfies (O5), then (4) implies (1), and so all 4 statements are equivalent in that case.

Moreover, if a is purely noncompact, then (3) holds for $k = 1$, and the converse holds if S satisfies (O5).

Proof. Let us show that (1) implies (2). So let I be an ideal of S and assume that $a_I \ll a_I$. We have to show that a multiple of a_I is properly infinite. Choose a rapidly increasing sequence $(a_n)_n \subset S$ such that $a = \sup_n a_n$. Then, in the quotient S/I, we have that $a_I = \sup_n (a_n)_I$.

Since a_I is compact, there is n such that $a_I \leq (a_n)_I$. Since $a_n \ll a$, we have by assumption that $a_n <_s a$. This means that there exists $k \in \mathbb{N}$ such that $(k+1)a_n \leq ka$. It follows that $(k+1)a_I = ka_I$, as desired.

Next, let us show that (2) implies (3). So assume that a is weakly purely noncompact, and let $a', x \in S$ satisfy $a' \ll a \leq a' + x$. Let $I = \{b \in S \mid b \leq \infty \cdot x\}$ be the ideal of S generated by x. Then, in the quotient S/I, we have that $a_I \leq a'_I \ll a_I$, whence by assumption there is $k \in \mathbb{N}$ such that $ka_I = (k+1)a_I = ka'_I$. This implies that $(k+1)a \leq ka' + \infty \cdot x$, as desired.

To show that (3) implies (4), let elements $a', x \in S$ be given and assume that $a' \ll a \leq a' + x$. By assumption, there is $k \in \mathbb{N}$ such that $(k+1)a \leq ka' + \infty \cdot x$. Then

$$(k+1)a' \ll (k+1)a \leq ka' + \infty \cdot x = \sup_{n \in \mathbb{N}} (ka' + nx).$$

It follows that there is $n \in \mathbb{N}$ such that $(k+1)a' \leq ka' + nx$. Let $m \in \mathbb{N}$ be the maximum of k and n. We get that

$$(k+1)a' \leq ka' + mx.$$

Adding $(m-k)a'$ on both sides, we obtain that

$$(m+1)a' \leq ma' + mx = m(a' + x),$$

and hence $a' <_s a' + x$, as desired.

Finally, let us show that (4) implies (1) under the assumption that S satisfies (O5). By Proposition 5.3.3, it is enough to show that $\hat{b} <_s \hat{a}$ for every $b \in S$ satisfying $b \ll a$. Let such b be given. Choose $c \in S$ such that $b \ll c \ll a$. Since S satisfies (O5), there exists $x \in S$ such that

$$b + x \leq a \leq c + x.$$

By assumption, we get that $c <_s c + x$. This means that there is $k \in \mathbb{N}$ such that

$$(k+1)c \leq kc + kx. \tag{5.1}$$

Let us show that $(k+1)\hat{b} \leq k\hat{a}$. So let $\lambda \in F(S)$ be a functional. If $\lambda(a) = \infty$, then there is nothing to show. So assume that $\lambda(a) < \infty$. Since $c \leq a$, it follows that $\lambda(c) < \infty$. Applying λ to the inequality (5.1), we obtain that

$$(k+1)\lambda(c) \leq k\lambda(c) + k\lambda(x).$$

Since $\lambda(c) < \infty$, we may cancel k summands of $\lambda(c)$ on both sides to get that

$$\lambda(c) \leq k\lambda(x).$$

Then, using that $b \leq c$ at the first step, and that $b + x \leq a$ at the last step, we deduce that

$$(k+1)\lambda(b) \leq k\lambda(b) + \lambda(c) \leq k\lambda(b) + k\lambda(x) \leq k\lambda(a).$$

This shows that $\hat{b} <_s \hat{a}$, as desired.

The implications concerning a purely noncompact element are obtained analogously. □

5.3.6. Let S be a Cu-semigroup. Let us denote the subsets of (weakly) purely noncompact elements in S by S_{wpnc} and S_{pnc}. We clearly have that $S_{\mathrm{pnc}} \subset S_{\mathrm{wpnc}}$, but the converse might fail. Indeed, if S is the elementary semigroup $E_k = \{0, 1, 2, \ldots, k, \infty\}$ as considered in Paragraph 5.1.16, then $S_{\mathrm{wpnc}} = S$ and $S_{\mathrm{pnc}} = \{0, \infty\}$.

Let us say that S satisfies condition (RQQ) if in every quotient of S, an element is properly infinite whenever a multiple of it is properly infinite. This property is a residual version (meaning to hold in all quotients) of property (QQ) as introduced in [**OPR12**, Remark 2.15], where it is also shown that (QQ) is connected to the (strong) Corona factorization property.

It is easy to see that for a Cu-semigroup satisfying (RQQ), we have that $S_{\mathrm{pnc}} = S_{\mathrm{wpnc}}$.

Lemma 5.3.7. *Let S be a* Cu*-semigroup that is almost unperforated. Then S satisfies (RQQ).*

Proof. Let I be an ideal of S, and let $a \in S$ be an element such that a multiple of a_I is properly infinite. This means that there is $k \in \mathbb{N}$ such that $(k+1)a_I = ka_I$. We need to show that $2a_I = a_I$.

Assuming that S is almost unperforated, it is straightforward to check that the quotient S/I is also almost unperforated. It follows from $(k+1)a_I = ka_I$ that $(k+n)a_I = ka_I$ for every $n \in \mathbb{N}$. In particular, we have that $(k+1)2a_I = ka_I$. By almost unperforation, it follows that $2a_I \leq a_I$. The converse inequality always holds, which shows that $2a_I = a_I$, as desired. □

In the next results, we will say that a Cu-semigroup S is *residually stably finite* if for every ideal I in S, the quotient Cu-semigroup S/I is stably finite. This is in accordance with the terminology used in C^*-algebra theory, see e.g. [**Bla06**, Definition V.2.1.3].

Lemma 5.3.8. *Let S be a residually stably finite* Cu*-semigroup satisfying (O5), and let $a \in S$. Then, the following statements are equivalent:*

(1) The element a is soft.
(2) For every $a' \in S$ satisfying $a' \ll a$, there exists $x \in S$ such that $a' + x \leq a$ and $a \leq \infty \cdot x$.
(3) For every a' and x in S satisfying $a' \ll a \leq a' + x$, we have that $a \leq \infty \cdot x$.

Proof. The proof is similar to that of Proposition 5.3.5. First, let us show that (1) implies (2). So let $a' \in S$ satisfy $a' \ll a$. Choose $b \in S$ such that $a' \ll b \ll a$. By (O5) in S, there exists $x \in S$ such that $a' + x \leq a \leq b + x$. Consider the ideal J of S generated by x. Then a_J is compact.

By Proposition 5.3.5, the element a is weakly purely noncompact. Thus, a multiple of a_J is properly infinite. Since S/J is stably finite, this implies that a_J is zero. Therefore $a \leq \infty \cdot x$, as desired.

Conversely, let us show that (2) implies (1). So let $a' \in S$ satisfy $a' \ll a$. We have to show that $a' <_s a$. By assumption, there exists $x \in S$ such that

$$a' + x \leq a \leq \infty \cdot x.$$

Since $a' \ll a$, there exists $n \in \mathbb{N}$ such that $a' \leq nx$. Then

$$(n+1)a' \leq na' + nx \leq na,$$

which shows that $a' <_s a$, as desired.

Next, let us show that (1) implies (3). So let a' and x in S satisfy $a' \ll a \leq a'+x$. Consider the ideal J of S generated by x. Then a_J is compact. As in the first part of the proof, we obtain that $a_J = 0$ and hence $a \leq \infty \cdot x$, as desired.

Finally, statement (3) is stronger than statement (3) of Proposition 5.3.5, which shows that it implies that a is soft. □

REMARK 5.3.9. Let S be a Cu-semigroup satisfying (O5), and let $a \in S$. Consider the statements (2) and (3) from Lemma 5.3.8. Even if S is not necessarily residually stably finite, then these imply that a is soft.

Conversely, if a is soft, then (3) of Lemma 5.3.8 might fail (one could have $x = 0$). It is not clear to the authors whether (2) might also fail, that is, it would be interesting to know whether a soft element satisfies (2) of Lemma 5.3.8 for a not necessarily residually stably finite Cu-semigroup.

COROLLARY 5.3.10. *Let S be a* Cu-*semigroup satisfying (O5), and let $a \in S$. Assume that S satisfies (RQQ) (e.g. S is almost unperforated) or that S is residually stably finite. Then, the following statements are equivalent:*

(1) The element a is soft.
(2) The element a is weakly purely noncompact.
(3) The element a is purely noncompact.

PROOF. Since S satisfies axiom (O5), the statements (1) and (2) are equivalent by Proposition 5.3.5. If S satisfies (RQQ), then the statements (2) and (3) are equivalent, as observed in Paragraph 5.3.6. Finally, if S is residually stably finite, it follows easily from Lemma 5.3.8 that (2) and (3) are equivalent. □

THEOREM 5.3.11. *Let S be a* Cu-*semigroup. Then:*

(1) The set S_{soft} of soft elements of S is a subsemigroup of S that is closed under passing to suprema of increasing sequences.
(2) The set S_{soft} is absorbing in the sense that for any $a, b \in S$ with $b \propto^* a$, we have that $a + b$ is soft whenever a is.

PROOF. To prove (1), let us first show that S_{soft} is closed under addition. So let $a, b \in S_{\mathrm{soft}}$ and let $x \in S$ satisfy $x \ll a + b$. We need to show that $x <_s a + b$.

Choose elements $a', b' \in S$ such that

$$x \leq a' + b', \quad a' \ll a, \quad b' \ll b.$$

Since a and b are soft, it follows that $a' <_s a$ and $b' <_s b$. By Proposition 5.2.8, this means that there exist $k_0, l_0 \in \mathbb{N}$ such that $(k+1)a' \leq ka$ for all $k \geq k_0$ and such that $(l+1)b' \leq lb$ for all $l \geq l_0$. Let $n \in \mathbb{N}$ be the maximum of k_0 and l_0. Then

$$(n+1)x \leq (n+1)(a' + b') \leq n(a + b),$$

which shows that $x <_s a + b$, as desired.

Next, let us show that S_{soft} is closed under suprema of increasing sequences. Let $(a_n)_{n\in\mathbb{N}} \subset S_{\mathrm{soft}}$ be an increasing sequence and set $a = \sup_n a_n$. Let $x \in S$ satisfy $x \ll a$. We need to show that $x <_s a$. Choose $\tilde{x} \in S$ such that $x \ll \tilde{x} \ll a$.

By definition of the way-below relation, we get that there is $n \in \mathbb{N}$ such that $\tilde{x} \leq a_n$. Then $x \ll a_n$. Since a_n is soft, it follows that $x <_s a_n$, and therefore $x <_s a$, as desired.

To prove (2), let $a \in S_{\mathrm{soft}}$ and $b \in S$ satisfy $b \propto^* a$. To show that $a + b$ is soft, let $x \ll a + b$. We need to show that $x <_s a + b$. Choose elements $a', b' \in S$ such that

$$x \leq a' + b', \quad a' \ll a, \quad b' \ll b.$$

Since $b \propto^* a$, it follows that there is $n \in \mathbb{N}$ such that $b' \leq na$. Moreover, since a is soft, we get that $a' <_s a$. By Proposition 5.2.8, this implies that there exists $k \in \mathbb{N}$ such that $(k+n+1)a' \leq ka$. Then:

$$\begin{aligned} (k+n+1)x &\leq (k+n+1)(a'+b') \\ &= (k+n+1)a' + (k+n)b' + b' \\ &\leq ka + (k+n)b + na = (k+n)(a+b), \end{aligned}$$

which shows that $x <_s a + b$, as desired. □

THEOREM 5.3.12. *Let S be an almost unperforated* Cu*-semigroup, and let $a, b \in S$. Assume that a is soft. Then $a \leq b$ if and only if $\hat{a} \leq \hat{b}$.*

PROOF. It is clear that $a \leq b$ implies that $\hat{a} \leq \hat{b}$. So assume that $\hat{a} \leq \hat{b}$. It is enough to show that $x \leq b$ for every $x \in S$ satisfying $x \ll a$. So let $x \in S$ such that $x \ll a$.

Since a is soft, we get that $x <_s a$. Then $\hat{x} <_s \hat{a}$, and together with the assumption we obtain that $\hat{x} <_s \hat{b}$. By Theorem 5.2.13, it follows that $x <_s^* b$. Since S is almost unperforated, this implies that $x \leq b$, as desired. □

REMARK 5.3.13. Theorems 5.3.11 and 5.3.12 are inspired by Proposition 6.4 and Theorem 6.6 in [**ERS11**]. Their results are concerned with purely noncompact elements in Cuntz semigroups of C^*-algebras, and their proofs use C^*-algebraic methods.

We generalize the mentioned results in [**ERS11**] in two ways. First, we consider abstract Cu-semigroups instead of concrete Cuntz semigroups coming from C^*-algebras. Therefore, our proofs are necessarily purely algebraic.

Second, we do not assume (O5), which is implicitly used to prove the results in [**ERS11**]. Note that axiom (O5) automatically holds for Cuntz semigroups of C^*-algebras, see Proposition 4.7, whence it is not an unreasonable assumption. We were able to obtain our results without using (O5), by considering soft elements instead of (weakly) purely noncompact elements.

It seems that soft elements form the right class to prove desirable results like Theorems 5.3.11 and 5.3.12. In the absence of (O5), it is unclear whether the same results hold for the class of (weakly) pruely noncompact elements. Moreover, as shown in Proposition 5.3.5, under the assumption of (O5) the class of soft and weakly purely noncompact elements coincide, so that then the results for (weakly) purely noncompact elements follow from that for soft elements.

PROBLEM 5.3.14. Given a Cu-semigroup S, is the subsemigroup S_{soft} of soft elements again a Cu-semigroup? Does this hold under the additional assumption that S satisfies (O5)? If so, does then S_{soft} satisfy (O5) as well?

REMARK 5.3.15. By Theorem 5.3.11, S_{soft} is a subsemigroup of S. It therefore inherits a natural structure as a positively ordered monoid. Moreover, axiom (O1) is satisfied. It is not clear, whether axiom (O2) holds. (If so, axioms (O3) and (O4) should follow immediately.)

The answer to Problem 5.3.14 is not even clear for Cuntz semigroups of C^*-algebras. In Proposition 7.3.14, we will provide a positive answer for semigroups with Z-multiplication, which includes in particular the Cuntz semigroups of $\mathcal{Z}$-stable C^*-algebras.

We end this section by showing that Problem 5.3.14 also has a positive answer for simple, stably finite Cu-semigroups satisfying (O5) and (O6). We first observe that, in this case, every noncompact element is automatically soft. This should be compared to [**Per01**, Lemma 3.4].

PROPOSITION 5.3.16. *Let S be a simple, stably finite* Cu*-semigroup satisfying (O5). Then a nonzero element in S is soft if and only if it is not compact.*

PROOF. In general, a finite compact element is never soft. Thus, let S be a simple, stably finite Cu-semigroup, and let $a \in S$ be a nonzero element that is not compact. We need to show that a is soft. By Lemma 5.3.8 and Remark 5.3.9, it is enough to show that for every $a', x \in S$ satisfying $a' \ll a \leq a' + x$, we have that $a \leq \infty \cdot x$. Given such a' and x, since $a' \ll a$ and a is not compact, we get that $a' \neq a$. Therefore, x is nonzero. Since S is simple, this implies that $a \leq \infty \cdot x$, as desired. □

LEMMA 5.3.17. *Let S be a* Cu*-semigroup, and let $B \subset S$ be a submonoid. Assume that for every $b \in B$, there exists a sequence $(b_k)_{k\in\mathbb{N}}$ in B such that $b = \sup_k b_k$ and such that $b_k \ll b_{k+1}$ in S for each k. Then*

$$\overline{B} = \left\{ \sup_k b_k \;\mid\; (b_k)_{k\in\mathbb{N}} \textit{ increasing sequence in } B \right\}$$

is a submonoid of S that is closed under passing to suprema of increasing sequences. Moreover, $\overline{B}$ is a Cu*-semigroup such that for each pair of elements $a, b \in \overline{B}$ we have $a \ll b$ in $\overline{B}$ if and only if $a \ll b$ in S.*

PROOF. We view $\overline{B}$ is a subset of S. It is easy to see that B is a subset of $\overline{B}$ and that $\overline{B}$ is a submonoid of S. Thus, endowed with the partial order induced by S, we have that $\overline{B}$ is a positively ordered monoid. Given an increasing sequence $(c_n)_n \subset \overline{B}$, let us show that the supremum $\sup_n c_n$ is an element of $\overline{B}$. For each $n \in \mathbb{N}$ we can, by assumption, choose a sequence $(c_{n,k})_k \subset B$ that is rapidly increasing in S and such that $c_n = \sup_k c_{n,k}$. As in the proof of Proposition 3.1.6, we can inductively choose indices k_n for $n \in \mathbb{N}$ such that

$$c_{1,k_1+n-1}, c_{2,k_2+n-2}, \ldots, c_{n,k_n} \leq c_{n+1,k_{n+1}}.$$

Then $(c_{n,k_n})_n$ is an increasing sequence of elements in B such that $\sup_n c_n = \sup_n c_{n,k_n}$. By definition, the element $\sup_n c_{n,k_n}$ belongs to $\overline{B}$. Thus, $\overline{B}$ is closed under passing to suprema of increasing sequences. Then, axioms (O1) and (O4) for $\overline{B}$ follow easily from their counterparts in S.

For clarity, let us denote the compact-containment relation with respect to S by $\ll_S$, and similarly for $\ll_{\overline{B}}$. Given $a, b \in \overline{B}$ satisfying $a \ll_S b$, let us prove that $a \ll_{\overline{B}} b$. Since $\overline{B}$ satisfies (O1), we need to show that for every increasing sequence $(b_k)_k$ in $\overline{B}$ satisfying $b \leq \sup_k b_k$ there exists $k \in \mathbb{N}$ such that $a \leq b_n$. Since the sequence $(b_k)_k$ has the same supremum in S as in $\overline{B}$, this follows directly form the assumption that $a \ll_S b$.

It follows that every element in B is the supremum of a sequence of elements in B that is rapidly increasing in $\overline{B}$. A diagonalization argument shows that the same holds for every element in $\overline{B}$. This verifies (O2) for $\overline{B}$.

Finally, given $a, b \in \overline{B}$ satisfying $a \ll_{\overline{B}} b$, let us show that $a \ll_S b$. Choose an element x in $\overline{B}$ such that $a \ll_{\overline{B}} x \ll_{\overline{B}} b$. Let $(b_k)_k \subset B$ be an increasing sequence such that $b = \sup_k b_k$. Then there exists $\bar{k} \in \mathbb{N}$ such that $x \leq b_{\bar{k}}$. By assumption, the element $b_{\bar{k}}$ is the supremum of an increasing sequence $(b_{\bar{k},n})_n$ in B such that $b_{\bar{k},n} \ll_S b_{\bar{k},n+1}$ for each n. Then

$$a \ll_{\overline{B}} x \leq b_{\bar{k}} = \sup_n b_{\bar{k},n},$$

which implies that there is $\bar{n}$ such that $a \leq b_{\bar{k},\bar{n}}$. Then

$$a \leq b_{\bar{k},\bar{n}} \ll_S b_{\bar{k},\bar{n}+1} \leq b,$$

which implies that $a \ll_S b$, as desired. Therefore, axiom (O3) for $\overline{B}$ follows since S satisfies (O3). □

Proposition 5.3.18. *Let S be a simple, stably finite* Cu*-semigroup satisfying (O5) and (O6). Then the subsemigroup S_{soft} is a* Cu*-semigroup satisfying (O5) and (O6).*

Proof. The statement holds if S is elementary. So we assume from now on that S is nonelementary. We want to apply Lemma 5.3.17 with $B = S_{\mathrm{soft}}$. By Theorem 5.3.11, we have that S_{soft} is closed under passing to suprema of increasing sequences. This implies that S satisfies (O1) and that $\overline{S_{\mathrm{soft}}} = S_{\mathrm{soft}}$.

Claim 1: For every nonzero element $a \in S$ there exists a nonzero element $b \in S_{\mathrm{soft}}$ satisfying $b \leq a$. To prove this claim, we inductively construct nonzero elements $a_n \in S$ for $n \in \mathbb{N}$ such that $2a_{n+1} \leq a_n$ for each n. We start by setting $a_0 = a$. Assuming that we have constructed a_k for all $k \leq n$, we apply [**Rob13a**, Proposition 5.2.1], see Proposition 5.4.1, to obtain a nonzero element $a_{n+1} \in S$ such that $2a_{n+1} \leq a_n$. Then, set

$$b = \sum_{n\in\mathbb{N}} a_n = \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} a_k.$$

We have $b \leq a$. Since S is stably finite, the element b cannot be compact. Therefore, by Proposition 5.3.16, it is a soft element. This proves the claim.

Claim 2: For every two elements $a \in S$ and $b \in S_{\mathrm{soft}}$ satisfying $a \ll b$, there exists an element $s \in S_{\mathrm{soft}}$ such that $a + s \ll b$. Note that by Theorem 5.3.11(2) this implies that $a + s$ is soft. To prove this claim, we first choose $b' \in S$ such that $a \ll b' \ll b$. By Lemma 5.3.8, there exists $x \in S$ such that $b' + x \leq b$ and $b \leq \infty \cdot x$. In particular, x is nonzero. Choose a nonzero element $x' \in S$ such that $x' \ll x$. By claim 1, there exists a nonzero element $s \in S_{\mathrm{soft}}$ such that $s \leq x'$. This implies that $s \ll x$. Moreover, we get that

$$a + s \ll b' + x \leq b,$$

which proves the claim.

Claim 3: For every $a \in S_{\mathrm{soft}}$, there exists a sequence $(a_k)_k \subset S_{\mathrm{soft}}$ that is rapidly increasing in S and such that $a = \sup_k a_k$. Since S satisfies (O2), it is enough to show that for every $a' \in S$ satisfying $a' \ll a$, there exists $b \in S_{\mathrm{soft}}$ such that $a' \leq b \ll a$. This follows directly from claim 2.

We can now apply Lemma 5.3.17 to deduce that S_{soft} is a Cu-semigroup. To verify (O5) for S_{soft}, let $a', a, b', b, c \in S_{\mathrm{soft}}$ satisfy

$$a + b \leq c, \quad a' \ll a, \quad b' \ll b.$$

Applying claim 2 for $a' \ll a$, there exists $s \in S_{\mathrm{soft}}$ such that $a' + s \ll a$. Then, using (O5) in S, we obtain an element $x \in S$ such that

$$(a' + s) + x \leq c \leq a + x, \quad b' \leq x.$$

Set $d = s + x$, which is soft by Theorem 5.3.11(2). Then

$$a' + d = a' + s + x \leq c \leq a + x \leq a + d, \quad b' \leq x \leq d,$$

which show that d has the desired properties to verify (O5) for S_{soft}.

Finally, to verify (O6) for S_{soft}, let $a', a, b, c \in S_{\mathrm{soft}}$ satisfy

$$a' \ll a \leq b + c.$$

Without loss of generality, we may assume that the elements a', a, b and c are nonzero. Using (O6) for S, there exist elements $e, f \in S$ such that

$$a' \leq e + f, \quad e \leq a, b, \quad f \leq a, c.$$

If e and f are soft, then these elements have the desired properties to verify (O6) for S_{soft}. So assume that e is not soft. By Proposition 5.3.16, this implies that e

is compact. Using (O5) for S, this implies that there are elements $x_1, x_2 \in S$ such that

$$e + x_1 = a, \quad e + x_2 = b.$$

Since a and b are not compact, we have that x_1 and x_2 are nonzero elements. By Lemma 5.1.17, there exists a nonzero element $\tilde{x} \in S$ satisfying $\tilde{x} \leq x_1, x_2$. Then, by claim 1, there exists a nonzero element $x \in S_{\mathrm{soft}}$ such that $x \leq \tilde{x}$ and hence $x \leq x_1, x_2$. By Theorem 5.3.11, the element $e + x$ is soft. Moreover, we get that

$$e + x \leq e + x_1 = a, \quad e + x \leq e + x_2 = b.$$

An analogous argument works in the case that f is not soft. □

5.4. Predecessors, after Engbers

In [**Eng14**], Engbers develops a theory of *predecessor* of compact elements in Cuntz semigroups of simple, stably finite C^*-algebras. Using algebraic methods, we obtain a weaker version of his results; see Theorem 5.4.5. First, we recall the following Glimm-halving result of Robert:

PROPOSITION 5.4.1 (Robert, [**Rob13a**, Proposition 5.2.1]). *Let S be a simple, nonelementary* Cu*-semigroup satisfying (O5) and (O6). Then, for every nonzero element $a \in S$, there exists a nonzero element b such that $2b \leq a$.*

PROPOSITION 5.4.2. *Let S be a simple, countably-based* Cu*-semigroup satisfying (O5) and (O6). Then, there exists a sequence $(g_n)_n \subset S$ of nonzero elements with the following properties:*

(1) The sequence is rapidly decreasing, that is, $g_n \gg g_{n+1}$ for each n.
(2) The sequence is cofinal among all nonzero elements, that is, for every nonzero element $a \in S$, there exists $n \in \mathbb{N}$ such that $g_n \leq a$.

PROOF. Since S is countably-based, we can choose a countable set of nonzero elements $\{a_n\}_{n \in \mathbb{N}} \subset S$ that is dense in the sense of Paragraph 2.1.1. We inductively construct the sequence $(g_n)_n$ such that for each n we have that

$$g_{n+1} \ll g_n, a_0, a_1, \ldots, a_{n+1}.$$

We start by letting $g_0 \in S$ be any nonzero element satisfying $g_0 \ll a_0$. Assume we have constructed g_k for all $k \leq n$. By Lemma 5.1.17, we can find g_{n+1} with the desired properties.

By construction, the sequence $(g_n)_n$ is rapidly decreasing. Finally, let $a \in S$ be a nonzero element. Since $\{a_n\}_n$ is a basis, there exists n such that $a_n \leq a$. It follows that $g_n \leq a$, as desired. □

In [**Eng14**], Engbers introduced the notion of a *predecessor* of a compact element p in a simple Cu-semigroup S. It is defined as

$$\gamma(p) = \max \{ x \in S \mid x < p \},$$

provided this maximum exists. Engbers shows the existence of predecessors for Cuntz semigroups of separable, simple and stably finite C^*-algebras, and he proves the following properties:

(1) For every nonzero element $z \in S$, we have $p \leq \gamma(p) + z$.
(2) For every noncompact element $y \in S$, we have $\gamma(p) + y = p + y$.
(3) For every $\lambda \in F(S)$, we have $\lambda(\gamma(p)) = \lambda(p)$.

With the new axiom (O5) we can almost recover this result in the algebraic setting, by showing that elements in $\{x \in S \mid x < p\}$ with these three properties do exist. However, we only get existence of the maximum (and thus uniqueness of predecessors) in the presence of weak cancellation or almost unperforation.

Since for a compact element p, the induced map $\hat{p}$ in $\mathrm{Lsc}(F(S))$ is continuous, the result gives us a *noncompact* element with the same property.

LEMMA 5.4.3. *Let S be a simple, countably-based, nonelementary, stably finite* Cu*-semigroup satisfying (O5) and (O6). Then, for every compact element $p \in S$, there exists a noncompact element $c \in S$ such that $c < p$ and such that $p \leq c + z$ for every nonzero element $z \in S$.*

PROOF. Without loss of generality, we may assume that $p \in S$ is a nonzero, compact element. Using Proposition 5.4.2, we can choose a rapidly decreasing sequence $(g_n)_n \subset S$ that is cofinal among the nonzero elements of S. By reindexing, if necessary, we may assume that $g_0 \leq p$. We will inductively construct elements $c_n, x_n, \tilde{c}_n \in S$ such that

$$c_n \ll x_n \ll \tilde{c}_n, \quad c_n \leq c_{n+1}, \quad x_n \leq \tilde{c}_{n+1}, \quad g_{n+1} + \tilde{c}_n \leq p \leq g_n + c_n,$$

for each $n \in \mathbb{N}$. First, we have that

$$g_1 \ll g_0 \leq p.$$

Applying axiom (O5), there exists an element $\tilde{c}_0$ such that

$$g_1 + \tilde{c}_0 \leq p \leq g_0 + \tilde{c}_0.$$

Since p is compact, there exists an element $c_0 \in S$ such that $c_0 \ll \tilde{c}_0$ and $p \leq g_0 + c_0$. Choose $x_0 \in S$ such that $c_0 \ll x_0 \ll \tilde{c}_0$.

For the induction, assume that we have constructed c_k, x_k and $\tilde{c}_k$ for $k \leq n$. Thus, we have that

$$g_{n+1} + \tilde{c}_n \leq p, \quad g_{n+2} \ll g_{n+1}, \quad x_n \ll \tilde{c}_n.$$

Then, by applying axiom (O5), there exists an element $\tilde{c}_{n+1}$ such that

$$g_{n+2} + \tilde{c}_{n+1} \leq p \leq g_{n+1} + \tilde{c}_{n+1}, \quad x_n \leq \tilde{c}_{n+1}.$$

Then $c_n \ll \tilde{c}_{n+1}$. Using also that p is a compact element, we can find an element $c_{n+1} \in S$ such that $c_n \ll c_{n+1} \ll \tilde{c}_{n+1}$ and $p \leq g_{n+2} + c_{n+1}$. Choose $x_{n+1} \in S$ such that $c_{n+1} \ll x_{n+1} \ll \tilde{c}_{n+1}$.

Note that the sequence $(c_n)_n$ is increasing. Therefore, we may set

$$c = \sup_n c_n.$$

Let us show that c has the desired properties.

We first observe that $c < p$. Indeed, it is clear that $c \leq p$. To obtain a contradiction, assume that $c = p$. Then, since p is compact and $c = \sup_n c_n$, we would have $p = c_n$ for some n. But we have that $g_{n+1} + c_n \leq p$ and g_{n+1} is nonzero. Thus, p would be an infinite compact element, which is not possible since S is stably finite.

Next, let $z \in S$ be a nonzero element. Then, there exists $n \in \mathbb{N}$ such that $z \geq g_n$. It follows that

$$c + z \geq c + g_n \geq c_n + g_n \geq p,$$

as desired.

Finally, let us show that c is not compact. Indeed, if c were compact, then by (O5) there would exist an element $y \in S$ such that $c + y = p$. Since $c < p$, the element y is nonzero. By Proposition 5.4.1, there exists a nonzero element $z \in S$ such that $2z \leq y$. As shown above, this implies that $p \leq c + z$. But then

$$p + z \leq c + z + z \leq p,$$

which is impossible since S is stably finite. $\square$

PROPOSITION 5.4.4. *Let S be a simple, nonelementary, stably finite* Cu*-semigroup satisfying (O5) and (O6). Let $p \in S$ be a compact element, and let $c \in S$ be a nonzero element such that $c < p$. Consider the following conditions:*

(1) For every nonzero element $z \in S$, we have $p \leq c + z$.
(2) The element c is noncompact, and for every noncompact element $y \in S$, we have $c + y = p + y$.
(3) For every $\lambda \in F(S)$, we have $\lambda(c) = \lambda(p)$.

Then, the following implications hold: '(1)⇔(2)⇒(3)'.

Moreover, if S has weak cancellation, then an element c satisfying (1) or (2) is equal to the maximum of the set $\{x \mid x < p\}$, and hence it is uniquely determined.

If S is almost unperforated, then all three conditions are equivalent and the element c satisfying (1)-(3) is uniquely determined.

PROOF. Let us show that (1) implies (2). So let $c \in S$ be an element satisfying the statement of (1). As shown at the end of the proof of Lemma 5.4.3, we have that c is necessarily noncompact.

Let y be a noncompact element in S. To show that $c + y = p + y$, we follow an argument similar to the one in [**Eng14**, Theorem 5.7], which we include for completeness. We clearly have $c + y \leq p + y$. For the converse inequality, it is enough to show that $p + y' \leq c + y'$ for every element $y' \in S$ satisfying $y' \ll y$. Given such an element y', choose an element $y'' \in S$ such that $y' \ll y'' \ll y$. Applying (O5) in S, there exists an element $z \in S$ such that

$$y' + z \leq y \leq y'' + z.$$

Notice that z is nonzero, as otherwise y would be compact. Using the assumption at the first step, we deduce that

$$y' + p \leq y' + c + z \leq y + c \leq y + p,$$

as desired.

Next, let us show that (2) implies (1). So let $z \in S$ be a nonzero element. We need to show that $p \leq c + z$. By assumption, this is clear if z is noncompact. So assume that z is compact. Choose a noncompact, nonzero element $y \in S$ with $y < z$. Then

$$p \leq p + y = c + y \leq c + z,$$

as desired.

Next, let us show that (2) implies (3). So let $\lambda \in F(S)$. We distinguish two cases. In the first case, we assume that $\lambda(p) < \infty$. Then $\lambda(c) < \infty$. Applying λ to the equality $p + c = 2c$, we get that

$$\lambda(p) + \lambda(c) = \lambda(c) + \lambda(c).$$

Then, we can cancel $\lambda(c)$ on both sides and obtain that $\lambda(p) = \lambda(c)$, as desired.

If the other case, we assume that $\lambda(p) = \infty$. It follows that λ is equal to λ_∞, the functional that takes value ∞ everywhere except at 0. Then, $\lambda(c) = \infty = \lambda(p)$, as desired.

Suppose now that S has weak cancellation, and that the element $c \in S$ satisfies (1)-(2). Assume that $x \in S$ satisfies $x < p$. We need to show that $x \leq c$. For this, it is enough to show that $x' \leq c$ for every $x' \in S$ satisfying $x' \ll x$.

So let such x' be given. By (O5), there exists $t \in S$ such that

$$x' + t \leq p \leq x + t.$$

Note that $t \neq 0$ as S is stably finite. Therefore, we get that

$$x' + t \leq p \ll p \leq c + t.$$

Applying weak cancellation, we obtain that $x' \leq c$, as desired.

Finally, assume that S is almost unperforated, and let us show that (3) implies (1). So let $z \in S$ be a nonzero element. By assumption, we have that $\hat{p} = \hat{c}$. Since S is simple and z is nonzero, it is straightforward to check that

$$\hat{p} <_s \widehat{c+z}.$$

By Theorem 5.2.13, we get that $p <_s^* c + z$. Since p is compact and S is almost unperforated, it follows that $p \leq c + z$. Moreover, the element is uniquely determined by Theorem 5.3.12. □

THEOREM 5.4.5 (cf. Engbers, [**Eng14**, Theorem 5.15]). *Let S be a simple, countably-based, nonelementary* Cu*-semigroup satisfying (O5), (O6) and weak cancellation. Then every compact element $p \in S$ has a predecessor $\gamma(p)$, uniquely determined by the property that $p \leq \gamma(p) + z$ for every nonzero element z.*

PROOF. This follows from Lemma 5.4.3 and Proposition 5.4.4. □

5.5. Algebraic semigroups

In this section, we study Cu-semigroup that have a basis of compact elements. Such semigroups are called 'algebraic'. Important examples are given by Cuntz semigroups or C^*-algebras with real rank zero.

Given a positively ordered monoid M, we show how to construct an algebraic Cu-semigroup $\mathrm{Cu}(M)$ such that the semigroup of compact elements in $\mathrm{Cu}(M)$ can be naturally identified with M, see Proposition 5.5.4. This establishes an equivalence between the category PoM of positively ordered monoids, and the full subcategory of Cu consisting of algebraic Cu-semigroups, see Proposition 5.5.5. In Proposition 5.5.8, we will see how certain properties of M translate to properties of $\mathrm{Cu}(M)$. Then, we provide a version of the Effros-Handelman-Shen theorem by showing that a Cu-semigroup is an inductive limit of simplicial Cu-semigroups if and only if it is weakly cancellative, unperforated, algebraic and satisfies (O5) and (O6), see Corollary 5.5.12. This also characterizes the Cuntz semigroups of separable AF-algebras.

DEFINITION 5.5.1. A Cu-semigroup S is *algebraic* if every element in S is the supremum of an increasing sequence of compact elements of S.

REMARKS 5.5.2. (1) Definition 5.5.1 is following the convention of domain theory to call a continuous partially ordered set *algebraic* if its compact elements form a basis; see [**GHK+03**, Definition I-4.2, p. 115].

(2) If A is a C^*-algebra with real rank zero, then $\mathrm{Cu}(A)$ is algebraic. For C^*-algebras with stable rank one, the converse holds; see [**CEI08**, Corollary 5].

5.5.3. Given a positively ordered monoid M, it is easy to see that the partial order $\leq$ is an auxiliary relation in the sense of Paragraph 2.1.1. In fact, it is the strongest auxiliary relation on M. Moreover, it is straightforward to check that $(M, \leq)$ is a W-semigroup. We denote its Cu-completion by $\mathrm{Cu}(M)$. In Proposition 5.5.4, we will see that $\mathrm{Cu}(M)$ is algebraic and that every algebraic Cu-semigroup arises this way.

Every PoM-morphism $f\colon M \to N$ between positively ordered monoids induces a Cu-morphism $\mathrm{Cu}(f)\colon \mathrm{Cu}(M) \to \mathrm{Cu}(N)$. Thus, we obtain a functor

$$\mathrm{Cu}\colon \mathrm{PoM} \to \mathrm{Cu}, \quad M \mapsto \mathrm{Cu}(M). \quad (M \in \mathrm{PoM})$$

Conversely, given a Cu-semigroup S, we let S_c denote the set of compact elements in S. It is easy to see that S_c is a submonoid of S and we equip it with the order induced by S. It follows that S_c is a positively ordered monoid. Moreover, every Cu-morphism $f\colon S \to T$ between Cu-semigroups restricts to a PoM-morphism from S_c to T_c. Hence, we obtain a functor

$$\mathrm{Cu} \to \mathrm{PoM}, \quad S \mapsto S_c. \quad (S \in \mathrm{Cu})$$

Proposition 5.5.4. *(1) Let M be a positively ordered monoid. Then $\mathrm{Cu}(M)$ as introduced in Paragraph 5.5.3 is an algebraic Cu-semigroup. Moreover, there is a natural identification of M with the positively ordered monoid of compact elements in $\mathrm{Cu}(M)$.*

(2) Let S be an algebraic Cu-semigroup. Consider the positively ordered monoid S_c of compact elements in S. Then there is a natural isomorphism $S \cong \mathrm{Cu}(S_c)$.

Proof. Let us show (1). Consider the natural map $\alpha\colon M \to \mathrm{Cu}(M)$ from M to its Cu-completion. Since $(M, \leq)$ is a W-semigroup, the map α is an order-embedding, see Remarks 3.1.9(2).

Given $a \in M$, we have $a \leq a$ and therefore $\alpha(a) \ll \alpha(a)$, showing that α maps M to the compact elements of S. On the other hand, let $s \in S$ be a compact element. By Theorem 3.1.8(1,ii), there exists $a \in M$ such that $s \leq \alpha(a) \leq s$, and hence $s = \alpha(a)$. This shows that α is an order-embedding that maps M onto S_c. It also follows from Theorem 3.1.8(1,ii) that every element in S is the supremum of an increasing sequence of compact elements, showing that S is an algebraic Cu-semigroup.

We leave the proof of (2) to the reader. $\square$

Proposition 5.5.5. *The two functors from Paragraph 5.5.3, assigning to a positively ordered monoid M its Cu-completion $\mathrm{Cu}(M)$, and assigning to an algebraic Cu-semigroup its positively ordered monoid of compact elements, establish an equivalence of the following categories:*

(1) The category PoM of positively ordered monoids, see Paragraph B.2.1.
(2) The full subcategory of Cu consisting of algebraic Cu-semigroups.

Remark 5.5.6. Let M be a positively ordered monoid. The Cu-completion $\mathrm{Cu}(M)$ has appeared in the literature before using different but equivalent constructions. First, recall that an interval in M is a nonempty, upwards directed, order-hereditary subset of M. In the literature, intervals are often called ideals or sometimes round ideals; see [**GHK$^+$03**, Definition 0-1.3, p. 3] and [**Law97**, Definition 2.1].

An interval I in M is *countably-generated* if there exists a countable cofinal subset for I. This is equivalent to saying that there is an increasing sequence $(a_n)_n$ in I such that

$$I = \{a \in M \mid a \leq a_n \text{ for some } n\}.$$

Countably generated intervals in M form a positively ordered monoid $\Lambda_\sigma(M)$, where addition of intervals I and J is the interval generated by $I + J$, and order is given by set inclusion; see [**Weh96**], and also [**Per97**].

Let us define a map $\Lambda_\sigma(M) \to \mathrm{Cu}(M)$. Given a countably generated ideal $I \in \Lambda_\sigma(M)$, let $(a_n)_n$ be a cofinal subsequence of I. Considering M as a submonoid of $\mathrm{Cu}(M)$, we may assign to I the element $\sup_n a_n$ in $\mathrm{Cu}(M)$. This induces a natural isomorphism $\Lambda_\sigma(M) \cong \mathrm{Cu}(M)$.

Similarly, if S is a Cu-semigroup, we may consider the natural map $\Phi\colon S \to \Lambda_\sigma(S_c)$, which sends an element $a \in S$ to the interval

$$\Phi(a) = \{x \in S_c \mid x \leq a\}.$$

If S is algebraic, then Φ is an isomorphism of positively ordered monoids, cf. [**ABP11**, Theorem 6.4].

We will now study how properties of a positively ordered monoid relate to properties of its Cu-completion. The results in Proposition 5.5.8 should be compared to Theorem 4.4.

Definition 5.5.7. Let M be a positively ordered monoid.

(1) We say that M has the *Riesz refinement property* if whenever there are $a_1, a_2, b_1, b_2 \in M$ with $a_1 + a_2 = b_1 + b_2$, then there exist $x_{i,j} \in M$ for $i = 1, 2$ such that $a_i = x_{i,1} + x_{i,2}$ for $i = 1, 2$ and $b_j = x_{1,j} + x_{2,j}$ for $j = 1, 2$.
(2) We say that M has the *Riesz decomposition property* if whenever there are $a, b, c \in M$ with $a \leq b + c$, then there exist $b', c' \in M$ such that $a = b' + c'$, $b' \leq b$ and $c' \leq c$.
(3) We say that M has the *Riesz interpolation property* if whenever there are $a_1, a_2, b_1, b_2 \in S$ such that $a_i \leq b_j$ for $i, j = 1, 2$, then there exists $c \in S$ such that $a_1, a_2 \leq c \leq b_1, b_2$.
(4) We say M has *cancellation* (or that M is *cancellative*) if for any $a, b, x \in M$, $a + x \leq b + x$ implies $a \leq b$.

The three Riesz properties are closely related but not equivalent in general. If M is algebraically ordered, then Riesz refinement implies Riesz decomposition. If M is cancellative and algebraically ordered, then all three Riesz properties are equivalent.

Proposition 5.5.8. *Let M be a positively ordered monoid.Then:*

(1) The monoid M is algebraically ordered if and only if $\mathrm{Cu}(M)$ satisfies (O5).
(2) The monoid M is cancellative if and only if $\mathrm{Cu}(M)$ is weakly cancellative.
(3) The monoid M has Riesz interpolation if and only if $\mathrm{Cu}(M)$ does.
(4) If M satisfies the Riesz decomposition property, then $\mathrm{Cu}(M)$ satisfies (O6). Conversely, if $\mathrm{Cu}(M)$ satisfies (O5), (O6) and weak cancellation, then M satisfies the Riesz decomposition property.

Proof. Let us show (1). By Theorem 4.4(1), $\mathrm{Cu}(M)$ satisfies (O5) if and only if M, considered as a W-semigroup $M = (M, \leq)$, satisfies (W5). First, assume that M is algebraically ordered. To show that M satisfies (W5), let $a', a, b', b, c, \tilde{c} \in M$ satisfy

$$a + b \leq c, \quad a' \leq a, \quad b' \leq b, \quad c \leq \tilde{c},$$

Since M is algebraically ordered, there exists $y \in M$ such that $a + b + y = c$. Set $x' = x = b + y$. One checks that x' and x have the desired properties to verify (W5) for the $M = (M, \leq)$.

Conversely, assume that M satisfies (W5). To show that M is algebraically ordered, let $a, c \in M$ satisfy $a \leq c$. Set $a' = a$, set $b' = b' = 0$ and set $\tilde{c} = c$. Since M satisfies (W5), there exist $x', x \in M$ such that

$$a' + x \leq \tilde{c}, \quad c \leq a + x', \quad x' \leq x.$$

Then

$$a + x = a' + x \leq \tilde{c} = c \leq a + x' \leq a + x,$$

showing that $a + x = c$. Thus, M is algebraically ordered.

Statement (2) follows directly from Theorem 4.4(3). Statement (3) is shown in [**Per97**, Proposition 2.12].

Finally, let us show (4). First, assume that M satisfies the Riesz decomposition property. By Theorem 4.4(2), it is enough to verify (W6) for M. Let $a', a, b, c \in M$ satisfy

$$a' \leq a \leq b + c.$$

By assumption, there exist $b', c' \in M$ such that

$$a' = b' + c', \quad b' \leq b, \quad c' \leq c.$$

Then $b' \leq a$ and $c' \leq a$, showing that b' and c' have the desired properties to verify (W6) for the W-semigroup $(M, \leq)$.

Conversely, assume that S satisfies (O5), (O6) and weak cancellation. By statements (1) and (2) and Theorem 4.4(2), we have that M is algebraically ordered, cancellative and satisfies (W6). To show that M has Riesz decomposition, let $a, b, c \in M$ satisfy $a \leq b + c$. Set $a' = a$. Since M satisfies (W6), there exist $e, f \in M$ such that

$$a' = a \leq e + f, \quad e \leq a, b, \quad f \leq a, c.$$

Since M is algebraically ordered, there exist $x, y, z \in M$ such that

$$a + x = e + f, \quad e + y = a, \quad f + z = a.$$

Then

$$a + x + y + z = e + f + y + z = 2a.$$

Since M is cancellative, we obtain that $a = x + y + z$. It follows that

$$y + [x + z] = a \leq a + x = f + [x + z],$$

which implies that $y \leq f$. Thus, we have that $a = e + y$ with $e \leq b$ and $y \leq c$, as desired. □

COROLLARY 5.5.9. *Let S be an algebraic* Cu-*semigroup satisfying (O5) and weak cancellation. Then the following are equivalent:*

(1) The Cu-*semigroup S satisfies (O6).*
(2) The Cu-*semigroup S has Riesz refinement.*
(3) The Cu-*semigroup S has Riesz decomposition.*
(4) The Cu-*semigroup S has Riesz interpolation.*
(5) The monoid of compact elements, S_c, has Riesz refinement (or equivalently, M has Riesz decomposition, or M has Riesz interpolation).

PROOF. Let S be an algebraic Cu-semigroup satisfying (O5) and weak cancellation. Let M be the positively ordered monoid of compact elements in S. As shown in Proposition 5.5.4, we have that S is isomorphic to the Cu-completion of M. By Proposition 5.5.8, M is algebraically ordered and cancellative. It follows that the three Riesz properties stated in (6) are equivalent for M.

By Proposition 5.5.8, we have that M has Riesz interpolation if and only S does. This shows the equivalence between (4) and (5). Similarly, we obtain the equivalence between (1) and (5) from Proposition 5.5.8.

It is easy to check that (3) implies (1). To see that (2) implies (1), let $a', a, b, c \in S$ satisfy $a' \ll a \leq b + c$. Since S is algebraic, there exists a compact element $x \in S$ such that $a' \leq x \leq a$. Since S satisfies (O5), there exists $y \in S$ such that $x + y = b + c$. Using Riesz refinement, there exist $r_{i,j} \in S$ for $i, j = 1, 2$ such that

$$x = r_{1,1} + r_{1,2}, \quad y = r_{2,1} + r_{2,2}, \quad b = r_{1,1} + r_{2,1}, \quad b = r_{1,2} + r_{2,2}.$$

Set $e = r_{1,1}$ and $f = r_{1,2}$. Then e and f have the desired properties to verify (O6) for S. Finally, it follows from Lemma 2.6(a) and Proposition 2.5 in [**Goo96**] that (5) implies (2) and (3). $\square$

We will now consider the class of algebraic Cu-semigroups that are Cu-completions of dimension groups. We first recall some definitions.

DEFINITION 5.5.10. Let M be a positively ordered monoid.

(1) We call M a *simplicial monoid* if it is isomorphic to the algebraically ordered monoid $\mathbb{N}^r$, for some $r \in \mathbb{N}_+$.
(2) We call M a *dimension monoid* if it is isomorphic to the inductive limit in PoM of simplicial monoids.

Let M be a positively ordered monoid. Recall that M is *unperforated* if for every $a, b \in M$ we have that $a \leq b$ whenever $na \leq nb$ for some $n \in \mathbb{N}_+$. Every simplicial monoid is algebraically ordered, cancellative, unperforated and satisfies the Riesz refinement property. It is easy to see that all these properties pass to inductive limits, whence they are satisfied by all dimension monoids. The converse is known as the Effros-Handelman-Shen theorem, [**EHS80**], which is formulated for partially ordered groups. The version given here for a positively ordered monoid M follows by passing to the Grothendieck completion G, from which M can be recovered as $M = G^+$. It is clear that for every separable AF-algebra A, the Murray-von Neumann semigroup $V(A)$ is a dimension monoid. The converse can for instance be found in [**Rør02**, Proposition 1.4.2, p.20].

THEOREM 5.5.11 (Effros, Handelman, Shen). *Let M be a countable positively ordered monoid. Then, the following are equivalent:*

(1) The monoid M is a dimension monoid.
(2) The monoid M is algebraically ordered, cancellative, unperforated and satisfies the Riesz refinement property.
(3) There is a separable AF-algebra A such that $M \cong V(A)$.

In order to formulate the analog of the Effros-Handelman-Shen theorem for Cu-semigroups, we will call a Cu-semigroup S a *simplicial* Cu-*semigroup* if it is isomorphic to the Cu-completion of a simplicial monoid, that is, if $S \cong \overline{\mathbb{N}}^r$ with the algebraic order, for some $r \in \mathbb{N}_+$.

COROLLARY 5.5.12. *Let S be a countably-based* Cu-*semigroup. Then, the following are equivalent:*

(1) The semigroup S is isomorphic to an inductive limit of simplicial Cu-*semigroups.*
(2) There is a dimension monoid M such that $S \cong \mathrm{Cu}(M)$.
(3) The semigroup S is weakly cancellative, unperforated, algebraic and satisfies (O5) and (O6).
(4) There is a separable AF-algebra A such that $S \cong \mathrm{Cu}(A)$.

PROOF. It easy to see that (1) and (2) are equivalent, and that (3) implies (2).

Let us show that (2) implies (3). So let M be a dimension monoid such that $S \cong \mathrm{Cu}(M)$. It follows directly from Theorem 5.5.11 and Proposition 5.5.8 that S is weakly cancellative, algebraic and satisfies (O5) and (O6). Let us check that S is unperforated. So let $a, b \in S$ and assume that $na \leq nb$ for some $n \in \mathbb{N}_+$. Since S is algebraic, we can choose increasing sequences $(a_k)_k$ and $(b_k)_k$ of compact elements in S, such that $a = \sup_k a_k$ and $b = \sup_k b_k$. For each k we have

$$na_k \ll na \leq nb = \sup_{l \in \mathbb{N}} nb_l.$$

Thus, there is $l \in \mathbb{N}$ such that $na_k \leq na_l$. Since M is unperforated and the natural map from M to S is an order-embedding, this implies that $a_k \leq b_l$. Thus, we have $a_k \leq b$ for each k, and therefore $a \leq b$, as desired.

Finally, let us show that (2) and (4) are equivalent. Given a separable AF-algebra A, the Cuntz semigroup of A is isomorphic to the Cu-completion of $V(A)$. Therefore, the desired equivalence follows from Theorem 5.5.11. $\square$

5.6. Nearly unperforated semigroups

In this section, we introduce the notion of 'near unperforation' for positively ordered monoids, see Definition 5.6.1. We study how this concept is connected to other notions like almost unperforation, separativity and cancellation properties. The main result of this section is Theorem 5.6.10 where we show that a simple, stably finite Cu-semigroup that satisfies (O5) is nearly unperforated if and only if it is weakly cancellative and almost unperforated.

In [**JS99**], the famous Jiang-Su algebra $\mathcal{Z}$ was introduced. Recall that it is a unital, separable, simple, nonelementary, nuclear C^*-algebra with stable rank one and unique tracial state. It is strongly self-absorbing and KK-equivalent to the complex numbers, which means that $K_0(\mathcal{Z}) \cong \mathbb{Z}$ and $K_1(\mathcal{Z}) = 0$. Therefore, tensoring with $\mathcal{Z}$ has no effect on the K-theory of a C^*-algebra, although it can change the ordering on the K_0-group (see e.g. [**GJS00**]). In the Elliott classification program, the Jiang-Su algebra is considered as the stably finite analog of the Cuntz algebra $\mathcal{O}_\infty$, which plays a central role in the classification of purely infinite C^*-algebras.

Given a C^*-algebra A that tensorially absorbs the Jiang-Su algebra $\mathcal{Z}$, it is well-known that the Cuntz semigroup $\mathrm{Cu}(A)$ is almost unperforated, [**Rør04**, Theorem 4.5]. Under the additional assumption that A is simple or that A has real rank zero and stable rank one, we obtain that $\mathrm{Cu}(A)$ is even nearly unperforated, see Corollary 5.6.15. We conjecture that the Cuntz semigroup of every $\mathcal{Z}$-stable C^*-algebra is nearly unperforated, see Conjecture 5.6.18.

Definition 5.6.1. Let M be a positively ordered monoid. We define a binary relation $\leq_p$ on M by setting $a \leq_p b$ for $a, b \in M$ if and only if there exists $k_0 \in \mathbb{N}$ such that $ka \leq kb$ for all $k \in \mathbb{N}$ satisfying $k \geq k_0$.

We say that M is *nearly unperforated* if for all $a, b \in M$ we have that $a \leq_p b$ implies $a \leq b$.

Note that $a \leq_p b$ if and only if there exists $k \in \mathbb{N}$ such that $ka \leq kb$ and $(k+1)a \leq (k+1)b$.

Lemma 5.6.2. *Let M be a positively ordered monoid. Then the following are equivalent:*

(1) The monoid M is nearly unperforated.
(2) For all $a, b \in M$, we have that $2a \leq 2b$ and $3a \leq 3b$ imply $a \leq b$.

Proof. It is easy to see that (1) implies (2). For the converse implication, let $a, b \in M$ satisfy $a \leq_p b$. Let $n \in \mathbb{N}$ be the smallest integer such that $ka \leq kb$ for all $k \geq n$. Arguing as in [**AGOP98**, Lemma 2.1], we will show that $(n-1)a \leq (n-1)b$ if $n \geq 2$. This shows that $n = 1$, and so $a \leq b$.

Assuming $n \geq 2$, we have that $2(n-1) \geq n$ and $3(n-1) \geq n$. It follows that

$$2(n-1)a \leq 2(n-1)b, \quad 3(n-1)a \leq 3(n-1)b.$$

By assumption, this implies $(n-1)a \leq (n-1)b$, as desired. $\square$

Let M be a positively ordered monoid. Recall that M is *unperforated* if for all elements $a, b \in M$ we have that $na \leq nb$ for some $n \in \mathbb{N}_+$ implies that $a \leq b$. Recall from Definition 5.2.11 that M is *almost unperforated* if $a <_s b$ implies $a \leq b$.

Let us say that M is *weakly separative* if for all elements a and b we have that $2a \leq a + b \leq 2b$ implies $a \leq b$. We warn the reader that different definitions of 'separativity' for (partially ordered) semigroups appear in the literature. However, in most of the recent literature, the notion of 'separativity' has been used for a concept which is stronger than the condition above, see e.g. [**Weh94**, Definition 1.2]. That is why we call the above condition 'weak separativity'.

Proposition 5.6.3. *Let M be a positively ordered monoid. Then, the following implications hold:*

$$\begin{array}{ccccc} M \text{ is unperforated} & \Longrightarrow & M \text{ is nearly unperforated} & \Longrightarrow & M \text{ is almost unperforated} \\ & & \Big\Downarrow & & \\ & & M \text{ is weakly separative}\,. & & \end{array}$$

Proof. It follows from Proposition 5.2.8 that the relation $a <_s b$ is stronger than the relation $\leq_p$. Therefore, for any pair of elements $a, b \in M$, the following implications hold:

$$na \leq nb \text{ for some } n \in \mathbb{N}_+ \quad \Longleftarrow \quad a \leq_p b \quad \Longleftarrow \quad a <_s b$$

This implies the horizontal implications of the diagram. It remains to show that near unperforation implies weak separativity. So let $a, b \in M$ satisfy $2a \leq a+b \leq 2b$. Then

$$2a \leq 2b, \quad 3a \leq a + 2b = (a + b) + b \leq 3b.$$

By Lemma 5.6.2, this implies that $a \leq b$, as desired. □

Lemma 5.6.4. *Let M be a positively ordered monoid, and let $a, b \in M$ and $k, l \in \mathbb{N}$. If $a + ka \leq b + ka$ and $a + lb \leq b + lb$, then $a \leq_p b$.*

Proof. Let a, b and k, l be as in the statement. Arguing as in [**AGOP98**, Lemma 2.1], it follows that

$$2a + ka = a + [a + ka] \leq a + [b + ka] \leq 2b + ka.$$

Inductively, we get that $ra + ka \leq rb + ka$ for all $r \in \mathbb{N}$. Analogously, we obtain that $lb + sa \leq lb + sb$ for all $s \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, we get that

$$(k + l + n)a \leq ka + (l + n)b = [ka + lb] + nb \leq [kb + lb] + nb = (k + l + n)b,$$

which shows that $a \leq_p b$, as desired. □

Definition 5.6.5. Let M be a positively pre-ordered monoid. We say that M is *preminimally ordered* if for all elements $a, b, x, y \in M$, we have that $a + x \leq b + x$ and $x \leq y$ imply that $a + y \leq b + y$.

We say that M is *simple* if for all elements $a, b \in M$ with b nonzero, we have that $a \propto b$, that is, there exists $n \in \mathbb{N}$ such that $a \leq nb$.

An element a in M is *finite* if $a < a + x$ for every nonzero element $x \in M$. We say that M is *stably finite* if each of its elements is finite.

Remarks 5.6.6. (1) The notion of being 'preminimally ordered' was introduced in [**Weh94**, Definition 1.2]. This concept is closely related to what has been called 'well-behaved' and 'strictly well-behaved' in [**Bla90**, Definition 2.2.1].

(2) Let M be a positively pre-ordered monoid. If M is cancellative, then it is stably finite. Indeed, given two elements a and x in M we always have $a \leq a + x$ and if $a = a + x$ then cancellation implies that $x = 0$.

(3) Let M be a conical monoid equipped with its algebraic pre-order. Then M is a positively pre-ordered monoid. If M is stably finite, then its algebraic pre-order is antisymmetric and hence M, with its algebraic order, becomes a partially ordered monoid.

(4) The notions of simplicity and stable finiteness have already been defined for Cu-semigroups, see Paragraph 5.2.2 and Definition 5.1.11. However, we warn the reader that for a Cu-semigroup S, theses notions do not coincide when considering S as a positively pre-ordered monoid. For instance, a nonzero Cu-semigroup always contains elements that are not finite. Moreover, a nonzero, simple Cu-semigroup is not simple as positively ordered monoid, since for a nonzero element $a \in S$ we need not have $\infty \propto a$, but only $\infty \propto^* a$.

One can, however, obtain a close connection as follows. Given a Cu-semigroup S, consider

$$S_0 = \{x \in S \mid x \ll \tilde{x} \text{ for some } \tilde{x} \in S\}.$$

Then S is simple (resp. stably finite) as a Cu-semigroup if and only if S_0 is simple (resp. stably finite) as a positively ordered monoid.

The next result shows that for Cu-semigroups, the axiom (O5) of almost algebraic order implies a suitable version of preminimality:

Lemma 5.6.7. *Let S be a* Cu*-semigroup satisfying (O5), and let $a, b, x, y \in S$. If $a + x \ll b + x$ and $x \leq y$, then $a + y \leq b + y$.*

Proof. Let a, b, x and y be as in the statement. Choose an element $x' \in S$ such that $x' \ll x$ and $a + x \ll b + x'$. Applying (O5) to the inequality $x' \ll x \leq y$, there exists $d \in S$ such that $x' + d \leq y \leq x + d$. Then

$$a + y \leq a + x + d \leq b + x' + d \leq b + y,$$

as desired. □

Proposition 5.6.8. *(1) Let M be a preminimal positively ordered monoid, and let $a, b, x \in M$. If $a + x \leq b + x$ and $x \propto a, b$, then $a \leq_p b$.*

(2) Let S be a Cu*-semigroup satisfying (O5), and let $a, b, x \in S$. If $a+x \ll b+x$ and $x \propto^* a, b$, then $a \leq_p b$.*

Proof. To show (2), let S be as in the statement, and let $a, b, x \in S$ satisfy $a + x \ll b + x$ and $x \propto^* a, b$. Choose an element $x' \in S$ such that $x' \ll x$ and $a + x \ll b + x'$. Then

$$a + x' \leq a + x \ll b + x'.$$

Moreover, we have $x' \propto a, b$, whence there are $k, l \in \mathbb{N}$ such that $x \leq ka$ and $x \leq lb$. By Lemma 5.6.7, we obtain that

$$a + ka \leq b + ka, \quad b + lb \leq b + lb.$$

Then, by Lemma 5.6.4, it follows that $a \leq_p b$. The proof of (1) is similar (and easier). □

The following result should be compared to [**Bla90**, Theorem 2.2.6].

Corollary 5.6.9. *(1) Let M be a simple, stably finite, preminimal positively ordered monoid, and let $a, b, x \in M$. If $a + x \leq b + x$, then $a \leq_p b$.*

(2) Let S be a simple, stably finite Cu*-semigroup satisfying (O5), and let $a, b, x \in S$. If $a + x \ll b + x$, then $a \leq_p b$.*

Proof. The argument for both statements is analogous. So let the elements a, b and x be as in the statements. The conclusion is clearly true if a is zero.

So assume that a is nonzero. Then, using stable finiteness in both cases, we get that b cannot be zero. Thus, we may assume that both a and b are nonzero. By simplicity, this implies that $x \propto a, b$ or $x \propto^* a, b$, respectively. Then, the conclusion follows from Proposition 5.6.8. $\square$

Recall from Definition 4.1 that a Cu-semigroup S is *weakly cancellative* if $a + x \ll b + x$ implies $a \leq b$, for any $a, b, x \in S$. It follows from the previous result that S is weakly cancellative whenever it is simple, stably finite, nearly unperforated and satisfies (O5). We remark that a simple, almost unperforated Cu-semigroup need not be weakly cancellative, see Chapter 9(8). See also Chapter 9(9) where we ask if this phenomenon is also possible for Cuntz semigroups of (simple) C^*-algebras.

By Proposition 5.6.3, near unperforation implies almost unperforation in general. The following result provides a converse.

Theorem 5.6.10. *Let S be a simple, stably finite* Cu*-semigroup satisfying (O5). Then S is nearly unperforated if and only if S is weakly cancellative and almost unperforated.*

Proof. It remains to check the 'if' part of the statement. So assume S is weakly cancellative and almost unperforated. It is clear that weak cancellation implies that S is stably finite. Let $a, b \in S$ satisfy $a \leq_p b$. By Proposition 5.3.16, an element in a simple, stably finite Cu-semigroup satisfying (O5) is either compact or nonzero and soft. We may therefore distinguish the following three cases.

Case 1: Assume that a is soft. Let $a' \in S$ satisfy $a' \ll a$. Since a is soft, it follows that $a' <_s a$ and therefore $a' <_s b$. Using that S is almost unperforated, we get that $a' \leq b$. Thus, we have shown that $a' \leq b$ for every $a' \in S$ satisfying $a' \ll a$, whence $a \leq b$.

Case 2: Assume that b is soft. Since $a \leq_p b$, there exists $n \in \mathbb{N}$ such that $na \leq nb$. Note that nb is also soft.

Let $a' \in S$ satisfy $a' \ll a$. Then $na' \ll nb$, and since nb is soft, it follows that $na' <_s nb$. This implies that $a' <_s b$, and hence $a' \leq b$ by almost unperforation. Again, as this holds for every $a' \in S$ satisfying $a' \ll a$, we get $a \leq b$.

Case 3: Assume that a and b are compact. If there is $n \in \mathbb{N}$ such that $na < nb$, then using (O5) for S, there exists a nonzero element $x \in S$ such that $na + x = nb$. Since S is simple and a is compact, we have that there exists $k \in \mathbb{N}$ such that $a \leq kx$. Then

$$(kn + 1)a \leq kna + kx = knb,$$

which shows that $a <_s b$. Since S is almost unperforated, we get that $a \leq b$.

In the other case, there exists $n \in \mathbb{N}$ with $na = nb$ and $(n+1)a = (n+1)b$. Let $x = na = nb$. Then $a + x \ll b + x$. It follows from weak cancellation that $a \ll b$.

Thus, in all three cases, it follows that $a \leq b$. This shows that S is nearly unperforated. $\square$

Proposition 5.6.11. *Let M be a simple, stably finite, algebraically ordered monoid. Then M is nearly unperforated if and only if M is cancellative and almost unperforated.*

Proof. It follows from Corollary 5.6.9 that every simple, stably finite, nearly unperforated, preminimal positively ordered monoid is cancellative. Moreover, by Proposition 5.6.3, near unperforation implies almost unperforation. This shows the 'only if' part of the statement.

For the converse, assume that M is a cancellative, almost unperforated, simple, algebraically ordered monoid. Let $a, b \in M$ satisfy $a \leq_p b$. If there is $n \in \mathbb{N}$ such that $na < nb$, then since M is algebraically ordered, there is a nonzero element $x \in M$ such that $na + x = nb$. As in case 3 in the proof of Theorem 5.6.10, this implies that $a \leq b$.

In the other case, there exists $n \in \mathbb{N}$ with $na = nb$ and $(n+1)a = (n+1)b$. By cancellation, it follows that $a \leq b$, as desired. □

PROPOSITION 5.6.12. *Let S be a* Cu*-semigroup satisfying (O5). Assume S is algebraic, almost unperforated and weakly cancellative. Then S is nearly unperforated.*

PROOF. Let S be as in the statement. To show that S is nearly unperforated, let $a, b \in S$ satisfy $a \leq_p b$. Since S is algebraic, we may assume without loss of generality that a and b are compact. Choose $n \in \mathbb{N}$ such that $na \leq nb$ and $(n+1)a \leq (n+1)b$. Since S satisfies (O5), there exist $x, y \in S$ such that

$$na + x = nb, \quad (n+1)a + y = (n+1)b.$$

Multiplying the first equation by $(n+1)$, and multiplying the second equation by n, we obtain that

$$n(n+1)a + (n+1)x = n(n+1)b, \quad n(n+1)a + ny = n(n+1)b.$$

Using that b is compact and that S is weakly cancellative, it follows that

$$(n+1)x = ny.$$

Then, since S is almost unperforated, we get that $x \leq y$. Using this at the second step, we get that

$$a + nb = a + na + x \leq a + na + y = b + nb.$$

Then, using that b is compact and that S has weak cancellation, it follows that $a \leq b$, as desired. □

PROBLEM 5.6.13. Let S be an almost unperforated Cu-semigroup. Which conditions are necessary and sufficient for S to be nearly unperforated? In particular, is it sufficient to assume that S satisfies weak cancellation and (O5)?

Concerning the second part of this problem, let S be an almost unperforated, weakly cancellative Cu-semigroup satisfying (O5). Then S is nearly unperforated if we additionally assume that S is simple or algebraic, see Theorem 5.6.10 and Proposition 5.6.12.

Let us draw some conclusions for Cuntz semigroups of C^*-algebras.

COROLLARY 5.6.14. *Let A be a C^*-algebra with stable rank one. Assume that A is either simple or has real rank zero. Then* $\mathrm{Cu}(A)$ *is nearly unperforated whenever it is almost unperforated.*

PROOF. By [**RW10**, Theorem 4.3], $\mathrm{Cu}(A)$ has weak cancellation. If A is simple, then so is $\mathrm{Cu}(A)$, see Corollary 5.1.12. If A has real rank zero, then $\mathrm{Cu}(A)$ is algebraic.

Then, the statement follows from Theorem 5.6.10 and Proposition 5.6.12 □

COROLLARY 5.6.15. *Let A be a $\mathcal{Z}$-stable C^*-algebra. Then* $\mathrm{Cu}(A)$ *is nearly unperforated if A is simple or has real rank zero and stable rank one.*

PROOF. Since A is $\mathcal{Z}$-stable, it follows from [**Rør04**, Theorem 4.5] that $\mathrm{Cu}(A)$ is almost unperforated.

We first assume that A is simple. Without loss of generality, we have $A \neq \{0\}$. Since A is $\mathcal{Z}$-stable, we can distinguish two cases. If A is purely infinite, then $\mathrm{Cu}(A) = \{0, \infty\}$, which is nearly unperforated. In the other case, we have that A

is stably finite, which by [**Rør04**, Theorem 6.7] implies that A has stable rank one. Then it follows from Corollary 5.6.14 that $\mathrm{Cu}(A)$ is nearly unperforated.

If A has real rank zero and stable rank one, then it follows also directly from Corollary 5.6.14 that $\mathrm{Cu}(A)$ is nearly unperforated. □

LEMMA 5.6.16. *(1) Let S be a nearly unperforated* PreW*-semigroup. Then, its* Cu*-completion $\gamma(S)$ is nearly unperforated.*

(2) Let (S_i, φ_i) be an inductive system of nearly unperforated semigroups in PoM*,* PreW *or* Cu*. Then $S = \lim_i S_i$ is nearly unperforated.*

PROOF. Let us show (1). Given $s \in S$, we denote by $\bar{s}$ its image in $\gamma(S)$. Let $a, b \in \gamma(S)$ such that $2a \leq 2b$ and $3a \leq 3b$. By properties of the Cu-completion, see Theorem 3.1.8, we can choose rapidly increasing sequences $(a_n)_n$ and $(b_n)_n$ in S such that $a = \sup_n \bar{a}_n$ and $b = \sup_n \bar{b}_n$.

Fix $n \in \mathbb{N}$. Then

$$2\bar{a}_n \ll 2b = \sup_k 2\bar{b}_k, \quad 3\bar{a}_n \ll 3b = \sup_k 3\bar{b}_k.$$

Thus, we can find indices k and l such that $2\bar{a}_n \leq 2\bar{b}_k$ and $3\bar{a}_n \leq 3\bar{b}_l$. Set $m = \max\{k, l\} + 1$. Then

$$2\bar{a}_n \ll 2\bar{b}_m, \quad 3\bar{a}_n \ll 3\bar{b}_m.$$

By properties of the Cu-completion, this implies that $2a_n \prec 2b_m$ and $3a_n \prec 3b_m$. Using that S is nearly unperforated, we obtain that $a_n \leq b_m$, and thus $\bar{a}_n \leq \bar{b}_m \leq b$. It follows that $a \leq b$, as desired.

Next, let us show (2). It is straightforward to check the statement for limits in PoM. Using that the limit in PreW has the same order structure as the limit in PoM, the result follows for limits in PreW. Recall that Corollary 3.1.11 shows that the limit of an inductive system in Cu is the Cu-completion of the limit of the same system considered in PreW. Therefore, the statement for Cu follows from (1). □

For the next result, recall that we say that a C^*-algebra A has *no K_1-obstructions*, if it has stable rank one and if $K_1(I) = \{0\}$ for any closed two-sided ideals I of A, see [**ABP13**] and [**ABPP14**].

PROPOSITION 5.6.17. *Let A be a separable $\mathcal{Z}$-stable C^*-algebra that has no K_1-obstructions. Then* $\mathrm{Cu}(A)$ *is nearly unperforated.*

PROOF. Recall from [**RW10**], that $\mathcal{Z}$ is isomorphic to a sequential inductive limit where each algebra in the inductive system is equal to the fixed generalized dimension drop algebra

$$Z_{2^\infty,3^\infty} := \{ f \in C([0,1], M_{2^\infty} \otimes M_{3^\infty}) \mid f(0) \in M_{2^\infty} \otimes 1,\ f(1) \in 1 \otimes M_{3^\infty} \}.$$

Since A is $\mathcal{Z}$-stable we have $A \cong \varinjlim_k A \otimes Z_{2^\infty,3^\infty}$. By Corollary 3.2.9, we obtain that

$$\mathrm{Cu}(A) \cong \varinjlim_k \mathrm{Cu}(A \otimes Z_{2^\infty,3^\infty}).$$

Therefore, by Lemma 5.6.16, it is enough to prove that $\mathrm{Cu}(A \otimes Z_{2^\infty,3^\infty})$ is nearly unperforated. We remark that so far the argument applies for every C^*-algebra A.

We identify $A \otimes Z_{2^\infty,3^\infty}$ with the C^*-algebra of continuous maps f from $[0,1]$ to $A \otimes M_{2^\infty} \otimes M_{3^\infty}$ such that

$$f(0) \in A \otimes M_{2^\infty} \otimes 1, \quad f(1) \in A \otimes 1 \otimes M_{3^\infty}.$$

We use ev_0 and ev_1 to denote the evaluation at the endpoints 0 and 1 of $[0,1]$, respectively. Then, we have a commutative pullback diagram:

$$\begin{tikzcd} A\otimes Z_{2^\infty,3^\infty} \arrow[r, dashed, "\mathrm{ev}_0\oplus\mathrm{ev}_1"] \arrow[d, dashed] & A\otimes M_{2^\infty}\otimes 1 \oplus A\otimes 1\otimes M_{3^\infty} \arrow[d, "\iota_3,\iota_2"] \\ C([0,1], A\otimes M_{2^\infty}\otimes M_{3^\infty}) \arrow[r, two heads, "\mathrm{ev}_0\oplus\mathrm{ev}_1"] & A\otimes M_{2^\infty}\otimes M_{3^\infty} \oplus A\otimes M_{2^\infty}\otimes M_{3^\infty}, \end{tikzcd}$$

where ι_3 and ι_2 denote the natural inclusion maps

$$\iota_3\colon A\otimes M_{2^\infty}\otimes 1 \to A\otimes M_{2^\infty}\otimes M_{3^\infty}, \quad \iota_2\colon A\otimes 1\otimes M_{3^\infty} \to A\otimes M_{2^\infty}\otimes M_{3^\infty}.$$

We identify M_{6^∞} with $M_{2^\infty}\otimes M_{3^\infty}$, and $A\otimes M_{2^\infty}\otimes 1$ with$A\otimes M_{2^\infty}$, and $A\otimes 1\otimes M_{3^\infty}$ with $A\otimes M_{3^\infty}$. Then, since A has no K_1-obstructions, we can apply [**APS11**, Theorem 3.5] to compute $\mathrm{Cu}(A\otimes Z_{2^\infty,3^\infty})$ as the pullback semigroup

$$\begin{tikzcd} \mathrm{Cu}(A\otimes Z_{2^\infty,3^\infty}) \arrow[r, dashed, "\mathrm{ev}_0\oplus\mathrm{ev}_1"] \arrow[d, dashed] & \mathrm{Cu}(A\otimes M_{2^\infty}) \oplus \mathrm{Cu}(A\otimes M_{3^\infty}) \arrow[d, "\mathrm{Cu}(\iota_3),\mathrm{Cu}(\iota_2)"] \\ \mathrm{Cu}(C([0,1], A\otimes M_{6^\infty})) \arrow[r, two heads, "\mathrm{ev}_0\oplus\mathrm{ev}_1"] & \mathrm{Cu}(A\otimes M_{6^\infty}) \oplus \mathrm{Cu}(A\otimes M_{6^\infty}). \end{tikzcd}$$

Given a Cu-semigroup S, we denote by $\mathrm{Lsc}([0,1],S)$ the semigroup of lower-semicontinuous functions from $[0,1]$ to S with pointwise order and addition. Again, using that A has no K_1-obstructions, by [**APS11**, Corollary 2.7] we have that

$$\mathrm{Cu}(C([0,1], A\otimes M_{6^\infty})) \cong \mathrm{Lsc}([0,1], \mathrm{Cu}(A\otimes M_{6^\infty})),$$

Now, let $a,b \in \mathrm{Cu}(A\otimes Z_{2^\infty,3^\infty})$ satisfy $a \leq_p b$. Using the pullback description above, we can choose $f,g \in \mathrm{Lsc}([0,1], \mathrm{Cu}(A\otimes M_{6^\infty}))$, and $x,u \in \mathrm{Cu}(A\otimes M_{2^\infty})$, and $y,v \in \mathrm{Cu}(A\otimes M_{3^\infty})$ such that

$$f(0)=x, \quad f(1)=y, \quad g(0)=u, \quad g(1)=v,$$

and so that a,b are identified as

$$a=(f,x,y), \quad b=(g,u,v).$$

It follows that $f \leq_p g$, and $x \leq_p u$, and $y \leq_p v$. By Corollary 7.4.15, the Cuntz semigroups $\mathrm{Cu}(C([0,1], A\otimes M_{6^\infty}))$, $\mathrm{Cu}(A\otimes M_{2^\infty})$ and $\mathrm{Cu}(A\otimes M_{3^\infty})$ are each nearly unperforated. Therefore, we obtain that $f \leq g$, and $x \leq u$, and $y \leq v$. Hence $a \leq b$, thus completing the proof. $\square$

Inspired by the previous results, we make the following conjecture.

CONJECTURE 5.6.18. Let A be a $\mathcal{Z}$-stable C^*-algebra. Then $\mathrm{Cu}(A)$ is nearly unperforated.

5.6.19. We have verified Conjecture 5.6.18 for several classes of C^*-algebras. Let A be a $\mathcal{Z}$-stable C^*-algebra. Then $\mathrm{Cu}(A)$ is nearly unperforated in the following cases:

(1) If A is simple, see Corollary 5.6.15.
(2) If A has real rank zero and stable rank one, see Corollary 5.6.15.
(3) If A is UHF-stable, see Corollary 7.4.15.
(4) If A is purely infinite (not necessarily simple), see Corollary 7.2.9.
(5) If A has no K_1-obstructions, see Proposition 5.6.17.

CHAPTER 6

Bimorphisms and tensor products

In this chapter, we first present a framework for a theory of tensor products in enriched categories. We focus on the categories PreW and Cu, which are both enriched over the category PoM, see Proposition 6.2.2.

In Section 6.2, we construct tensor products in PreW. Given two PreW-semigroups S and T, we consider the tensor product $S \otimes_{\mathrm{PoM}} T$ of the underlying positively ordered monoids as constructed in Section B.2, and we equip it with a natural auxiliary relation $\prec$, see Definition 6.2.9. We show that

$$S \otimes_{\mathrm{PreW}} T = (S \otimes_{\mathrm{PoM}} T, \prec)$$

is a PreW-semigroup that has the universal properties of a tensor product, see Theorem 6.2.10. We then show that this gives PreW the structure of a symmetric, monoidal category, see Paragraph 6.2.11.

In Section 6.3, we show the existence of tensor products in Cu by combining the result for PreW with the fact that Cu is a reflective subcategory of PreW. More precisely, given two Cu-semigroups S and T, their tensor product in Cu is given as

$$S \otimes_{\mathrm{Cu}} T = \gamma(S \otimes_{\mathrm{PreW}} T),$$

which is the Cu-completion of $S \otimes_{\mathrm{PreW}} T$, see Theorem 6.3.3.

Given two C^*-algebras A and B, there is a natural Cu-morphism

$$\tau_{A,B} \colon \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\max} B).$$

It is a natural to ask when this map is an isomorphism. In Proposition 6.4.13, we provide a positive answer if one of the C^*-algebras is an AF-algebra. The crucial observation is that the tensor product in PreW and Cu are continuous functors in each variable, see Proposition 6.4.1.

In Proposition 7.2.3, we show that for every Cu-semigroup S, the tensor product of S with $\{0, \infty\}$ is naturally isomorphic to $\mathrm{Lat}_{\mathrm{f}}(S)$, the Cu-semigroup of singly-generated ideals of S as considered in Proposition 5.1.7. It follows that for every two Cu-semigroups S and T, there is a natural isomorphism

$$\mathrm{Lat}_{\mathrm{f}}(S \otimes_{\mathrm{Cu}} T) \cong \mathrm{Lat}_{\mathrm{f}}(S) \otimes_{\mathrm{Cu}} \mathrm{Lat}_{\mathrm{f}}(T).$$

In Corollary 7.2.13, we apply these results for the Cuntz semigroup of a separable C^*-algebra A and obtain that there are natural isomorphisms

$$\mathrm{Cu}(A \otimes \mathcal{O}_2) \cong \mathrm{Lat}(A) \cong \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \{0, \infty\} \cong \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(\mathcal{O}_2),$$

where $\mathcal{O}_2$ denotes the Cuntz algebra generated by two isometries with range projections adding up to the unit. The same result holds when $\mathcal{O}_2$ is replaced by any simple, purely infinite C^*-algebra.

6.1. Tensor product as representing object

In this section, we give a general categorial setup for tensor product, which is in part inspired by the approach in [**BN76**]. When constructing the tensor product of objects with a certain structure, the notion of a bimorphism is a crucial ingredient.

In some categories, a bimorphism between three objects X, Y and Z is simply a map $X \times Y \to Z$ that is a morphism in each variable; for instance in the categories Mon and PoM, see Paragraphs B.1.1 and B.2.1. In other cases, a bimorphism is a map $X \times Y \to Z$ that is only required to be a 'generalized' morphism in each variable but additionally has to satisfy a condition taking both variables into account; for instance in the category of C^*-algebras C^*, see Example 6.1.3, and the categories PreW and Cu, see Definitions 6.2.3 and 6.3.1, see also Lemmas 6.2.4 and 6.3.2.

With the notion of bimorphisms at hand, the tensor product of two objects X and Y can often be defined as an object that represents the functor $Z \mapsto \mathrm{Bimor}(X \times Y, Z)$. This means, the tensor product $X \otimes Y$ satisfies:

$$\mathrm{Bimor}(X \times Y, Z) \cong \mathrm{Mor}(X \otimes Y, Z).$$

Of course, whether the functor $\mathrm{Bimor}(X \times Y, \_)$ is representable or not depends heavily on the considered categories and the objects X and Y.

If the (bi)morphism sets carry additional structure, we want that it is preserved by the above identification. This can be made precise using the language of enriched categories and functors. The basic theory of monoidal and enriched categories can be found in Appendix A. For details we refer the reader to [**Mac71**] and [**Kel05**].

6.1.1 (Representable functor). Let $\mathcal{C}$ be a category that is enriched over the closed, symmetric, monoidal category $\mathcal{V}$. Each object X in $\mathcal{C}$ defines a $\mathcal{V}$-functor

$$\mathcal{C}(X, \_)\colon \mathcal{C} \to \mathcal{V}$$

as follows: An object Z in $\mathcal{C}$ is sent to the object $\mathcal{C}(X, Z)$ in $\mathcal{V}$. Further, given objects Z and Z' in $\mathcal{C}$, the $\mathcal{V}$-morphism

$$C(X, \_)_{Z,Z'}\colon \mathcal{C}(Z, Z') \to \mathcal{C}(X, Z')^{\mathcal{C}(X,Z)}$$

is the one corresponding to $M_{X,Z,Z'}$ (defining the composition of morphisms in $\mathcal{C}$) under the identification

$$\mathcal{V}\left(\mathcal{C}(Z, Z') \otimes \mathcal{C}(X, Z), \mathcal{C}(X, Z')\right) \cong \mathcal{V}\left(\mathcal{C}(Z, Z'), \mathcal{C}(X, Z')^{\mathcal{C}(X,Z)}\right).$$

The $\mathcal{V}$-functor $\mathcal{C}(X, \_)$ is called the *representable functor* corresponding to X.

6.1.2 (Bimorphism functor). Let $\mathcal{C}$ be a category that is enriched over the closed, symmetric, monoidal category $\mathcal{V}$. Given objects X and Y in $\mathcal{C}$, we assume that there is a $\mathcal{V}$-functor

$$\mathrm{Bi}\mathcal{C}(X \times Y, \_)\colon \mathcal{C} \to \mathcal{V}.$$

This means that for each object Z in $\mathcal{C}$ there is an object $\mathrm{Bi}\mathcal{C}(X \times Y, Z)$ in $\mathcal{V}$, representing the bimorphisms from $X \times Y$ to Z. Moreover, for each pair of objects Z and Z' in $\mathcal{C}$, there is a $\mathcal{V}$-morphism

$$\mathrm{Bi}\mathcal{C}(X \times Y, \_)_{Z,Z'}\colon \mathcal{C}(Z, Z') \to \mathrm{Bi}\mathcal{C}(X \times Y, Z')^{\mathrm{Bi}\mathcal{C}(X\times Y,Z)}.$$

We remark that the notation '$X \times Y$' appearing in the bimorphism functor does not refer to the product of the objects X and Y. In general, we do not need to assume that the considered category has products. The notation is chosen since for the concrete cases considered in this paper, a bimorphism is a map from the Cartesian product of the underlying sets of X and Y to the underlying set of Z.

Example 6.1.3. Let $\mathcal{C}$ be one of the concrete categories considered in this paper (e.g. Cu), and let X, Y and Z be objects in $\mathcal{C}$. Then X, Y and Z are sets with additional structure, and a $\mathcal{C}$-morphism from X to Z is just a map $X \to Z$ preserving this structure. Similarly, a $\mathcal{C}$-bimorphism from $X \times Y$ to Z is just a map $X \times Y \to Z$ satisfying certain conditions.

Consider for example the category C^* of C^*-algebras, which is enriched over CGHTop, see Examples A.0.7. Given C^*-algebras A, B and C, the set of *-homomorphisms from A to B has a natural topology, giving $C^*(A,B)$ the structure of a compactly generated, Hausdorff space. The representable functor

$$C^*(A,\_)\colon C^* \to \mathrm{CGHTop},$$

sends a C^*-algebra C to the space $C^*(A,C)$ in CGHTop, and for each pair of C^*-algebras, C and C', the CGHTop-morphism

$$C^*(A,\_)_{C,C'}\colon C^*(C,C') \to C^*(A,C')^{C^*(A,C)}$$

is given by

$$C^*(A,\_)_{C,C'}\colon \alpha \mapsto [\varphi \mapsto \alpha\circ\varphi]. \quad (\alpha \in C^*(C,C'), \varphi \in C^*(A,C))$$

Given three C^*-algebras A, B and C, a C^*-bimorphism from $A\times B$ to C is a map $\varphi\colon A\times B \to C$ satisfying the following conditions:

(i) The map φ is bounded and linear in each variable.
(ii) We have $\varphi(a^*,b^*) = \varphi(a,b)^*$ for each $a\in A$ and $b \in B$.
(iii) We have $\varphi(a_1a_2, b_1b_2) = \varphi(a_1,b_1)\varphi(a_2,b_2)$ for each $a_1,a_2\in A$ and $b_1,b_2\in B$.

We equip the set $\mathrm{Bi}C^*(A\times B, C)$ of all C^*-bimorphisms from $A\times B$ to C with the topology of point-norm convergence.

Then, given two C^*-algebras A and B, we define the bimorphism functor

$$\mathrm{Bi}C^*(A\times B,\_)\colon C^* \to \mathrm{CGHTop}$$

as follows: A C^*-algebra C is sent to the space $\mathrm{Bi}C^*(A\times B, C)$ in CGHTop, and for each pair of C^*-algebras, C and C', the CGHTop-morphism

$$\mathrm{Bi}C^*(A\times B,\_)_{C,C'}\colon C^*(C,C') \to \mathrm{Bi}C^*(A\times B,C')^{\mathrm{Bi}C^*(A\times B,C)}$$

is given by

$$\mathrm{Bi}C^*(A\times B,\_,)_{C,C'}\colon \alpha \mapsto [\varphi \mapsto \alpha\circ\varphi], \quad (\alpha \in C^*(C,C'), \varphi \in \mathrm{Bi}C^*(A\times B, C))$$

6.1.4. Let $\mathcal{C}$ be a category that is enriched over the closed, symmetric, monoidal category $\mathcal{V}$. Assume that for each pair of objects X and Y in $\mathcal{C}$, there is a bimorphism $\mathcal{V}$-functor $\mathrm{Bi}\mathcal{C}(X\times Y,\_)$. Let Q be an object in $\mathcal{C}$, and let φ be an element of $\mathrm{Bi}\mathcal{C}(X\times Y, Q)$. Let us show that this induces a $\mathcal{V}$-natural transformation

$$\Phi\colon\ \mathcal{C}(Q,\_)\ \Rightarrow\ \mathrm{Bi}\mathcal{C}(X\times Q,\_).$$

Given an object Z in $\mathcal{C}$, we need to define an element

$$\Phi_Z \in \mathrm{Bi}\mathcal{C}(X\times Y, Z)^{\mathcal{C}(Q,Z)}.$$

Elements in $\mathrm{Bi}\mathcal{C}(X\times Y,Z)^{\mathcal{C}(Q,Z)}$ correspond naturally to elements in the set

$$\mathcal{V}\left(\mathcal{C}(Q,Z), \mathrm{Bi}\mathcal{C}(X\times Y, Z)\right).$$

Thus, we can think of Φ_Z as a $\mathcal{V}$-morphism from $\mathcal{C}(Q,Z)$ to $\mathrm{Bi}\mathcal{C}(X\times Y,Z)$. To define Φ_Z, we use the $\mathcal{V}$-morphism defining the bimorphism functor

$$\mathrm{Bi}\mathcal{C}(X\times Y,\_)_{Q,Z}\colon \mathcal{C}(Q,Z) \to \mathrm{Bi}\mathcal{C}(X\times Y,Z)^{\mathrm{Bi}\mathcal{C}(X\times Y,Q)},$$

which naturally corresponds to a $\mathcal{V}$-morphism

$$G_{Q,Z}\colon \mathcal{C}(Q,Z)\otimes \mathrm{Bi}\mathcal{C}(X\times Y,Q) \to \mathrm{Bi}\mathcal{C}(X\times Y,Z).$$

Then, Φ_Z is the $\mathcal{V}$-morphism given as the following composition:

$$\mathcal{C}(Q,Z) \xrightarrow{\cong} \mathcal{C}(Q,Z)\otimes I \xrightarrow{\mathrm{id}\otimes\varphi} \mathcal{C}(Q,Z)\otimes \mathrm{Bi}\mathcal{C}(X\times Y,Q) \xrightarrow{G_{Q,Z}} \mathrm{Bi}\mathcal{C}(X\times Y,Z).$$

Definition 6.1.5. With the notation from Paragraph 6.1.4, we say that the pair (Q,φ) is a *tensor product* of X and Y in the enriched category $\mathcal{C}$, if Φ is a natural isomorphism, that is, if

$$\Phi_Z\colon \mathcal{C}(Q,Z) \to \mathrm{Bi}\mathcal{C}(X\times Y,Z)$$

is a $\mathcal{V}$-isomorphism for each object Z in $\mathcal{C}$.

Remark 6.1.6. We retain the notation from Paragraph 6.1.4. Recall that a representation of a $\mathcal{V}$-functor $F\colon \mathcal{C}\to\mathcal{V}$ is an object Q in $\mathcal{C}$ together with a natural isomorphism Φ from the representable functor $\mathcal{C}(Q,\_)$ to F. Thus, a tensor product (Q,φ) for X and Y induces a representation (Q,Φ) of the $\mathcal{V}$-functor $\mathrm{Bi}\mathcal{C}(X\times Y,\_)$.

Conversely, assume that the $\mathcal{V}$-functor $\mathrm{Bi}\mathcal{C}(X\times Y,\_)$ is represented by the object Q and the natural isomorphism Φ. Then Φ_Q is a $\mathcal{V}$-isomorphism

$$\Phi_Q\colon \mathcal{C}(Q,Q) \xrightarrow{\cong} \mathrm{Bi}\mathcal{C}(X\times Y,Q).$$

Under this isomorphism, the identity element $\mathrm{id}_Q\in\mathcal{C}(Q,Q)$ corresponds to an element $\varphi\in\mathrm{Bi}\mathcal{C}(X\times Y,Q)$. It is straightforward to check that (Q,φ) induces the $\mathcal{V}$-natural isomorphism (Q,Φ). Thus, (Q,φ) is a tensor product of X and Y.

To summarize, we have a natural correspondence between the following two classes:

(1) Concrete tensor products (Q,φ) of X and Y, where Q is an object in $\mathcal{C}$, and where φ is an element in $\mathrm{Bi}\mathcal{C}(X\times Y,Q)$.
(2) Representations of the $\mathcal{V}$-functor $\mathrm{Bi}\mathcal{C}(X\times Y,\_)$.

Any two tensor products of X and Y are isomorphic, which justifies to write $X\otimes Y$. We also write $X\otimes_{\mathcal{C}} Y$ if we need to clarify the category where the tensor product is taken.

Example 6.1.7. Consider the category C^* of C^*-algebras, which is enriched over CGHTop. Let A and B be two C^*-algebras and consider the C^*-bimorphism functor

$$\mathrm{Bi}C^*(A\times B,\_)\colon C^*\to \mathrm{CGHTop}$$

from Example 6.1.3. Let $A\otimes_{\max} B$ denote the maximal tensor product of A and B. We refer the reader to [**Bla06**, § II.9] for an introduction and details of the rich theory of tensor products of C^*-algebras. Consider the map

$$\varphi_{A,B}\colon A\times B\to A\otimes_{\max} B,\quad (a,b)\mapsto a\otimes b.\quad (a\in A, b\in B)$$

It is easy to see that $\varphi_{A,B}$ is a C^*-bimorphism. For each C^*-algebra C, the assignment

$$C^*(A\otimes_{\max} B,C)\to \mathrm{Bi}C^*(A\times B,C),\quad \tau\mapsto\tau\circ\varphi_{A,B},$$

is a homeomorphism, that is, an isomorphism in CGHTop. This means that the maximal tensor product of C^*-algebras represents the C^*-bimorphism functor.

6.1.8. Let $\mathcal{C}$ be a category that is enriched over the closed, symmetric, monoidal category $\mathcal{V}$. Assume that $\mathcal{C}$ has a bimorphism functor that is also functorial in the first two variables. This means, there is a $\mathcal{V}$-multifunctor

$$\mathrm{Bi}\mathcal{C}(\_\times\_,\_)\colon \mathcal{C}^{\mathrm{op}}\times\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{V}.$$

Let us also assume that for each pair of objects X and Y in $\mathcal{C}$ the tensor product exists, that is, there is an object $X\otimes Y$ in $\mathcal{C}$ and a universal bimorphism

$$\varphi_{X,Y}\in\mathrm{Bi}\mathcal{C}(X\times Y,X\otimes Y).$$

The functoriality of $\mathrm{Bi}\mathcal{C}(\_\times\_,\_)$ in the first two variables induces a $\mathcal{V}$-bifunctor

$$\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}.$$

A pair of objects X and Y in $\mathcal{C}$ is sent to the object $X \otimes Y$ in $\mathcal{C}$. Given objects X, X', Y and Y' in $\mathcal{C}$, let us see how the required $\mathcal{V}$-morphism

$$\mathcal{C}(X,X') \times \mathcal{C}(Y,Y') \to \mathcal{C}(X \otimes Y, X' \otimes Y')$$

is obtained. We consider the composition of $\mathcal{V}$-morphisms in the following diagram, where the first morphism is obtained using that $\mathrm{Bi}\mathcal{C}(\_ \times \_, X' \otimes Y')$ is a (contravariant) $\mathcal{V}$-bifunctor, and where the second morphism is obtained by applying the $\mathcal{V}$-morphism $\varphi_{X',Y'} \colon I \to \mathrm{Bi}\mathcal{C}(X' \times Y', X' \otimes Y')$ in the first variable to the internal hom-bifunctor, and where the last isomorphism is obtained using that $X \otimes Y$ represents the functor $\mathrm{Bi}\mathcal{C}(X \times Y, \_)$:

$$\begin{aligned} \mathcal{C}(X,X') \times \mathcal{C}(Y,Y') &\to \mathrm{Bi}\mathcal{C}(X \times Y, X' \otimes Y')^{\mathrm{Bi}\mathcal{C}(X' \times Y', X' \otimes Y')} \\ &\to \mathrm{Bi}\mathcal{C}(X \times Y, X' \otimes Y')^{I} \\ &\cong \mathrm{Bi}\mathcal{C}(X \times Y, X' \otimes Y') \\ &\cong \mathcal{C}(X \otimes Y, X' \otimes Y'). \end{aligned}$$

6.2. The tensor product in PreW

6.2.1. Let us show that PreW is enriched over the closed, monoidal category PoM. Given PreW-semigroups S and T, recall that we denote by $\mathrm{W}(S,T)$ the set of W-morphisms from S to T. Equipped with pointwise order and addition, $\mathrm{W}(S,T)$ has a natural structure as a positively ordered monoid.

Given three PreW-semigroups S, T and R, it is easy to see that the composition of morphisms

$$C_{S,T,R} \colon \mathrm{W}(T,R) \times \mathrm{W}(S,T) \to \mathrm{W}(S,R), \quad (g,f) \mapsto g \circ f,$$

is a PoM-bimorphism. By Proposition B.2.5, $C_{S,T,R}$ factors through the PoM-tensor product. This means that there exists a PoM-morphism

$$M_{S,T,R} \colon \mathrm{W}(T,R) \otimes_{\mathrm{PoM}} \mathrm{W}(S,T) \to \mathrm{W}(S,R)$$

such that $g \circ f = M_{S,T,R}(g \otimes f)$ for every $f \in \mathrm{W}(S,T)$ and $g \in \mathrm{W}(T,R)$.

One can prove that this structure defines an enrichment of PreW over PoM. Since the categories W and Cu are full subcategories of PreW, they inherit the enrichment over PoM.

PROPOSITION 6.2.2. *The categories* PreW, W *and* Cu *are enriched over* PoM. *Moreover, the two reflection functors* $\mu \colon \mathrm{PreW} \to \mathrm{W}$ *from Paragraph 2.1.4 and* $\gamma \colon \mathrm{PreW} \to \mathrm{Cu}$ *from Theorem 3.1.10 are* PoM-*functors.*

PROOF. We have already observed in Paragraph 6.2.1 that the three categories are enriched over PoM. Let us show that the reflection functor $\gamma \colon \mathrm{PreW} \to \mathrm{Cu}$ is compatible with the enrichment. So let S and T be PreW-semigroups. We need to define a PoM-morphism

$$\gamma_{S,T} \colon \mathrm{W}(S,T) \to \mathrm{Cu}(\gamma(S), \gamma(T)).$$

Let $\alpha_T \colon T \to \gamma(T)$ be the Cu-completion of T, see Definition 3.1.7. Given f in $\mathrm{W}(S,T)$, we consider the composition $\alpha_T \circ f \colon S \to \gamma(T)$. Using this assignment at the first step, and using Theorem 3.1.8 to obtain the natural identification at the second step, we obtain the following composition:

$$\mathrm{W}(S,T) \xrightarrow{f \mapsto \alpha_T \circ f} \mathrm{W}(S, \gamma(T)) \cong \mathrm{Cu}(\gamma(S), \gamma(T)).$$

It is easy to see that these maps respect the PoM-structure of the involved morphism sets. It is then straightforward to check that γ is a PoM-functor.

Analogously, one shows that the reflection functor μ preserves the enrichment. $\square$

Definition 6.2.3. Let S, T and R be PreW-semigroups, and let $f\colon S\times T\to R$ be a PoM-bimorphism. We say that f is a W-*bimorphism* if it satisfies the following two conditions:

(i) The map f is continuous in the following sense: For every $a\in S$, $b\in T$ and $r\in R$ satisfying $r\prec f(a,b)$, there exist $a'\in S$ and $b'\in T$ such that $a'\prec a$, $b'\prec b$ and $r\leq f(a',b')$.

(ii) If $a'\prec a$ and $b'\prec b$, then $f(a',b')\prec f(a,b)$, for any $a',a\in S$ and $b',b\in T$.

We denote the set of all W-bimorphisms by $\mathrm{BiW}(S\times T,R)$.

If the PoM-bimorphism f is only required to satisfy condition (i) then we call it a *generalized* W-*bimorphism*. We denote the collection of all generalized W-bimorphisms by $\mathrm{BiW}[S\times T,R]$.

Lemma 6.2.4. *Let S, T and R be* PreW*-semigroups, and let $f\colon S\times T\to R$ be a* PoM*-bimorphism. Then, the following conditions are equivalent:*

(1) The map f is a generalized W*-bimorphism.*

(2) In each variable, f is a generalized W*-morphism.*

Proof. To show that (1) implies (2), fix an element $b\in T$ and consider the map

$$S\to R,\quad a\mapsto f(a,b).$$

This map is clearly a PoM-morphism. It remains to show that it is continuous. So let $a\in S$ and $r\in R$ satisfy $r\prec f(a,b)$. We need to find $a'\in S$ such that $a'\prec a$ and $r\leq f(a',b)$.

By assumption, there exist $a'\in S$ and $b'\in T$ such that

$$a'\prec a,\quad b'\prec b,\quad r\leq f(a',b').$$

Since $f(a',b')\leq f(a',b)$, we see that a' has the desired properties. The analogous result holds in the second variable.

To show that (2) implies (1), let $a\in S$, $b\in T$ and $r\in R$ satisfy $r\prec f(a,b)$. Since R satisfies (W1), there exists $\tilde{r}\in R$ such that

$$r\prec\tilde{r}\prec f(a,b).$$

Since f is continuous in the first variable, there exists $a'\in S$ such that $a'\prec a$ and $\tilde{r}\leq f(a',b)$. It follows that $r\prec f(a',b)$. Using that f is continuous in the second variable, we obtain an element $b'\in T$ such that $b'\prec b$ and $r\leq f(a',b')$, as desired. $\square$

6.2.5. Let S and T be PreW-semigroups. Let us show that the W-bimorphisms define a PoM-functor

$$\mathrm{BiW}(S\times T,\_)\colon \mathrm{PreW}\to\mathrm{PoM}.$$

Given a PreW-semigroup R, the set $\mathrm{BiW}(S\times T,R)$ has a natural structure of a positively ordered monoid when endowed with pointwise addition and order. This defines an assignment from the objects in PreW to the objects in PoM. Moreover, given a pair of PreW-semigroups R and R', we define a PoM-morphism

$$\mathrm{BiW}(S\times T,\_)_{R,R'}\colon \mathrm{W}(R,R')\to\mathrm{PoM}\left(\mathrm{BiW}(S\times T,R),\mathrm{BiW}(S\times T,R')\right),$$

as follows: A W-morphism $f\in\mathrm{W}(R,R')$ is sent to the PoM-morphism

$$\mathrm{BiW}(S\times T,R)\to\mathrm{BiW}(S\times T,R'),\quad \tau\mapsto f\circ\tau.\quad (\tau\in\mathrm{BiW}(S\times T,R))$$

It is straightforward to check that this defines a PoM-functor. In Theorem 6.2.10, we show that the bimorphism functor is representable.

6.2.6 (Auxiliary relation on (bi)morphism sets). Let S, T and R be PreW-semigroups. We define an auxiliary relation $\prec$ on the set $\mathrm{W}[S,T]$ of generalized W-morphisms as follows: For $f,g\in\mathrm{W}[S,T]$, we set $f\prec g$ if and only if $f(a')\prec g(a)$ whenever two elements $a',a\in S$ satisfy $a'\prec a$.

Similarly, we define an auxiliary relation $\prec$ on the set $\mathrm{BiW}[S\times T,R]$ of generalized W-bimorphisms as follows: For $f,g\in\mathrm{BiW}[S\times T,R]$, we set $f\prec g$ if and only if $f(a',b')\prec f(a,b)$ whenever four elements $a',a\in S$ and $b',b\in T$ satisfy $a'\prec a$ and $b'\prec b$. We have that

$$\mathrm{W}(S,T)=\{f\in\mathrm{W}[S,T]\ \mid\ f\prec f\},$$

and

$$\mathrm{BiW}(S\times T,R)=\{f\in\mathrm{BiW}[S\times T,R]\ \mid\ f\prec f\}.$$

In this way, we can think of the W-(bi)morphisms as the 'compact' generalized W-(bi)morphisms.

It is clear that the auxiliary relation $\prec$ on $\mathrm{W}[S,T]$ satisfies (W3). In some cases, $\prec$ also satisfies (W1) and (W4), but this seems not to be the case in general. Thus, we do not claim that the pair $(\mathrm{W}[S,T],\prec)$ is a PreW-semigroup. The same remark applies to $(\mathrm{BiW}[S\times T,R],\prec)$.

The following Definition 6.2.7 and Lemma 6.2.8 are rather technical. We encourage the reader not interested in all details to skip ahead to Definition 6.2.9, where we define a natural auxiliary relation on the PoM-tensor product $S\otimes_{\mathrm{PoM}}T$ of two PreW-semigroups S and T. The idea is that for simple tensors in $S\otimes_{\mathrm{PoM}}T$ we have $a'\otimes b'\prec a\otimes b$ whenever $a'\prec a$ and $b'\prec b$.

For the next definition, we need to recall some notation for the construction of tensor products in PoM from Proposition B.2.5. Let S and T be PoM-semigroups. We denote by $S^\times$ the submonoid of S consisting of nonzero elements.

On the free abelian monoid $F=\mathbb{N}[S^\times\times T^\times]$, we consider the congruence relation $\cong$ from Paragraph B.1.2. Then $S\otimes_{\mathrm{Mon}}T=F/{\cong}$ is the tensor product of the underlying monoids. Further, recall the binary relation $\leq'$ on F from Paragraph B.2.4. Let $\leq$ be the relation on F generated by $\cong$ and $\leq'$. Then $\leq$ is a pre-order on F.

Recall from Paragraph B.1.2 that for a pair of elements $a\in S^\times$ and $b\in T^\times$, we write $a\odot b$ for the generator in F indexed by (a,b). Then, every element $f\in F$ can be written as a finite sum $f=\sum_{i\in I}a_i\odot b_i$ for some finite index set I and pairs $(a_i,b_i)\in S^\times\times T^\times$ for $i\in I$. Note that we do not require that a_i and a_j are distinct for different indices i and j.

Definition 6.2.7. Let S and T be PreW-semigroups. We define a relation $\lessdot$ on the free abelian monoid $F=\mathbb{N}[S^\times\times T^\times]$ as follows:

For f and $g=\sum_{j\in J}a_j\odot b_j$ in F we set $f\lessdot g$ if and only if there exist a subset $J'\subset J$ and elements $a_j'\in S^\times$ and $b_j'\in T^\times$ for $j\in J'$ such that

$$f\leq\sum_{j\in J'}a_j'\odot b_j',\qquad\text{and}\qquad a_j'\prec a_j,\ b_j'\prec b_j\text{ for each }j\in J'.$$

Lemma 6.2.8. *Let S and T be* PreW*-semigroups. Then, the relation $\lessdot$ from Definition 6.2.7 satisfies the following statements:*

1. *If $f\lessdot g$, then $f\leq g$, for any $f,g\in F$.*
2. *If $f'\leq f\lessdot g$, then $f'\lessdot g$, for any $f',f,g\in F$.*
3. *If $f\lessdot g\leq g'$, then $f\lessdot g'$, for any $f,g,g'\in F$.*
4. *We have that $0\lessdot f$, for any $f\in F$.*
5. *For each g in F, there exists a sequence $(g_k)_k$ in F such that $g_k\lessdot g_{k+1}$ for each k and such that for any $f\in F$ satisfying $f\lessdot g$ there exists an index k such that $f\leq g_k$.*

(6) *If $f_1 \lessdot g_1$ and $f_2 \lessdot g_2$, then $f_1 + f_2 \lessdot g_1 + g_2$, for any $f_1, f_2, g_1, g_2 \in F$.*
(7) *If f, g_1 and g_2 in F satisfy $f \lessdot g_1 + g_2$, then there exist g_1' and g_2' in F such that $f \leq g_1' + g_2'$ and $g_1' \lessdot g_1$ and $g_2' \lessdot g_2$.*

Proof. Statements (1), (2), (4) and (6) are straightforward to show. To prove (7), let $f, g_1, g_2 \in F$ satisfy $f \lessdot g_1 + g_2$. Then there exist finite disjoint index sets I_1 and I_2, and elements $a_i \in S^\times$ and $b_i \in T^\times$ for $i \in I_1 \cup I_2$ such that

$$g_1 = \sum_{i \in I_1} a_i \odot b_i, \quad g_2 = \sum_{i \in I_2} a_i \odot b_i.$$

Since $f \lessdot g_1 + g_2$, there exist a subset $I' \subset I_1 \cup I_2$ and elements $a_i' \in S^\times$ and $b_i' \in T^\times$ for $i \in I'$ such that

$$f \leq \sum_{i \in I'} a_i' \odot b_i', \quad \text{and} \quad a_i' \prec a_i,\ b_i' \prec b_i \text{ for each } i \in I'.$$

Set

$$g_1' = \sum_{i \in I_1 \cap I'} a_i' \odot b_i', \quad g_2' = \sum_{i \in I_2 \cap I'} a_i' \odot b_i'.$$

Then it is easy to check that

$$f \leq g_1' + g_2', \quad g_1' \lessdot g_1, \quad g_2' \lessdot g_2,$$

as desired.

Next, let us show (3). So let $f, g, g' \in F$ satisfy $f \lessdot g \leq g'$. Since the relation $\leq$ is the transitive closure of the relation generated by $\to$, $\leftarrow$ and $\leq'$, it is enough to consider the cases where $g \to g'$, or $g \leftarrow g'$, or $g \leq' g'$. We may assume that f, g and g' are nonzero. Using statements (6) and (7), it is furthermore enough to consider the following three cases:

Case 1: Assume that $g \to^0 g'$. This means that there exist elements $a \in S^\times$ and $b \in T^\times$, and nonempty finite index sets J and K, and elements $a_j \in S^\times$ for $j \in J$, and elements $b_k \in T^\times$ for $k \in K$ such that

$$a = \sum_{j \in J} a_j, \quad b = \sum_{k \in K} b_k, \quad g = a \odot b, \quad g' = \sum_{j \in J, k \in K} a_j \odot b_k.$$

Since $f \lessdot g$, there exist elements $a' \in S^\times$ and $b' \in T^\times$ such that

$$f \leq a' \odot b', \quad a' \prec a, \quad b' \prec b.$$

Using that S satisfies (W4) and that

$$a' \prec a = \sum_{j \in J} a_j,$$

we obtain elements $a_j' \in S$ for $j \in J$ such that

$$a' \leq \sum_{j \in J} a_j', \quad \text{and} \quad a_j' \prec a_j \text{ for each } j \in J.$$

Similarly, we obtain elements $b_k' \in T$ for $k \in K$ such that

$$b' \leq \sum_{k \in K} b_k', \quad \text{and} \quad b_k' \prec b_k \text{ for each } k \in K.$$

Set $J' = \{j \in J \mid a_j' \neq 0\}$ and $K' = \{k \in K \mid b_k' \neq 0\}$. Then it is easy to check that

$$f \leq \sum_{j \in J', k \in K'} a_j' \odot b_k'.$$

This shows that $f \lessdot g'$, as desired.

Case 2: Assume that $g \,{}^0\!\!\leftarrow g'$. This means that there exist elements $a \in S^\times$ and $b \in T^\times$, and nonempty finite index sets J and K, and elements $a_j \in S^\times$ for $j \in J$, and elements $b_k \in T^\times$ for $k \in K$ such that

$$a = \sum_{j \in J} a_j, \quad b = \sum_{k \in K} b_k, \quad g = \sum_{j \in J, k \in K} a_j \odot b_k, \quad g' = a \odot b.$$

Since $f \lessdot g$, there exist a subset $L \subset J \times K$ and elements $a'_{j,k} \in S^\times$ and $b'_{j,k} \in T^\times$ for $(j,k) \in L$ such that

$$f \leq \sum_{(j,k) \in L} a'_{j,k} \odot b'_{j,k}, \quad \text{and} \quad a'_{j,k} \prec a_j,\, b'_{j,k} \prec b_k \text{ for each } (j,k) \in L.$$

For $j^\sharp \in J$ and $k^\sharp \in K$ set

$$J_{k^\sharp} = \left\{ j \in J \mid (j, k^\sharp) \in L \right\}, \quad K_{j^\sharp} = \left\{ k \in K \mid (j^\sharp, k) \in L \right\}.$$

Moreover, set

$$J' = \left\{ j^\sharp \in J \mid K_{j^\sharp} \neq \emptyset \right\}, \quad K' = \left\{ k^\sharp \in K \mid J_{k^\sharp} \neq \emptyset \right\}.$$

Let $j^\sharp \in J'$. Then for each $k \in K_{j^\sharp}$ we have that

$$a'_{j^\sharp,k} \prec a_{j^\sharp}.$$

Using that S satisfies (W1), there exists an element $a'_{j^\sharp} \in S$ such that $a'_{j^\sharp} \prec a_{j^\sharp}$ and such that $a'_{j^\sharp,k} \leq a'_{j^\sharp}$ for each $k \in K_{j^\sharp}$. Note that $a'_{j^\sharp}$ is necessarily nonzero. Similarly, we obtain for each $k^\sharp \in K'$ an element $b'_{k^\sharp} \in T^\times$ such that $b'_{k^\sharp} \prec b_{k^\sharp}$ and such that $b'_{j,k^\sharp} \leq b'_{k^\sharp}$ for each $j \in J_{k^\sharp}$.

It follows that

$$f \leq \sum_{(j,k) \in L} a'_{j,k} \odot b'_{j,k} \leq' \sum_{(j,k) \in L} a'_j \odot b'_k.$$

Set

$$a' = \sum_{j \in J'} a'_j, \quad b' = \sum_{k \in K'} b'_k.$$

Since S and T satisfy (W3), we get $a' \prec a$ and $b' \prec b$. Then

$$f \leq \sum_{(j,k) \in L} a'_j \odot b'_k \leq \sum_{(j,k) \in I' \times J'} a'_j \odot b'_k \cong a' \odot b' \lessdot a \odot b = g',$$

which shows that $f \lessdot g'$, as desired.

Case 3: Assume that $g \leq^0 g'$. This means that there exist elements $a, \tilde{a} \in S^\times$ and $b, \tilde{b} \in T^\times$ such that

$$a \leq \tilde{a}, \quad b \leq \tilde{b}, \quad g = a \odot b, \quad g' = \tilde{a} \odot \tilde{b}.$$

Since $f \lessdot g$, there exist elements $a' \in S^\times$ and $b' \in T^\times$ such that

$$f \leq a' \odot b', \quad a' \prec a, \quad b' \prec b.$$

Since $\prec$ is an auxiliary relation for S and T, we deduce that $a' \prec \tilde{a}$ and $b' \prec \tilde{b}$. Therefore, we immediately get that $f \lessdot g'$, as desired.

Finally, let us prove (5). So let $g = \sum_{i \in I} a_i \odot b_i \in F$. Given $a \in S$, we write $a^\prec$ for the set $\{x \in S \mid x \prec a\}$, and similarly for elements in T. Since S satisfies (W1), for each $i \in I$ there exists a sequence $(a_{i,k})_{k \in \mathbb{N}} \subset S$ that is cofinal in $a_i^\prec$ and such that $a_{i,k} \prec a_{i,k+1}$ for each k. Similarly, for each i there exists a $\prec$-increasing sequence $(b_{i,k})_{k \in \mathbb{N}} \subset T$ that is cofinal in $b_i^\prec$. Set

$$I' = \left\{ i \in I \mid a_i^\prec \neq \{0\}, b_i^\prec \neq \{0\} \right\}.$$

For $i \in I \setminus I'$ we have $a_{i,k} = 0$ for each $k \in \mathbb{N}$ or $b_{i,k} = 0$ for each $k \in \mathbb{N}$. For $i \in I'$, we may assume that $a_{i,k}$ and $b_{i,k}$ are nonzero for each $k \in \mathbb{N}$. Then, we set

$$g_k = \sum_{i \in I'} a_{i,k} \odot b_{i,k},$$

which is an element of F. It is clear that $g_k \lessdot g_{k+1}$ for each k.

Let $f \in F$ satisfy $f \lessdot g$. We need to show that there is $n \in \mathbb{N}$ such that $f \leq g_n$. Since $f \lessdot g$, there exist a subset $J \subset I$ and elements $a'_j \in S^\times$ and $b'_j \in T^\times$ for $j \in J$ such that

$$f \leq \sum_{j \in J} a'_j \odot b'_j, \quad \text{ and } \quad a'_j \prec a_j,\, b'_j \prec b_j \text{ for each } j \in J.$$

Note that J is necessarily a subset of I'.

Since S and T satisfy (W1), for each $j \in J$ there exist indices $k(j)$ and $l(j)$ in $\mathbb{N}$ such that

$$a'_j \leq a_{j,k(j)}, \quad b'_j \leq b_{j,l(j)}.$$

Set

$$n = \max\left\{ k(j), l(j) \ \mid \ j \in J \right\}.$$

Then

$$f \leq \sum_{j \in J} a'_j \odot b'_j \leq \sum_{j \in J} a_{j,n} \odot b_{j,n} \leq \sum_{i \in I'} a_{i,n} \odot b_{i,n} = g_n,$$

as desired. □

For the next definition, recall that for PreW-semigroups S and T, and for an element $f \in F = \mathbb{N}[S^\times \times T^\times]$, we denote the congruence class of f in $S \otimes_{\mathrm{PoM}} T = F/\!\cong$ by $[f]$.

Definition 6.2.9 (Auxiliary relation on $S \otimes_{\mathrm{PoM}} T$). Let S and T be PreW-semigroups, and let $\lessdot$ be the relation on $\mathbb{N}[S^\times \times T^\times]$ introduced in Definition 6.2.7.

We let $\prec$ be the binary relation on the tensor product $S \otimes_{\mathrm{PoM}} T$ of the underlying positively ordered monoids that is induced by $\lessdot$. That is, for elements x and y in $S \otimes_{\mathrm{PoM}} T$ we set $x \prec y$ if and only if there exist representatives f and g in $\mathbb{N}[S^\times \times T^\times]$ such that

$$x = [f], \quad f \lessdot g, \quad [g] = y.$$

Theorem 6.2.10. *Let S and T be* PreW*-semigroups. Let*

$$\omega\colon S \times T \to S \otimes_{\mathrm{PoM}} T,$$

be the tensor product of the underlying positively ordered monoids, as constructed in Proposition B.2.5. Then, the relation $\prec$ on $S \otimes_{\mathrm{PoM}} T$ from Definition 6.2.9 is an auxiliary relation and $(S \otimes_{\mathrm{PoM}} T, \prec)$ is a PreW*-semigroup, denoted by $S \otimes_{\mathrm{PreW}} T$. Moreover, the map ω becomes a* W*-bimorphism.*

Furthermore, for every PreW*-semigroup R, the following universal properties hold:*

(1) For every (generalized) W*-bimorphism $f\colon S \times T \to R$, there exists a (generalized)* W*-morphism $\tilde{f}\colon S \otimes_{\mathrm{PreW}} T \to R$ such that $f = \tilde{f} \circ \omega$.*

(2) We have $g_1 \circ \omega \leq g_2 \circ \omega$ if and only if $g_1 \leq g_2$, for any pair of generalized W*-morphisms $g_1, g_2\colon S \otimes_{\mathrm{PreW}} T \to R$.*

(3) We have $g_1 \circ \omega \prec g_2 \circ \omega$ if and only if $g_1 \prec g_2$, for any pair of generalized W*-morphisms $g_1, g_2\colon S \otimes_{\mathrm{PreW}} T \to R$.*

Thus, for every PreW*-semigroup R, we obtain a situation as shown in the commutative diagram below. In the top row, the map ω induces an isomorphism of the sets of generalized* W*-(bi)morphisms with their structure as positively ordered monoids and with their additional auxiliary relations from Paragraph 6.2.6. When* *restricting to* W*-(bi)morphisms, as in the bottom row of the diagram, the map* ω *induces a* PoM*-isomorphism of respective* W*-(bi)morphisms sets.*

$$
\begin{array}{ccc}
\mathrm{W}[S \otimes_{\mathrm{PreW}} T, R] & \xrightarrow[f \mapsto f\circ\omega]{\cong} & \mathrm{BiW}[S \times T, R] \\
\cup & & \cup \\
\mathrm{W}(S \otimes_{\mathrm{PreW}} T, R) & \xrightarrow{\cong} & \mathrm{BiW}(S \times T, R).
\end{array}
$$

In particular, the pair $(S \otimes_{\mathrm{PreW}} T,\ \omega)$ *represents the bimorphism* PoM*-functor* $\mathrm{BiW}(S \times T, \_)$.

Proof. Set $F = \mathbb{N}[S^\times \times T^\times]$, and let $\leq$ be the pre-order on F introduced in Paragraph B.2.4. Let $\sim$ be the binary relation on F defined by symmetrizing $\leq$, that is, for f and g in F we set $f \sim g$ if and only if $f \leq g$ and $g \leq f$. Then $\sim$ is a congruence relation on F and $S \otimes_{\mathrm{PoM}} T = F/{\sim}$. For an element $f \in F$, we denote by $[f]$ the congruence class of f in $S \otimes_{\mathrm{PoM}} T$. Let $\lessdot$ be the relation on F from Definition 6.2.7.

It follows from statements (2) and (3) in Lemma 6.2.8 that $\lessdot$ does only depend on the $\sim$-equivalence class of elements in F. Thus, if x and y are elements in $S \otimes_{\mathrm{PoM}} T$, then the following are equivalent:

(1) We have $x \prec y$ in the sense of Definition 6.2.9, that is, there are $f, g \in F$ such that $x = [f]$, $y = [g]$ and $f \lessdot g$.
(2) For each $f, g \in F$ satisfying $x = [f]$ and $y = [g]$, we have that $f \lessdot g$.

It follows easily from (1)-(4) in Lemma 6.2.8 that $\prec$ is an auxiliary relation on $S \otimes_{\mathrm{PoM}} T$. Moreover, statements (5)-(7) in Lemma 6.2.8 imply that $(S \otimes_{\mathrm{PoM}} T, \prec)$ satisfies (W1), (W3) and (W4), showing that it is a PreW-semigroup, denoted by $S \otimes_{\mathrm{PreW}} T$.

Let us show that the map

$$\omega\colon S \times T \to S \otimes_{\mathrm{PreW}} T,$$

is a W-bimorphism. It is clear that ω is a PoM-bimorphism respecting the auxiliary relations. Thus, it remains to show that ω is continuous in the sense of Definition 6.2.3. So let $a \in S$, $b \in T$ and $x \in S \otimes_{\mathrm{PreW}} T$ satisfy $x \prec \omega(a, b)$. We may assume that a, b and x are nonzero. Then $f := a \odot b$ is an element in F such that $\omega(a, b) = [f]$. As explained at the beginning of the proof, it follows from $x \prec [f]$ that there exists $f' \in F$ such that $x \leq [f']$ and $f' \lessdot f$. Therefore, by definition of the relation $\lessdot$ and since x is nonzero, there exist elements $a' \in S^\times$ and $b' \in T^\times$ such that

$$f' \leq a' \odot b', \quad a' \prec a, \quad b' \prec b.$$

Then

$$x \leq [f'] \leq [a' \odot b'] = \omega(a', b'),$$

showing that a' and b' have the desired properties to verify the continuity of ω.

Let $\mathfrak{F}\colon \mathrm{PreW} \to \mathrm{PoM}$ be the forgetful functor, which associates to a PreW-semigroup X the underlying positively ordered monoid (also denoted by X, by abusing notation). Every W-morphism is already given as a PoM-morphism of the underlying positively ordered monoids (satisfying additional properties). Thus, $\mathfrak{F}$ is faithful.

Given PreW-semigroups X, Y and Z, the functor $\mathfrak{F}$ induces maps

$$
\begin{aligned}
\mathfrak{F}_{X,Y}&\colon \mathrm{W}[X, Y] \to \mathrm{PoM}(X, Y),\\
\mathfrak{F}_{X\times Y,Z}&\colon \mathrm{BiW}[X \times Y, Z] \to \mathrm{BiPoM}(X \times Y, Z),
\end{aligned}
$$

by mapping a generalized W-(bi)morphism to the same map considered as a PoM-(bi)morphism. It is clear that $\mathfrak{F}_{X,Y}$ and $\mathfrak{F}_{X\times Y,Z}$ are order-embeddings when the respective (bi)morphism sets are equipped with their natural structure as positively ordered monoids.

To check the universal properties, let R be a PreW-semigroup. Consider the map

$$\Omega_R\colon \mathrm{PoM}(S\otimes_{\mathrm{PoM}} T, R)\to \mathrm{BiPoM}(S\times T, R),\quad f\mapsto f\circ\omega.$$

Since $S\otimes_{\mathrm{PoM}} T$ and ω have the universal property of a tensor product in PoM, the map Ω_R is a isomorphism of the (bi)morphism sets with their structure as objects in PoM, see Proposition B.2.5. In particular, Ω_R is an order-embedding.

Since ω is also a W-bimorphism, the same assignment maps (generalized) W-morphisms to (generalized) W-bimorphisms. We denote this map by

$$\Phi_R\colon \mathrm{W}[S\otimes_{\mathrm{PreW}} T, R]\to \mathrm{BiW}[S\times T, R],\quad f\mapsto f\circ\omega.$$

The situation is shown in the following commutative diagram of PoM-morphisms:

$$\begin{tikzcd}
\mathrm{PoM}(S\otimes_{\mathrm{PoM}} T, R) \arrow[r, "\Omega_R", "\cong"'] & \mathrm{BiPoM}(S\times T, R) \\
\mathrm{W}[S\otimes_{\mathrm{PreW}} T, R] \arrow[u, hook, "\mathfrak{F}_{S\otimes T,R}"] \arrow[r, "\Phi_R"] & \mathrm{BiW}[S\times T, R] \arrow[u, hook, "\mathfrak{F}_{S\times T,R}"']
\end{tikzcd}$$

Since $\mathfrak{F}_{S\otimes T,R}$ and $\mathfrak{F}_{S\times T,R}$ are order-embeddings, and since the map Ω_R is a PoM-isomorphism, it follows that Φ_R is an order-embedding. This shows the universal property (2).

For (1), we need to show that Φ_R is surjective. Thus, let

$$f\colon S\times T\to R$$

be a generalized W-bimorphism. Considering f as a PoM-bimorphism and using that Ω_R is an isomorphism, there exists a unique PoM-morphism

$$\tilde{f}\colon S\otimes_{\mathrm{PoM}} T\to R$$

such that $f=\tilde{f}\circ\omega$. We need to show that $\tilde{f}$ is continuous.

So let $x\in S\otimes_{\mathrm{PreW}}T$ and $r\in R$ satisfy $r\prec\tilde{f}(x)$. We need to find $x'\in S\otimes_{\mathrm{PreW}}T$ such that $x'\prec x$ and $r\leq\tilde{f}(x')$. Choose a finite index set I and elements $a_i\in S$ and $b_i\in T$ for $i\in I$ such that $x=\sum_{i\in I}a_i\otimes b_i$. Then

$$r\prec\tilde{f}(x)=\sum_{i\in I}\tilde{f}(a_i\otimes b_i).$$

Using that R satisfies (W4), there exist elements r_i in R for $i\in I$ such that

$$r\leq\sum_{i\in I}r_i,\qquad\text{and}\qquad r_i\prec\tilde{f}(a_i\otimes b_i)=f(a_i,b_i)\text{ for each }i\in I.$$

Since f is continuous, for each $i\in I$ there exist elements $a_i'\in S$ and $b_i'\in T$ such that

$$a_i'\prec a_i,\quad b_i'\prec b_i,\quad r_i\leq f(a_i',b_i').$$

Set $x'=\sum_{i\in I}a_i'\otimes b_i'$. Then $x'\prec x$, and

$$r\leq\sum_{i\in I}r_i\leq\sum_{i\in I}f(a_i',b_i')=\tilde{f}(x'),$$

as desired.

Finally, to prove (3), we need to show that for any two generalized W-morphisms f and g in $\mathrm{W}[S\otimes_{\mathrm{PreW}} T, R]$, we have that $f\prec g$ if and only if $\Phi_R(f)\prec\Phi_R(g)$. This is left to the reader. $\square$

6.2.11 (PreW is a symmetric, monoidal category). It is straightforward to check that the bimorphism-functor on PreW from Paragraph 6.2.5 is also functorial in the first entries. Thus, we have a PoM-multifunctor

$$\mathrm{BiW}(\_ \times \_, \_)\colon \mathrm{PreW}^{\mathrm{op}} \times \mathrm{PreW}^{\mathrm{op}} \times \mathrm{PreW} \to \mathrm{PoM}.$$

By Theorem 6.2.10, the tensor product for every two PreW-semigroups exists. Therefore, as explained in Paragraph 6.1.8, it follows that the tensor product in PreW induces a PoM-bifunctor

$$\otimes\colon \mathrm{PreW} \times \mathrm{PreW} \to \mathrm{PreW}.$$

We use this to define a monoidal structure on PreW.

Recall that the unit object of PoM is given by $\mathbb{N}$ with its usual structure as an algebraically ordered monoid. We equip $\mathbb{N}$ with the auxiliary relation that is equal to the partial order. Then $\mathbb{N}$ is PreW-semigroup.

Let S be a PreW-semigroup. Since $\mathbb{N}$ is the unit object of PoM, there are natural isomorphisms for the underlying positively ordered monoid of S:

$$\mathbb{N} \otimes_{\mathrm{PoM}} S \cong S \cong S \otimes_{\mathrm{PoM}} \mathbb{N}.$$

It is straightforward to check that these isomorphisms preserve the auxiliary relations and are therefore isomorphisms in PreW. Thus, $\mathbb{N}$ is the unit object in PreW. In the same way, associativity and symmetry of the tensor product in PreW follow from the respective properties in PoM.

6.3. The tensor product in Cu

In this section, we will use the construction of tensor products in PreW and the fact that Cu is a reflective subcategory of PreW to show that the category Cu has a symmetric, monoidal structure.

Before we make this concrete in Theorem 6.3.3, let us consider the natural notion of bimorphisms in the category Cu, cf. [**ABP13**, Definition 4.3].

Definition 6.3.1. Let S, T and R be Cu-semigroups, and let $f\colon S \times T \to R$ be a PoM-bimorphism. We say that f is a Cu-*bimorphism* if it satisfies the following two conditions:

(i) We have that $\sup_k f(a_k, b_k) = f(\sup_k a_k, \sup_k b_k)$, for any pair of increasing sequences $(a_k)_k \subset S$ and $(b_k)_k \subset T$.
(ii) If $a' \ll a$ and $b' \ll b$, then $f(a', b') \ll f(a, b)$, for any $a', a \in S$ and $b', b \in T$.

We denote the set of all Cu-bimorphisms by $\mathrm{BiCu}(S \times T, R)$.

If f is only required to satisfy condition (i) then we call it a *generalized* Cu-*bimorphism*. We denote the collection of all generalized Cu-bimorphisms by $\mathrm{BiCu}[S \times T, R]$.

The next result is the analog of Lemma 3.1.4 for (generalized) bimorphisms. It shows that for Cu-semigroups, the notions of (generalized) W-bimorphism and (generalized) Cu-bimorphism agree.

Lemma 6.3.2. *Let S, T and R be* Cu*-semigroups, and let $f\colon S \times T \to R$ be a* PoM*-bimorphism. Then, the following are equivalent:*

(1) The map f is a generalized Cu*-bimorphism.*
(2) In each variable, f is a generalized Cu*-morphism.*
(3) The map f is a generalized W*-bimorphism.*

Proof. The equivalence of (2) and (3) follows by combining Lemma 6.2.4 and Lemma 3.1.4. Moreover, it is clear that (1) implies (2), and the converse is straightforward to show. □

Given Cu-semigroups S and T, the Cu-bimorphisms from $S \times T$ to a Cu-semigroup R form a positively ordered monoid. This can be used to define a PoM-functor

$$\mathrm{BiCu}(S \times T, \_)\colon \mathrm{Cu} \to \mathrm{PoM}.$$

Indeed, by Lemma 6.3.2, this is just the restriction of the PoM-functor $\mathrm{BiW}(S \times T, \_)$ from Paragraph 6.2.5 to the full subcategory Cu of PreW. In the next result, we will show that this bimorphism functor is representable.

Given three Cu-semigroups S, T and R, we equip the set of generalized Cu-(bi)morphisms $\mathrm{Cu}[S, R]$ and $\mathrm{BiCu}[S \times T, R]$ with the same auxiliary relation as defined in Paragraph 6.2.6. For example, for $f, g \in \mathrm{Cu}[S, R]$, we set $f \prec g$ if and only if $f(a') \ll g(a)$ for any pair of elements $a', a \in S$ satisfying $a' \ll a$.

Theorem 6.3.3. *Let S and T be* Cu*-semigroups. Consider the tensor product*

$$\omega\colon S \times T \to S \otimes_{\mathrm{PreW}} T,$$

as constructed in Theorem 6.2.10. Applying the completion of Proposition 3.1.6, we obtain a Cu*-semigroup $\gamma(S \otimes_{\mathrm{PreW}} T)$, which we also denote by $S \otimes_{\mathrm{Cu}} T$, and a universal* W*-morphism $\alpha\colon S \otimes_{\mathrm{PreW}} T \to S \otimes_{\mathrm{Cu}} T$.*

Then, the composed map $\varphi := \alpha \circ \omega$

$$S \times T \xrightarrow{\omega} S \otimes_{\mathrm{PreW}} T \xrightarrow{\alpha} \gamma(S \otimes_{\mathrm{PreW}} T) = S \otimes_{\mathrm{Cu}} T$$

is a Cu*-bimorphism. For every* Cu*-semigroup R, it satisfies the following universal properties:*

(1) *For every (generalized)* Cu*-bimorphism $f\colon S \times T \to R$, there exists a (generalized)* Cu*-morphism $\tilde{f}\colon S \otimes_{\mathrm{Cu}} T \to R$ such that $f = \tilde{f} \circ \varphi$.*
(2) *We have $g_1 \circ \varphi \leq g_2 \circ \varphi$ if and only if $g_1 \leq g_2$, for any pair of generalized* Cu*-morphisms $g_1, g_2\colon S \otimes_{\mathrm{Cu}} T \to R$.*
(3) *We have $g_1 \circ \varphi \prec g_2 \circ \varphi$ if and only if $g_1 \prec g_2$, for any pair of generalized* Cu*-morphisms $g_1, g_2\colon S \otimes_{\mathrm{Cu}} T \to R$.*

Thus, for every Cu*-semigroup R, we obtain a situation as shown in the commutative diagram below, analogous to that in Theorem 6.2.10.*

$$\begin{array}{ccc} \mathrm{Cu}[S \otimes_{\mathrm{Cu}} T, R] & \xrightarrow[f \mapsto f \circ \varphi]{\cong} & \mathrm{BiCu}[S \times T, R] \\ \cup & & \cup \\ \mathrm{Cu}(S \otimes_{\mathrm{Cu}} T, R) & \xrightarrow{\cong} & \mathrm{BiCu}(S \times T, R). \end{array}$$

In particular, we have that the pair $(S \otimes_{\mathrm{Cu}} T, \varphi)$ represents the bimorphism PoM*-functor* $\mathrm{BiCu}(S \times T, \_)$.

Proof. It is clear that φ is a W-bimorphism. Therefore, it follows from Lemma 6.3.2 that it is a Cu-bimorphism. To check the universal properties, let R be a Cu-semigroup. In the diagram below, the horizontal maps on the left are PoM-isomorphisms by Theorem 3.1.8 and the horizontal maps on the right are PoM-isomorphisms by Theorem 6.2.10.

$$\begin{array}{ccccc} \mathrm{Cu}[S \otimes_{\mathrm{Cu}} T, R] & \xrightarrow[f \,\mapsto\, f \circ \alpha]{\cong} & \mathrm{W}[S \otimes_{\mathrm{PreW}} T, R] & \xrightarrow[g \,\mapsto\, g \circ \omega]{\cong} & \mathrm{BiW}[S \times T, R] \\ \cup & & \cup & & \cup \\ \mathrm{Cu}(S \otimes_{\mathrm{Cu}} T, R) & \xrightarrow{\cong} & \mathrm{W}(S \otimes_{\mathrm{PreW}} T, R) & \xrightarrow{\cong} & \mathrm{BiW}(S \times T, R). \end{array}$$

This establishes the universal properties (1) and (2). It is also straightforward to check that the isomorphism between $\mathrm{Cu}[S \otimes_{\mathrm{Cu}} T, R]$ and $\mathrm{BiCu}[S \times T, R]$ preserves the auxiliary relations, which establishes (3). $\square$

REMARK 6.3.4 (Tensor product in W). Analogous to the above Theorem 6.3.3, one can construct tensor products in the category W. Given W-semigroups S and T, one first considers the tensor product in PreW, $S \otimes_{\mathrm{PreW}} T$. Then, one uses the reflection PreW $\to$ W from Paragraph 2.1.4 to obtain the tensor product in W.

THEOREM 6.3.5. *Let S and T be* PreW*-semigroups. Then there is a natural* Cu*-isomorphism*

$$\gamma(S) \otimes_{\mathrm{Cu}} \gamma(T) \cong \gamma(S \otimes_{\mathrm{PreW}} T)\,.$$

PROOF. Let R be a Cu-semigroup. Using Theorem 6.2.10 at the first step, and using that Cu is a reflective subcategory of PreW at the second step, we have natural isomorphisms of the following (bi)morphism sets

$$\mathrm{BiW}(S \times T, R) \cong \mathrm{W}(S \otimes_{\mathrm{PreW}} T, R) \cong \mathrm{Cu}(\gamma(S \otimes_{\mathrm{PreW}} T), R).$$

On the other hand, using Lemma 6.3.2 at the first step, and Theorem 6.3.3 at the second step, we obtain natural isomorphisms

$$\mathrm{BiW}(S \times T, R) \cong \mathrm{BiCu}(\gamma(S) \times \gamma(T), R) \cong \mathrm{Cu}(\gamma(S) \otimes_{\mathrm{Cu}} \gamma(T), R).$$

Hence, the Cu-semigroups $\gamma(S) \otimes_{\mathrm{Cu}} \gamma(T)$ and $\gamma(S \otimes_{\mathrm{PreW}} T)$ both represent the same functor, which implies that they are naturally isomorphic. □

COROLLARY 6.3.6. *Let S, T and R be* Cu*-semigroups. Then, there is a natural isomorphism identifying the different iterated tensor products:*

$$S \otimes_{\mathrm{Cu}} (T \otimes_{\mathrm{Cu}} R) \;\cong\; (S \otimes_{\mathrm{Cu}} T) \otimes_{\mathrm{Cu}} R.$$

PROOF. Using Theorem 6.3.5 at the second and last step, and that $\otimes_{\mathrm{PreW}}$ is a associative (see Paragraph 6.2.11) at the third step, we obtain that

$$\begin{aligned} S \otimes_{\mathrm{Cu}} (T \otimes_{\mathrm{Cu}} R) &\cong \gamma(S) \otimes_{\mathrm{Cu}} \gamma(T \otimes_{\mathrm{PreW}} R) \\ &\cong \gamma(S \otimes_{\mathrm{PreW}} (T \otimes_{\mathrm{PreW}} R)) \\ &\cong \gamma((S \otimes_{\mathrm{PreW}} T) \otimes_{\mathrm{PreW}} R) \cong (S \otimes_{\mathrm{Cu}} T) \otimes_{\mathrm{Cu}} R\,, \end{aligned}$$

and all isomorphisms are natural. □

6.3.7 (Cu is a symmetric, monoidal category). Similar as in Paragraph 6.2.11, it follows that that tensor product in Cu extends to a bifunctor

$$\otimes\colon \mathrm{Cu} \times \mathrm{Cu} \to \mathrm{Cu}.$$

We showed in Corollary 6.3.6 that this functor is associative. Let us show that the Cu-semigroup $\overline{\mathbb{N}}$ is a unit object for Cu. Note that $\overline{\mathbb{N}}$ is the reflection in Cu of the unit object $\mathbb{N}$ of PreW. Let S be a Cu-semigroup S. Using Theorem 6.3.5 at the first step, we obtain natural isomorphisms

$$S \otimes_{\mathrm{Cu}} \overline{\mathbb{N}} \cong \gamma(S \otimes_{\mathrm{PreW}} \mathbb{N}) \cong \gamma(S) \cong S,$$

and analogously $\overline{\mathbb{N}} \otimes_{\mathrm{Cu}} S \cong S$. Similarly, symmetry of the tensor product in Cu follows from symmetry of the tensor product in PreW. Thus, the category Cu has a symmetric, monoidal structure.

6.4. Examples and Applications

In this subsection, we are mainly concerned with the following two problems: Under which conditions do the axioms (O5), (O6) and weak cancellation pass to tensor products of Cu-semigroups, see Problem 6.4.2. Secondly, for C^*-algebras A and B, what can we say about the natural Cu-morphism from $\mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B)$ to $\mathrm{Cu}(A \otimes_{\max} B)$, see Problem 6.4.11.

The following result is a useful tool to solve particular cases of both problems.

Proposition 6.4.1. *The tensor products in* PreW *and* Cu *are continuous in each variable. More precisely, let* $(S_i, \varphi_{i,j})$ *be an inductive system in* PreW*, and let* T *be a* PreW*-semigroup. Then there is a natural isomorphism*

$$\mathrm{PreW}\text{-}\varinjlim_i (S_i \otimes_{\mathrm{PreW}} T) \cong (\mathrm{PreW}\text{-}\varinjlim_i S_i) \otimes_{\mathrm{PreW}} T.$$

The analogous statement holds for the second variable and for the tensor product in Cu.

Proof. We first note that the tensor product in PoM is a continuous functor in each variable. This follows from the fact that $\_ \otimes_{\mathrm{PoM}} T$ is a left adjoint to $\mathrm{PoM}(T, \_)$. To simplify notation, in the first part of this proof we will write $\otimes$ for $\otimes_{\mathrm{PreW}}$ and $\varinjlim$ for PreW-$\varinjlim$.

Let $(S_i, \varphi_{i,j})$ be an inductive system in PreW indexed over the directed set I, and let T be a PreW-semigroup. This induces an inductive system $(S_i \otimes T, \varphi_{i,j} \otimes \mathrm{id}_T)$. For $i \in I$, we denote the respective W-morphisms into the inductive limits by:

$$\varphi_{i,\infty} \colon S_i \to \varinjlim_i S_i, \quad \lambda_{i,\infty} \colon S_i \otimes T \to \varinjlim_i (S_i \otimes T).$$

The W-morphisms $\varphi_{i,\infty} \otimes \mathrm{id}_T$ induce a W-morphism

$$\psi \colon \varinjlim_i (S_i \otimes T) \to (\varinjlim_i S_i) \otimes T,$$

such that $\psi \circ \lambda_{i,\infty}(s \otimes t) = \varphi_{i,\infty}(s) \otimes t$ for every i and $s \in S_i$ and $t \in T$. These maps are shown in the following commutative diagram.

$$\begin{tikzcd}
S_i \otimes T \arrow[r, "\varphi_{i,\infty} \otimes \mathrm{id}_T"] \arrow[d, "\lambda_{i,\infty}"'] & (\varinjlim_i S_i) \otimes T \\
\varinjlim_i (S_i \otimes T) \arrow[ur, "\psi"'] &
\end{tikzcd}$$

The inductive limit of PreW-semigroups is simply the inductive limit in PoM of the underlying positively ordered monoids equipped with a natural auxiliary relation, see Theorem 2.1.8. Similarly, the tensor product of two PreW-semigroups is the PoM-tensor product of the underlying positively ordered monoids equipped with a natural auxiliary relation, see Theorem 6.2.10.

Thus, since the tensor product in PoM is continuous in each variable, the map ψ is a PoM-isomorphism. Moreover, ψ preserves the auxiliary relation since it is a W-morphism. Hence, to show that ψ is a W-isomorphism, it remains to prove that $x \prec y$ whenever $\psi(x) \prec \psi(y)$ for any x and y in the domain of ψ.

Given such x and y, we can choose an index i and $n \in \mathbb{N}$ and elements $s_k \in S_i$ and $t_k \in T$ for $k = 1, \ldots, n$ such that $y = \lambda_{i,\infty}(\sum_k s_k \otimes t_k)$. Then

$$\psi(x) \prec \psi(y) = \sum_{k=1}^{n} \varphi_{i,\infty}(s_k) \otimes t_k.$$

By definition of the auxiliary relation for tensor products in PreW, there are elements $a'_k \in \varinjlim_i S_i$ and $t'_k \in T$ such that

$$\psi(x) \leq \sum_{k=1}^{n} a'_k \otimes t'_k, \quad \text{and} \quad a'_k \prec \varphi_{i,\infty}(s_k),\ t'_k \prec t_k \text{ for each } k = 1, \ldots, n.$$

It follows from the definition of the auxiliary relation for inductive limits in PreW, that there is an index $j \geq i$ and elements $s'_k \in S_j$ such that

$$a'_k = \varphi_{j,\infty}(s'_k),\ s'_k \prec \varphi_{i,j}(s_k) \text{ for each } k = 1, \ldots, n.$$

Set $y' = \lambda_{j,\infty}(\sum_{k=1}^n s'_k \otimes t'_k)$. Then

$$\psi(x) \leq \sum_{k=1}^n a'_k \otimes t'_k = \sum_{k=1}^n \varphi_{j,\infty}(s'_k) \otimes t'_k = \psi(y').$$

Since ψ is an order-embedding, we have $x \leq y'$. It is easily checked that $y' \prec y$. Thus, $x \prec y$, as desired.

The continuity in the second variable is proven analogously. The result for tensor products in Cu follows from that for PreW. More precisely, let $(S_i, \varphi_{i,j})$ be an inductive system in Cu, and let T be a Cu-semigroup. Using Corollary 3.1.11 at the first and last step, using Theorem 6.3.5 at the second to last step, and using the result for PreW at the second step, we obtain natural Cu-isomorphisms:

$$\begin{aligned} \text{Cu-}\varinjlim_i(S_i \otimes_{\mathrm{Cu}} T) &\cong \gamma\left(\text{PreW-}\varinjlim_i(S_i \otimes_{\mathrm{PreW}} T)\right) \\ &\cong \gamma\left((\text{PreW-}\varinjlim_i S_i) \otimes_{\mathrm{PreW}} T\right) \\ &\cong \gamma\left(\text{PreW-}\varinjlim_i S_i\right) \otimes_{\mathrm{Cu}} \gamma(T) \cong (\text{Cu-}\varinjlim_i S_i) \otimes_{\mathrm{Cu}} T. \end{aligned}$$

This finishes the proof. □

Problem 6.4.2. Given Cu-semigroups S and T that satisfy (O5) (respectively (O6), weak cancellation). When does $S \otimes_{\mathrm{Cu}} T$ satisfy (O5) (respectively (O6), weak cancellation)?

6.4.3. In general, (O5) does not pass to tensor products, see Proposition 6.4.4. However, for given Cu-semigroups S and T, we obtain the following partial positive answers to Problem 6.4.2:

(1) If S or T is an inductive limit of simplicial Cu-semigroups, then each of the axioms (O5), (O6) and weak cancellation passes to $S \otimes_{\mathrm{Cu}} T$, see Proposition 6.4.6.
(2) If S and T are algebraic Cu-semigroups, then axiom (O5) passes to $S \otimes_{\mathrm{Cu}} T$, see Corollary 6.4.9.

With view to the above mentioned example and results, this raises the following refined version of Problem 6.4.2: Does (O5) pass to tensor products where one of the Cu-semigroups is algebraic? Do the axioms pass to tensor products of simple Cu-semigroups?

For the next result, we use Z to denote the Cuntz semigroup of the Jiang-Su algebra $\mathcal{Z}$. We have that $Z = \mathbb{N} \sqcup (0, \infty]$, see Paragraph 7.3.2. We let $\mathrm{Lsc}\big([0,1], \overline{\mathbb{N}}\big)$ denote the set of lower-semicontinuous functions from $[0,1]$ to $\overline{\mathbb{N}}$, which is known to be isomorphic to the Cuntz semigroup of the C^*-algebra $C([0,1])$. It follows from Proposition 4.7 that Z and $\mathrm{Lsc}\big([0,1], \overline{\mathbb{N}}\big)$ satisfy (O5), but this is also easy to see directly. The next result shows that (O5) does in general not pass to tensor products.

Proposition 6.4.4. *The* Cu*-semigroup* $Z \otimes_{\mathrm{Cu}} \mathrm{Lsc}\big([0,1], \overline{\mathbb{N}}\big)$ *does not satisfy axiom (O5).*

Proof. Consider the Cu-semigroup $S = \mathrm{Lsc}([0,1], Z)$ which clearly has Z-multiplication. Let S_{00} be the smallest submonoid of S containing all elements of the form $z \cdot f$ for $z \in Z$ and $f \in \mathrm{Lsc}\big([0,1], \overline{\mathbb{N}}\big)$. Let us verify that the assumptions of Lemma 5.3.17 are satisfied for S_{00}. Given $z \in Z$ and $f \in \mathrm{Lsc}\big([0,1], \overline{\mathbb{N}}\big)$, we choose rapidly increasing sequences $(z_n)_n$ in Z and $(f_n)_n$ in $\mathrm{Lsc}\left([0,1],\overline{\mathbb{N}}\right)$ such that $z=\sup_n z_n$ and $f=\sup_n f_n$. Then, by the properties of Z-multiplication, $(z_n\cdot f_n)_n$ is a rapidly increasing sequence of elements in S_{00} such that

$$z\cdot f=\sup_n z_n\cdot f_n.$$

Applying Lemma 5.3.17, we obtain that the sup-closure of S_{00} in S, denoted by

$$S_0:=\overline{S_{00}}=\overline{\mathrm{span}}\left\{z\cdot f \;\mid\; z\in Z, f\in\mathrm{Lsc}\left([0,1],\overline{\mathbb{N}}\right)\right\},$$

is a Cu-semigroup. We first show that S_0 does not satisfy (O5).

Given an open set U in $[0,1]$, we denote by 1_U the indicator function of U, which is an element in $\mathrm{Lsc}\left([0,1],\overline{\mathbb{N}}\right)\subseteq S$. Given $a\in S$, we let $\mathrm{supp}(s)$ denote the support of a, which is open since a is lower semicontinuous. Set

$$a'=\tfrac{1}{4}\cdot 1_{(3/4,1]},\quad a=\tfrac{1}{2}\cdot 1_{(1/2,1]},\quad b=1_{[0,1]}.$$

These are all elements in S_0, and it is clear that $a'\ll a\leq b$. In order to obtain a contradiction, suppose that there exists $c\in S_0$ such that

$$a'+c\leq b\leq a+c. \tag{6.1}$$

(Note that an element c with this property can easily be found in S; in fact, its existence is guaranteed since S is the Cuntz semigroup of a C^*-algebra and therefore satisfies (O5).)

Choose a sequence $(c_n)_{n\in\mathbb{N}}$ in S_{00} such that $c=\sup_n c_n$. By evaluation the inequality (6.1) at each point in $[0,1]$, it is clear that $\mathrm{supp}(c)=[0,1]$. Since $\mathrm{supp}(c)=\cup_n\mathrm{supp}(c_n)$, there exists $N\in\mathbb{N}$ such that $\mathrm{supp}(c_N)=[0,1]$. Without loss of generality, we may assume that $N=0$.

By evaluating (6.1) at $\frac{1}{2}$, we have that $c\left(\frac{1}{2}\right)=1$. Since the element 1 in Z is compact, it follows from $c\left(\frac{1}{2}\right)=\sup_n c_n\left(\frac{1}{2}\right)$ that there exists $N\in\mathbb{N}$ such that $c_N\left(\frac{1}{2}\right)=1$. Again, without loss of generality, we may assume that $N=0$.

Note that every f in $\mathrm{Lsc}([0,1],\overline{\mathbb{N}})$ has a (canonical) decomposition as $f=\sum_{k=0}^\infty 1_{f^{-1}((k,\infty])}$, where $f^{-1}((k,\infty])$ are open sets. Applying this to the element c_0, and using that $c_0\leq 1$, it follows that there exist a finite index set I and nonempty open subsets $U_i\subseteq[0,1]$ and nonzero elements $z_i\in Z$ for $i\in I$ such that

$$c_0=\sum_{i\in I}z_i\cdot 1_{U_i}.$$

We have that $c_0\left(\frac{1}{2}\right)$ is the sum of the elements z_i for which $\frac{1}{2}\in U_i$. Since $1=c_0\left(\frac{1}{2}\right)$ is a minimal compact element in Z, and since the noncompact elements in Z are soft and therefore absorbing, we deduce that $\frac{1}{2}$ belongs to exactly one of the sets U_i. Let i_0 be the unique index in I such that $\frac{1}{2}\in U_{i_0}$. We necessarily have that $z_{i_0}=1$. Set

$$V=U_{i_0},\quad W=\bigcup_{i\neq i_0}U_i.$$

We have that V and W are open subset of $[0,1]$. Since $\mathrm{supp}(c_0)=[0,1]$, it follows that $V\cup W=[0,1]$. Sine c_0 is strictly less than $1_{[0,1]}$, we have that V is a proper subset of $[0,1]$. Therefore, the intersection $V\cap W$ is nonempty. For each $t\in V\cap W$ we have that $c_0(t)>1$, which clearly is a contradiction.

Next, consider the map

$$\tau\colon Z\times\mathrm{Lsc}\left([0,1],\overline{\mathbb{N}}\right)\to S_0\subset\mathrm{Lsc}([0,1],Z),\quad (z,f)\mapsto z\cdot f,$$

which is easily seen to be a Cu-bimorphism. By Theorem 6.3.3, there exists a Cu-morphism

$$\tilde{\tau}\colon Z\otimes_{\mathrm{Cu}}\mathrm{Lsc}\left([0,1],\overline{\mathbb{N}}\right)\to S_0,$$

such that $\tilde{\tau}(z\otimes f)=\tau(z,f)$ for each $z\in Z$ and $f\in\mathrm{Lsc}\left([0,1],\overline{\mathbb{N}}\right)$.

Since $1_{(3/4,1]} \ll 1_{(1/4,1]} \leq 1$ in $\mathrm{Lsc}\left([0,1],\overline{\mathbb{N}}\right)$, and $\frac{1}{4} \ll \frac{1}{2} \leq 1_{[0,1]}$ in Z, we have

$$\tfrac{1}{4} \otimes 1_{(3/4,1]} \ll \tfrac{1}{2} \otimes 1_{(1/2,1]} \leq 1 \otimes 1_{[0,1]},$$

in $Z \otimes_{\mathrm{Cu}} \mathrm{Lsc}\left([0,1],\overline{\mathbb{N}}\right)$. Note that

$$a' = \tilde{\tau}\left(\tfrac{1}{4} \otimes 1_{(3/4,1]}\right), \quad a = \tilde{\tau}\left(\tfrac{1}{2} \otimes 1_{(1/2,1]}\right), \quad b = \tilde{\tau}\left(1 \otimes 1_{[0,1]}\right).$$

Thus, if there exists $d \in Z \otimes_{\mathrm{Cu}} \mathrm{Lsc}\left([0,1],\overline{\mathbb{N}}\right)$ such that

$$\tfrac{1}{4} \otimes 1_{(3/4,1]} + d \leq 1 \otimes 1_{[0,1]} \leq \tfrac{1}{2} \otimes 1_{(1/2,1]} + d,$$

then the element $c := \tilde{\tau}(d)$ would satisfy $a' + c \leq b \leq a + c$, which is not possible by the first part of the proof. Therefore, $Z \otimes_{\mathrm{Cu}} \mathrm{Lsc}\left([0,1],\overline{\mathbb{N}}\right)$ does not satisfy (O5). $\square$

Corollary 6.4.5. *We have that* $\mathrm{Lsc}([0,1],Z) \not\cong Z \otimes_{\mathrm{Cu}} \mathrm{Lsc}\left([0,1],\overline{\mathbb{N}}\right)$.

Proof. By Proposition 6.4.4, we have that $Z \otimes_{\mathrm{Cu}} \mathrm{Lsc}\left([0,1],\overline{\mathbb{N}}\right)$ does not satisfy axiom (O5). On the other hand, we have that

$$\mathrm{Lsc}([0,1],Z) \cong \mathrm{Cu}(C([0,1],\mathcal{Z})),$$

by e.g. [**APS11**, Theorem 3.4], which in combination with Proposition 4.7 shows that $\mathrm{Lsc}([0,1],Z)$ satisfies (O5). $\square$

For the next result, recall from Definition 5.5.10 that a Cu-semigroup is *simplicial* if it is isomorphic to to the algebraically ordered Cu-semigroup $\overline{\mathbb{N}}^r$ for some r in $\mathbb{N}$. In Corollary 5.5.12, we have seen that a countably-based Cu-semigroup S is an inductive limit of simplicial Cu-semigroups if and only if there exists a separable AF-algebra A such that $S \cong \mathrm{Cu}(A)$.

Proposition 6.4.6. *Let S be an inductive limit of simplicial* Cu*-semigroups. Then, taking the tensor product with S preserves (O5), (O6) and weak cancellation.*

Proof. Let $(S_i, \varphi_{i,j})$ be an inductive system of simplicial Cu-semigroups, indexed over the a directed set I. Then there are numbers $r_i \in \mathbb{N}$ such that $S_i \cong \overline{\mathbb{N}}^{r_i}$ for each $i \in I$. Let T be a Cu-semigroup. For $r \in \mathbb{N}$, let T^r be the set of r-tuples with entries in T, equipped with pointwise addition and order. It is easily checked that T^r is a Cu-semigroup and that there is a natural isomorphism $S_i \otimes_{\mathrm{Cu}} T \cong T^{r_i}$ for each i. It follows from Proposition 6.4.1 that $S \otimes_{\mathrm{Cu}} T \cong \mathrm{Cu}\text{-}\varinjlim_i T^{r_i}$.

Assume now that T satisfies (O5). It follows easily that T^r satisfies (O5) for each $r \in \mathbb{N}$. Then, $S \otimes_{\mathrm{Cu}} T$ satisfies (O5) by Theorem 4.5. It is proved analogously that weak cancellation and (O6) pass from T to $S \otimes_{\mathrm{Cu}} T$. $\square$

For the next result, recall that for a positively ordered monoid M, we denote by $\mathrm{Cu}(M)$ the Cu-completion of the PreW-semigroup $(M, \leq)$, see Paragraph 5.5.3.

Proposition 6.4.7. *Let M and N be two positively ordered monoids. Then, there is a canonical isomorphism*

$$\mathrm{Cu}(M) \otimes_{\mathrm{Cu}} \mathrm{Cu}(N) \cong \mathrm{Cu}(M \otimes_{\mathrm{PoM}} N).$$

Proof. We will write $(M, \leq)$ and $(N, \leq)$ for the W-semigroups associated to M and N, see Proposition 5.5.4. It follows easily from the construction of the tensor product in PreW that

$$(M, \leq) \otimes_{\mathrm{PreW}} (N, \leq) \cong (M \otimes_{\mathrm{PoM}} N, \leq).$$

Now the result follows from Theorem 6.3.5. $\square$

Corollary 6.4.8. *If S and T are algebraic* Cu*-semigroups, then so is $S \otimes_{\mathrm{Cu}} T$.*

Corollary 6.4.9. *Let S and T be algebraic* Cu*-semigroups. If S and T satisfy axiom (O5), then so does $S \otimes_{\mathrm{Cu}} T$.*

PROOF. Let S_c and T_c denote the positively ordered monoid of compact elements in S and T, respectively. By Proposition 5.5.4 and Proposition 6.4.7, there are natural isomorphisms

$$S \cong \mathrm{Cu}(S_c), \quad T \cong \mathrm{Cu}(T_c), \quad S \otimes_{\mathrm{Cu}} T \cong \mathrm{Cu}(S_c \otimes_{\mathrm{PoM}} T_c).$$

Assume now that S and T satisfy (O5). By Proposition 5.5.8, this implies that S_c and T_c are algebraically ordered. It follows that $S_c \otimes_{\mathrm{PoM}} T_c$ is algebraically ordered, see Proposition B.2.8. Using Proposition 5.5.8 again, we deduce that $S \otimes_{\mathrm{Cu}} T$ satisfies (O5). $\square$

6.4.10. Let A and B be C^*-algebras, and let $A \otimes_{\max} B$ be their maximal tensor product. Given positive elements $x \in M_\infty(A)_+$ and $y \in M_\infty(A)_+$, the simple tensor $x \otimes y$ is a positive element in $M_\infty(A \otimes_{\max} B)_+$. This induces a W-bimorphism from $W(A) \times W(B)$ to $W(A \otimes_{\max} B)$, which in turn induces a natural W-morphism

$$W(A) \otimes_{\mathrm{W}} W(B) \to W(A \otimes_{\max} B).$$

Similarly, there is a natural Cu-morphism

$$\tau^{\max}_{A,B}\colon \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\max} B),$$

such that $\tau^{\max}_{A,B}([x] \otimes [y]) = [x \otimes y]$ for every pair of positive elements $x \in (A \otimes \mathrm{K})_+$ and $y \in (B \otimes \mathrm{K})_+$.

The natural quotient *-homomorphism from $A \otimes_{\max} B$ to $A \otimes_{\min} B$ induces a surjective Cu-morphism

$$\mathrm{Cu}(A \otimes_{\max} B) \to \mathrm{Cu}(A \otimes_{\min} B).$$

By composing the map $\tau^{\max}_{A,B}$ with this Cu-morphism, we obtain a natural Cu-morphism

$$\tau^{\min}_{A,B}\colon \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\min} B).$$

PROBLEM 6.4.11. Let A and B be two C^*-algebras. When is the map $\tau^{\min}_{A,B}$ a Cu-isomorphism? When is it surjective? When is it an order-embedding? Similarly, when is the map $\tau^{\min}_{A,B}$ an isomorphism, surjective, an order-embedding?

6.4.12. Let A and B be C^*-algebras. It is clear that $\tau^{\max}_{A,B}$ is an order-embedding whenever $\tau^{\min}_{A,B}$ is. Similarly, if $\tau^{\max}_{A,B}$ is surjective, then so is $\tau^{\min}_{A,B}$.

If A or B is nuclear, then the natural map from $A \otimes_{\max} B$ to $A \otimes_{\min} B$ is an isomorphism. In that case, the maps $\tau^{\min}_{A,B}$ and $\tau^{\max}_{A,B}$ are equal, and we simply write $\tau_{A,B}$ for this map.

It is easy to find examples of C^*-algebras A and B for which the natural map $\tau^{\max}_{A,B}$ is not surjective. For instance, this is the case for $A = C([0,1])$ and $B = \mathcal{Z}$, as shown in Proposition 6.4.4; see also Corollary 6.4.5.

Other (counter)examples can be found using K-theory. If A is a unital, simple, stably finite C^*-algebra, then $K_0(A)$ is determined by the Cuntz semigroup of A via the formula

$$K_0(A) = \mathrm{Gr}(\mathrm{Cu}(A)_c), \tag{6.2}$$

where $\mathrm{Cu}(A)_c$ denotes the submonoid of compact elements in $\mathrm{Cu}(A)$, and where Gr denotes the Grothendieck completion.

Now, let A and B be two unital, simple, stably finite C^*-algebras. Assume that A is nuclear, whence we can unambiguously write $\otimes$ instead of $\otimes_{\max}$ for tensor products with A. Then the tensor product $A \otimes B$ is also a unital, simple, stably finite C^*-algebra. Assume that the map

$$\tau_{A,B}\colon \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes B)$$

is surjective. Using that A, B and $A \otimes B$ are simple and stably finite, it follows that the natural map

$$\mathrm{Cu}(A)_c \otimes_{\mathrm{PoM}} \mathrm{Cu}(B)_c \to \mathrm{Cu}(A \otimes B)_c$$

is surjective as well. Passing to Grothendieck completions, and using (6.2) at the second step, we obtain a surjective map

$$\mathrm{Gr}(\mathrm{Cu}(A)_c \otimes_{\mathrm{PoM}} \mathrm{Cu}(B)_c) \to \mathrm{Gr}(\mathrm{Cu}(A \otimes B)_c) \cong K_0(A \otimes B).$$

Since taking the Grothendieck completion commutes with tensor products (see Proposition B.1.7), we have that

$$K_0(A) \otimes K_0(B) \cong \mathrm{Gr}(\mathrm{Cu}(A)_c) \otimes_{\mathrm{PoM}} \mathrm{Gr}(\mathrm{Cu}(B)_c) \cong \mathrm{Gr}(\mathrm{Cu}(A)_c \otimes_{\mathrm{PoM}} \mathrm{Cu}(B)_c).$$

Thus, we have shown that the natural map

$$K_0(A) \otimes K_0(B) \to K_0(A \otimes B),$$

is surjective.

Let us further assume that A is a C^*-algebra in the bootstrap class, see [**Bla06**, V.1.5.4, p.437]. Then A and B satisfy the 'Künneth formula for tensor products in K-theory', see [**Bla06**, Theorem V.1.5.10, p.440], which means that there is a short exact sequence

$$0 \to \bigoplus_{i=0,1} K_i(A) \otimes K_i(B) \to K_0(A \otimes B) \to \bigoplus_{i=0,1} \mathrm{Tor}_1^{\mathbb{Z}}(K_i(A), K_{1-i}(B)) \to 0.$$

Since our assumptions on A and B imply that the natural map from $K_0(A) \otimes K_0(B)$ to $K_0(A \otimes B)$ is surjective, we deduce from the Künneth formula that

$$K_1(A) \otimes K_1(B) = 0, \quad \mathrm{Tor}_1^{\mathbb{Z}}(K_i(A), K_{1-i}(B)) = 0, \text{ for } i = 0, 1.$$

In conclusion, we get that the map $\tau_{A,B}$ is not surjective whenever A and B are unital, simple, stably finite C^*-algebras in the bootstrap class for which $K_1(A) \otimes K_1(B) \neq 0$ or for which $\mathrm{Tor}_1^{\mathbb{Z}}(K_i(A), K_{1-i}(B)) \neq 0$ for $i = 0$ or $i = 1$.

On the other hand, we have that $\tau_{A,B}$ is an isomorphism in the following cases:

(1) If A or B is an AF-algebra, see Proposition 6.4.13.
(2) If A or B is nuclear and $\mathcal{O}_\infty$-stable, see Proposition 7.2.11.

PROPOSITION 6.4.13. *Let A and B be C^*-algebras. Assume that at least one of the algebras is an AF-algebra. Then, the natural map*

$$\tau_{A,B}\colon \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \xrightarrow{\cong} \mathrm{Cu}(A \otimes_{\max} B)$$

is an isomorphism.

PROOF. Without loss of generality, we may assume that A is an AF-algebra. Then there is an inductive system of finite-dimensional C^*-algebras A_i such that $A \cong \varinjlim_i A_i$. For each i, there is $r_i \in \mathbb{N}$ such that A_i is isomorphic to a direct sum of r_i matrix algebras. Then $A_i \otimes \mathrm{K} \cong \mathrm{K}^{r_i}$ and $\mathrm{Cu}(A_i) \cong \overline{\mathbb{N}}^{r_i}$. Moreover, $A_i \otimes B \otimes \mathrm{K} \cong (B \otimes \mathrm{K})^{r_i}$ and there are isomorphisms

$$\mathrm{Cu}(A_i) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \cong \overline{\mathbb{N}}^{r_i} \otimes \mathrm{Cu}(B) \cong \mathrm{Cu}(B)^{r_i} \cong \mathrm{Cu}(A_i \otimes B),$$

for each i.

Since the maximal tensor product commutes with inductive limits of C^*-algebras, see [**Bla06**, II.9.6.5, p.200], there is a natural isomorphism $\varinjlim_i (A_i \otimes_{\max} B) \cong$

$A \otimes_{\max} B$. Using Corollary 3.2.9 at the first and last step, and using Proposition 6.4.1 at the second step, we obtain that

$$\begin{aligned}\mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) &\cong \left(\mathrm{Cu}\text{-}\varinjlim_i \mathrm{Cu}(A_i) \right) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \\ &\cong \mathrm{Cu}\text{-}\varinjlim_i \left(\mathrm{Cu}(A_i) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \right) \\ &\cong \mathrm{Cu}\text{-}\varinjlim_i \mathrm{Cu}(A_i \otimes B) \cong \mathrm{Cu}(A \otimes B),\end{aligned}$$

as desired. □

6.4.14. Recall that the set of functionals on a Cu-semigroup S is defined as the set of generalized Cu-morphisms $S \to [0, \infty]$, see Paragraph 5.2.1. Now, let S and T be two Cu-semigroups. Then, there is a natural map

$$F(S) \times F(T) \to F(S \otimes_{\mathrm{Cu}} T),$$

defined as follows: Given two functionals $\lambda \in F(S)$ and $\mu \in F(T)$, consider the map

$$f\colon S \times T \to [0, \infty], \quad (a, b) \mapsto \lambda(a)\mu(b). \quad (a \in S, b \in T)$$

Here, the product of two elements $x, y \in [0, \infty)$ is defined as the usual product of real numbers, and $0y = x0 = 0$ for all $x, y \in [0, \infty]$, and $\infty y = x\infty = \infty$ for all $x, y \in (0, \infty]$. (This is the Cu-product on $[0, \infty]$ when considered with its structure as a Cu-semiring, see Example 7.1.7.) It is easily checked that f is a generalized Cu-bimorphism. By Theorem 6.3.3, f induces a generalized Cu-morphism

$$\tilde{f}\colon S \otimes_{\mathrm{Cu}} T \to [0, \infty].$$

This means that $\tilde{f} \in F(S \otimes_{\mathrm{Cu}} T)$, satisfying $\tilde{f}(a \otimes b) = \lambda(a)\mu(b)$ for every $a \in S$ and $b \in T$.

The following is a version of [**Rør02**, Theorem 4.1.10, p.69] for Cu-semigroups.

Proposition 6.4.15. *Let S and T be two simple, nonelementary* Cu*-semigroups satisfying (O5) and (O6). Then:*

(1) If S and T are stably finite, then so is $S \otimes_{\mathrm{Cu}} T$.
(2) If S or T is not stably finite, then $S \otimes_{\mathrm{Cu}} T \cong \{0, \infty\}$.

Proof. To show (1), assume that S and T are simple and stably finite. Set $R = S \otimes_{\mathrm{Cu}} T$. There are nontrivial functionals $\lambda \in F(S)$ and $\mu \in F(T)$, see Proposition 5.2.5. By Paragraph 6.4.14, this induces a functional $\delta \in F(R)$ such that $\delta(a \otimes b) = \lambda(a)\mu(b)$ for every $a \in S$ and $b \in T$. It is clear that δ is nontrivial, which implies that R is stably finite.

To show (2), we may assume without loss of generality that T is not stably finite. By Corollary 7.2.5, $S \otimes_{\mathrm{Cu}} T$ is simple. Thus, there is a unique element $\infty \in S \otimes_{\mathrm{Cu}} T$, such that $\infty = \sup_{n \in \mathbb{N}} nx$ for every nonzero element $x \in S \otimes_{\mathrm{Cu}} T$. Let $a \in S$ and $b \in T$ be nonzero elements. We will show that $a \otimes b = \infty$.

Let ∞_T denote the infinite element of T. By Proposition 5.2.5, ∞_T is compact. Thus, there is $n \in \mathbb{N}$ such that $nb = \infty_T$. By [**Rob13a**, Proposition 5.2.1], see Proposition 5.4.1, there is a nonzero element $x \in S$ such that $nx \leq a$. It follows that

$$a \otimes b \geq (nx) \otimes b = x \otimes (nb) = x \otimes \infty_T = \infty.$$

It is also clear that $a \otimes b \leq \infty$, and therefore $a \otimes b = \infty$ whenever a and b are nonzero. This implies that $S \otimes_{\mathrm{Cu}} T \cong \{0, \infty\}$, as desired. □

CHAPTER 7

Cu-semirings and Cu-semimodules

In Section 7.1, we introduce the concepts of Cu-semirings and their semimodules. Natural examples are given by Cuntz semigroups of C^*-algebras that are strongly self-absorbing and of C^*-algebras that tensorially absorb such a C^*-algebra, respectively; see Proposition 7.1.4.

We say that a Cu-semiring R is solid if the multiplication map $R \times R \to R$ induces an isomorphism $R \otimes_{\mathrm{Cu}} R \xrightarrow{\cong} R$, see Definition 7.1.5. This is analogous to the concept of solidity for rings as introduced in [**BK72**, Definition 2.1; 2.4]. This property can also be interpreted as an algebraic analog of being strongly self-absorbing. The Cuntz semigroup of every known strongly self-absorbing C^*-algebra is a solid Cu-semiring, see Paragraph 7.6.1.

Given a solid Cu-semiring R, we say that a Cu-semigroup S has R-multiplication if it is a semimodule over R, in a suitable sense, and it is very interesting to study the class of such Cu-semigroups. We have that every generalized Cu-morphism between two Cu-semigroups with R-multiplication is automatically R-linear, see Proposition 7.1.6. We deduce that a Cu-semigroup S has at most one R-multiplication. In other words, either there is no way to give S the structure of a semimodule over R, or there is a unique such structure. This also means that, as we have already remarked in the Introduction, being a semimodule over R is a *property* of S, rather than extra structure, see Remark 7.1.9.

It may therefore seem that Cu-semigroups with R-multiplication are rare. However, we show in Lemma 7.1.10 that for every Cu-semiring S, the tensor product $R \otimes_{\mathrm{Cu}} S$ has R-multiplication. We obtain that S has R-multiplication if and only if S is naturally isomorphic to $R \otimes_{\mathrm{Cu}} S$, see Theorem 7.1.12.

We refer to Chapter 8 for a detailed study of the structure of Cu-semirings, including a complete classification of solid Cu-semirings in Section 8.3.

In Sections 7.2 through 7.5, we study Cu-semimodules over the following solid Cu-semirings:

(1) If A is a purely infinite, strongly self-absorbing C^*-algebra, e.g. the Cuntz algebra $\mathcal{O}_\infty$, then $\mathrm{Cu}(A) = \{0, \infty\}$. In Section 7.2, we study Cu-semigroups that have $\{0, \infty\}$-multiplication. This can be considered as a theory of 'purely infinite Cuntz semigroups'.

Indeed, it is clear that the Cuntz semigroup of every $\mathcal{O}_\infty$-stable C^*-algebra is a $\{0, \infty\}$-semimodule. More generally, we show that a (not necessarily simple) C^*-algebra is purely infinite if and only if its Cuntz semigroup has $\{0, \infty\}$-multiplication.

(2) The Jiang-Su algebra $\mathcal{Z}$ is a strongly self-absorbing C^*-algebra, whose Cuntz semigroup is a solid Cu-semiring, denoted by Z. In Section 7.3, we study Cu-semigroups that have Z-multiplication.

The analogy between $\mathcal{Z}$-stable C^*-algebras and Cuntz semigroups with Z-multiplication is however not as close. In one direction, we clearly have that the Cuntz semigroup of every $\mathcal{Z}$-stable C^*-algebra is a Z-semimodule. However, the converse is not true in general. We show in Theorem 7.3.8 that a Cu-semigroup has Z-multiplication if and only if it is almost unperforated and almost divisible. On the other hand, it seems that the Cuntz semigroup of every $\mathcal{Z}$-stable C^*-algebra is nearly unperforated, see Conjecture 5.6.18.

(3) Every strongly self-absorbing UHF-algebra is of the form M_q for some supernatural number q satisfying $q = q^2$ and $q \neq 1$. We denote the Cuntz semigroup of M_q by R_q, which is a solid Cu-semiring. In Section 7.4, we study Cu-semigroups that have R_q-multiplication. This can be considered as a theory of 'UHF-absorbing Cuntz semigroups'. Given a Cu-semigroup S, we also think of $R_q \otimes_{\mathrm{Cu}} S$ as the 'rationalization' of S.

(4) The Jacelon-Razak algebra $\mathcal{R}$ is a stably projectionless C^*-algebra, whence it does not satisfy the definition of a strongly self-absorbing C^*-algebra (which are required to be unital). However, the Cuntz semigroup of $\mathcal{R}$ is $[0,\infty]$, which is a solid Cu-semiring. Moreover, for every C^*-algebra A we have that $\mathrm{Cu}(\mathcal{R} \otimes A) \cong [0,\infty] \otimes_{\mathrm{Cu}} \mathrm{Cu}(A)$. In particular, if a C^*-algebra tensorially absorbs $\mathcal{R}$, then its Cuntz semigroup has $[0,\infty]$-multiplication.

In Section 7.4, we study Cu-semigroups that have $[0,\infty]$-multiplication. This can be considered as a theory of '$\mathcal{R}$-absorbing Cuntz semigroups'. Given a Cu-semigroup S, we also think of $[0,\infty] \otimes_{\mathrm{Cu}} S$ as the 'realification' of S, a term that was introduced by Robert.

In the following table, we summarize some results of this chapter. The middle column contains the characterizations when a Cu-semigroup S has R-multiplication for the solid Cu-semiring listed in the left column. The column on the right characterizes the effect that 'stabilizing' with R has on the order-structure of the Cu-semigroup.

R	Characterization when S is a Cu-semimodule over R.	For $a, b \in S$, characterization when $1 \otimes a \leq 1 \otimes b$ in $R \otimes_{\mathrm{Cu}} S$.
$\{0,\infty\}$	$a = 2a$ for each $a \in S$; Theorem 7.2.2.	$a \propto^* b$; Theorem 7.2.6.
Z	S is almost unperforated and almost divisible; Theorem 7.3.8.	Unclear; Problem 7.3.10.
R_q	S is q-unperforated and q-divisible; Theorem 7.4.10.	For each $a' \ll a$, there exists n dividing q such that $na' \leq nb$; Theorem 7.4.11.
$[0,\infty]$	S is unperforated, divisible and every element is soft; Theorem 7.5.4.	$\hat{a} \leq \hat{b}$ in $\mathrm{Lsc}(F(S))$; Theorem 7.5.11.

7.1. Strongly self-absorbing C*-algebras and solid Cu-semirings

Definition 7.1.1. A Cu-*semiring* is a Cu-semigroup R together with a Cu-bimorphism, $(a,b) \mapsto ab$, and a distinguished element 1 in R such that for all $a, b, c \in R$ we have

$$ab = ba, \quad a(bc) = (ab)c, \quad 1a = a = a1.$$

The Cu-bimorphism $R \times R \to R$ is also called the Cu-*product* of R.

Remarks 7.1.2. (1) A Cu-semiring is a commutative, unital semiring (see Section B.4) with a compatible partial order turning it into a ω-continuous ω-**dcpo** in the sense of lattice theory, see Remarks 3.1.3.

(2) It is natural to assume that a Cu-semiring has no zero divisors. Indeed, if two elements a and b satisfy $ab = 0$, then $st = 0$ for all elements s and t satisfying $s \leq \infty \cdot a$ and $t \leq \infty \cdot b$. Thus, if $ab = 0$, then the multiplication is trivial on the ideals generated by a and b. In particular, a simple Cu-semiring with zero divisors is isomorphic to $\{0\}$.

(3) Let R be a Cu-semiring. Recall from Paragraph 5.1.1 that an ideal I in R is an order-hereditary submonoid that is closed under passing to suprema of increasing sequences. This notion of ideal is a-priori not connected to the ring-theoretic notion of an ideal, which means that $ab \in I$ for any $a \in I$ and $b \in R$.

However, if the unit of R is full (that is, if it is also an order unit), then every ideal is also a ring-theoretic ideal. Indeed, given $a \in I$ and $b \in R$, we have $b \leq \infty \cdot 1$, and therefore $ab \leq a(\infty \cdot 1) = \sup k(a1) = \infty \cdot a \in I$. Then $ab \in I$, as desired.

It is unclear to the authors, whether the unit of a Cu-semiring is automatically full. Possibly, this only holds under additional assumptions on the Cu-semiring.

The following definition is an adoption of the terminology introduced by Robert, [**Rob13a**, Definition 3.1.2].

Definition 7.1.3. Let S be a Cu-semigroup, and let R be a Cu-semiring. An *R-multiplication* on S is a Cu-bimorphism $R \times S \to S$, $(r, s) \mapsto rs$ such that for all $r_1, r_2 \in R$ and $s \in S$, we have

$$(r_1 r_2)s = r_1(r_2 s), \quad 1s = s.$$

In this case we also say that S is a Cu-*semimodule* over R.

One motivation for the definition of a Cu-semiring comes from strongly self-absorbing C^*-algebras, as defined in [**TW07**, Definition 1.3]. Every such algebra is simple and nuclear, and either purely infinite or stably finite with a unique tracial state, [**TW07**, 1.6, 1.7]. The Cuntz semigroup of a simple, purely infinite C^*-algebra is isomorphic to $\{0, \infty\}$. Thus, we will focus our attention on the stably finite case.

Proposition 7.1.4. *Let D be a unital, separable, stably finite, strongly self-absorbing C^*-algebra. Then:*

(1) The Cuntz semigroup $\mathrm{Cu}(D)$ *has a natural* Cu-*product giving it the structure of a countably-based, simple* Cu-*semiring satisfying (O5) and (O6), and with a unique normalized functional.*

(2) If A is a D-absorbing C^*-algebra (i.e., $A \cong A \otimes D$), then $\mathrm{Cu}(A)$ *has a natural* $\mathrm{Cu}(D)$*-multiplication.*

Proof. We use the symbol '$\approx$' to denote approximate unitary equivalence. For positive elements in a C^*-algebra, this is a stronger equivalence relation than Cuntz equivalence.

(1) By definition, there is a *-isomorphism $\psi\colon D \to D \otimes D$ such that $\psi \approx \mathrm{id}_D \otimes 1_D$. Consider the natural Cu-bimorphism $\mathrm{Cu}(D) \times \mathrm{Cu}(D) \to \mathrm{Cu}(D \otimes D)$ from Paragraph 6.4.10. Composed with $\mathrm{Cu}(\psi^{-1})$, this yields a Cu-bimorphism

$$\varphi\colon \mathrm{Cu}(D) \times \mathrm{Cu}(D) \to \mathrm{Cu}(D).$$

We will show that φ together with $1 = [1_A]$ gives $\mathrm{Cu}(D)$ the structure of a Cu-semiring.

We know from [**TW07**, Corollary 1.11] that D has approximately inner flip. Thus, for any positive elements x and y in A, we have $x \otimes y \approx y \otimes x$ in $A \otimes A$. It follows that

$$\varphi([x],[y]) = [\psi^{-1}(x\otimes y)] = [\psi^{-1}(y\otimes x)] = \varphi([y],[x]).$$

The analogous computation holds for Cuntz classes of positive elements in $A\otimes \mathrm{K}$, which implies that φ defines a commutative multiplication.

To show associativity of the product, consider positive elements x, y and z. Using the approximately inner flip, we get $x\otimes y\otimes z \approx y\otimes z\otimes x$. Applying $\psi^{-1}\otimes \mathrm{id}_D$, it follows that $\psi^{-1}(x\otimes y)\otimes z \approx \psi^{-1}(y\otimes z)\otimes x$, and therefore

$$\begin{aligned}\varphi(\varphi([x],[y]),[z]) &= [\psi^{-1}(\psi^{-1}(x\otimes y)\otimes z)]\\ &= [\psi^{-1}(\psi^{-1}(y\otimes z)\otimes x)]\\ &= [\psi^{-1}(x\otimes \psi^{-1}(y\otimes z))] = \varphi([x],\varphi([y],[z])).\end{aligned}$$

The analogous computation in $A\otimes \mathrm{K}$ imply that the product is associative.

Using that $\mathrm{id}_D\otimes 1_D \approx \psi$ and $\psi \approx 1_D\otimes \mathrm{id}_D$, we obtain for every positive element x that

$$\varphi([x],1) = [\psi^{-1}(x\otimes 1)] = [x] = [\psi^{-1}(1\otimes x)] = \varphi(1,[x]).$$

This finishes the proof that $\mathrm{Cu}(D)$ is a Cu-semiring.

The Cuntz semigroup of every separable C^*-algebra is countably-based and satisfies (O5) and (O6), see Proposition 3.2.3 and Proposition 4.7. Moreover, since D is simple and with unique (2-quasi)tracial state, we get that $\mathrm{Cu}(D)$ is simple and has a unique normalized functional, see Corollary 5.1.12 and Corollary 5.2.17.

(2) By [**TW07**, Theorem 2.3], there is a *-isomorphism $\phi\colon A\to D\otimes A$ such that to $\phi\approx 1_D\otimes \mathrm{id}_A$ (note that the condition of D being K_1-injective is automatic by [**Win11**, Theorem 3.1, Remark 3.3]). Arguing as in (1), the natural map $D\times A\to D\otimes A$ induces a Cu-bimorphism $\mathrm{Cu}(D)\times\mathrm{Cu}(A)\to\mathrm{Cu}(D\otimes A)$ and this, composed with $\mathrm{Cu}(\phi^{-1})$, yields a Cu-bimorphism $\varphi\colon\ \mathrm{Cu}(D)\times\mathrm{Cu}(A)\to\mathrm{Cu}(A)$.

Given $x\in A_+$, we have $\phi(x)\approx 1_D\otimes x$ and therefore $\varphi([1_D],[x])=[x]$. Given also $d_1,d_2\in D_+$, we have

$$d_1\otimes d_2\otimes\phi(x)\approx d_1\otimes d_2\otimes 1_D\otimes x\approx d_1\otimes 1_D\otimes d_2\otimes x,$$

in $D\otimes D\otimes D\otimes A$. Applying $\phi^{-1}\circ(\psi^{-1}\otimes\phi^{-1})$ to the above relation, we get

$$\phi^{-1}(\psi^{-1}(d_1\otimes d_2)\otimes x)\approx\phi^{-1}(\psi^{-1}(d_1\otimes 1_D)\otimes\phi^{-1}(d_2\otimes x)).$$

Therefore, $\varphi([d_1]\cdot[d_2],[x]) = \varphi([d_1],\varphi([d_2],[x]))$. The same computations hold for positive elements in the stabilizations, which implies that φ defines a $\mathrm{Cu}(D)$-multiplication on $\mathrm{Cu}(A)$. $\square$

A ring R is called *solid* if the multiplication map induces an isomorphism $R\otimes R\cong R$; see [**BK72**, Definition 2.1; 2.4], where it is pointed out that solidity of the ring R is equivalent to the requirement that $a\otimes 1 = 1\otimes a$ for every $a\in R$. Here, we use the usual tensor product of (discrete) groups and rings, and every ring is understood to be unital and commutative. See Section B.4 for more details.

As pointed out in [**Gut13**], solid rings have also been called T-rings and $\mathbb{Z}$-epimorphs; see [**BS77**, Definition 1.6] and [**DS84**]. We define solid Cu-semirings in analogy to the case for rings.

Definition 7.1.5. A Cu-semiring R is *solid* if the Cu-bimorphism $\varphi\colon R\times R\to R$ defining the multiplication induces an isomorphism $R\otimes_{\mathrm{Cu}} R\cong R$.

The next result shows that for a Cu-semiring, there are many conditions equivalent to being solid. This is analogous to the case for rings. Indeed, for a ring, all of the conditions in Proposition 7.1.6, when suitably interpreted, are equivalent to solidity of the ring. This is known, and most of it is shown in the references mentioned at the beginning of the section.

PROPOSITION 7.1.6. *Let R be a* Cu*-semiring. Then, the following conditions are equivalent:*

(1) *The* Cu*-semiring R is solid.*
(2) *Whenever S is a* Cu*-semimodule over R, then the R-multiplication on S induces an isomorphism $R \otimes_{\mathrm{Cu}} S \cong S$.*
(3) *Whenever S_1 and S_2 are* Cu*-semimodules over R, and $\tau\colon S_1 \times S_2 \to T$ is a generalized* Cu*-bimorphism, then $\tau(ra_1, a_2) = \tau(a_1, ra_2)$ for all $r \in R$ and $a_i \in S_i$.*
(4) *Every generalized* Cu*-morphism $S_1 \to S_2$ between* Cu*-semimodules S_1 and S_2 over R is automatically R-linear.*
(5) *For all $a, b \in R$, we have $a \otimes b = b \otimes a$ in $R \otimes_{\mathrm{Cu}} R$.*
(6) *For every $a \in R$, we have $a \otimes 1 = 1 \otimes a$ in $R \otimes_{\mathrm{Cu}} R$.*

PROOF. The implications '(2) $\Rightarrow$ (1) $\Rightarrow$ (5) $\Rightarrow$ (6)' are clear.

Let us prove that (6) implies (2). So let S be a Cu-semimodule over R. Given $r \in R$ and $s \in S$, we will show that $r \otimes s = 1 \otimes rs$ in $R \otimes_{\mathrm{Cu}} S$. To that end, consider the map

$$\tau_s\colon R \times R \to R \otimes_{\mathrm{Cu}} S, \quad \tau_s(a, b) = a \otimes bs. \quad (a, b \in R)$$

It is straightforward to check that this is a generalized Cu-bimorphism, whence there is a generalized Cu-morphism $\bar{\tau}_s\colon R \otimes_{\mathrm{Cu}} R \to R \otimes_{\mathrm{Cu}} S$ such that $\tau_s(a, b) = \bar{\tau}_s(a \otimes b)$. Using the assumption at the third step, we obtain that

$$r \otimes s = \tau_s(r, 1) = \bar{\tau}_s(r \otimes 1) = \bar{\tau}_s(1 \otimes r) = \tau_s(1, r) = 1 \otimes rs.$$

Let $\varphi\colon R \times S \to S$ be the Cu-bimorphism defining the R-multiplication. This induces a Cu-morphism $\bar{\varphi}\colon R \otimes_{\mathrm{Cu}} S \to S$. We let $\psi\colon S \to R \otimes S$ be the generalized Cu-morphism defined by $\psi(s) = 1 \otimes s$. We clearly have $\bar{\varphi} \circ \psi = \mathrm{id}_S$. On the other hand, for every $r \in R$ and $s \in S$, using the formula of the previous paragraph at the last step, we have

$$\psi \circ \bar{\varphi}(r \otimes s) = \psi(rs) = 1 \otimes rs = r \otimes s.$$

It follows that $\psi \circ \bar{\varphi} = \mathrm{id}_{R \otimes_{\mathrm{Cu}} S}$, and so $R \otimes_{\mathrm{Cu}} S \cong S$.

Next, we prove that (2) implies (3). It is enough to show that for every $r \in R$ and $s_1 \in S_1$ and $s_2 \in S_2$, we have that $rs_1 \otimes s_2 = s_1 \otimes rs_2$ in $S_1 \otimes_{\mathrm{Cu}} S_2$. To that end, we use the isomorphisms $S_1 \cong S_1 \otimes_{\mathrm{Cu}} R$ and $R \otimes_{\mathrm{Cu}} S_2 \cong S_2$ given by assumption, and the natural isomorphism of associativity of the tensor product from Corollary 6.3.6, to obtain the identifications shown in the following diagram:

$$\begin{array}{ccccccc}
S_1 \otimes_{\mathrm{Cu}} S_2 & \cong & (S_1 \otimes_{\mathrm{Cu}} R) \otimes_{\mathrm{Cu}} S_2 & \cong & S_1 \otimes_{\mathrm{Cu}} (R \otimes_{\mathrm{Cu}} S_2) & \cong & S_1 \otimes_{\mathrm{Cu}} S_2 \\
\cup & & \cup & & \cup & & \cup \\
rs_1 \otimes s_2 & \longleftrightarrow & \begin{array}{c} (rs_1 \otimes 1) \otimes s_2 \\ = (s_1 \otimes r) \otimes s_2 \end{array} & \longleftrightarrow & \begin{array}{c} s_1 \otimes (1 \otimes rs_2) \\ = s_1 \otimes (r \otimes s_2) \end{array} & \longleftrightarrow & s_1 \otimes rs_2,
\end{array}$$

as desired.

Next, we prove that (3) implies (4). So let S_1 and S_2 be Cu-semimodules over R, and let $\alpha\colon S_1 \to S_2$ be a generalized Cu-morphism. Consider he map

$$\tau\colon R \times S_1 \to S_2, \quad \tau(r, s) = r\alpha(s), \quad (r \in R, s \in S_2)$$

which is easily seen to be a generalized Cu-bimorphism. We consider R with the R-multiplication given by its Cu-semiring structure. Given $r \in R$ and $s \in S$, we use the assumption at the second step to obtain that

$$\alpha(rs) = \tau(1, rs) = \tau(r, s) = r\alpha(s).$$

Thus, the map α is R-linear, as desired.

Finally, let us show that (4) implies (6). We endow the Cu-semigroup $R\otimes_{\mathrm{Cu}} R$ with two (a priori different) R-multiplications induced by

$$r\cdot(r_1\otimes r_2)=rr_1\otimes r_2,\quad r\cdot(r_1\otimes r_2)=r_1\otimes rr_2.\quad (r,r_1,r_2\in R)$$

Now, we consider the identity Cu-morphism $\mathrm{id}\colon R\otimes_{\mathrm{Cu}} R\to R\otimes_{\mathrm{Cu}} R$, but we equip the source and target with the two different R-multiplications. By assumption, the map id_R is R-linear. Then, given any $r\in R$, we compute the product of the element $1\otimes 1\in R\otimes_{\mathrm{Cu}} R$ with r using the different R-multiplications. This gives $r1\otimes 1=1\otimes r1$, as desired. □

EXAMPLE 7.1.7. Consider the Cu-semigroup $[0,\infty]$. Given two finite elements that are both not equal to ∞, we define their product as for real numbers. Moreover, we set $\infty\cdot 0=0$ and $\infty\cdot a=\infty$ for every nonzero $a\in[0,\infty]$. It is easy to check that this defines a Cu-product on $[0,\infty]$.

Thus, $[0,\infty]$ is a Cu-semiring. Let us show that it is solid. By Proposition 7.1.6, it is enough to show that $1\otimes a=a\otimes 1$ for every $a\in[0,\infty]$. Given $k,n\in\mathbb{N}$ with $n\neq 0$, we consider the element $\frac{k}{n}\in[0,\infty]$ and we compute that

$$1\otimes\tfrac{k}{n}=\tfrac{n}{n}\otimes\tfrac{k}{n}=kn\left(\tfrac{1}{n}\otimes\tfrac{1}{n}\right)=\tfrac{k}{n}\otimes\tfrac{n}{n}=\tfrac{k}{n}\otimes 1.$$

Since rational elements are dense in $[0,\infty]$, we get that $[0,\infty]$ is solid.

COROLLARY 7.1.8. *Let R be a solid* Cu-*semiring, and let S be a* Cu-*semigroup. Then, any two R-multiplications on S are equal.*

PROOF. Consider the identity morphism $S\to S$, where the range and target are equipped with the two R-multiplications in question. By Proposition 7.1.6, this map is R-linear, which means exactly that the two R-multiplications are equal. □

REMARK 7.1.9. Let R be a solid Cu-semiring, and let S be a Cu-semigroup. By Corollary 7.1.8, S has at most one R-multiplication. Thus, having an R-multiplication is a *property* rather than an additional *structure*.

LEMMA 7.1.10. *Let R be a* Cu-*semiring, and let S, T be* Cu-*semigroups. Assume that S has an R-multiplication. Then $S\otimes_{\mathrm{Cu}} T$ also has an R-multiplication that satisfies*

$$r(s\otimes t)=(rs)\otimes t,$$

for all $r\in R$, $s\in S$ and $t\in T$.

PROOF. Given $r\in R$, consider the map

$$\alpha_r\colon S\times T\to S\otimes_{\mathrm{Cu}} T,\quad (s,t)\mapsto rs\otimes t.\quad (s\in S,t\in T)$$

This is a generalized Cu-bimorphism. Therefore, it induces a generalized Cu-morphism $\bar\alpha_r\colon S\otimes_{\mathrm{Cu}} T\to S\otimes_{\mathrm{Cu}} T$ such that $\alpha_r(s\otimes t)=(rs)\otimes t$. Using the universal properties of the tensor product, see Theorem 6.3.3, one shows that the map

$$R\times(S\otimes_{\mathrm{Cu}} T)\to S\otimes_{\mathrm{Cu}} T,\quad (r,x)\mapsto\bar\alpha_r(x),\quad (r\in R,x\in S\otimes_{\mathrm{Cu}} T)$$

is a Cu-bimorphism defining an R-multiplication on $S\otimes T$. □

LEMMA 7.1.11. *Let R be a solid* Cu-*semiring, let S be a* Cu-*semigroup, and let $\varphi\colon R\times S\to S$ be a generalized* Cu-*bimorphism. Assume that $\varphi(1_R,a)=a$ for every $a\in S$. Then φ defines an R-multiplication on S. Thus, φ is a* Cu-*bimorphism satisfying*

$$\varphi(r_1r_2,a)=\varphi(r_1,\varphi(r_2,a)),$$

for every $r_1,r_2\in R$ and $a\in S$.

Proof. Let $\varphi\colon R\times S\to S$ be a map as in the statement. Let $\bar{\varphi}\colon R\otimes_{\mathrm{Cu}}S\to S$ be the generalized Cu-morphism induced by φ. We will show that $\bar{\varphi}$ is a Cu-isomorphism. Consider the map

$$\rho\colon S\to R\otimes_{\mathrm{Cu}}S,\quad \rho(a)=1_R\otimes a.\quad (a\in S)$$

It is clear that ρ is a generalized Cu-morphism. Using the assumption on φ at the third step, we deduce for each $a\in S$ that

$$\bar{\varphi}\circ\rho(a)=\bar{\varphi}(1_R\otimes a)=\varphi(1_R,a)=a.$$

Thus, $\bar{\varphi}\circ\rho=\mathrm{id}_S$. For the converse, consider the generalized Cu-morphism $\rho\circ\bar{\varphi}\colon R\otimes_{\mathrm{Cu}}S\to R\otimes_{\mathrm{Cu}}S$. For each $r\in R$ and $a\in S$ we have that

$$\rho\circ\bar{\varphi}(r\otimes a)=1_R\otimes\varphi(r,a).$$

By Lemma 7.1.10, $R\otimes S$ has an R-multiplication such that $r_1(r_2\otimes a)=(r_1r_2)\otimes a$ for all $r_1,r_2\in R$ and $a\in S$. It follows from Proposition 7.1.6 that $\rho\circ\bar{\varphi}$ is R-linear. Using this at the second step, we obtain for each $r\in R$ and $a\in S$ that

$$\rho\circ\bar{\varphi}(r\otimes a)=\rho\circ\bar{\varphi}(r\cdot(1_R\otimes a))=r\cdot(\rho\circ\bar{\varphi}(1_R\otimes a))=r\cdot(1_R\otimes a)=r\otimes a.$$

This shows that $\rho\circ\bar{\varphi}$ is the identity map on simple tensors in $R\otimes_{\mathrm{Cu}}S$. It follows that $\rho\circ\bar{\varphi}=\mathrm{id}_{R\otimes S}$.

In general, every PoM-isomorphism between Cu-semigroups automatically preserves the way-below relation and suprema of increasing sequences, since these notions are completely encoded in the order-structure. We clearly have that $\bar{\varphi}$ is an isomorphism of positively ordered monoids. Therefore it is also a Cu-isomorphism. It follows that φ is a Cu-bimorphism.

Let $r_1,r_2\in R$ and $a\in S$. Since $\rho\circ\bar{\varphi}=\mathrm{id}_{R\otimes S}$, we have that $r_2\otimes a=1_R\otimes\varphi(r_2,a)$. Using the R-multiplication of $R\otimes S$ to multiply by r_1, we deduce that

$$(r_1r_2)\otimes a=r_1\cdot(r_2\otimes a)=r_1\cdot(1_R\otimes\varphi(r_2,a))=r_1\otimes\varphi(r_2,a).$$

Applying $\bar{\varphi}$, we have that

$$\varphi(r_1r_2,a)=\bar{\varphi}((r_1r_2)\otimes a)=\bar{\varphi}(r_1\otimes\varphi(r_2,a))=\varphi(r_1,\varphi(r_2,a)),$$

as desired. □

Theorem 7.1.12. *Let R be a solid* Cu-*semiring, and let S be a* Cu-*semigroup. Then the following are equivalent:*

(1) *The* Cu-*semigroup S has an R-multiplication.*
(2) *There exists a* Cu-*isomorphism between $R\otimes_{\mathrm{Cu}}S$ and S.*
(3) *The map $S\to R\otimes S$ that sends a in S to $1_R\otimes a$ is a* Cu-*isomorphism.*

Proof. It is clear that (3) implies (2). It follows from Lemma 7.1.10 that (2) implies (1). To show that (1) implies (3), assume that S has an R-multiplication $\varphi\colon R\times S\to S$. By Proposition 7.1.6, the induced Cu-morphism $\bar{\varphi}\colon R\otimes_{\mathrm{Cu}}S\to S$ is an isomorphism. It is straightforward to check that the inverse of $\bar{\varphi}$ sends a in S to $1_R\otimes a$. □

The following result should be compared with an analogous result for strongly self-absorbing C^*-algebras in [**TW07**, Proposition 5.12].

Proposition 7.1.13. *Let R be a solid* Cu-*semiring, and let S be another* Cu-*semiring. Then the following are equivalent:*

(1) *There is a* Cu-*isomorphism between $R\otimes_{\mathrm{Cu}}S$ and S.*
(2) *There exists a unital, multiplicative, generalized* Cu-*morphism $R\to S$.*
(3) *There exists a unital, generalized* Cu-*morphism $R\to S$.*

Moreover, if a map as in (3) exists, then it is unique (and therefore automatically multiplicative). Furthermore, if 1_S is a compact element, then any map as in (3) is automatically a Cu*-morphism.*

Proof. It is clear that (2) implies (3). Let $\alpha\colon R \to S$ be a unital, generalized Cu-morphism. This induces a map

$$\varphi_\alpha\colon R \times S \to S, \quad (r, a) \mapsto \alpha(r) \cdot a. \quad (r \in R, a \in S)$$

It follows easily from the properties of α that φ_α is a generalized Cu-bimorphism satisfying $\varphi_\alpha(1_R, a) = a$ for every $a \in S$. Then, it follows from Lemma 7.1.11 that φ_α is a Cu-bimorphism defining an R-multiplication on S.

Thus, using Theorem 7.1.12, we obtain that (3) implies (1). Moreover, by Corollary 7.1.8, any two R-multiplications on S are equal. This implies that a map satisfying (3) is unique (if it exists).

Let us show that (1) implies (2). By Lemma 7.1.10, the semigroup $R \otimes_{\mathrm{Cu}} S$ has an R-multiplication. Therefore, by assumption, S has an R-multiplication $\varphi\colon R \times S \to S$. Consider the map

$$\alpha\colon R \to S, \quad r \mapsto \varphi(r, 1_S). \quad (r \in R)$$

It is clear that α is a unital, generalized Cu-morphism. In order to show that α is multiplicative, we consider the following map

$$\psi\colon R \times S \to S, \quad (r, a) \mapsto \varphi(r, 1_S)a. \quad (r \in R, a \in S)$$

It is easy to see that ψ is a generalized Cu-bimorphism satisfying $\psi(1_R, a) = a$ for every $a \in S$. By Lemma 7.1.11, we have that $\psi = \varphi$. Using this at the third step, we deduce that

$$\begin{aligned}\alpha(r_1 r_2) = \varphi(r_1 r_2, 1_S) &= \varphi(r_1, \varphi(r_2, 1_S)) \\ &= \psi(r_1, \varphi(r_2, 1_S)) = \varphi(r, 1_S)\varphi(r_2, 1_S) = \alpha(r_1)\alpha(r_2).\end{aligned}$$

Thus, α is multiplicative, as desired.

Finally, if 1_S is compact, then it is also clear from the definition that α is a Cu-morphism. □

7.2. Cuntz semigroups of purely infinite C*-algebras

In this section, we study Cu-semigroups that are semimodules over the Cu-semiring $\{0, \infty\}$. If A is a purely infinite, strongly self-absorbing C^*-algebra (e.g. $\mathcal{O}_2$, $\mathcal{O}_\infty$, or the tensor product of $\mathcal{O}_\infty$ with a UHF-algebra of infinite type), then $\mathrm{Cu}(A) \cong \{0, \infty\}$.

In Theorem 7.2.2, we characterize the Cu-semimodules over $\{0, \infty\}$ as the Cu-semigroups that are idempotent. We show that that tensor product of a given Cu-semigroup S with $\{0, \infty\}$ is canonically isomorphic to $\mathrm{Lat}_{\mathrm{f}}(S)$, the semigroup of singly-generated ideals in S; see Proposition 7.2.3.

In Proposition 7.2.8, we apply our results to Cuntz semigroups of C^*-algebras by showing that a (not necessarily simple) C^*-algebra A is purely infinite if and only if

$$\mathrm{Cu}(A) \cong \{0, \infty\} \otimes_{\mathrm{Cu}} \mathrm{Cu}(A).$$

We deduce that for every separable C^*-algebra A, there are natural isomorphisms of the following Cu-semigroups:

$$\mathrm{Cu}(\mathcal{O}_\infty \otimes A) \cong \mathrm{Lat}(A) \cong \mathrm{Lat}(\mathrm{Cu}(A)) \cong \{0, \infty\} \otimes_{\mathrm{Cu}} \mathrm{Cu}(A),$$

see Corollary 7.2.13.

The following is easy to prove and hence we omit the details:

LEMMA 7.2.1. *The* Cu*-semiring* $\{0, \infty\}$ *is solid.*

Recall that a commutative semigroup S is called *idempotent* if each of its elements is idempotent, that is, if $2a = a$ for every $a \in S$. In the literature, an idempotent, commutative semigroup S is also called a *commutative band*, or a *semilattice* (with 'join' in the semilattice corresponding to addition in the semigroup).

THEOREM 7.2.2. *Let S be a* Cu*-semigroup. Then, the following are equivalent:*

(1) We have $S \cong \{0, \infty\} \otimes_{\mathrm{Cu}} S$.
(2) The Cu*-semigroup S has a $\{0, \infty\}$-multiplication.*
(3) The semigroup S is idempotent.

PROOF. Since $\{0, \infty\}$ is a solid Cu-semiring, the equivalence of (1) and (2) follows from Theorem 7.1.12. To show that (2) implies (3), let a be an element of S. Since ∞ is the unit of the Cu-semiring $\{0, \infty\}$ and since $2\infty = \infty$ in $\{0, \infty\}$, we obtain that

$$a = \infty \cdot a = (2\infty) \cdot a = 2(\infty \cdot a) = 2a,$$

as desired.

Next, let us show that (3) implies (2). So assume that S is an idempotent Cu-semigroup. Consider the map

$$\varphi\colon \{0, \infty\} \times S \to S, \quad (0, a) \mapsto 0, \quad (\infty, a) \mapsto a. \quad (a \in S)$$

We have that φ is a generalized Cu-morphism in the second variable. It is also clear that φ preserves zero, order and suprema of increasing sequences (there are no nontrivial ones) in the first variable. Using that S is idempotent, it follows that φ is additive in the first variable. By Lemma 6.3.2, we have that φ is a generalized Cu-bimorphism. Then, we obtain from Lemma 7.1.11 that S has $\{0, \infty\}$-multiplication, as desired. □

Recall that, for a given Cu-semigroup S, we denote by $\mathrm{Lat}_{\mathrm{f}}(S)$ the Cu-semigroup of singly-generated ideals in S, as considered in Proposition 5.1.7.

PROPOSITION 7.2.3. *Let S be a* Cu*-semigroup. Then, there is a natural* Cu*-isomorphism*

$$\{0, \infty\} \otimes_{\mathrm{Cu}} S \cong \mathrm{Lat}_{\mathrm{f}}(S)$$

identifying $\infty \otimes a$ with $\mathrm{Idl}(a)$.

PROOF. Let S be a Cu-semigroup, and let $\varphi\colon S \times \{0, \infty\} \to S \otimes_{\mathrm{Cu}} \{0, \infty\}$ denote the universal Cu-bimorphism. Consider the map

$$\tau\colon S \times \{0, \infty\} \to \mathrm{Lat}_{\mathrm{f}}(S), \quad \tau(a, 0) = 0, \quad \tau(a, \infty) = \mathrm{Idl}(a). \quad (a \in S).$$

It follows from Proposition 5.1.7 that τ is Cu-bimorphism. Then, there is a Cu-morphism

$$\tilde{\tau}\colon S \otimes_{\mathrm{Cu}} \{0, \infty\} \to \mathrm{Lat}_{\mathrm{f}}(S),$$

such that $\tau = \tilde{\tau} \circ \varphi$. It is clear that $\tilde{\tau}$ is a surjective Cu-morphism.

As shown in Paragraph 5.1.6, every ideal I in $\mathrm{Lat}_{\mathrm{f}}(S)$ contains a largest element, denoted by $\bigvee I$. We may therefore define a map

$$\psi\colon \mathrm{Lat}_{\mathrm{f}}(S) \to S \otimes_{\mathrm{Cu}} \{0, \infty\}, \quad I \mapsto \Big(\bigvee I\Big) \otimes \infty.$$

It is easy to see that ψ is a Cu-morphism. Given $a \in S$, we clearly have

$$\psi \circ \tilde{\tau} \circ \varphi(a, 0) = 0 = \varphi(a, 0).$$

We can also deduce that

$$\psi \circ \tilde{\tau} \circ \varphi(a, \infty) = \Big(\bigvee \mathrm{Idl}(a)\Big) \otimes \infty = (\infty \cdot a) \otimes \infty = a \otimes \infty = \varphi(a, 0).$$

Thus, we have shown that $\psi \circ \tilde{\tau} \circ \varphi = \varphi$, which implies that $\psi \circ \tilde{\tau}$ is the identity on $S \otimes_{\mathrm{Cu}} \{0, \infty\}$. Therefore, the map $\tilde{\tau}$ is an order-embedding, and therefore an isomorphism. □

Corollary 7.2.4. *Let S be a simple, nonzero* Cu*-semigroup. Then*

$$\{0, \infty\} \otimes_{\mathrm{Cu}} S \cong \{0, \infty\}.$$

Corollary 7.2.5. *Let S and T be two* Cu*-semigroups. Then there is a natural isomorphism*

$$\mathrm{Lat}_{\mathrm{f}}(S \otimes_{\mathrm{Cu}} T) \cong \mathrm{Lat}_{\mathrm{f}}(S) \otimes_{\mathrm{Cu}} \mathrm{Lat}_{\mathrm{f}}(T).$$

Proof. It is clear that $\{0, \infty\} \cong \{0, \infty\} \otimes_{\mathrm{Cu}} \{0, \infty\}$. We then obtain, using Proposition 7.2.3 at the first and last step, and using the associativity and symmetry of the tensor product (see Corollary 6.3.6 and Paragraph 6.3.7) at the third step, that

$$\begin{aligned}\mathrm{Lat}_{\mathrm{f}}(S \otimes_{\mathrm{Cu}} T) &\cong (S \otimes_{\mathrm{Cu}} T) \otimes_{\mathrm{Cu}} \{0, \infty\} \\ &\cong (S \otimes_{\mathrm{Cu}} \{0, \infty\}) \otimes_{\mathrm{Cu}} (T \otimes \{0, \infty\}) \cong \mathrm{Lat}_{\mathrm{f}}(S) \otimes_{\mathrm{Cu}} \mathrm{Lat}_{\mathrm{f}}(T),\end{aligned}$$

as desired. □

Theorem 7.2.6. *Let S be a* Cu*-semigroup, and let $a, b \in S$. Then, the following are equivalent:*

(1) We have $1 \otimes a \leq 1 \otimes b$ in $\{0, \infty\} \otimes_{\mathrm{Cu}} S$, where $1 = \infty$ is the unit of $\{0, \infty\}$.
(2) We have $a \propto^* b$.

Proof. Let $a, b \in S$. By Proposition 7.2.3, there is an isomorphism between $\{0, \infty\} \otimes_{\mathrm{Cu}} S$ and $\mathrm{Lat}_{\mathrm{f}}(S)$ that identifies the simple tensor $1 \otimes a$ with $\mathrm{Idl}(a)$. Then, we have $1 \otimes a \leq 1 \otimes b$ in $\{0, \infty\} \otimes_{\mathrm{Cu}} S$ if and only if $\mathrm{Idl}(a) \subset \mathrm{Idl}(b)$ in $\mathrm{Lat}_{\mathrm{f}}(S)$, which in turn happens if and only if $a \propto^* b$, as desired. □

Proposition 7.2.7. *Let S be a* Cu*-semigroup. Then $\{0, \infty\} \otimes_{\mathrm{Cu}} S$ is unperforated, divisible and satisfies (O5). Moreover, if S satisfies (O6), then so does $\{0, \infty\} \otimes_{\mathrm{Cu}} S$.*

Proof. Set $T = \{0, \infty\} \otimes_{\mathrm{Cu}} S$, which by Theorem 7.2.2 is an idempotent Cu-semigroup. To show that it is unperforated, let $a, b \in T$ satisfy $na \leq nb$ for some $n \in \mathbb{N}_+$. Since $a = na$ and $b = nb$, we immediately get $a \leq b$. Similarly, T is divisible. The statements about (O5) and (O6) follow directly by combining Proposition 7.2.3 with Proposition 5.1.7. □

A not necessarily simple C^*-algebra A is *purely infinite* if A has no characters and if for any two positive elements x and y in A we have $x \precsim y$ whenever x is contained in the ideal of A generated by y; see [**KR00**, Definition 4.1], see also [**Bla06**, p.450ff]. By [**KR00**, Theorem 4.23], if A is purely infinite, then so is $A \otimes \mathrm{K}$.

An nonzero element x in A_+ is *properly infinite* if $x \oplus x \precsim x$, considered in $M_2(A)$. If we denote by $[x]$ the class of x in $\mathrm{Cu}(A)$, then x is properly infinite if and only if $2[x] = [x]$ and $[x] \neq 0$. By [**KR00**, Theorem 4.16], a C^*-algebra is purely infinite if and only each of its nonzero elements is properly infinite. Using Theorem 7.2.2, we may reformulate the result of Kirchberg and Rørdam as follows:

Proposition 7.2.8. *Let A be a C^*-algebra. Then A is purely infinite if and only if* $\mathrm{Cu}(A) \cong \{0, \infty\} \otimes_{\mathrm{Cu}} \mathrm{Cu}(A)$.

It follows from Proposition 7.2.7 that the Cuntz semigroup of every purely infinite C^*-algebra is unperforated. This verifies Conjecture 5.6.18 for the class for purely infinite C^*-algebras.

COROLLARY 7.2.9. *Let A be a purely infinite C^*-algebra. Then* $\mathrm{Cu}(A)$ *is nearly unperforated.*

Let A and B be two C^*-algebras. In [**KR00**, Question 5.12], Kirchberg and Rørdam ask if the minimal tensor product of A and B is purely infinite as soon as one of A or B is purely infinite. Let us assume that A is purely infinite. By Proposition 7.2.8, we have $\mathrm{Cu}(A) \cong \{0,\infty\} \otimes_{\mathrm{Cu}} \mathrm{Cu}(A)$. Using the associativity of the tensor product in Cu, this implies that

$$\begin{aligned}\mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) &\cong \big(\{0,\infty\} \otimes_{\mathrm{Cu}} \mathrm{Cu}(A)\big) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \\ &\cong \{0,\infty\} \otimes_{\mathrm{Cu}} \big(\mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B)\big).\end{aligned}$$

Recall from Paragraph 6.4.10 that there is a natural Cu-morphism

$$\tau_{A,B} \colon \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\max} B).$$

If we compose with the Cu-morphism induced by the quotient *-homomorphism from $A \otimes_{\max} B$ to $A \otimes_{\min} B$, we obtain a natural Cu-bimorphism

$$\tau_{A,B}^{\min} \colon \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\min} B).$$

Then, we may reformulate [**KR00**, Question 5.12] as follows:

PROBLEM 7.2.10. Let A and B be two C^*-algebras. Assuming that A or B is purely infinite, does it follows that the Cu-bimorphism

$$\tau_{A,B}^{\min} \colon \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\min} B).$$

is an isomorphism?

Using the results in [**KR00**], we obtain the following result, which is a partial answer to Problem 6.4.11.

PROPOSITION 7.2.11. *Let A and B be two C^*-algebras. Assume that at least one of the algebras is nuclear and $\mathcal{O}_\infty$-stable. Then, the natural map*

$$\tau_{A,B} \colon \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\max} B).$$

is an isomorphism.

COROLLARY 7.2.12. *Let A be a C^*-algebra and let B be a simple, nuclear, purely infinite C^*-algebra. Then, there are natural isomorphisms:*

$$\mathrm{Cu}(A \otimes B) \cong \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \cong \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \{0,\infty\}.$$

COROLLARY 7.2.13. *Let A be a separable C^*-algebra. Then there are natural isomorphisms between the following* Cu*-semigroups:*

$$\mathrm{Cu}(\mathcal{O}_\infty \otimes A) \cong \mathrm{Lat}(A) \cong \mathrm{Lat}(\mathrm{Cu}(A)) \cong \{0,\infty\} \otimes_{\mathrm{Cu}} \mathrm{Cu}(A).$$

REMARKS 7.2.14. Note that it follows from our observations that a C^*-algebra A is purely infinite if and only if $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathcal{O}_\infty)$.

7.3. Almost unperforated and almost divisible Cu-semigroups

In this section, we study Cu-semigroups that are semimodules over the Cu-semiring of the Jiang-Su algebra $\mathcal{Z}$. We use Z to denote $\mathrm{Cu}(\mathcal{Z})$ and we begin by showing that Z is a solid Cu-semiring, see Proposition 7.3.3. The main result of this section is Theorem 7.3.8, where we characterize the Cu-semimodules over Z as the Cu-semigroups that are almost unperforated and almost divisible. This can be interpreted as a verification of the Cu-semigroup version of the Toms-Winter conjecture, see Remark 7.3.9.

Let A be a C^*-algebra. Recall that $V(A)$ denotes the Murray-von Neumann semigroup of equivalence classes of projections in matrices over A. We use $\mathrm{QT}_2(A)$ to denote the set of 2-quasitraces on A, see Section 5.2. By a famous result of Haagerup, [**Haa14**], every 2-quasitrace on a unital, exact C^*-algebra is a trace. We let $\mathrm{Lsc}(\mathrm{QT}_2(A))$ denote the set of lower-semicontinuous linear functions from the cone $\mathrm{QT}_2(A)$ to $[0,\infty]$. The Cu-semigroup $L(\mathrm{QT}_2(A))$ is defined as a certain subset of $\mathrm{Lsc}(\mathrm{QT}_2(A))$, see Paragraph 5.2.1; see [**ERS11**] for more details. If A is simple and unital, then $L_b(\mathrm{QT}_2(A))$ is defined as the elements in $L(\mathrm{QT}_2(A))$ that are bounded by a finite multiple of the function $\hat{1} \in L(\mathrm{QT}_2(A))$ associated to the unit of A.

The following result is the combination of work of many people and has appeared in several (partial) versions in the literature. In the formulation presented here, it can be found as Corollary 6.8 and Remark 6.9 in [**ERS11**]. Equivalent results and previous partial results can be found in Theorems 4.4 and 6.5 in [**PT07**], Theorem 2.6 in [**BT07**], Theorems 6.2 and 6.3 in [**ABP11**], Theorem 5.27 in [**APT11**], and Theorem 5.5 in [**BPT08**].

Proposition 7.3.1 (A number of people). *Let A be a unital, separable, simple, finite, $\mathcal{Z}$-stable C^*-algebra. Then the (pre)completed Cuntz semigroup of A can be computed as:*

$$W(A) \cong V(A)^\times \sqcup L_b(\mathrm{QT}_2(A)), \quad \mathrm{Cu}(A) \cong V(A)^\times \sqcup L(\mathrm{QT}_2(A)).$$

In particular, if A is exact and has a unique tracial state, then

$$W(A) \cong V(A) \sqcup (0,\infty), \quad \mathrm{Cu}(A) \cong V(A) \sqcup (0,\infty].$$

7.3.2. Now let Z be the (completed) Cuntz semigroup of the Jiang-Su algebra $\mathcal{Z}$. Using Proposition 7.3.1, we can compute Z as

$$Z = \mathbb{N} \sqcup (0,\infty],$$

where the elements of $\mathbb{N} \subset Z$ are compact and the elements of $(0,\infty] \subset Z$ are soft; for the concrete case of the Cuntz semigroup of the Jiang-Su algebra, this computation has also appeared in [**PT07**, Theorem 3.1]. (We are assuming here that the set $\mathbb{N}$ contains 0.)

Using this decomposition into two parts, the addition, multiplication and order on Z are defined as usual in each of the parts; see also Example 7.1.7. Given a compact, nonzero element $n \in \mathbb{N} \subset Z$, we let $n' \in (0,\infty]$ denote the associated soft element with the same number. Then, given $n \in \mathbb{N}$ and $a \in (0,\infty]$, we define $n+a = n'+a$ and $na = n'a$. Thus, the soft part of Z is additively and multiplicatively absorbing. For a compact element $n \in Z$ and a soft element $a \in Z$ we have $n \leq a$ if and only if $n' \leq a$; and we have $a \leq n$ if and only if $a < n'$.

Proposition 7.3.3. *The* Cu*-semiring $Z = \mathbb{N} \sqcup (0,\infty]$ is solid.*

Proof. By Proposition 7.1.6, it is enough to show that $1 \otimes a = a \otimes 1$ for every $a \in Z$. This follows easily for compact elements in Z, since they are multiples of the unit. In the other case, if $a \in Z$ is a soft element, we can use the same argument that was used in Example 7.1.7 to show that $[0,\infty]$ is solid. □

Definition 7.3.4. Let S be a positively ordered monoid, let $a \in S$, and let $k \in \mathbb{N}_+$. We say that a is *almost k-divisible* if there exists $x \in S$ such that $kx \leq a \leq (k+1)x$. We say that a is *almost divisible* if it is almost n-divisible for every $n \in \mathbb{N}_+$. We say that S is *almost divisible* if each of its elements is.

Recall from Definition 5.2.11 that a Cu-semigroup S is *almost unperforated* if for all $a, b \in S$, $a \leq b$ whenever $(n+1)a \leq nb$ for some $n \in \mathbb{N}$. It is easily checked that Z is almost unperforated and almost divisible. It then follows from the next result that every Cu-semigroup with Z-multiplication is almost divisible and almost unperforated. The converse is shown in Theorem 7.3.8. The following lemma is also used in the proof of Theorem 8.1.6, which is why we formulate it more general than needed in this section.

LEMMA 7.3.5. *Let S be a* Cu*-semimodule over the* Cu*-semiring R. Assume that the unit element of R is almost divisible. Then S is almost unperforated and almost divisible. In particular, R itself is almost unperforated and almost divisible.*

PROOF. Let $n \in \mathbb{N}$. Since 1_R is almost divisible, there exists $z \in R$ such that $nz \leq 1 \leq (n+1)z$. Then, for any $a \in S$, we have

$$n(za) \leq a \leq (n+1)(za),$$

which shows that S is almost divisible.

To show that S is almost unperforated, let $a, b \in S$ be two elements satisfying $(n+1)a \leq nb$ for some $n \in \mathbb{N}$. Choose $z \in R$ such that $nz \leq 1 \leq (n+1)z$. Then

$$a \leq z(n+1)a \leq znb \leq b,$$

as desired. □

To prepare the proof of Theorem 7.3.8, we first provide some results that are also of interest in themselves. We need to introduce some notation. Given $a \in S$ and $k, n \in \mathbb{N}_+$, we set

$$\mu((k,n),a) = \{x \in S \mid nx \leq ka \leq (n+1)x\}.$$

Note that an element $a \in S$ is almost divisible if and only if $\mu((k,n),a)$ is nonempty for every $k, n \in \mathbb{N}_+$. We think of $\mu((k,n),a)$ as the set of those elements obtained by 'approximately multiplying' a by $\frac{k}{n}$. With this idea in mind, the following Lemma asserts that for almost unperforated semigroups this 'approximate multiplication' is 'approximately ordered'.

LEMMA 7.3.6. *Let S be an almost unperforated* Cu*-semigroup, let $a, b \in S$, and let $k_1, n_1, k_2, n_2 \in \mathbb{N}_+$ such that $\frac{k_1}{n_1} < \frac{k_2}{n_2+1}$.*

(1) If $a \leq b$, then $x \leq y$ for every $x \in \mu((k_1,n_1),a)$ and $y \in \mu((k_2,n_2),b)$.
(2) If $a \ll b$, then $x \ll y$ for every $x \in \mu((k_1,n_1),a)$ and $y \in \mu((k_2,n_2),b)$.

PROOF. Let k_1, n_1, k_2, n_2 be as in the statement, and let $x \in \mu((k_1,n_1),a)$ and $y \in \mu((k_2,n_2),b)$. Then

$$n_1 x \leq k_1 a, \quad k_2 b \leq (n_2+1)y.$$

Multiplying the first equations by k_2 and the second by k_1, we obtain that

$$k_2 n_1 x \leq k_2 k_1 a, \quad k_1 k_2 b \leq k_1(n_2+1)y.$$

If $a \leq b$, then it follows that $k_2 n_1 x \leq k_1(n_2+1)y$. Since $k_1(n_2+1) < k_2 n_1$ and S is almost unperforated, we obtain that $x \leq y$.

In the second case, assuming $a \ll b$, it follows that $k_2 n_1 x \ll k_1(n_2+1)y$. Choose y' such that $y' \ll y$ and $k_2 n_1 x \ll k_1(n_2+1)y'$. As in the first case, we obtain that $x \leq y'$. Then $x \ll y$, as desired. □

PROPOSITION 7.3.7. *Let S be an almost unperforated* Cu*-semigroup, and let $a \in S$. Then, there exists a generalized* Cu*-morphism $\alpha_a \colon Z \to S$ with $\alpha_a(1) = a$ if and only if a is almost divisible.*

If the map α_a exists, then it is unique. Moreover, it is a Cu*-morphism if and only if a is compact.*

PROOF. If there exists a generalized Cu-morphism $\alpha_a\colon Z\to S$ with $\alpha_a(1)=a$, then for each $n\in\mathbb{N}_+$ we have that

$$n\cdot\alpha_a(\tfrac{1}{n})=\alpha_a(1')\leq\alpha_a(1)=a\leq\alpha_a(\tfrac{n+1}{n})=(n+1)\alpha_a(\tfrac{1}{n}),$$

which shows that a is almost divisible.

For the converse, let $a\in S$ be an almost divisible element. We define a map $\alpha_a\colon Z\to S$ by considering the decomposition $Z=\mathbb{N}\sqcup(0,\infty]$. For $n\in\mathbb{N}\subset Z$, we set $\alpha_a(n)=na=a+\overset{n}{\dots}+a$. For $t\in(0,\infty]\subset Z$, we set

$$\alpha_a(t)=\sup\left\{x\in\mu((k,n),a)\mid\tfrac{k}{n}<t\right\}.$$

We first prove that this supremum exists, by finding an increasing cofinal sequence. Choose numbers $k_d,n_d\in\mathbb{N}_+$ for $d\in\mathbb{N}_+$ such that

$$\tfrac{k_d}{n_d}<\tfrac{k_{d+1}}{n_{d+1}+1},\qquad\text{and}\qquad\sup_d\tfrac{k_d}{n_d}=t.$$

Since a is almost divisible, for each d we can choose an element $x_d\in\mu((k_d,n_d),a)$. By Lemma 7.3.6, the sequence $(x_d)_d\subset S$ is increasing. Moreover, it is easily checked that for every $\frac{k}{n}\in\mathbb{Q}_+$ with $\frac{k}{n}<t$ and for every $x\in\mu((k,n),a)$ there is an index d such that $x\leq x_d$. It follows that $\alpha_a(t)=\sup_d x_d$, which exists by (O1).

To show uniqueness, let $\beta\colon Z\to S$ be a generalized Cu-morphism with $\beta(1)=a$. It is clear that $\beta(n)=\alpha(n)$ for all elements $n\in\mathbb{N}\subset Z$. Consider now $t\in(0,\infty]\subset Z$. As above, for each $d\in\mathbb{N}_+$ choose numbers $k_d,n_d\in\mathbb{N}_+$ and an element x_d such that

$$\tfrac{k_d}{n_d}<\tfrac{k_{d+1}}{n_{d+1}+1},\qquad\text{and}\qquad\sup_d\tfrac{k_d}{n_d}=t,\qquad\text{and}\qquad x_d\in\mu((k_d,n_d),a).$$

It is easy to see that $\beta(\frac{k_d}{n_d})\in\mu((k_d,n_d),a)$ for each $d\in\mathbb{N}_+$. By Lemma 7.3.6, we deduce for each d that

$$x_d\leq\beta(\tfrac{k_{d+1}}{n_{d+1}})\leq x_{d+2}.$$

Using this at the second step, and that β preserves suprema of increasing sequences at the first step, it follows that

$$\beta(t)=\sup_d\beta(\tfrac{k_d}{n_d})=\sup_d x_d=\alpha_a(t).$$

It is left to the reader to check that α_a preserves the zero element, the order, and suprema of increasing sequences. It remains to prove that α_a is additive. This is clear for sums of two elements in $\mathbb{N}\subset Z$. So let $t_1,t_2\in(0,\infty]\subset Z$.

For each $d\in\mathbb{N}_+$, choose numbers $k_d^{(1)},k_d^{(2)},n_d\in\mathbb{N}_+$ such that for $i=1,2$:

$$\tfrac{k_d^{(i)}}{n_d}<\tfrac{k_{d+1}^{(i)}}{n_{d+1}+1},\qquad\text{and}\qquad\sup_d\tfrac{k_d^{(i)}}{n_d}=t_i.$$

For each d and $i=1,2$, choose $x_n^{(i)}\in\mu((k_d^{(i)},n_d),a)$. Then $\alpha_a(t_i)=\sup_d x_d^{(i)}$ for $i=1,2$. Moreover, we get that

$$\tfrac{k_d^{(1)}+k_d^{(2)}}{n_d}<\tfrac{k_{d+1}^{(1)}+k_{d+1}^{(2)}}{n_{d+1}+1},\qquad\text{and}\qquad\sup_d\tfrac{k_d^{(1)}+k_d^{(2)}}{n_d}=t_1+t_2.$$

Thus, for any sequence of elements $(y_d)_d$ with $y_d\in\mu((k_d^{(1)}+k_d^{(1)},n_d),a)$ we will get $\alpha_a(t_1+t_2)=\sup_d y_d$. However, it is easily seen that $x_d^{(1)}+x_d^{(2)}\in\mu((k_d^{(1)}+k_d^{(2)},n_d),a)$. Using (O4) at the second step, we obtain that

$$\alpha_a(t_1+t_2)=\sup_d(x_d^{(1)}+x_d^{(2)})=\sup_d x_d^{(1)}+\sup_d x_d^{(2)}=\alpha_a(t_1)+\alpha_a(t_2).$$

It is left to the reader to show that α_a preserves the sum of an element in $\mathbb{N}\subset Z$ with an element in $(0,\infty]\subset Z$.

Finally, let us show that α_a is a Cu-morphism if and only if a is compact. Assume first that α_a preserves the way-below relation. Since the unit element of Z is compact, it follows easily that

$$a = \alpha_a(1) \ll \alpha_a(1) = a.$$

For the converse, assume that a is a compact element. We need to show that for any two elements $x, y \in Z$, we have that $x \ll y$ implies that $\alpha_a(x) \ll \alpha_a(y)$. This is clear if x or y is an element in $\mathbb{N} \subset Z$. So assume that $x, y \in (0, \infty] \subset Z$.

Without loss of generality we may assume that $x < y$. Choose $\frac{k}{n} \in \mathbb{Q}_+$ and elements u, v such that

$$x \leq \tfrac{k}{n+3}, \ \tfrac{k}{n} < b, \quad u \in \mu((k, n+2), a), \ v \in \mu((k, n), a).$$

Since $a \ll a$, it follows from Lemma 7.3.6 that $u \ll v$. It also follows from Lemma 7.3.6 and the definition of α_a that $\alpha_a(x) \leq u$ and $v \leq \alpha_a(y)$. Therefore

$$\alpha_a(x) \leq u \ll v \leq \alpha_a(y),$$

as desired. $\square$

Theorem 7.3.8. *Let S be a* Cu*-semigroup. Then, the following are equivalent:*

(1) We have $S \cong Z \otimes_{\mathrm{Cu}} S$.
(2) The semigroup S has Z-multiplication.
(3) The semigroup S is almost unperforated and almost divisible.

Proof. By Proposition 7.3.3, the Cu-semiring Z is solid. Therefore, the equivalence between (1) and (2) follows from Theorem 7.1.12. Since the unit of Z is almost divisible, we obtain from Lemma 7.3.5 that every Cu-semigroup with Z-multiplication is almost unperforated and almost divisible. This shows that (2) implies (3).

To show that (3) implies (2), suppose that S is almost unperforated and almost divisible. Using Proposition 7.3.7, we define $\alpha\colon Z \times S \to S$ by $\alpha(z, a) = \alpha_a(z)$ for each $a \in S$ and $z \in Z$. We claim that α is a Cu-bimorphism.

By Proposition 7.3.7, $\alpha(\_, a)$ is a generalized Cu-morphism for each $a \in S$. For the other variable, it is also clear that $\alpha(n, \_)$ is a Cu-morphism for each $n \in \mathbb{N} \subset Z$. So let $t \in (0, \infty] \subset Z$. To show that $\alpha(t, \_)$ preserves order, let $a, b \in S$ satisfy $a \leq b$. By definition, we have

$$\alpha(t, a) = \sup \left\{ x \in \mu((k, n), a) \mid \tfrac{k}{n} < t \right\}.$$

Thus, given any $k, n \in \mathbb{N}_+$ satisfying $\frac{k}{n} < t$ and given any element $x \in \mu((k, n), a)$, we need to show that $x \leq \alpha(t, b)$. Choose $k', n' \in \mathbb{N}_+$ and an element y such that

$$\tfrac{k}{n} < \tfrac{k'}{n'+1}, \ \tfrac{k'}{n'} < t, \quad y \in \mu((k', n'), b).$$

By Lemma 7.3.6, we have $x \leq y$. Therefore

$$x \leq \sup \left\{ z \in \mu((c, d), b) \mid \tfrac{c}{d} < t \right\} = \alpha(t, b),$$

from which we deduce that $\alpha(t, a) \leq \alpha(t, b)$, as desired.

To show additivity in the second variable, let $z \in Z$ and let $a, b \in S$. Consider the following two maps from Z to S given by

$$\alpha_{a+b} = (z \mapsto \alpha(z, a+b)), \quad \alpha_a + \alpha_b = (z \mapsto \alpha(z, a) + \alpha(z, b)).$$

It is clear that both maps are generalized Cu-morphisms that send the unit of Z to the element $a + b$. By Proposition 7.3.7, the map with this property is unique, and therefore $\alpha(z, a+b) = \alpha(z, a) + \alpha(z, b)$. Analogously, one proves that $\alpha(z, \sup_n a_n) = \sup_n \alpha(z, a_n)$ for every $z \in Z$ and every increasing sequence $(a_n)_n \subset S$.

Thus, $\alpha\colon Z\times S\to S$ is a generalized Cu-bimorphism. It is clear that $\alpha(1,a)=a$ for every $a\in S$ and it is straightforward to check that

$$(*)\qquad \alpha(z_1z_2,a)=\alpha(z_1,\alpha(z_2,a)),$$

for every $z_1,z_2\in Z$ and $a\in S$.

It remains to show that for any $t_1,t_2\in Z$ with $t_1\ll t_2$ and for any $a,b\in S$ with $a\ll b$, we have that $\alpha(t_1,a)\ll\alpha(t_2,b)$. This is clear if t_1 or t_2 is an element in $\mathbb{N}\subset Z$. Thus, we consider the case that $t_1,t_2\in(0,\infty]$, and without loss of generality we may assume $t_1<t_2$. Then t_1 is necessarily finite, and $1_Z\leq t_1^{-1}t_2$ in Z. In order to show $\alpha(t_1,a)\ll\alpha(t_2,b)$, let $(x_n)_n\subset S$ be an increasing sequence with $\alpha(t_2,b)\leq\sup_n x_n$. Using this at the fifth step, and using (*) at the third step, we obtain that

$$\begin{aligned} a\ll b=\alpha(1_Z,b) &\leq \alpha(t_1^{-1}t_2,b)\\ &=\alpha(t_1^{-1},\alpha(t_2,b))\\ &\leq \alpha(t_1^{-1},\sup_n x_n)=\sup_n(\alpha(t_1^{-1},x_n)). \end{aligned}$$

Therefore, there exists an index n_0 such that $a\leq\alpha(t_1^{-1},x_{n_0})$. Then

$$\alpha(t_1,a)\leq\alpha(t_1,\alpha(t_1^{-1},x_{n_0}))=\alpha(1',x_{n_0})\leq\alpha(1,x_{n_0})=x_{n_0}.$$

Hence $\alpha(t_1,a)\ll\alpha(t_2,b)$, as desired. This finishes the proof that S has Z-multiplication. $\square$

Remark 7.3.9. The Toms-Winter conjecture, see [**TW09**, Remarks 3.5] and [**Win12**, Conjecture 0.1], predicts that for every unital, separable, simple, nonelementary, nuclear C^*-algebra A, the following conditions are equivalent:

(1) The algebra A is $\mathcal{Z}$-stable, that is, we have $A\cong\mathcal{Z}\otimes A$.
(2) The Cuntz semigroup $\mathrm{Cu}(A)$ is almost unperforated.
(3) The algebra A has finite nuclear dimension.

We can interpret Theorem 7.3.8 as the verification of the Cu-semigroup version of the Toms-Winter conjecture. The analog of '$\mathcal{Z}$-stability' for a Cu-semigroup S is the property that $S\cong Z\otimes_{\mathrm{Cu}}S$, which is (1) of Theorem 7.3.8. The second condition of the Toms-Winter conjecture is already formulated for Cu-semigroups. However, in Theorem 7.3.8(3) we not only require that the Cu-semigroup be almost unperforated but also almost divisible. We remark that not every Cuntz semigroup of a simple C^*-algebra is almost divisible, see [**DHTW09**]. On the other hand, it seems possible that the Cuntz semigroup of a simple C^*-algebra is automatically almost divisible whenever it is almost unperforated. Indeed, if the Toms-Winter conjecture holds true, then this would be a consequence for at least the class of nuclear C^*-algebras.

It is not clear what the analog of condition (3) of the Toms-Winter conjecture for Cu-semigroups should be. This would entail to define nuclearity and dimension concepts for Cu-semigroups, which is not pursued here.

The following problem asks if there is an analog of Theorems 7.2.6, 7.4.11 and 7.5.11 for tensor products with Z.

Problem 7.3.10. Let S be a Cu-semigroup, and let $a,b\in S$. Characterize when $1\otimes a\leq 1\otimes b$ in $Z\otimes_{\mathrm{Cu}}S$.

Problem 7.3.11. When does axiom (O5), (O6) or weak cancellation pass from a Cu-semigroup S to the tensor product $Z\otimes_{\mathrm{Cu}}S$?

7.3.12. In general, axiom (O5) does not pass to tensor products with Z, see Proposition 6.4.4. We have that Z satisfies (O5), (O6) and weak cancellation itself. Therefore, if S is inductive limit of simplicial Cu-semigroups, then $Z \otimes_{\mathrm{Cu}} S$ satisfies the three axioms as well, seeProposition 6.4.6.

It seems likely that Problem 7.3.11 has a positive answer if S is assumed to be algebraic.

We end this section with some structure results about Cu-semigroups with Z-multiplication. For the next result, recall that $1' \in Z$ denotes the soft 'one'.

PROPOSITION 7.3.13. *Let S be a* Cu*-semigroup with Z-multiplication. Then:*

(1) An element $a \in S$ is soft if and only if $a = 1'a$.
(2) For every functional $\lambda \in F(S)$ and every $a \in S$, we have $\lambda(a) = \lambda(1'a)$.

PROOF. Let us first show (1). So let $a \in S$. To prove that $1'a$ is soft, let $a' \in S$ satisfy $a' \ll a$. We need to show that $a' <_s 1'a$. Consider the increasing sequence $(\frac{k-1}{k})_k$ of noncompact elements in $(0, \infty] \subset Z$. Since $1' = \sup_k \frac{k-1}{k}$ in Z, we get that

$$a' \ll 1'a = \sup_k \tfrac{k-1}{k} a.$$

Thus, there exists $n \in \mathbb{N}$ such that $a' \leq \frac{n-1}{n} a$. It is easy to verify that $\frac{n-1}{n} a <_s 1'a$. It follows that $a' <_s 1'a$, as desired

Conversely, assume that $a \in S$ is a soft element. It is clear that $1'a \leq a$. To show the converse inequality, it is enough to show that $a' \leq 1'a$ for every $a' \in S$ satisfying $a' \ll a$. Given such a', it follows from softness of a that there exists $k \in \mathbb{N}$ such that $(k+1)a' \leq ka$. Using this at the third step, we obtain that

$$a' \leq \tfrac{k+1}{k} a' = \tfrac{1}{k}((k+1)a') \leq \tfrac{1}{k}(ka) = 1'a,$$

as desired.

To show (2), let $\lambda \in F(S)$ and $a \in S$. The Cu-semigroup $[0, \infty]$ has a Z-multiplication. Since λ is a generalized Cu-morphism from S to $[0, \infty]$, it follows from Proposition 7.1.6 that λ is Z-linear. Using this at the first step, and using that every element of $[0, \infty]$ is soft, we deduce that

$$\lambda(1's) = 1'\lambda(s) = \lambda(s),$$

as desired. $\square$

The following result provides a partial answer to Problem 5.3.14.

PROPOSITION 7.3.14. *Let S be a* Cu*-semigroup with Z-multiplication. Then the subsemigroup S_{soft} of soft elements is a* Cu*-semigroup. If S satisfies (O5) ((O6), weak cancellation), then so does S_{soft}.*

PROOF. By Theorem 5.3.11, S_{soft} is a subsemigroup of S that is closed under suprema of increasing sequences. This show that S_{soft} satisfies (O1).

Claim 1: For every $a \in S_{\mathrm{soft}}$ there exists an increasing sequence $(a_k)_k \subset S_{\mathrm{soft}}$ such that $a = \sup_k a_k$ and such that $a_k \ll a_{k+1}$ in S for each k.

To show this claim, let $a \in S_{\mathrm{soft}}$. Since S satisfies (O2), there exists a rapidly increasing sequence $(s_k)_k \subset S$ such that $a = \sup_k s_k$. For each $k \in \mathbb{N}_+$, set

$$a_k = \tfrac{k-1}{k} s_k.$$

It is easy to check that $1'a_k = a_k$, which by Proposition 7.3.13 implies that $a_k \in S_{\mathrm{soft}}$. Moreover, for each k we have that $\frac{k-1}{k} \ll \frac{k}{k+1}$ in Z and that $s_k \ll s_{k+1}$ in S. Since the Z-multiplication on S is given by a Cu-bimorphism, we obtain that

$$a_k = \tfrac{k-1}{k} s_k \ll \tfrac{k}{k+1} s_{k+1} = a_{k+1},$$

for each k. It is clear that $a = \sup_k a_k$, which finished the proof of the claim.

The same argument which showed claim 3 in the proof of Proposition 5.3.18 implies the following: For every two elements $a, b \in S_{\mathrm{soft}}$ satisfying $a \ll b$ in S, we have that $a \ll b$ in S_{soft}. Together with claim 1, this verifies (O2) for S_{soft}. Then, axioms (O3) and (O4) for S_{soft} follow from their counterparts in S.

Next, let us show that S_{soft} satisfies (O5), assuming that S does. So let a', a, b', b and c be elements in S_{soft} that satisfy

$$a + b \leq c, \quad a' \ll a, \quad b' \ll b.$$

Using that S satisfies (O5), there exists $x \in S$ such that

$$a' + x \leq c \leq a + x, \quad b' \leq x.$$

Set $y = 1'x$, which by Proposition 7.3.13 is an element in S_{soft}. We claim that y has the desired properties to verify (O5) for S_{soft}. Indeed, using Proposition 7.3.13 again, we have that $a' = 1'a'$, $a = 1'a$, $c = 1'c$ and $b' = 1'b'$. Therefore

$$a' + y = 1'(a' + x) \leq 1'c = c \leq 1'(a + x) = a + y, \quad b' = 1'(b') \leq 1'(x) = y,$$

as desired.

In the same way, one shows that S_{soft} inherits (O6) from S. Finally, it is straightforward to check that S_{soft} is weakly cancellative whenever S is. $\square$

7.4. The rationalization of a semigroup

In this section, we study Cu-semigroups that are semimodules over the Cu-semiring of a strongly self-absorbing UHF-algebra. Given a supernatural number q satisfying $q^2 = q$ and $q \neq 1$, we let M_q be the associated UHF-algebra, see Paragraph 7.4.2. We use R_q to denote $\mathrm{Cu}(M_q)$. In Proposition 7.4.4, we show that R_q is a solid Cu-semiring.

In Definition 7.4.6, we recall the natural notions of q-unperforation and q-divisibility for semigroups. The main result of this section is Theorem 7.4.10, where we characterize the Cu-semimodules over R_q as the Cu-semigroups that are q-unperforated and q-divisible.

In Proposition 7.4.14, we apply the results to the Cuntz semigroup of a C^*-algebra A. In particular, we obtain that $\mathrm{Cu}(M_q \otimes A) \cong \mathrm{Cu}(A)$ if and only if $\mathrm{Cu}(A)$ is q-divisible and q-unperforated. We also deduce that the Cuntz semigroup of a C^*-algebra A is nearly unperforated whenever A tensorially absorbs a strongly self-absorbing UHF-algebra, see Corollary 7.4.15. This verifies Conjecture 5.6.18 in that case.

7.4.1. A *supernatural number* q is a formal product

$$q = \prod_{k \in \mathbb{N}} p_k^{n_k},$$

where $p_0, p_1, p_2, \ldots$ is an enumeration of all prime numbers and where each n_k is a number in $\{0, 1, 2, \ldots, \infty\}$ that denotes the multiplicity with which the prime p_k occurs in q. By definition, zero is not a supernatural number. Given supernatural numbers $q = \prod_k p_k^{m_k}$ and $r = \prod_k p_k^{n_k}$, their (formal) product is given by $qr = \prod_k p_k^{m_k + n_k}$. Analogously one can naturally define the product of infinitely many (super)natural numbers $\prod_{n \in \mathbb{N}} q_n$ in the obvious way. If $q = q^2$, then each n_k is either 0 or ∞.

We identify the nonzero natural numbers with the supernatural numbers of the form $\prod_{k \in \mathbb{N}} p_k^{n_k}$ where $\sum_{k \in \mathbb{N}} n_k < \infty$. In particular, the number 'one' is the supernatural number $\prod_{k \in \mathbb{N}} p_k^{n_k}$ where each n_k is zero.

Let q be a supernatural number satisfying $q = q^2$. We write $\mathbb{Z}\left[\frac{1}{q}\right]$ for the ring obtained by inverting in $\mathbb{Z}$ all primes that divide q, that is:

$$\mathbb{Z}\left[\tfrac{1}{q}\right] = \mathbb{Z}\left[\left\{\tfrac{1}{p} \mid p \text{ prime}, p|q\right\}\right].$$

Then, we let K_q denote the subsemiring of nonnegative numbers in $\mathbb{Z}\left[\frac{1}{q}\right]$, that is:

$$K_q = \mathbb{Q}_+ \cap \mathbb{Z}\left[\tfrac{1}{q}\right].$$

For example, we have

$$K_1 = \mathbb{N} = \{0, 1, 2, \ldots\}, \quad K_{2^\infty} = \mathbb{Q}_+ \cap \mathbb{Z}\left[\tfrac{1}{2}\right] = \mathbb{N}\left[\tfrac{1}{2}\right].$$

Then K_q is a unital subsemiring of $\mathbb{Q}_+$, and all unital subsemirings of $\mathbb{Q}_+$ arise this way.

For the rest of the paragraph, we fix a supernatural number q satisfying $q = q^2$ and $q \neq 1$. We equip K_q with the natural algebraic order. Recall that, for a positively ordered monoid M, we denote by $\mathrm{Cu}(M)$ the Cu-completion of the PreW-semigroup $(M, \leq)$, see Paragraph 5.5.3. Then, we define

$$R_q = \mathrm{Cu}(K_q).$$

It follows from the results about algebraic Cu-semigroups in Section 5.5 that R_q is a weakly cancellative Cu-semigroup satisfying (O5) and (O6), and whose submonoid of compact elements is canonically identified with K_q. It is then straightforward to check that there is a decomposition of R_q as

$$R_q = K_q \sqcup (0, \infty],$$

where $K_q \subset R_q$ are the compact elements in R_q, and where $(0, \infty] \subset R_q$ is the submonoid of nonzero soft elements in R_q.

Using that K_q is a semiring, we can define a product on R_q. In Paragraph 8.2.1, this construction will be carried out in greater generality. Here, we only consider the concrete case of R_q.

The order and semiring-structure of R_q are so that the inclusion of K_q in R_q and the inclusion of $(0, \infty]$ in R_q are order-embeddings and semiring-homomorphisms. We let $\iota\colon K_q \to [0, \infty]$ be the natural inclusion map. Let $a \in K_q$ and $t \in (0, \infty]$. Then their sum in R_q is given as $a + t = \iota(a) + t \in (0, \infty]$. If $a = 0$, then $at = 0 \in K_q$. If a is nonzero, then the product of a and t in R_q is given as $at = \iota(a)t \in (0, \infty]$. Moreover, we have $a \leq t$ in R_q if and only if $\iota(a) < t$, and we have $t \leq a$ in R_q if and only if $t \leq \iota(a)$.

Thus, the submonoid of soft elements in R_q is additively and multiplicatively absorbing. It is straightforward to check directly that the product on R_q is a Cu-bimorphism and that the unit element of K_q is also a unit for R_q. This gives R_q the structure of a Cu-semiring.

7.4.2. Given a nonzero supernatural number q, one associates a UHF-algebra M_q as follows: If q is finite, then M_q denotes the C^*-algebra of q by q matrices. If q is infinite, then we choose a sequence $n_0, n_1, n_2, \ldots$ of prime numbers such that q is equal to the product $\prod_{k=0}^{\infty} n_k$. Then, we set

$$M_q = \bigotimes_{k=0}^{\infty} M_{n_k}.$$

The isomorphism type of M_q does not depend on the choice of sequence $(n_k)_k$.

Let q_1 and q_2 be two nonzero supernatural numbers. Then $q_1 = q_2$ if and only if $M_{q_1} \cong M_{q_2}$. Moreover,

$$M_{q_1} \otimes M_{q_2} \cong M_{q_1 q_2}.$$

The UHF-algebra M_q is said to be of *infinite type* if $M_q \cong M_q \otimes M_q$ and $M_q \neq \mathbb{C}$. Equivalently, we have $q = q^2$ and $q \neq 1$. It is known that every UHF-algebra of infinite type is strongly self-absorbing, see [**TW07**].

PROPOSITION 7.4.3. *Let q be a supernatural number satisfying $q = q^2$ and $q \neq 1$. Then*

$$R_q \cong \mathrm{Cu}(M_q).$$

PROOF. It is well-known that M_q is a unital, separable, simple, $\mathcal{Z}$-stable C^*-algebra with stable rank one and unique tracial state. The K_0-group of M_q is isomorphic to $\mathbb{Z}\left[\frac{1}{q}\right]$. Since M_q has stable rank one, the positive part of the ordered K_0-group is naturally isomorphic with $V(M_q)$. We therefore have that $V(M_q) = \mathbb{Q}_+ \cap \mathbb{Z}\left[\frac{1}{q}\right] = K_q$. Then it follows from Proposition 7.3.1 that

$$\mathrm{Cu}(M_q) \cong V(M_q) \sqcup (0, \infty] \cong K_q \sqcup (0, \infty] = R_q,$$

as desired. $\square$

PROPOSITION 7.4.4. *Let q be a supernatural number satisfying $q = q^2$ and $q \neq 1$. Then R_q is a solid* Cu*-semiring.*

PROOF. By Proposition 7.1.6, it is enough to show that $1 \otimes a = a \otimes 1$ for every $a \in R_q$. If a is a nonzero, compact element in R_q, then there are $k, n \in \mathbb{N}_+$ such that $n|q$ and $a = \frac{k}{n}$. Then, using only compact elements, we have

$$1 \otimes a = \tfrac{n}{n} \otimes \tfrac{k}{n} = (nk)\tfrac{1}{n} \otimes \tfrac{1}{n} = \tfrac{k}{n} \otimes \tfrac{n}{n} = a \otimes 1.$$

For a soft element in R_q, one can apply the same argument that was used in Example 7.1.7 to show that $[0, \infty]$ is solid. $\square$

We remark that a more general result will be proved in Theorem 8.3.13.

The following result follows by combining the observations in Paragraph 7.4.2 with Proposition 7.4.3 and Proposition 6.4.13.

PROPOSITION 7.4.5. *Let q and r be supernatural numbers satisfying $q = q^2 \neq 1$ and $r = r^2 \neq 1$. Then $R_q \otimes_{\mathrm{Cu}} R_r \cong R_{qr}$.*

The concepts of n-unperforation and n-divisibility of a positively ordered monoid are well-known for a natural number n. The following definition is a straightforward generalization to supernatural numbers.

DEFINITION 7.4.6. Let S be a positively ordered monoid, and let q be a nonzero supernatural number. We say that S is *q-unperforated* if for every finite number n dividing q, and for every two elements $a, b \in S$ we have $a \leq b$ whenever $na \leq nb$.

We say that S is *q-divisible* if for every finite number n dividing q, and for every $a \in S$, there exists $x \in S$ such that $a = nx$.

REMARKS 7.4.7. Let S be a positively ordered monoid, and let n be a nonzero natural number. We let $\mu_n \colon S \to S$ be the map that multiplies each element in S by n.

(1) We have that S is n-divisible if and only if μ_n is surjective.

(2) We have that S is n-unperforated if and only if μ_n is an order-embedding.

(3) Let q be an nonzero supernatural number, and let q^∞ denote its infinite product with itself. Then S is q-divisible if and only if S is q^∞-divisible. Similarly, S is q-unperforated if and only if S is q^∞-unperforated.

(4) Let q_∞ be the largest supernatural number, for which each prime has infinite multiplicity. A positively ordered monoid S is *divisible* if it is n-divisible for every $n \in \mathbb{N}_+$, which is equivalent to being q_∞-divisible. Similarly, S is *unperforated* if it is n-unperforated for every $n \in \mathbb{N}_+$, or, equivalently, if if is q_∞-unperforated.

LEMMA 7.4.8. *Let S be a positively ordered monoid, and let q be a supernatural number with $q \neq 1$.*

(1) If S is q-divisible, then S is almost divisible.

(2) If S is q-unperforated, then S is nearly unperforated and therefore also almost unperforated.

PROOF. To show the first statement, assume that S is a q-divisible, positively ordered monoid. Let $a \in S$ and let $n \in \mathbb{N}_+$. We need to find $x \in S$ such that $nx \leq a \leq (n+1)x$.

Choose a number $d \geq 2$ that divides q. Since the set $\left\{ \frac{r}{d^k} \mid r, k \in \mathbb{N}_+ \right\}$ is dense in $\mathbb{Q}_+$, we can find r and k in $\mathbb{N}_+$ such that $\frac{1}{n+1} < \frac{r}{d^k} < \frac{1}{n}$. Since S is d-divisible, there exists $x \in S$ such that $d^k x = a$. Then

$$n(rx) \leq d^k x = a \leq (n+1)(rx),$$

which shows that the element rx has the desired properties to verify that S is almost divisible.

To prove the second statement, assume that S is a q-unperforated, positively ordered monoid. Choose a number $d \geq 2$ that divides q. To show that S is nearly unperforated, let a and b be two elements in S satisfying $a \leq_p b$. This means that there exists $n_0 \in \mathbb{N}$ such that $na \leq nb$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Choose $k \in \mathbb{N}_+$ such that $d^k \geq n_0$. Then $d^k a \leq d^k b$. As observed in Remarks 7.4.7, we have that S is d^k-unperforated. Thus, we obtain that $a \leq b$, as desired.

We have seen in Proposition 5.6.3 that near unperforation implies almost unperforation in general. □

LEMMA 7.4.9. *Let S be a* Cu*-semimodule over the* Cu*-semiring R, and let q be a supernatural number with $q \neq 1$. Assume that the unit element of R is q-divisible. Then S is q-unperforated and q-divisible. In particular, R itself is q-unperforated and q-divisible.*

PROOF. The proof is analogous to that of Lemma 7.3.5 and is left to the reader. □

THEOREM 7.4.10. *Let S be a* Cu*-semigroup, and let q be an supernatural number satisfying $q = q^2$ and $q \neq 1$. Then, the following are equivalent:*

(1) We have $S \cong R_q \otimes_{\mathrm{Cu}} S$.

(2) The Cu*-semigroup S has R_q-multiplication.*

(3) The Cu*-semigroup S is q-divisible and q-unperforated.*

PROOF. By Proposition 7.4.4, the Cu-semiring R_q is solid. Therefore, the equivalence between (1) and (2) follows from Theorem 7.1.12. The unit of R_q is clearly q-divisible. Therefore, it follows from Lemma 7.4.9 that (2) implies (3).

Finally, let us show that (3) implies (2). So suppose that S is n-divisible and n-unperforated for every $n \in \mathbb{N}_+$ that divides q. It follows from Lemma 7.4.8 and Theorem 7.3.8 that S has Z-multiplication. By Lemma 7.1.11, it is enough to define a generalized Cu-bimorphism

$$\varphi\colon R_q \times S \to S$$

such that $\varphi(1, a) = a$ for each $a \in S$. Recall that $R_q = K_q \sqcup (0, \infty]$, where K_q is a unital subsemiring of $\mathbb{Q}_+$. For $r \in (0, \infty] \subset R_q$ we use the Z-multiplication on S to define $\varphi(r, \_)$. So let $r \in K_q$. Then there exist unique coprime integers $n, k \in \mathbb{N}_+$ such that $r = \frac{k}{n}$ and n divides q. Consider the map $\mu_n\colon S \to S$ that multiplies each element in S by n. Since S is n-divisible and n-unperforated, the map μ_n is an PoM-isomorphism and therefore a Cu-isomorphism. Given $a \in S$, we set

$$\varphi(r, a) = k\mu_n^{-1}(a).$$

It is now straightforward to check that φ is a Cu-bimorphism. It is also clear that $\varphi(1,a)=a$ for each $a\in S$. Therefore, we may apply Lemma 7.1.11 to deduce that S has R_q-multiplication. □

THEOREM 7.4.11. *Let S be a* Cu*-semigroup, let $a,b\in S$, and let q be a supernatural number satisfying $q=q^2$ and $q\neq 1$. Then, the following are equivalent:*

(1) We have $1\otimes a\leq 1\otimes b$ in $R_q\otimes_{\mathrm{Cu}} S$.
(2) For each $a'\in S$ satisfying $a'\ll a$, there exists $n\in\mathbb{N}_+$ dividing q such that $na'\leq nb$ in S.

PROOF. First, let us show that (1) implies (2). By definition, R_q is the Cu-completion of the (algebraically ordered) W-semigroup $(K_q,\leq)$. By Theorem 6.3.5, we have that

$$R_q\otimes_{\mathrm{Cu}} S=\gamma\left(K_q\otimes_{\mathrm{PreW}} S\right).$$

Let $\alpha\colon K_q\otimes_{\mathrm{PreW}} S\to R_q\otimes_{\mathrm{Cu}} S$ denote the universal Cu-morphism of the Cu-completion. The underlying positively ordered monoid of $K_q\otimes_{\mathrm{PreW}} S$ is

$$K_q\otimes_{\mathrm{PoM}} S=\mathbb{N}\left[\tfrac{1}{q}\right]\otimes_{\mathrm{PoM}} S.$$

Then, given two elements $x,y\in S$, it is easy to see that $1\otimes x\leq 1\otimes y$ in $K_q\otimes_{\mathrm{PoM}} S$ if and only if there exists a natural number n dividing q such that $nx\leq ny$ in S. Now, let $a,b\in S$ satisfy $1\otimes a\leq 1\otimes b$ in $R_q\otimes_{\mathrm{Cu}} S$, and let $a'\in S$ satisfy $a'\ll a$. Using at the second step that the unit of R_q is a compact element, it follows that

$$\alpha(1\otimes a')=1\otimes a'\ll 1\otimes b=\alpha(1\otimes b)$$

in $R_q\otimes_{\mathrm{Cu}} S$. By properties of the Cu-completion, we deduce that $1\otimes a'\prec 1\otimes b$ in $K_q\otimes_{\mathrm{PreW}} S$. Hence, $1\otimes a'\leq 1\otimes b$ in $K_q\otimes_{\mathrm{PoM}} S$. As observed above, this implies that there exists $n\in\mathbb{N}_+$ dividing q such that $na'\leq nb$ in S. This verifies (2).

Next, let us show that (2) implies (1). Choose a rapidly increasing sequence $(a_k)_k\subset S$ such that $a=\sup_k a_k$. By assumption, for each k there exists $n_k\in\mathbb{N}_+$ that divides q and such that $n_ka_k\leq n_kb$. Since n_k is divisible in R_q, we deduce that

$$1\otimes a_k=\left(n_k\tfrac{1}{n_k}\right)\otimes a_k=\tfrac{1}{n_k}\otimes(n_ka_k)\leq\tfrac{1}{n_k}\otimes(n_kb)=\left(n_k\tfrac{1}{n_k}\right)\otimes b=1\otimes b.$$

Since this holds for each k, and since $1\otimes a=\sup_k\left(1\otimes a_k\right)$ in $R_q\otimes_{\mathrm{Cu}} S$, we obtain that $1\otimes a\leq 1\otimes b$, as desired. □

PROPOSITION 7.4.12. *Let q be a supernatural number satisfying $q=q^2$ and $q\neq 1$. Let $(d_k)_{k\in\mathbb{N}}$ be a sequence of natural numbers such that $q=\prod_{k\in\mathbb{N}}d_k$. Then R_q is isomorphic to the limit of the following inductive system of simplicial* Cu*-semigroups:*

$$\overline{\mathbb{N}}\xrightarrow{d_0}\overline{\mathbb{N}}\xrightarrow{d_1}\overline{\mathbb{N}}\xrightarrow{d_2}\ldots.$$

Consequently, if we have given a Cu*-semigroup S, then $R_q\otimes_{\mathrm{Cu}} S$ is isomorphic to the limit of the inductive system*

$$S\xrightarrow{d_0}S\xrightarrow{d_1}S\xrightarrow{d_2}\ldots$$

PROOF. Consider the following inductive system, where the map at the k-th step is multiplication by d_k:

$$\mathbb{N}\xrightarrow{d_0}\mathbb{N}\xrightarrow{d_1}\mathbb{N}\xrightarrow{d_2}\ldots.$$

It is straightforward to check that the inductive limit of this system in PoM is $\mathbb{N}\left[\frac{1}{q}\right]$. If we endow $\mathbb{N}$ and $\mathbb{N}\left[\frac{1}{q}\right]$ with auxiliary relations equal to their partial order, then we also have that

$$\mathbb{N}\left[\tfrac{1}{q}\right] \cong \text{W-}\varinjlim\left(\mathbb{N} \xrightarrow{d_0} \mathbb{N} \xrightarrow{d_1} \mathbb{N} \xrightarrow{d_2} \dots\right).$$

Applying the reflection functor $\gamma\colon \mathrm{PreW} \to \mathrm{Cu}$, and using also Corollary 3.1.11, we obtain that

$$\begin{aligned} R_q \cong \gamma(\mathbb{N}\left[\tfrac{1}{q}\right]) &\cong \gamma\left(\text{W-}\varinjlim\left(\mathbb{N} \xrightarrow{d_0} \mathbb{N} \xrightarrow{d_1} \mathbb{N} \xrightarrow{d_2} \dots\right)\right) \\ &\cong \text{Cu-}\varinjlim\left(\overline{\mathbb{N}} \xrightarrow{d_0} \overline{\mathbb{N}} \xrightarrow{d_1} \overline{\mathbb{N}} \xrightarrow{d_2} \dots\right), \end{aligned}$$

as desired. The result for $R_q \otimes_{\mathrm{Cu}} S$ follows from the limit presentation for R_q in combination with Proposition 6.4.1. $\square$

Corollary 7.4.13. *Let S be a* Cu*-semigroup, and let q be a supernatural number satisfying $q = q^2$ and $q \neq 1$. If S satisfies (O5) (resp. (O6), weak cancellation), then so does $S \otimes R_q$.*

Proof. By Proposition 6.4.6, each of the axioms (O5), (O6) and weak cancellation is preserved by taking the tensor product with a Cu-semigroup that is an inductive limit of simplicial Cu-semigroups. Therefore, the result follows from Proposition 7.4.12. $\square$

Proposition 7.4.14. *Let A be a C^*-algebra, and let q be a supernatural number satisfying $q = q^2$ and $q \neq 1$. Then there are natural isomorphisms*

$$\mathrm{Cu}(M_q \otimes A) \cong \mathrm{Cu}(M_q) \otimes_{\mathrm{Cu}} \mathrm{Cu}(A) \cong R_q \otimes_{\mathrm{Cu}} \mathrm{Cu}(A).$$

In particular, we have $\mathrm{Cu}(A \otimes M_q) \cong \mathrm{Cu}(A)$ *if and only if* $\mathrm{Cu}(A)$ *is q-unperforated and q-divisible.*

Proof. The isomorphism on the left follows from Proposition 6.4.13 since M_q is an AF-algebra. By Proposition 7.4.3, we have $\mathrm{Cu}(M_q) \cong R_q$, which gives the isomorphism on the right. $\square$

Corollary 7.4.15. *Let A be a C^*-algebra. If A tensorially absorbs a UHF algebra of infinite type, then* $\mathrm{Cu}(A)$ *is nearly unperforated.*

Proof. Let q be a supernatural number such that $q^2 = q$ and $q \neq 1$ and $A \cong M_q \otimes A$. By Proposition 7.4.14, we have that $\mathrm{Cu}(A)$ is q-unperforated. Then, it follows from Lemma 7.4.8 that $\mathrm{Cu}(A)$ is nearly unperforated. $\square$

7.5. The realification of a semigroup

In this section, we study Cu-semigroups that are semimodules over the Cu-semiring $[0,\infty]$. We have already shown in Example 7.1.7 that $[0,\infty]$ is a solid Cu-semiring. It is also known that $[0,\infty]$ is the Cuntz semigroup of a C^*-algebra, called the Jacelon-Razak algebra $\mathcal{R}$, see Remark 7.5.1.

In Theorem 7.5.4, we characterize the Cu-semimodules over $[0,\infty]$ as the Cu-semigroups that are unperforated, divisible and that contain only soft elements. We observe in Remark 7.5.2 that a Cu-semigroup has $[0,\infty]$-multiplication if and only if it has 'real multiplication' in the sense of Robert, [**Rob13a**]. Given a Cu-semigroup S, Robert defines a 'realification' S_R, which is a Cu-semigroup with real multiplication satisfying a natural universal property. In [**Rob13a**, Remark 3.1.5], Robert suggests that the realification of a Cu-semigroup can be considered as the tensor product of S with $[0,\infty]$. We verify this in Proposition 7.5.9.

REMARK 7.5.1. The Cu-semiring $[0,\infty]$ is the Cuntz semigroup of the stably projectionless C^*-algebra known as the Jacelon-Razak algebra. This algebra has been studied in [**Jac13**], where it is denoted by $\mathcal{W}$. Following Robert, we denote the Jacelon-Razak algebra by $\mathcal{R}$, see [**Rob13a**].

Using the result in [**Jac13**], the Cuntz semigroup of $\mathcal{R}$ was computed by Robert, [**Rob13a**, § 5] as

$$\mathrm{Cu}(\mathcal{R}) \cong [0,\infty].$$

REMARK 7.5.2. Let S be a Cu-semigroup. In [**Rob13a**, Definition 3.1.2], Robert defines S to have *real multiplication* if there exists a map

$$(0,\infty] \times S \to S, \quad (t,a) \mapsto t \cdot a, \quad (t \in (0,\infty], a \in S)$$

that preserves addition, order and suprema of increasing sequences in each variable, and such that $1 \cdot a = a$ for every $a \in S$. It is clear that such a map extends uniquely to a generalized Cu-bimorphism

$$\varphi\colon [0,\infty] \times S \to S,$$

satisfying $\varphi(1,a) = a$ for each $a \in S$. As observed in Example 7.1.7, we have that $[0,\infty]$ is a solid Cu-semiring. Thus, we may apply Lemma 7.1.11 to deduce that S has a $[0,\infty]$-multiplication in the sense of Definition 7.1.3.

To summarize, a Cu-semigroup has real multiplication in the sense of Robert if and only if it is a Cu-semimodule over the solid Cu-semiring $[0,\infty]$.

LEMMA 7.5.3. *Let S be a* Cu-*semigroup with Z-multiplication. Then the map*

$$\varphi\colon [0,\infty] \times S \to S, \quad (t,a) \mapsto t \cdot a, \quad (t \in [0,\infty], a \in S)$$

is a Cu-*bimorphism that induces an isomorphism*

$$\bar{\varphi}\colon [0,\infty] \otimes_{\mathrm{Cu}} S \xrightarrow{\cong} S_{\mathrm{soft}}.$$

PROOF. The map φ is a restriction of the Cu-bimorphism defining the Z-multiplication. Therefore, φ is a Cu-bimorphism. By Proposition 7.3.14, we have that the submonoid S_{soft} of soft elements in S is a Cu-semigroup. Moreover, by Proposition 7.3.13, an element a in S is soft if and only if $1'a = a$, where $1'$ denotes the 'one' in the submonoid $[0,\infty] = Z_{\mathrm{soft}} \subset Z$ of soft elements in Z. Therefore, $\bar{\varphi}$ maps into S_{soft}. We define a map, which will turn out to be the inverse of $\bar{\varphi}$, as follows:

$$\psi\colon S_{\mathrm{soft}} \to [0,\infty] \otimes_{\mathrm{Cu}} S, \quad a \mapsto 1' \otimes a. \quad (a \in S_{\mathrm{soft}})$$

It is easy to see that ψ is a generalized Cu-morphism and that $\bar{\varphi} \circ \psi$ is the identity on S_{soft}. Let $t \in [0,\infty]$ and $a \in S$. Using that $[0,\infty]$ and S have Z-multiplication, it follows from Lemma 7.1.10 (and also Corollary 7.1.8) that $1' \otimes (t \cdot a) = t \otimes a$ in $[0,\infty] \otimes_{\mathrm{Cu}} S$. Using this at the third step, we deduce that

$$\psi \circ \varphi(t,a) = \psi(t \cdot a) = 1' \otimes (t \cdot a) = t \otimes a.$$

This implies that $\psi \circ \bar{\varphi}$ is the identity on $[0,\infty] \otimes_{\mathrm{Cu}} S$, and hence $\bar{\varphi}$ is an isomorphism, as desired. □

THEOREM 7.5.4. *Let S be a* Cu-*semigroup. Then, the following are equivalent:*

(1) We have $S \cong [0,\infty] \otimes_{\mathrm{Cu}} S$.
(2) The semigroup S has $[0,\infty]$-multiplication.
(3) The semigroup S is almost unperforated and almost divisible, and every element of S is soft.
(4) The semigroup S is unperforated and divisible, and every element of S is soft.

Proof. Since $[0,\infty]$ is a solid Cu-semiring, the equivalence between (1) and (2) follows from Theorem 7.1.12. It is clear that (4) implies (3). Let us show that (2) implies (4). To show that S is unperforated, let $a,b\in S$ such that $na\leq nb$ for some $n\in\mathbb{N}_+$. Since in $[0,\infty]$ we have $1=\frac{1}{n}n$, we obtain that

$$a=(\tfrac{1}{n}n)\cdot a=\tfrac{1}{n}\cdot(na)\leq\tfrac{1}{n}\cdot(nb)=b.$$

It is also clear that S is divisible.

Next, let us show that (3) implies (1). By Theorem 7.3.8, we have that S has Z-multiplication. Using Lemma 7.5.3 to obtain the first isomorphism, and using the assumption for the second equality, we obtain that

$$[0,\infty]\otimes_{\mathrm{Cu}}S\cong S_{\mathrm{soft}}=S,$$

as desired. □

Lemma 7.5.5. *Let S be a* Cu*-semigroup. Then there is a natural isomorphism*

$$[0,\infty]\otimes_{\mathrm{Cu}}S\cong(R_{2^\infty}\otimes_{\mathrm{Cu}}S)_{\mathrm{soft}}.$$

Proof. Since R_{2^∞} has Z-multiplication, it follows from Lemma 7.1.10 that $R_{2^\infty}\otimes_{\mathrm{Cu}}S$ has Z-multiplication. Then, using Lemma 7.5.3 to obtain the last isomorphism, and using that $[0,\infty]\cong[0,\infty]\otimes_{\mathrm{Cu}}R_{2^\infty}$ at the first step (which follows from a second usage of Lemma 7.5.3), we obtain that

$$\begin{aligned}[0,\infty]\otimes_{\mathrm{Cu}}S&\cong([0,\infty]\otimes_{\mathrm{Cu}}R_{2^\infty})\otimes_{\mathrm{Cu}}S\\&\cong[0,\infty]\otimes_{\mathrm{Cu}}(R_{2^\infty}\otimes_{\mathrm{Cu}}S)\cong(R_{2^\infty}\otimes_{\mathrm{Cu}}S)_{\mathrm{soft}},\end{aligned}$$

as desired. □

Proposition 7.5.6. *Let S be a* Cu*-semigroup. If S satisfies (O5) (resp. (O6), weak cancellation), then so does $[0,\infty]\otimes_{\mathrm{Cu}}S$.*

Proof. Assume that S is a Cu-semigroup satisfying (O5). By Corollary 7.4.13, the tensor product $R_{2^\infty}\otimes_{\mathrm{Cu}}S$ satisfies (O5). Since $R_{2^\infty}\otimes_{\mathrm{Cu}}S$ has Z-multiplication, it follows from Proposition 7.3.14 that the subsemigroup of soft elements in the tensor product $R_{2^\infty}\otimes_{\mathrm{Cu}}S$ is a Cu-semigroup satisfying (O5). Now the desired result follows from Lemma 7.5.5. It is proved analogously that axiom (O6) and weak cancellation pass from S to $[0,\infty]\otimes_{\mathrm{Cu}}S$. □

7.5.7. Let S be a Cu-semigroup. Recall from Paragraph 5.2.1, that there is a natural map

$$S\to\mathrm{Lsc}(F(S)),\quad a\mapsto\hat{a},\quad(a\in S)$$

where $F(S)$ is the cone of functionals on S.

In [**Rob13a**, § 3.1], Robert defines the *realification* of S as the smallest subsemigroup of $\mathrm{Lsc}(F(S))$ that is closed under passing to suprema of increasing sequences, and which contains all elements of the form $\frac{1}{n}\hat{a}$ for some $n\in\mathbb{N}_+$ and some $a\in S$. We denote the realification of S by S_R. In [**Rob13a**, Proposition 3.1.1], it is shown that S_R is a Cu-semigroup. Moreover, if S satisfies (O5'), the original version of the almost algebraic order axiom, then so does S_R. In Proposition 7.5.9, we show that S_R is naturally isomorphic to $[0,\infty]\otimes_{\mathrm{Cu}}S$. Then, it follows from Proposition 7.5.6 that S_R satisfies (O5) whenever S does.

Lemma 7.5.8. *Let S be a* Cu*-semigroup. Consider the map $\vartheta\colon S\to[0,\infty]\otimes_{\mathrm{Cu}}S$ that sends a in S to $1\otimes a$. Then, given a functional $\lambda\in F([0,\infty]\otimes_{\mathrm{Cu}}S)$, the composition $\lambda\circ\vartheta$ is a functional in $F(S)$. Moreover, the assignment*

$$\theta\colon F([0,\infty]\otimes_{\mathrm{Cu}}S)\to F(S),\quad\lambda\mapsto\lambda\circ\vartheta,$$

is an isomorphism of topological cones.

Proof. We first define a map that will turn out to be the inverse of θ. Given a functional μ in $F(S)$, consider the map

$$[0,\infty] \times S \to [0,\infty], \quad (t,a) \mapsto t \cdot \mu(a).$$

It is straightforward to check that this is a generalized Cu-bimorphism, which therefore induces a generalized Cu-morphism

$$\tilde{\mu}\colon [0,\infty] \otimes_{\mathrm{Cu}} S \to [0,\infty].$$

This means that $\tilde{\mu}$ is a functional in $F([0,\infty] \otimes_{\mathrm{Cu}} S)$ such that $\tilde{\mu}(t \otimes a) = t \cdot \mu(a)$ for each $t \in [0,\infty]$ and $a \in S$. This defines a map

$$\psi\colon F(S) \to F([0,\infty] \otimes_{\mathrm{Cu}} S), \quad \mu \mapsto \tilde{\mu}. \quad (\mu \in F(S))$$

Given μ in $F(S)$ and $a \in S$, we deduce that

$$\theta \circ \psi(\mu)(a) = \theta(\tilde{\mu})(a) = \tilde{\mu}(1 \otimes a) = 1 \cdot \mu(a) = \mu(a).$$

Thus, $\theta \circ \psi$ is the identity on $F(S)$. Conversely, let λ be a functional in $F([0,\infty] \otimes_{\mathrm{Cu}} S)$. Since $[0,\infty]$ is solid, it follows from Proposition 7.1.6 that λ is automatically $[0,\infty]$-linear. Using this at the last step, we deduce for each $t \in [0,\infty]$ and $a \in S$ that

$$\psi \circ \theta(\lambda)(t \otimes a) = t \cdot \theta(\lambda)(a) = t\lambda(1 \otimes a) = \lambda(t \otimes a).$$

It follows that $\psi \circ \theta$ is the identity on $F([0,\infty] \otimes_{\mathrm{Cu}} S)$. It is straightforward to check that θ and ψ are continuous and linear, which shows the desired result. $\square$

Proposition 7.5.9. *Let S be a* Cu*-semigroup. Then the natural map*

$$\varphi\colon [0,\infty] \times S \to S_R \subset \mathrm{Lsc}(F(S)), \quad (t,a) \mapsto t \cdot \hat{a}, \quad (t \in [0,\infty], a \in S)$$

is a Cu*-bimorphism that induces an isomorphism*

$$\bar{\varphi}\colon [0,\infty] \otimes_{\mathrm{Cu}} S \xrightarrow{\cong} S_R.$$

Proof. It is straightforward to check that φ is a generalized Cu-bimorphism. By universal properties of the tensor product, the induced map $\bar{\varphi}$ is a generalized Cu-morphism. It follows easily from the definition of S_R that $\bar{\varphi}$ is surjective.

Next, we show that $\bar{\varphi}$ is an order-embedding. Consider the isomorphism $\theta\colon F([0,\infty] \otimes_{\mathrm{Cu}} S) \to F(S)$ from Lemma 7.5.8. It induces an isomorphism of positively ordered monoids

$$\theta^*\colon \mathrm{Lsc}(F(S)) \to \mathrm{Lsc}(F([0,\infty] \otimes_{\mathrm{Cu}} S)), \quad f \mapsto f \circ \theta.$$

We let

$$\Gamma\colon [0,\infty] \otimes_{\mathrm{Cu}} S \to \mathrm{Lsc}(F([0,\infty] \otimes_{\mathrm{Cu}} S)),$$

denote the canonical map that sends $x \in [0,\infty] \otimes_{\mathrm{Cu}} S$ to $\hat{x}$. Then, for $t \in [0,\infty]$ and $a \in S$ and $\lambda \in F([0,\infty] \otimes_{\mathrm{Cu}} S)$, we have that

$$\Gamma(t \otimes a)(\lambda) = \lambda(t \otimes a) = t \cdot \theta(\lambda)(a) = \varphi(t,a)(\theta(\lambda)) = (\theta^* \circ \bar{\varphi})(t \otimes a)(\lambda).$$

This implies that $\Gamma = \theta^* \circ \bar{\varphi}$. The situation is shown in the following commutative diagram:

$$\begin{tikzcd}
{[0,\infty] \otimes_{\mathrm{Cu}} S} \arrow[r, "\bar{\varphi}"] \arrow[dr, "\Gamma"'] & \mathrm{Lsc}(F(S)) \arrow[d, "\theta^*", "\cong"'] \\
& \mathrm{Lsc}(F([0,\infty] \otimes_{\mathrm{Cu}} S))
\end{tikzcd}$$

It follows from Theorem 7.5.4 that the Cu-semigroup $[0,\infty] \otimes_{\mathrm{Cu}} S$ is almost unperforated and that each of its elements is soft. Then Theorem 5.3.12 implies that the map Γ is an order-embedding. Since θ^* is an order-isomorphism, it follows that $\bar{\varphi}$ is an order-embedding. Thus, $\bar{\varphi}$ is an isomorphism of positively ordered monoids, and consequently a Cu-isomorphism. $\square$

Corollary 7.5.10. *Let S be a* Cu*-semigroup. Then, there are canonical isomorphisms*

$$[0,\infty] \otimes_{\mathrm{Cu}} S \cong S_R \cong (Z \otimes_{\mathrm{Cu}} S)_{\mathrm{soft}}.$$

In particular, if S has Z-multiplication, then there are isomorphisms

$$[0,\infty] \otimes_{\mathrm{Cu}} S \cong S_R \cong S_{\mathrm{soft}}.$$

Proof. This follows by combining Lemma 7.5.3 with Proposition 7.5.9. (Notice that $[0,\infty] \otimes_{\mathrm{Cu}} Z \cong Z_{\mathrm{soft}} = [0,\infty]$.) □

Theorem 7.5.11. *Let S be a* Cu*-semigroup, let $a, b \in S$. Then, the following are equivalent:*

(1) We have $1 \otimes a \leq 1 \otimes b$ in $[0,\infty] \otimes_{\mathrm{Cu}} S$.
(2) We have $\hat{a} \leq \hat{b}$ in $\mathrm{Lsc}(F(S))$.
(3) For every $a' \in S$ satisfying $a' \ll a$, and every $\varepsilon > 0$, there exist $k, n \in \mathbb{N}_+$ such that $(1-\varepsilon) < \frac{k}{n}$ and $ka' \leq nb$ in S.

Proof. As shown in Proposition 7.5.9, there is an isomorphism between the tensor product $[0,\infty]\otimes_{\mathrm{Cu}} S$ and S_R that identifies $1\otimes a$ with $\hat{a}$ and $1\otimes b$ with $\hat{b}$. This shows the equivalence between (1) and (2). The equivalence between statement (2) and (3) is shown in Proposition 5.2.14. □

Let A be a C^*-algebra. It is shown in [**Rob13a**, Theorem 5.1.2] there there is a natural isomorphism between $\mathrm{Cu}(A \otimes \mathcal{R})$ and $\mathrm{Cu}(A)_R$. Using Proposition 7.5.9 we can rephrase the result of Robert as follows:

Proposition 7.5.12. *Let A be a C^*-algebra. Then there are natural isomorphisms*

$$\mathrm{Cu}(A \otimes \mathcal{R}) \cong \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(\mathcal{R}) \cong \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} [0,\infty].$$

In particular, we have $\mathrm{Cu}(A \otimes \mathcal{R}) \cong \mathrm{Cu}(A)$ if and only if $\mathrm{Cu}(A)$ is unperforated, divisible and each element in $\mathrm{Cu}(A)$ is soft (or equivalently, purely noncompact).

7.6. Examples and Applications

7.6.1. Let D be a unital, separable, strongly self-absorbing C^*-algebra. As mentioned before in Proposition 7.1.4, it is known that D is nuclear and simple and that D is either purely infinite, in which case we have $\mathrm{Cu}(D) \cong \{0,\infty\}$, or that D is stably finite with unique tracial state. The only known examples of purely infinite, strongly self-absorbing C^*-algebras are the Cuntz algebras $\mathcal{O}_\infty$ and $\mathcal{O}_2$, and the tensor products of $\mathcal{O}_\infty$ with a UHF-algebra of infinite type. It follows from the Kirchberg-Phillips classification theorem that these are the only purely infinite, strongly self-absorbing C^*-algebras satisfying the the Universal coefficient theorem (UCT).

Let us assume that D is stably finite. In that case, the only known examples are the Jiang-Su algebra $\mathcal{Z}$ and the UHF-algebras of infinite type M_q, see Paragraph 7.4.2. Each of these algebras satisfies the UCT. We also have that D is $\mathcal{Z}$-stable, by [**Win11**, Theorem 3.1], see also Proposition 8.1.8. Therefore, it follows from Proposition 7.3.1 that the Cuntz semigroup of D can be computed as

$$\mathrm{Cu}(D) \cong V(D) \sqcup (0,\infty].$$

By [**Rør04**, Theorem 6.7], D has stable rank one. Therefore, $V(D)$ is a cancellative, algebraically ordered monoid that is isomorphic to the positive part of $K_0(D)$.

We conclude that the only known Cuntz semigroups realized by stably finite, strongly self-absorbing C^*-algebras are the following:

$$Z = \mathbb{N} \sqcup (0, \infty] = \mathrm{Cu}(\mathcal{Z}), \quad R_q = \mathbb{N}\left[\tfrac{1}{q}\right] \sqcup (0, \infty] = \mathrm{Cu}(M_q).$$

It follows from the K-theory computations in [**TW07**, Proposition 5.1] that the Cuntz semigroups Z and R_q are the only Cuntz semigroups of stably finite, strongly self-absorbing C^*-algebras that satisfy the UCT.

We have seen that Z and R_q (and also $\{0, \infty\}$) are solid Cu-semiring; see e.g. Proposition 7.3.3 and Proposition 7.4.4. Therefore, the Cuntz semigroup of every strongly self-absorbing C^*-algebras satisfying the UCT is a solid Cu-semiring.

It is an open problem whether every nuclear C^*-algebra satisfies the UCT. It is also unclear if there exist strongly self-absorbing C^*-algebras that do not satisfy the UCT. More modestly, we ask the following question:

Problem 7.6.2. Given a strongly self-absorbing C^*-algebra D, is the Cuntz semiring $\mathrm{Cu}(D)$ a solid Cu-semiring?

As noted in Paragraph 7.6.1, the answer is 'yes' for every strongly self-absorbing C^*-algebra satisfying the UCT. In Section 8.3, we provide a complete classification of solid Cu-semirings. We remark that even when excluding Cu-semirings that are elementary or have noncompact unit, there exist solid Cu-semirings that are not the Cuntz semigroup of any known strongly self-absorbing C^*-algebra, see Theorem 8.3.13.

We can summarize Corollary 7.2.13, Proposition 7.4.14 and Proposition 7.5.12 as follows:

Proposition 7.6.3. *Let A and D be two C^*-algebras. Assume that either $D = \mathcal{R}$ or that D is a (unital) separable, strongly self-absorbing C^*-algebra satisfying the UCT but not equal to $\mathcal{Z}$. Then, there is a natural isomorphism*

$$\mathrm{Cu}(A \otimes D) \cong \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(D).$$

7.6.4. Recall that a C^*-algebra A is said to have *strict comparison of positive elements* if for any two positive elements $x, y \in M_\infty(A)$ the following holds: If x is contained in $\overline{AyA}$, the closed two-sided ideal generated by y, and if $d(x) < d(y)$ for every dimension function d on A satisfying $d(y) = 1$, then $x \precsim y$ (x is Cuntz subequivalent to y). It was shown by Rørdam that a C^*-algebra A has strict comparison of positive elements if and only if its pre-completed Cuntz semigroup $W(A)$ is almost unperforated, [**Rør04**, Proposition 3.2]; see also [**APT11**, Lemma 5.7]. It is easy to see that $W(A)$ is almost unperforated if and only if $\mathrm{Cu}(A)$ is.

For a unital, simple C^*-algebra A, and positive elements $x, y \in M_\infty(A)$, the condition $x \in \overline{AyA}$ is automatically satisfied whenever y is nonzero, and it is moreover enough to consider lower-semicontinuous dimension functions. In this form, the notion of strict comparison of positive elements in simple C^*-algebras was introduced by Blackadar as the 'Fundamental Comparability Question (FCQ4)' in [**Bla88**, § 6.4.7].

In [**RW10**, Question 5.3], Rørdam and Winter ask whether the Jiang-Su algebra unitally embeds into any unital C^*-algebra A such that the class of the unit is almost divisible in $\mathrm{W}(A)$. It is easy to see that the class of the unit is almost divisible in $W(A)$ if and only if it is in $\mathrm{Cu}(A)$. Thus, the implication '(2)⇒(4)' of the following Proposition 7.6.5 provides a positive answer to the question of Rørdam and Winter for C^*-algebras that have stable rank one and strict comparison of positive elements.

PROPOSITION 7.6.5. *Let A be a unital C^*-algebra with stable rank one and with strict comparison of positive elements. Then, the following statements are equivalent:*

(1) For each $n \in \mathbb{N}$, there exists a unital *-homomorphism $Z_{n,n+1} \to A$, where $Z_{n,n+1}$ is the dimension drop algebra.
(2) The element $[1_A]$ is almost divisible in $\mathrm{Cu}(A)$.
(3) There exists a Cu-morphism $Z \to \mathrm{Cu}(A)$ that sends 1_Z to $[1_A]$.
(4) There exists a unital *-homomorphism $\mathcal{Z} \to A$.

PROOF. The equivalence between (1) and (2) is shown in [**RW10**, Proposition 5.1]. The equivalence between (2) and (3) follows from Proposition 7.3.7. Finally, it is clear that (4) implies (3). The converse follows from Theorems 1.0.1 and 3.2.2 in [**Rob12**]. □

Let S be a simple, stably finite Cu-semigroup satisfying (O5). Recall that we denote by S_c (resp. $S_c^\times$) the subsemigroup of (nonzero) compact elements in S. As shown in Proposition 5.3.16, an element $a \in S$ is soft whenever it is not compact. The only element that is both compact and soft is the zero element. Thus, there is a natural decomposition

$$S = S_c^\times \sqcup S_{\mathrm{soft}}.$$

If we additionally assume that S has Z-multiplication, then we can apply Corollary 7.5.10 to compute S_{soft} as S_R, the realification of S. We obtain

$$S \cong S_c^\times \sqcup S_R.$$

Since S also satisfies (O5), it follows from [**Rob13a**, Theorem 3.2.1] that S_R is isomorphic to $L(F(S))$. The point is that the semigroup $L(F(S))$ only depends on $F(S)$, the cone of functionals on S. On the other hand, it is not clear if S_R only depends on $F(S)$, see Problem 7.6.7.

We summarize our representation result for simple Cu-semigroups with Z-multiplication in the following Theorem. For Cuntz semigroups of C^*-algebras, the analogous result has appeared in [**BT07**, Theorem 2.6] and [**ABP11**, Theorem 6.3].

THEOREM 7.6.6. *Let S be a simple, stably finite* Cu*-semigroup satisfying (O5) and with Z-multiplication. Then, the soft part of S is isomorphic to $L(F(S))$. Thus, there is a natural isomorphism*

$$S \cong S_c^\times \sqcup L(F(S)).$$

Let S be a Cu-semigroup. By definition, S_R and $L(F(S))$ are submonoids of $\mathrm{Lsc}(F(S))$. It follows from [**Rob13a**, Proposition 3.1.6] that S_R is a subset of $L(F(S))$. (The implicit assumption of (O5) in [**Rob13a**] is not needed in the proof of [**Rob13a**, Proposition 3.1.6].)

It is natural to ask whether S_R is in fact equal to $L(F(S))$. Under the assumption of (O5) this is indeed the case, see [**Rob13a**, Theorem 3.2.1]. Thus, we ask if the result of Robert holds without the assumption of (O5).

PROBLEM 7.6.7. Let S be a Cu-semigroup. Is it true that $S_R = L(F(S))$?

CHAPTER 8

Structure of Cu-semirings

In this chapter, we study the structure of certain classes of Cu-semirings that satisfy (O5). The main result is Theorem 8.1.6, where we show that every simple, nonelementary Cu-semiring is automatically almost unperforated and almost divisible. Together with Theorem 7.3.8, we obtain that every simple, nonelementary Cu-semiring has Z-multiplication, which can be interpreted as the Cu-semiring version of Winter's result that every strongly self-absorbing C^*-algebra is $\mathcal{Z}$-stable, see Corollary 8.1.7. We also use our findings to give an alternative proof of Winter's result, see Proposition 8.1.8.

In Section 8.2, we study algebraic Cu-semirings. We establish an equivalence between the category of weakly cancellative, algebraic Cu-semirings and the category of directed, partially ordered rings; see Proposition 8.2.2. We also give several characterizations when a simple Cu-semiring with unique normalized functional is algebraic; see Proposition 8.2.11.

In Section 8.3, we analyse the structure of solid Cu-semirings. The main result is Theorem 8.3.13, where we classify all solid, nonelementary Cu-semirings satisfying (O5).

8.1. Simple Cu-semirings

Recall that a simple Cu-semigroup is *elementary* if it contains a minimal nonzero element, see Paragraph 5.1.16.

Lemma 8.1.1. *Let R be a simple Cu-semiring satisfying (O5). Then:*

(1) If R is nonelementary, then there exist nonzero elements c and d in R such that $c+d\leq 1$.
(2) Assume that $R\neq\{0\}$. Then R is elementary if and only if the unit 1_R is a minimal nonzero element.
(3) If 1_R is properly infinite (i.e., $2\cdot 1_R=1_R$), then $R\cong\{0,\infty\}$ or $R\cong\{0\}$.

Proof. We first assume that $1_R\neq 0_R$ and that 1_R is not a minimal nonzero element. Then there exists a nonzero element x in R such that $x\leq 1$ and $x\neq 1$. Choose a nonzero element c in R satisfying $c\ll x$. Using that R satisfies (O5), there exists $d\in R$ such that

$$c+d\leq 1\leq x+d.$$

Since $x\neq 1$, we have that d is nonzero. This immediately implies statement (1).

To show (2), note that the assumption $R\neq\{0\}$ implies that $1_R\neq 0_R$. It follows that R has no zero divisors, see Remarks 7.1.2. If 1_R is a minimal, nonzero element, then R is elementary by definition. Conversely, assume 1_R is not minimal. As shown at the beginning of the proof, there exist nonzero elements c and d such that $c+d\leq 1$. We have to prove that R is nonelementary.

So assume a is a nonzero element in R. Consider the elements ca and da, which are nonzero since R has no zero divisors. Moreover, we have that $ca+da\leq a$. If $ca\neq a$ or $da\neq a$, then a is not a minimal, nonzero element. Otherwise, we can deduce that $a=2a$ and hence $a=\infty$. Since $1\leq\infty$ and 1 is assumed not to be a minimal, nonzero element, we obtain that a is not a minimal, nonzero element in either case. Thus, R is nonelementary.

Finally, statement (3) is easily verified. $\square$

EXAMPLE 8.1.2. The elementary Cu-semigroups $\overline{\mathbb{N}}$ and $E_k = \{0, 1, 2, \dots, k, \infty\}$ have natural (and unique) Cu-products giving them the structure of solid Cu-semirings. These are the only simple, elementary Cu-semirings satisfying (O5) and (O6); see Paragraph 5.1.16.

Without (O6), there are other examples of simple, elementary Cu-semirings: Consider $S = \{0, 1, 1', 2, 3, 4, \dots, \infty\}$, with addition and multiplication among the un-apostrophized elements as usual and such that $1' + k = 1 + k$ and $k \cdot 1' = k$ for each $k \in \mathbb{N}$. The elements 1 and $1'$ are incomparable.

For proving Theorem 8.1.6, we need several lemmas. Given a positively ordered monoid S and $k \in \mathbb{N}_+$, recall that an element a in S is *almost k-divisible* if there exists $x \in S$ such that $kx \leq a \leq (k+1)x$, see Definition 7.3.4. If this holds for every $k \in \mathbb{N}_+$, we say that the element is *almost divisible*. Moreover, S is *almost divisible* if each of its elements is.

LEMMA 8.1.3. *Let R be a simple* Cu*-semiring. Let c and d be nonzero elements in R such that $c + d \leq 1$. Then, for every x in R satisfying $x \ll \infty$ there exists $n \in \mathbb{N}$ such that $xc^n \leq 1$.*

PROOF. We inductively show that

$$[(k+1)d]c^k \leq 1, \tag{8.1}$$

for every $k \in \mathbb{N}$. For $k = 0$ this is clear. For the induction step, assume that (8.1) is satisfied for some $k \in \mathbb{N}$. Multiplying both sides of (8.1) by c, we obtain that

$$[(k+1)d]c^{k+1} \leq c.$$

Using this at the second step, we deduce

$$[(k+2)d]c^{k+1} \leq [(k+1)d]c^{k+1} + d \leq c + d \leq 1.$$

Since R is simple, and since $d \neq 0$ and $x \ll \infty$, there is $n \in \mathbb{N}$ such that $x \leq (n+1)d$. Then $xc^n \leq [(n+1)d]c^n \leq 1$, as desired. $\square$

LEMMA 8.1.4. *Let R be a simple* Cu*-semiring, and let $k \in \mathbb{N}_+$. Assume that there exist nonzero elements c and d in R such that $c + d \leq 1$. Then there exists a nonzero element a in R such that $ka \ll (k+1)a \leq 1$.*

PROOF. Let c, d and k be as in the statement. We may assume that $c \ll \infty$ (by replacing it by some nonzero element $c' \in R$ satisfying $c' \ll c$, if necessary). We construct the element a in two steps.

Step 1: Let c_1 be a nonzero element in R satisfying $c_1 \ll c$. Since $c \ll \infty$, there exists $n \in \mathbb{N}$ such that $c \leq nc_1$. By Lemma 8.1.3, there exists $m \in \mathbb{N}$ such that

$$[k(k+1)n^2]c^m \leq 1.$$

Choose elements c_i in R for $i = 2, \dots, m$ such that

$$c_1 \ll c_2 \ll \dots \ll c_m \ll c.$$

Set $b = c_1c_2 \cdots c_m$ (the product). We compute, using at the second step that compact containment is preserved under multiplication:

$$b = [c_1c_2 \cdots c_{m-1}]c_m \ll [c_2c_3 \cdots c_m]c \leq [c_2c_3 \cdots c_m]nc_1 = nb.$$

Moreover, we have $b \leq c^m$, and therefore $k(k+1)n^2b \leq 1$.

Step 2: Choose elements b_i in R for $i = 1, \dots, kn$ such that

$$b \ll b_1 \ll b_2 \ll \dots \ll b_{kn} \ll nb.$$

Set $a = b_1 + \ldots + b_{kn}$. We compute, using at the second step that compact containment is preserved under addition:

$$a = [b_1 + b_2 + \ldots + b_{kn-1}] + b_{kn} \ll [b_2 + b_3 + \ldots + b_{kn}] + nb \leq a + nb.$$

Multiplying this inequality by k, and using that $knb \leq a$ at the last step, we obtain that

$$ka \ll ka + knb \leq (k+1)a.$$

Moreover, we have $a \leq kn(nb)$, and thus $(k+1)a \leq (k+1)kn^2b \leq 1$, as desired. $\square$

Lemma 8.1.5. *Let R be a simple* Cu*-semiring satisfying (O5), and let $k \in \mathbb{N}_+$. Suppose that there exists a nonzero element a in R such that $ka \ll (k+1)a \leq 1$. Then the unit element 1 in R is almost k-divisible.*

Proof. Using that R satisfies (O5), there exists t in R such that

$$2ka + t \leq 1 \leq (2k+1)a + t.$$

We think of this inequality as '$2ka \leq (1-t) \leq (2k+1)a$'. Then, we want to multiply by $(1-t)^{-1}$, which we think of as $(1 + t + t^2 + \ldots)$. Let us make this precise, set

$$z = 1 + t + t^2 + \ldots = \sup_{n \in \mathbb{N}} \left(1 + t + t^2 + \ldots + t^n\right).$$

We show that 1 is almost k-divisible in two steps.

Step 1: We show that $2kaz \leq 1$. To obtain this, we first show that if x and y are two elements in R satisfying $x + y \leq 1$, then

$$x\left(\sum_{n=0}^{\infty} y^n\right) \leq 1.$$

Indeed, multiplying the inequality $x + y \leq 1$ by y, we obtain that $xy + y^2 \leq y$. Using this at the second step, we deduce that

$$x(1+y) + y^2 = x + (xy + y^2) \leq x + y \leq 1.$$

Inductively, we obtain for all $n \in \mathbb{N}_+$ that

$$x(1 + y + y^2 + \ldots + y^n) + y^{n+1} \leq 1,$$

and therefore $x\left(\sum_{n=0}^{\infty} y^n\right) \leq 1$, as desired. Applying this to $2ka + t \leq 1$, we obtain $2kaz \leq 1$.

Step 2: We show that $1 \leq (2k+2)az$. Choose a rapidly increasing sequence $(w_r)_{r \in \mathbb{N}} \subset R$ such that $1 = \sup_r w_r$. (If the unit element is compact, the following argument can be simplified.) For each fixed $r \in \mathbb{N}$, since $w_r \ll 1 \leq (2k+1)a + t$, there exists t_r in R such that

$$t_r \ll t, \quad w_r \leq (2k+1)a + t_r.$$

We have that $1 \leq (2k+1)a + t$. Multiplying this inequality by t_r, we obtain that

$$t_r \leq (2k+1)at_r + tt_r \leq (2k+1)at + tt_r.$$

Since $w_r \leq (2k+1)a + t_r$, it follows that

$$w_r \leq (2k+1)a(1+t) + tt_r.$$

Inductively, we obtain for all $n \in \mathbb{N}$ that

$$w_r \leq (2k+1)a(1 + t + \ldots + t^n) + t^n t_r.$$

Since $t_r \ll \infty$ and $a \neq 0$, there exists $m_r \in \mathbb{N}$ such that $t_r \leq (m_r+1)a$. Then

$$t^{m_r}t_r \leq t^{m_r}(m_r+1)a = a(t^{m_r} + \ldots + t^{m_r}) \leq a(1 + \cdots + t^{m_r}),$$

This implies that

$$\begin{aligned} w_r &\leq (2k+1)a(1+t+\ldots+t^{m_r})+t^{m_r}t_r \\ &\leq (2k+2)a(1+t+\cdots+t^{m_r}) \\ &\leq (2k+2)a(1+t+\cdots+t^{m_r}+\ldots) \\ &= (2k+2)az. \end{aligned}$$

Since this holds for all $r \in \mathbb{N}$ and since $1 = \sup_r w_r$, we obtain that $1 \leq (2k+2)az$. Thus, $k(2az) \leq 1 \leq (k+1)(2az)$, which finishes the proof. □

The following theorem is the main structure result for simple Cu-semirings.

Theorem 8.1.6. *Let R be a simple, nonelementary* Cu*-semiring satisfying (O5). Then R is almost unperforated and almost divisible.*

Proof. By Lemma 7.3.5, it is enough to show that the unit of R is almost divisible. So let k be a natural number with $k \geq 1$. By Lemma 8.1.1, there exist nonzero elements $c, d \in R$ such that $c + d \leq 1$. Thus, we may apply Lemma 8.1.4 (for $2k$) to obtain a nonzero element a such that $2ka \ll (2k+1)a \leq 1$. Now it follows from Lemma 8.1.5 that the unit of R is almost k-divisible. □

Combining the above result with Theorem 7.3.8, we obtain the Cu-semigroup version of Winter's result that strongly self-absorbing C^*-algebras are $\mathcal{Z}$-stable, [**Win11**, Theorem 3.1].

Corollary 8.1.7. *Let R be a simple, nonelementary* Cu*-semiring satisfying axiom (O5). Then R has Z-multiplication and is therefore 'Z-stable' in the sense that $R \cong Z \otimes_{\mathrm{Cu}} R$.*

Using Corollary 8.1.7, we obtain an alternative proof of [**Win11**, Theorem 3.1], as follows.

Proposition 8.1.8 (Winter). *Let D be a unital, separable, strongly self-absorbing C^*-algebra. Then D is $\mathcal{Z}$-stable, that is, $D \cong \mathcal{Z} \otimes D$.*

Proof. The famous $\mathcal{O}_\infty$-absorption theorem states that every unital, separable, nuclear, purely infinite, simple C^*-algebra A satisfies $\mathcal{O}_\infty \otimes A \cong A$, see [**KP00**, Theorem 3.15]. Thus, if D is purely infinite, then it is $\mathcal{O}_\infty$-stable and therefore also $\mathcal{Z}$-stable.

Assume now that D is stably finite. Let R be the Cuntz semigroup of D. By Proposition 7.1.4, R is a simple Cu-semiring satisfying (O5). Since R is also nonelementary, we obtain from Theorem 8.1.6 that R is almost unperforated and almost divisible.

Almost unperforation of R means that D has has strict comparison of positive elements. We also have that D has a (unique) tracial state. Given $k \in \mathbb{N}_+$, we have that the unit of D is almost k-divisible. We can now apply [**DT10**, Theorem 3.6] to deduce that there exists a unital *-homomorphism from the dimension drop algebra $Z_{k,k+1}$ to D. Since the Jiang-Su algebra $\mathcal{Z}$ is an inductive limit of dimension drop algebras $Z_{k,k+1}$, it follows from [**TW08**, Proposition 2.2] that D is $\mathcal{Z}$-stable, as desired. □

Corollary 8.1.9. *Let R be a simple* Cu*-semiring satisfying (O5). Then:*

(1) If R is nonelementary, then R is stably finite.
(2) Either, the unit 1_R is a compact element, or R contains no nonzero compact elements.

Proof. To show (1), let R be a simple, nonelementary Cu-semiring satisfying (O5). By Theorem 8.1.6, we have that R is almost divisible. The rest of the proof is similar to that of Proposition 6.4.15.

Assume that R is not stably finite. As shown in Proposition 5.2.5, it follows that ∞ is a compact element. Since R is nonelementary, we have that 1_R is nonzero, which implies that there exists $k \in \mathbb{N}$ such that $k1_R = \infty$. Given a nonzero element a in R, let us show that $a = \infty$. Since R is almost divisible, there exists an element t in R such that $kt \leq a \leq (k+1)t$. This implies that t is nonzero. Then

$$\infty = \infty \cdot t = k1_R \cdot t \leq a \leq \infty.$$

Thus, we have shown that $a = \infty$ for every nonzero element $a \in R$. This implies that $R \cong \{0, \infty\}$, which is a contradiction since R was assumed to be nonelementary.

Let us show (2). The statement is clearly true if $R \cong \{0\}$ or if 1_R is a compact element. So we may assume from now on that $R \neq \{0\}$ and that 1_R is not compact.

Let us show that R is nonelementary. To reach a contradiction, assume the opposite. Then, by Lemma 8.1.1, we have that 1_R is a minimal, nonzero element. This implies that 1_R is compact, a contradiction.

Hence R is nonelementary, and therefore stably finite by statement (1). Then, by Proposition 5.3.16, every nonzero element of R is either soft or compact. Thus, the unit element 1_R is soft. It follows from Corollary 8.1.7 that R has Z-multiplication. By Proposition 7.3.13, an element a in R is soft if and only if $a = 1'_Z a$, where $1'_Z$ is the 'soft' unit of Z. We deduce that $1_R = 1'_Z 1_R$. Given a nonzero element a in R, we obtain that

$$a = 1_R a = 1'_Z 1_R a = 1'_Z a.$$

Using Proposition 5.3.16 again, it follows that a is noncompact, as desired. □

Next, we study simple Cu-semirings that have a unique functional normalized at the unit. As we will see in Section 8.3, in particular Theorem 8.3.1, this class includes all solid Cu-semirings. It also includes the Cuntz semigroups of stably finite, strongly self-absorbing C^*-algebras, see Proposition 7.1.4.

We first study the multiplicativity of functionals. The result is inspired by [**Han13**, Corollary 3]. The requirement that the function $\widehat{1} \in \mathrm{Lsc}(F(R))$ be continuous is not very restrictive. It is automatically satisfied if 1 is a compact element or if R has only finitely many extremal functionals. This will for instance be used in Corollary 8.1.11.

Proposition 8.1.10. *Let R be simple, nonelementary* Cu*-semiring satisfying (O5). Assume that $\widehat{1}$ is continuous. Let $F_1(R)$ denote the functionals of R that are normalized at* 1*. Then:*

(1) *A functional $\lambda \in F_1(R)$ is multiplicative if and only if it is an extreme point of $F_1(R)$.*
(2) *The space $F_1(R)$ is a Bauer simplex, i.e., a Choquet simplex with closed extreme boundary.*
(3) *Every functional $\lambda \in F_1(R)$ satisfies $\lambda(ab)^2 \leq \lambda(a^2)\lambda(b^2)$ for all $a, b \in R$. In particular, we have $\lambda(a)^2 \leq \lambda(a^2)$ for every $a \in R$.*

Proof. The assumption that $\widehat{1}$ is continuous implies that $F_1(R)$ is a closed subset of $F(R)$. (In fact, this is equivalent to $\widehat{1}$ being continuous.) Since $F(R)$ is compact, it follows that $F_1(R)$ is a compact, convex set. We denote the subset of its extreme points by $\partial F_1(R)$.

We note that $F_1(R)$ is a Choquet simplex. This follows for instance from [**Rob13a**, Proposition 3.2.3, Theorem 4.1.2]. Note first that $F_1(R)$ is a basis for

$$F_0(R) = \{\lambda \in F(R) \mid \lambda(a) < \infty \text{ for all } a \ll \infty\}.$$

Let $V(F_0(R))$ denote the vector space of linear, real-valued, continuous functions on $F_0(R)$. Then $F_0(R)$ is a lattice-ordered, strict, convex cone in the vector space of linear functionals on $V(F_0(R))$.

Let λ be an extreme point in $F_1(R)$. We show that λ is multiplicative. To reach a contradiction, assume that $\lambda(ab) \neq \lambda(a)\lambda(b)$ for some $a, b \in R$. By Corollaries 8.1.9 and 8.1.7, we know that R is stably finite and that R has Z-multiplication. By Proposition 7.3.13, we have $\lambda(x) = \lambda(1'x)$ for every $x \in R$. Thus, we may assume that a and b are soft elements, by replacing a with $1'a$ and by replacing b with $1'b$, if necessary.

Since a is the supremum of a rapidly increasing sequence and since functionals preserve suprema of increasing sequences, we may also assume that $a \ll \infty$. Then there is $n \in \mathbb{N}$ such that $a \ll n1'$.

Since R satisfies (O5), we may apply [**Rob13a**, Theorem 3.2.1] to deduce that $R_{\mathrm{soft}} \cong L(F(R))$. By the definition of $L(F(R))$, there exists a sequence $(x_k)_k$ in $L(F(R))$ such that $x_k \lhd x_{k+1}$ for each $k \in \mathbb{N}$ and such that $a = \sup_k x_k$. Since

$$\sup_k \lambda(x_k b) = \lambda(ab) \neq \lambda(a)\lambda(b) = \sup_k \lambda(x_k)\lambda(b),$$

we can choose $k \in \mathbb{N}$ such that $\lambda(x_k b) \neq \lambda(x_k)\lambda(b)$. Set $x = x_k$.

We have that $x \lhd x_{k+1} \ll n1'$. By [**Rob13a**, Lemma 3.3.2], there exists $y \in L(F(R)) \cong R_{\mathrm{soft}}$ such that $x + y = n1'$. Without loss of generality, we may assume that $x, y \neq 0$. Then we can consider the maps $\lambda_i \colon R \to [0, \infty]$ given by

$$\lambda_0(s) = \lambda(x)^{-1}\lambda(xs), \quad \lambda_1(s) = \lambda(y)^{-1}\lambda(ys),$$

for $s \in R$. It is easy to check that λ_0 and λ_1 are functionals on R and that $\lambda_0(b) \neq \lambda_1(b)$. Since

$$\lambda = \tfrac{\lambda(x)}{n}\lambda_0 + \tfrac{\lambda(y)}{n}\lambda_1,$$

we have shown that λ is not an extreme point of $F_1(R)$, as desired.

Let us show (3). Since $F_1(R)$ is a Choquet simplex, there is a measure μ on its extreme boundary $\partial F_1(R)$ such that

$$\lambda(x) = \int_{\partial F_1(R)} \varphi(x) d\mu(\varphi), \tag{8.2}$$

for every element x in R for which $\widehat{x}$ is continuous.

We claim that (8.2) holds for every element x in R. This is clear if x is compact, since then $\widehat{x}$ is continuous. If x is soft, then it follows from [**Rob13a**, Proposition 3.1.6] that there is an increasing sequence $(x_k)_k$ in R such that $x = \sup_k x_k$ and $\widehat{x_k}$ is continuous for each $k \in \mathbb{N}$. Using the theorem of monotone convergence at the third step, we obtain that

$$\begin{aligned}\lambda(x) = \sup_k \lambda(x_k) &= \sup_k \int_{\partial F_1(R)} \varphi(x_k) d\mu(\varphi) \\ &= \int_{\partial F_1(R)} \sup_k \varphi(x_k) d\mu(\varphi) \;=\; \int_{\partial F_1(R)} \varphi(x) d\mu(\varphi),\end{aligned}$$

which verifies (8.2).

Now, given two elements a and b in R, we use the Cauchy-Schwarz inequality at the second step to deduce that

$$\begin{aligned}\lambda(ab)^2 &= \left(\int_{\partial F_1(R)} \varphi(a)\varphi(b) d\mu(\varphi) \right)^2 \\ &\leq \int_{\partial F_1(R)} \varphi(a)^2 d\mu(\varphi) \int_{\partial C} \varphi(b)^2 d\mu(\varphi) \;=\; \lambda(a^2)\lambda(b^2).\end{aligned}$$

Let us show (1). We have already seen that every functional in $\partial F_1(R)$ is multiplicative. To show the converse, assume that $\lambda = \frac{1}{2}(\lambda_0 + \lambda_1)$ for two different functionals λ_0 and λ_1. Choose a in R such that $\lambda_0(a) \neq \lambda_1(a)$. By switching the role of λ_0 and λ_1, if necessary, we may assume that there is $\varepsilon > 0$ such that

$$\lambda_0(a) = \lambda(a) - \varepsilon, \quad \lambda_1(a) = \lambda(a) + \varepsilon.$$

Then, using (3) at the second step, it follows that

$$\lambda(a^2) = \frac{1}{2}(\lambda_0(a^2) + \lambda_1(a^2)) \geq \frac{1}{2}(\lambda_0(a)^2 + \lambda_1(a)^2) = \lambda(a)^2 + \varepsilon^2,$$

which shows that $\lambda(a^2) \neq \lambda(a)^2$ and so λ is not multiplicative.

Finally, it follows easily from (1) that $\partial F_1(R)$ is closed in $F_1(R)$. This verifies (2). □

Corollary 8.1.11. *Let R be simple, nonelementary* Cu*-semiring satisfying (O5). Assume that R has a unique functional λ that is normalized at* 1*. Then λ is multiplicative.*

Proof. This follows directly from Proposition 8.1.10, since $\widehat{1}$ is automatically continuous. □

Corollary 8.1.12. *Let R be a simple* Cu*-semiring satisfying (O5). Assume that R has a unique functional normalized at 1_R. Then $R \cong [0,\infty]$ if and only if 1_R is not compact.*

Proof. It is clear that the unit element of $[0,\infty]$ is not compact. Conversely, assume that R satisfies the conditions of the statement and that 1_R is not compact. By Corollary 8.1.9, this means that 0 is the only compact element of R, that is, $R_c = \{0\}$. Using Lemma 8.1.1, this also implies that R is nonelementary. Therefore, R is almost unperforated and almost divisible by Theorem 8.1.6. Now, the result follows from Theorem 7.6.6. □

8.2. Algebraic Cu-semirings

Recall from Definition 5.5.1 that a Cu-semigroup S is *algebraic* if every element in S is the supremum of an increasing sequence of compact elements.

8.2.1. Let K be a cancellative, conical semiring. We equip K with the algebraic order. Then K is a positively ordered monoid and we may apply the construction of Section 5.5 to the underlying additive monoid of K. We denote by S the resulting Cu-completion of K. Then S is an algebraic Cu-semigroup whose compact elements can be identified with K. We therefore think of K as a submonoid of S.

The multiplication on K can be extended to S as follows: First, we define the product of an element in K with an element in S. So let $a \in K$ and $b \in S$. Choose an increasing sequence $(b_k)_k \subset K$ with $b = \sup_k b_k$. Then, the sequence $(ab_k)_k \subset K$ is increasing and we may set $ab = \sup_k(ab_k)$. It is straightforward to check that this is independent of the choice of the sequence $(b_k)_k$. Moreover, if two elements a' and a in K satisfy $a' \leq a$, then $a'b \leq ab$ for every $b \in S$.

Now we can define the product of two arbitrary elements a and b in S as follows. Choose an increasing sequence $(a_k)_k \subset K$ with $a = \sup_k a_k$. For each k, the product $a_k b$ is already defined. Moreover, the resulting sequence $(a_k b)_k$ is increasing and we may set $ab = \sup_k(a_k b)$. It is easy to check that this defines a Cu-product on S. We denote the resulting Cu-semiring by $\mathrm{Cu}(K)$.

By Proposition 5.5.8, $\mathrm{Cu}(K)$ is a weakly cancellative, algebraic Cu-semiring satisfying (O5). Moreover, there is a natural isomorphism between K and the semiring of compact elements in $\mathrm{Cu}(K)$.

Given two cancellative, conical semirings K and L, it is clear that every semiring homomorphism $f\colon K \to L$ induces a multiplicative Cu-morphism from $\mathrm{Cu}(K)$ to $\mathrm{Cu}(L)$. This defines a functor from the category $\mathrm{ConSrg}_{\mathrm{canc}}$ of cancellative, conical semirings to the category of algebraic Cu-semirings.

Conversely, for every Cu-semiring S with compact unit, the compact elements in S form a subsemiring. The assignment $S \mapsto S_c$ can be extended to a functor from the category of algebraic Cu-semirings to the category of conical semirings.

Let S be a weakly cancellative, algebraic Cu-semiring satisfying (O5). By Proposition 5.5.8, the subset S_c is a cancellative, conical, algebraically ordered semiring. Moreover, the Cu-semiring S is naturally isomorphic to $\mathrm{Cu}(S_c)$.

Proposition 8.2.2. *The functors from Paragraphs 8.2.1 and B.4.3 establish equivalences between the following categories:*

(1) Directed, partially ordered rings, together with ring homomorphisms.
(2) Cancellative, conical semirings, together with semiring homomorphisms.
(3) Weakly cancellative, algebraic Cu-*semirings satisfying (O5), together with multiplicative* Cu-*morphisms.*

The following notion of weak divisibility was introduced in [**OPR11**, Definition 2.2] (see also [**PR04**] and [**AGPS10**]). This property has also been called *quasi-divisible* in [**Weh98**, Definition 3.2].

Definition 8.2.3. A monoid M is *weakly divisible* if for every $s \in M$, there are $a, b \in M$ such that $s = 2a + 3b$.

Definition 8.2.4. A conical semiring R is *nonelementary* if there exist nonzero elements s and t in R such that $1 = s + t$.

Remarks 8.2.5. (1) Let S be a conical semiring. Then the underlying additive monoid of S is weakly divisible if and only if there exist elements s and t in S such that $1 = 2s + 3t$.

(2) Let S be a conical semiring. If we equip S with its algebraic pre-order, then S is nonelementary if and only if $S \neq \{0\}$ and the unit is not a minimal nonzero element.

It is easily seen that every (nonzero) weakly divisible, conical semiring is nonelementary. In the next result, we show that the converse holds for simple, conical semirings. Part of the argument in the proof of the next result is inspired by [**DR09**].

Proposition 8.2.6. *Let R be a nonelementary, conical semiring that is simple for its algebraic pre-order. Then R is weakly divisible.*

Proof. In this proof, we will write $\leq$ for the algebraic pre-order on R. Then, simplicity of R means that for every two elements $x, y \in R$ with $y \neq 0$, there exists $n \in \mathbb{N}$ such that $x \leq ny$.

We first observe that R contains no zero divisors. Indeed, assume two nonzero elements $x, y \in R$ satisfy $xy = 0$. Since R is simple, there exist $x', y' \in R$ and positive numbers $n, m \in \mathbb{N}$ such that $1 + x' = nx$ and $1 + y' = my$. This implies $1 + x' + y' + x'y' = 0$. Since R is conical, it follows that the unit of R is zero, whence $R = \{0\}$, which contradicts that R is nonelementary.

To prove weak divisibility of R, it is enough to show that there are elements a and b in R such that $1 = 2a + 3b$. Since R is nonelementary, there exist nonzero elements $s, t \in R$ such that $1 = s + t$. Then $1 = s^2 + t^2 + 2st$. Set

$$f = s^2 + t^2, \quad e = st.$$

Note that both e and f are nonzero elements. Then $1 = f + 2e$, which implies that $f = f^2 + 2ef$. It follows that $1 = f^2 + 2e(1 + f)$. Inductively, we obtain for each $m \in \mathbb{N}$ that

$$1 = f^m + 2e(1 + f + \cdots + f^{m-1}).$$

By simplicity of R, there is m in $\mathbb{N}_+$ such that $f \leq me$. Then

$$f^m \leq mef^{m-1} \leq e(1 + f + \cdots + f^{m-1}).$$

Set $b = f^m$ and $a' = e(1 + f + \cdots + f^{m-1})$, so that $1 = b + 2a'$. Since $b \leq a'$, there exists an element a in R such that $b + a = a'$. Then

$$1 = b + 2a' = b + 2b + 2a = 2a + 3b,$$

as desired. □

Lemma 8.2.7. *Let R be a weakly divisible semiring. Let M be a positively ordered monoid that is a semimodule over R. Then M is nearly unperforated.*

In particular, if R has a compatible positive order, then it is nearly unperforated itself.

Proof. By Lemma 5.6.2, it is enough to show that $2a \leq 2b$ and $3a \leq 3b$ imply $a \leq b$, for any $a, b \in M$. Let such a and b be given. By weak divisibility of R, there are elements $s, t \in R$ such that $1 = 2s + 3t$. Then

$$a = (2s + 3t)a = s(2a) + t(3a) \leq s(2b) + t(3b) = (2s + 3t)b = b,$$

as desired. □

Proposition 8.2.8. *Let R be a nonelementary, conical semiring that is simple and stably finite for its algebraic (pre)order. Then R is cancellative.*

Proof. The assumptions imply that the algebraic pre-order of R is antisymmetric, see Remarks 5.6.6. Thus, the underlying monoid of R is a simple, stably finite, partially ordered monoid (with its algebraic order). By Proposition 8.2.6 and Lemma 8.2.7, the semiring R is nearly unperforated. Then we may apply Proposition 5.6.11 to deduce that R is cancellative. □

Lemma 8.2.9. *Let R be a simple algebraic* Cu*-semiring satisfying (O5). Then, the following conditions are equivalent:*

(1) R is nonelementary as a Cu*-semiring.*
(2) R_c is stably finite, 1_R is compact and R_c is a nonelementary semiring.
(3) R_c is stably finite, 1_R is compact and not minimal.

Proof. To see this, assume condition (1). Then $R_c \neq \{0\}$ as R is algebraic, and so by Corollary 8.1.9 we see that 1_R is compact, and also that R (hence also R_c) is stably finite. Thus R_c is a (conical) semiring. Since R is nonelementary and algebraic, there exists a compact element $a \in R$ such that $a < 1_R$, and by (O5) we obtain a nonzero (compact) element b with $1_R = a + b$. Assume now (2). Then $1_R = a + b$ for nonzero compact elements a and b. If now c is a minimal nonzero element in R, then $c \in R_c$ and we have that $c = ca + cb$, with both ca and cb nonzero as R does not have zero divisors. Note that $ca < c$ because R_c is stably finite, and this contradicts the minimality of c. Finally, it is clear that (2) and (3) are equivalent. □

Corollary 8.2.10. *Let R be a simple, nonelementary, algebraic* Cu*-semiring satisfying (O5). Then R is weakly divisible and weakly cancellative.*

Proof. Let K denote the subsemiring of compact elements in R. Notice that K is conical. By Lemma 8.2.9, K is a nonelementary semiring (with unit 1_R) whose underlying additive monoid is stably finite.

Since R satisfies (O5), we may apply Proposition 5.5.8 to deduce that the partial order on R induces the algebraic order on K.

Using that R is simple and algebraic, it is straightforward to check that K is simple. Then, by Proposition 8.2.6, we have that K is weakly divisible. It follows easily that R is weakly divisible as well. (Indeed, write $1_R = 2a + 3b$ for a and b in K, and then for any $c \in R$, we have $c = 2ac+3bc$.) Moreover, by Proposition 8.2.8, we have that K is cancellative. Using Proposition 5.5.8 again, we obtain that R is weakly cancellative. □

The following proposition is a Cu-semigroup version of results for C^*-algebras that have appeared in [**TW07**, Proposition 5.8] and [**DR09**, Theorem 2.5].

Proposition 8.2.11. *Let R be a simple, nonelementary* Cu*-semiring satisfying (O5) and with a unique normalized functional λ. Then, the following conditions are equivalent:*

(1) There exists a compact element $p \in R$ with $0 < p < 1_R$.
(2) There exists a compact element $p \in R$ with $\lambda(p) \notin \mathbb{N}$.
(3) The set $\lambda(R_c)$ is dense in $\mathbb{R}_+$.
(4) The Cu*-semiring R is weakly divisible.*
(5) The Cu*-semiring R is algebraic.*

Proof. By Theorem 8.1.6, we have that R is almost unperforated. Then, by Proposition 5.2.15, for any two elements a and b in R we have that $a < b$ if and only if $\lambda(a) < \lambda(b)$.

The implications '(4) $\Rightarrow$ (3) $\Rightarrow$ (2)' are clear. By Corollary 8.2.10, we have that (5) implies (4). To show that (2) implies (1), choose a compact element $a \in R$ satisfying $\lambda(a) \notin \mathbb{N}$. Let $n \in \mathbb{N}$ such that $n < \lambda(a) < n + 1$. Then

$$\lambda(n1_R) = n < \lambda(a) < n + 1 = \lambda((n + 1)1_R).$$

As explained at the beginning of the proof, it follows that $n1_R < a < (n + 1)1_R$. Since R satisfies (O5), there exists a compact element p in S such that $a + p = (n + 1)1_R$. Then $0 < \lambda(p) < 1$, which implies that $0 < p < 1_R$, as desired.

Next, we show that (3) implies (5). Let $a \in R$ be an element. We need to show that a is the supremum of an increasing sequence of compact elements. This is clear if a is compact itself.

Thus, we may assume that a is noncompact. By Proposition 5.3.16, we get that a is soft. By assumption, there is a sequence $(b_k)_k$ of compact elements such that $\lambda(b_k)_k$ is strictly increasing with $\lambda(a) = \sup_k \lambda(b_k)$. As observed at the beginning of the proof, it follows that the sequence $(b_k)_k$ is increasing.

Set $b := \sup_k b_k$. Then $\lambda(b) = \lambda(a)$. Since R is stably finite, the element b is noncompact and therefore soft. Then Theorem 5.3.12 implies that $a = b$. Thus, a is the supremum of the increasing sequence $(b_k)_k$ of compact elements, as desired.

Finally, let us show that (1) implies (3). Since λ is multiplicative, we have that $\lambda(R_c)$ is a subsemiring of $[0, \infty]$, which must be dense as it contains arbitrarily small elements.

□

Corollary 8.2.12 (Dadarlat-Rørdam, [**DR09**, Theorem 2.5]). *Let D be a strongly self-absorbing C^*-algebra. Then, D has real rank zero if, and only if, it contains a nontrivial projection.*

8.3. Classification of solid Cu-semirings

We now study the structure of general solid Cu-semirings. The goal is the classification result in Theorem 8.3.13.

THEOREM 8.3.1. *Let R be a solid* Cu*-semiring. Then R is simple. Moreover, if R is stably finite, then it has a unique functional λ that is normalized at the unit element* 1*. This functional is moreover multiplicative.*

PROOF. We first show that R is simple. Given an ideal I in R, consider the map

$$\tau_I\colon R\times R\to\{0,\infty\},\quad \tau_I(a,b)=\begin{cases}0, & \text{if } a\in I \text{ or } b=0\\ \infty, & \text{otherwise,}\end{cases}\qquad (a,b\in R)$$

which is easily checked to be a generalized Cu-bimorphism. Since R is solid, the map τ_I factors through multiplication in R. This means that there exists a generalized Cu-morphism $\widetilde{\tau}_I\colon R\to\{0,\infty\}$ such that $\widetilde{\tau}_I(ab)=\tau_I(a,b)$ for all $a,b\in R$.

Consider the case that I is the ideal generated by 1. Then

$$0=\tau_I(1,a)=\widetilde{\tau}_I(a)=\tau_I(a,1),$$

for all $a\in R$. This implies that $I=R$ and so 1 is a full element.

Now let J be an ideal in R satisfying $J\neq R$. Since 1 is full, this implies $1\notin J$. Let $a\in J$. We deduce that

$$0=\tau_J(a,1)=\widetilde{\tau}_J(a)=\tau_J(1,a).$$

This implies that $a=0$, and so $J=\{0\}$. Thus, we have shown that R is simple.

To show that R has a unique normalized functional, let λ_1 and λ_2 be two functionals on R such that $\lambda_1(1)=\lambda_2(1)=1$. Consider the map

$$\tau\colon R\times R\to[0,\infty],\quad \tau(a,b)=\lambda_1(a)\lambda_2(b).\quad (a,b\in R)$$

It is clear that τ is a generalized Cu-bimorphism. Since R is solid, there exists a generalized Cu-morphism $\widetilde{\tau}\colon R\to[0,\infty]$ such that $\widetilde{\tau}(ab)=\lambda_1(a)\lambda_2(b)$ for all $a,b\in R$. Then, we obtain for all a in R that

$$\lambda_1(a)=\tau(a,1)=\widetilde{\tau}(a1)=\widetilde{\tau}(1a)=\tau(1,a)=\lambda_2(a).$$

Thus, $\lambda_1=\lambda_2$, which shows that R has at most one normalized functional.

Let us now assume that R is stably finite. It follows from Lemma 5.2.3 that there is a functional λ on R such that $\lambda(1)=1$. Thus, the set $F_1(R)$ of functionals that are normalized at 1 is nonempty. By the argument above, we obtain that $F_1(R)=\{\lambda\}$. Hence, the functional λ is an extreme point of $F_1(R)$ and therefore multiplicative, by Proposition 8.1.10.

□

COROLLARY 8.3.2. *Let R be a nonelementary, solid* Cu*-semiring satisfying axiom (O5). Then $R\cong Z\otimes_{\mathrm{Cu}}R$.*

PROOF. By Theorem 8.3.1, we have that R is a simple Cu-semiring. Then, it follows from Corollary 8.1.7 that R satisfies $R\cong Z\otimes_{\mathrm{Cu}}R$. □

8.3.3. Let us recall the classification of solid rings from [**BK72**, Proposition 3.5] and [**BS77**, Proposition 1.10]. Every unital subring of the rational numbers $\mathbb{Q}$ is a (torsion-free) solid ring. Conversely, every torsion-free, solid ring is isomorphic to a unital subring of $\mathbb{Q}$.

Given a set of primes J, we let $\mathbb{Z}\left[J^{-1}\right]$ denote the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and the numbers $\frac{1}{p}$ for every $p \in J$. We associate to J the supernatural number $q_J = \prod_{p\in J} p^\infty$. Then, using the notation from Paragraph 7.4.1, we have that

$$\mathbb{Z}\left[J^{-1}\right] = \mathbb{Z}\left[\left\{\tfrac{1}{p} \mid p \in J\right\}\right] = \mathbb{Z}\left[\tfrac{1}{q_J}\right].$$

Every unital subring of $\mathbb{Q}$ is of the form $\mathbb{Z}\left[J^{-1}\right]$ for some set of primes J.

Given a ring R, we let $t(R)$ denote the torsion part of R. If R is a solid ring, then $R/t(R)$ is a torsion-free, solid ring. It is possible that $R/t(R) = \{0\}$, which happens precisely when the unit of R is a torsion element.

Let us now assume that R is a solid ring whose unit is not torsion. Then $R/t(R)$ is a unital subring of $\mathbb{Q}$, and consequently there is a set of primes J such that

$$R/t(R) \cong \mathbb{Z}[J^{-1}].$$

Furthermore, it is known that the order of every torsion element in R is divides $R/t(R)$. More precisely, one can prove that there is a subset $K \subset J$ and integers $e(p)$ for $p \in K$ such that

$$t(R) \cong \bigoplus_{p\in K} \mathbb{Z}_{p^{e(p)}}.$$

If R is a solid ring whose unit is not torsion and $R/t(R) \cong \mathbb{Z}$, then $t(R) = \{0\}$ and hence $R \cong \mathbb{Z}$. Indeed, if R has nonzero p-torsion elements for a prime p, then p becomes invertible in $R/t(R)$, which is impossible.

Definition 8.3.4. Let R be a solid ring whose unit is not torsion. As explained in Paragraph 8.3.3, there exists a canonical embedding of $R/t(R)$ into $\mathbb{Q}$. Let $\lambda\colon R \to \mathbb{Q}$ be the ring homomorphism obtained by composing the quotient map $R \to R/t(R)$ with the embedding $R/t(R) \subset \mathbb{Q}$. We define R_+ as the set

$$R_+ := \{r \in R \mid \lambda(r) > 0\} \cup \{0\}.$$

Definition 8.3.5. A semiring R is *solid* if for every $a \in R$ we have that the equality

$$a \otimes 1 = 1 \otimes a$$

holds in $R \otimes_{\mathrm{Srg}} R$.

Lemma 8.3.6. *(1) Let S be a solid semiring. Then the Grothendieck-completion $\mathrm{Gr}(S)$ is a solid ring.*

(2) Let R be a solid ring whose unit is not a torsion element. Then the subset R_+ from Definition 8.3.4 is a unital, conical subsemiring of R. Moreover, the semiring R_+ is cancellative, and solid in the sense of Definition 8.3.5, and the algebraic order of R_+ is almost unperforated.

Proof. To show the first part of the statement, let S be a solid semiring. Let R denote the Grothendieck-completion of S, and let

$$\delta\colon S \to R = \mathrm{Gr}(S)$$

denote the natural map. To show that R is solid, let a be an element in R. We need to show that $a \otimes 1 = 1 \otimes a$ in $R \otimes R$.

By properties of the Grothendieck-completion, there exist elements x and y in S such that $a = \delta(x) - \delta(y)$. Using that S is solid at the second step, we deduce that

$$a \otimes 1 = (\delta\otimes\delta)(x\otimes 1) - (\delta\otimes\delta)(y\otimes 1) = (\delta\otimes\delta)(1\otimes x) - (\delta\otimes\delta)(1\otimes y) = 1 \otimes a,$$

as desired.

To show the second part of the statement, let R be a solid ring with non-torsion unit. It is straightforward to check that R_+ is a unital, conical subsemiring of R. Let us show that R_+ is a solid semiring. If $R \cong \mathbb{Z}$, then $R_+ \cong \mathbb{N}$, which is obviously a solid semiring. Therefore, we may assume that $R \not\cong \mathbb{Z}$. Let us denote the inclusion of R_+ into R by $\iota\colon R_+ \to R$. Let $\lambda\colon R \to \mathbb{Q}$ denote the canonical ring homomorphism associated to R, as introduced in Paragraph 8.3.3. By abuse of notation, we denote the composition $\lambda \circ \iota\colon R_+ \to \mathbb{Q}$ also by λ.

We endow R_+ with the algebraic order. Then, for any $a, b \in R_+$ we have

$$a < b \quad \text{if and only if} \quad \lambda(a) < \lambda(b).$$

Note that for every nonzero element a in R_+, we have $\lambda(a) > 0$. It follows easily that R_+ is a simple semiring. This implies that $R_+ \otimes_{\mathrm{Srg}} R_+$ is a simple semiring, as well. It is easy to see that λ is a state when considering it as a map $\lambda\colon R_+ \to \mathbb{R}$. This induces a state $\lambda \otimes \lambda$ on $R_+ \otimes_{\mathrm{Srg}} R_+$, such that $(\lambda \otimes \lambda)(x) > 0$ for every nonzero element $x \in R_+ \otimes_{\mathrm{Srg}} R_+$. It follows that $R_+ \otimes_{\mathrm{Srg}} R_+$ is conical and stably finite.

Since $R \not\cong \mathbb{Z}$, then $R/t(R) \cong \mathbb{Z}[J^{-1}]$ for some, non-empty set of primes J. Then it is easy to see that R_+ contains two nonzero elements c and d such that $1 = c + d$. Note that the elements $1 \otimes c$ and $1 \otimes d$ in $R_+ \otimes_{\mathrm{Srg}} R_+$ are nonzero. Thus, the unit of $R_+ \otimes_{\mathrm{Srg}} R_+$ is equal to $1 \otimes c + 1 \otimes d$, the sum of two nonzero elements. By Proposition 8.2.8, the semiring $R_+ \otimes_{\mathrm{Srg}} R_+$ is cancellative. The following commutative diagram shows the (semi)rings and maps to be considered.

$$\begin{tikzcd}
R_+ \otimes_{\mathrm{Srg}} R_+ \arrow[r, phantom, "="] & R_+ \otimes_{\mathrm{Mon}} R_+ \arrow[r, hook] \arrow[d, "\iota\otimes\iota"] & \mathrm{Gr}(R_+ \otimes_{\mathrm{Mon}} R_+) \arrow[d, "\cong"] \\
R \otimes R \arrow[r, phantom, "="] & R \otimes_{\mathrm{Mon}} R \arrow[r, phantom, "\cong"] & \mathrm{Gr}(R_+) \otimes_{\mathrm{Mon}} \mathrm{Gr}(R_+)
\end{tikzcd}$$

The tensor product of two (semi)rings is just the tensor product of the underlying monoids, equipped with a natural multiplication; see Section B.4. We want to show that the map $\iota \otimes \iota$ is injective. This does not follow directly from the injectivity of ι, since in general the tensor product of two injective morphisms need not be injective again. However, we have shown above that $R_+ \otimes_{\mathrm{Srg}} R_+$ is cancellative. Therefore, the map to the Grothendieck-completion is injective, as indicated by the upper-right horizontal arrow in the diagram.

In general, given two monoids M and N, there is a natural isomorphism between $\mathrm{Gr}(M \otimes_{\mathrm{Mon}} N)$, the Grothendieck-completion of their tensor product, and $\mathrm{Gr}(M) \otimes_{\mathrm{Mon}} \mathrm{Gr}(N)$, the tensor product of their respective Grothendieck-completions; see [**Ful70**, Proposition 17], see also Proposition B.1.7.

It is clear that R is canonically isomorphic to the Grothendieck-completion of R_+. It follows from the commutativity of the above diagram that the map $\iota \otimes \iota$ is injective.

Now let $a \in R_+$ be given. Using that R is a solid ring at the second step, we deduce that

$$(\iota \otimes \iota)(a \otimes 1) = \iota(a) \otimes 1 = 1 \otimes \iota(a) = (\iota \otimes \iota)(1 \otimes a),$$

in $R \otimes R$. Since the map $\iota \otimes \iota$ is injective, this implies that

$$a \otimes 1 = 1 \otimes a,$$

in $R_+ \otimes_{\mathrm{Srg}} R_+$, as desired. It is left to the reader to check that the algebraic order of R_+ is almost unperforated. □

Remark 8.3.7. Let R be a solid ring whose unit is not a torsion element. By Lemma 8.3.6, the subset R_+ of R is a unital, conical, subsemiring. It follows that R has the structure of a partially ordered ring such that the positive cone is given by R_+, see Paragraph B.4.3. It is clear that R is directed. This means that every solid ring with non-torsion unit has a canonical structure as a directed, partially ordered ring.

Lemma 8.3.8. *Let R be a solid ring whose unit is not a torsion element, and let a be an element in R_+. Then there exist k and n in $\mathbb{N}$ such that $na = k1$.*

Proof. We may assume that a is nonzero. Let $\lambda\colon R \to \mathbb{Q}$ be the canonical ring homomorphism introduced in Definition 8.3.4 such that

$$R_+ = \{r \in R \mid \lambda(r) > 0\} \cup \{0\}.$$

Choose positive $k_0, n_0 \in \mathbb{N}$ such that $\lambda(a) = \frac{k_0}{n_0}$. Then

$$\lambda(n_0 a - k_0 1) = 0.$$

Therefore, $n_0 a - k_0 1$ is a torsion element of R. Let m be its order. Then

$$(mn_0)a = (mk_0)1,$$

which shows that $n := mn_0$ and $k := mk_0$ are numbers with the desired properties. □

Proposition 8.3.9. *There is a natural one-to-one correspondence between the following classes:*

(1) Solid rings whose unit is not a torsion element.
(2) Solid, cancellative, conical semirings for which the algebraic order is almost unperforated.

The correspondence is given by associating to a solid ring R with non-torsion unit the solid semiring R_+ from Definition 8.3.4, and conversely by associating to a solid semiring S its Grothendieck-completion $\mathrm{Gr}(S)$.

Proof. It follows easily from Lemma 8.3.6 that the assignments of the statement are well-defined. Thus, it remains to show that the assignments are inverse to each other. Given a solid ring R with non-torsion unit, it is easy to check that the Grothendieck-completion of R_+ is (canonically) isomorphic to R.

Conversely, let S be a solid, cancellative, conical semiring for which the algebraic order is almost unperforated. Let R be the Grothendieck-completion of S. Since S is cancellative, we can consider S as a unital subsemiring of R. We need to show that $S = R_+$.

Let $\lambda\colon R \to \mathbb{Q}$ be the ring homomorphism such that

$$R_+ = \{r \in R \mid \lambda(r) > 0\} \cup \{0\}.$$

We first show that $S \subset R_+$. Since S is conical, it contains no torsion element. Thus, every element a of S satisfies either $\lambda(a) \neq 0$ or $a = 0$.

Assume that an element $a \in S$ satisfies $\lambda(a) < 0$. Choose $k, n \in \mathbb{N}$ such that $\lambda(a) = -\frac{k}{n}$. Then

$$\lambda(na + k1) = 0.$$

Since 1 is an element of S, we have that $na + k1$ is an element of S, and therefore $na + k1 = 0$. This contradicts conicality of S. Thus, we have that $\lambda(a) \geq 0$ for each $a \in S$. It follows that $S \subset R_+$, as desired.

To show that $R_+ \subset S$, let a be an element in R_+. We may assume that a is nonzero. Since R is the Grothendieck-completion of S, there exist $x, y \in S$ such that $a + x = y$. Then y is nonzero, and we may clearly assume that x is also nonzero. Let us show that $x <_s y$. It follows from the proof of Lemma 8.3.8, that there are n, k_1 and k_2 in $\mathbb{N}$ such that

$$nx = k_1 1, \quad ny = k_2 1.$$

Since a is nonzero, we have $k_1 < k_2$. Then $k_2 n x = k_1 k_2 1 = k_1 n y$, with $k_1 n < k_2 n$, and thus $x <_s y$. Since S is almost unperforated (with the algebraic order), we have that $x \leq y$. Using that the order of S is algebraic and that S is cancellative, we deduce that a belongs to S, as desired. □

Remark 8.3.10. In Proposition B.4.4, we recall the natural one-to-one correspondence between directed, partially ordered rings and cancellative, conical semirings, given by assigning to a partially ordered ring its positive cone, and conversely by associating to a conical semiring its Grothendieck-completion. Every solid ring whose unit is not a torsion element has a canonical structure as a directed partially ordered ring, see Remark 8.3.7. Then, Proposition 8.3.9 shows that the above correspondence restricts to a natural identification between directed, partially ordered rings coming from solid rings and cancellative, conical semirings that are solid and whose algebraic order is almost unperforated.

Lemma 8.3.11. *(1) Let K be a solid, cancellative, conical semiring. Then its* Cu*-completion* $\mathrm{Cu}(K)$ *as constructed in Paragraph 8.2.1 is a solid* Cu*-semiring.*

(2) Let R be a solid, nonelementary, algebraic Cu*-semiring satisfying (O5). Then the subsemiring of compact elements R_c is a solid, nonelementary, cancellative, conical semiring for which the algebraic order is almost unperforated.*

Proof. Let K be a cancellative, conical semiring. Consider the tensor square $K \otimes_{\mathrm{Mon}} K$ of K in the category Mon of monoids. Equipped with the natural multiplication, the monoid $K \otimes_{\mathrm{Mon}} K$ becomes a semiring, denoted by $K \otimes_{\mathrm{Srg}} K$, which is the tensor square of K in the category Srg of (unital, commutative) semirings, see Paragraph B.4.1.

As explained in Paragraph 5.5.3, we obtain a W-semigroup $(K, \leq)$ if we equip the monoid K with the auxiliary relation that is equal to its partial order. As shown in Paragraph 8.2.1, it follows that the Cu-completion of $(K, \leq)$ is a Cu-semiring which we denote by $\mathrm{Cu}(K)$. We denote the universal W-morphism to the Cu-completion by

$$\alpha\colon K \to \mathrm{Cu}(K).$$

Considering K and $\mathrm{Cu}(K)$ as semirings, the map α is a semiring homomorphism and an order-embedding that identifies K with the compact elements of $\mathrm{Cu}(K)$, see Remarks 3.1.9 and Proposition 5.5.4. We therefore think of K as as subset of $\mathrm{Cu}(K)$ and identify a with $\alpha(a)$.

The map

$$K \times K \to \mathrm{Cu}(K) \otimes_{\mathrm{Cu}} \mathrm{Cu}(K), \quad (a, b) \mapsto a \otimes b, \quad (a, b \in K)$$

is a monoid bimorphism and therefore induces a monoid homomorphism

$$\varphi\colon K \otimes_{\mathrm{Mon}} K \to \mathrm{Cu}(K) \otimes_{\mathrm{Cu}} \mathrm{Cu}(K)$$

such that $\varphi(a \otimes b) = a \otimes b$ for each $a, b \in K$.

To show the first part of the statements, assume that K is a solid, cancellative, conical semiring. Let a be an element of $\mathrm{Cu}(K)$. In order to prove that $\mathrm{Cu}(K)$ is a solid Cu-semiring, we need to show by Proposition 7.1.6 that $1 \otimes a = a \otimes 1$ in $\mathrm{Cu}(K) \otimes_{\mathrm{Cu}} \mathrm{Cu}(K)$. Assume first that a is a compact element. Then a is an element of K. Using that K is solid at the second step, we obtain that

$$1 \otimes a = \varphi(1 \otimes a) = \varphi(a \otimes 1) = a \otimes 1.$$

If a is a not necessarily compact element, then there exists an increasing sequence $(a_k)_k$ of compact elements in $\mathrm{Cu}(K)$ such that $a = \sup_k a_k$. Then

$$1 \otimes a = 1 \otimes (\sup_k a_k) = \sup_k (1 \otimes a_k) = \sup_k (a_k \otimes 1) = a \otimes 1.$$

Next, let us show the second part of the statement. So let R be a solid, nonelementary, algebraic Cu-semiring satisfying (O5). By Theorem 8.3.1, R is simple. Then, it follows from Corollary 8.2.10 that R is weakly divisible and weakly cancellative.

We set $K = R_c$, the subsemiring of compact elements, and we identify R with $\mathrm{Cu}(K)$. It follows from Proposition 5.5.8 that K is a conical, cancellative semiring such that the order on K induced by R is the algebraic order. We know from Lemma 8.2.9 that K is a stably finite nonelementary semiring. Moreover, by Theorem 8.3.1, R has a unique normalized functional, which we denote by λ. We have that $\lambda(a) > 0$ for every nonzero element of R. The map λ is a state on K, which induces a state $\lambda\otimes\lambda$ on $K\otimes_{\mathrm{Srg}}K$ with the property that $(\lambda\otimes\lambda)(x) > 0$ for every nonzero element x of $K\otimes_{\mathrm{Srg}}K$. It follows that $K\otimes_{\mathrm{Srg}}K$ is stably finite. Since K is a nonelementary semiring we see, as in the proof of Lemma 8.3.6, that $K\otimes_{\mathrm{Srg}}K$ is also nonelementary. It is also straightforward to deduce that $K\otimes_{\mathrm{Srg}}K$ is simple and weakly divisible. Therefore, by Proposition 8.2.8, the semiring $K\otimes_{\mathrm{Srg}}K$ is cancellative. It follows that its algebraic pre-order is partial. Thus, we have shown that the natural quotient map

$$K\otimes_{\mathrm{Mon}}K \to K\otimes_{\mathrm{PoM}}K$$

is an isomorphism.

We obtain a W-semigroup by equipping $K\otimes_{\mathrm{Mon}}K$ with the auxiliary relation that is equal to its partial order. The tensor square of $(K,\leq)$ in the category PreW is given as the tensor square in PoM of the underlying monoids together with a naturally defined auxiliary relation, see Definition 6.2.9. It follows that there is a natural isomorphism

$$(K\otimes_{\mathrm{Mon}}K,\leq)\cong(K,\leq)\otimes_{\mathrm{PreW}}(K,\leq)$$

Applying Cu-completions to both sides and using Theorem 6.3.5, we deduce that there is a natural isomorphism

$$\mathrm{Cu}(K\otimes_{\mathrm{Srg}}K)\cong\mathrm{Cu}(K)\otimes_{\mathrm{Cu}}\mathrm{Cu}(K).$$

We denote the universal W-morphism to the Cu-completion of $K\otimes_{\mathrm{Srg}}K$ by

$$\beta\colon K\otimes_{\mathrm{Srg}}K\to\mathrm{Cu}(K\otimes_{\mathrm{Srg}}K).$$

By Remarks 3.1.9, the map β is an order-embedding. In conclusion, we have that the map φ from the beginning of the proof is an order-embedding.

Then, if a is an element of K, we have

$$\beta(1\otimes a)=1\otimes\alpha(a)=\alpha(a)\otimes 1=\beta(a\otimes 1).$$

Since β is an oder-embedding, we obtain that $1\otimes a=a\otimes 1$ in $K\otimes_{\mathrm{Srg}}K$. Thus, K is a solid semiring. It is easy to check that K is almost unperforated. □

THEOREM 8.3.12. *There is a natural one-to-one correspondence between each of the following classes:*

(1) Solid rings whose unit is not a torsion element and that are not isomorphic to $\mathbb{Z}$.
(2) Solid, nonelementary, cancellative, conical semirings for which the algebraic order is almost unperforated.
(3) Solid, nonelementary, algebraic Cu*-semirings satisfying (O5).*

The correspondence between (1) and (2) is given by associating to a solid ring R with non-torsion unit the solid semiring R_+ from Definition 8.3.4, and conversely by associating to a solid semiring S its Grothendieck-completion $\mathrm{Gr}(S)$*.*

The correspondence between (2) and (3) is given by associating to a solid semiring K the Cu*-semiring* $\mathrm{Cu}(K)$ *as constructed in Paragraph 8.2.1, and conversely by associating to a solid, algebraic* Cu*-semiring S its subsemiring of compact elements.*

Proof. The correspondence between the classes (1) and (2) follows directly from Proposition 8.3.9. Let us show the correspondence between the classes (2) and (3), It follows easily from Lemma 8.3.11 that the assignments of the statement are well-defined. Thus, it remains to show that the assignments are inverse to each other. This follows directly from Proposition 5.5.4. □

Theorem 8.3.13. *Let S be a solid* Cu*-semiring satisfying (O5). If S is nonelementary, then exactly one of the following statements holds:*

(1) We have $S \cong [0,\infty]$.
(2) We have $S \cong Z$.
(3) There is a solid ring R with non-torsion unit such that $R \ncong \mathbb{Z}$ and such that $S \cong \mathrm{Cu}(R_+)$.

If S is elementary and satisfies (O6), then exactly one of the following conditions holds:

(4) We have $S \cong \mathbb{N}$.
(5) There is $k \in \mathbb{N}$ such that $S \cong E_k = \{0,1,2,\ldots,k,\infty\}$.

The Cu*-semiring S is algebraic if and only if it satisfies (3), (4) or (5).*

Proof. We have observed that all Cu-semirings in statements (1)-(5) are solid. It is also clear that a solid ring can satisfy at most one of the statements.

So let S be a solid Cu-semiring. We will show that S satisfies one of the statements.

Case 1: Assume S is nonelementary. By Theorem 8.3.1, we obtain that S is simple. Therefore, by Theorem 8.1.6 and Corollary 8.1.9, S is almost unperforated, almost divisible and stably finite. Then, using Theorem 8.3.1 again, we have that S has a unique normalized functional. We obtain from Theorem 7.6.6 that there is a canonical decomposition

$$S = S_c \sqcup (0,\infty].$$

If S contains no nonzero compact element, then $S \cong [0,\infty]$ by Corollary 8.1.12. Otherwise, by Corollary 8.1.9, the unit of S is compact.

Let $\lambda\colon S \to [0,\infty]$ denote the unique normalized functional on S. It follows from Proposition 8.2.11 that either $\lambda(S_c) \subset \mathbb{N}$ or that S is algebraic. In the latter case, S satisfies (3), see Theorem 8.3.12. Thus, let us assume that $\lambda(S_c) \subset \mathbb{N}$. Then we consider the map

$$\alpha\colon S = S_c \sqcup (0,\infty] \to Z = \mathbb{N} \sqcup (0,\infty],$$

which maps a compact element r in S to the compact element $\lambda(r) \in \mathbb{N} \subset Z$, and which maps a soft element in $S_{\mathrm{soft}} = (0,\infty]$ to the same in $Z_{\mathrm{soft}} = (0,\infty]$. It is straightforward to check that α is a unital, multiplicative Cu-morphism. It follows from Proposition 7.1.13 that $Z \otimes_{\mathrm{Cu}} S \cong Z$. By Corollary 8.3.2, we also have $S \cong S \otimes_{\mathrm{Cu}} Z$. It follows that $S \cong Z$, which shows that S satisfies (2).

Case 2: Assume that R is elementary and satisfies (O6). Then S satisfies (4) or (5), see Example 8.1.2. □

Remark 8.3.14. We remark that, if S is a nonelementary solid Cu-semiring satisfying (O5), then by the classification above, we see that (O6) is also satisfied. Indeed, only the case where $S = \mathrm{Cu}(R_+)$ for a solid ring R with nontorsion unit needs verification. In this situation, by Proposition 5.5.8, it is enough to show that R_+, endowed with the algebraic order, is a Riesz semigroup. This is easy to check once we note that $R/t(R) \cong \mathbb{Z}[J^{-1}]$ for a nonemtpy set of primes J.

EXAMPLE 8.3.15. Let $\mathbb{Q}$ be the solid ring of rational numbers. We obtain a corresponding solid Cu-semiring, which we denote by Q. Thus

$$Q = \mathrm{Cu}(\mathbb{Q}_+) \cong \mathbb{Q}_+ \sqcup (0, \infty].$$

In Proposition 8.3.17, we will see that Q is the terminal object in a suitable category.

We end this section with a result about initial and terminal objects among solid Cu-semirings. This can be considered as a Cu-semigroups version of [**Win11**, Corollary 3.2], which characterizes the Jiang-Su algebra $\mathcal{Z}$ as the initial object in the category of unital, strongly self-absorbing C^*-algebras with *-homomorphisms up to approximate unitary equivalence.

LEMMA 8.3.16. *Let S be a solid, stably finite,* Cu*-semiring satisfying (O5), and let λ be its unique normalized functional. Then, $\lambda(S_c) \subseteq \mathbb{Q}_+$.*

PROOF. If S is elementary (and stably finite), then clearly $\lambda(S_c) = \mathbb{N}$. Otherwise, if S is nonelementary, then by Proposition 8.2.11 either $\lambda(S_c) \subseteq \mathbb{N}$ or S is algebraic. Finally, if S is algebraic, by Lemma 8.3.11 S_c is a conical, cancellative, solid semiring for which the algebraic order is almost unperforated. Now the result follows from Proposition 8.3.9 and Lemma 8.3.8. □

We remark that for every simple, nonelementary Cu-semiring S that satisfies (O5) and has a compact unit, there exists a multiplicative Cu-morphism from Z to S. This follows from Corollary 8.1.7. If S has a unique normalized functional, then the map is unique.

PROPOSITION 8.3.17. *Let S be a solid, nonelementary* Cu*-semiring satisfying (O5) and for which the unit is a compact element. Let Q be the solid* Cu*-semiring from Example 8.3.15. Then, there are unique unital, multiplicative* Cu*-morphisms*

$$Z \to S \to Q.$$

Thus, $Z \otimes_{\mathrm{Cu}} S \cong S$ and $S \otimes_{\mathrm{Cu}} Q \cong Q$.

This means that Z and Q are the initial and final objects of the category of considered Cu*-semirings with unital, multiplicative* Cu*-morphisms.*

PROOF. The existence and uniqueness of the map $Z \to S$ is observed in the paragraph before this proposition. Let S be as in the statement. As in the beginning of the proof of Theorem 8.3.13, we obtain that there is a natural decomposition

$$S = S_c \sqcup (0, \infty].$$

By Theorem 8.3.1 and Lemma 8.3.16, S has a unique normalized functional λ and $\lambda(S_c) \subset \mathbb{Q}_+$. By Proposition 8.1.10, λ is multiplicative. Thus, we may consider the map

$$\alpha\colon S = S_c \sqcup (0, \infty] \to Q = \mathbb{Q}_+ \sqcup (0, \infty],$$

which maps a compact element r in S_c to the compact element $\lambda(r) \in \mathbb{Q}_+ \subset Q$, and which maps a soft element in $S_{\mathrm{soft}} = (0, \infty]$ to the same in $Q_{\mathrm{soft}} = (0, \infty]$. It is easy to see that α is a unital, multiplicative Cu-morphism, as desired.

It is left to the reader to show uniqueness of the maps $Z \to S$ and $S \to Q$. The results about tensorial absorption follow from Proposition 7.1.13. □

CHAPTER 9

Concluding remarks and Open Problems

In this chapter we list some problems that we believe to be open and that have appeared in the course of our investigations.

(1) Problem 6.4.11: Let A and B be two C^*-algebras. When are the natural Cu-morphisms

$$\tau_{A,B}^{\min}\colon \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\min} B),$$
$$\tau_{A,B}^{\max}\colon \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\max} B),$$

from Paragraph 6.4.10 surjective, or order-embeddings, or isomorphisms? More generally, what is the relation between $\mathrm{Cu}(A)\otimes_{\mathrm{Cu}}\mathrm{Cu}(B)$ and $\mathrm{Cu}(A\otimes B)$?

In Paragraph 6.4.12, we mention some partial result concerning this problem. This problem asks for a more general formula of a Künneth type flavor. It looks plausible that such a formula will have to take the K_1-groups of the involved C^*-algebras into account. One possible invariant is $\mathrm{Cu}_{\mathbb{T}}(-)$, as introduced in [**ADPS14**], which for a C^*-algebra A is defined as

$$\mathrm{Cu}\,(C(\mathrm{T}) \otimes A).$$

In significant cases, this invariant records both the Cuntz semigroups of A and the K_1-group of A.

(2) Is Cu a closed category?

This is a natural question given that Cu is a symmetric, monoidal category. A positive answer to this problem would provide additional structure to the morphism sets in Cu, and this is a potentially useful tool in connection with the current development of a bivariant version of the Cuntz semigroup (see [**BTZ14**]).

(3) Problem 6.4.2: Given Cu-semigroups S and T that satisfy (O5) (respectively (O6), weak cancellation). When does $S \otimes_{\mathrm{Cu}} T$ satisfy (O5) (respectively (O6), weak cancellation)?

In Paragraph 6.4.3, we mention some partial result concerning this problem. A particular variant of this problem is:

(4) Problem 7.3.11: When does axiom (O5), (O6) or weak cancellation pass from a Cu-semigroup S to the tensor product $Z \otimes_{\mathrm{Cu}} S$?

(5) Let X be a finite-dimensional, compact, Hausdorff space, and let S be a Cu-semigroup. It was proved in [**APS11**, Theorem 5.15] that the semigroup of lower semicontinuous functions from X to S, denoted by $\mathrm{Lsc}(X, S)$, is a Cu-semigroup. Under which conditions on S and X does $\mathrm{Lsc}(X, S)$ satisfy (O5) (respectively (O6), weak cancellation)?

We remark that if $\mathrm{Lsc}(X, S)$ satisfies (O5) (respectively (O6), weak cancellation), then so does S. The natural test case is $X = [0, 1]$. A positive answer seems likely if S is algebraic.

(6) Let X be a finite-dimensional, compact, Hausdorff space, and let S be a Cu-semigroup. When does it hold that $\mathrm{Lsc}(X,S) = \mathrm{Lsc}(X,\overline{\mathbb{N}}) \otimes_{\mathrm{Cu}} S$?

We showed in Corollary 6.4.5 that this question has a negative answer for $X = [0,1]$ and $S = Z$. On the other hand, a positive answer seems likely if S is algebraic.

(7) Range problem: Under which conditions can a Cu-semigroup S be realized as the Cuntz semigroup $\mathrm{Cu}(A)$ of a C^*-algebra A?

Necessarily, such a Cu-semigroup satisfies (O5) and (O6). Thus, we are asking for additional conditions on Cu-semigroups beyond these two axioms which would guarantee that a Cu-semigrouip is realized by a C^*-algebra.

As we have already mentioned, L. Robert showed in [**Rob13b**] that if X is a compact Hausdorff space whose covering dimension is at least 3, then there is no C^*-algebra A such that $\mathrm{Cu}(A) \cong \mathrm{Lsc}(X,\overline{\mathbb{N}})$. It was shown by J. Bosa (private communication), that none of the elementary semigroups E_k as described in Paragraph 5.1.16 for $k \geq 1$ are realized as the Cuntz semigroup of a C^*-algebra. (The Cu-semigroup $E_0 = \{0,\infty\}$ is of course the Cuntz semigroup of any purely infinite simple C^*-algebra.)

(8) Consider the Cu-semigroup Z' defined as follows:

$$Z' = \{0,1,1',2,3,4,\ldots\} \sqcup (0,\infty],$$

with addition as in Z, except that $k+1' = k+1$ for any $k \in \mathbb{N}_+$. It is easy to check that Z' is a simple, stably finite Cu-semigroup satisfying (O5) and (O6), but it is not weakly cancellative as $1+1' = 1'+1'$ but $1 \neq 1'$. It is also easy to prove that Z' has Z-multiplication, so that $Z' \cong Z \otimes_{\mathrm{Cu}} Z'$.

A particularly interesting instance of the range problem is the following: Does there exist a (separable, unital, simple, stably finite) C^*-algebra A such that $\mathrm{Cu}(A) \cong Z'$?

Note that if such a C^*-algebra A exists, then it is necessarily simple, not $\mathcal{Z}$-stable, and not nuclear. For if A is $\mathcal{Z}$-stable, then it has stable rank one and hence its Cuntz semigroup has weak cancellation. Similarly, if A is nuclear, then as Z' has only one normalized functional, we would get that A is monotracial. In that situation, the solution of the Toms-Winter conjecture (see [**MS12**]) would imply that A is $\mathcal{Z}$-stable, a contradiction.

The Cu-semigroup Z' seems to be the simplest example that is not weakly cancellative and has Z-multiplication. A more general question is then:

(9) Does there exist a finite, simple C^*-algebra A, such that $\mathrm{Cu}(A)$ has Z-multiplication, but is not weakly cancellative?

Let A be such a C^*-algebra. If A is nuclear, then we could deduce as above that A is not $\mathcal{Z}$-stable although $\mathrm{Cu}(A)$ is almost unperforated. Therefore, the Toms-Winter Conjecture predicts that A cannot be nuclear.

It is natural to seek for additional axioms that allow us to rule out Z' as a semigroup in a future reformulation of the category Cu. We make this explicit with the following question.

(10) Under what additional axioms (besides (O5) and (O6)) is a simple Cu-semigroup with Z-multiplication necessarily weakly cancellative?

This question also refers to structural properties of Cu-semigroups. In this direction, Conjecture 5.6.18 bears repeating. Let us recall from Definition 5.6.1 that a Cu-semigroup S is nearly unperforated if and only if

$a \leq b$ whenever $a \leq_p b$ for any pair of elements a and b in S. Equivalently, by Lemma 5.6.2, we have $a \leq b$ whenever $2a \leq 2b$ and $3a \leq 3b$.

(11) Conjecture 5.6.18: Let A be a $\mathcal{Z}$-stable C^*-algebra. Then $\mathrm{Cu}(A)$ is nearly unperforated.

(12) Problem 5.1.5: Let S be a Cu-semigroup, and let I be an ideal in S. Assume that I and S/I satisfy (O5) (resp. (O6), weak cancellation). Under what assumptions does this imply that S itself satisfies the respective axiom?

(13) Problem 5.3.14: Given a Cu-semigroup S, is the subsemigroup S_{soft} of soft elements again a Cu-semigroup? Does this hold under the additional assumption that S satisfies (O5)? If so, does then S_{soft} satisfy (O5) as well?

(14) Problem 5.6.13: Let S be an almost unperforated Cu-semigroup. Which conditions are necessary and sufficient for S to be nearly unperforated? In particular, is it sufficient to assume that S satisfies weak cancellation and (O5)?

(15) Problem 7.2.10: Let A and B be two C^*-algebras. Assuming that A or B is purely infinite, does it follows that the C^*-bimorphism

$$\tau_{A,B}^{\mathrm{min}} \colon \mathrm{Cu}(A) \otimes_{\mathrm{Cu}} \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes_{\mathrm{min}} B).$$

is an isomorphism?

(16) Problem 7.3.10: Let S be a Cu-semigroup, and let $a, b \in S$. Characterize when $1 \otimes a \leq 1 \otimes b$ in $Z \otimes_{\mathrm{Cu}} S$.

(17) Problem 7.6.2: Given a strongly self-absorbing C^*-algebra D, is the Cuntz semiring $\mathrm{Cu}(D)$ a solid Cu-semiring?

Should the answer to this problem be positive, our Theorem 8.3.13 (see also Remark 8.3.14) would yield a complete list of the possible Cuntz semigroups for stably finite, strongly self-absorbing C^*-algebras, and this would be valuable information towards finding a possible non-UCT example, if such exists.

(18) Problem 7.6.7: Let S be a Cu-semigroup. Is it true that $S_R = L(F(S))$?

APPENDIX A

Monoidal and enriched categories

In this appendix, we will recall the basic theory of monoidal and enriched categories. For details we refer the reader to [**Kel05**] and [**Mac71**].

A.0.1 (Monoidal categories). A *monoidal category* $\mathcal{V}$ consists of the following data: An underlying category $\mathcal{V}_0$; and a bifunctor

$$\otimes \colon \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0;$$

and a unit object I in $\mathcal{V}_0$; and for each triple of objects X, Y and Z in $\mathcal{V}_0$ a natural isomorphism

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z);$$

and for each object X in $\mathcal{V}_0$ two natural isomorphisms

$$I \otimes X \cong X, \quad X \otimes I \cong X.$$

Moreover, certain coherence axioms need to be satisfied.

By an object in $\mathcal{V}$ we mean an object of the underlying category, and similarly for $\mathcal{V}$-morphisms. Given objects X and Y of $\mathcal{V}$, we let $\mathcal{V}(X, Y)$ denote the collection of $\mathcal{V}$-morphisms from X to Y, and we will always assume that it is a set. (This means that $\mathcal{V}$ is *locally small*.)

The monoidal category $\mathcal{V}$ is *symmetric* if for each pair of objects X and Y in $\mathcal{V}_0$ there is a natural isomorphism

$$X \otimes Y \cong Y \otimes X.$$

A.0.2 (Concrete monoidal categories). Let $\mathcal{V}$ be a monoidal category with unit object I. Since $\mathcal{V}$ is assumed to be locally small, the representable functor $\mathcal{V}(I, \_)$ is a functor from $\mathcal{V}_0$ to the category of sets. We denote this functor by

$$V \colon \mathcal{V}_0 \to \mathrm{Set}.$$

We say that $\mathcal{C}$ is a *concrete monoidal category* if V is faithful. In that case we can think of objects in $\mathcal{V}$ as sets with additional structure, and we can think of morphisms in $\mathcal{V}$ as maps preserving that structure.

Let X be an object in $\mathcal{V}$. Then, an *element x of X* is an element in $V(X)$, that is, a $\mathcal{V}$-morphism $x \colon I \to X$. We write $x \in X$ to denote that x is an element of X. This terminology is even used when V is not necessarily faithful.

Let X and Y be a pair of objects in $\mathcal{V}$, and let $x \in X$ and $y \in Y$. Then, the composed morphism

$$I \xrightarrow{\cong} I \otimes I \xrightarrow{x \otimes y} X \otimes Y,$$

is an element of $X \otimes Y$, which we will also denote by $x \otimes y$.

Example A.0.3. The category Top of topological spaces is a concrete, symmetric monoidal category. The tensor product of two spaces is their Cartesian product with the product topology. The unit object is the one-element space.

A.0.4 (Enriched categories). Let $\mathcal{V}$ be a monoidal category. A *$\mathcal{V}$-category $\mathcal{C}$* (also called a category *enriched over $\mathcal{V}$*) consists of the following data: A collection of objects in $\mathcal{C}$; and for each pair of objects X and Y in $\mathcal{C}$ an object $\mathcal{C}(X, Y)$ in $\mathcal{V}$, playing the role of the collection of morphisms from X to Y; and for each object X in $\mathcal{C}$ an element id_X in $\mathcal{C}(X,X)$, called the identity on X and playing the role of the identity morphism on X; and for each triple of objects X, Y and Z in $\mathcal{C}$ a $\mathcal{V}$-morphism

$$M_{X,Y,Z} \colon \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \to \mathcal{C}(X,Z),$$

implementing the composition of morphisms. This structure is required to satisfy certain conditions expressing for example the associativity of composition of morphisms.

Recall that the element id_X in $\mathcal{C}(X,X)$ is, by definition, a $\mathcal{V}$-morphism

$$\mathrm{id}_X \colon I \to \mathcal{C}(X,X).$$

Given two $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$, a *$\mathcal{V}$-functor* F from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data: an assignment F from the objects of $\mathcal{C}$ to the objects of $\mathcal{D}$; and for each pair of objects X and Y in $\mathcal{C}$ a $\mathcal{V}$-morphism

$$F_{X,Y} \colon \mathcal{C}(X,Y) \to \mathcal{D}\left(F(X), F(Y)\right).$$

It is required that certain diagrams, which for instance express compatibility with composition of morphisms, commute. (Again, we refer to [**Kel05**] for details.)

Given two $\mathcal{V}$-functors F and G from $\mathcal{C}$ to $\mathcal{D}$, a *$\mathcal{V}$-natural transformation* from F to G, denoted by $F \Rightarrow G$, is a collection of elements $\alpha_X \in \mathcal{D}(F(X), G(X))$, indexed by the objects X in $\mathcal{C}$, such that certain natural conditions are satisfied.

A.0.5 (Concrete enriched categories). Let $\mathcal{C}$ be a category that is enriched over the concrete monoidal category $\mathcal{V}$. We can use the faithful functor V to associate to $\mathcal{C}$ an ordinary underlying category $\mathcal{C}_0$ as follows: The objects of $\mathcal{C}_0$ are the same as the objects of $\mathcal{C}$; and the $\mathcal{C}_0$-morphisms between two objects X and Y in $\mathcal{C}_0$ are given as

$$\mathcal{C}_0(X,Y) := V\left(\mathcal{C}(X,Y)\right) = \mathcal{V}\left(I, \mathcal{C}(X,Y)\right);$$

and for each object X in $\mathcal{C}_0$, the identity morphism id_X in $\mathcal{C}_0$ is just the $\mathcal{V}$-morphism

$$\mathrm{id}_X \colon I \to \mathcal{C}(X,X),$$

considered as an element of $\mathcal{C}_0(X,X)$; and for each pair of $\mathcal{C}_0$-morphisms $f \in \mathcal{C}_0(X,Y)$ and $g \in \mathcal{C}_0(Y,Z)$, which by definition are $\mathcal{V}$-morphisms

$$f \colon I \to \mathcal{C}(X,Y), \quad g \colon I \to \mathcal{C}(Y,Z),$$

we consider the composed $\mathcal{V}$-morphism

$$I \xrightarrow{g \otimes f} \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \xrightarrow{M_{X,Y,Z}} \mathcal{C}(X,Z),$$

which is an element of $\mathcal{C}_0(X,Z)$ defining the composition $g \circ f$ in $\mathcal{C}_0$. Using the coherence axioms for monoidal and enriched structures, one can show that the laws of a category are fulfilled for $\mathcal{C}_0$.

We can then think of $\mathcal{C}(X,Y)$ as the set of morphisms $\mathcal{C}_0(X,Y)$ endowed with additional structure making it into an object in $\mathcal{V}$.

A.0.6 (Closed Categories). A monoidal category $\mathcal{V}$ is *closed* if, for each object Y in $\mathcal{V}$, the functor

$$\_ \otimes Y \colon \mathcal{V} \to \mathcal{V}$$

has a right adjoint, which we will denote by $(\_)^Y$. Thus, in a closed, monoidal category $\mathcal{V}$, for any three objects X, Y and Z, there is a bijection (natural in X and Z) between the following hom-sets:

$$\mathcal{V}(X \otimes Y, Z) \cong \mathcal{V}\left(X, Z^Y\right). \tag{A.1}$$

Given $\mathcal{V}$-morphisms $f\colon Y_2 \to Y_1$ and $g\colon Z_1 \to Z_2$, there is an induced $\mathcal{V}$-morphism $(f^*, g)\colon Z_1^{Y_1} \to Z_2^{Y_2}$. This defines a bifunctor

$$(\_)^{(\_)}\colon \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V},$$

called the *internal hom-bifunctor* of $\mathcal{V}$.

Let Z and Y be two objects in $\mathcal{V}$. The evaluation morphism

$$\varepsilon_Y^Z\colon Z^Y \otimes Y \to Z,$$

is defined as the $\mathcal{V}$-morphism that corresponds to the identity morphism under the natural identification

$$\mathcal{V}\left(Z^Y \otimes Y, Z\right) \cong \mathcal{V}\left(Z^Y, Z^Y\right).$$

Let $\mathcal{V}$ be a closed, symmetric, monoidal category. Then $\mathcal{V}$ is enriched over itself. For each pair of objects X and Y in $\mathcal{V}$, we use the internal hom-bifunctor to obtain an object Y^X in $\mathcal{V}$ playing the role of morphisms from X to Y; and for each object X in $\mathcal{V}$, the identity morphism $\mathrm{id}_X \in X^X$ is defined as the $\mathcal{V}$-morphism $\mathrm{id}_X\colon I \to X^X$ corresponding to the identity morphism from X to X under the identification

$$\mathcal{V}(I, X^X) \cong \mathcal{V}(I \otimes X, X) \cong \mathcal{V}(X, X);$$

and for each triple of objects X, Y and Z in $\mathcal{V}$, the map

$$M_{X,Y,Z}\colon Z^Y \otimes Y^X \to Z^X,$$

implementing the composition of morphisms is the $\mathcal{V}$-morphism that under the identification

$$\mathcal{V}\left(Z^Y \otimes Y^X, Z^X\right) \cong \mathcal{V}\left(\left(Z^Y \otimes Y^X\right) \otimes X, Z\right)$$

corresponds to the composition

$$\left(Z^Y \otimes Y^X\right) \otimes X \cong Z^Y \otimes \left(Y^X \otimes X\right) \xrightarrow{\mathrm{id}_{Z^Y} \otimes \varepsilon_X^Y} Z^Y \otimes Y \xrightarrow{\varepsilon_Y^Z} Z.$$

Examples A.0.7. (1) We will see in Paragraphs B.1.5 and B.2.6 that the category Mon of monoids and the category PoM of positively ordered monoids are both closed, symmetric, monoidal categories.

(2) The category Top is not closed. However, it contains several full, symmetric, monoidal subcategories that are closed.

A Hausdorff, topological space X is *compactly generated* if a subset $M \subset X$ is closed whenever $M \cap K$ is closed for every compact subset K of X. The class of compactly generated, Hausdorff spaces contains for example all metric space and all locally compact, Hausdorff spaces. We let CGHTop denote the full subcategory of Top consisting of compactly generated, Hausdorff spaces. It is known that CGHTop is a reflective subcategory of Top. Given a topological space X, we let X_k denote its reflection in CGHTop.

Let X and Y be two compactly generated, Hausdorff spaces. Their Cartesian product need not be compactly generated. Therefore, the 'correct' tensor product of X and Y in the category CGHTop is $(X \times Y)_k$. Consider the set $C(X, Y)$ of continuous maps from X to Y equipped with the compact open topology. Again, $C(X, Y)$ need not be compactly generated. Nevertheless, CGHTop is a closed category with internal hom-bifunctor given by

$$\mathrm{CGHTop}(X, Y) = (C(X, Y))_k.$$

Then, for every three spaces X, Y and Z in CGHTop, we have a natural isomorphism (i.e. homeomorphism) between the following spaces:

$$\mathrm{CGHTop}((X \times Y)_k, Z) \cong \mathrm{CGHTop}(X, \mathrm{CGHTop}(Y, Z)).$$

(3) It is shown in [**DP71**] that the category C^* of C^*-algebras is enriched over CGHTop. For any pair of C^*-algebras A and B, the set of *-homomorphisms from

A to B is denoted by $C^*(A,B)$ and it is endowed with the topology of pointwise convergence. This means that a net $(\varphi_i)_i$ in $C^*(A,B)$ converges to a *-homomorphism $\varphi\colon A \to B$ if and only if $\varinjlim_i \|\varphi_i(x) - \varphi(x)\| = 0$ for each $x \in A$. With this topology, $C^*(A,B)$ is a compactly generated, Hausdorff space.

APPENDIX B

Partially ordered monoids, groups and rings

In this appendix we will only consider commutative structures. Therefore, every monoid, semigroup and group is written with operation of addition, and multiplication in every semiring and ring is commutative.

In Section B.1, we study the category Mon of (abelian) monoids, and the full, reflective subcategories Gp of groups, and Con of conical monoids. We recall the construction of tensor products in Mon and we show that Mon is a closed, symmetric, monoidal category.

In Section B.2, we study the category PrePoM of positively pre-ordered monoids and its full subcategory PoM of positively ordered monoids. We show that PoM is a reflective subcategory of PrePoM. Both categories have a tensor product giving them a closed, symmetric, monoidal structure.

We also show that Con can be identified with the full (and reflective) subcategory of PrePoM consisting of algebraically pre-ordered monoids.

In Section B.3, we study the category PoGp of partially ordered groups. In particular, we recall the equivalence between the categories of directed, partially ordered groups and the category of cancellative, conical monoids.

Finally, in Section B.4, we study semirings and (partially ordered) rings.

For further details, the reader is referred to, e.g. [**Ful70**], [**Gol99**], [**Goo86**], [**Gri69**], [**Weh96**].

B.1. The category Mon of monoids

B.1.1. In this paper, by a *monoid* we always mean an abelian monoid, written additively and with zero element denoted by 0.

Let M, N and R be monoids. A *monoid homomorphism* from M to R is a map $f\colon M \to R$ that preserves addition and the zero element. We denote the collection of such maps by $\mathrm{Mon}(M,R)$. We let Mon denote the category whose objects are all monoids and whose morphisms are monoid homomorphisms.

We can endow the set $\mathrm{Mon}(M,R)$ with a natural monoid structure as follows. Given $f,g \in \mathrm{Mon}(M,R)$, we define their sum $f+g$ by pointwise addition, that is, we have $(f+g)(a) = f(a)+g(a)$ for each $a \in M$. The zero element in $\mathrm{Mon}(M,R)$ is given by the zero map that sends every a in M to the zero element in R.

It is easy to see that the homomorphism-monoid $\mathrm{Mon}(M,R)$ is functorial in both variables. Therefore, we obtain a bifunctor

$$\mathrm{Mon}(\_,\_)\colon \mathrm{Mon}^{\mathrm{op}} \times \mathrm{Mon} \to \mathrm{Mon},$$

called the *internal hom-bifunctor* of Mon.

A *monoid bimorphism* from $M \times N$ to R is a map $f\colon M \times N \to R$ which is a monoid homomorphism in each variable, i.e., for fixed $a \in M$ and $b \in N$ the two maps

$$M \to R, \quad x \mapsto f(x,b), \quad N \to R, \quad y \mapsto f(a,y), \quad (x \in M, y \in N)$$

are monoid homomorphism. We denote the set of these monoid bimorphisms by $\mathrm{BiMon}(M\times N,R)$. It becomes a monoid when endowed with pointwise addition. As for the monoid morphisms, it is easy to see that the monoid bimorphisms are functorial in all three entries. We therefore have a multifunctor

$$\mathrm{BiMon}(\_\times\_,\_)\colon \mathrm{Mon}^{\mathrm{op}}\times\mathrm{Mon}^{\mathrm{op}}\times\mathrm{Mon}\to\mathrm{Mon}.$$

Next, we recall the construction of tensor products in the category Mon. The construction is based on the tensor product of abelian semigroups as studied by Grillet, [**Gri69**]. The tensor product in Mon has also been studied in [**Ful70**], where it is denoted by $\otimes_0$.

B.1.2. Let M and N be two monoids. Consider the free abelian monoid $F:=\mathbb{N}[M\times N]$ whose basis is the cartesian product $M\times N$. For $a\in M$ and $b\in N$ we let $a\odot b$ denote the element in F that takes value 1 at (a,b) and that takes value 0 elsewhere. Then, for every element $f\in F$ there exist a finite index set I and elements $a_i\in M$ and $b_i\in N$ for $i\in I$ such that

$$f=\sum_{i\in I}a_i\odot b_i.$$

We do not require that $a_i\neq a_j$ for distinct indices i and j. Moreover, the presentation of f in this way is essentially unique (up to permutation of the index set).

Following the notation in [**Weh96**, Section 2], we define two binary relations $\to^0$ and $\to$ on F. We also define a binary relation $\cong_0$ on F. Let f and g be two elements in F. Then:

(1) We set $f\to^0 g$ if and only if there exists a pair $(a,b)\in M\times N$, and nonempty finite index sets I and J, and elements $a_i\in M$ for $i\in I$ and $b_j\in N$ for $j\in J$ such that

$$a=\sum_{i\in I}a_i,\quad b=\sum_{j\in J}b_j,\quad f=a\odot b,\quad g=\sum_{i\in I,j\in J}a_i\odot b_j.$$

(2) We set $f\to g$ if and only if $f=g=0$ or if there are $n\in\mathbb{N}$ and $f_k,g_k\in F$ for $k=0,\ldots,n$ such that

$$f=\sum_{k=0}^n f_k,\quad g=\sum_{k=0}^n g_k,\quad \text{and}\quad f_k\to^0 g_k,\ \text{for each } k.$$

(3) For any $a\in M$ and $b\in N$, we set $0\cong_0 a\odot 0$ and $0\cong_0 0\odot b$.

A binary relation $\mathcal{R}$ on M is called *additive* if for any four elements $a,b,c,d\in M$ we have $(a+c,b+d)\in\mathcal{R}$ whenever $(a,b)\in\mathcal{R}$ and $(c,d)\in\mathcal{R}$. It follows easily from [**Weh96**, Lemma 2.1] that the relation $\to$ is reflexive and additive.

A *congruence relation* on a monoid is an additive equivalence relation. We let $\cong$ be the congruence relation on F generated by $\to$, and $\leftarrow$, and $\cong_0$. We set

$$M\otimes N=F_{/\cong},$$

which is the set of $\cong$-congruence classes in F. It is easy to check that $M\otimes N$ is a monoid, see [**Ful70**].

Given $(a,b)\in M\times N$, we write $a\otimes b$ for the congruence class of $a\odot b$ in $M\otimes N$. In particular, we have $a\otimes 0=0$ and $0\otimes b=0$ for every $a\in M$ and $b\in N$. We define a map

$$\omega\colon M\times N\to M\otimes N\quad \omega(a,b)=a\otimes b,\quad (a\in M,b\in N)$$

which is easily seen to be a monoid bimorphism.

Remark B.1.3. We denote the tensor product in Mon by $\otimes$. If we need to specify the category in which the tensor product is taken, we will also write $\otimes_{\mathrm{Mon}}$.

As we will see below, the tensor product in Mon restricts to the tensor product in the categories of conical monoids, groups, semirings and rings. Therefore, the tensor product in Mon seems most universal and that is why we will usually drop the subscript to shorten notation.

Proposition B.1.4. *Let M and N be two monoids. Then, the monoid $M\otimes N$ and the monoid bimorphism*

$$\omega\colon M\times N\to M\otimes N$$

constructed in Paragraph B.1.2 have the following universal property:

For every monoid R and for every monoid bimorphism $f\colon M\times N\to R$, there exists a unique monoid homomorphism $\tilde{f}\colon M\otimes N\to R$ such that $\tilde{f}\circ\omega=f$.

Thus, the assignment $g\mapsto g\circ\omega$ defines a map

$$\mathrm{Mon}(M\otimes N,R)\xrightarrow{\cong}\mathrm{BiMon}(M\times N,R),$$

which is a monoid isomorphism when considering the (bi)morphism sets as monoids.

Proof. Let M and N be monoids. To check the universal property of ω, let R and f be as in the statement. Since $F=\mathbb{N}[M\times N]$ is the free abelian monoid on the set $M\times N$, there is a unique monoid homomorphism $f'\colon F\to R$ such that $f'(a\odot b)=f(a,b)$ for every $(a,b)\in M\times N$. It is easy to check that f' is constant on the congruence classes of $\cong$. Therefore, f' induces a map

$$\tilde{f}\colon M\otimes N=F_{/\cong}\to R.$$

It is clear that $\tilde{f}$ is a monoid homomorphism. The rest of the statement is easy to check. $\square$

B.1.5. Using that the monoid bimorphisms are functorial in the first two entries, the tensor product in Mon induces a bifunctor

$$\otimes\colon\mathrm{Mon}\times\mathrm{Mon}\to\mathrm{Mon}.$$

In Paragraph 6.1.8, we explain this in more detail in the setting of enriched categories.

We remark that the category Mon is enriched over the symmetric, closed, monoidal category of sets. With this viewpoint, the tensor product in Mon fits into the framework developed in Section 6.1.

Let M,N and R be three monoids. It is easy to verify that there is a natural isomorphism

$$M\otimes N\cong N\otimes M,$$

identifying the simple tensor $a\otimes b$ with $b\otimes a$, for any $a\in M$ and $b\in N$. Similarly, there is a natural isomorphism

$$(M\otimes N)\otimes R\cong M\otimes(N\otimes R),$$

identifying the simple tensor $(a\otimes b)\otimes c$ with $a\otimes(b\otimes c)$ for $a\in M$, $b\in N$ and $c\in R$.

The monoid $\mathbb{N}$ acts as a unit for the tensor product, that is, there are natural isomorphisms

$$\mathbb{N}\otimes M\cong M\cong M\otimes\mathbb{N}.$$

One can show that this gives Mon the structure of a symmetric, monoidal category.

For each pair of monoids M and N, we have seen that $\mathrm{Mon}(M,N)$ has a natural structure as a monoid. Thus, we can consider $\mathrm{Mon}(M,N)$ as an object in Mon. It follows that the category Mon has an internal hom-bifunctor, which is equal to the (given) hom-bifunctor $\mathrm{Mon}(\_,\_)$.

Moreover, the category Mon is closed. This means that, given a monoid N, the internal hom-bifunctor $\mathrm{Mon}(N, \_)$ is right adjoint to the functor $\_ \otimes_{\mathrm{Mon}} N$. Indeed, given two more monoids M and R, there are natural isomorphisms between the following monoids:

$$\mathrm{BiMon}(M \times N, R) \cong \mathrm{Mon}(M \otimes N, R) \cong \mathrm{Mon}(M, \mathrm{Mon}(N, R)).$$

Next, we consider two important subcategories of Mon: The category Gp of (abelian) groups, and the category Con of conical monoids.

B.1.6. We let Gp denote the category of (abelian) groups. A map between two groups is a group homomorphism if and only if it is a monoid homomorphism. Therefore, by considering a group as a monoid, we think of Gp as a full subcategory of Mon.

There are two important observations:

(1) The category Gp is reflective in Mon.
(2) The category Gp is closed under the tensor product in Mon.

Indeed, given a monoid M, its reflection in Gp is the Grothendieck-completion $\mathrm{Gr}(M)$ of M. This induces a reflection functor

$$\mathrm{Gr}\colon \mathrm{Mon} \to \mathrm{Gp}.$$

Given two groups G and H, it is easy to see that their tensor product as monoids, $G \otimes_{\mathrm{Mon}} H$, is in fact a group. For a pair $(a, b) \in M \times N$, the inverse of the simple tensor $a \otimes b$ is equal to $(-a) \otimes b$.

In general, the tensor product of a monoidal category induces a tensor product in every reflective subcategory. Given two objects in the subcategory, their tensor product in the subcategory is the reflection of their tensor product in the containing monoidal category.

In the concrete case of Gp and Mon, this agrees with the tensor product of Gp constructed above. Indeed, given two groups G and H, their (abstract) tensor product is defined as

$$\mathrm{Gr}(G \otimes_{\mathrm{Mon}} H),$$

the Grothendieck-completion of their tensor product as monoids. However, as observed above, the monoid $G \otimes_{\mathrm{Mon}} H$ is automatically a group and therefore

$$G \otimes_{\mathrm{Gp}} H = \mathrm{Gr}(G \otimes_{\mathrm{Mon}} H) = G \otimes_{\mathrm{Mon}} H.$$

Therefore, it is unambiguous to write $G \otimes H$ for the tensor product of G and H.

More generally, given two monoids M and N, there is a natural isomorphism between $\mathrm{Gr}(M \otimes N)$, the Grothendieck-completion of their tensor product as monoids, and $\mathrm{Gr}(M) \otimes \mathrm{Gr}(N)$, the tensor product of their respective Grothendieck-completions.

Proposition B.1.7 (Fulp, [**Ful70**, Proposition 17]). *Let M and N be two monoids. Then*

$$\mathrm{Gr}(M \otimes N) \cong \mathrm{Gr}(M) \otimes \mathrm{Gr}(N).$$

B.1.8. A monoid M is *conical* if for any two elements a and b in M we have that $a + b = 0$ implies $a = b = 0$. Equivalently, the subset $M^{\times}$ of nonzero elements is a subsemigroup. This property has appeared in the literature under many different names, see [**Weh96**, p.268].

We let Con denote the full subcategory of Mon consisting of conical (abelian) monoids. Analogous to the category of groups, we have the following facts:

(1) The category Con is reflective in Mon.
(2) The category Con is closed under the tensor product in Mon.

Given a monoid M, we let $U(M)$ denote the subgroup of units, that is

$$U(M) = \{a \in M \ \mid \ a + b = 0 \text{ for some } b \in M\}.$$

Then M is conical if and only if $U(M) = \{0\}$.

We define a binary relation $\sim$ on M by setting $a \sim b$ if and only if there exist $x, y \in U(M)$ such that $a + x = b + y$, for any pair $a, b \in M$. It is easy to check that $\sim$ is a congruence relation on M. We set

$$M_{\mathrm{Con}} = M_{/\sim},$$

the set of congruence classes in M. Then M_{Con} is a conical monoid, which is the reflection of M in Mon. This induces a reflection functor from Mon to Con.

Given two monoids M and N, it is shown in [**Ful70**, Corollary 8] that

$$U(M \otimes N) \cong U(M) \otimes U(N).$$

Thus, given two conical monoids M and N, their tensor product in Mon is conical. Therefore, it is unambiguous to write $M \otimes N$ for the tensor product of M and N.

B.1.9. Let us recall a different construction of the tensor product of conical monoids, as considered by Wehrung in [**Weh96**]. Let M and N be two conical monoids. Set $M^\times = M \setminus \{0\}$. Since M is conical, we have that $M^\times$ is a subsemigroup of M. Analogously, $N^\times := N \setminus \{0\}$ is a subsemigroup of N.

In Paragraph B.1.2, we considered the binary relations $\to^0$ and $\to$ on the free monoid $\mathbb{N}[M \times N]$. Now, we consider the free monoid

$$F = \mathbb{N}[M^\times \times N^\times].$$

We can define binary relations $\to^0$ and $\to$ on F as in Paragraph B.1.2, and this is in fact the original definition of $\to^0$ and $\to$ as in [**Weh96**, Section 2].

We let $\cong$ be the congruence relation on F generated by $\to$ and $\leftarrow$. Thus, for two elements f and g in F we have $f \cong g$ if and only if there are $n \in \mathbb{N}$ and elements $f_k, f_k' \in F$ for $k = 0, \ldots, n$ such that $f = f_0$ and $f_n = g$ and such that $f_k \to f_k' \leftarrow f_{k+1}$ for each $k < n$:

$$f = f_0 \to f_0' \leftarrow f_1 \to f_1' \leftarrow \ldots f_n \to f_n' = g.$$

We set

$$M \otimes_{\mathrm{Con}} N = F_{/\cong}.$$

It is clear that $M \otimes_{\mathrm{Con}} N$ is an abelian semigroup. Using that F is conical, it follows easily from the definition of $\leftarrow$ and $\to$ that $f \to 0$ or $0 \to f$ implies $f = 0$, for any $f \in F$. Therefore, the congruence class of the element 0 contains only 0 itself. It follows that $F \setminus \{0\}$ is a subsemigroup of F that is closed under the congruence relation. Thus, $M \otimes_{\mathrm{Con}} N$ is a conical monoid.

The natural map from $\mathbb{N}[M^\times \times N^\times]$ to $\mathbb{N}[M \times N]$ induces a map

$$M \otimes_{\mathrm{Con}} N = \mathbb{N}[M^\times \times N^\times]_{/\langle \leftarrow, \to \rangle} \to \mathbb{N}[M \times N]_{/\langle \leftarrow, \to, \cong_0 \rangle} = M \otimes N,$$

which is easily checked to be an isomorphism.

B.2. The categories PrePoM and PoM of positively (pre)ordered monoids

B.2.1. A *partially ordered monoid* is a monoid M with a partial order $\leq$ such that $a \leq b$ implies $a + c \leq b + c$, for any $a, b, c \in M$. If, in addition, we have $0 \leq a$ for every $a \in M$, then we call M a *positively ordered monoid*.

If the order is not necessarily antisymmetric, we speak of a *(positively) preordered monoid*. This terminology follows Wehrung, [**Weh92**]. Note, however, that in [**Weh92**] a 'positively ordered monoid' (abbreviated by P.O.M. there) is only assumed to be pre-ordered. We include the assumption of antisymmetry in our definition of 'positively ordered monoid' since our focus is on partially ordered structures.

A PoM-*morphism* is an order-preserving monoid homomorphism. Let PrePoM denote the category of positively pre-ordered monoids together with PoM-morphisms. We let PoM be the full subcategory of positively ordered monoids.

Let M, N and R be positively pre-ordered monoids. Given two PoM-morphisms $f, g\colon M \to R$, we set $f \leq g$ if and only if $f(a) \leq g(a)$ for each $a \in M$. This defines a positive pre-order on M. Together with pointwise addition, this endows the set $\mathrm{PoM}(M, R)$ of PoM-morphisms with the structure of a positively pre-ordered monoid. If the order of R is antisymmetric, then so is the order of $\mathrm{PoM}(M, R)$. One can extend the assignment $R \mapsto \mathrm{PoM}(M, R)$ to the following two functors

$$\mathrm{PoM}(M, \_)\colon \mathrm{PrePoM} \to \mathrm{PrePoM}, \quad \mathrm{PoM}(M, \_)\colon \mathrm{PoM} \to \mathrm{PoM}.$$

The functor on the left is the internal hom-bifunctor of PrePoM. If M is in PoM, then the functor on the right is the internal hom-bifunctor of PoM.

A PoM-*bimorphism* from $M \times N$ to R is a map $f\colon M \times N \to R$ that is a PoM-morphism in each variable. We denote the set of such bimorphisms by $\mathrm{BiPoM}(M \times N, R)$. It is a positively pre-ordered monoid when endowed with pointwise order and addition. If the order of R is antisymmetric, then so is the order of $\mathrm{BiPoM}(M \times N, R)$. One can extend the assignment $R \mapsto \mathrm{BiPoM}(M \times N, R)$ to the following two functors

$$\mathrm{BiPoM}(M \times N, \_)\colon \mathrm{PrePoM} \to \mathrm{PrePoM}, \quad \mathrm{BiPoM}(M \times N, \_)\colon \mathrm{PoM} \to \mathrm{PoM}.$$

Since our focus is on the category PoM, we denote the (bi)morphisms in both PoM and PrePoM as PoM-(bi)morphisms.

B.2.2. Let M be a positively pre-ordered monoid. We define a relation on M by setting $a \equiv b$ if and only if $a \leq b$ and $b \leq a$, for any $a, b \in M$. Then $\equiv$ is a congruence relation. We set $\mu(M) = M_{/\equiv}$ and we let $\beta\colon M \to \mu(M)$ denote the quotient map. The pre-order on M induces a partial order on $\mu(M)$. It is easy to check that this gives $\mu(M)$ the structure of a positively ordered monoid and that β is a PoM-morphism.

The assignment $M \mapsto \mu(M)$ extends to a functor

$$\mu\colon \mathrm{PrePoM} \to \mathrm{PoM}.$$

It is straightforward to check that this is a left adjoint to the inclusion of PoM in PrePoM. More precisely, for any positively ordered monoid R, the following universal properties hold:

(1) For every PoM-morphism $f\colon M \to R$, there is a unique PoM-morphism $\tilde{f}\colon \mu(M) \to R$ such that $\tilde{f} \circ \beta = f$.
(2) If $g_1, g_2\colon \mu(M) \to R$ are two PoM-morphisms, then $g_1 \leq g_2$ if and only if $g_1 \circ \beta \leq g_2 \circ \beta$.

Thus, the assignment $g \mapsto g \circ \beta$ defines a map

$$\mathrm{PoM}(\mu(M), R) \xrightarrow{\cong} \mathrm{PoM}(M, R),$$

which is a PoM-isomorphism when considering the (bi)morphism sets as positively ordered monoids.

Proposition B.2.3. *The category* PoM *is a full, reflective subcategory of the category* PrePoM*.*

B.2.4 (Tensor product in PrePoM). Let M and N be positively pre-ordered monoids. We first consider the tensor product of the underlying monoids as constructed in Paragraph B.1.2. Set $F = \mathbb{N}[M \times N]$.

We define binary relations on F as follows:

(1) We set $f \leq^0 g$ if and only if there are (a, b) and $(\tilde{a}, \tilde{b})$ in $M \times N$ such that
$$a \leq \tilde{a}, \quad b \leq \tilde{b}, \quad f = a \odot b, \quad g = \tilde{a} \odot \tilde{b}.$$

(2) We set $f \leq' g$ if and only if $f = 0$ or if there are $n \in \mathbb{N}$ and $f_k, g_k \in F$ for $k = 0, \ldots, n$ and such that
$$f = \sum_{k=0}^{n} f_k, \quad g = \sum_{k=0}^{n} g_k, \quad \text{and} \quad f_k \leq^0 g_k \text{ for each } k.$$

Recall that $\cong$ is the congruence relation on F generated by $\leftarrow$, $\rightarrow$ and $\cong_0$. We let $\leq$ be the transitive relation on F generated by $\cong$ and $\leq'$. Thus, for two elements $f, g \in F$ we have $f \leq g$ if and only if there are $n \in \mathbb{N}$ and elements $f_k, f_k' \in F$ such that $f = f_0$ and $f_n = g$ and for each $k < n$:
$$f_k \leq' f_k' \cong f_{k+1}.$$

It is easy to see that $\leq$ is a positive pre-order on F. This induces a positive pre-order on $M \otimes_{\mathrm{Mon}} N = F_{/\cong}$. We denote the resulting positively pre-ordered monoid by $M \otimes_{\mathrm{PrePoM}} N$.

By construction, the universal monoid-bimorphism
$$\omega \colon M \times N \to M \otimes N$$
is order-preserving in each variable. We may therefore consider ω as a PoM-bimorphism from $M \times N$ to $M \otimes_{\mathrm{PrePoM}} N$.

Proposition B.2.5. *Let M and N be two positively pre-ordered monoids. Then, the positively pre-ordered monoid $M \otimes_{\mathrm{PrePoM}} N$ and the* PoM*-bimorphism*
$$\omega \colon M \times N \to M \otimes_{\mathrm{PrePoM}} N.$$
constructed in Paragraph B.2.4 satisfy the following universal properties for each positively pre-ordered monoid R:

(1) For every PoM*-bimorphism $f \colon M \times N \to R$, there exists a unique* PoM*-morphism $\tilde{f} \colon M \otimes_{\mathrm{PrePoM}} N \to R$ such that $\tilde{f} \circ \omega = f$.*
(2) If $g_1, g_2 \colon M \otimes_{\mathrm{PrePoM}} N \to R$ are two PoM*-morphisms, then $g_1 \leq g_2$ if and only if $g_1 \circ \omega \leq g_2 \circ \omega$.*

Thus, the assignment $g \mapsto g \circ \omega$ defines a map
$$\mathrm{PoM}(M \otimes_{\mathrm{PrePoM}} N, R) \xrightarrow{\cong} \mathrm{BiPoM}(M \times N, R),$$
which is a PoM*-isomorphism when considering the (bi)morphism sets as positively pre-ordered monoids.*

Moreover, the reflection $\mu \colon \mathrm{PrePoM} \to \mathrm{PoM}$ induces a tensor product in PoM*. More precisely, given two positively ordered monoids M and N, we set*
$$M \otimes_{\mathrm{PoM}} N = \mu(M \otimes_{\mathrm{PrePoM}} N).$$
The composition $\beta \circ \omega \colon M \times N \to M \otimes_{\mathrm{PoM}} N$ is a PoM*-bimorphism which has the analogous universal properties of the tensor product in* PoM.

Proof. Let M and N be positively pre-ordered monoids. To check the universal property of ω, let R and f be as in the statement. Since the underlying monoid of $M \otimes_{\mathrm{PrePoM}} N$ is the tensor product in Mon, there is a unique monoid homomorphism $\tilde{f} \colon M \otimes_{\mathrm{Mon}} N \to R$. It follows easily from the definition of the pre-order on $M \otimes_{\mathrm{Mon}} N$ that $\tilde{f}$ is order-preserving. This proves (1). The proof of statement (2) is left to the reader.

Now, let M and N be positively ordered monoids. Define $M \otimes_{\mathrm{PoM}} N$ as the reflection of $M \otimes_{\mathrm{PrePoM}} N$ in PoM. To show that this has the analogous universal properties, let R be a positively ordered monoid. Given a PoM-morphism $f\colon M \otimes_{\mathrm{PoM}} N \to R$, we consider the maps $f \circ \beta$ and $f \circ \beta \circ \omega$, which are shown in the following commutative diagram:

$$\begin{tikzcd}
M \otimes_{\mathrm{PoM}} N \arrow[r, equal] & \mu(M \otimes_{\mathrm{PrePoM}} N) \arrow[dr, "f"] & M \otimes_{\mathrm{PrePoM}} N \arrow[l, "\beta"'] \arrow[d, "f\circ\beta"] & M \times N \arrow[l, "\omega"'] \arrow[dl, "f\circ\beta\circ\omega"] \\
 & & R &
\end{tikzcd}$$

It follows from Paragraph B.2.2 and the universal property of the tensor product in PrePoM that this induces bijective maps

$$\mathrm{PoM}(M \otimes_{\mathrm{PoM}} N, R) \xrightarrow{\cong} \mathrm{PoM}(M \otimes_{\mathrm{PrePoM}} N, R) \xrightarrow{\cong} \mathrm{BiPoM}(M \times N, R),$$

which are PoM-isomorphism when considering the (bi)morphism sets as positively ordered monoids. □

B.2.6. Analogous to Paragraph B.1.5, we obtain that PrePoM and PoM are closed, symmetric, monoidal categories.

B.2.7. Let us clarify the connection between the four categories Mon, Con, PrePoM and PoM. We have already observed that Con is a full, reflective subcategory of Mon and that PoM is a full, reflective subcategory of PrePoM.

To every positively pre-ordered monoid, we may associate its underlying additive monoid. This induces the forgetful functor

$$\mathfrak{F}\colon \mathrm{PrePoM} \to \mathrm{Mon}.$$

Conversely, let M be a monoid. The *algebraic pre-order* on M is defined as follows: For two elements $a, b \in M$, we set $a \leq_{\mathrm{alg}} b$ if and only if there exists $x \in M$ such that $a + x = b$. It is clear that $\leq_{\mathrm{alg}}$ is a positive pre-order on M. Given two monoids M and N, every monoid homomorphism $f\colon M \to N$ becomes a PoM-morphism when M and N are equipped with their respective algebraic pre-orders. This defines a functor

$$\mathfrak{A}\colon \mathrm{Mon} \to \mathrm{PrePoM},$$

which assigns to a monoid M the positively pre-ordered monoid $(M, \leq_{\mathrm{alg}})$.

We say that a positively pre-ordered monoid M is *algebraically pre-ordered*, or we simply say that M is an *algebraically pre-ordered monoid*, if it is equipped with the algebraic pre-order of the underlying monoid. It is easy to check that $\mathfrak{A}$ is a fully faithful functor. Thus, we may identify Mon with the full subcategory of PrePoM consisting of algebraically pre-ordered monoids.

Moreover, we have the following:

(1) The category Mon is reflective in PrePoM.
(2) The property of being algebraically pre-ordered is closed under tensor products in PrePoM, see Proposition B.2.8. Thus, the category Mon considered as a subcategory of PrePoM is closed under the tensor product in PrePoM.

To see that Mon is a reflective subcategory of PrePoM, let us show that the forgetful functor $\mathfrak{F}$ is a left adjoint to the inclusion $\mathfrak{A}$. Indeed, given a monoid M and a positively pre-ordered monoid R, it is easy to check that every monoid homomorphism $f\colon M \to \mathfrak{F}(R)$ is automatically order-preserving as a map from M (with the algebraic pre-order) to R. Thus, we have a natural bijection of the following morphism-sets:

$$\mathrm{Mon}(M, \mathfrak{F}R) \cong \mathrm{PoM}(\mathfrak{A}(M), R).$$

Proposition B.2.8. *Let M and N be algebraically pre-ordered monoids. Then $M \otimes_{\mathrm{PrePoM}} N$ is algebraically pre-ordered.*

Proof. We use the notation that was introduced in Paragraph B.1.2 and Paragraph B.2.4. Thus, we consider the monoid $F = \mathbb{N}[M \times N]$ and the congruence relation $\cong$ on F generated by $\leftarrow$, $\rightarrow$ and $\cong_0$.

Claim 1: If $f, g \in F$ satisfy $f \leq^0 g$, then there exists $h \in F$ such that $f + h \cong g$. To prove the claim, assume that $f, g \in F$ satisfying $f \leq^0 g$ are given. By definition, there are (a, b) and $(\tilde{a}, \tilde{b})$ in $M \times N$ such that

$$a \leq \tilde{a}, \quad b \leq \tilde{b}, \quad f = a \odot b, \quad g = \tilde{a} \odot \tilde{b}.$$

Since M and N are algebraically pre-ordered, it follows that there exist $x \in M$ and $y \in N$ such that

$$a + x = \tilde{a}, \quad b + y = \tilde{b}.$$

Set

$$h = x \odot b + \tilde{a} \odot y.$$

Then

$$f + h = a \odot b + x \odot b + \tilde{a} \odot y \cong \tilde{a} \odot b + \tilde{a} \odot y \cong \tilde{a} \odot \tilde{b} = g,$$

as desired.

Claim 2: If $f, g \in F$ satisfy $f \leq' g$, then there exists $h \in F$ such that $f + h \cong g$. This follows easily from claim 1, since $\leq'$ is defined as the additive closure of $\leq_0$, and since $\cong$ is an additive relation.

Next, let us show that the pre-order of $M \otimes_{\mathrm{PrePoM}} N$ is algebraic. The underlying monoid of $M \otimes_{\mathrm{PrePoM}} N$ is equal to $M \otimes_{\mathrm{Mon}} N = F_{/\cong}$. The pre-order $\leq$ of F is defined as the transitive relation generated by $\leq'$ and $\cong$. This induces the pre-order of $M \otimes_{\mathrm{PrePoM}} N$, which by abuse of notation is also denoted by $\leq$. Now, let $x, y \in M \otimes_{\mathrm{PrePoM}} N$ satisfy $x \leq y$. Choose representatives f and g in F such that $x = [f]$ and $y = [g]$. Then $f \leq g$. This means that there are $n \in \mathbb{N}$ and elements $f_k, f_k' \in F$ for $k \leq n$ such that

$$f = f_0 \leq' f_0' \cong f_1 \leq' f_1' \cong \ldots \cong f_n \leq' f_n' = g.$$

For each $k \leq n$, we have $f_k \leq' f_k'$. By claim 2, there exists $h_k \in F$ such that $f_k + h_k \cong f_k'$. Then

$$f_k + h_k \cong f_{k+1},$$

for each $k \leq n$. Set $h = h_0 + \ldots + h_n$. It follows that

$$f + h = f_0 + \sum_{k=0}^{n} h_k \cong f_1 + \sum_{k=1}^{n} h_k \cong f_2 + \sum_{k=2}^{n} h_k \cong \ldots \cong g.$$

Thus, $f + h \cong g$, which implies that $x + [h] = y$ in $M \otimes_{\mathrm{PrePoM}} N$, as desired. □

B.2.9. The underlying monoid of a positively ordered monoid is conical. Therefore, the forgetful functor $\mathfrak{F}\colon \mathrm{PrePoM} \to \mathrm{Mon}$ considered in Paragraph B.2.7 restricts to a functor

$$\mathfrak{F}\colon \mathrm{PoM} \to \mathrm{Con},$$

which by abuse of notation is also denoted by $\mathfrak{F}$.

However, the algebraic pre-order on a conical monoid is not necessarily anti-symmetric. Therefore, the functor $\mathfrak{A}\colon \mathrm{Mon} \to \mathrm{PrePoM}$ from Paragraph B.2.7 does not restrict to a functor from Mon to PoM.

We say that a positively ordered monoid M is *algebraically ordered*, or we simply say that M is an *algebraically ordered monoid*, if it is equipped with the algebraic partial order of the underlying monoid.

B.3. The category PoGp of partially ordered groups

B.3.1. A *partially ordered group* is an (abelian) group with a partial order $\leq$ such that $a \leq b$ implies $a+c \leq b+c$ for any group elements a, b and c. Given partially ordered groups G and R, a PoGp-*morphism* from G to R is an order-preserving group homomorphism. We denote the collection of such maps by $\mathrm{PoGp}(G, R)$. We let PoGp denote the category of partially ordered groups and PoGp-morphisms.

Given another partially ordered group H, a PoGp-*bimorphism* from $G \times H$ to R is a map $f\colon G \times H \to R$ that is a group homomorphism in each variable and such that $f(a_1, b_1) \leq f(a_2, b_2)$ whenever $a_1, a_2 \in G$ and $b_1, b_2 \in H$ are elements satisfying $a_1 \leq a_2$ and $b_1 \leq b_2$. We denote the collection of such bimorphisms by $\mathrm{BiPoGp}(G \times H, R)$.

Note that for a PoGp-bimorphism $f\colon G \times H \to R$ and a fixed $a \in G$, the map

$$H \to R, \quad b \mapsto f(a, b), \quad (b \in H)$$

is not necessarily order-preserving, unless $a \geq 0$.

B.3.2. To clarify the connection between PoGp and Mon, we need to recall the following notion. We say that a positively ordered monoid M has *cancellation* (or that M is *cancellative*) if for any $a, b, x \in M$ we have that $a + x \leq b + x$ implies $a \leq b$. Let $\mathrm{Mon}_{\mathrm{canc}}$ denote the full subcategory of Mon of cancellative monoids. Similarly, we let $\mathrm{Con}_{\mathrm{canc}}$ be the full subcategory of Con of cancellative, conical monoids.

Let G be a partially ordered group. The *positive cone* of G is defined as

$$G_+ = \{a \in G \mid 0 \leq a\}.$$

It is easy to check that G_+ satisfies

$$G_+ \cap (-G_+) = \{0\}, \quad G_+ + G_+ \subset G_+.$$

Therefore, G_+ is a conical submonoid of G. Moreover, the order of G induces the algebraic order on G_+. Thus, considering G_+ as an algebraically ordered monoid, the inclusion of G_+ in G is an order-embedding.

Now, let G be a group and let P be a conical submonoid of G. This defines a partial order on G by setting $a \leq b$ if and only if there exists $x \in P$ such that $a + x = b$, for any $a, b \in G$. For every group G, this establishes a natural one-to-one correspondence between:

(1) Partial orders on G such that $(G, \leq)$ is a partially ordered group.
(2) Conical submonoids of G.

Since the positive cone of a partially ordered group is automatically cancellative, there is a functor

$$\mathfrak{P}\colon \mathrm{PoGp} \to \mathrm{Con}_{\mathrm{canc}},$$

that assigns to a partially ordered group G its positive cone G_+.

Recall that a partially ordered group G is *directed*, if $G = G_+ - G_+$. We let $\mathrm{PoGp}_{\mathrm{dir}}$ denote the full subcategory of PoGp consisting of directed, partially ordered groups.

Now, let M be a monoid. Consider the Grothendieck completion $\mathrm{Gr}(M)$ together with the universal map

$$\delta\colon M \to \mathrm{Gr}(M).$$

We have that δ is injective if and only if M is cancellative. The image of δ is a submonoid of $\mathrm{Gr}(M)$. If M is conical, then so is $\delta(M)$, whence it gives $\mathrm{Gr}(M)$ the structure of a partially ordered group whose positive cone is $\delta(M)$. Moreover, $\mathrm{Gr}(M)$ is directed. This induces a functor

$$\mathrm{Gr}\colon \mathrm{Con} \to \mathrm{PoGp}_{\mathrm{dir}}.$$

A partially ordered group G is in the image of the functor Gr if and only if it is directed, in which case $G \cong \mathrm{Gr}(G_+)$. Conversely, a conical monoid M is in the image of the functor $\mathfrak{P}$ if and only if it is cancellative, in which case $M \cong \mathrm{Gr}(M)_+$. We summarize this as follows:

Proposition B.3.3. *The functors* $\mathfrak{P}$ *and* Gr *from Paragraph B.3.2 establish an equivalence between the following categories:*

(1) The category PoGp_{dir} *of directed, partially ordered groups.*
(2) The category Con_{canc} *of cancellative, conical monoids.*

B.3.4. Let G and H be partially ordered groups. Consider the tensor product $G \otimes H$ of the underlying groups. The map

$$G_+ \times H_+ \to G \otimes H, \quad (a,b) \mapsto a \otimes b, \quad (a \in G_+, b \in H_+)$$

is a monoid bimorphism. It therefore induces a monoid homomorphism

$$\delta \colon G_+ \otimes H_+ \to G \otimes H.$$

Since G_+ and H_+ are conical, so is $G_+ \otimes H_+$, see Paragraph B.1.8.

The image of the map δ is a conical submonoid of $G \otimes H$. As explained in Paragraph B.3.2, this induces a partial order on $G \otimes H$ whose positive cone is the image of δ. We denote the resulting partially ordered group by $G \otimes_{\mathrm{PoGp}} H$, which is the tensor product of G and H in the category PoGp, see [**Weh96**, Proposition 1.1].

In Proposition B.3.3, we have seen that the full subcategory $\mathrm{PoGp}_{\mathrm{dir}}$ of directed, partially ordered groups is equivalent to $\mathrm{Con}_{\mathrm{canc}}$. This equivalence is also compatible with the tensor product. Thus, given two directed partially ordered groups G and H, we have

$$G \otimes_{\mathrm{PoGp}} H \cong \mathrm{Gr}(G_+ \otimes H_+).$$

Conversely, if M and N are two cancellative, conical monoids, then the monoid

$$\left(\mathrm{Gr}(M) \otimes_{\mathrm{PoGp}} \mathrm{Gr}(N)\right)_+$$

is isomorphic to the reflection of $M \otimes N$ in the subcategory of cancellative, conical monoids. A proof of these statements can be found in [**Weh96**, Proposition 1.2].

It is natural to ask whether the tensor product of two cancellative, conical monoids is again cancellative. As it turns out, the answer to this question is negative. We thank Fred Wehrung for showing us a counterexample.

In [**Weh96**, Examples 1.4 and 1.5], examples of partially ordered abelian groups with Riesz interpolation G and H such that $G \otimes_{\mathrm{PoGp}} H$ does not have interpolation are given. If we let $M = G_+$ and $N = H_+$, then M and N are conical, cancellative, monoids that satisfy the Riesz refinement property. It then follows from [**Weh96**, Theorem 2.9] that $M \otimes N$ also satisfies the Riesz refinement property. Since

$$G \otimes_{\mathrm{PoGp}} H = \mathrm{Gr}(M) \otimes_{\mathrm{PoGp}} \mathrm{Gr}(N) \cong \mathrm{Gr}(M \otimes N)$$

we conclude that $\mathrm{Gr}(M \otimes N)_+$ does not satisfy the Riesz refinement property, and hence $M \otimes N$ is not a cancellative monoid (as otherwise would be isomorphic to the positive cone of its Grothendieck completion).

B.4. The category PoRg of partially ordered rings

B.4.1. A *semiring* is a monoid R together with a commutative multiplication satisfying distributivity and such that there is a unit element, denoted by 1. A semiring homomorphism is a multiplicative monoid homomorphism preserving the unit element. We let Srg denote the category of semirings together with semiring homomorphisms.

Let R and S be two semirings. Consider the tensor product $R \otimes S$ of the underlying monoids. Given $r_1, r_2 \in R$ and $s_1, s_2 \in S$, we define the product of simple tensors as

$$(r_1 \otimes s_1)(r_2 \otimes s_2) = (r_1 r_2) \otimes (s_1 s_2).$$

This extends to a well-defined, commutative multiplication on $R \otimes S$. The element $1 \otimes 1$ is a unit element. It follows that $R \otimes S$ has a natural structure as a semiring. If we want to stress that $R \otimes S$ has a semiring-structure, we write $R \otimes_{\mathrm{Srg}} S$. This is the tensor product of R and S in the category of semirings.

All *rings* will be unital and commutative. Equivalently, a ring will be a semiring such that the underlying additive monoid is a group. Given two rings R and S, the tensor product $R \otimes S$ of the underlying groups has a natural multiplication such that $R \otimes S$ is a ring. This is the tensor product of R and S in the category of rings.

B.4.2. A *partially ordered ring* is a ring R together with a partial order $\leq$ such that $0 \leq 1$ and such that

(i) If $a \leq b$, then $a + c \leq b + c$, for any $a, b, c \in R$.
(ii) If $a \leq b$ and $0 \leq c$, then $ac \leq bc$, for any $a, b, c \in R$.

In particular, the underlying group of R together with $\leq$ is a partially ordered group. This induces a forgetful functor

$$\mathfrak{F} \colon \mathrm{PoRg} \to \mathrm{PoGp}.$$

Let R be a partially ordered ring. As for partially ordered groups, we define the *positive cone* of R as

$$R_+ = \{a \in R \mid 0 \leq a\}.$$

As in the group case, we have $R_+ + R_+ \subset R_+$ and $R_+ \cap (-R_+) = \{0\}$. Moreover, we have $R_+ \cdot R_+ \subset R_+$. Thus, the positive cone is a cancellative, conical semiring.

Now let R and S be partially ordered rings. Consider the tensor product $R \otimes S$ of the underlying rings. As for partially ordered groups, there is a natural partial order on $R \otimes S$ with positive cone given as the image of $R_+ \otimes S_+$ in $R \otimes S$. This partial order is compatible with the multiplication, and we denote the resulting partially ordered ring by $R \otimes_{\mathrm{PoRg}} S$. This is the tensor product of R and S in the category PoRg.

Equivalently, we can consider $R \otimes_{\mathrm{PoRg}} S$ as the tensor product $R \otimes_{\mathrm{PoGp}} S$ of the underlying partially ordered groups equipped with a product as in Paragraph B.4.1.

B.4.3. Recall that Srg denotes the category of semirings. We let ConSrg denote the full subcategory of conical semirings. Further, we let $\mathrm{Srg}_{\mathrm{canc}}$ (resp. $\mathrm{ConSrg}_{\mathrm{canc}}$) denote the full subcategories of cancellative (conical) semirings. We have a functor

$$\mathfrak{P} \colon \mathrm{PoRg} \to \mathrm{ConSrg}_{\mathrm{canc}},$$

which assigns to a partially ordered ring R its positive cone R_+.

Conversely, it is easy to see that the Grothendieck completion of a semiring S has a natural multiplication giving it the structure of a ring. If S is conical, the image of S in $\mathrm{Gr}(S)$ defines a partial order on $\mathrm{Gr}(S)$. Let $\mathrm{PoRg}_{\mathrm{dir}}$ be the full subcategory of PoRg consisting of directed partially ordered rings. Then, the Grothendieck-completion induces a functor

$$\mathrm{Gr} \colon \mathrm{ConSrg} \to \mathrm{PoRg}_{\mathrm{dir}}.$$

The situation is completely analogous to the connection between (cancellative) conical monoids and (directed) partially ordered groups that was discussed in Paragraph B.3.2. Therefore, we have the following analog of Proposition B.3.3:

Proposition B.4.4. *The functors* $\mathfrak{P}$ *and* Gr *from Paragraph B.4.3 establish an equivalence between the following categories:*

(1) *The category* PoRg_{dir} *of directed partially ordered rings.*
(2) *The category* ConSrg_{canc} *of cancellative, conical semirings.*

B.4.5. In Proposition B.4.4, we have seen that the full subcategory $\mathrm{PoRg}_{\mathrm{dir}}$ of directed, partially ordered rings is equivalent to $\mathrm{ConSrg}_{\mathrm{canc}}$ of cancellative, conical semirings. This equivalence is also compatible with the tensor product in the following sense: Given two directed partially ordered rings R and S, we have

$$R \otimes_{\mathrm{PoRg}} S \cong \mathrm{Gr}(R_+ \otimes S_+).$$

Conversely, if M and N are two cancellative, conical semirings, then the semiring

$$(\mathrm{Gr}(M) \otimes_{\mathrm{PoGp}} \mathrm{Gr}(N))_+$$

is isomorphic to the reflection of $M \otimes N$ in the subcategory of cancellative, conical semirings.

Bibliography

[ABP11] R. Antoine, J. Bosa, and F. Perera, *Completions of monoids with applications to the Cuntz semigroup*, Internat. J. Math. **22** (2011), no. 6, 837–861. MR 2812090 (2012d:46146)

[ABP13] ———, *The Cuntz semigroup of continuous fields*, Indiana Univ. Math. J. **62** (2013), no. 4, 1105–1131. MR 3179685

[ABPP14] R. Antoine, J. Bosa, F. Perera, and H. Petzka, *Geometric structure of dimension functions of certain continuous fields*, J. Funct. Anal. **266** (2014), no. 4, 2403–2423. MR 3150165

[ADPS14] R. Antoine, M. Dadarlat, F. Perera, and L. Santiago, *Recovering the Elliott invariant from the Cuntz semigroup*, Trans. Amer. Math. Soc. **366** (2014), no. 6, 2907–2922. MR 3180735

[AGOP98] P. Ara, K. R. Goodearl, K. C. O'Meara, and E. Pardo, *Separative cancellation for projective modules over exchange rings*, Israel J. Math. **105** (1998), 105–137. MR 1639739 (99g:16006)

[AGPS10] G. Aranda Pino, K. R. Goodearl, F. Perera, M. Siles Molina, *Non-simple purely infinite rings*, Amer. J. Math. **132** (2010), 563–610. MR 2666902 (2011e:16064)

[APS11] R. Antoine, F. Perera, and L. Santiago, *Pullbacks, $C(X)$-algebras, and their Cuntz semigroup*, J. Funct. Anal. **260** (2011), no. 10, 2844–2880. MR 2774057 (2011m:46110)

[APT11] P. Ara, F. Perera, and A. S. Toms, *K-theory for operator algebras. Classification of C^*-algebras*, Aspects of operator algebras and applications, Contemp. Math., vol. 534, Amer. Math. Soc., Providence, RI, 2011, pp. 1–71. MR 2767222 (2012g:46092)

[BBSTWW14] J. Bosa, N. P. Brown, Y. Sato, A. Tikuisis, S. White, W. Winter, *Covering dimension of C^*-algebras and the classification of maps by traces*, Preprint 2014.

[BC09] N. P. Brown, A. Ciuperca, *Isomorphism of Hilbert modules over stably finite C^*-algebras*, J. Funct. Anal. **257** (2009), no. 1, 332–339. MR 2523343 (2010d:46077)

[BGR77] L. G. Brown, P. Green, and M. A. Rieffel, *Stable isomorphism and strong Morita equivalence of C^*-algebras*, Pacific J. Math. **71** (1977), no. 2, 349–363. MR 0463928 (57 #3866)

[BH82] B. Blackadar and D. Handelman, *Dimension functions and traces on C^*-algebras*, J. Funct. Anal. **45** (1982), no. 3, 297–340. MR 650185 (83g:46050)

[BK72] A. K. Bousfield and D. M. Kan, *The core of a ring*, J. Pure Appl. Algebra **2** (1972), 73–81. MR 0308107 (46 #7222)

[BK04] E. Blanchard and E. Kirchberg, *Non-simple purely infinite C^*-algebras: the Hausdorff case*, J. Funct. Anal. **207** (2004), no. 2, 461–513. MR 2032998 (2005b:46136)

[Bla88] B. Blackadar, *Comparison theory for simple C^*-algebras*, Operator algebras and applications, Vol. 1, London Math. Soc. Lecture Note Ser., vol. 135, Cambridge Univ. Press, Cambridge, 1988, pp. 21–54. MR 996438 (90g:46078)

[Bla90] ———, *Rational C^*-algebras and nonstable K-theory*, Proceedings of the Seventh Great Plains Operator Theory Seminar (Lawrence, KS, 1987), vol. 20, 1990, pp. 285–316. MR 1065831 (92e:46135)

[Bla98] ———, *K-theory for operator algebras*, second ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998. MR 1656031 (99g:46104)

[Bla06] ———, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III. MR 2188261 (2006k:46082)

[BN76] B. Banaschewski and E. Nelson, *Tensor products and biomorphisms*, Canad. Math. Bull. **19** (1976), no. 4, 385–402. MR 0442059 (56 #447)

[Bor94] F. Borceux, *Handbook of categorical algebra. 1*, Encyclopedia of Mathematics and its Applications, vol. 50, Cambridge University Press, Cambridge, 1994, Basic category theory. MR 1291599 (96g:18001a)

[BPT08] N. P. Brown, F. Perera, and A. S. Toms, *The Cuntz semigroup, the Elliott conjecture, and dimension functions on C^*-algebras*, J. Reine Angew. Math. **621** (2008), 191–211. MR 2431254 (2010a:46125)

[BR92] B. Blackadar and M. Rørdam, *Extending states on preordered semigroups and the existence of quasitraces on C^*-algebras*, J. Algebra **152** (1992), no. 1, 240–247. MR 1190414 (93k:46049)

[BS77] R. A. Bowshell and P. Schultz, *Unital rings whose additive endomorphisms commute*, Math. Ann. **228** (1977), no. 3, 197–214. MR 0498691 (58 #16768)

[BT07] N. P. Brown and A. S. Toms, *Three applications of the Cuntz semigroup*, Int. Math. Res. Not. IMRN (2007), no. 19, Art. ID rnm068, 14. MR 2359541 (2009a:46104)

[BTZ14] J. Bosa, G. Tornetta, and J. Zacharias, *The bivariant Cuntz semigroup*, preprint, 2014.

[CEI08] K. T. Coward, G. A. Elliott, and C. Ivanescu, *The Cuntz semigroup as an invariant for C^*-algebras*, J. Reine Angew. Math. **623** (2008), 161–193. MR 2458043 (2010m:46101)

[Ciu08] A. Ciuperca, *Some properties of the Cuntz semigroup and an isomorphism theorem for a certain class of non-simple C^*-algebras*, ProQuest LLC, Ann Arbor, MI, 2008, Thesis (Ph.D.)–University of Toronto (Canada). MR 2712681

[CMR07] J. Cuntz, R. Meyer, and J. M. Rosenberg, *Topological and bivariant K-theory*, Oberwolfach Seminars, vol. 36, Birkhäuser Verlag, Basel, 2007. MR 2340673 (2008j:19001)

[CRS10] A. Ciuperca, L. Robert, and L. Santiago, *The Cuntz semigroup of ideals and quotients and a generalized Kasparov stabilization theorem*, J. Operator Theory **64** (2010), no. 1, 155–169. MR 2669433 (2012b:46129)

[Cun78] J. Cuntz, *Dimension functions on simple C^*-algebras*, Math. Ann. **233** (1978), no. 2, 145–153. MR 0467332 (57 #7191)

[DHTW09] M. Dadarlat, I. Hirshberg, A. S. Toms, and W. Winter, *The Jiang-Su algebra does not always embed*, Math. Res. Lett. **16** (2009), no. 1, 23–26. MR 2480557 (2010b:46130)

[DP71] E. J. Dubuc and H. Porta, *Convenient categories of topological algebras, and their duality theory*, J. Pure Appl. Algebra **1** (1971), no. 3, 281–316. MR 0301079 (46 #237)

[DR09] M. Dadarlat and M. Rørdam, *Strongly self-absorbing C^*-algebras which contain a nontrivial projection*, Münster J. Math. **2** (2009), 35–44. MR 2545606 (2010h:46094)

[DS84] W. Dicks and W. Stephenson, *Epimorphs and dominions of Dedekind domains*, J. London Math. Soc. (2) **29** (1984), no. 2, 224–228. MR 744090 (85i:16054)

[DT10] M. Dadarlat and A. S. Toms, *Ranks of operators in simple C^*-algebras*, J. Funct. Anal. **259** (2010), no. 5, 1209–1229. MR 2652186 (2012e:46111)

[EHS80] E. G. Effros, D. E. Handelman, and C. L. Shen, *Dimension groups and their affine representations*, Amer. J. Math. **102** (1980), no. 2, 385–407. MR 564479 (83g:46061)

[Eng14] M. Engbers, *Decomposition of simple Cuntz semigroups*, Ph.D Thesis (draft), 2014.

[ER06] G. A. Elliott and M. Rørdam, *Perturbation of Hausdorff moment sequences, and an application to the theory of C^*-algebras of real rank zero*, Operator Algebras: The Abel Symposium 2004, Abel Symp., vol. 1, Springer, Berlin, 2006, pp. 97–115. MR 2265045 (2007m:46085)

[ERS11] G. A. Elliott, L. Robert, and L. Santiago, *The cone of lower semicontinuous traces on a C^*-algebra*, Amer. J. Math. **133** (2011), no. 4, 969–1005. MR 2823868 (2012f:46120)

[ET08] G. A. Elliott and A. S. Toms, *Regularity properties in the classification program for separable amenable C^*-algebras*, Bull. Amer. Math. Soc. (N.S.) **45** (2008), no. 2, 229–245. MR 2383304 (2009k:46111)

[Ful70] R. Fulp, *Tensor and torsion products of semigroups*, Pacific J. Math. **32** (1970), 685–696. MR 0272928 (42 #7809)

[GH76] K. R. Goodearl and D. E. Handelman, *Rank functions and K_0 of regular rings*, J. Pure Appl. Algebra **7** (1976), no. 2, 195–216. MR 0389965 (52 #10794)

[GH82] ———, *Stenosis in dimension groups and AF C^*-algebras*, J. Reine Angew. Math. **332** (1982), 1–98. MR 656856 (83m:46101)

[GHK+03] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *Continuous lattices and domains*, Encyclopedia of Mathematics and its Applications, vol. 93, Cambridge University Press, Cambridge, 2003. MR 1975381 (2004h:06001)

[GJS00] G. Gong, X. Jiang and H. Su, *Obstructions to $\mathcal{Z}$-stability for unital simple C^*-algebras.*, Canad. Math. Bull., **43** (2000), no. 4, 418–426. MR 1793944 (2001k:46086)

[Gol99] J. Golan, *Semirings and their applications*, Kluwer Academic Publishers, Dordrecht (1999), xii+381, Updated and expanded version of *The theory of semirings, with applications to mathematics and theoretical computer science* [Longman Sci. Tech., Harlow, 1992; MR 1163371 (93b:16085)] MR 1746739 (2001c:16082)

[Goo86] K. R. Goodearl, *Partially ordered abelian groups with interpolation*, Mathematical Surveys and Monographs, no. 20, xxii+336, American Mathematical Society, Providence, RI, 1986. MR 0845783 (88f:06013)

[Goo96] ———, *K_0 of multiplier algebras of C^*-algebras with real rank zero*, K-Theory **10** (1996), no. 5, 419–489. MR 1404412 (98e:46086)

[Gri69] P.-A. Grillet, *The tensor product of commutative semigroups*, Trans. Amer. Math. Soc. **138** (1969), 281–293. MR 0237688 (38 #5969)

[Gut13] J. J. Gutiérrez, *On solid and rigid monoids in monoidal categories*, preprint (arXiv:1303.5265 [math.CT]), 2013.

[Haa14] U. Haagerup, *Quasitraces on exact C^*-algebras are traces*, preprint (arXiv:1403.7653 [math.OA]), 2014.

[Han13] D. Handelman, *Simple Archimedean dimension groups*, Proc. Amer. Math. Soc. **141** (2013), no. 11, 3787–3792. MR 3091768

[Jac13] B. Jacelon, *A simple, monotracial, stably projectionless C^*-algebra*, J. Lond. Math. Soc. (2) **87** (2013), no. 2, 365–383. MR 3046276

[JS99] X. Jiang and H. Su, *On a simple unital projectionless C^*-algebra*, Amer. J. Math. **121** (1999), no. 2, 359–413. MR 1680321 (2000a:46104)

[Kel05] G. M. Kelly, *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ. (2005), no. 10, vi+137, Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714]. MR 2177301

[KL09] K. Keimel and J. D. Lawson, *D-completions and the d-topology*, Ann. Pure Appl. Logic **159** (2009), no. 3, 292–306. MR 2522623 (2010g:54020)

[KP00] E. Kirchberg and N. C. Phillips, *Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$*, J. Reine Angew. Math. **525** (2000), 17–53. MR 1780426 (2001d:46086a)

[KR00] E. Kirchberg and M. Rørdam, *Non-simple purely infinite C^*-algebras*, Amer. J. Math. **122** (2000), no. 3, 637–666. MR 1759891 (2001k:46088)

[KR12] ———, *Central sequence C^*-algebras and tensorial absorption of the Jiang-Su algebra*, to appear in J. Reine Angew. Math. (arXiv:1209.5311 [math.OA]), 2012.

[Law97] J. D. Lawson, *The round ideal completion via sobrification*, Proceedings of the 12th Summer Conference on General Topology and its Applications (North Bay, ON, 1997), vol. 22, 1997, pp. 261–274. MR 1718887 (2000k:06007)

[Mac71] S. MacLane, *Categories for the working mathematician*, Springer-Verlag, New York-Berlin, 1971, Graduate Texts in Mathematics, Vol. 5. MR 0354798 (50 #7275)

[Mey99] R. Meyer, *Analytic cyclic cohomology.*, Münster: Univ. Münster, Fachbereich Mathematik und Informatik, 1999 (English).

[MS12] H. Matui and Y. Sato, *Strict comparison and $\mathcal{Z}$-absorption of nuclear C^*-algebras*, Acta Math. **209** (2012), no. 1, 179–196. MR 2979512

[OPR11] E. Ortega, F. Perera, and M. Rørdam, *The corona factorization property and refinement monoids*, Trans. Amer. Math. Soc. **363** (2011), no. 9, 4505–4525. MR 2806681 (2012e:46130)

[OPR12] ———, *The corona factorization property, stability, and the Cuntz semigroup of a C^*-algebra*, Int. Math. Res. Not. IMRN (2012), no. 1, 34–66. MR 2874927 (2012m:46068)

[ORT11] E. Ortega, M. Rørdam, and H. Thiel, *The Cuntz semigroup and comparison of open projections*, J. Funct. Anal. **260** (2011), no. 12, 3474–3493. MR 2781968 (2012d:46148)

[Pal01] T. W. Palmer, *Banach algebras and the general theory of $*$-algebras. Vol. 2*, Encyclopedia of Mathematics and its Applications, vol. 79, Cambridge University Press, Cambridge, 2001, $*$-algebras. MR 1819503 (2002e:46002)

[Per97] F. Perera, *The structure of positive elements for C^*-algebras with real rank zero*, Internat. J. Math. **8** (1997), no. 3, 383–405. MR 1454480 (98i:46058)

[Per01] ———, *Ideal structure of multiplier algebras of simple C^*-algebras with real rank zero*, Canad. J. Math. **53** (2001), no. 3, 592–630. MR 1827822 (2002c:46111)

[PR04] F. Perera, M. Rørdam, *AF-embeddings into C^*-algebras of real rank zero*, J. Funct. Anal. **217** (2004), 142–170. MR 2097610 (2005g:46109)

[PT07] F. Perera and A. S. Toms, *Recasting the Elliott conjecture*, Math. Ann. **338** (2007), no. 3, 669–702. MR 2317934 (2008g:46110)

[Rob12] L. Robert, *Classification of inductive limits of 1-dimensional NCCW complexes*, Adv. Math. **231** (2012), no. 5, 2802–2836. MR 2970466

[Rob13a] ———, *The cone of functionals on the Cuntz semigroup*, Math. Scand. **113** (2013), no. 2, 161–186. MR 3145179

[Rob13b] ———, *The Cuntz semigroup of some spaces of dimension at most two*, C. R. Math. Acad. Sci. Soc. R. Can. **35** (2013), no. 1, 22–32. MR 3098039

[Rør92] M. Rørdam, *On the structure of simple C^*-algebras tensored with a UHF-algebra. II*, J. Funct. Anal. **107** (1992), no. 2, 255–269. MR 1172023 (93f:46094)

[Rør02] ———, *Classification of nuclear, simple C^*-algebras*, Classification of nuclear C^*-algebras. Entropy in operator algebras, Encyclopaedia Math. Sci., vol. 126, Springer, Berlin, 2002, pp. 1–145. MR 1878882 (2003i:46060)

[Rør03] ———, *A simple C^*-algebra with a finite and an infinite projection*, Acta Math. **191** (2003), no. 1, 109–142. MR 2020420 (2005m:46096)

[Rør04] ———, *The stable and the real rank of $\mathcal{Z}$-absorbing C^*-algebras*, Internat. J. Math. **15** (2004), no. 10, 1065–1084. MR 2106263 (2005k:46164)

[RS10] L. Robert and L. Santiago, *Classification of C^*-homomorphisms from $C_0(0,1]$ to a C^*-algebra*, J. Funct. Anal. **258** (2010), no. 3, 869–892. MR 2558180 (2010j:46108)

[RW10] M. Rørdam and W. Winter, *The Jiang-Su algebra revisited*, J. Reine Angew. Math. **642** (2010), 129–155. MR 2658184 (2011i:46074)

[Sat12] Y. Sato, *Trace spaces of simple nuclear C^*-algebras with finite-dimensional extreme boundary*, preprint (arXiv:1209.3000 [math.OA]), 2012.

[SWW14] Y. Sato, S. White, and W. Winter, *Nuclear dimension and $\mathcal{Z}$-stability*, preprint, (arXiv:1403.0747, [math.OA]), 2014.

[Tik11] A. Tikuisis, *The Cuntz semigroup of continuous functions into certain simple C^*-algebras*, Internat. J. Math. **22** (2011), no. 8, 1051–1087. MR 2826555 (2012h:46102)

[Tom05] A. S. Toms, *On the independence of K-theory and stable rank for simple C^*-algebras*, J. Reine Angew. Math. **578** (2005), 185–199. MR 2113894 (2005k:46189)

[Tom08] ———, *On the classification problem for nuclear C^*-algebras*, Ann. of Math. (2) **167** (2008), no. 3, 1029–1044. MR 2415391 (2009g:46119)

[TW07] A. S. Toms and W. Winter, *Strongly self-absorbing C^*-algebras*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3999–4029. MR 2302521 (2008c:46086)

[TW08] ———, *$\mathcal{Z}$-stable ASH algebras*, Canad. J. Math. **60** (2008), no. 3, 703–720. MR 2414961 (2009m:46089)

[TW09] ———, *The Elliott conjecture for Villadsen algebras of the first type*, J. Funct. Anal. **256** (2009), no. 5, 1311–1340. MR 2490221 (2010m:46092)

[TWW12] A. S. Toms, S. White, and W. Winter, *$\mathcal{Z}$-stability and finite dimensional tracial boundaries*, to appear in Int. Math. Res. Not. (arXiv:1209.3292, [math.OA]), 2012.

[Weh92] F. Wehrung, *Injective positively ordered monoids. I, II*, J. Pure Appl. Algebra **83** (1992), no. 1, 43–82, 83–100. MR 1190444 (93k:06023)

[Weh94] ———, *Restricted injectivity, transfer property and decompositions of separative positively ordered monoids*, Comm. Algebra **22** (1994), no. 5, 1747–1781. MR 1264740 (94m:06017)

[Weh96] ———, *Tensor products of structures with interpolation*, Pacific J. Math. **176** (1996), no. 1, 267–285. MR 1433994 (98e:06010)

[Weh98] ———, *Embedding simple commutative monoids into simple refinement monoids*, Semigroup Forum **56** (1998), no. 1, 104–129. MR 1490558 (99b:20092)

[Win11] W. Winter, *Strongly self-absorbing C^*-algebras are $\mathcal{Z}$-stable*, J. Noncommut. Geom. **5** (2011), no. 2, 253–264. MR 2784504 (2012e:46132)

[Win12] ———, *Nuclear dimension and $\mathcal{Z}$-stability of pure C^*-algebras*, Invent. Math. **187** (2012), no. 2, 259–342. MR 2885621

[WZ09] W. Winter and J. Zacharias, *Completely positive maps of order zero*, Münster J. Math. **2** (2009), 311–324. MR 2545617 (2010g:46093)