Asymptotic Flocking Dynamics for the kinetic Cucker-Smale model

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Abstract. In this paper, we analyse the asymptotic behavior of solutions of the continuous kinetic version of flocking by Cucker and Smale [16], which describes the collective behavior of an ensemble of organisms, animals or devices. This kinetic version introduced in [24] is here obtained starting from a Boltzmann-type equation. The large-time behavior of the distribution in phase space is subsequently studied by means of particle approximations and a stability property in distances between measures. A continuous analogue of the theorems of [16] is shown to hold for the solutions on the kinetic model. More precisely, the solutions will concentrate exponentially fast their velocity to their mean while in space they will converge towards a translational flocking solution.

Keywords. Flocking, nonlinear friction equations, mass transportation methods.

1 Introduction

The description of emerging collective behaviors and self–organization in multi-agent interactions has gained increasing interest from various research communities in biology, ecology, robotics and control theory, as well as sociology and economics. In the biological context, the emergent behavior of bird flocks, fish schools or bacteria aggregations, among others, is a major research topic in population and behavioral biology and ecology [1, 4, 14, 15, 16, 18, 30, 33, 34]. Likewise, the coordination and cooperation among multiple mobile agents (robots or sensors) have been playing central roles in sensor networking, with broad applications in environmental control [22, 13, 28]. Emergent economic behaviors, such as distribution of wealth in a modern society [12, 20], or the formation of choices and opinions [21, 32], are also challenging problems studied in recent years in which emergence of universal equilibria is shown.

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In particular, the recent mathematical work of Cucker and Smale [16], connected with the emergent behaviors of flocks, obtained a noticeable resonance in the mathematical community. In analogy with physics, the study of idealized models can often shed light on various observed patterns in the real world, if such models can indeed catch the very essence. In biology and physics, the main goal of flocking simulation is to be able to interpret and predict different flocking or multi-agent aggregating behavior. A relevant part of the existing works, however, have been focusing on modelling and simulation [33, 34, 37]. Quantitative analysis [16, 22] on the asymptotic rates of emergence and convergence, on the other hand, are relatively rare. Mathematical efforts are gradually gaining strength in this multidisciplinary area. In the continuum limit, for example, there have been several recent efforts [33, 34, 7], in which global swarming (i.e., with densely populated agents) patterns are modelled and analyzed via suitable partial differential equations involving both diffusion and interaction via pairwise attraction/repulsion potentials. Patterns for discrete models have been classified in [19] for typical interaction potentials and studied by hydrodynamic [14] and kinetic equations [8].

As far as the emergence of collective behavior in the socio-economic context is studied, the mathematical aspects are more pronounced. In this area, in fact, starting from the leading idea that collective behaviors of a group composed by a sufficiently large number of individuals (agents) could be hopefully described using the laws of statistical mechanics as it happens in a physical system composed of many interacting particles, various methods from statistical physics have been introduced both to model and study the underlying phenomena. In particular, powerful methods borrowed from kinetic theory of rarefied gases have been fruitfully employed to construct kinetic equations of Boltzmann type which describe the emergency of universal structures through their equilibria, see [25] and the references therein.

Our first aim is to use an analogous methodology to describe the work of Cucker and Smale [16] on flocking analysis by means of kinetic equations. Making use of a collisional mechanism between individuals similar to the change in velocity of birds population introduced in [16], we derive a dissipative spatially dependent Boltzmann-type equation which describes the behavior of the flock in terms of a density $f = f(x, v, t)$. This equation is reminiscent of the modification of the Boltzmann equation due to Povzner [29]. Next, in the asymptotic procedure known as *grazing collision* limit, we obtain a simpler equation in divergence form, which retains all properties of the underlying Boltzmann equation, and in addition can be studied in detail. This equation has been derived recently in the work of Ha and Tadmor [24] and further analysed in [23] by Ha and Liu.

In the particle model proposed by Cucker and Smale, the particles influence each other according to a decreasing function of their mutual space distance; summarizing the results in the finite particle model, they prove that all the particles tend exponentially fast to move with their global mean velocity whenever the mutual interaction was strong enough at far distance, independently of the initial conditions. This situation is called *unconditional flocking*.

In the work [24], the authors are able to show that the kinetic energy is a Lyapunov functional for classical solutions of the kinetic equation and it vanishes sub-exponentially fast in time, however with more restrictive conditions than for the finite particle model on the strength of the long-range interaction between particles in order to achieve the unconditional flocking.

The second goal of the present paper is to extend these latter results and provide a unconditional flocking theorem with the same strength estimates valid for the finite particle models, not only for classical solutions, but also for measure valued solutions.

The way we proceed is first by reformulating the finite particle model in terms of atomic

measures and by proving the unconditional flocking theorem in this measure setting. In particular, we show in Proposition 5 the existence of an atomic measure fully concentrated on the mean velocity to which the atomic measure valued solution of the kinetic equation converges exponentially fast in time in a dual distance with respect to bounded Lipschitz functions. In particular, the support in space of the measure valued solution keeps bounded all over the process. Our rate of convergence in time and our support estimates do not depend on the number of particles. Therefore, in Theorem 6, we are able to extend the mentioned properties of the support by an approximation argument to any measure valued solution with initial compactly supported measure datum. There we combine the particle result with a stability property provided in [23] (we report the stability result below in Theorem 3.D). As an immediate consequence, we obtain a refinement of the result of Ha and Tadmor, by showing that the kinetic energy vanishes with exponential rate. We complete the picture and we fully extend the results valid for the Cucker and Smale finite particle model, by proving our unconditional flocking Theorem 9, which states the convergence of any measure valued solution with compactly supported initial datum to a compactly supported measure fully concentrated on the mean velocity.

The paper is organized as follows. In Section 2 we recall the finite particle model of Cucker and Smale, and their main results concerning unconditional flocking. We propose a kinetic equation which integrates the smeared collisional rules of the finite particle model of Cucker and Smale. Via grazing collision limit we eventually derive a nonlinear friction equation in divergence form which essentially describes the large time behavior of the Boltzmann-type model. Section 3 is dedicated to reviewing the known results from [24, 23] on the latter equation which was independently proposed as a natural mean field limit of the finite particle model. Section 4 collects our main results which extend the work [24, 23] and fully generalize the finite particle model.

2 A Boltzmann type equation for flocking

2.1 The Cucker-Smale model

In [16] Cucker and Smale studied the phenomenon of flocking in a population of birds, whose members are moving in the physical space $\mathbb{E} = \mathbb{R}^3$. The goal was to prove that under certain communication rates between the birds, the state of the flock converges to one in which all birds fly with the same velocity. The main hypothesis justifying the behavior of the population is that every bird adjusts its velocity to a weighted average of the relative velocity with respect to the other birds. That is, given a population of N birds, at time $t \in \mathbb{N}$, for the i-th bird,

$$v_i(t+1) - v_i(t) = \frac{\gamma}{N} \sum_{i=1}^{N} a_{ij} \left(v_j(t) - v_i(t) \right), \tag{2.1}$$

where the weights a_{ij} quantify the way the birds influence each other, communication rate, independently of their total number N and γ measures the interaction strength. In [16], it is assumed that the communication rate is a function of the distance between birds, namely

$$a_{ij} = \frac{1}{\left(1 + \|x_i - x_j\|^2\right)^{\beta}} \tag{2.2}$$

for some $\beta \geq 0$. For $x, v \in I\!E^N$, denote

$$\Gamma(x) = \frac{1}{2} \sum_{i \neq j} \|x_i - x_j\|^2, \tag{2.3}$$

and

$$\Lambda(v) = \frac{1}{2} \sum_{i \neq j} \|v_i - v_j\|^2. \tag{2.4}$$

Then, under suitable restrictions on β and γ, and certain initial configurations (see [16] for details), it is proven that there exists a constant B_0 such that $\Gamma(x(t)) \leq B_0$ for all $t \in \mathbb{R}_+$, while $\Lambda(v(t))$ converges towards zero as $t \to \infty$, and the vectors $x_i - x_j$ tend to a limit vector $\widehat{x}_{ij}$, for all $i, j \leq N$.

In particular, and rather remarkably, when $\beta < 1/2$, no restrictions on γ and initial configurations are needed [17, Theorem 1] and [16]. In this case, called unconditional flocking, the behavior of the population of birds is perfectly specified. All birds tend to fly exponentially fast with the same velocity, while their relative distances tend to remain constant. This result has been recently improved in several works [23, 24, 30], where other weights or communication rates are studied and sharper rate of convergence are obtained in certain cases.

Despite the theoretical and fundamental nature of this result, surprising and remarkable applications of the Cucker-Smale principle have recently been found in spacecraft flight control [28] in the context of the ESA-mission DARWIN. Darwin will be a flotilla of four or five free-flying spacecrafts that will search for Earth-like planets around other stars and analyse their atmospheres for the chemical signature of life. The fundamental problem is to ensure that, with a minimal amount of fuel expenditure, the spacecraft fleet keep remaining in flight (flock), without loosing mutual radio contact, and eventually scattering.

Back to mathematical terms, condition (2.1) can be fruitfully rephrased in a different way, which will be helpful in the following. Suppose that we have a population composed by two birds, say i and j. Assume that their velocities are modified according to (2.1)

$$v_i(t+1) = (1 - \gamma a_{ij}) v_i(t) + \gamma a_{ij} v_j(t), \tag{2.5a}$$

$$v_j(t+1) = \gamma a_{ij} v_i(t) + (1 - \gamma a_{ij}) v_j(t). \tag{2.5b}$$

while the momentum is preserved after the interaction, then

$$v_i(t+1) + v_j(t+1) = v_i(t) + v_j(t),$$

the energy increases or decreases according to the value of γ

$$v_i^2(t+1) + v_j^2(t+1) = v_i^2(t) + v_j^2(t) - 2\gamma a_{ij} \left(1 - \gamma a_{ij}\right) \left(v_i(t) - v_j(t)\right)^2. \tag{2.6}$$

If $\gamma < 1$, the energy is dissipated. Note that in this case the relative velocity is decreasing, since

$$|v_i(t+1) - v_j(t+1)| = |1 - 2\gamma a_{ij}||v_i(t) - v_j(t)| < |v_i(t) - v_j(t)|, \tag{2.7}$$

and the velocities of the two birds tend towards the mean velocity $(v_i + v_j)/2$. In case $\gamma < 1$, the interaction (2.5) is similar to a binary interaction between molecules of a dissipative gas, see [11] and the references therein.

In the general case of a population of N birds, the binary law (2.5) is taken into account together with the assumption that the i-th bird modifies its velocity giving the same weight

to all the other velocities. In consequence of this,

$$v_i(t+1) = \frac{1}{N}\sum_{j=1}^{N}\left\{(1-\gamma\, a_{ij})v_i(t) + \gamma\, a_{ij}v_j(t)\right\}, \tag{2.8}$$

that is a different way to write formula (2.1).

2.2 A Boltzmann-type equation for Cucker-Smale flocking

Unlike the control of a finite number of agents, the numerical simulation of a rather large population of interacting agents can constitute a serious difficulty which stems from the accurate solution of a possibly very large system of ODEs. Borrowing the strategy from the kinetic theory of gases, we may want instead to consider a density distribution of agents, depending on spatial position, velocity, and time evolution, which interact with stochastic collisions – in this case the collisions are spatially "smeared" since two birds do interact also when they are far apart. Hence, instead of simulating the behavior of each individual agent, we would like to describe the collective behavior encoded by the density distribution whose evolution is governed by one sole mesoscopic partial differential equation.

Let $f(x,v,t)$ denote the density of birds in the position $x \in \mathbb{R}^d$ with velocity $v \in \mathbb{R}^d$ at time $t \geq 0$, $d \geq 1$. The kinetic model for the evolution of $f = f(x,v,t)$ can be easily derived by standard methods of kinetic theory, considering that the change in time of $f(x,v,t)$ depends both on transport (birds fly freely if they do not interact with others), and interactions with other birds. Discarding other effects, this change in density depends on a balance between the gain and loss of birds with velocity v due to binary interactions. Let us assume that two birds with positions and velocities (x,v) and (y,w) modify their velocities after the interaction, according to (2.1)

$$v^* = (1-\gamma a(x-y))v + \gamma a(x-y)w, \tag{2.9a}$$

$$w^* = \gamma a(x-y)v + (1-\gamma a(x-y))w. \tag{2.9b}$$

where now the communication rate function a takes the form

$$a(x) = \frac{1}{(1+|x|^2)^{\beta}}, \quad x \in \mathbb{R}^d, \tag{2.10}$$

and $\gamma < 1/2$. This leads to the following integro-differential equation of Boltzmann type,

$$\left(\frac{\partial f}{\partial t} + v\cdot\nabla_x f\right)(x,v,t) = Q(f,f)(x,v,t), \tag{2.11}$$

where

$$Q(f,f)(x,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{1}{J}f(x,v_*)f(y,w_*) - f(x,v)f(y,w)\right)dw\,dy. \tag{2.12}$$

In (2.12) (v_*, w_*) are the pre-collisional velocities that generate the couple (v,w) after the interaction. In (2.11) $J = (1-2\gamma a)^d$ is the Jacobian of the transformation of (v,w) into (v^*, w^*). Note that, since we fixed $\gamma < 1/2$, the Jacobian J is always positive. The bilinear operator Q in (2.12) is the analogous of the Boltzmann equation for Maxwell molecules [5, 11], where the collision frequency is assumed to be constant. In a number of different kinetic equations, the evolution of the density f is driven by collisions and the rate of

change is defined through the collision term Q. One of the assumptions in the derivation of the Boltzmann collision operator is that only pair collisions are significant and that each separate collision between two molecules occurs at one point in space. Povzner [29] proposed a modified Boltzmann collision operator considering a smearing process for the pair collisions. This modified Povzner collision operator looks as follows

$$Q_P(f,f)(x,v) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} B(x-y, v-w)\left(f(x,v_*)f(y,w_*) - f(x,v)f(y,w)\right) dw\, dy. \quad (2.13)$$

In (2.13) B is the collision kernel and the post-collision velocities (v^*, w^*) are given by

$$v^* = (I - A(|x-y|))v + A(|x-y|)w, \quad (2.14a)$$

$$w^* = A(|x-y|)v + (I - A(|x-y|))w. \quad (2.14b)$$

where A is a 3×3 matrix and I the identity matrix. These last relations imply the conservation of momentum. We remark that, differently from interactions (2.9), in Povzner's equation the matrix A is such that also the energy is preserved in a collision. It is clear that equation (2.11) can be viewed like a Povzner type equation with dissipative interactions, where the matrix $A(|x-y|) = \gamma a(x-y)I$. It is remarkable that, while the Povzner equation was first introduced for purely mathematical reasons and usually ignored by the physicists, related kinetic equations can be fruitfully introduced to model many agents systems in biology and ecology.

A first important consequence of the interaction mechanism given by (2.9) is that the support of the allowed velocities can not increase. In fact, since $0 \le a \le 1$ and $0 < \gamma < \frac{1}{2}$,

$$|v^*| = |(1-\gamma a(|x-y|))v + \gamma a(|x-y|)w| \le (1-\gamma a)|v| + \gamma a|w| \le \max\{|v|, |w|\}.$$

The presence of the Jacobian in the collision operator (2.12) can be avoided by considering a weak formulation. By a weak solution of the initial value problem for equation (2.11), corresponding to the initial density $f_0(x,v)$, we shall mean any density satisfying the weak form of (2.11)-(2.12) given by

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^{2d}} \phi(x,v)f(x,v,t)\, dv\, dx + \int_{\mathbb{R}^{2d}} (v\cdot\nabla_x\phi(x,v))f(x,v,t)\, dv\, dx = \int_{\mathbb{R}^{4d}} (\phi(x,v^*) - \phi(x,v))f(x,v,t)f(y,w,t) dv\, dx\, dw\, dy \quad (2.15)$$

for $t > 0$ and all smooth functions ϕ with compact support, and such that

$$\lim_{t\to 0}\int_{\mathbb{R}^{2d}} \phi(x,v)f(x,v,t)\, dx\, dv = \int_{\mathbb{R}^{2d}} \phi(x,v)f_0(x,v)\, dx\, dv. \quad (2.16)$$

The form (2.15) is easier to handle, and it is the starting point to explore the evolution of macroscopic quantities (moments).

2.3 A kinetic equation for flocking

While the time-evolution of the population density is described in details by the Boltzmann equation (2.11), a precise description of the phenomenon of flocking is mainly related to the large-time behavior of the solution. On the other hand, this large-time behavior has to depend mainly from the type of collisions (2.9), and not from the size of the parameter γ which determines the *strongness* of the interaction itself. In this situation, an accurate

description can be furnished as well by resorting to simplified models, which turn out to be valid exactly for large times.

This idea has been first used in dissipative kinetic theory by McNamara and Young [26] to recover from the Boltzmann equation in a suitable asymptotic procedure, simplified models of nonlinear frictions for the evolution of the gas density [2, 35]. Similar asymptotic procedures have been subsequently used to recover Fokker-Planck type equations for wealth distribution [6], or opinion formation [36].

Let us assume that the parameter γ, which measures the intensity of the velocity change in the binary interactions is small ($\gamma \ll 1$). In this case, by expanding $\phi(x, v^*)$ in Taylor's series of $v^* - v$ up to the second order the weak form of the collision integral takes the form

$$\begin{aligned}
&\int_{\mathbb{R}^{4d}} (\phi(x,v^*) - \phi(x,v)) f(x,v,t) f(y,w,t)\, dx\, dv\, dy\, dw \\
&= \gamma \int_{\mathbb{R}^{4d}} (\nabla_v \phi(x,v) \cdot (w-v))\, a(x-y) f(x,v,t)\, f(y,w,t)\, dx\, dv\, dy\, dw \\
&\quad + \frac{\gamma^2}{2} \underbrace{\int_{\mathbb{R}^{4d}} \left[\sum_{i,j=1}^{d} \frac{\partial^2 \phi(x,\tilde{v})}{\partial v_i^2} (w_j - v_j)^2 \right] a(x-y)^2 f(x,v) f(y,w)\, dx\, dv\, dy\, dw}_{:=I},
\end{aligned} \tag{2.17}$$

with $\tilde{v} = \theta v + (1-\theta) v^*$, $0 \le \theta \le 1$. If the collisions are nearly grazing, $\gamma \ll 1$, we can cut the expansion (2.17) after the first–order term. Note that, due to the fact that the second moment of the solution to the Boltzmann equation is non-increasing

$$\int_{\mathbb{R}^{2d}} |v|^2 f(x,v,t)\, dx\, dv \le \int_{\mathbb{R}^{2d}} |v|^2 f_0(x,v)\, dx\, dv,$$

while $a(x) \le 1$,

$$|I| \le 2 \|\phi(x,v)\|_{C_0^2} \int_{\mathbb{R}^{2d}} |v|^2 f_0(x,v)\, dx\, dv. \tag{2.18}$$

Hence, we have a uniform in time upper bound for the remainder term of order γ^2. It follows that, in the regime of small γ, the Boltzmann collision operator $Q(f,f)$ can be approximated formally by the dissipative operator $I_\gamma(f,f)$, in strong divergence form,

$$I_\gamma(f,f) = \nabla_v \cdot \{ f(x,v,t) \left[(H(x) \nabla_v W(v)) * f \right](x,v,t) \} \tag{2.19}$$

where $W(v) = \frac{1}{2}|v|^2$ and $H(x) = \gamma a(x)$ and $*$ is the (x,v)-convolution. Notice that, as remarked by McNamara and Young in their pioneering paper [26], the operator $I_\gamma(f,f)$ maintains the same dissipation properties of the full Boltzmann collision operator.

3 A review on the nonlinear friction equation

The aim of this Section is to study the large-time behavior of the solution to the approximation of the Boltzmann equation obtained in the previous section. For convenience, let us fix $\gamma = 1$ in the rest of this work. This approximated equation corresponds to a nonlinear-type friction equation for the density $f = f(x,v,t)$, which reads

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot [\xi(f)(x,v,t) f(x,v,t)] \tag{3.1}$$

where

$$\xi(f)(x,v,t) = \left[(H(x)\nabla_v W(v)) * f\right](x,v,t) = \int_{\mathbb{R}^{2d}} \frac{v-w}{(1+|x-y|^2)^\beta} f(y,w,t)\,dy\,dw. \quad (3.2)$$

The same equation has been derived and analyzed by Ha and Tadmor [24] as the mean-field limit of the discrete and finite dimensional model (2.1) by Cucker and Smale. Their main result [24, Theorem 4.3] states that the following functional

$$\Lambda(f)(t) = \int_{\mathbb{R}^{2d}} |v-m|^2 f(x,v,t)\,dx\,dv, \text{ where} \quad (3.3)$$
$$m(f) = \int_{\mathbb{R}^{2d}} v f(x,v,t)\,dx\,dv = \int_{\mathbb{R}^{2d}} v f_0(x,v)\,dx\,dv,$$

is a Lyapunov functional for classical solutions of (3.1), and it converges to zero *sub-exponentially* for $0 \le \beta \le \frac{1}{4}$, i.e.,

$$\Lambda(f)(t) \le C\Lambda(f_0) \times \begin{cases} e^{-\kappa t^{1-4\beta}} & , \quad 0 \le \beta < \frac{1}{4} \\ (1+t)^{-\kappa'} & , \quad \beta = \frac{1}{4}, \end{cases}$$

where the constants C, κ, and κ' are positive and depend on $\beta > 0$. Classical solutions are constructed for initial data $f_0 \in C^1 \cap W^{1,\infty}(\mathbb{R}^{2d})$, compactly supported in (x,v).

The results obtained in [24] are sub-optimal though, in the sense that they do not reproduce at the continuous level the analysis in [16, 17]; no uniform bound of the spatial support of the density is provided, resulting in the sub-optimal estimate for $\beta \le \frac{1}{4}$, instead of $\beta \le \frac{1}{2}$ valid for the discrete and finite dimensional model. Moreover, while the functional Λ defined in (3.3) encodes information about the velocities, no dependence on the space is explicitly given.

On the other hand, in [23] the authors give a well-posedness result of measure-valued solutions for the Cauchy problem to (3.1). Their results show three important consequences: particles can be seen as measure-valued solutions to (3.1), the particle method converges towards a measure solution to (3.1), and finally the constructed solutions are unique in a sense to be specified below.

However, again the results in [23] are suboptimal concerning the asymptotic behavior and the qualitative properties of the constructed measure-valued solutions. They provide estimates on the growth of the support of the solutions in position and in velocity [23, Lemma 5.4]. These increasing time bounds do not reflect the qualitative picture of flocking in the discrete model [16, 17] implying convergence in velocity towards its mean and bounded growth in position variables. Moreover, they improved the results in the discrete original Cucker-Smale model for the exponent $\beta = \frac{1}{2}$ for which they prove unconditional flocking [23, Proposition 4.3].

Here, we will concentrate on three main goals. We will show an improvement in the estimates of the evolution of the support in (x,v). In fact, we will show that the support in velocity shrinks towards its mean velocity exponentially fast while the support in position is bounded around the position of the center of mass that increases linearly due to the constant mean velocity. This result is crucial for the rest and is valid for measure valued solutions.

Based on these improvements on the support evolution, we have two main consequences, the convergence towards flocking behavior for measure solutions and the improved exponential convergence to zero of the Lyapunov functional (3.3) for classical solutions in the whole range $\beta \le 1/2$.

Let us start by reminding the notion of solution to (3.1) we will be dealing with. Let us denote by $\mathcal{M}(\mathbb{R}^{2d})$ the set of positive Radon measures and let fix $T > 0$.

Definition 1 *We say that $\mu \in L^\infty((0,T);\mathcal{M}(\mathbb{R}^{2d}))$ is a weak measure-valued solution of* (3.1) *with initial data $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ in the time interval $[0,T]$, if it verifies that*

1. *$\mu \in \mathcal{C}_w([0,T],\mathcal{M}(\mathbb{R}^{2d}))$ with $\mu(0)=\mu_0$.*
2. *Given*
$$\xi(\mu)(x,v,t) := \int_{\mathbb{R}^{2d}} \frac{v-w}{(1+|x-y|^2)^\beta}\,\mathrm{d}\mu(t)(y,w) \tag{3.4}$$
then, for any $\varphi \in \mathcal{C}_0^\infty([0,T)\times\mathbb{R}^{2d}_+)$, we have that
$$\int_{\mathbb{R}^{2d}} \varphi\,\mathrm{d}\mu(t)(x,v) - \int_{\mathbb{R}^{2d}} \varphi\,\mathrm{d}\mu(0)(x,v) = \int_0^t \int_{\mathbb{R}^{2d}} \left[\frac{\partial\varphi}{\partial t} + v\cdot\nabla_x\varphi - \nabla_v\varphi\cdot\xi(\mu)\right]\mathrm{d}\mu(s)(x,v)\,ds.$$

Remark 2

1. *As it is pointed out in [23], if $f \in \mathrm{L}^1(\mathbb{R}^{2d}\times[0,T))$ is a weak solution in the distributional sense to* (3.1)*, then $f(x,v,t)\,\mathrm{d}x\mathrm{d}v$ is a measure valued solution to* (3.1)*. On the other hand, a measure valued solution to* (3.1) *which is also absolutely continuous with respect to the Lebesgue measure is a weak solution to* (3.1) *on the sense of distributions.*
2. *Particle solutions: given any solution to the discrete system $x_i(0)=x_i^0$, $v_i(0)=v_i^0$, and*
$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N_p} m_j H(|x_i-x_j|)\,(v_j-v_i)\,, \end{cases} \tag{3.5}$$
then the measure curve given by
$$\mu(x,v,t) = \sum_{i=1}^{N_p} m_i\,\delta(x-x_i(t))\,\delta(v-v_i(t)), \quad \sum_{i=1}^{N_p} m_i = M, \text{ and } m_i>0,$$
is a weak measure-valued solution to (3.1)*.*

Given the set $\mathrm{Lip}_b(\mathbb{R}^N)$ of bounded and Lipschitz functions φ on $\mathbb{R}^N$ endowed with the norm
$$\|\varphi\|_{\mathrm{Lip}_b(\mathbb{R}^N)} := \|\varphi\|_{L^\infty(\mathbb{R}^N)} + \mathrm{Lip}(\varphi),$$
we define the bounded Lipschitz distance between two measures $\mu,\nu \in \mathcal{M}(\mathbb{R}^N)$ as
$$d_{\mathbb{R}^N}(\mu,\nu) = \sup\left\{\left|\int_{\mathbb{R}^N}\varphi(z)\,d\mu(z) - \int_{\mathbb{R}^N}\varphi(z)\,d\nu(z)\right|;\ \|\varphi\|_{\mathrm{Lip}_b(\mathbb{R}^N)} \le 1\right\}. \tag{3.6}$$
This distance was classically used in particle limits for the Vlasov equation in [27, 31]. We refer to [38] for comments and relations of these dual distances to optimal transport distances.

Let us summarize the main findings of the paper of Ha and Liu about the measure-valued solutions to (3.1) in the following theorem.

Theorem 3 [23] *Given $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ compactly supported, then there exists a unique measure-valued solution μ to* (3.1) *satisfying the following properties:*

A. The total mass of the system M and its mean velocity m are conserved:

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}} \mathrm{d}\mu(x,v,t) = \frac{d}{dt}\int_{\mathbb{R}^{2d}} v\,\mathrm{d}\mu(x,v,t) = 0,$$

while its center of mass $x_c(t)$ is linearly increasing

$$\frac{d}{dt}x_c(t) = \frac{d}{dt}\int_{\mathbb{R}^{2d}} x\,\mathrm{d}\mu(x,v,t) = \int_{\mathbb{R}^{2d}} v\,\mathrm{d}\mu(x,v,t) = m.$$

B. Propagation of the support: the solution is compactly supported for all times $t \geq 0$ and there exists increasing functions of time $R^x(t) \simeq t^2$ and $R^v(t) \simeq t$ such that

$$supp\ \mu(t) \subset B(x_c(0) + mt, R^x(t)) \times B(m, R^v(t))$$

for all $t \geq 0$ with $x_c(0)$ the initial center of mass.

C. The flow map $(X(t), V(t))$ associated to the characteristic equations:

$$\begin{cases} \dfrac{d}{dt}X(t;x,v) = V(t;x,v) & , X(0;,x,v) = x \\ \dfrac{d}{dt}V(t;x,v) = \xi(\mu)(X(t;x,v), V(t;x,v), t) & , V(0;x,v) = v, \end{cases}, \tag{3.7}$$

is a well-defined homeomorphism for each fixed time t and is a C^1-function of time since $\xi(\mu)$ is continuous in t and Lipschitz continuous in (x,v). Moreover, the unique solution to (3.1) is given by $\mu(x,v,t) = (X(t;x,v), V(t;x,v))\#\mu_0$ in the mass transportation notation, i.e.,

$$\int_{\mathbb{R}^{2d}} \zeta(x,v)\,\mathrm{d}\mu(t)(x,v) = \int_{\mathbb{R}^{2d}} \zeta(X(t;x,v), V(t;,x,v))\,\mathrm{d}\mu_0(x,v), \tag{3.8}$$

for all $\zeta \in \mathcal{C}_b^0(\mathbb{R}^{2d})$.

D. Given any other measure valued solution ν to equation (3.1), the following stability estimate

$$d_{\mathbb{R}^{2d}}(\mu(t), \nu(t)) \leq \alpha(t)\, d_{\mathbb{R}^{2d}}(\mu_0, \nu_0)$$

holds for all $t \in [0,T]$ and all $T > 0$ where the increasing function $\alpha(t) \simeq e^{e^t}$ as $t \to \infty$ with $\alpha(0) = 1$.

4 Exponential in time collapse of the velocity support

We first start by showing the result on the system of particles. Let us consider a N_p-particle system following the dynamics (3.5):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i & , x_i(0) = x_i^0 \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N_p} m_j H(|x_i - x_j|)\,(v_j - v_i) & , v_i(0) = v_i^0, \end{cases}.$$

Since the equation is translational invariant, let us assume without loss of generality that the mean velocity is zero and thus the center of mass is preserved along the evolution, i.e.,

$$\sum_{i=1}^{N_p} m_i v_i(t) = 0 \qquad \text{and} \qquad \sum_{i=1}^{N_p} m_i x_i(t) = x_c$$

for all $t \geq 0$ and $x_c \in \mathbb{R}^d$. Then, let us fix any $R_0^x > 0$ and $R_0^v > 0$, such that all the initial velocities lie inside the ball $B(0, R_0^v)$ and all positions inside $B(x_c, R_0^x)$. Now, the solutions to this system are $C^1([0,\infty), \mathbb{R}^{2d})$ for both positions and velocities and for any label i. Let us define the function $R^v(t)$ to be

$$R^v(t) := \max_{i=1,\dots,N_p} |v_i(t)|.$$

Since the number of particles is finite and the curves are smooth in time, there exist at most countable number of increasing times t_k such that on any time interval (t_k, t_{k+1}), we can choose an index i (depending on k) such that $R^v(t) = |v_i(t)|$ for all $t \in (t_k, t_{k+1})$. Now, we can compute

$$\frac{d}{dt} R^v(t)^2 = \frac{d}{dt}|v_i|^2 = -2 \sum_{j \neq i} m_j \left[(v_i - v_j) \cdot v_i\right] H(|x_i - x_j|) \,. \tag{4.1}$$

Because of the choice of the label i, we have that $(v_i - v_j) \cdot v_i \geq 0$ for all $j \neq i$ that together with $H \geq 0$ imply that $R^v(t)$ is a non-increasing function. Hence, $R^v(t) \leq R_0^v$ for all $t \geq 0$. Now coming back to the equation for the positions, we deduce that

$$|x_i(t) - x_i^0| \leq R_0^v t \quad \text{for all } t \geq 0 \text{ and all } i = 1, \dots, N_p.$$

We infer that $|x_i - x_j| \leq 2R_0^x + 2R_0^v t$ and thus

$$H(|x_i - x_j|) \geq \frac{1}{[1 + 4R_0^2(1+t)^2]^\beta} \quad \text{for all } t \geq 0 \text{ and all } i, j = 1, \dots, N_p,$$

with $R_0 = \min(R_0^x, R_0^v)$. Again, we come back to the equation for the maximal velocity (4.1), and we deduce

$$\begin{aligned}
\frac{d}{dt} R^v(t)^2 &= -2 \sum_{j \neq i} m_j \left[(v_i - v_j) \cdot v_i\right] H(|x_i - x_j|) \\
&\leq -\frac{2}{[1 + 4R_0^2(1+t)^2]^\beta} \sum_{j \neq i} m_j \left[(v_i - v_j) \cdot v_i\right] \\
&= -\frac{2}{[1 + 4R_0^2(1+t)^2]^\beta} R^v(t)^2 := -f(t)\, R^v(t)^2,
\end{aligned}$$

from which we obtain by direct integration or Gronwall's lemma:

$$R^v(t) \leq R_0^v \exp\left\{ -\frac{1}{2} \int_0^t f(s)\, ds \right\}.$$

It is immediate to check that

$$\lim_{t \to \infty} t^{2\beta} f(t) = \left(\frac{1}{2R_0^2} \right)^\beta,$$

then, for $\beta \leq 1/2$, the function $f(t)$ is not integrable at ∞ and therefore

$$\lim_{t\to\infty} \int_0^t f(s)\, ds = +\infty$$

and $R^v(t) \to 0$ as $t \to \infty$ giving the convergence to a single point, its mean velocity, of the support for the velocity.

Now, let us estimate the position variables again, we deduce that

$$\int_0^t |v_i(s)|\, ds \leq R_0^v \int_0^t \exp\left\{-\frac{1}{2}\int_0^s f(\tau)\, d\tau\right\} ds. \tag{4.2}$$

Let us distinguish two cases. It is trivial to check that for $0 < \beta < 1/2$

$$\lim_{t\to\infty} (1+t) f(t) = +\infty,$$

then for any $C > 0$, there exists $\epsilon > 0$ such that if $s \geq \epsilon^{-1}$ then $(1+s)f(s) \geq C$. We conclude that

$$-\int_0^t f(s)\, ds \leq -\int_0^{1/\epsilon} f(s)\, ds - C\ln(1+t) + C\ln(1+\epsilon^{-1})$$

and thus,

$$\exp\left\{-\frac{1}{2}\int_0^t f(s)\, ds\right\} \leq A(\epsilon)\,(1+t)^{-C/2}.$$

Plugging into (4.2), we deduce

$$\int_0^t |v_i(s)|\, ds \leq R_0^v \int_0^t \exp\left\{-\frac{1}{2}\int_0^s f(\tau)\, d\tau\right\} ds \leq R_0^v\, A(\epsilon) \int_0^t (1+s)^{-C/2}\, ds$$

and since the constant C can be chosen arbitrarily large by choosing ϵ small, we conclude that the integral is bounded in $t > 0$, and that there exists $R_1^x > 0$ such that

$$|x_i(t) - x_i^0| \leq R_1^x$$

for all $t \geq 0$ and $i = 1, \ldots, N_p$. This implies that $x_i(t) \in B(x_c, \bar{R}^x)$ for all $t \geq 0$ and $i = 1, \ldots, N_p$ with $\bar{R}^x = R_1^x + R_0^x$.

Moreover, coming back again to the velocities, we deduce now that $H(|x_i(t) - x_j(t)|) \geq H(2\bar{R}^x)$. So that from (4.1), we get

$$\begin{aligned}\frac{d}{dt} R^v(t)^2 =& -2\sum_{j\neq i} m_j\, [(v_i - v_j)\cdot v_i]\; H(|x_i - x_j|)\\ \leq& -2H(2\bar{R}^x)\sum_{j\neq i} m_j\, [(v_i - v_j)\cdot v_i] = -2H(2\bar{R}^x) R^v(t)^2\end{aligned}$$

from which we finally deduce the exponential decay to zero of $R^v(t)$.

Now again, if we come back to the position variables since the velocity curve is integrable on time due to the exponential decay, we deduce that

$$\lim_{t\to\infty} x_i(t) = x_i^\infty$$

and that the lengths of the curves followed by each of particles is finite once subtracted the translational movement due to the constant mean velocity.

Now, let us come back to the case $\beta = \frac{1}{2}$. Coming back to (4.2), we can now compute the integral explicitly implying that

$$\frac{1}{2}\int_0^t f(s)\,ds = \ln\left[2R_0(1+t) + \sqrt{1+4R_0^2(1+t)^2}\right] - \ln\left[2R_0 + \sqrt{1+4R_0^2}\right].$$

It is straightforward to check that

$$\int_0^t |v_i(s)|\,ds \le C\int_0^t \frac{1}{1+s}\,ds = C\ln(1+t),$$

with C depending only on R_0. In this case, we do another loop of going to the position variables to deduce as before that

$$\frac{d}{dt}R^v(t)^2 \le -\frac{2}{[1+C(1+\ln(1+t))^2]^\beta}R^v(t)^2 := -g(t)R^v(t)^2.$$

It is obvious that

$$\lim_{t\to\infty}(1+t)g(t) = +\infty$$

and thus we come back to the same situation as for $\beta < \frac{1}{2}$ and thus, giving the desired result for $\beta = \frac{1}{2}$.

Remark 4 *The unconditional flocking result in the whole range $0 < \beta \le \frac{1}{2}$ was already obtained in Section 3 and 4 of* [23] *with a completely different argument. The authors give an alternative proof of the unconditional flocking of the Cucker-Smale model for any $\beta < \frac{1}{2}$ and prove it for $\beta = \frac{1}{2}$ based on estimates of the dissipation of the dynamical system with euclidean norms.*

It is completely crucial to note that the constants we obtain in the exponential rate to zero of the support in velocity of the particles do not depend either on the number of particles nor on their masses. They only depend on the initial values of R_0^v and R_0^x, i.e., on the initial values of the particles with the largest velocity and the furthest away from the center of mass.

The importance of this presented alternative proof of the unconditional flocking is that the estimates are independent of the number of particles. This was not the case in [23] since their argument is based on the euclidean norm of the velocity and position vectors and not in the bounded norm. Some of the above ideas are reminiscent of arguments used for continuum models of swarming in [13, 3].

We summarize the previous discussion in the following theorem written in terms of solutions of the equation (3.1).

Proposition 5 *For any $\tilde{\mu}_0 \in \mathcal{M}(\mathbb{R}^{2d})$ composed by finite number of particles, i.e., an atomic measure with N_p atoms, there exists $\{x_1^\infty, \dots, x_{N_p}^\infty\}$ such that the unique measure-valued solution to* (3.1) *with $\beta \le 1/2$, given by*

$$\tilde{\mu}(t) = \sum_{i=1}^{N_p} m_i\,\delta(x - x_i(t))\,\delta(v - v_i(t)),$$

where the curves are given by the ODE system (3.5)*, satisfies that*

$$\lim_{t\to\infty} d\left(\tilde{\mu}(t), \tilde{\mu}^\infty\right) = 0$$

with

$$\tilde{\mu}^\infty = \sum_{i=1}^{N_p} m_i\, \delta(x - x_i^\infty - mt)\, \delta(v - m)$$

with m the initial mean velocity of the particles. Moreover, given the largest velocity $R^v(t)$ defined as

$$R^v(t) := \max_{i=1,\dots,N_p} |v_i(t) - m|.$$

and the most distant space location $R^x(t)$ with respect to the initial center of mass x_c,

$$R^x(t) := \max_{i=1,\dots,N_p} |x_i(t) - x_c - mt|,$$

then

$$R^x(t) \le \bar{R}^x \qquad \text{and} \qquad R^v(t) \le R_0^v\, e^{-\lambda t}$$

for all $t \ge 0$, with $\bar{R}^x$ depending only on the initial value of $R_0 = \max\{R^v(0), R^x(0)\}$ and $\lambda = H(2\bar{R}^x)$.

Let us point out that the proof is trivial noting that the bounded Lipschitz norm between finite atomic measures is bounded by a sum of Euclidean distances for any permutation of the points of one of them, see [38] for instance. We also wrote them as solutions of the partial differential equation instead of the particle system since this will be useful for general measure solutions.

Once we have the control on the support of particles independent of the number of particles given in the previous theorem, it is trivial to deduce the main result of this work.

Theorem 6 *Given $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ compactly supported, then the unique measure-valued solution to* (3.1) *with $\beta \le 1/2$, satisfies the following bounds on their supports:*

$$\text{supp } \mu(t) \subset B(x_c(0) + mt, R^x(t)) \times B(m, R^v(t))$$

for all $t \ge 0$, with

$$R^x(t) \le \bar{R} \qquad \text{and} \qquad R^v(t) \le R_0\, e^{-\lambda t}$$

with $\bar{R}^x$ depending only on the initial value of $R_0 = \max\{R^x(0), R^v(0)\}$ and $\lambda = H(2\bar{R})$.

`Proof` As for particles, we can assume $m = 0$ and $x_c(t) = x_c(0)$ for all $t \ge 0$ without loss of generality. Given any compactly supported measure μ_0 in $B(x_c(0), R^x(0)) \times B(0, R^v(0))$ and any $\eta > 0$, we can find a number of particles $N_p = N_p(\eta)$, set of positions $\{x_1^0, \dots, x_{N_p}^0\} \subset B(c_M(0), R^x(0))$, a set of velocities $\{v_1^0, \dots, v_{N_p}^0\} \subset B(0, R^v(0))$, and masses $\{m_1, \dots, m_{N_p}\}$, such that

$$d_{\mathbb{R}^{2d}}\left(\mu_0, \sum_{i=1}^{N_p} m_i \delta(x - x_i^0)\delta(v - v_i^0)\right) \le \eta.$$

Let us denote by $\mu_\eta(t)$ the particle solution associated to the initial datum

$$\mu_\eta(0) = \sum_{i=1}^{N_p} m_i\, \delta(x - x_i^0)\, \delta(v - v_i^0).$$

Using Proposition 5, we have that

$$\text{supp } \mu_\eta(t) \subset B(x_c(0) + mt, R^x(t)) \times B(m, R^v(t))$$

with $R^x(t)$ and $R^v(t)$ verifying the stated properties for all small η since the result was independent of the number of particles.

By the stability result in Theorem 3 included in [23], we obtain

$$d_{\mathbb{R}^{2d}}(\mu(t),\mu_\eta(t)) \le \alpha(t)\, d_{\mathbb{R}^{2d}}\left(\mu_0, \sum_{i=1}^{N_p} m_i\, \delta(x-x_i^0)\,\delta(v-v_i^0)\right) \le \alpha(t)\eta.$$

Since t is fixed and η can be arbitrarily small, we conclude that $\mu_\eta(t) \to \mu(t)$ weakly-$*$ as measures when $\eta \to 0$ for all $t \ge 0$. Since the support of a measure is stable under weak-$*$ limits, we conclude the proof.

■

Let us remark that the same strategy of proof has been used for continuum models of aggregation [7]. An immediate consequence is to control directly the decay of the Lyapunov functional used by Ha and Tadmor in [24].

Corollary 7 *Given $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ compactly supported, then the unique measure-valued solution to* (3.1) *with $\beta \le 1/2$, satisfies*

$$\Lambda(\mu)(t) \le R_0^2\, e^{-2\lambda t}$$

with R_0 and λ given in Theorem 6.

`Proof` Since the solution by Theorem 6 is supported in velocity in $B(m, R^v(t))$, then

$$\Lambda(\mu)(t) = \int_{\mathbb{R}^{2d}} |v-m|^2\, d\mu(t)(x,v) = \int_{|v-m|\le R^v(t)} |v-m|^2\, d\mu(t)(x,v) \le R^v(t)^2$$

concluding the proof.

■

Remark 8 *Let us point out that there is another interesting functional associated to the system*

$$\mathscr{F}(\mu)(t) = \frac{1}{2}\int_{\mathbb{R}^{4d}} \frac{|v-w|^2}{(1+|x-y|^2)^\beta}\, d\mu(t)(x,v)\, d\mu(t)(y,w). \tag{4.3}$$

This functional comes naturally as a Lyapunov functional for classical solutions of the equation

$$\frac{\partial f}{\partial t} = \nabla_v \cdot \left[\xi(f)(x,v,t) f(x,v,t)\right], \tag{4.4}$$

which stems from (3.1) *if the transport term $v \cdot \nabla_x f$ is dropped [9, 10]. As with the other functional it is trivial to check that since the solution by Theorem 6 is supported in velocity in $B(m, R^v(t))$ and in position in $B(x_c(0)+mt, R^x(t))$, then $\mathscr{F}(\mu)(t) \le R^v(t)^2$, for any weak measure-valued solution μ of* (3.1)*. Moreover, due to the uniform space support bound given in Theorem 6, the functionals $\Lambda(f)$ and $\mathscr{F}(f)$ are equivalent in the sense that*

$$C_{\mathscr{F}}\Lambda(\mu) \le \mathscr{F}(\mu) \le C^{\mathscr{F}}\Lambda(\mu),$$

for $C_{\mathscr{F}} \le C^{\mathscr{F}}$ positive constants independent of t. Actually, using again Theorem 6, the solution is supported in velocity in $B(m, R^v(t))$ and in position in $B(x_c(0)+mt, R^x(t))$ and

thus

$$\begin{aligned}\mathscr{F}(\mu)(t) &= \frac{1}{2}\int_{\mathbb{R}^{4d}} \frac{|v-w|^2}{(1+|x-y|^2)^\beta}\, d\mu(t)(x,v)\, d\mu(t)(y,w)\\ &\geq \frac{1}{2(1+\bar{R}^2)^\beta}\int_{\mathbb{R}^{4d}} \left(|v-m|^2+|w-m|^2\right)\, d\mu(t)(x,v)\, d\mu(t)(y,w)\\ &= \frac{2}{(1+\bar{R}^2)^\beta}\Lambda(\mu)(t)\end{aligned}$$

and trivially $\mathscr{F}(\mu)(t) \leq 2\Lambda(\mu)(t)$.

Finally, let us obtain some more information about the asymptotic limit. Using the characterization of solutions by characteristics in Theorem 3, then we have that $\mu(x,v,t) = (X(t;x,v), V(t;x,v))\#\mu_0$ where the characteristics $(X(t;x,v), V(t;x,v))$ satisfy (3.7). It is then clear that

$$|V(t;x,v)-m| \leq R_0\, e^{-\lambda t} \text{ for all } v \in B(m,R_0)\,,\ x \in B(x_c(0),R_0).$$

But using the equation for the position variables we find

$$\frac{d}{dt}\left[X(t;x,v)-mt\right] = V(t;x,v)-m$$

with a right hand side whose components are exponentially decaying in time and thus, integrable at infinity. As a consequence, for all (x,v) initially in the support of μ_0, we have

$$\lim_{t\to\infty}[X(t;x,v)-mt] = x + \int_0^\infty [V(s;x,v)-m]\, ds.$$

This can be rephrased in terms of the density in position associated to the solutions. We need a bit of notation: given a measure $\mu \in \mathcal{M}(\mathbb{R}^{2d})$, we define its translate μ^h with vector $h \in \mathbb{R}^d$ by:

$$\int_{\mathbb{R}^d} \zeta(x,v)\, \mathrm{d}\mu^h(x,v) = \int_{\mathbb{R}^{2d}} \zeta(x-h,v)\, \mathrm{d}\mu(x,v),$$

for all $\zeta \in \mathcal{C}_b^0(\mathbb{R}^{2d})$. We will also denote by μ_x the marginal in the position variable, that is,

$$\int_{\mathbb{R}^{2d}} \zeta(x)\, \mathrm{d}\mu(x,v) = \int_{\mathbb{R}^d} \zeta(x)\, \mathrm{d}\mu_x(x),$$

for all $\zeta \in \mathcal{C}_b^0(\mathbb{R}^d)$. With this we can write the main conclusion about the asymptotic behavior, i.e., the convergence in relative to the center of mass variables to a fixed density characterized by the initial data and its unique solution.

Theorem 9 *Given* $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ *compactly supported, then the unique measure-valued solution to* (3.1) *with* $\beta \leq 1/2$, *satisfies*

$$\lim_{t\to\infty} d_{\mathbb{R}^d}(\mu_x^{mt}(t), L_\infty(\mu_0)) = 0,$$

where the measure $L_\infty(\mu_0)$ *is defined as*

$$\int_{\mathbb{R}^d} \zeta(x)\, \mathrm{d}L_\infty(\mu_0)(x) = \int_{\mathbb{R}^{2d}} \zeta\left(x + \int_0^\infty [V(s;x,v)-m]\, ds\right) \mathrm{d}\mu_0(x,v),$$

for all $\zeta \in \mathcal{C}_b^0(\mathbb{R}^d)$.

`Proof` Given a test function $\zeta \in \mathcal{C}_b^0(\mathbb{R}^d)$, we compute

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \zeta(x)\,\mathrm{d}\mu_x^{mt}(t)(x) - \int_{\mathbb{R}^d} \zeta(x)\,\mathrm{d}L_\infty(\mu_0)(x)\right| \\
&= \left|\int_{\mathbb{R}^{2d}} \zeta(x)\,\mathrm{d}\mu^{mt}(t)(x,v) - \int_{\mathbb{R}^d} \zeta(x)\,\mathrm{d}L_\infty(\mu_0)(x)\right| \\
&= \left|\int_{\mathbb{R}^{2d}} \zeta(x-mt)\,\mathrm{d}\mu(t)(x,v) - \int_{\mathbb{R}^d} \zeta(x)\,\mathrm{d}L_\infty(\mu_0)(x)\right| \\
&= \left|\int_{\mathbb{R}^{2d}} \left[\zeta(X(t;x,v)-mt) - \zeta\left(x + \int_0^\infty [V(s;x,v)-m]\,ds\right)\right]\,\mathrm{d}\mu_0(x,v)\right| \\
&\leq \mathrm{Lip}(\zeta) \int_{\mathbb{R}^{2d}} \left|(X(t;x,v)-mt) - \left(x + \int_0^\infty [V(s;x,v)-m]\,ds\right)\right|\,\mathrm{d}\mu_0(x,v).
\end{aligned}$$

An easy application of the Lebesgue dominate convergence theorem gives the result where one uses the uniform in time bound on the characteristics above.

■

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