# Capacities associated with scalar signed Riesz kernels, and analytic capacity

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#### Abstract

The real and imaginari parts of the Cauchy kernel in the plane are scalar Riesz kernels of homogeneity -1. One can associate with each of them a natural notion of capacity related to bounded potentials. The main result of the paper asserts that these capacities are comparable to classical analytic capacity, thus stressing the real variables nature of analytic capacity. Higher dimensional versions of this result are also considered.

# 1 Introduction

The analytic capacity of a compact subset E of the plane is defined by

$$\gamma(E) = \sup |f'(\infty)|$$

where the supremum is taken over those analytic functions on  $\mathbb{C} \setminus E$  such that  $|f(z)| \leq 1, z \in \mathbb{C} \setminus E$ . Sets of zero analytic capacity are exactly the removable sets for bounded analytic functions, as it is easily seen, and thus  $\gamma(E)$  quantifies the non-removability of E. Early work on analytic capacity used basically one complex variable methods (see, e.g., [A], [G1] and [Vi]). Analytic capacity may be written as

$$\gamma(E) = \sup |\langle T, 1 \rangle| \tag{1}$$

where the supremum is taken over all complex distributions T supported on Ewhose Cauchy potential f = 1/z \* T is in the closed unit ball of  $L^{\infty}(\mathbb{C})$ . The transition from f to T and viceversa is performed through the formulae  $T = \frac{1}{\pi} \overline{\partial} f$ and f = 1/z \* T.

Expression (1) shows that analytic capacity is formally an analogue of classical logarithmic capacity, in which the logarithmic kernel has been replaced by the complex kernel 1/z. This suggests that real variables techniques could help in studying analytic capacity, in spite of the fact that the Cauchy kernel is complex. In fact, significant progress in the understanding of analytic capacity was achieved when real variables methods, in particular the Calderón-Zygmund theory of the Cauchy singular integral, were systematically used ([C], [Da], [MaMeV], [MTV], [T2] and [T4]). A

striking result of Tolsa [T2] asserts that analytic capacity is comparable to a smaller quantity, called positive analytic capacity, which is defined on compact sets E by

$$\gamma_+(E) = \sup \mu(E)$$

where the supremum is taken over those positive measures supported on E whose Cauchy potential  $1/z * \mu$  is in the closed unit ball of  $L^{\infty}(\mathbb{C})$ . In other words, there exists a positive constant C such that

$$\gamma(E) \le C \,\gamma_+(E),\tag{2}$$

for each compact subset E of the plane. This implies, in particular, that analytic capacity is comparable to planar Lipschitz harmonic capacity. The Lipschitz harmonic capacity of a compact subset of  $\mathbb{R}^n$  is defined by

$$\kappa(E) = \sup |\langle T, 1 \rangle| \tag{3}$$

where the supremum is taken over those real distributions T supported on E such that the vector field  $\frac{x}{|x|^n} * T$  is in the unit ball of  $L^{\infty}(\mathbb{R}^n, \mathbb{R}^n)$ . The terminology stems from the fact that  $\kappa(E)$  vanishes if and only if E is removable for harmonic functions on  $\mathbb{R}^n \setminus E$  satisfying a global Lipschitz condition. Notice that the fact that analytic capacity and Lipschitz harmonic capacity in the plane are comparable cannot be deduced just by inspection from (1) and (3). The reason is that the distributions involved in the supremum in (1) are complex.

In this paper we continue the study of the real variables nature of analytic capacity. For a compact subset E of  $\mathbb{R}^n$  and  $1 \le i \le n$  set

$$\kappa_i(E) = \sup |\langle T, 1 \rangle| \tag{4}$$

where the supremum is taken over those real distributions T such that the scalar signed *i*-th Riesz potential

$$\frac{x_i}{|x|^n} * T \tag{5}$$

is in the unit ball of  $L^{\infty}(\mathbb{R}^n)$ . Although there are obvious formal similarities between the definitions of the set functions in (1) and (4), very little is known about  $\kappa_i$ . The reader will find in section 6.3 a proof of the elementary fact that  $\kappa_i(E)$  is finite for each compact subset E of  $\mathbb{R}^n$ . The reason why  $\kappa_i$  is difficult to understand is that boundedness of the potential (5) does not provide any linear growth condition on T. Concretely, it is not true that boundedness of (5) implies that for each cube Q one has

$$|\langle T, \varphi_Q \rangle| \le C \, l(Q),\tag{6}$$

for each test function  $\varphi_Q \in \mathcal{C}_0^{\infty}(Q)$  satisfying  $\|\partial^s \varphi_Q\|_{\infty} \leq C_s l(Q)^{-|s|}$  for all multiindexes s. Here l(Q) stands for the side length of Q and we are adopting the standard notation related to multi-indexes, that is,  $s = (s_1, \ldots, s_n)$ , where each coordinate  $s_j$  is a non-negative integer and  $|s| = s_1 + \cdots + s_n$ . The reader will find in section 5 two exemples of such phenomenon. On the other hand, recall that if T is a compactly supported distribution with bounded Cauchy potential then

$$|\langle T, \varphi_Q \rangle| = \left| \left\langle T, \frac{1}{\pi z} * \overline{\partial} \varphi_Q \right\rangle \right| = \left| \left\langle \frac{1}{\pi z} * T, \overline{\partial} \varphi_Q \right\rangle \right|$$

$$\leq C \left\| \frac{1}{z} * T \right\|_{\infty} \| \overline{\partial} \varphi_Q \|_{L^1(Q)} \leq C \, l(Q).$$
(7)

We say that a distribution T has linear growth provided that

$$G(T) = \sup_{\varphi_Q} \frac{|\langle T, \varphi_Q \rangle|}{l(Q)} < \infty,$$
(8)

where the supremum is taken over all  $\varphi_Q \in \mathcal{C}_0^{\infty}(Q)$  satisfying the normalization inequalities

$$\|\partial^{s}\varphi_{Q}\|_{L^{1}(Q)} \leq l(Q), \quad |s| = n - 1.$$
 (9)

The above special normalization in the  $L^1(Q)$  norm agrees with (7) and, in fact, is the right condition to impose, as will become clear later on. For positive Radon measures  $\mu$  in  $\mathbb{R}^n$  the preceding notion of linear growth is equivalent to the usual one (see (15) below). In subsection 6.5 complete details on this fact are provided.

For a compact set E in  $\mathbb{R}^n$  we define g(E) as the set of all distributions supported on E having linear growth with constant G(T) at most 1. For each coordinate k set

$$\Gamma_k(E) = \sup \left\{ |\langle T, 1 \rangle| : T \in g(E) \text{ and } \left\| \frac{x_k}{|x|^2} * T \right\|_{\infty} \le 1 \right\}.$$

The requirement of the growth condition in the preceding definition is vital in obtaining the localization result (19). In subsection 6.4 we show that a growth condition is necessary for a localization estimate in  $L^{\infty}$ .

Our main result reads as follows.

**Theorem 1.** There exists a positive constant C such that for each compact set  $E \subset \mathbb{R}^2$  and k = 1, 2

$$C^{-1}\Gamma_k(E) \le \gamma(E) \le C\Gamma_k(E).$$
(10)

Thus analytic capacity is the capacity associated with any component of the Cauchy kernel in which a natural growth condition on the admissible distributions is required. Observe that the second inequality in (10) follows readily from the definitions of  $\gamma$  and  $\Gamma_k$ .

Our next result is a higher dimensional version of Theorem 1. For a compact  $E \subset \mathbb{R}^n$  set

$$\Gamma(E) = \sup \left\{ |\langle T, 1 \rangle| : \operatorname{spt} T \subset E \quad \text{and} \quad \left\| \frac{x}{|x|^2} * T \right\|_{\infty} \le 1 \right\},$$

so that  $\Gamma(E) = \kappa(E)$  for n = 2. Finally, for  $1 \le k \le n$ , set

$$\Gamma_{\hat{k}}(E) = \sup\left\{ |\langle T, 1 \rangle| : T \in g(E) \quad \text{and} \quad \left\| \frac{x_i}{|x|^2} * T \right\|_{\infty} \le 1, \ 1 \le i \le n, \ i \ne k \right\}.$$

Thus we require the boundedness of n-1 components of the vector valued potential  $x/|x|^2 * T$  with Riesz kernel of homogeneity -1. Our extension of Theorem 1 to  $\mathbb{R}^n$  is the following.

**Theorem 2.** There exists a positive constant C such that for each compact set  $E \subset \mathbb{R}^n$  and  $1 \leq k \leq n$ 

$$C^{-1}\Gamma_{\hat{k}}(E) \le \Gamma(E) \le C\Gamma_{\hat{k}}(E).$$
(11)

The second inequality in (11) follows immediately from the definitions of  $\Gamma$  and  $\Gamma_{\hat{k}}$ , because any distribution T with bounded vector valued Riesz potential has linear growth (see Lemma 3.2 in [Pr1]).

The paper is organized as follows. In section 2 we present a sketch of the proof of Theorem 2. It becomes clear that the proof depends on two facts: the close relationship between the quantities one obtains after symmetrization of the kernels  $x/|x|^2$ and  $x_i/|x|^2$  and a localization  $L^{\infty}$  estimate for the scalar kernels  $x_i/|x|^2$ . In section 3 we deal with the symmetrization issue and in section 4 with the localization estimate. In section 5 we discuss two examples showing that boundedness of the scalar signed Riesz potential  $x_i/|x|^2 * T$  does not imply a linear growth estimate on T. In section 6 we present various additional results and examples. We show that  $\kappa_i(E)$  is finite for each compact E. We present counter-examples to two natural inequalities. The first shows that the obvious extension of Theorem 2 to the vector valued Riesz kernels  $x/|x|^{1+\alpha}$  and scalar kernels  $x_i/|x|^{1+\alpha}$  of homogeneity  $\alpha$ ,  $0 < \alpha < 1$ , fails. The second counter-example shows that the obvious extension of Theorem 2 to kernels of homogeneity -d, where d is an integer greater than 1, also fails. Finally we point out that a growth condition is necessary to have localization inequalities in  $L^{\infty}$ .

Our notation and terminology are standard. For instance,  $C_0^{\infty}(E)$  denotes the set of all infinitely differentiable functions with compact support contained in the set E. Cubes will always be supposed to have sides parallel to the coordinate axis, l(Q) is the side length of the cube Q and  $|Q| = l(Q)^n$  its volume.

We remind the reader that the convolution of two distributions T and S is well defined if T has compact support. In this case the action of T \* S on the test function  $\varphi$  is

$$\langle T * S, \varphi \rangle = \langle T, S * \varphi \rangle,$$

which makes sense because  $S * \varphi$  is an infinitely differentiable function on  $\mathbb{R}^n$ .

# 2 Sketch of the proof of Theorem 2

As we remarked before, one only has to prove that

$$\Gamma_{\hat{k}}(E) \le C \,\Gamma(E). \tag{12}$$

Clearly  $\Gamma(E)$  is larger than or equal to

$$\Gamma_{+}(E) = \sup \mu(E) \tag{13}$$

where the supremum is taken over those positive measures  $\mu$  supported on E whose vector valued Riesz potential  $x/|x|^2 * \mu$  lies in the closed unit ball of  $L^{\infty}(\mathbb{R}^n, \mathbb{R}^n)$ . Now,  $\Gamma_+(E)$  is comparable to yet another quantity  $\Gamma_{\rm op}(E)$ , that is, for some positive constant C one has

$$C^{-1}\Gamma_{\rm op}(E) \le \Gamma_{+}(E) \le C\Gamma_{\rm op}(E), \tag{14}$$

for each compact set  $E \subset \mathbb{R}^n$  (see [T1]). Before giving the definition of  $\Gamma_{\text{op}}(E)$  we need to introduce the Riesz transform with respect to an underlying positive Radon measure  $\mu$  satisfying the linear growth condition

$$\mu(B(x,r)) \le C r, \quad x \in \mathbb{R}^n, \quad r \ge 0.$$
(15)

Given  $\epsilon > 0$  we define the truncated Riesz transform at level  $\epsilon$  as

$$R_{\epsilon}(f\,\mu)(x) = \int_{|y-x|>\epsilon} \frac{x-y}{|x-y|^2} f(y) \, d\mu(y), \quad x \in \mathbb{R}^n, \tag{16}$$

for  $f \in L^2(\mu)$ . The growth condition on  $\mu$  insures that each  $R_{\epsilon}$  is a bounded operator on  $L^2(\mu)$  with operator norm  $||R_{\epsilon}||_{L^2(\mu)}$  possibly depending on  $\epsilon$ . We say that the Riesz transform is bounded on  $L^2(\mu)$  when

$$||R||_{L^2(\mu)} = \sup_{\epsilon > 0} ||R_\epsilon||_{L^2(\mu)} < \infty,$$

or, in other words, when the truncated Riesz transforms are uniformly bounded on  $L^2(\mu)$ . Call L(E) the set of positive Radon measures supported on E which satisfy (15) with C = 1. One defines  $\Gamma_{\text{op}}(E)$  by

$$\Gamma_{\rm op}(E) = \sup\{\mu(E) : \mu \in L(E) \text{ and } \|R\|_{L^2(\mu)} \le 1\}.$$

From the first inequality in (14) we get that, for some constant C and all compact sets E,

$$\Gamma_{\rm op}(E) \le C \Gamma(E).$$

We remind the reader that the first inequality in (14) depends on a simple but ingenious duality argument due to Davie and Oksendal (see [DO, p.139], [Ch, Theorem 23, p.107] and [V3, Lemma 4.2]). To prove (12) we have to estimate  $\Gamma_{\hat{k}}(E)$  by a constant times  $\Gamma_{op}(E)$ . The natural way to perform that is to introduce the quantity  $\Gamma_{\hat{k},op}(E)$  and try the two estimates

$$\Gamma_{\hat{k}}(E) \le C \,\Gamma_{\hat{k},\mathrm{op}}(E) \tag{17}$$

and

$$\Gamma_{\hat{k},\mathrm{op}}(E) \le C \,\Gamma_{\mathrm{op}}(E). \tag{18}$$

We define the truncated scalar Riesz transform  $R_{\varepsilon}^{i}(f \mu)(x)$  associated with the *i*-th coordinate as in (16) with the vector valued Riesz kernel replaced by the scalar Riesz kernel  $\frac{x_{i} - y_{i}}{|x - y|^{2}}$ . We also set

$$||R^i||_{L^2(\mu)} = \sup_{\epsilon>0} ||R^i_{\varepsilon}||_{L^2(\mu)},$$

and

$$\Gamma_{\hat{k}, \text{op}}(E) = \sup\{\mu(E) : \mu \in L(E) \text{ and } \|R^i\|_{L^2(\mu)} \le 1, 1 \le i \le n, i \ne k\}.$$

One proves (18) by checking that symmetrization of a scalar Riesz kernel is controlled by the symmetrization of the scalar Riesz kernels associated with all other variables. Here the fact that we are dealing with kernels of homogeneity -1 plays a key role, because, as it is well-known, they enjoy a special positivity property which is missing in general. See section 3 for complete details. For other homogeneities, either the corresponding statements are false or open (see section 6).

The proof of (17) depends on Tolsa's approach to the proof of (2), which extends without any significant change to the higher dimensional setting to give

$$\Gamma(E) \le C \,\Gamma_+(E).$$

The main technical point missing in our setting is a localization result for scalar Riesz potentials. This turns out to be a delicate issue, which we deal with in section 4. Specifically, we prove that there exists a positive constant C such that, for each compactly supported distribution T and for each coordinate i, we have

$$\left\|\frac{x_i}{|x|^2} * \varphi_Q T\right\|_{\infty} \le C\left(\left\|\frac{x_i}{|x|^2} * T\right\|_{\infty} + G(T)\right)$$
(19)

for each cube Q and each  $\varphi_Q \in \mathcal{C}^{\infty}_0(Q)$  satisfying  $\|\partial^s \varphi_Q\|_{\infty} \leq l(Q)^{-|s|}, 0 \leq |s| \leq n-1.$ 

This improves significantly a previous localization result in [MPrVe], which, in particular, yields

$$\left\|\frac{x}{|x|^2} * \varphi_Q T\right\|_{\infty} \le C \left\|\frac{x}{|x|^2} * T\right\|_{\infty},\tag{20}$$

for  $\varphi_Q$  as above. Inequality (19) implies (20) because boundedness of the vector valued potential  $x/|x|^2 * T$  provides a growth condition on T. Indeed one has (see Lemma 3.2 in [Pr1])

$$G(T) \le C \left\| \frac{x}{|x|^2} * T \right\|_{\infty}$$

Once (19) is at our disposition Tolsa's machinery applies straightforwardly as was already explained in [MPrVe, Section 2.2]. However we will again describe the main steps in the proof of inequality (17) at the end of section 4.

# $3 \quad \text{Proof of } \Gamma_{\hat{k}, \text{op}}(E) \leq C \, \Gamma_{\text{op}}(E)$

The symmetrization process for the Cauchy kernel introduced in [Me] has been succesfully applied to many problems of analytic capacity and  $L^2$  boundedness of the Cauchy integral operator (see [MeV], [MaMeV] and the book [P], for example) and also to problems concerning the capacities,  $\gamma_{\alpha}$ ,  $0 < \alpha < 1$ , (which are related to the vector valued Riesz kernels  $x/|x|^{1+\alpha}$ ) and the  $L^2$  boundedness of the  $\alpha$ -Riesz transforms (see [Pr1], [MPrVe], [Pr2] and [Pr3]). Given 3 distinct points in the plane,  $z_1$ ,  $z_2$  and  $z_3$ , one finds out, by an elementary computation that

$$c(z_1, z_2, z_3)^2 = \sum_{\sigma} \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}}$$
(21)

where the sum is taken over the permutations of the set  $\{1, 2, 3\}$  and  $c(z_1, z_2, z_3)$  is *Menger curvature*, that is, the inverse of the radius of the circle through  $z_1$ ,  $z_2$  and  $z_3$ . In particular (21) shows that the sum on the right hand side is a non-negative quantity.

In  $\mathbb{R}^n$  and for  $1 \leq i \leq n$  the quantity

$$\sum_{\sigma} \frac{x_{\sigma(2)}^{i} - x_{\sigma(1)}^{i}}{|x_{\sigma(2)} - x_{\sigma(1)}|^{2}} \frac{x_{\sigma(3)}^{i} - x_{\sigma(1)}^{i}}{|x_{\sigma(3)} - x_{\sigma(1)}|^{2}}$$
(22)

where the sum is taken over the permutations of the set  $\{1, 2, 3\}$ , is the obvious analogue of the right hand side of (21) for the *i*-th coordinate of the Riesz kernel  $x/|x|^2$ . Notice that (22) is exactly

$$2p_i(x_1, x_2, x_3),$$

where  $p_i(x_1, x_2, x_3)$  is defined as the sum in (22) taken only on the three permutations (1, 2, 3), (3, 1, 2) and (2, 1, 3).

In Lemma 3, we will show that, given three distinct points  $x_1, x_2, x_3 \in \mathbb{R}^n$ , the quantity  $p_i(x_1, x_2, x_3)$ ,  $1 \leq i \leq n$ , is also non-negative. We will use this remarkable fact to study the  $L^2$  boundedness of the operators associated with the scalar Riesz kernels  $x_i/|x|^2$ .

The relationship between the quantity  $p_i(x_1, x_2, x_3)$ ,  $1 \le i \le n$ , and the  $L^2$  estimates of the operator with kernel  $x_i/|x|^2$  is as follows. Take a positive finite Radon measure  $\mu$  in  $\mathbb{R}^n$  with linear growth. Given  $\varepsilon > 0$  consider the truncated scalar Riesz transform  $R^i_{\varepsilon}(\mu)$ . Then we have (see in [MeV] the argument for the Cauchy integral operator)

$$\left| \int |R^i_{\varepsilon}(\mu)(x)|^2 \, d\mu(x) - \frac{1}{3} p_{i,\varepsilon}(\mu) \right| \le C \|\mu\|,\tag{23}$$

C being a positive constant depending only on n, and

$$p_{i,\varepsilon}(\mu) = \iiint_{S_{\varepsilon}} p_i(x, y, z) \, d\mu(x) \, d\mu(y) \, d\mu(z),$$

with

$$S_{\varepsilon} = \{(x, y, z) : |x - y| > \varepsilon, |x - z| > \varepsilon \text{ and } |y - z| > \varepsilon\}$$

**Lemma 3.** For  $1 \leq i \leq n$ , and any three different points  $x_1, x_2, x_3 \in \mathbb{R}^n$  we have

$$p_i(x_1, x_2, x_3) \ge 0$$

Moreover,

1. If  $p_i(x_1, x_2, x_3) = 0$  for n - 1 values of  $i \in \{1, 2, ..., n\}$ , then  $x_1, x_2, x_3$  are aligned.

2. If the three points  $x_1$ ,  $x_2$ ,  $x_3$  are aligned, then  $p_i(x_1, x_2, x_3) = 0$  for  $1 \le i \le n$ .

*Proof.* Write  $a = x_2 - x_1$  and  $b = x_3 - x_2$ . Then

$$p_i(x_1, x_2, x_3) = \frac{a_i(a_i + b_i)|b|^2 - b_i a_i |a + b|^2 + b_i(a_i + b_i)|a|^2}{|a|^2|b|^2|a + b|^2}$$
$$= \frac{a_i b_i \left(-2\sum_{j=1}^n a_j b_j\right) + \sum_{j=1}^n a_i^2 b_j^2 + b_i^2 a_j^2}{|a|^2|b|^2|a + b|^2}$$
$$= \frac{\sum_{j=1}^n (a_i b_j - b_i a_j)^2}{|a|^2|b|^2|a + b|^2} = \frac{\sum_{j\neq i} (a_i b_j - b_i a_j)^2}{|a|^2|b|^2|a + b|^2} \ge 0.$$

Therefore, given three pairwise different points  $x_1$ ,  $x_2$ ,  $x_3$ , the permutations  $p_i(x_1, x_2, x_3) \equiv 0$  if and only if  $a_i b_j = b_i a_j$  for all  $1 \leq j \leq n$ .

Without loss of generality, assume that  $p_i(x_1, x_2, x_3) = 0$  for  $1 \le i \le n-1$ . Then the following n(n-1)/2 conditions hold

$$a_i b_j = a_j b_i \quad 1 \le i \le n - 1, \quad i + 1 \le j \le n.$$

These conditions imply that  $a = \lambda b$ , for some  $\lambda \in \mathbb{R}$ , which means the three points  $x_1, x_2, x_3$  lie on the same line.

Assume now that the three points are aligned. Without loss of generality set  $x_1 = 0, x_2 = y$  and  $x_3 = \lambda y$  for some  $\lambda > 0$ , and  $y \in \mathbb{R}^n$ . Then for  $i, j \in \{1, 2, ..., n\}$ , we have

$$a_i b_j = y_i (\lambda - 1) y_j = (\lambda - 1) y_i y_j = b_i a_j,$$

hence  $p_i(x_1, x_2, x_3) = 0$  for  $1 \le i \le n$ .

If we are in the plane, then Menger curvature can be written as

$$c(x_1, x_2, x_3) = \frac{4A}{|x_1 - x_2||x_1 - x_3||x_3 - x_2|},$$

where A denotes the area of the triangle determined by the points  $x_1$ ,  $x_2$ ,  $x_3$ . A consequence of Lemma 3 and its proof is the following.

**Corollary 4.** Given three different points  $x_1, x_2, x_3 \in \mathbb{R}^2$ , we have

$$p_1(x_1, x_2, x_3) = p_2(x_1, x_2, x_3) = \frac{1}{4}c(x_1, x_2, x_3)^2.$$

Hence, the quantities  $p_1(x_1, x_2, x_3)$  and  $p_2(x_1, x_2, x_3)$  are non-negative, and vanish if and only if  $x_1, x_2, x_3$  are aligned.

In the plane the singular Cauchy transform  $C(\mu)$  may be written as  $C(\mu) = R^1(\mu) - iR^2(\mu)$ . By Corollary 4 and the T(1)-Theorem , we see that  $C(\mu)$  is bounded on  $L^2(\mu)$  if and only if one of its real components, no matter which one, is bounded on  $L^2(\mu)$ . We state this, for emphasis, as a corollary.

**Corollary 5.** If  $\mu$  is a compactly supported positive measure in the plane having linear growth, the Cauchy transform of  $\mu$  is bounded on  $L^2(\mu)$  if and only if  $R^i(\mu)$  is bounded on  $L^2(\mu)$  for one  $i \in \{1, 2\}$ .

For a positive measure  $\mu$  with linear growth we have, by (23),

$$\begin{split} \|R_{\varepsilon}(\mu)\|_{L^{2}(\mu)}^{2} &= \sum_{j=1}^{n} \int |R_{\varepsilon}^{j}(\mu)(x)|^{2} \, d\mu(x) \\ &= \frac{1}{3} \sum_{j=1}^{n} \iiint_{S_{\varepsilon}} p_{j}(x,y,z) \, d\mu(x) \, d\mu(y) \, d\mu(z) + O(\|\mu\|) \\ &\leq \frac{2}{3} \sum_{\substack{j=1\\ j \neq i}}^{n} \iiint_{S_{\varepsilon}} p_{j}(x,y,z) \, d\mu(x) \, d\mu(y) \, d\mu(z) + O(\|\mu\|), \end{split}$$

where the last inequality follows easily from the formula

$$p_i(x_1, x_2, x_3) = \frac{\sum_{j \neq i} \left( (x_2^i - x_1^i)(x_3^j - x_2^j) - (x_2^j - x_1^j)(x_3^i - x_2^i) \right)^2}{|x_2 - x_1|^2 |x_3 - x_2|^2 |x_3 - x_1|^2}, \quad 1 \le i \le n.$$

The above estimate can be localized replacing  $\mu$  by  $\chi_B \mu$  for each ball B. Therefore, appealing to the T(1)-Theorem for non necessarily doubling measures [NTV1], if n-1 components  $R^j(\mu)$  are bounded on  $L^2(\mu)$  (no matter which n-1 components), then the whole vector valued operator  $R(\mu)$  is bounded on  $L^2(\mu)$ .

**Theorem 6.** Let  $\mu$  be a non-negative measure with compact support in  $\mathbb{R}^n$  and linear growth. Then the vector valued Riesz operator  $R(\mu)$  is bounded on  $L^2(\mu)$  provided any set of n-1 components  $R^j(\mu)$  of  $R(\mu)$  are bounded on  $L^2(\mu)$ .

The inequality (18) is an immediate consequence of Theorem 6.

# 4 Proof of $\Gamma_{\hat{k}}(E) \leq C \ \Gamma_{\hat{k}, \mathrm{op}}(E)$

The proof of the inequality  $\Gamma_{\hat{k}}(E) \leq C \Gamma_{\hat{k},op}(E)$  is based in two ingredients, the localization of scalar Riesz potentials and the exterior regularity of  $\Gamma_{\hat{k}}$ , which we discuss below.

#### 4.1 Localization of scalar Riesz potentials

When analyzing the argument for the proof of (2) (see Theorem 1.1 in [T2]) one realizes that one of the technical tools used is the fact that the Cauchy kernel 1/zlocalizes in the uniform norm. By this we mean that if T is a compactly supported distribution such that 1/z \* T is a bounded measurable function, then  $1/z * (\varphi T)$  is also bounded measurable for each compactly supported  $C^1$  function  $\varphi$ . This is an old result, which is simple to prove because 1/z is related to the differential operator  $\overline{\partial}$ (see [G1, Chapter V]). The same localization result can be proved easily in any dimension for the kernel  $x/|x|^n$ , which is, modulo a multiplicative constant, the gradient of the fundamental solution of the Laplacian. Again the proof is reasonably straightforward because the kernel is related to a differential operator (see [Pa] and [V1]).

In [MPrVe, Lemma 3.1] we were concerned with the localization of the vector valued  $\alpha$ -Riesz kernel  $x/|x|^{1+\alpha}$ ,  $0 < \alpha < n$ . For general values of  $\alpha$  there is no differential operator in the background and consequently the corresponding localization result becomes far from obvious (see Lemma 3.1 in [MPrVe]).

We now state the new localization lemma we need.

**Lemma 7.** Let T be a compactly supported distribution in  $\mathbb{R}^n$ , with linear growth, such that  $(x_i/|x|^2) * T$  is in  $L^{\infty}(\mathbb{R}^n)$  for some  $i, 1 \leq i \leq n$ . Let Q be a cube and assume that  $\varphi_Q \in C_0^{\infty}(Q)$  satisfies  $\|\partial^s \varphi_Q\|_{\infty} \leq l(Q)^{-|s|}, 0 \leq |s| \leq n-1$ . Then  $(x_i/|x|^2) * \varphi_Q T$  is in  $L^{\infty}(\mathbb{R}^n)$  and

$$\left\|\frac{x_i}{|x|^2} * \varphi_Q T\right\|_{\infty} \le C \left( \left\|\frac{x_i}{|x|^2} * T\right\|_{\infty} + G(T) \right),$$

for some positive constant C = C(n) depending only on n.

With analogous techniques and replacing G(T) by  $G_{\alpha}(T)$  (see section 6 for a definition) one can prove that the above lemma also holds in  $\mathbb{R}^n$  for the scalar  $\alpha$ -Riesz potentials

$$\frac{x_i}{|x|^{1+\alpha}} * T, \quad 0 < \alpha < n, \ \alpha \in \mathbb{Z}.$$

For the proof of Lemma 7 we need the following.

**Lemma 8.** Let T be a compactly supported distribution in  $\mathbb{R}^n$  with linear growth and assume that Q is a cube and  $\varphi_Q \in \mathcal{C}_0^{\infty}(Q)$  satisfies  $\|\partial^s \varphi_Q\|_{L^1(Q)} \leq l(Q), \|s\| = n - 1$ .

Then, for each coordinate *i*, the distribution  $(x_i/|x|^2) * \varphi_Q T$  is a locally integrable function and there exists a point  $x_0 \in \frac{1}{4}Q$  such that

$$\left| \left( \frac{x_i}{|x|^2} * \varphi_Q T \right) (x_0) \right| \le C G(T),$$

where C = C(n) is a positive constant depending only on n.

Proof. Without loss of generality set i = 1 and write  $k^1(x) = x_1/|x|^2$ . Since  $k^1 * \varphi_Q T$  is infinitely differentiable off the closure of Q, we only need to show that  $k^1 * \varphi_Q T$  is integrable on 2Q. We will actually prove a stronger statement, namely, that  $k^1 * \varphi_Q T$  is in  $L^p(2Q)$  for each p in the interval  $1 \le p < n$ . Indeed, fix any q satisfying  $n/(n-1) < q < \infty$  and call p the dual exponent, so that  $1 . We need to estimate the action of <math>k^1 * \varphi_Q T$  on functions  $\psi \in \mathcal{C}_0^\infty(2Q)$  in terms of  $\|\psi\|_q$ . We clearly have

$$\langle k^1 * \varphi_Q T, \psi \rangle = \langle T, \varphi_Q (k^1 * \psi) \rangle.$$

We claim that, for an appropriate dimensional constant C, the test function

$$\frac{\varphi_Q(k^1 * \psi)}{C \, l(Q)^{\frac{n}{p} - 1} \|\psi\|_q}$$

satisfies the normalization inequalities (9) in the definition of G(T). Once this is proved, by the definition of G(T) we get

$$|\langle k^1 * \varphi_Q T, \psi \rangle| \le C \, l(Q)^{\frac{n}{p}} \|\psi\|_q \, G(T),$$

and so

$$||k^1 * \varphi_Q T||_{L^p(2Q)} \le C l(Q)^{\frac{n}{p}} G(T).$$

Hence

$$\begin{aligned} \frac{1}{|\frac{1}{4}Q|} \int_{\frac{1}{4}Q} |(k^1 * \varphi_Q T)(x)| \, dx &\leq 4^n \frac{1}{|Q|} \int_Q |(k^1 * \varphi_Q T)(x)| \, dx \\ &\leq 4^n \left(\frac{1}{|Q|} \int_Q |(k^1 * \varphi_Q T)(x)|^p \, dx\right)^{\frac{1}{p}} \\ &\leq C \, G(T), \end{aligned}$$

which completes the proof of Lemma 8.

To prove the claim we have to show that

$$\|\partial^{s} \left(\varphi_{Q} \left(k^{1} * \psi\right)\right)\|_{L^{1}(Q)} \leq C \, l(Q)^{\frac{n}{p}} \|\psi\|_{q}, \quad |s| = n - 1.$$
(24)

By Leibnitz formula

$$\partial^{s} \left( \varphi_{Q} \left( k^{1} * \psi \right) \right) = \sum_{|r|=0}^{n-1} c_{s,r} \, \partial^{r} \varphi_{Q} \, \partial^{s-r} (k^{1} * \psi)$$
$$= \varphi_{Q} \, \partial^{s} (k^{1} * \psi) + \sum_{|r|=1}^{n-1} c_{s,r} \, \partial^{r} \varphi_{Q} \, \partial^{s-r} (k^{1} * \psi)$$
$$\equiv A + B,$$

where the last identity is a definition of A and B.

To estimate the function B we remark that, since |s| = n - 1,

$$|\partial^{s-r}k^1(x)| \le C |x|^{-(n-|r|)}, \quad 1 \le |r| \le n-1,$$

and then, by Hölder, for each  $1 \le |r| \le n - 1$ ,

$$\begin{split} \int_{Q} |\partial^{r} \varphi_{Q}| |\partial^{s-r} k^{1} * \psi | dx &\leq C \, \|\partial^{r} \varphi_{Q}\|_{L^{1}(Q)} \|\psi\|_{q} \left( \int_{Q} \frac{dy}{|x-y|^{p(n-|r|)}} \right)^{1/p} \\ &\leq C \, l(Q)^{\frac{n}{p}-n+|r|} \|\partial^{r} \varphi_{Q}\|_{L^{1}(Q)} \|\psi\|_{q}. \end{split}$$

For |s| = n - 1 and  $0 \le |r| \le n - 1$ , an inequality by Maz'ya (see [Mz, 1.1.4, p. 15] and [Mz, 1.2.2, p. 24]) tells us that

$$\|\partial^r \varphi_Q\|_{\frac{n}{1+|r|}} \le C \int |\nabla^{n-1} \varphi_Q|,$$

where  $\nabla^{n-1}\varphi_Q$  denotes the vector of all derivatives  $\partial^s\varphi_Q$  of order |s| = n - 1.

Thus using Hölder and the fact that  $\|\partial^s \varphi_Q\|_{L^1(Q)} \le l(Q)$  for |s| = n - 1, we get

$$\int |\partial^r \varphi_Q| \le \|\partial^r \varphi_Q\|_{\frac{n}{1+|r|}} \, l(Q)^{n-1-|r|} \le C l(Q) l(Q)^{n-1-|r|} = C l(Q)^{n-|r|}.$$

We therefore conclude

$$\int_{Q} |B| \le C \sum_{|r|=1}^{n-1} \int_{Q} |\partial^{r} \varphi_{Q}| |\partial^{s-r} k^{1} \ast \psi| dx \le C l(Q)^{\frac{n}{p}} \|\psi\|_{q}$$

We turn now to the term A. We remark that, for |s| = n - 1,

$$\partial^s k^1 * \psi = c \,\psi + S(\psi),\tag{25}$$

where S is a smooth homogeneous convolution Calderón-Zygmund operator and c a constant depending on s. This can be seen by computing the Fourier transform of  $\partial^s k^1$  and then using that each homogeneous polynomial can be decomposed in

terms of homogeneous harmonic polynomials of lower degrees (see [St, 3.1.2 p. 69]). Since Calderón-Zygmund operators preserve  $L^q(\mathbb{R}^n)$ ,  $1 < q < \infty$ , we get, using again Hölder,

$$\int_{Q} |A| \, dx \le C \, \|\varphi_Q\|_p \|\psi\|_q$$

The Sobolev imbedding theorem, case p = 1 (see [St]), tells us that

$$\|\varphi_Q\|_n \le C \int_Q |\nabla^{n-1}\varphi_Q(x)| dx,$$

therefore, by Hölder,

$$\|\varphi_Q\|_p \le C \left(\int_Q |\varphi_Q|^n\right)^{\frac{1}{n}} l(Q)^{\frac{n-p}{p}} \le C \, l(Q)^{\frac{n}{p}}.$$

This finishes the estimate of term A and the proof of (24).

Proof of Lemma 7. Without loss of generality take i = 1. Since  $k^1 * \varphi_Q T$  is a harmonic function off the closure of Q, by the maximum principle we only need to estimate  $|(k^1 * \varphi_Q T)(x)|$  for  $x \in \frac{3}{2}Q$ . Since  $k^1 * T$  and  $\varphi_Q$  are bounded functions, we can write

$$|(k^{1} * \varphi_{Q}T)(x)| \leq |(k^{1} * \varphi_{Q}T)(x) - \varphi_{Q}(x)(k^{1} * T)(x)| + \|\varphi_{Q}\|_{\infty} \|k^{1} * T\|_{\infty}.$$

Let  $\psi_Q \in \mathcal{C}_0^{\infty}(\mathbb{R}^n)$  be such that  $\psi_Q \equiv 1$  in 2Q,  $\psi_Q \equiv 0$  in  $(4Q)^c$  and  $\|\partial^s \psi_Q\|_{\infty} \leq C_s l(Q)^{-|s|}$ , for each multi-index s. Then one is tempted to write

$$|(k^{1} * \varphi_{Q}T)(x) - \varphi_{Q}(x)(k^{1} * T)(x)| \leq |\langle T, \psi_{Q}(y)(\varphi_{Q}(y) - \varphi_{Q}(x))k^{1}(x - y)\rangle| + \|\varphi_{Q}\|_{\infty}|\langle T, (1 - \psi_{Q}(y))k^{1}(x - y)\rangle|.$$

The problem is that the first term in the right hand side above does not make any sense because T is acting on a function of y which is not necessarily differentiable at the point x. To overcome this difficulty one needs to use a standard regularization process. Take  $\chi \in \mathcal{C}^{\infty}(B(0,1))$  such that  $\int \chi(x) dx = 1$  and set  $\chi_{\varepsilon}(x) = \varepsilon^{-n} \chi(x/\varepsilon)$ . The plan is to estimate, uniformly on x and  $\epsilon$ ,

$$|(\chi_{\varepsilon} * k^{1} * \varphi_{Q}T)(x) - \varphi_{Q}(x)(\chi_{\varepsilon} * k^{1} * T)(x)|.$$
(26)

Clearly (26) tends, as  $\varepsilon$  tends to zero, to

$$|(k^1 * \varphi_Q T)(x) - \varphi_Q(x)(k^1 * T)(x)|,$$

for almost all  $x \in \mathbb{R}^n$ , which allows the transfer of uniform estimates. We now have

$$\begin{aligned} |(\chi_{\varepsilon} * k^{1} * \varphi_{Q}T)(x) - \varphi_{Q}(x)(\chi_{\varepsilon} * k^{1} * T)(x)| \\ &\leq |\langle T, \psi_{Q}(y)(\varphi_{Q}(y) - \varphi_{Q}(x))(\chi_{\varepsilon} * k^{1})(x - y)\rangle| \\ &+ \|\varphi_{Q}\|_{\infty} |\langle T, (1 - \psi_{Q}(y))(\chi_{\varepsilon} * k^{1})(x - y)\rangle| \\ &= A_{1} + A_{2}, \end{aligned}$$

where the last identity is the definition of  $A_1$  and  $A_2$ . To deal with term  $A_1$  set

$$k_{\varepsilon}^{1,x}(y) = (\chi_{\varepsilon} * k^1)(x - y).$$

We claim that, for an appropriate dimensional constant C, the test function

$$f(y) = Cl(Q)\psi_Q(y)(\varphi_Q(y) - \varphi_Q(x))k_{\varepsilon}^{1,x}(y),$$

satisfies the normalization inequalities (9) in the definition of G(T), with  $\varphi_Q$  replaced by f and Q by 4Q. If this is the case, then

$$A_1 \le Cl(Q)^{-1} |\langle T, f \rangle| \le C G(T).$$

To prove the claim we first notice that the regularized kernel  $\chi_{\varepsilon}*k^1$  satisfies the inequalities

$$|(\chi_{\varepsilon} * \partial^s k^1)(x)| \le \frac{C}{|x|^{1+|s|}}, \quad x \in \mathbb{R}^n \setminus \{0\} \quad \text{and} \quad 0 \le |s| < n-1,$$
(27)

where C is a dimensional constant, which, in particular, is independent of  $\epsilon$ . This can be proved by standard estimates which we omit. For |s| = n - 1 the situation is a little bit more complicated. By (25) we have

$$(\chi_{\varepsilon} * \partial^s k^1)(x) = c \,\chi_{\varepsilon}(x) + (\chi_{\varepsilon} * S)(x),$$

where S is a smooth homogeneous convolution Calderón-Zygmund operator. As such, its kernel H satisfies the usual growth condition  $|H(x)| \leq C/|x|^n$ . From this is not difficult to show that

$$|(\chi_{\varepsilon} * S)(x)| \le \frac{C}{|x|^n}, \quad x \in \mathbb{R}^n \setminus \{0\},$$
(28)

for a dimensional constant C.

By Leibnitz formula, for |s| = n - 1,

$$\partial^{s} \left( \psi_{Q}(\varphi_{Q} - \varphi_{Q}(x))k_{\varepsilon}^{1,x} \right) = \psi_{Q} \left( \varphi_{Q} - \varphi_{Q}(x) \right) \partial^{s} k_{\varepsilon}^{1,x} + \sum_{|r|=1}^{n-1} c_{r,s} \,\partial^{r} \left( \psi_{Q}(\varphi_{Q} - \varphi_{Q}(x)) \right) \,\partial^{s-r} \,k_{\varepsilon}^{1,x},$$
<sup>(29)</sup>

and so

$$\begin{aligned} \|\partial^{s} f\|_{L^{1}(4Q)} &\leq Cl(Q) \int_{4Q} |\psi_{Q}(y) \left(\varphi_{Q}(y) - \varphi_{Q}(x)\right) \partial^{s} k_{\varepsilon}^{1,x}(y) | \, dy \\ &+ Cl(Q) \sum_{|r|=1}^{n-1} \int_{4Q} |\partial^{r} \left(\psi_{Q}(\varphi_{Q} - \varphi_{Q}(x)) \, \partial^{s-r} k_{\varepsilon}^{1,x}\right)(y) | \, dy = A_{11} + A_{12}. \end{aligned}$$

Making use of (27) one obtains

$$A_{12} \le Cl(Q) \sum_{|r|=1}^{n-1} \frac{1}{l(Q)^{|r|}} \int_{4Q} |(\partial^{s-r} k_{\varepsilon}^{1,x})(y)| \, dy$$
  
$$\le Cl(Q).$$

To estimate  $A_{11}$  we resort to (28), which yields

$$A_{11} = Cl(Q) \int_{4Q} |\psi_Q(y)(\varphi_Q(y) - \varphi_Q(x))\partial^s k_{\varepsilon}^{1,x}(y)| \, dy$$
  
$$\leq Cl(Q) \left( \int \chi_{\varepsilon}(y-x) \, dy + \|\partial\varphi_Q\|_{\infty} \int_{4Q} \frac{dy}{|y-x|^{n-1}} \, dy \right)$$
  
$$\leq Cl(Q).$$

We now turn to  $A_2$ . By Lemma 8, there exists a point  $x_0 \in Q$  such that  $|(k^1 * \psi_Q T)(x_0)| \leq C G(T)$ . Then

$$|(k^1 * (1 - \psi_Q)T)(x_0)| \le C (||k^1 * T||_{\infty} + G(T)).$$

The analogous inequality holds as well for the regularized potentials appearing in  $A_2$ , uniformly in  $\epsilon$ , and therefore

$$A_2 \le C |\langle T, (1 - \psi_Q)(k_{\varepsilon}^{1,x} - k_{\varepsilon}^{1,x_0}) \rangle| + C (||k^1 * T||_{\infty} + G(T)).$$

To estimate  $|\langle T, (1 - \psi_Q)(k_{\varepsilon}^{1,x} - k_{\varepsilon}^{1,x_0})\rangle|$ , we decompose  $\mathbb{R}^n \setminus \{x\}$  into a union of rings

$$R_j = \{ z \in \mathbb{R}^n : 2^j \, l(Q) \le |z - x| \le 2^{j+1} \, l(Q) \}, \quad j \in \mathbb{Z},$$

and consider functions  $\varphi_j$  in  $\mathcal{C}_0^{\infty}(\mathbb{R}^n)$ , with support contained in  $\frac{3}{2}R_j$ , such that  $\|\partial^s \varphi_j\|_{\infty} \leq C \left(2^j l(Q)\right)^{-|s|}, |s| \geq 0$ , and  $\sum_j \varphi_j = 1$  on  $\mathbb{R}^n \setminus \{x\}$ . Then, since  $x \in \frac{3}{2}Q$  and  $1 - \psi_Q \equiv 0$  in 2Q, the smallest ring  $R_j$  that may intersect  $(2Q)^c$  is  $R_{-2}$ . Therefore

we have

$$\begin{split} |\langle T, (1-\psi_Q)(k_{\varepsilon}^{1,x}-k_{\varepsilon}^{1,x_0})\rangle| &= \left| \left\langle T, \sum_{j\geq -2} \varphi_j (1-\psi_Q)(k_{\varepsilon}^{1,x}-k_{\varepsilon}^{1,x_0}) \right\rangle \right| \\ &\leq \left| \left\langle T, \sum_{j\in I} \varphi_j (1-\psi_Q)(k_{\varepsilon}^{1,x}-k_{\varepsilon}^{1,x_0}) \right\rangle \right| \\ &+ \sum_{j\in J} |\langle T, \varphi_j (k_{\varepsilon}^{1,x}-k_{\varepsilon}^{1,x_0}) \rangle|, \end{split}$$

where I denotes the set of indices  $j \ge -2$  such that the support of  $\varphi_j$  intersects 4Q and J the remaining indices, namely those  $j \ge -2$  such that the support of  $\varphi_j$  is contained in the complement of 4Q. Notice that the cardinality of I is bounded by a dimensional constant.

Set

$$g = C l(Q) \sum_{j \in I} \varphi_j (1 - \psi_Q) \left( k_{\varepsilon}^{1,x} - k_{\varepsilon}^{1,x_0} \right),$$

and for  $j \in J$ 

$$g_j = C \, 2^{2j} \, l(Q) \, \varphi_j \, (k_{\varepsilon}^{1,x} - k_{\varepsilon}^{1,x_0}).$$

We show now that the test functions g and  $g_j$ ,  $j \in J$ , satisfy the normalization inequalities (9) in the definition of G(T). Once this is available, using the linear growth condition of T we obtain

$$\begin{aligned} |\langle T, (1 - \psi_Q)(k_{\varepsilon}^{1,x} - k_{\varepsilon}^{1,x_0})\rangle| &\leq Cl(Q)^{-1} |\langle T, g\rangle| \\ &+ C \sum_{j \in J} (2^{2j}l(Q))^{-1} |\langle T, g_j\rangle| \\ &\leq C G(T) + C \sum_{j \geq 2} 2^{-j} G(T) \leq C G(T), \end{aligned}$$

which completes the proof of Lemma 7.

Checking the normalization inequalities for g and  $g_j$  is easy. For g one uses that  $\|\partial^s(1-\psi_Q)\|_{\infty} \leq Cl(Q)^{-|s|}, \|\partial^s\varphi_j\|_{\infty} \leq C(2^j l(Q))^{-|s|}, 0 \leq |s| \leq n-1, (27)$ , the fact that  $x, x_0 \in \frac{3}{2}Q, y \in (2Q)^c$ , and a gradient estimate. For  $g_j$  we use in addition Leibnitz formula and a gradient estimate to conclude that, for  $j \in J$  and |s| = n-1,

$$\|\partial^s g_j\|_{\infty} \le C \, 2^{2j} \, l(Q) \sum_{|r|=0}^{n-1} \frac{1}{(2^j \, l(Q))^{|r|}} \frac{l(Q)}{(2^j \, l(Q))^{n+1-|r|}} \le C \, (2^j \, l(Q))^{-(n-1)}. \qquad \Box$$

## 4.2 A continuity property for the capacity $\Gamma_{\hat{k}}$

In this section we prove a continuity property for the capacity  $\Gamma_{\hat{k}}$ ,  $1 \leq k \leq n$ , which will be used in the proof of inequality (17). Although we state the result only for the capacities  $\Gamma_{\hat{k}}$ ,  $1 \leq k \leq n$ , Lemma 9 below holds for the capacities  $\kappa_i$ ,  $1 \leq i \leq n$ , defined in the Introduction, because the proof does not use any growth condition on distributions with bounded scalar Riesz potential.

**Lemma 9.** Let  $\{E_j\}_j$  be a decreasing sequence of compact sets, with intersection the compact set  $E \subset \mathbb{R}^n$ . Then, for  $1 \leq k \leq n$ ,

$$\Gamma_{\hat{k}}(E) = \lim_{j \to \infty} \Gamma_{\hat{k}}(E_j).$$

*Proof.* Since, by definition, the set function  $\Gamma_{\hat{k}}$  in non-decreasing

$$\lim_{j \to \infty} \Gamma_{\hat{k}}(E_j) \ge \Gamma_{\hat{k}}(E),$$

and the limit clearly exists. For each  $j \ge 1$ , let  $T_j$  be a distribution such that the potentials  $x_i/|x|^2 * T_j$  are in the unit ball of  $L^{\infty}(\mathbb{R}^n)$ ,  $i \ne k$ , and

$$\Gamma_{\hat{k}}(E_j) - \frac{1}{j} < |\langle T_j, 1 \rangle| \le \Gamma_{\hat{k}}(E_j).$$

We want to show that for each test function  $\varphi$ ,

$$\langle T_j, \varphi \rangle \xrightarrow[j \to \infty]{} \langle T, \varphi \rangle,$$
 (30)

for some distribution T whose potentials  $x_i/|x|^2 * T$  are in the unit ball of  $L^{\infty}(\mathbb{R}^n)$ for  $i \neq k$ . If (30) holds and  $\varphi$  is a test function satisfying  $\varphi \equiv 1$  in a neighbourhood of E, then

$$\lim_{j \to \infty} \Gamma_{\hat{k}}(E_j) = \lim_{j \to \infty} |\langle T_j, 1 \rangle| = \lim_{j \to \infty} |\langle T_j, \varphi \rangle| = |\langle T, \varphi \rangle| \le \Gamma_{\hat{k}}(E).$$

To show (30), fix  $i \neq k$  and assume, without loss of generality, that i = 1. Set  $k^1(x) = x_1/|x|^2$  and  $f_j = k^1 * T_j$ . Write a point  $x \in \mathbb{R}^n$  as  $x = (x_1, x_2)$ , with  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}^{n-1}$ . Finally notice that  $c k^1 = \partial_1 E$  where E is the fundamental solution of the Laplacian in  $\mathbb{R}^n$  and c is a constant. Therefore, for each test function  $\varphi$ ,

$$(T_j * \varphi)(x_1, x_2) = \int_{-\infty}^{x_1} \partial_1(T_j * \varphi)(t, x_2) dt = c \int_{-\infty}^{x_1} \Delta(\varphi * f_j)(t, x_2) dt$$

Setting  $\overline{\varphi}(x) = \varphi(-x)$  we get

$$\langle T_j, \varphi \rangle = (T_j * \overline{\varphi})(0, 0) = c \int_{-\infty}^0 \Delta(\overline{\varphi} * f_j)(t, 0) dt.$$
 (31)

We remark, incidentally, that the above formula tells us how to recover a distribution from one of its scalar Riesz potentials.

Passing to a subsequence, we can assume that  $f_j \longrightarrow f$  in the weak \* topology of  $L^{\infty}(\mathbb{R}^n)$ . But then  $(f_j * \Delta \varphi)(x) \longrightarrow (f * \Delta \varphi)(x), x \in \mathbb{R}^n$ . This pointwise convergence is bounded because  $|(f_j * \Delta \varphi)(x)| \leq ||\Delta \varphi||_1 ||f_j||_{\infty} \leq ||\Delta \varphi||_1$ . Hence the dominated convergence theorem yields

$$\lim_{j \to \infty} \langle T_j, \varphi \rangle = c \lim_{j \to \infty} \int_{-\infty}^0 \Delta(\overline{\varphi} * f_j)(t, 0) \, dt = c \int_{-\infty}^0 \Delta(\overline{\varphi} * f)(t, 0) \, dt.$$

Define the distribution T by

$$\langle T, \varphi \rangle = c \int_{-\infty}^{0} \Delta(\overline{\varphi} * f)(t, 0) dt$$

Now we want to show that  $f = k^1 * T$ . For that we regularize  $f_j$  and  $T_j$ . Take  $\chi \in C_0^{\infty}(B(0,1))$  with  $\int \chi(x) dx = 1$  and set  $\chi_{\varepsilon}(x) = \varepsilon^{-n} \chi(x/\varepsilon)$ . Then we have, as  $j \to \infty$ ,

$$\left(\chi_{\varepsilon} * k^{1} * T_{j}\right)(x) = \left(\chi_{\varepsilon} * f_{j}\right)(x) \longrightarrow \left(\chi_{\varepsilon} * f\right)(x), \quad x \in \mathbb{R}^{n},$$

because  $f_j$  converges to f weak \* in  $L^{\infty}(\mathbb{R}^n)$ . On the other hand, since  $\chi_{\varepsilon} * k_1 \in \mathcal{C}^{\infty}(\mathbb{R}^n)$  and  $T_j$  tends to T in the weak topology of distributions, with controlled supports, we have

$$\left(\chi_{\varepsilon} * k^{1} * T_{j}\right)(x) \longrightarrow \left(\chi_{\varepsilon} * k^{1} * T\right)(x), \quad x \in \mathbb{R}^{n}.$$

Hence

$$\chi_{\varepsilon} * k^1 * T = \chi_{\varepsilon} * f, \quad \varepsilon > 0,$$

and so, letting  $\varepsilon \to 0$ ,  $k^1 * T = f$ .

## 4.3 End of the proof of the inequality $\Gamma_{\hat{k}} \leq C \Gamma_{\hat{k}, \text{op}}$

We claim that the inequality in the title of this subsection can be proved by adapting the scheme of the proof of Theorems 1.1 in [T2] and 7.1 in [T3]. As Lemma 9 shows, the capacities  $\Gamma_{\hat{k}}$ ,  $1 \leq k \leq n$ , enjoy the exterior regularity property. This is also true for the capacities  $\Gamma_{\hat{k},+}$ ,  $1 \leq k \leq n$ , defined by

$$\Gamma_{\hat{k},+}(E) = \sup\left\{\mu(E) : \mu \in L(E), \ \left\|\frac{x_j}{|x|^2} * \mu\right\|_{\infty} \le 1, \ 1 \le j \le n, \ j \ne k\right\},\$$

just by the weak  $\star$  compactness of the set of positive measures with total variation not exceeding 1. We can approximate a general compact set E by sets which are finite unions of cubes of the same side length in such a way that the capacities  $\Gamma_{\hat{k}}$  and  $\Gamma_{\hat{k},+}$  of the approximating sets are as close as we wish to those of E. As in (14), one has, using the Davie-Oksendal Lemma for several operators [MaPa, Lemma 4.2],

$$C^{-1}\Gamma_{\hat{k},\mathrm{op}}(E) \le \Gamma_{\hat{k},+}(E) \le C\Gamma_{\hat{k},\mathrm{op}}(E).$$
(32)

Thus we can assume, without loss of generality, that E is a finite union of cubes of the same size. This will allow to implement an induction argument on the size of certain (n-dimensional) rectangles. The first step involves rectangles of diameter comparable to the side length of the cubes whose union is E.

The starting point of the general inductive step in the proof of Tolsa's Theorem in [T2] (and [T3]) consists in the construction of a positive Radon measure  $\mu$  supported on a compact set F which approximates E in an appropriate sense. The construction of F and  $\mu$  gives readily that  $\Gamma_{\hat{k}}(E) \leq C \mu(F)$ , and  $\Gamma_{\hat{k},+}(F) \leq C \Gamma_{\hat{k},+}(E)$ , which tells us that F is not too small but also not too big. However, one cannot expect, in the context of [T2] and [T3], the Cauchy singular integral to be bounded on  $L^2(\mu)$ . In our case one cannot expect the operators  $R^j(\mu)$  to be bounded on  $L^2(\mu)$ , for  $1 \leq j \leq n, j \neq k$ . One has to carefully look for a compact subset G of F such that  $\mu(F) \leq C \mu(G)$ , the restriction  $\mu_G$  of  $\mu$  to G has linear growth and the operators  $R^j(\mu_G), 1 \leq j \leq n, j \neq k$ , are bounded on  $L^2(\mu_G)$  with dimensional constants. This completes the proof because then

$$\Gamma_{\hat{k}}(E) \le C \,\mu(F) \le C \,\mu(G) \le C \,\Gamma_{\hat{k},\mathrm{op}}(G) \le C \,\Gamma_{\hat{k},\mathrm{op}}(F)$$
$$\le C \,\Gamma_{\hat{k},+}(F) \le C \,\Gamma_{\hat{k},+}(E) \le C \,\Gamma_{\hat{k},\mathrm{op}}(E).$$

In [T2] and [T3] the set F is defined as the union of a special family of cubes  $\{Q_i\}_{i=1}^N$  that cover the set E and approximate E at an appropriate intermediate scale. One then sets

$$F = \bigcup_{i=1}^{N} Q_i.$$

This part of the proof extends without any obstruction to our case because of the positivity properties of the symmetrization of the scalar Riesz kernels (see section 3). As in Lemma 7.2 in [T3], just by how the approximating set F is constructed, one gets  $\Gamma_{\hat{k},+}(F) \leq C \Gamma_{\hat{k},+}(E)$ . By the definition of  $\Gamma_{\hat{k}}(E)$  it follows that there exists a real distribution  $T_0$  supported on E such that

- 1.  $\Gamma_{\hat{k}}(E) \leq 2|\langle T_0, 1 \rangle|.$
- 2.  $T_0$  has linear growth and  $G(T_0) \leq 1$ .
- 3.  $\|\frac{x_j}{|x|^2} * T_0\|_{\infty} \le 1, \quad 1 \le j \le n, \quad j \ne k.$

Consider now functions  $\varphi_i \in \mathcal{C}_0^{\infty}(2Q_i), 0 \leq \varphi_i \leq 1, \|\partial^s \varphi_i\|_{\infty} \leq C l(Q_i)^{-|s|}, 0 \leq |s| \leq n-1$  and  $\sum_{i=1}^N \varphi_i = 1$  on  $\bigcup_i Q_i$ . We define now simultaneously the measure  $\mu$ 

and an auxiliary measure  $\nu$ , which should be viewed as a model for  $T_0$  adapted to the family of squares  $\{Q_i\}_{i=1}^N$ . For each cube  $Q_i$  take a concentric segment  $\Sigma_i$  of length a small fixed fraction of  $\Gamma_{\hat{k}}(E \cap Q_i)$  and set

$$\mu = \sum_{i=1}^{N} \mathcal{H}^{1}_{|\Sigma_{i}}$$

and

$$\nu = \sum_{i=1}^{N} \frac{\langle T_0, \varphi_i \rangle}{\mathcal{H}^1(\Sigma_i)} \mathcal{H}^1_{|\Sigma_i|}.$$

We have  $d\nu = bd\mu$ , with  $b = \frac{\langle \varphi_i, \nu_0 \rangle}{\mathcal{H}^1(\Sigma_i)}$  on  $\Sigma_i$ . At this point we need to show that our function b is bounded, to apply later a suitable T(b) Theorem. To estimate  $||b||_{\infty}$  we use the localization inequalities

$$\left\|\frac{x_j}{|x|^2} * \varphi_i T_0\right\|_{\infty} \le C, \quad 1 \le j \le n, \quad j \ne k, \quad 1 \le i \le N.$$

This was proved in Lemma 7 of Section 4.1. Since it is easily seen that  $\varphi_i T_0$  has linear growth and  $G(\varphi_i T_0) \leq C$ , we obtain, by the definition of  $\Gamma_{\hat{k}}$ ,

$$|\langle T_0, \varphi_i \rangle| \le C \,\Gamma_{\hat{k}}(2Q_i \cap E), \quad \text{for } 1 \le i \le N.$$
(33)

It is now easy to see why  $\Gamma_{\hat{k}}(E) \leq C \mu(F)$ :

$$\Gamma_{\hat{k}}(E) \leq 2 |\langle T_0, 1 \rangle| = 2 \left| \sum_{i=1}^{N} \langle T_0, \varphi_i \rangle \right|$$

$$\leq C \sum_{i=1}^{N} \Gamma_{\hat{k}}(2Q_i \cap E) = C \,\mu(F).$$
(34)

We do not insist in summarizing the intricate details, which can be found in [T2] and [T3], of the definition of the set G and of the application of the T(b) Theorem of [NTV2].

## 5 Counter-examples to the growth estimate

As it was shown in , Let T be a compactly supported distribution whose Riesz potential  $x/|x|^{1+\alpha} * T$  is in  $L^{\infty}(\mathbb{R}^n, \mathbb{R}^n)$ . The proof of lemma 3.2 in [Pr1] shows that T satisfies the growth condition

$$\langle T, \varphi_Q \rangle | \leq C \, l(Q)^{\alpha},$$

for each cube Q and each  $\varphi_Q \in \mathcal{C}_0^{\infty}(Q)$  satisfying the normalization condition  $\|\partial^s \varphi_Q\|_{L^1(Q)} \leq l(Q)^{[\alpha]}, \ |s| = n - [\alpha]$ . Here  $\alpha$  is any number, not necessarily integer, with  $0 < \alpha < n$ , and  $[\alpha]$  is its integer part. A similar result does not hold in the context of this paper. In  $\mathbb{R}^2$ , boundedness of one scalar Riesz potential of a distribution does not imply  $G(T) < \infty$  (see (8) for the definition of G(T)).

**Proposition 10.** There exist a compactly supported real Radon measure  $\mu$  in the plane, such that  $x_1/|x|^2 * \mu$  is in  $L^{\infty}(\mathbb{R}^2)$  and  $G(\mu) = \infty$ .

*Proof.* The idea of the proof is that there is no relation, in general, between the derivative with respect to the first variable and the derivative with respect to the second variable. Set  $z = (x, y) \in \mathbb{R}^2$  and let h(z) = f(x)g(y) with

$$f(x) = \begin{cases} x+1 & \text{if } -1 \le x \le 0\\ -x+1 & \text{if } 0 \le x \le 1 \end{cases}$$

•

To define g on  $I_n = [2^{-n-1}, 2^{-n}], n \ge 0$ , let  $\mu_n = 3/2^{n+2}$  be the center of  $I_n$  and set

$$g(y) = \begin{cases} \frac{2^{n+2}}{n^2} \left( y - \frac{1}{2^{n+1}} \right), & \text{if } \frac{1}{2^{n+1}} \le y \le \mu_n \\ -\frac{2^{n+2}}{n^2} \left( y - \frac{1}{2^n} \right), & \text{if } \mu_n \le y \le \frac{1}{2^n} \end{cases}$$

Define  $\mu = \Delta h$ , the Laplacian of h. Then

$$\mu = \left(\delta_{-1} - 2\delta_0 + \delta_1\right)(x)g(y) + f(x)\sum_{n=1}^{\infty} \frac{2^{n+2}}{n^2} \left(\delta_{\frac{1}{2^n}} - 2\delta_{\mu_n} + \delta_{\frac{1}{2^{n+1}}}\right)(y).$$

Write  $k^1(z) = x/|z|^2$  and  $k^2(z) = y/|z|^2$ . Notice that  $k^1 = c \partial_1 E$ , where E is the fundamental solution of the Laplacian and c a constant. Then

$$||k^{1} * T||_{\infty} = ||k^{1} * \Delta h||_{\infty} = ||c \partial_{1}(\Delta h * E)||_{\infty} = ||\partial_{1}h||_{\infty} = ||f'g||_{\infty} \le 1.$$

Since

$$g'(y) = \begin{cases} \frac{2^{n+2}}{n^2}, & \text{if } \frac{1}{2^{n+1}} \le y \le \mu_n \\ -\frac{2^{n+2}}{n^2}, & \text{if } \mu_n \le y \le \frac{1}{2^n} \end{cases}$$

we have

$$||k^{2} * T||_{\infty} = ||k_{2} * \Delta h||_{\infty} = ||c \partial_{2}(\Delta h * E)||_{\infty} = ||\partial_{2}h||_{\infty} = ||fg'||_{\infty} = \infty.$$

In fact  $g' \in L^1(\mathbb{R}) \setminus L^{\infty}(\mathbb{R})$ .

For  $n \ge 0$ , consider the square  $Q_n = I_n \times I_n$ . Then, since f is a linear function on  $I_n$ ,

$$\langle \mu, \chi_{Q_n} \rangle = -2 \frac{2^{n+2}}{n^2} \int_{I_n} f(x) \, dx = -2 \, \frac{2^{n+2}}{n^2} \, l(I_n) f(\mu_n)$$

 $\frac{|\langle \mu, \chi_{Q_n} \rangle|}{l(Q_n)} = 2 \frac{2^{n+2}}{n^2} (1 - \mu_n) \underset{n \to \infty}{\longrightarrow} \infty.$ 

Now we need to regularize  $\chi_{Q_n}$ . Take a non-negative  $\chi \in \mathcal{C}^{\infty}(B(0,1))$  such that  $\int \chi(x) \, dx = 1$  and set  $\chi_{\varepsilon}(x) = \varepsilon^{-n} \chi(x/\varepsilon)$ . Then, for  $\varepsilon$  small enough,  $\chi_{Q_n} * \chi_{\varepsilon}$  is supported in  $2Q_n$  and satisfies,  $|\langle \mu, \chi_{Q_n} * \chi_{\varepsilon} \rangle| \ge |\langle \mu, \chi_{Q_n} \rangle| - \frac{1}{2}$ . Also  $\|\chi_{Q_n} * \chi_{\varepsilon}\|_{\infty} \le 1$  and  $\|\nabla(\chi_{Q_n} * \chi_{\varepsilon})\|_{L^1(\mathbb{R}^2)}$  does not exceed the total variation of the measure  $\nabla\chi_{Q_n}$ , which is less than or equal to  $Cl(Q_n)$ . Therefore  $G(T) = \infty$ .

We do also have a counterexample in the setting of positive measures, based on a completely different idea.

**Proposition 11.** There exists a positive Radon measure  $\mu$  such that  $x_1/|x|^2 * \mu$  is in  $L^{\infty}(\mathbb{R}^2)$  and  $G(\mu) = \infty$ .

*Proof.* Consider the function  $f(t) = \log^+ \frac{1}{|t|}$ ,  $t \in \mathbb{R}$ . Then  $f \in BMO(\mathbb{R}) \setminus L^{\infty}(\mathbb{R})$ and f is supported on the interval [-1, 1]. Then write

$$\left(\frac{i}{\pi z} * f\right)(x, y) = \frac{1}{\pi} (k^2 * f)(x, y) + \frac{i}{\pi} (k^1 * f)(x, y)$$
$$= \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} f(t) dt + \frac{i}{\pi} \int_{\mathbb{R}} \frac{x-t}{(x-t)^2 + y^2} f(t) dt$$
$$= (P_y f)(x) + i(Q_y f)(x),$$

where  $P_y f(x)$  and  $Q_y f(x)$  are the Poisson transform and the conjugate Poisson transform of f respectively.

Therefore, if  $Hf = \frac{1}{\pi} \text{p.v.} \frac{1}{x} * f$  is the Hilbert transform of f,  $(k^1 * fdt)(x, y) = (Q_y f)(x) = P_y(Hf)(x).$ 

We claim that

$$H(f) \in L^{\infty}(\mathbb{R}). \tag{35}$$

If (35) holds, then the positive measure  $\mu = f(t) dt$  satisfies

$$(k^{1} * \mu)(x, y) = P_{y}(Hf)(x) \in L^{\infty}(\mathbb{R}^{2})$$
 (36)

and  $\mu$  has not linear growth, just because f is unbounded.

To show (35), we distinguish two cases.

Thus

• Case |x| > 1. Since our function H(f) is odd, without loss of generality we can assume that x > 1. Then an integration by parts together with the fact that the principal value integral of 1/t on [-1, 1] is zero, yield

$$\pi H f(x) = \text{p.v.} \int_{-1}^{1} \log \frac{1}{|t|} \frac{dt}{x-t}$$
$$= \text{p.v.} \int_{-1}^{1} \left( \log \frac{1}{x-t} + \log x \right) \frac{dt}{t}$$
$$= \text{p.v.} \int_{-1}^{1} \log \frac{x}{x-t} \frac{dt}{t} = \text{p.v.} \int_{-1/x}^{1/x} \log \frac{1}{1-u} \frac{du}{u}$$

Since x > 1, then 0 < 1/x < 1. Therefore

$$|\pi Hf(x)| \le \int_{-1}^{1} \left|\log \frac{1}{1-u}\right| \frac{du}{|u|} < +\infty.$$

• Case |x| < 1. Since for x = 0, Hf(0) = 0, we can assume that 0 < |x| < 1.

$$\pi Hf(x) = \text{p.v.} \int_{-1}^{1} \log \frac{1}{|t|} \frac{dt}{x-t}$$
  
= p.v.  $\int_{-\infty}^{\infty} \log \frac{1}{|t|} \frac{dt}{x-t} - \text{p.v.} \int_{|t|>1} \log \frac{1}{|t|} \frac{dt}{x-t}$   
=  $A + B$ .

Notice that since  $H^2 = -I$  and  $\log |x| = H(\pi \operatorname{sgn})(x)$ , then

$$H\left(\log\frac{1}{|t|}\right)(x) = H^2(-\pi\operatorname{sgn})(x) = \pi\operatorname{sgn}(x).$$

Therefore we only need to estimate B. Making the change of variable u = 1/t we get

$$B = \text{p.v.} \int_{|t|>1} \log \frac{1}{|t|} \frac{dt}{x-t} = \text{p.v.} \int_{-1}^{1} \log \frac{1}{|u|} \frac{\frac{1}{x}}{u-\frac{1}{x}} \frac{du}{u}$$
$$= \text{p.v.} \int_{-1}^{1} \log \frac{1}{|u|} \frac{du}{u-\frac{1}{x}} - \int_{-1}^{1} \log \frac{1}{|u|} \frac{du}{u}$$
$$= \text{p.v.} \int_{-1}^{1} \log \frac{1}{|u|} \frac{du}{u-\frac{1}{x}} < +\infty,$$

as in case 1.

## 6 Miscellaneous related results

As we have seen in the previous sections, the fact that the Cauchy kernel is complex is not as relevant as the fact that it is odd and has homogeneity -1. Indeed, Theorem 1 shows that one recovers the theory of analytic capacity by replacing the Cauchy kernel 1/z by any of the real kernels  $\operatorname{Re}(1/z)$  or  $\operatorname{Im}(1/z)$  and adding appropriate growth conditions on the admissible distributions.

A natural question is how one can extend this kind of results to the higher dimensional real variable setting in which the kernel  $x/|x|^2$  is replaced by the vector valued Riesz kernels

$$k_{\alpha}(x) = \frac{x}{|x|^{1+\alpha}}, \quad x \in \mathbb{R}^n, \quad 0 < \alpha < n,$$

and the capacity associated with this kernel is defined by (see [Pr1])

$$\Gamma_{\alpha}(E) = \sup\left\{ |\langle T, 1 \rangle| : \operatorname{spt}(T) \subset E, \left\| \frac{x}{|x|^{1+\alpha}} * T \right\|_{\infty} \le 1 \right\}.$$

The case  $\alpha = n - 1$ ,  $n \ge 2$ , is especially interesting, because it gives Lipschitz harmonic capacity (see (3)).

Unfortunately, as we show in subsections 6.1 and 6.2 below, the most obvious analogues of Theorems 1 and 2 in higher dimensions fail.

#### 6.1 Capacities associated with scalar $\alpha$ -Riesz potentials

Let T be a compactly supported distribution in  $\mathbb{R}^n$  and  $0 < \alpha < n$ . We say that the distribution T has growth  $\alpha$  provided that

$$G_{\alpha}(T) = \sup_{\varphi_Q} \frac{|\langle T, \varphi_Q \rangle|}{l(Q)^{\alpha}} < \infty,$$
(37)

where the supremum is taken over all  $\varphi_Q \in \mathcal{C}_0^{\infty}(Q)$  satisfying the normalization inequalities

$$\|\partial^s \varphi_Q\|_{L^1(Q)} \le l(Q)^{[\alpha]}, \quad |s| = n - [\alpha].$$

Here  $[\alpha]$  stands for the integer part of  $\alpha$ . For a compact set E in  $\mathbb{R}^n$  we define  $g_{\alpha}(E)$  as the set of all distributions T supported on E having growth  $\alpha$  with constant  $G_{\alpha}(T)$  at most 1. For each coordinate k set

$$\Gamma_{\alpha,\hat{k}}(E) = \sup\{|\langle T,1\rangle|\},\$$

where the supremum is taken over those distributions  $T \in g_{\alpha}(E)$ , such that the *j*-th component of the  $\alpha$ -Riesz potential  $x_j/|x|^{1+\alpha} * T$  is in the unit closed ball of  $L^{\infty}(\mathbb{R}^n)$ , for  $1 \leq j \leq n, j \neq k$ .

The proof of Lemma 3.2 in [Pr1] tells us that if  $k_{\alpha} * T$  is in the unit ball  $L^{\infty}(\mathbb{R}^n, \mathbb{R}^n)$ , then the distribution T has  $\alpha$ -growth and  $G_{\alpha}(T) \leq C$ . Hence  $\Gamma_{\alpha}(E) \leq C$ 

 $C \Gamma_{\alpha,\hat{k}}(E)$ . In this section we will show that for  $0 < \alpha < 1$ , there exists a set  $E \subset \mathbb{R}^n$  with  $\Gamma_{\alpha}(E) = 0$  and  $\Gamma_{\alpha,\hat{k}}(E) > 0$ . Therefore  $\Gamma_{\alpha}$  and  $\Gamma_{\alpha,\hat{k}}$  are not comparable and thus the direct analogue of Theorem 2 fails in this setting.

We proceed now to symmetrize the scalar  $\alpha$ -Riesz kernels in order to get a better understanding of the capacities  $\Gamma_{\alpha,\hat{k}}$ , for  $1 \leq k \leq n$  and  $0 < \alpha < 1$ .

For  $0 < \alpha < n$  and  $1 \le i \le n$  the quantity

$$\sum_{\sigma} \frac{x_{\sigma(2)}^{i} - x_{\sigma(1)}^{i}}{|x_{\sigma(2)} - x_{\sigma(1)}|^{1+\alpha}} \frac{x_{\sigma(3)}^{i} - x_{\sigma(1)}^{i}}{|x_{\sigma(3)} - x_{\sigma(1)}|^{1+\alpha}}$$
(38)

where the sum is taken over the permutations of the set  $\{1, 2, 3\}$ , is the analogue of the right hand side of (22) for the *i*-th coordinate of the Riesz kernel  $k_{\alpha}$ . Notice that (38) is exactly

$$2p_{\alpha,i}(x_1, x_2, x_3)$$

where  $p_{\alpha,i}(x_1, x_2, x_3)$  is defined as the sum in (38) only taken on the three permutations (1, 2, 3), (2, 3, 1) and (3, 1, 2).

We will now show that given three distinct points  $x_1, x_2, x_3 \in \mathbb{R}^n$ , for  $1 \leq i \leq n$ and  $0 < \alpha \leq 1$ , the quantity  $p_{\alpha,i}(x_1, x_2, x_3)$  is non-negative. We will use this to study the  $L^2$  boundedness of the scalar Riesz integral operator of homogeneity  $-\alpha$ .

The relationship between the quantity  $p_{\alpha,i}(x, y, z)$ ,  $0 < \alpha \leq 1$ ,  $1 \leq i \leq n$ , and the  $L^2$  estimates of the operator with kernel  $k^i_{\alpha} = x_i/|x|^{1+\alpha}$  is as in (23). That is, if  $\mu$  is a positive finite Radon measure in  $\mathbb{R}^n$  with  $\alpha$ -growth,  $\varepsilon > 0$  and we set

$$R^{i}_{\alpha,\varepsilon}(\mu)(x) = \int_{|y-x|>\varepsilon} k^{i}_{\alpha}(y-x) \, d\mu(y),$$

then (see in [MeV] the argument for the Cauchy singular integral operator)

$$\left| \int |R^{i}_{\alpha,\varepsilon}(\mu)(x)|^{2} d\mu(x) - \frac{1}{3} p_{\alpha,i,\varepsilon}(\mu) \right| \le C \|\mu\|,$$
(39)

C being a positive constant depending only on n and  $\alpha$ , and

$$p_{\alpha,i,\varepsilon}(\mu) = \iiint_{S_{\varepsilon}} p_{\alpha,i}(x,y,z) \, d\mu(x) \, d\mu(y) \, d\mu(z),$$

with

$$S_{\varepsilon} = \{(x, y, z) : |x - y| > \varepsilon, |x - z| > \varepsilon \text{ and } |y - z| > \varepsilon \}.$$

**Lemma 12.** Let  $0 < \alpha < 1$  and  $x_1, x_2, x_3$  three different points in  $\mathbb{R}^n$ . For  $1 \le i \le n$  we have

$$\frac{(2-2^{\alpha})m^2}{L(x_1,x_2,x_3)^{2+2\alpha}} \le p_{\alpha,i}(x_1,x_2,x_3) \le \frac{3m^2}{L(x_1,x_2,x_3)^{2+2\alpha}},\tag{40}$$

where  $m = \max(|x_2^i - x_1^i|, |x_3^i - x_2^i|, |x_3^i - x_1^i|)$  and  $L(x_1, x_2, x_3)$  is the length of the largest side of the triangle determined by the three points  $x_1, x_2, x_3$ .

Moreover,  $p_{\alpha,i}(x_1, x_2, x_3) = 0$  if and only if the three points lie on a (n-1)-hypersurface perpendicular to the *i* axis, *i.e.*  $x_1^i = x_2^i = x_3^i$ .

*Proof.* Without loss of generality fix i = 1. Write  $a = x_2 - x_1$  and  $b = x_3 - x_2$ , then  $a + b = x_3 - x_1$ . A simple computation yields

$$p_{\alpha,1}(x_1, x_2, x_3) = \frac{a_1^2 |b|^{1+\alpha} + b_1^2 |a|^{1+\alpha} + a_1 b_1 \left( |b|^{1+\alpha} + |a|^{1+\alpha} - |a+b|^{1+\alpha} \right)}{|a|^{1+\alpha} |b|^{1+\alpha} |a+b|^{1+\alpha}}, \quad (41)$$

which makes the second inequality in (40) obvious. To prove the first inequality in (40), assume without loss of generality, that  $1 = |a| \le |b| \le |a + b|$ . Then

$$p_{\alpha,1}(x_1, x_2, x_3) = \frac{1}{|b|^{1+\alpha}|a+b|^{1+\alpha}} \left( a_1^2 |b|^{1+\alpha} + b_1^2 + a_1 b_1 (1+|b|^{1+\alpha} - |a+b|^{1+\alpha}) \right).$$

We distinguish now two cases,

• Case  $a_1b_1 \leq 0$ . Notice that since  $|b| \leq |a+b|$ ,

$$a_1b_1(1+|b|^{1+\alpha}-|a+b|^{1+\alpha}) \ge a_1b_1.$$

Then, since  $|b| \ge 1$ ,

$$p_{\alpha,1}(x_1, x_2, x_3) = \frac{1}{|b|^{1+\alpha}|a+b|^{1+\alpha}} \left(a_1^2 |b|^{1+\alpha} + b_1^2 + a_1 b_1 (1+|b|^{1+\alpha} - |a+b|^{1+\alpha})\right)$$
  

$$\geq \frac{a_1^2 |b|^{1+\alpha} + b_1^2 + a_1 b_1}{|b|^{1+\alpha}|a+b|^{1+\alpha}} \geq \frac{a_1^2 + b_1^2 + a_1 b_1}{|b|^{1+\alpha}|a+b|^{1+\alpha}}$$
  

$$= \frac{1}{2} \frac{(a_1 + b_1)^2 + a_1^2 + b_1^2}{|b|^{1+\alpha}|a+b|^{1+\alpha}}.$$

• Case  $a_1b_1 > 0$ . Then  $\max\{a_1^2, b_1^2, (a_1 + b_1)^2\} = (a_1 + b_1)^2$ . Write  $t = |b| \ge 1$ and

$$f(t) = a_1^2 t^{1+\alpha} + b_1^2 + a_1 b_1 \left( 1 + t^{1+\alpha} - (1+t)^{1+\alpha} \right).$$

By the triangle inequality,

$$p_{\alpha,1}(x_1, x_2, x_3) \ge \frac{f(t)}{|b|^{1+\alpha}|a+b|^{1+\alpha}} \ge \frac{\min_{t\ge 1} f(t)}{|b|^{1+\alpha}|a+b|^{1+\alpha}}$$

Our function f has a minimum at the point  $t^* = \left(\left(\frac{a_1}{b_1}+1\right)^{1/\alpha}-1\right)^{-1}$ .

1. If  $a_1/b_1 \ge 2^{\alpha} - 1$ , then  $t^* \le 1$ . Therefore

$$p_{\alpha,1}(x_1, x_2, x_3) \ge \frac{f(1)}{|b|^{1+\alpha}|a+b|^{1+\alpha}}$$
  
=  $\frac{a_1^2 + b_1^2 + 2a_1b_1(1-2^{\alpha})}{|b|^{1+\alpha}|a+b|^{1+\alpha}}$   
=  $(2^{\alpha} - 1)\frac{(a_1 - b_1)^2}{|b|^{1+\alpha}|a+b|^{1+\alpha}} + (2 - 2^{\alpha})\frac{a_1^2 + b_1^2}{|b|^{1+\alpha}|a+b|^{1+\alpha}}$   
 $\ge \frac{2 - 2^{\alpha}}{2}\frac{(a_1 + b_1)^2}{|b|^{1+\alpha}|a+b|^{1+\alpha}}.$ 

2. If  $a_1/b_1 < 2^{\alpha} - 1$ , then  $t^* > 1$ . Hence,

$$p_{\alpha,1}(x_1, x_2, x_3) \ge \frac{f(t^*)}{|b|^{1+\alpha}|a+b|^{1+\alpha}}$$

Since

$$f(t^*) = b_1^2 \left( 1 + \frac{a_1}{b_1} \right) \left( 1 - \frac{a_1}{\left( (a_1 + b_1)^{1/\alpha} - b_1^{1/\alpha} \right)^{\alpha}} \right),$$

then

$$f(t^*) \ge b_1^2 \min_{a_1 < b_1(2^{\alpha} - 1)} \left( 1 - \frac{a_1}{\left( (a_1 + b_1)^{1/\alpha} - b_1^{1/\alpha} \right)^{\alpha}} \right)$$
$$= b_1^2 (2 - 2^{\alpha}) \ge \frac{2 - 2^{\alpha}}{2^{2\alpha}} (a_1 + b_1)^2,$$

since the function

$$g(x) = 1 - \frac{x}{\left((x+b_1)^{1/\alpha} - b_1^{1/\alpha}\right)^{\alpha}}$$

is decreasing and  $(a_1 + b_1)^2 \le (2^{\alpha} b_1)^2$ .

Now, If  $x_1^1 = x_2^1 = x_3^1$ , then  $a_1 = b_1 = 0$ . Hence (41) gives us  $p_{\alpha,1}(x_1, x_2, x_3) = 0$ . On the other hand, if  $p_{\alpha,1}(x_1, x_2, x_3) = 0$ , inequality (40) gives us  $\max((x_2^i - x_1^i)^2, (x_3^i - x_2^i)^2, (x_3^i - x_1^i)^2) = 0$ , hence  $a_1^2 = b_1^2 = (a_1 + b_1)^2 = 0$ , which implies  $x_1^1 = x_2^1 = x_3^1$ .  $\Box$ 

We are now ready to prove the existence of a compact set  $E \subset \mathbb{R}^n$  with  $\Gamma_{\alpha}(E) = 0$ but  $\Gamma_{\alpha,\hat{1}}(E) > 0$ . Take a compact subset E of the  $x_1$ -axis with positive finite  $\alpha$ -dimensional Hausdorff measure. Then by [Pr1, Theorem 1.1],  $\Gamma_{\alpha}(E) = 0$ . It remains to show that  $\Gamma_{\alpha,\hat{1}}(E) > 0$ . For this let  $\mu$  be  $\alpha$ -dimensional Hausdorff measure restricted to E. It is enough to show that the singular integral operator  $R^i_{\alpha}$  associated with the scalar kernel  $k^i_{\alpha} = x_i/|x|^{1+\alpha}$ ,  $i \neq 1$ , is bounded on  $L^2(\mu)$ . This is so by a lemma of Davie and Oksendal (see [DO, p.139], [Ch, Theorem 23] or [V3, Lemma 4.2]). By Lemma 12 we have  $p_{\alpha,i}(x_1, x_2, x_3) = 0$  for  $x_1, x_2$  and  $x_3$  in E and  $i \neq 1$  and thus (39) yields

$$\int |R^i_{\alpha,\varepsilon}(\mu)(x)|^2 d\mu(x) \le C \, \|\mu\|, \quad \epsilon > 0.$$

Replacing in the above inequality  $\mu$  by  $\chi_B \mu$  where B is any ball we get

$$\int_{B} |R^{i}_{\alpha,\varepsilon}(\chi_{B}\,\mu)(x)|^{2}\,d\mu(x) \leq C\,\mu(B), \quad \epsilon > 0.$$

By the non-doubling T(1)-Theorem of [NTV1] we conclude that  $R^i_{\alpha}$  is bounded on  $L^2(\mu)$ .

# 6.2 Lipschitz harmonic capacity is not comparable to the capacity associated with a scalar Riesz-potential

Theorem 1 says that in the plane, analytic capacity can be characterized in terms of either capacity  $\Gamma_i$ , i = 1, 2. In particular this implies a weaker qualitative statement, namely, that if E is a compact set in the plane and there exists a non-zero distribution T supported on E with linear growth and bounded potential  $x_i/|x|^2 * T$ , for i = 1 or i = 2, then there exists another non-zero distribution S supported on E with bounded potentials  $x_i/|x|^2 * S$ , i = 1, 2.

In  $\mathbb{R}^n$  Lipschitz harmonic capacity is an excellent replacement for analytic capacity. Thus one may ask whether Lipschitz harmonic capacity can be described in terms of one of the capacities associated with a component of the kernel  $x/|x|^n$  in which the growth condition n-1 has been required on the distributions involved. In a qualitative way we ask the following question. Assume that E is a compact set in  $\mathbb{R}^n$  and that there exists a non-zero distribution T supported on E with growth n-1 and bounded potential  $x_n/|x|^n * T$ . Is it true that there exists another non-zero distribution S supported on E with bounded vector valued potential  $x/|x|^n * T$ ? The answer is no for  $n \geq 3$ . We describe the example in  $\mathbb{R}^3$ . We thank X. Tolsa for the right suggestion.

**Proposition 13.** There exists a compact set  $E \subset \mathbb{R}^3$  which supports a non-zero distribution T with growth 2 and bounded scalar Riesz potential  $x_3/|x|^3 * T$ , but does not support any non-zero distribution S with bounded vector valued Riesz potential  $x/|x|^3 * S$ .

*Proof.* Let  $K \subset H = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}$  be the classical 1-dimensional planar Cantor defined by taking the "corner quarters" at each generation. Then K has finite positive length but zero analytic capacity (see [G1], [G2] or [I]). In

particular, K has zero Lipschitz harmonic capacity and by [MaPa] the same happens to  $E = K \times [-1, 1]$ . Thus E does not support any distribution S with bounded vector valued Riesz potential  $x/|x|^3 * S$ .

Let  $\mu$  denote 2-dimensional Hausdorff measure restricted to  $K \times \mathbb{R} \subset \mathbb{R}^3$  and let  $\nu$  denote the restriction of  $\mu$  to E. It is a simple matter to check that  $\mu$  satisfies the growth condition

$$\mu(B(x,r)) \le C r^2, \quad x \in K \times \mathbb{R}, \quad 0 < r.$$

Although the reverse inequality does not hold for large r,  $\mu$  is a doubling measure. Indeed,  $\mu(B(x,r))$  is comparable to  $r^2$  for  $0 < r \leq 1$  and to r for  $1 \leq r$ . Our goal is to show that the scalar Riesz singular integral operator  $R^3$  with kernel  $k^3(x) = x_3/|x|^3$  is bounded on  $L^2(\nu)$ . Once this is established the Davie-Oksendal lemma (see [Ch, Theorem 33 ] or [V3, Lemma 4.2]) provides a non-negative function  $b \in L^{\infty}(\nu)$  such that  $x_3/|x|^3 * b\nu$  is in  $L^{\infty}(\mathbb{R}^3)$ , which completes the proof.

We claim that, indeed,  $R^3$  is bounded on  $L^2(\mu)$ . To show this we check that  $R^3(1) = 0$  and then we apply the standard T(1)-Theorem for doubling measures. The computation of  $R^3(1)$  is performed as follows. Set  $K(x, \epsilon) = \{(y_1, y_2) \in K : |x_1 - y_1| > \epsilon \text{ and } |x_2 - y_2| > \epsilon\}$ , Then

$$R^{3}(1)(x) = \lim_{\epsilon \to 0} \int_{|y-x| > \epsilon} \frac{x_{3} - y_{3}}{|x - y|^{3}} d\nu(y)$$
  
= 
$$\lim_{\epsilon \to 0} \int_{K(x,\epsilon)} \left( \int_{|y_{3} - x_{3}| > \epsilon} \frac{x_{3} - y_{3}}{|x - y|^{3}} dy_{3} \right) dH^{1}(y_{1}, y_{2}) = 0,$$
  
$$x \in K \times \mathbb{R}.$$

for each  $x \in K \times \mathbb{R}$ .

#### Remarks

- Notice that in the above example one obtains that  $R^3(\nu)$  is bounded on  $L^2(\nu)$ , while the whole vector  $R(\nu)$  is not bounded on  $L^2(\nu)$ . Therefore, the above example shows that corollary 5 does not hold if  $n \ge 3$ , namely, we cannot get  $L^2(\nu)$  boundedness of the vector valued Riesz operator  $R_{n-1}(\nu)$  from  $L^2(\nu)$ boundedness of only one component  $R_{n-1}^i(\nu)$ .
- It is an open question to decide whether, for  $n \ge 3$ , Lipschitz harmonic capacity is comparable to the capacities associated with (n-1)-components of the vector valued Riesz potential  $x/|x|^n * T$ .

#### 6.3 Finiteness of the capacities $\kappa_i$

Indeed, we give a proof of a more general result, stating that for compact sets  $E \subset \mathbb{R}^n$ ,  $0 < \alpha < n$  and  $1 \le i \le n$ , the capacities

$$\kappa_{\alpha,i}(E) = \sup\left\{ |\langle T, 1 \rangle| : \operatorname{spt}(T) \subset E, \left\| \frac{x_i}{|x|^{1+\alpha}} * T \right\|_{\infty} \le 1 \right\},\$$

are finite.

**Lemma 14.** For any cube  $Q \subset \mathbb{R}^n$ ,  $0 < \alpha < n$  and  $1 \leq i \leq n$ , we have

$$\kappa_{\alpha,i}(Q) \le Cl(Q)^{\alpha}.$$

*Proof.* Without loss of generality assume i = 1. Assume also momentarily that the dimension n is odd, say n = 2k + 1. Our argument uses a reproduction formula for test functions involving the kernel  $k^i(y) = y_i/|y|^{1+\alpha}$ ,  $1 \le i \le n$ , [Pr1, Lemma 3.1]. For a test function g, the formula reads

$$g(x) = c_{n,\alpha} \sum_{j=1}^{n} \left( \Delta^k \partial_j g * \frac{1}{|y|^{n-\alpha}} * k^j \right) (x), \tag{42}$$

for some constant  $c_{n,\alpha}$  depending only on the dimension n and on  $\alpha$ . For n = 2k, there is an analogous reproduction formula that settles the even case [Pr1, Lemma 3.1].

Let T be a real distribution supported on Q such that  $k^1 * T \in L^{\infty}(\mathbb{R}^n)$ . Write the cube Q as  $Q = I_1 \times Q'$ , with  $I_1$  being an interval in  $\mathbb{R}$  and Q' an n-1 dimensional cube in  $\mathbb{R}^{n-1}$ , and let  $\varphi_Q \in \mathcal{C}_0^{\infty}(2Q)$  be such that  $\|\partial^s \varphi_Q\|_{\infty} \leq C_s l(Q)^{-|s|}$  and

$$\varphi_Q(x) = \varphi_1(x_1)\varphi_2(x_2,\ldots,x_n)$$

with  $\varphi_1(x_1) = 1$  on  $I_1$ ,  $\varphi_1(x_1) = 0$  on  $(2I_1)^c$  and  $\int_{-\infty}^{\infty} \varphi_1 = 0$ , and  $\varphi_2 \ge 0$ ,  $\varphi_2 \equiv 1$  on Q' and  $\varphi_2 \equiv 0$  on  $(2Q')^c$ . Then, since our distribution T is supported on Q, using the reproduction formula (42),

$$\begin{split} |\langle T,1\rangle| &= |\langle T,\varphi_Q\rangle| \le C\sum_{j=1}^n \left| \left\langle T,\Delta^k \partial_j \varphi_Q * \frac{1}{|y|^{n-\alpha}} * k^j \right\rangle \right| \\ &= C \left| \left\langle k^1 * T,\Delta^k \partial_1 \varphi_Q * \frac{1}{|y|^{n-\alpha}} \right\rangle \right| + C\sum_{j=2}^n \left| \left\langle T,\Delta^k \partial_j \varphi_Q * \frac{1}{|y|^{n-\alpha}} * k^j \right\rangle \right| \\ &= A + B. \end{split}$$

We first estimate the term A. We have

$$\begin{split} \int (k^1 * T)(x) \,\Delta^k \partial_1 \varphi_Q * \frac{1}{|y|^{n-\alpha}}(x) \,dx &= \int_{3Q} (k^1 * T)(x) \,(\Delta^k \partial_1 \varphi_Q * \frac{1}{|y|^{n-\alpha}})(x) \,dx \\ &+ \int_{\mathbb{R}^n \setminus 3Q} (k^1 * T)(x) \,(\varphi_Q * \Delta^k \partial_1(\frac{1}{|y|^{n-\alpha}}))(x) \,dx. \end{split}$$

Let  $Q_0$  be the unit cube centered at 0. Dilating to bring the integrals on  $3Q_0$  and  $2Q_0$ , and using  $|\partial^s \varphi_Q| \leq C_s l(Q)^{-|s|}$ , we get

$$\begin{split} A &\leq \|k^1 * T\|_{\infty} \left( \int_{3Q} \int_{2Q} \frac{|\Delta^k \partial_1 \varphi_Q(y)|}{|x - y|^{n - \alpha}} \, dy \, dx + \int_{\mathbb{R}^n \setminus 3Q} \int_{2Q} \frac{|\varphi_Q(y)|}{|x - y|^{2n - \alpha}} \, dy \, dx \right) \\ &\leq Cl(Q)^{\alpha} \left( \int_{3Q_0} \int_{2Q_0} \frac{dy \, dx}{|x - y|^{n - \alpha}} + \int_{\mathbb{R}^n \setminus 3Q_0} \int_{2Q_0} \frac{dy \, dx}{|x - y|^{2n - \alpha}} \right) \\ &\leq Cl(Q)^{\alpha}. \end{split}$$

We turn now to the estimate of B. The homogeneous differential operator  $\Delta^k$  can be written as  $\Delta^k = \sum_{|s|=2k} a_s \partial^s$ , for certain constants  $a_s$ . Divide the set of multi-indexes s of length 2k into two classes I and J according to whether  $s_1 \ge 1$  or  $s_1 = 0$ . In other words,  $s \in I$  if  $\partial^s$  contains at least one partial derivative with respect to first variable. Thus  $\Delta^k = \sum_{s \in I} a_s \partial^s + \sum_{s \in J} a_s \partial^s$ , and so  $B = B_1 + B_2$  where

$$B_1 = C \sum_{j=2}^n \left| \left\langle T, \sum_{s \in I} a_s \partial^s \partial_j \varphi_Q * \frac{1}{|y|^{n-\alpha}} * k^j \right\rangle \right|$$

and

$$B_2 = C \sum_{j=2}^n \left| \left\langle T, \sum_{s \in J} a_s \, \partial^s \partial_j \varphi_Q * \frac{1}{|y|^{n-\alpha}} * k^j \right\rangle \right|.$$

To estimate  $B_1$  we bring in each term of the sum in  $s \in I$  one derivative with respect to the first variable into the kernel  $k^j$  and use  $\partial_1 k^j = \partial_j k^1$  to take back a derivative with respect to j into  $\varphi_Q$ . The effect of these moves is to replace  $k^j$  by  $k^1$ . Therefore

$$B_1 = C \sum_{j=2}^n \left| \left\langle k^1 * T, \sum_{|s|=2k} b_s \,\partial^s \partial_j \varphi_Q * \frac{1}{|y|^{n-\alpha}} \right\rangle \right|,$$

for some numbers  $b_s$ . This expression can be estimated as we did before with A.

To estimate  $B_2$  we need to replace in some way the kernel  $k^j$  by  $k^1$ . We do this by showing that, for each j there exists a function  $\psi_Q^j \in \mathcal{C}_0^\infty(2Q)$  satisfying

$$k^j * \varphi_Q = k^1 * \psi_Q^j, \quad 1 \le j \le n, \tag{43}$$

and  $\|\partial^s \psi_Q^j\|_{\infty} \leq C_s l(Q)^{-|s|}$ . Before proving (43) we show how to estimate  $B_2$ .

By (43)

$$B_{2} = C \sum_{j=2}^{n} \left| \left\langle T, \sum_{s \in J} a_{s} \partial^{s} \partial_{j} \varphi_{Q} * \frac{1}{|y|^{n-\alpha}} * k^{j} \right\rangle \right|$$
$$= C \sum_{j=2}^{n} \left| \left\langle T, \sum_{s \in J} a_{s} \partial^{s} \partial_{j} \psi_{Q}^{j} * \frac{1}{|y|^{n-\alpha}} * k^{1} \right\rangle \right|$$
$$= C \sum_{j=2}^{n} \left| \left\langle k^{1} * T, \sum_{s \in J} a_{s} \partial^{s} \partial_{j} \psi_{Q}^{j} * \frac{1}{|y|^{n-\alpha}} \right\rangle \right|,$$

which can be estimated as the term A.

We are left with proving (43). Taking Fourier transforms in (43) we obtain for some constant a,

$$a\,\hat{\varphi}_Q(\xi)\xi_j = \psi_Q^j(\xi)\xi_1,$$

which becomes

$$a\,\partial_j\varphi_Q=\partial_1\psi_Q^j.$$

Hence, for the non-trivial case  $2 \le j \le n$ ,

$$\psi_Q^j(x) = a \int_{-\infty}^{x_1} \partial_j \varphi_Q(t, x_2, \dots, x_n) dt = a \partial_j \varphi_2(x_2, \dots, x_n) \int_{-\infty}^{x_1} \varphi_1(t) dt,$$

and the key remark is that the function above has compact support because the integral of  $\varphi_1$  on the real line vanishes.

We conclude with the following corollary.

**Corollary 15.** For any compact set  $E \subset \mathbb{R}^n$ ,  $0 < \alpha < n$  and  $1 \leq i \leq n$ , we have  $\kappa_{\alpha,i}(E) \leq C \operatorname{diam}(E)^{\alpha}$ .

We do not know whether in the preceding inequality the diameter of E can be replaced by the Hausdorff content of E.

#### 6.4 Localization and growth

The growth assumption on the distribution T in the localization lemma (Lemma 7) cannot be completely dispensed with. Indeed, if for the *i*-th coordinate one has the inequality

$$\left\|\frac{x_i}{|x|^2} * \varphi_Q T\right\|_{\infty} \le C \left\|\frac{x_i}{|x|^2} * T\right\|_{\infty},\tag{44}$$

for all  $\varphi_Q$  satisfying the normalization conditions (9), then necessarily T has linear growth. This can be shown by an argument very close to that of the previous subsection. We only deal with the details of the case n = 2. The case of even dimensions is very similar, while the case of odd dimensions needs some additional care. We also assume i = 1.

Let Q be square and  $\varphi_Q$  a function in  $\mathcal{C}_0^{\infty}(Q)$  satisfying the normalization conditions (9). Set  $Q = I_1 \times I_2$  and  $\psi(x_1, x_2) = \psi_1(x_1)\psi(x_2)$ , where, for j = 1, 2,  $\psi_j \in \mathcal{C}_0^{\infty}(I_j), \ \psi_j = 1 \text{ on } I_j, \ \int_{-\infty}^{\infty} \psi(x_1) \, dx_1 = 0 \text{ and } \|d^k \psi_j / (dx_j)^k\|_{\infty} \leq C \, l(I_j)^{-k}, 0 \leq k \leq 2$ . We then have

$$\langle T, \varphi_Q \rangle = \langle \varphi_Q T, 1 \rangle = \langle \varphi_Q T, \psi \rangle.$$

We want now to find a function  $\chi$  such that  $\psi = k^1 * \chi$ , where  $k^1 = x_1/|x|^2$ . Taking the Fourier transform we get  $\hat{\psi}(\xi) = a(\xi_1/|\xi|^2) \hat{\chi}(\xi)$  for some constant a. Hence  $\partial_1 \chi = b \Delta \psi$ , for some other constant b. Thus

$$\chi = b \int_{-\infty}^{x_1} \Delta \psi(t, x_2) dt$$
$$= b \left( \partial_1 \psi_1(x_1) \psi_2(x_2) + \left( \int_{-\infty}^{x_1} \psi_1(t) dt \right) \partial_2^2 \psi_2(x_2) \right)$$

Notice that  $\chi$  is supported on Q and  $\|\chi\|_{\infty} \leq C l(Q)^{-1}$ . Therefore by Lemma 8

$$|\langle T, \varphi_Q \rangle| = |\langle k^1 * \varphi_Q T, \chi \rangle| \le C \, \|k^1 * \varphi_Q T\|_{L^1(Q)} \, \|\chi\|_{\infty} \le C \, l(Q). \qquad \Box$$

#### 6.5 The growth condition for positive measures

We start by showing that the usual linear growth condition for a positive Radon measure is equivalent to the linear growth condition for distributions as defined in (8). Later on we treat also the case of the  $\alpha$ -growth condition for  $0 < \alpha < n$ .

Given a positive Radon measure  $\mu$  set

$$L(\mu) = \sup_{Q} \frac{\mu(Q)}{l(Q)},$$

where the supremum is taken over all squares Q with sides parallel to the coordinate axis.

If  $\varphi \in \mathcal{C}^{\infty}_0(\mathbb{R}^n)$ , then by an inequality of Mazya [Mz, 1.2.2, p. 24]

$$|\langle \mu, \varphi \rangle| = |\int \varphi \, d\mu| \le \int |\varphi| \, d\mu \le C \, L(\mu) \, \int |\nabla^{n-1} \varphi(x)| \, dx,$$

where  $\nabla^{n-1}\varphi$  denotes the vector of all derivatives  $\partial^s \varphi$  of order |s| = n - 1. Thus

$$G(\mu) \le C L(\mu).$$

The reverse inequality is immediate. Indeed, given a square Q let  $\varphi_Q$  be a function in  $\mathcal{C}_0^{\infty}(2Q)$  such that  $1 \leq \varphi_Q$  on Q and  $\|\partial^s \varphi_Q\|_{\infty} \leq C_s l(Q)^{-|s|}, |s| \geq 0$ . Then

$$\mu(Q) \le \int \varphi_Q \, d\mu = |\langle \mu, \varphi_Q \rangle| \le C \, G(\mu) \, l(Q),$$

because  $\int |\nabla^{n-1}\varphi_Q(x) dx| \leq C l(Q).$ 

We proceed now to treat the case of a general  $\alpha$ -growth condition,  $0 < \alpha < n$ . Set

$$L_{\alpha}(\mu) = \sup_{Q} \frac{\mu(Q)}{l(Q)^{\alpha}},$$

where the supremum is taken over all squares Q with sides parallel to the coordinate axis. The inequality  $L_{\alpha}(\mu) \leq C G_{\alpha}(\mu)$  is proven exactly as above. The definition of  $G_{\alpha}$  is in (37).

For the reverse inequality is convenient to distinguish several cases.

•  $\alpha$  is integer. The argument is exactly as in the case  $\alpha = 1$ . If  $\varphi \in \mathcal{C}_0^{\infty}(\mathbb{R}^n)$ , then by an inequality of Mazya [Mz, 1.2.2, p. 24]

$$|\langle \mu, \varphi \rangle| = |\int \varphi \, d\mu| \le \int |\varphi| \, d\mu \le C \, L_{\alpha}(\mu) \, \int |\nabla^{n-[\alpha]} \varphi(x)| \, dx,$$

where  $\nabla^{n-[\alpha]}\varphi$  denotes the vector of all derivatives  $\partial^s\varphi$  of order  $|s| = n - [\alpha]$ .

•  $\alpha$  is not integer and  $n - [\alpha]$  is odd. Set  $n - [\alpha] = 2m + 1$ . Then, for a constant c, we have

$$\varphi = c \sum_{i=1}^{n} \Delta^{m} \partial_{i} \varphi * \frac{x_{i}}{|x|^{[\alpha]+1}}, \qquad (45)$$

for each test function  $\varphi$ . This can be easily checked by taking the Fourier transform. Let  $\varphi_Q \in \mathcal{C}_0^{\infty}(Q)$  satisfy the normalization condition  $\|\partial^s \varphi_Q\|_{L^1(Q)} \leq l(Q)^{[\alpha]}, \quad |s| = n - [\alpha]$ . Then

$$\begin{aligned} |\langle \mu, \varphi_Q \rangle| &\leq \int |\varphi_Q| \, d\mu \leq C \, \int_Q \left| \sum_{i=1}^n \Delta^m \partial_i \varphi_Q * \frac{x_i}{|x|^{[\alpha]+1}} \right| \, d\mu \\ &\leq C \, \sum_{i=1}^n \int_Q \int \frac{|\Delta^m \partial_i \varphi_Q(y)|}{|x-y|^{[\alpha]}} \, dy \, d\mu(x) \\ &= C \, \sum_{i=1}^n \int |\Delta^m \partial_i \varphi_Q(y)| \left( \int_Q \frac{d\mu(x)}{|x-y|^{[\alpha]}} \right) \, dy \\ &\leq C \, \sum_{i=1}^n \|\Delta^m \partial_i \varphi_Q\|_{L^1(Q)} \|\chi_Q \mu * \frac{1}{|x|^{[\alpha]}}\|_{L^\infty(Q)}. \end{aligned}$$

The estimate of the  $L^{\infty}(Q)$  norm of  $\chi_Q \mu * (1/|x|^{[\alpha]})$  is standard. If  $x \in Q$  and we set  $\mu(r) \equiv \mu_x(r) = \mu(B(x,r))$ , then we get (d(Q) stands for the diameter

of Q)

$$\int_{Q} \frac{d\mu(y)}{|x-y|^{[\alpha]}} \leq \int_{0}^{d(Q)} \frac{d\mu(r)}{r^{[\alpha]}}$$
$$= \left[\frac{\mu(r)}{r^{[\alpha]}}\right]_{0}^{d(Q)} + [\alpha] \int_{0}^{d(Q)} \frac{\mu(r)}{r^{[\alpha]+1}} dr$$
$$\leq C L_{\alpha}(\mu) l(Q)^{\alpha-[\alpha]},$$

and therefore  $G_{\alpha}(\mu) \leq CL_{\alpha}(\mu)$ .

•  $\alpha$  is not integer and  $n - [\alpha]$  is even. We still need too distinguish two subcases:  $[\alpha] = 0$  and  $[\alpha] \ge 1$ . If  $[\alpha] = 0$ , take a cube Q and  $\varphi_Q \in \mathcal{C}_0^{\infty}(Q)$  satisfying the normalization condition  $\|\partial^s \varphi_Q\|_{L^1(Q)} \le 1$ , |s| = n. Then by [AH, 7.6.10, p. 212]

$$\|\varphi_Q\|_{\infty} \le C \|\nabla^n \varphi_Q\|_{L^1(Q)},$$

and so

$$|\langle \mu, \varphi_Q \rangle| \le \int |\varphi_Q| \, d\mu \le C \, \|\nabla^n \varphi_Q\|_{L^1(Q)} \mu(Q) \,,$$

which yields  $G_{\alpha}(\mu) \leq CL_{\alpha}(\mu)$ .

Assume now that  $[\alpha] \ge 1$ . Set  $n - [\alpha] = 2m$  for some integer m. For each test function  $\varphi$  we have the identity

$$\varphi = c \,\Delta^m \varphi * \frac{1}{|x|^{[\alpha]}},$$

where c is an appropriate constant. Let  $\varphi_Q \in \mathcal{C}_0^{\infty}(Q)$  satisfy the normalization condition  $\|\partial^s \varphi_Q\|_{L^1(Q)} \leq l(Q)^{[\alpha]}, \quad |s| = n - [\alpha]$ . Then

$$\begin{aligned} |\langle \mu, \varphi_Q \rangle| &\leq \int |\varphi_Q| \, d\mu \leq C \, \int_Q \left| \Delta^m \varphi_Q * \frac{1}{|x|^{[\alpha]}} \right| \, d\mu \\ &\leq C \, \int_Q \int \frac{|\Delta^m \varphi_Q(y)|}{|x - y|^{[\alpha]}} \, dy \, d\mu(x) \\ &= C \, \int |\Delta^m \varphi_Q(y)| \left( \int_Q \frac{d\mu(x)}{|x - y|^{[\alpha]}} \right) \, dy \\ &\leq C \, \|\Delta^m \varphi_Q\|_{L^1(Q)} \, \|\chi_Q \mu * \frac{1}{|x|^{[\alpha]}}\|_{L^{\infty}(Q)}. \end{aligned}$$

As it was shown above  $\|\chi_Q\mu * \frac{1}{|x|^{[\alpha]}}\|_{L^{\infty}(Q)} \leq C L_{\alpha}(\mu) l(Q)^{\alpha-[\alpha]}$ , which yields  $G_{\alpha}(\mu) \leq C L_{\alpha}(\mu)$ .

Acknowledgements. The authors are indebted to X.Tolsa for several useful conversations, which led to the example in section 6.2. The authors were partially supported by grants 2009-SGR-420 (Generalitat de Catalunya) and MTM2007-60062 (Spanish Ministry of Science).

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