Curvature and isoperimetric inequality

Julià Cufí[*], Agustí Reventós[†], Carlos J. Rodríguez

Abstract

We prove an inequality involving the length of a plane curve and the integral of its radius of curvature, that has as a consequence the 2-dimensional Ros' inequality. We also study the corresponding inequality for the case the curve is a polygon. For this we must introduce a new concept of radius of curvature at the vertices of a polygon. Finally we provide the link between the discrete and the continuous settings.

1 Introduction

The starting point of this note is the following inequality: if $C = \partial K$ is the boundary of a compact, convex set K of area A in $\mathbb{R}^2$, then

$$A \leq \frac{1}{2} \int_C \rho(s)\, ds, \tag{1}$$

where $\rho = \rho(s) < \infty$ is the radius of curvature on C and ds signifies arclength measure on C. Equality holds if and only if C is a circle. A proof of (1) is given in [1].

On the other hand one has the isoperimetric inequality

$$A \leq \frac{L^2}{4\pi},$$

where L is the length of C, with equality if and only if C is a circle.

So it is natural to try to compare the two quantities $L^2/4\pi$ and $\frac{1}{2}\int_C \rho(s)\, ds$.

In this paper we prove that

$$\frac{L^2}{4\pi} \leq \frac{1}{2} \int_C \rho(s) ds,$$

with equality if and only if C is a circle (Theorem 1).

Next we study this inequality for the case the curve C is a polygon. For this we need to introduce a notion of radius of curvature at the vertices of a polygon (Definition 1). This radius of curvature is a good approximation

[*]Partially supported by FEDER/Micinn MTM 2008-05561-C02-02.
[†]Partially supported by FEDER/Micinn MTM2009-0759.

of the radius of curvature of a smooth curve (Theorem 2). We prove the following result:

$$\frac{L^2}{4\pi} \leq \frac{1}{2}\sum_{k=1}^{n} l_k \frac{\rho_k + \rho_{k+1}}{2},$$

where L is the length of the polygon, l_k the length of its sides, and ρ_k the curvature at its vertices. Equality holds if and only if the polygon is umbilical (Theorem 3).

As a consequence we obtain a discrete proof of Theorem 1 (Theorem 4) an a geometrical proof of an interesting analytical property stated in Lemma 1.

Inequality (1) is the two-dimensional analogue of Ros' inequality:

$$V \leq \frac{1}{3}\int_S \frac{1}{H} dA,$$

where $H > 0$ is the mean curvature of the boundary S of a compact domain of volume V in $\mathbb{R}^3$, and dA signifies area measure on S. Equality holds if and only if S is a standard sphere (see [2] and [3]).

2 On the integral of the radius of curvature

We begin with the following analytical result, suggested by the geometrical problems we study here.

Lemma 1. *Let $p : \mathbb{R} \longrightarrow \mathbb{R}$ be a periodic function of class $\mathcal{C}^2$ with period 2π. Then*

$$\left(\int_0^{2\pi} p(\phi)\, d\phi\right)^2 \leq 2\pi \int_0^{2\pi} \left(p(\phi) + p''(\phi)\right)^2 d\phi.$$

Equality holds if and only if $p(\phi) + p''(\phi)$ is constant.

Proof. Note first that

$$\begin{aligned} & 2\pi \int_0^{2\pi} (p + p'')^2\, d\phi - (\int_0^{2\pi} p\, d\phi)^2 \\ = \quad & 2\pi \left[\int_0^{2\pi} p^2\, d\phi + \int_0^{2\pi} (p'')^2\, d\phi - 2\int_0^{2\pi} (p')^2\, d\phi\right] - (\int_0^{2\pi} p\, d\phi)^2. \quad (2) \end{aligned}$$

By developing $p = p(\phi)$ in Fourier series expansion,

$$p(\phi) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi),$$

we have, by Parseval's identity,

$$\frac{1}{2\pi}\int_0^{2\pi} p^2\, d\phi = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2),$$

and similar expressions for p' and p''.

Substituting in (2) we have

$$2\pi \int_0^{2\pi} (p+p'')^2 \, d\phi - (\int_0^{2\pi} p \, d\phi)^2$$
$$= 4\pi^2 \left[a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2) + \frac{1}{2}\sum_{n=1}^{\infty} n^4(a_n^2+b_n^2) - \sum_{n=1}^{\infty} n^2(a_n^2+b_n^2) \right] - 4\pi^2 a_0^2$$
$$= 2\pi^2 \sum_{n=2}^{\infty}(a_n^2+b_n^2)(n^2-1)^2 \geq 0.$$

Moreover, equality holds if and only if $a_n = b_n = 0$, for $n \geq 2$. That is, if and only if $p(\phi) = a_0 + a_1 \cos\phi + b_1 \sin\phi$. Equivalently, $p + p'' = a_0$. □

Recall that the boundary of a plane compact convex set can be parametrized by its support function $p(\phi)$. The angle $\phi - \frac{\pi}{2}$ is the angle between the tangent line at a given point of the boundary and the x axis, and $p(\phi)$ is the distance from this tangent line to the origin.

Theorem 1. *If the boundary $C = \partial K$ of a convex set K in the plane is a $\mathcal{C}^2$-curve of length L, then*

$$\frac{L^2}{4\pi} \leq \frac{1}{2}\int_C \rho(s) ds,$$

where $\rho(s)$ is the radius of curvature of C, and ds signifies arclength measure on C.

Equality holds if and only if C is a circle.

Proof. Let $p(\phi)$ be the support function of K. The following relations are well known (see [4], p. 3):

$$\begin{aligned} L &= \int_0^{2\pi} p(\phi)\, d\phi \\ \rho(\phi) &= p(\phi) + p''(\phi). \end{aligned}$$

The relation between s and ϕ is given by $ds = (p(\phi) + p''(\phi))d\phi$. Now the theorem follows directly by applying Lemma 1 to the support function $p(\phi)$. □

Note that, by the isoperimetric inequality, we have proved

$$A \leq \frac{1}{2}\int_C \rho(s) ds,$$

where A denotes the area of K. This is the 2-dimensional analogous of Ros' inequality and the difference $\frac{1}{2}\int_C \rho(s)ds - A$ was studied in [1].

We also have an estimation of the isoperimetric deficit:

Corollary 1. *If the boundary $C = \partial K$ of a plane convex set K of area A, is a $\mathcal{C}^2$-curve of length L, then*

$$0 \leq \frac{L^2}{4\pi} - A \leq -A_e$$

where $A_e \leq 0$ is the (algebraic) area of the domain bounded by the evolute of C. Equality holds if and only if C is a circle.

Proof. It is a consequence of Theorem 1 and the fact that

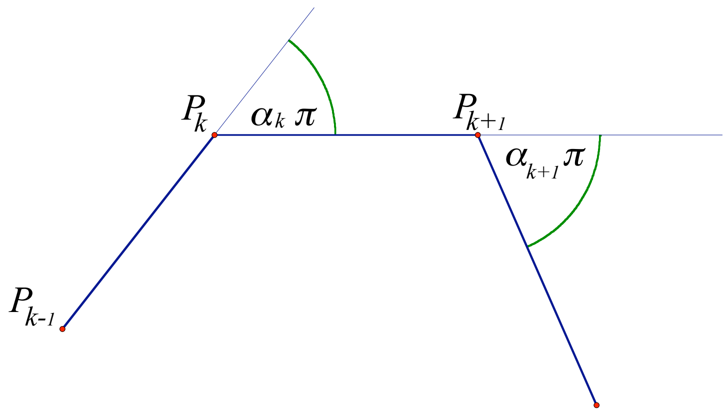
$$\int_C \rho(s)ds = 2(A - A_e)$$

(see Theorem 1, in [1]). □

3 Curvature for polygons

Given a plane convex polygon of vertices $P_1, P_2, \dots P_n$, we denote by $l_k = |\overrightarrow{P_k P_{k+1}}|$ the length of its sides and by $\alpha_k \pi$ the measure of its external angles. Of course we have $\sum_{k=1}^{n} \alpha_k = 2$, with $0 < \alpha_k < 1$, and

$$\overrightarrow{P_{k-1}P_k} \cdot \overrightarrow{P_k P_{k+1}} = l_{k-1} \cdot l_k \cdot \cos(\alpha_k \pi)$$



Definition 1. *Given a plane convex polygon of vertices $P_1, P_2, \dots P_n$, and sides of lengths $l_1, l_2, \dots, l_n$, we define the radius of curvature at the vertex P_k by*

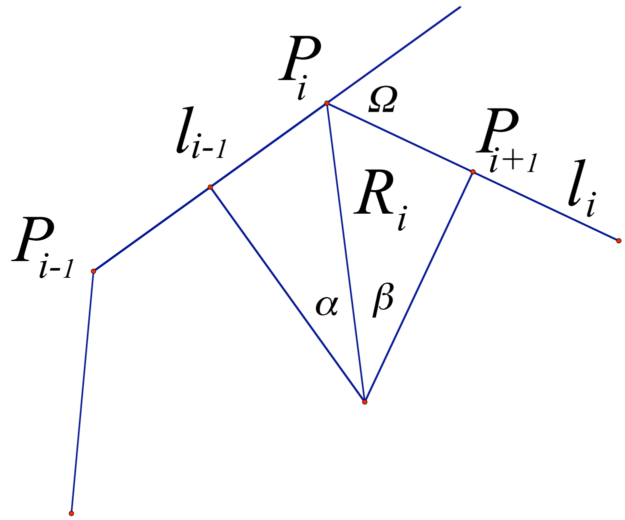
$$\rho_k = \frac{l_{k-1} + l_k}{2\alpha_k \pi}.$$

In particular, the curvature at the vertex P_k is given by

$$\kappa_k = \frac{1}{\rho_k} = \frac{\alpha_k \pi}{\frac{l_{k-1}}{2} + \frac{l_k}{2}},$$

an expression that essentially agrees with the classical definition of curvature as the ratio of the angle to the length. Note also that $l_0 = l_n$.

Note 1. Another natural definition of radius of curvature of a polygon (that we will not use in this note) is the following: If $P_{i-1}P_iP_{i+1}$ are consecutive vertices of a polygon, the radius of curvature R_i at P_i is the radius of the circumscribed circle around the triangle $P_{i-1}P_iP_{i+1}$ (see [5]).



The relation between R_i and ρ_i is (with the notation of the figure)

$$\rho_i = R_i \frac{\sin\alpha + \sin\beta}{\alpha + \beta}.$$

In particular, since $\Omega = \alpha + \beta$, R_i tends to ρ_i when the external angles of the polygon tend to zero.

Now we shall see (Theorem 2) that the radius of curvature, as defined in Definition 1, converges to the ordinary radius of curvature of a smooth curve when this curve is approximated by polygons. We shall consider dyadic approximations only for convenience. To be precise we give the following definition:

Definition 2. *Let $\gamma : [0, L] \longrightarrow \mathbb{R}^2$ be the parametrization by the arc length of a closed curve C of class $\mathcal{C}^1$. The n-th* dyadic polygon *$\mathcal{P}_n$ associated to C is the polygon given by the consecutive vertices*

$$P_k^{(n)} = \gamma(s_k), \qquad s_k = k\frac{L}{2^n} \in [0, L], \quad k = 1, 2, 3, \ldots, 2^n.$$

We shall need the following lemma:

Lemma 2 (Cauchy's mean value theorem)**.** *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be continuous functions, differentiable on the open interval (a, b) and let $\xi \in (a, b)$, such that*

$$(f(b) - f(a))g'(\xi) = (g(b) - g(a))f'(\xi). \tag{3}$$

If the curve $(f(t), g(t))$ has curvature $k > 0$, and we put

$$\xi = a + \theta h, \quad h = b - a, \quad 0 < \theta < 1$$

then

$$\lim_{h\to 0} \theta = \frac{1}{2}.$$

Proof. Note that

$$\begin{aligned} f(b)-f(a) &= f'(a)h+\frac{f''(\eta_1)}{2}h^2, \qquad a<\eta_1<b,\\ f'(\xi) &= f'(a)+f''(\eta_2)\theta h, \qquad a<\eta_2<\xi, \end{aligned}$$

and similar expressions for g, with corresponding η_3, η_4. Substituting these expressions in Cauchy's equality (3) we get

$$\theta(f'(a)g''(a)-g'(a)f''(a))=\frac{1}{2}(f'(a)g''(\eta_3))-f''(\eta_1)g'(a))+o(1), h\to 0.$$

Taking limits we obtain the result. □

Theorem 2. *Let $\gamma : [0,L] \longrightarrow \mathbb{R}^2$ be the parametrization by the arc length s of a closed strictly convex curve C of class $\mathcal{C}^2$. Let $\rho_k^{(n)}$ be the radius of curvature at the vertex $P_k^{(n)}$ of the n-th dyadic polygon $\mathcal{P}_n$. Then, for all $\epsilon>0$, there exists $n_\epsilon\in\mathbb{N}$ such that for all $n>n_\epsilon$,*

$$|\rho_k^{(n)}-\rho(s_k^{(n)})|<\epsilon, \qquad k=1,2,\ldots,2^n.$$

Proof. Since $\rho_k^{(n)}$ can be approximated by $\dfrac{l_{k-1}^{(n)}+l_k^{(n)}}{2\sin\Omega_k^{(n)}}$ (the sine approximation for small angles), it is sufficient to prove that, given $\epsilon>0$, and for n big enough, we have

$$\left|\frac{l_{k-1}^{(n)}+l_k^{(n)}}{2\sin\Omega_k^{(n)}}-\rho(s_k^{(n)})\right|<\epsilon, \qquad k=1,2,\ldots,2^n \tag{4}$$

where $\Omega_k^{(n)}=\alpha_k^{(n)}\pi$ is the exterior angle of the polygon $\mathcal{P}_n$ at vertex $P_k^{(n)}$.

The angle $\Omega_k^{(n)}$ can be computed using Cauchy's mean value theorem. In fact, there are points $\eta_{k-1}\in[s_{k-1}^{(n)},s_k^{(n)}]$, $\eta_k\in[s_k^{(n)},s_{k+1}^{(n)}]$, such that

$$\frac{y_k^{(n)}-y_{k-1}^{(n)}}{x_k^{(n)}-x_{k-1}^{(n)}}=\frac{y'(\eta_{k-1})}{x'(\eta_{k-1})}, \qquad \frac{y_{k+1}^{(n)}-y_k^{(n)}}{x_{k+1}^{(n)}-x_k^{(n)}}=\frac{y'(\eta_k)}{x'(\eta_k)}.$$

In particular, we have

$$\gamma'(\eta_{k-1})\cdot\gamma'(\eta_k)=\cos\Omega_k.$$

Equivalently

$$\begin{aligned} \sin\Omega_k &= x'(\eta_{k-1})y'(\eta_k)-y'(\eta_{k-1})x'(\eta_k)\\ &= x'(\eta_{k-1})\cdot(y'(\eta_k)-y'(\eta_{k-1}))-y'(\eta_{k-1})\cdot(x'(\eta_k)-x'(\eta_{k-1}))\\ &= (\eta_k-\eta_{k-1})\cdot[x'(\eta_{k-1})y''(\tau_{k-1})-y'(\eta_{k-1})x''(\nu_{k-1})], \end{aligned}$$

where $\tau_{k-1},\nu_{k-1}\in[\eta_{k-1},\eta_k]$.

On the other hand it is clear that the sum $l_{k-1}^{(n)}+l_k^{(n)}$ can be written as

$$l_{k-1}^{(n)}+l_k^{(n)}=L^{(n)}\left(\sqrt{x'(a_{k-1})^2+y'(b_{k-1})^2}+\sqrt{x'(a_k)^2+y'(b_k)^2}\right)$$

where $a_{k-1},b_{k-1}\in[s_{k-1}^{(n)},s_k^{(n)}]$, $a_k,b_k\in[s_k^{(n)},s_{k+1}^{(n)}]$.

Hence

$$\frac{l_{k-1}^{(n)}+l_k^{(n)}}{2\sin\Omega_k^{(n)}}=\frac{\sqrt{x'(a_{k-1})^2+y'(b_{k-1})^2}+\sqrt{x'(a_k)^2+y'(b_k)^2}}{2(x'(\eta_{k-1})y''(\tau_{k-1})-y'(\eta_{k-1})x''(\nu_{k-1}))}\cdot\frac{L^{(n)}}{\eta_k-\eta_{k-1}}.$$

Let us denote

$$B(k,n)=\frac{\sqrt{x'(a_{k-1})^2+y'(b_{k-1})^2}+\sqrt{x'(a_k)^2+y'(b_k)^2}}{2(x'(\eta_{k-1})y''(\tau_{k-1})-y'(\eta_{k-1})x''(\nu_{k-1}))}.$$

Since C is strictly convex, its curvature is a strictly positive continuous function on a compact set (the interval $[0,L]$). Hence, there is a constant M such that $0<\rho(s)<M$.

It is clear that $B(k,n)$ converges to the radius of curvature. In fact we have, for all $\epsilon>0$, and for n big enough,

$$|B(k,n)-\rho(s_k^{(n)})|<\epsilon/2, \qquad k=1,2,\ldots,2^n. \tag{5}$$

Moreover, since $\rho(s)$ is bounded, there exists $N>0$ such that $|B(k,n)|<N$.

On the other hand, the fraction $\dfrac{L^{(n)}}{\eta_k-\eta_{k-1}}$ converges uniformly to 1. In fact, by Lemma 2 applied to the functions $x(t),y(t)$, over the intervals $[s_{k-1},s_k]$ and $[s_k,s_{k+1}]$, and putting

$$\begin{aligned}\eta_{k-1} &= s_{k-1}+\theta_{k-1}L^{(n)}, \quad 0<\theta_{k-1}<1,\\ \eta_k &= s_k+\theta_kL^{(n)}, \quad 0<\theta_k<1,\end{aligned}$$

we have

$$\left|\frac{L^{(n)}}{\eta_k-\eta_{k-1}}-1\right|=\left|\frac{L^{(n)}}{L^{(n)}+L^{(n)}(\theta_{k-1}-\theta_k)}-1\right|<\epsilon, \qquad k=1,2,\ldots,2^n. \tag{6}$$

From inequalities (5) and (6) we get easily inequality (4), and theorem is proved. □

4 A discrete version of Theorem 1

In this section we shall give a discrete version of Theorem 1. For this, we shall need the following result.

Lemma 3. *Let $a_1,\ldots,a_n\in\mathbb{R}^+$ and let $f:(\mathbb{R}^+)^n\longrightarrow\mathbb{R}$ be the function given by*

$$f(x_1,\ldots,x_n)=\frac{a_1^2}{x_1}+\cdots+\frac{a_n^2}{x_n}.$$

If $x_1+\cdots+x_n=2$, then

$$f(x_1,\ldots,x_n)\geq\frac{1}{2}\left(\sum_{i=1}^n a_i\right)^2.$$

Proof. Following the method of Lagrange multipliers, we find the critical points of the function

$$g(x_1,\dots,x_n) = f(x_1,\dots,x_n) + \lambda(x_1 + \cdots + x_n - 2).$$

We obtain

$$x_k = \frac{2a_k}{\sum_{i=1}^n a_i}, \qquad k = 1,\dots,n.$$

The value of $f(x)$ at this point is

$$f(x_1,\dots,x_n) = \frac{a_1^2}{x_1} + \cdots + \frac{a_n^2}{x_n} = (\sum_{i=1}^n a_i)\frac{a_1}{2} + \cdots + (\sum_{i=1}^n a_i)\frac{a_n}{2} = \frac{1}{2}\left(\sum_{i=1}^n a_i\right)^2.$$

Since $f(x_1,\dots,x_n) > 0$, and it is not bounded above, the point

$$(\frac{2a_1}{\sum_{i=1}^n a_i},\dots,\frac{2a_n}{\sum_{i=1}^n a_i})$$

is a minimum. This completes the proof of the lemma. □

Definition 3. *A convex polygon is called umbilical if the radius of curvature at its vertices is constant.*

Note that, in this case, this constant must be equal to $L/2\pi$, where L is the length of the polygon. This fact is easily demonstrated by simply adding the equalities

$$l_{k-1} + l_k = 2\alpha_k \pi \rho, \qquad k = 1,\dots,n,$$

where ρ is the constant radius of curvature.

Theorem 3. *Let L be the length of a convex polygon. Then we have*

$$\frac{L^2}{4\pi} \le \frac{1}{2}\sum_{k=1}^n l_k \frac{\rho_k + \rho_{k+1}}{2}.$$

Equality holds if and only if the polygon is umbilical.

Proof. By definition of ρ_k, the second term of this inequality is

$$\begin{aligned}
\frac{1}{2}\sum_{k=1}^n l_k \frac{\rho_k + \rho_{k+1}}{2} &= \frac{1}{8\pi}\sum_{k=1}^n l_k \left(\frac{l_{k-1} + l_k}{\alpha_k} + \frac{l_k + l_{k+1}}{\alpha_{k+1}}\right) \\
&= \frac{1}{8\pi}\sum_{k=1}^n \frac{(l_k + l_{k+1})^2}{\alpha_k}.
\end{aligned}$$

Since $\alpha_1 + \cdots + \alpha_n = 2$, we can apply Lemma 3 and we obtain

$$\frac{1}{8\pi}\sum_{k=1}^n \frac{(l_k + l_{k+1})^2}{\alpha_k} \ge \frac{1}{8\pi}\frac{1}{2}(2L)^2.$$

Hence

$$\frac{1}{2}\sum_{k=1}^n l_k \frac{\rho_k + \rho_{k+1}}{2} \ge \frac{1}{4\pi} L^2,$$

and the inequality of the theorem is proved.

By Lemma 3, equality is attained when

$$\alpha_k = \frac{2(l_k + l_{k+1})}{\sum_{i=1}^n (l_i + l_{i+1})} = \frac{l_k + l_{k+1}}{L} = \frac{2\alpha_k \pi \rho_k}{L}.$$

Hence

$$\rho_k = \frac{L}{2\pi}, \qquad k = 1, \ldots, n$$

and the polygon is umbilical. □

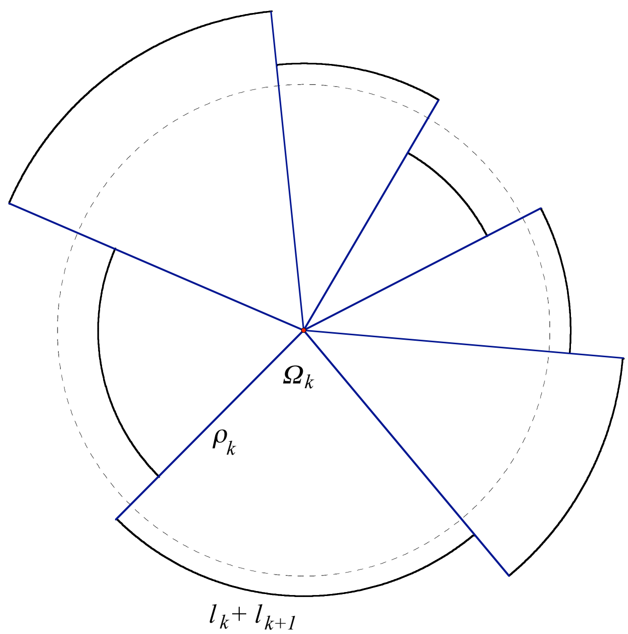
Corollary 2. *Let A be the area of a convex polygon. Then we have*

$$A \leq \frac{1}{2} \sum_{k=1}^{n} l_k \frac{\rho_k + \rho_{k+1}}{2}.$$

Proof. It is a direct consequence of the isoperimetric inequality

$$4\pi A - L^2 \leq 0. \qquad \square$$

Note 2. Note that the term $\frac{1}{2}\sum_{k=1}^{n} l_k \frac{\rho_k + \rho_{k+1}}{2}$ can be interpreted as the area of a rosette composed by isosceles triangles of sides ρ_k and angles $\alpha_k \pi$.



In particular, equality in Theorem 3 holds if and only if the rosette is a circle. The defect

$$\frac{1}{2} \sum_{k=1}^{n} l_k \frac{\rho_k + \rho_{k+1}}{2} - \frac{L^2}{4\pi}$$

coincides with the difference between the area of the rosette and the area of a circle of radius $L/2\pi$.

5 Approximation by polygons

In this section we justify why Theorem 3 can be viewed as a discrete version of Theorem 1.

Lemma 4. *Let C be a closed convex curve of class $\mathcal{C}^1$ in the plane. Let $l_k^{(n)}$ be the length of the side $P_k^{(n)}P_{k+1}^{(n)}$ of the n-th dyadic polygon $\mathcal{P}_n$ associated to C. Then, for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n > n_0$, and for all $k = 1, 2, \ldots, 2^n$ we have*

$$\left| 2^n \cdot \frac{l_k^{(n)}}{L} - 1 \right| < \epsilon. \tag{7}$$

Proof. The proof is standard, using the mean value theorem. □

Theorem 4. *Let $\gamma : [0, L] \longrightarrow \mathbb{R}^2$ be the parametrization by the arc length s of a closed strictly convex curve C of class $\mathcal{C}^1$. Let $l_k^{(n)}$ be the length of the side $P_k^{(n)}P_{k+1}^{(n)}$ of the n-th dyadic polygon $\mathcal{P}_n$, and let $\rho_k^{(n)}$ be the radius of curvature at the vertex $P_k^{(n)}$.*

Denoting by

$$\tilde{\rho_k}^{(n)} = \frac{\rho_k^{(n)} + \rho_{k+1}^{(n)}}{2}$$

the arithmetic mean of two consecutive radius of curvature, we have

$$\lim_{n\to\infty} \left(\sum_{k=1}^{2^n} l_k^{(n)} \tilde{\rho_k}^{(n)} \right) = \int_C \rho(s)\, ds. \tag{8}$$

Proof. By definition of Riemann integral, in order to prove equality (8) we must only prove

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \left(l_k^{(n)} \tilde{\rho_k}^{(n)} - \rho(s_k^{(n)}) L^{(n)} \right) = 0.$$

Equivalently

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} L^{(n)} \left(\frac{l_k^{(n)}}{L^{(n)}} \tilde{\rho_k}^{(n)} - \rho(s_k^{(n)}) \right) = 0.$$

Since

$$|\tilde{\rho_k}^{(n)} - \rho(s_k^{(n)})| = \frac{1}{2} \left| (\rho_k^{(n)} - \rho(s_k^{(n)})) + (\rho_{k+1}^{(n)} - \rho(s_{k+1}^{(n)})) + (\rho(s_{k+1}^{(n)}) - \rho(s_k^{(n)})) \right|$$

and $\rho(s)$ is continuous, we can assume, by Theorem 2, that for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that for all $n > n_\epsilon$,

$$|\tilde{\rho_k}^{(n)} - \rho(s_k^{(n)})| < \epsilon/2, \qquad k = 1, 2, \ldots, 2^n.$$

Note that the radii of curvature $\rho_k^{(n)}$ are uniformly bounded. In fact, since the given curve C is strictly convex, there is a constant M such that $0 < \rho(s) < M$. So it is clear that, for n big enough, there exists $N > 0$ such that $\rho_k^{(n)} < N$.

By Lemma 4, given $\epsilon > 0$, and for n big enough, we have

$$1 - \frac{\epsilon}{2N} < \frac{l_k^{(n)}}{L^{(n)}} < 1 + \frac{\epsilon}{2N}.$$

From this it follows easily that

$$-\epsilon < \frac{l_k^{(n)}}{L^{(n)}} \cdot \tilde{\rho_k}^{(n)} - \rho(s_k^{(n)}) < \epsilon,$$

and hence

$$\left| \sum_{k=1}^{2^n} L^{(n)} \left(\frac{l_k^{(n)}}{L^{(n)}} \tilde{\rho_k}^{(n)} - \rho(s_k^{(n)}) \right) \right| < L\epsilon,$$

and theorem is proved. □

Applying Theorem 3 to a sequence of dyadic polygons associated to a closed convex curve, and taking limits (the length of the polygons converge to the length of the curve) we obtain, by Theorem 4, a discrete proof of Theorem 1.

As this proof does not use Lemma 1, and each 2π-periodic function $p(\phi)$ with $p + p'' > 0$ is the support function of a convex set, we have given in fact a geometrical proof of this lemma. Note that the condition $p + p'' > 0$ is not a restriction since the addition of a constant to $p(\phi)$ leaves invariant the inequality of Lemma 1.

References

[1] Carlos A. Escudero and Agustí Reventós. An interesting property of the evolute. *Amer. Math. Monthly*, 114(7):623–628, 2007.

[2] Robert Osserman. Curvature in the eighties. *Amer. Math. Monthly*, 97(8):731–756, 1990.

[3] Antonio Ros. Compact hypersufaces with constant scalar curvature and congruence theorem. *J. of Diff. Geom.*, 27:215–220, 1988.

[4] Luis A. Santaló. *Integral geometry and geometric probability*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. With a foreword by Mark Kac, Encyclopedia of Mathematics and its Applications, Vol. 1.

[5] Sergei Tabachnikov. A four vertex theorem for polygons. *Amer. Math. Monthly*, 107(9):830–833, 2000.

Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona
Catalunya (Spain)

jcufi@mat.uab.cat, agusti@mat.uab.cat, crodri@mat.uab.cat.