Characterization of the weak foci and centers of quadratic systems via invariant theory

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Abstract

In this article the following problem is solved: Give necessary and sufficient conditions in order that a real quadratic differential system possesses a weak focus of order i, where $i \in \{1, 2, 3\}$, or a center. We note that such singular point does not need to be at the origin. The conditions are stated in terms of affine invariant polynomials in the 12-dimensional space of the coefficients and give the exact number of singularities of the above types that the differential system has.

1 Introduction and the statement of the main result

The notion of the center was introduced by Poincaré in his 1885 article [24]. Poincaré gave an algorithm for deciding when a polynomial differential system has a center. An algebraic version of this algorithm was stated by Shi Song Ling [32]. Placing the singular point at the origin, the necessary and sufficient conditions to have a center at the origin is the annihilation of an infinite number of polynomials in the coefficients of the system. In view of Hilbert's basis theorem, this amounts to the annihilation of only a finite number of them. These finite number of polynomials in the coefficients of the systems are called the Poincaré-Lyapunov constants as Lyapunov generalized Poincaré's results for analytic systems.

The next result on centers was Dulac's theorem [18] saying that a quadratic system possessing a center is integrable in finite terms. Dulac gave a finite number of conditions for such a system to have a center.

Dulac's notion of center is for complex systems: a non-degenerate singularity is a center if and only if the quotient λ of its eigenvalues is negative and rational and the system has a local nonconstant analytic first integral. Dulac had much insight in working with this notion. However, as his work was for complex systems, his canonical form for the case $\lambda = -1$ was for systems with a saddle and his conditions were not readily applicable to real systems. Kapteyn's work [20] dealt with real systems and he obtained conditions for the center in a more compact form for systems in the normal form which now bears his name.

In 1939, using Kapteyn's normal form, Bautin proved his now well known result [10] which says that a center in a quadratic system has cyclicity three . He gave necessary and sufficient conditions

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for a singular point of a quadratic system with a center for its linearization to have cyclicity one two and three.

In [33] (see also [34]) Sibirsky gave necessary and sufficient conditions for the origin to be a center (respectively a weak focus of given order i, where $i \in \{1, 2, 3\}$) in terms of algebraic invariants under the action of the linear group $GL(2, \mathbb{R})$.

The above mentioned set of conditions for the center was given for a system possessing a singular point placed at the origin. The question remained as to how to state conditions for a system to possess a center (respectively a weak focus of given order i, where $i \in \{1, 2, 3\}$) irrespectively of its position in the plane. In other words, how to formulate the conditions for existence of a center or weak focus independent of how system may be presented. This problem was solved for the case of the center in [14] and later the equivalent affine invariant conditions have been constructed in [35].

However in [14] the obtained conditions for the existence of one or two centers are associated to several canonical forms which were constructed using some invariant algebraic equalities and have not a relevant geometrical meaning. The conditions constructed in [35] are associated to the configurations of the finite singularities (real and/or complex, simple and/or multiple) and therefore there are needed additional conditions in order to determine such concrete configuration.

We remark that the class of quadratic systems with a weak focus at the origin of coordinates is examined by many authors. Thus in the paper [21] (respectively in [2], [3]) using the algebro-geometrical concepts of divisor and zero-cycle the topological classification of the class of quadratic systems possessing a weak focus of the third (respectively second) order is provided and the respective bifurcation diagrams are constructed. These classifications are based on the corresponding normal forms and in order to extend them in the whole space $\mathbb{R}^{12}$ of the coefficients of quadratic systems the affine invariant conditions for the existence of a weak focus of a given order are necessary.

Applying the theory of algebraic invariants of differential equations developed by K. Sibirsky and his disciples (cf. [34], [37], [25], [9], [17]) we propose here a complete characterization of weak foci and of centers for the class of non-degenerate quadratic systems. And this characterization is double global: the conditions are given in the whole coefficient space $\mathbb{R}^{12}$ and at the same time they describe the existence and the order of weak foci (and centers) arbitrarily located on the whole phase plane.

Our main result is the following one.

Main Theorem. *Consider a non-degenerate quadratic system.*

(a) *If* $\mathcal{T}_1 \neq 0$ *then this system has neither a weak focus nor a center.*

(b) *If* $\mathcal{T}_1 = 0$ *and* $\mathcal{T}_2 \neq 0$ *then this system has either one weak focus of the indicated order, or one center if and only if* $\mathcal{T}_2\mathcal{F} < 0$ *and the following respective set of conditions holds:*

$$
\begin{array}{llll}
(b_1) & \textit{1st order} & \Leftrightarrow & \mathcal{F}_1 \neq 0; \\
(b_2) & \textit{2nd order} & \Leftrightarrow & \mathcal{F}_1 = 0,\ \mathcal{F}_2 \neq 0; \\
(b_3) & \textit{3rd order} & \Leftrightarrow & \mathcal{F}_1 = \mathcal{F}_2 = 0,\ \mathcal{F}_3\mathcal{F}_4 \neq 0; \\
(b_4) & \textit{center} & \Leftrightarrow & \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3\mathcal{F}_4 = 0.
\end{array}
$$

(c) *If* $\mathcal{T}_1 = \mathcal{T}_2 = 0$ *and* $\mathcal{T}_3 \neq 0$ *then this system possesses either one or two weak foci of first order, or one or two centers if and only if* $\mathcal{F} = 0$, $\mathcal{E} < 0$ *and the following respective set of* *conditions holds:*

(c_1) *1 focus* $\Leftrightarrow$ $\mathcal{F}_1 \neq 0,\ \mathcal{T}_3 > 0;$
(c_2) *2 foci* $\Leftrightarrow$ $\mathcal{F}_1 \neq 0,\ \mathcal{T}_3 < 0,\ \mathcal{H} < 0;$
(c_3) *1 center* $\Leftrightarrow$ $\mathcal{F}_1 = 0,\ \mathcal{T}_3 > 0;$
(c_4) *2 centers* $\Leftrightarrow$ $\mathcal{F}_1 = 0,\ \mathcal{T}_3 < 0,\ \mathcal{H} < 0.$

(d) *If* $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = 0$ *and* $\mathcal{T}_4 \neq 0$ *then this system possesses either one weak focus of first order, or one center if and only if the following respective set of conditions holds:*

(d_1) *1 focus* $\Leftrightarrow$ $\mathcal{F}_1 \neq 0,\ \mathcal{F} = 0,\ \mathcal{H} < 0;$
(d_2) *1 center* $\Leftrightarrow$ $\mathcal{F}_1 = 0,\ \mathcal{F} = 0,\ \mathcal{H} < 0.$

(e) *If* $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = \mathcal{T}_4 = 0$ *and* $\sigma(a,x,y) \neq 0$ *then this system possesses either one weak focus of first order, or one center if and only if the following respective set of conditions holds:*

(e_1) *1 focus* $\Leftrightarrow$ $\mathcal{H} = \mathcal{E}_1 = 0,\ \mathcal{F}_1 \neq 0,\ \mathcal{E}_2 < 0;$
(e_2) *1 center* $\Leftrightarrow$ *either* $\mathcal{H} = \mathcal{E}_1 = \mathcal{F}_1 = 0,\ \mathcal{E}_2 < 0,$ *or* $\mathcal{H} < 0,\ \mathcal{E} < 0.$

(f) *If* $\sigma(a,x,y) = 0$ *then this system is Hamiltonian and it possesses two centers if and only if the conditions*

(f_1) $\mu_0 > 0,\ \mathbf{D} < 0,\ \mathbf{R} > 0,\ \mathbf{S} > 0$

hold; and it possesses one center if and only if one of the following sets of conditions holds:

(f_2) $\mu_0 < 0,\ \mathbf{D} < 0,\ \mathbf{R} > 0,\ \mathbf{S} > 0;$
(f_3) $\mu_0 > 0,\ \mathbf{D} > 0;$
(f_4) $\mu_0 > 0,\ \mathbf{D} = 0,\ \mathbf{T} < 0;$
(f_5) $\mu_0 > 0,\ \mathbf{D} = \mathbf{T} = \mathbf{P} = 0,\ \mathbf{R} \neq 0;$
(f_6) $\mu_0 = 0,\ \mathbf{D} < 0,\ \mathbf{R} \neq 0;$
(f_7) $\mu_0 = \mathbf{R} = 0,\ \mathbf{P} \neq 0,\ \mathbf{U} > 0,\ K = 0.$

Here the invariant polynomials are defined in Subsections 2.3 and 2.4.

This article is organized as follows:

In Section 2 we construct the necessary invariant polynomials and functions and prove some needed auxiliary results. More precisely, in Subsection 2.1 we define some new GL-invariants which are responsible for the existence of a weak focus and its order (or a center) at the origin of coordinates. These invariant polynomials are more convenient by two reasons: *(i)* they have served as a base for the construction of the respective affine invariant conditions; *(ii)* they could be applied to characterize also the degree of the weakness of the saddle at the origin (see Lemma 4.1).

In Subsection 2.2 the whole class of non-degenerate quadratic systems (2.1) is split in invariant way in several subfamilies according to the number and multiplicities of the finite singularities (real and/or complex, simple and/or multiple).

In Subsection 2.3 we define the *trace function* (see Definition 2.2) $\mathfrak{T}(w)$ and the associated affine invariants $\mathcal{T}_1$–$\mathcal{T}_4$ which are responsible for the number of finite singularities having zero traces. These invariant polynomials served as a fundament for the partition of the coefficient space $\mathbb{R}^{12}$ of quadratic systems (2.1), on which the Main Theorem is based.

Subsection 2.4 is dedicated to the construction of the affine invariant polynomials $\mathcal{F}_1$–$\mathcal{F}_4$ associated to Poincaré-Lyapunov constants. More precisely, when the singular point with zero trace is located at the origin of coordinate, the polynomials $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3\mathcal{F}_4$ become equivalent to $\mathcal{F}_1$–$\mathcal{F}_4$ Poincaré-Lyapunov constants (dual Poincaré-Lyapunov constants) if this point is a focus (respectively a saddle).

In Section 3 we prove the Main Theorem.

In Section 4 we discuss the invariant criteria for the existence of weak saddles. We prove that the GL-invariants constructed in Subsection 2.1 indeed characterize the weak saddle located at the origin of coordinate. We show that the statement (b) of Main Theorem could be generalized and formulated for both types of singularities (foci and saddles, see Theorem 4.1). In such a way the necessary and sufficient affine invariant conditions for the existence of a saddle of the order i ($i \in \{2, 3\}$) are also determined.

2 Preliminary

Consider real quadratic systems of the form:

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dx}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y) \end{aligned} \tag{2.1}$$

with homogeneous polynomials p_i and q_i $(i = 0, 1, 2)$ of degree i in x, y:

$$\begin{aligned} p_0 = a_{00}, \quad & p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) + a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 = a_{00}, \quad & q_1(x,y) = a_{10}x + a_{01}y, \quad q_2(x,y) + a_{20}x^2 + 2a_{11}xy + a_{02}y^2. \end{aligned}$$

Let $\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ be the 12-tuple of the coefficients of system (2.1) and denote $\mathbb{R}[\tilde{a}, x, y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, x, y]$.

2.1 Local GL-invariant conditions

In [34] the necessary and sufficient GL-invariant conditions for the existence at the origin of coordinates of a weak focus of the given order or a center are constructed. However we shall construct here other GL-invariant conditions which are equivalent to them from [34, Theorem 34.3] but are more convenient by two reasons: *(i)* the defined GL-invariants could serve as a base for the construction of the respective affine invariant conditions; *(ii)* they could be also applied to characterize the degree of weakness of the saddle at the origin of coordinate (see Lemma 4.1).

We single out the following five polynomials, basic ingredients in constructing invariant polynomials for systems (2.1):

$$\begin{aligned} C_i(\tilde{a},x,y) &= yp_i(x,y) - xq_i(x,y), \ (i = 0,1,2) \\ D_i(\tilde{a},x,y) &= \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y}, \ (i = 1,2). \end{aligned} \tag{2.2}$$

Using the so-called *transvectant of index k* (see [22]) of two polynomials f, $g \in \mathbb{R}[\tilde{a}, x, y]$

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \ \frac{\partial^k g}{\partial x^h \partial y^{k-h}},$$

we construct the following GL—comitants of the second degree with respect to the coefficients of the initial system

$$
\begin{array}{lll}
T_1 = (C_0, C_1)^{(1)}, & T_2 = (C_0, C_2)^{(1)}, & T_3 = (C_0, D_2)^{(1)}, \\
T_4 = (C_1, C_1)^{(2)}, & T_5 = (C_1, C_2)^{(1)}, & T_6 = (C_1, C_2)^{(2)}, \\
T_7 = (C_1, D_2)^{(1)}, & T_8 = (C_2, C_2)^{(2)}, & T_9 = (C_2, D_2)^{(1)}.
\end{array}
\tag{2.3}
$$

Denoting $\mathcal{A} = \begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix}$ we define the following GL-invariants:

$$
\begin{aligned}
G_0 &= \frac{\partial p_1}{\partial x} + \frac{\partial q_1}{\partial y} \equiv D_1(a) = \mathrm{tr}\,(\mathcal{A}), \\
G_1 &= (C_1 T_7,\, C_2)^{(3)}, \\
G_2 &= \big(5(T_8,\, T_6)^{(1)} - 2\big(7T_8 + 8T_9,\, T_7\big)^{(1)},\, D_2\big)^{(1)}, \\
G_3 &= \big((T_8,\, T_6)^{(1)} + 2(T_8 + 8T_9,\, T_7)^{(1)},\, D_2\big)^{(1)}, \\
G_4 &= T_4\left[(T_8,\, C_1)^{(2)} + 8(T_7 - T_6,\, D_2)^{(1)}\right] - 3\big((T_6,\, C_1)^{(1)},\, T_6\big)^{(1)}, \\
G_5 &= (p_1,\, q_1)^{(1)} = \det(\mathcal{A}).
\end{aligned}
\tag{2.4}
$$

In what follows we shall use the next two useful lemmas.

Lemma 2.1. *A quadratic system*

$$
\begin{aligned}
\dot{x} &= a_{10}x + a_{01}y + a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\
\dot{y} &= b_{10}x + b_{01}y + b_{20}x^2 + 2b_{11}xy + b_{02}y^2,
\end{aligned}
\tag{2.5}
$$

has at the origin of coordinates a weak focus of the the indicated order or a center if and only if $G_0 = 0$, $G_5 > 0$ *and the following respective set of conditions holds:*

$$
\begin{array}{lll}
\textit{1st order} & \Leftrightarrow & G_1 \neq 0; \\
\textit{2nd order} & \Leftrightarrow & G_1 = 0,\ G_2 \neq 0; \\
\textit{3rd order} & \Leftrightarrow & G_1 = G_2 = 0,\ G_3 G_4 \neq 0; \\
\textit{center} & \Leftrightarrow & G_1 = G_2 = G_3 G_4 = 0.
\end{array}
\tag{2.6}
$$

Proof: It is known that the matrix $\mathcal{A}$ via a linear transformation and a time rescaling can be brought to the form $a_{10} = 0$, $a_{01} = -1$, $b_{10} = 1$ and $b_{01} = 0$ if and only if $G_0 = \mathrm{tr}\,(\mathcal{A}) = 0$ and $G_5 = \det(\mathcal{A}) > 0$ So, in what follows we assume that these conditions hold. Then the corresponding eigenvalues of the matrix $\mathcal{A}$ are purely imaginary and according to [10] systems (2.5) via a linear transformation and time rescaling can be brought to the systems

$$
\begin{aligned}
\dot{x} &= -y - mx^2 + (2a + l)xy + ny^2, \\
\dot{y} &= x + ax^2 + (2m + b)xy - ay^2,
\end{aligned}
\tag{2.7}
$$

for which the Lyapunov quantities are as follows:

$$
\begin{aligned}
L_1 &= l(n - m), \\
L_2 &= ab(n - m)(5n - 5m - b), \\
L_3 &= ab(m - n)^2[a^2 + n(2n - m)].
\end{aligned}
$$

On the other hand for system (2.7) calculations yield $G_0 = 0$, $G_5 = 1 > 0$ and

$$\begin{aligned}
&G_1 = 72l(n-m) = 72L_1,\\
&G_2 = 864(n-m)\big[10l(a^2+m^2) + b^2(a+l) + 5ab(m-n) + l(4al+7bm)\big],\\
&G_3 = 864(n-m)\big[b^2(a+l) + ab(m-n) + 2l(a^2+m^2) + 3blm\big],\\
&G_4 = 288\big[(4a+l)^2 + (b+5m-n)(b+m+3n)\big].
\end{aligned} \tag{2.8}$$

Evidently, the condition $G_1 \neq 0$ is equivalent to $L_1 \neq 0$. Assume $G_1 = 0$, i.e. $l(n-m) = 0$. The condition $m - n = 0$ yields $G_1 = G_2 = G_3G_4 = 0$ and simultaneously $L_1 = L_2 = L_3 = 0$, i.e. we have a center. If $m - n \neq 0$ then $l = 0$ and considering (2.8) we obtain

$$G_2 = -864L_2, \qquad G_3G_4 = -2^{10}3^5\big[64L_3 - L_2\big(16a^2 + (m-5n)^2 + b^2 + 2bm\big)\big]$$

and this completes the proof of Lemma 2.1. ∎

Lemma 2.2. *Assume that a quadratic system* (2.1) *possesses two real singular points* M_1 *and* M_2 *with zero traces, i.e.* $\mathrm{tr}\,(\mathcal{A}^{(1)}) = \mathrm{tr}\,(\mathcal{A}^{(2)}) = 0$. *Then if* $\det(\mathcal{A}^{(1)}) > 0$ *(respectively* $\det(\mathcal{A}^{(2)}) > 0$*) the point* M_1 *(respectively* M_2*) is a weak focus of the first order if* $\mathcal{F}_1 \neq 0$ *and it is a center if* $\mathcal{F}_1 = 0$, *where* $\mathcal{F}_1$ *is the affine invariant defined in* (2.21).

Proof: If a quadratic system (2.1) possesses two real singular points then via an affine transformation these points can be localized at $M_1(0,0)$ and $M_2(1,0)$, respectively. In this case we obtain the systems

$$\begin{aligned}
\dot{x} &= cx + dy - cx^2 + 2hxy + ky^2,\\
\dot{y} &= ex + fy - ex^2 + 2mxy + ny^2,
\end{aligned} \tag{2.9}$$

and for the singular points $M_1(0,0)$ and $M_2(1,0)$ we have $\rho_1 = c + f$ and $\rho_2 = -c + f + 2m$. Hence the conditions $\rho_1 = \rho_2 = 0$ yield $f = -c$ and $m = c$.

In order to apply the local conditions of Lemma 2.1 we shall examine the systems above in two forms: one of them having at the origin the point M_1 and another one having at the origin the point M_2. The first form evidently will be obtained from (2.9) by setting $f = -c$ and $m = c$:

$$\begin{aligned}
\dot{x} &= cx + dy - cx^2 + 2hxy + ky^2,\\
\dot{y} &= ex - cy - ex^2 + 2cxy + ny^2.
\end{aligned} \tag{2.10}$$

For these systems calculations yield:

$$\begin{aligned}
&G_0^{(1)} = 0, \quad G_1^{(1)} = -144(h+n)(2c^3 - 2ceh - e^2k + cen) = -72\mathcal{F}_1,\\
&G_2^{(1)} = 6(5d + 8h + 3n)\mathcal{F}_1, \quad G_3^{(1)} = 6(d-n)\mathcal{F}_1,\\
&G_5^{(1)} = -(c^2 + de) = \det\big(\mathcal{A}^{(1)}\big)
\end{aligned}$$

Replacing the point $M_2(1,0)$ at the origin due to a translation systems (2.10) become:

$$\begin{aligned}
\dot{x} &= -cx + (d+2h)y - cx^2 + 2hxy + ky^2,\\
\dot{y} &= -ex + cy - ex^2 + 2cxy + ny^2,
\end{aligned} \tag{2.11}$$

and for these systems we calculate again the values of G_i, attached to the point M_2 (which is located at the origin of coordinate now):

$$\begin{aligned}
&G_0^{(2)} = 0, \quad G_1^{(2)} = -144(h+n)(2c^3 - 2ceh - e^2k + cen) = -72\mathcal{F}_1,\\
&G_2^{(2)} = -12(3n - 5d - 2h)\mathcal{F}_1, \quad G_3^{(2)} = 12(d + 2h + n)\mathcal{F}_1,\\
&G_5^{(2)} = -c^2 + de + 2eh = \det\big(\mathcal{A}^{(2)}\big).
\end{aligned}$$

We observe that $\mathcal{F}_1 \neq 0$ implies $G_1^{(1)} G_1^{(2)} \neq 0$ whereas $\mathcal{F}_1 = 0$ implies $G_i^{(1)} = G_i^{(2)} = 0$, $i = 0, 1, 2, 3$. So according to Lemma 2.1 if $\det\big(\mathcal{A}^{(i)}\big) > 0$ $(i = 1, 2)$ then the singular point M_i is a weak focus of the first order if $\mathcal{F}_1 \neq 0$ and it is a center if $\mathcal{F}_1 = 0$. ■

Remark 2.1. *We note that one of the points either M_1 or M_2 could be not simple (for example $\det\big(\mathcal{A}^{(1)}\big) = 0$) but the statement of Lemma 2.2 regarding the point M_2 is still valid.*

In what follows the next remark will be useful.

Remark 2.2. *Assume that we have obtained a normal form of a family of quadratic systems moving a simple singular point to the origin of coordinates and fixing the position of all other singular points, even depending on some parameters. Then any geometrical propriety of any simply finite singular point can be considered to be hold by the origin point.*

2.2 Canonical forms associated to the finite singularities

We shall use the notion of zero–cycle in order to describe the number and multiplicity of singular points of a quadratic system. This notion as well as its particular case, the notion of *divisor*, were used for classification purposes of planar quadratic differential systems by Pal and Schlomiuk [23], Llibre and Schlomiuk [21], Schlomiuk and Vulpe [27] and by Artes and Llibre and Schlomiuk [3].

Definition 2.1. *We consider formal expressions $\mathcal{D} = \sum n(w)w$ where $n(w)$ is an integer and only a finite number of $n(w)$ are nonzero. Such an expression is called a zero–cycle of $\mathbf{P}_2(\mathbb{C})$ if all w appearing in $\mathcal{D}$ are points of $\mathbf{P}_2(\mathbb{C})$. We call degree of the zero–cycle $\mathcal{D}$ the integer $\deg(\mathcal{D}) = \sum n(w)$. We call support of $\mathcal{D}$ the set $\mathrm{Supp}\,(\mathcal{D})$ of w's appearing in $\mathcal{D}$ such that $n(w) \neq 0$.*

We note that $\mathbf{P}_2(\mathbb{C})$ denotes the complex projective space of dimension 2. For a system (S) belonging to the family (2.1) we denote $\nu(P,Q) = \{w \in \mathbb{C}_2 \mid P(w) = Q(w) = 0\}$ and we define the following zero–cycle $\mathcal{D}_S(P,Q) = \sum_{w\in\nu(P,Q)} I_w(P,Q)w$, where $I_w(P,Q)$ is the intersection number or multiplicity of intersection at w. It is clear that for a non–degenerate quadratic system $\deg(\mathcal{D}_S) \leq 4$ as well as $\mathrm{Supp}\,(\mathcal{D}_S) \leq 4$. For the degenerate system the zero–cycle $\mathcal{D}_S(P,Q)$ is undefined.

Consider now the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ (see [7]) acting on $\mathbb{R}[\tilde{a}, x, y]$, where

$$\mathbf{L}_1 = 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}};$$
$$\mathbf{L}_2 = 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}.$$

Using this operator we construct the following important set of invariant polynomials:

$$\begin{aligned} \mu_0(\tilde{a}) \;&=\; \mathrm{Res}_{\,x}\big(p_2(x,y), q_2(x,y)\big)/y^4, \\ \mu_i(\tilde{a},x,y) &= \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\;\; i = 1,..,4, \end{aligned} \tag{2.12}$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$. These polynomials are in fact invariant polynomials of systems (2.1) with respect to the group $GL(2,\mathbb{R})$ (see [7]). Their geometrical meaning is revealed in the following two lemmas:

Lemma 2.3. ([7]) *The total multiplicity of all finite singularities of a quadratic system* (2.1) *equals k if and only if for every $i \in \{0, 1, \ldots, k-1\}$ we have $\mu_i(\tilde{a}, x, y) = 0$ in $\mathbb{R}[\tilde{a}, x, y]$ and $\mu_k(\tilde{a}, x, y) \neq 0$. Moreover a system* (2.1) *is degenerate (i.e.* $\gcd(P,Q) \neq \mathrm{constant}$*) if and only if $\mu_i(\tilde{a}, x, y) = 0$ in $\mathbb{R}[x, y]$ for every $i = 0, 1, 2, 3, 4$.*

Lemma 2.4. ([8]) The point $M_0(0,0)$ is a singular point of multiplicity k $(1 \leq k \leq 4)$ for a quadratic system (2.1) if and only if for every $i \in \{0, 1, \ldots, k-1\}$ we have $\mu_{4-i}(\tilde{a}, x, y) = 0$ in $\mathbb{R}[\tilde{a}, x, y]$ and $\mu_{4-k}(\tilde{a}, x, y) \neq 0$.

Using the invariant polynomials μ_i $(i = 0, 1, \ldots, 4)$ we could construct the invariant polynomials $\mathbf{D}$, $\mathbf{P}$, $\mathbf{R}$, $\mathbf{S}$, $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$, which are responsible for the number and multiplicities of finite singularities of a non-degenerate quadratic system. We note that these polynomials were constructed (using another way) and applied in [6, 7] (see also [4]). Here they are constructed as follows:

$$
\begin{aligned}
\mathbf{D} =& 3\big((\mu_3, \mu_3)^{(2)},\ \mu_2\big)^{(2)} - \big(6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\ \mu_4\big)^{(4)},\\
\mathbf{P} =& \mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\\
\mathbf{R} =& 3\mu_1^2 - 8\mu_0\mu_2,\\
\mathbf{S} =& \mathbf{R}^2 - 16\mu_0^2\mathbf{P},\\
\mathbf{T} =& 18\mu_0^2(3\mu_3^2 - 8\mu_2\mu_4) + 3\mu_0(2\mu_2^3 - 9\mu_1\mu_2\mu_3 + 27\mu_1^2\mu_4) - \mathbf{PR},\\
\mathbf{U} =& \mu_3^2 - 4\mu_2\mu_4,\\
\mathbf{V} =& \mu_4.
\end{aligned}
\tag{2.13}
$$

The geometrical meaning of the invariant polynomials above is revealed in the next proposition:

Proposition 2.1. ([7]) *Necessary and sufficient conditions for the number and multiplicities of finite singular points of a non-degenerate quadratic system* (2.1) *are given by Table 1.*

Using this partition of the coefficient space $\mathbb{R}^{12}$ of the family of non-degenerate quadratic systems (2.1) the respective canonical forms could be associated. We have the next result.

Table 1

No.	Zero–cycle $\mathcal{D}_S(P,Q)$	Invariant criteria	No.	Zero–cycle $\mathcal{D}_S(P,Q)$	Invariant criteria
1	$p+q+r+s$	$\mu_0 \neq 0, \mathbf{D} < 0, \mathbf{R} > 0, \mathbf{S} > 0$	10	$p+q+r$	$\mu = 0, \mathbf{D} < 0, \mathbf{R} \neq 0$
2	$p+q+r^c+s^c$	$\mu_0 \neq 0, \mathbf{D} > 0$	11	$p+q^c+r^c$	$\mu = 0, \mathbf{D} > 0, \mathbf{R} \neq 0$
3	$p^c+q^c+r^c+s^c$	$\mu_0 \neq 0, \mathbf{D} < 0, \mathbf{R} \leq 0$ $\mu_0 \neq 0, \mathbf{D} < 0, \mathbf{S} \leq 0$	12	$2p+q$	$\mu = \mathbf{D} = 0, \mathbf{PR} \neq 0$
4	$2p+q+r$	$\mu \neq 0, \mathbf{D} = 0, \mathbf{T} < 0$	13	$3p$	$\mu_0 = \mathbf{D} = \mathbf{P} = 0, \mathbf{R} \neq 0$
5	$2p+q^c+r^c$	$\mu_0 \neq 0, \mathbf{D} = 0, \mathbf{T} > 0$	14	$p+q$	$\mu_0 = \mathbf{R} = 0, \mathbf{P} \neq 0, \mathbf{U} > 0$
6	$2p+2q$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{PR} > 0$	15	p^c+q^c	$\mu_0 = \mathbf{R} = 0, \mathbf{P} \neq 0, \mathbf{U} < 0$
7	$2p^c+2q^c$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{PR} < 0$	16	$2p$	$\mu_0 = \mathbf{R} = 0, \mathbf{P} \neq 0, \mathbf{U} = 0$
8	$3p+q$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{P} = 0, \mathbf{R} \neq 0$	17	p	$\mu_0 = \mathbf{R} = 0, \mathbf{P} = 0, \mathbf{U} \neq 0$
9	$4p$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{P} = \mathbf{R} = 0$	18	0	$\mu_0 = \mathbf{R} = \mathbf{P} = 0, \mathbf{U} = 0, \mathbf{V} \neq 0$

Proposition 2.2. *Any non-degenerate quadratic system in dependence of the number and multiplicities of finite singularities could be brought via an affine transformation and time rescaling to one of the canonical forms given by Table 2.*

Table 2

Value of $D_S(P,Q)$		Canonical system	Finite singularities and conditions on parameters
$p+q+$ $r+s$	1)	$\dot{x}=cx+dy-cx^2+2hxy-dy^2,$ $\dot{y}=ex+fy-ex^2+2mxy-fy^2$	$M_1(0,0), M_2(1,0), M_3(0,1),$ $M_4\Big(\frac{\beta(\beta-2\gamma)}{\beta^2-4\alpha\gamma},\ \frac{\beta(\beta-2\alpha)}{\beta^2-4\alpha\gamma}\Big)$, where $\alpha=eh-cm,\quad \beta=cf-de,$ $\gamma=dm-fh;\ \ \beta^2-4\alpha\gamma\neq 0$
$p+q+$ r^c+s^c	2)	$\dot{x}=a-(a+g)x+gx^2+2hxy+ay^2,$ $\dot{y}=b-(b+l)x+lx^2+2mxy+by^2$	$M_1(0,i),\ M_2(0,-i),\ M_3(1,0),$ $M_4\Big(\frac{\beta^2+4\gamma^2}{\beta^2-4\alpha\gamma},\ \frac{2\beta(\alpha+\gamma)}{4\alpha\gamma-\beta^2}\Big)$, where $\alpha=gm-hl,\quad \beta=bg-al,$ $\gamma=bh-am;\ \ \beta^2-4\alpha\gamma\neq 0$
p^c+q^c+ r^c+s^c	$3a)$	$\dot{x}=\frac{g(u^2+1)}{v^2+1}(1+y^2)-2(gu+hv)x$ $+gx^2+2hxy,\ \dot{y}=\frac{l(u^2+1)}{v^2+1}(1+y^2)$ $-2(lu+mv)x+lx^2+2mxy$	$M_{1,2}(0,\pm i),\ M_{3,4}(u\pm i,v)$ $gm-lh\neq 0$
	$3b)$	$\dot{x}=a+[a(u^2-1)-g]x+gx^2+ay^2,$ $\dot{y}=b+[b(u^2-1)-l]x+lx^2+by^2$	$M_{1,2}(0,\pm i),\ M_{3,4}(1,\pm iu)$ $al-bg\neq 0$
$2p+q+r$	4)	$\dot{x}=cx+cuy-cx^2+2hxy-cuy^2,$ $\dot{y}=ex+euy-ex^2+2mxy-euy^2$	$M_{1,2}(0,0)-double, M_3(1,0), M_4(0,1)$ $u(cm-eh)\neq 0$
$2p+$ q^c+r^c	5)	$\dot{x}=cm\,x+cn\,y+g\,x^2-cn\,xy+$ $+(g+cm)\,y^2,\ \ \dot{y}=em\,x+en\,y+$ $+l\,x^2-en\,xy+(l+em)\,y^2$	$M_{1,2}(0,0)-double,\ M_{3,4}(1,\pm i)$ $(cl-eg)(m^2+n^2)\neq 0$
$2p+2q$	6)	$\dot{x}=cx+cuy-cx^2+2cvxy+ky^2,$ $\dot{y}=ex+euy-ex^2+2evxy+ny^2$	$M_{1,2}(0,0),\ M_{3,4}(1,0)$ $cn-ek\neq 0$
$2p^c+2q^c$	7)	$\dot{x}=a+aux+gx^2+2avxy+ay^2,$ $\dot{y}=b+bux+lx^2+bvxy+by^2$	$M_{1,2}(0,i),\ M_{3,4}(0,-i)$ $al-bg\neq 0$
$3p+q$	8)	$\dot{x}=cx+cuy-cx^2+2hxy+$ $+(2hu+cv)y^2,\ \ \dot{y}=ex+euy-$ $-ex^2+2mxy+(2mu+ev)y^2$	$M_{1,2,3}(0,0)$*–triple*, $M_4(1,0)$ $(u^2-v)(cm-eh)\neq 0$
$4p$	$9a)$	$\dot{x}=ky^2,$ $\dot{y}=fy+lx^2+2mxy+ny^2$	$M_0(0,0)$*–multiplicity 4* $fkl\neq 0$
	$9b)$	$\dot{x}=dy+gx^2+2hxy+ky^2,$ $\dot{y}=ny^2$	$M_0(0,0)$*–multiplicity 4* $dgn\neq 0$
	$9c)$	$\dot{x}=gx^2+2hxy,$ $\dot{y}=lx^2+2mxy+ny^2$	$M_0(0,0)$*–multiplicity 4* $g^2n^2-4hn(gm-hl)\neq 0$

Proof: All the canonical forms corresponding to the configurations of finite singularities given by Table 1, were constructed in [4] and [5] except the configurations corresponding to the cases 9 ($\mathcal{D}_S(P,Q)=4p$) and 18 ($\mathcal{D}_S(P,Q)=0$). We note that some of canonical forms corresponding to

Table 2 *(continued)*

Value of $D_S(P,Q)$	Canonical system		Finite singularities and conditions on parameters
$p+q+r$	10)	$\dot{x} = cx+dy-cx^2+2hxy,$ $\dot{y} = ex+fy-ex^2+2mxy$	$M_1(0,0), M_2(1,0),$ $M_4\Big(\frac{\beta}{2\alpha}, \frac{\beta(2\alpha-\beta)}{4\alpha\gamma}\Big)$, where $\alpha = eh-cm, \quad \beta = cf-de,$ $\gamma = dm-fh; \alpha\beta\gamma(2\alpha-\beta)\neq 0$
$p+q^c+r^c$	11)	$\dot{x} = 2(h-gu)x+g(u^2+1)y+$ $+gx^2-2hxy, \ \dot{y} = 2(m-lu)x+$ $+l(u^2+1)y+lx^2-2mxy$	$M_1(0,0), \quad M_{2,3}(u\pm i,1)$ $gm-hl\neq 0$
$2p+q$	12)	$\dot{x} = cx+cuy-cx^2+2hxy,$ $\dot{y} = ex+euy-ex^2+2mxy$	$M_{1,2}(0,0)-double, M_3(1,0)$ $u(cm-eh)\neq 0$
$3p$	13)	$\dot{x} = guy+gx^2+2hxy,$ $\dot{y} = luy+lx^2+2mxy$	$M_{1,2,3}(0,0)-triple$ $u(gm-lh)\neq 0$
$p+q$	14a)	$\dot{x} = cx+dy-cx^2+2duxy,$ $\dot{y} = ex+fy-ex^2+2fuxy$	$M_1(0,0), \quad M_2(1,0)$ $(cf-de)(2u+1)\neq 0$
	14b)	$\dot{x} = -(g+ku^2)x+-2hy+gx^2+$ $+2hxy+ky^2, \quad \dot{y} = ux+y$	$M_1(0,0), \quad M_2(1,-u)$ $g-2hu+ku^2\neq 0$
p^c+q^c	15a)	$\dot{x} = a+hux+2hxy+ay^2,$ $\dot{y} = b+mux+2mxy+by^2$	$M_{1,2}(0,\pm i)$ $am-bh\neq 0$
	15b)	$\dot{x} = a+cx+gx^2+2hxy+ay^2, \ \dot{y} = ex$	$M_{1,2}(0,\pm i) \ \ ae\neq 0$
$2p$	16a)	$\dot{x} = dy+gx^2+2duxy,$ $\dot{y} = fy+lx^2+2fuxy$	$M_{1,2}(0,0)-double$ $u(fg-dl)\neq 0$
	16b)	$\dot{x} = cx+dy, \ \dot{y} = lx^2+2mxy+ny^2$	$M_{1,2}(0,0)-double$ $c^2n-2cdm+dl^2\neq 0$
p	17a)	$\dot{x} = cx+dy+(2c+du)x^2+2dxy,$ $\dot{y} = ex+fy+(2e+fu)x^2+2fxy$	$M_1(0,0)$ $u(cf-de)\neq 0$
	17b)	$\dot{x} = x+dy, \quad \dot{y} = ex+fy+$ $+lx^2+2mxy+(2dm-ld^2)y^2$	$M_1(0,0)$ $de-f\neq 0$
0	18a)	$\dot{x} = hu+gx^2+2hxy,$ $\dot{y} = mu+lx^2+2mxy$	$u(hl-gm)\neq 0$
	18b)	$\dot{x} = y, \ \ \dot{y} = 1+fy+2mxy+ny^2$	$m^2+n^2\neq 0$
	18c)	$\dot{x} = x, \quad \dot{y} = 1+lx^2+2mxy$	$l^2+m^2\neq 0$
	18d)	$\dot{x} = 1, \quad \dot{y} = ex+fy+$ $+lx^2+2mxy+ny^2$	$(e,f\in\{0,1\})$ $l^2+m^2+n^2\neq 0$

these cases (and to others from Table 1) were constructed earlier in different papers (see for example [11, 12, 13]. However we shall construct here the needed canonical forms more convenient for our propose.

1) *Systems with zero-cycle* $\mathcal{D}_S(P,Q)=4d$. In this case systems have one finite singular point of multiplicity 4 and via a translation we may locate this singular at the origin of coordinates. Clearly we could have either a semi-elementary (with one non-zero eigenvalue), or a non-elementary (with two zero eigenvalues) singular point .

***a*)** In the case of semi-elementary singular point it is known (cf. for example,[1]) that via a linear transformation a quadratic system in this case can be transformed to the canonical

$$\dot{x} = gx^2 + 2hxy + ky^2, \quad \dot{y} = fy + lx^2 + 2mxy + ny^2,$$

with parameter $f \neq 0$. By Lemma 2.4 the singular point $M_0(0,0)$ has multiplicity 4 if and only if $\mu_4 = \mu_3 = \mu_2 = \mu_1 = 0$ and $\mu_0 \neq 0$ For the systems above we calculate

$$\mu_4 = \mu_3 = 0, \quad \mu_2 = f^2 g(gx^2 + 2hxy + ky^2)$$

and since $f \neq 0$ the condition $\mu_2 = 0$ yields $g = 0$. Then we have:

$$\mu_1 = 2fhl(2hx + ky) = 0, \quad \mu_0 = l(k^2 l - 4hkm + 4h^2 n) \neq 0$$

that implies $h = 0$ and then we have $\mu_0 = k^2 l^2 \neq 0$. This leads to the canonical form $9a$) from Table 2.

***b*)** Assume that the singular point $M_0(0,0)$ is non-elementary. If these systems are homogeneous then the canonical form is trivial and could be considered as $19c$).

Suppose finally, that linear matrix is not zero identically. Then via a linear transformation these systems can be brought to the form

$$\dot{x} = dy + gx^2 + 2hxy + ky^2, \quad \dot{y} = lx^2 + 2mxy + ny^2$$

with $d \neq 0$. Considering Lemma 2.4 we calculate

$$\mu_4 = \mu_3 = 0, \quad \mu_2 = ld^2(lx^2 + 2mxy + ny^2)$$

and the condition $\mu_2 = 0$ yields $l = 0$. Then we have:

$$\mu_1 = 2dgm(2mx + ny) = 0, \quad \mu_0 = g(4km^2 - 4hmn + gn^2)$$

and due to $d\mu_0 \neq 0$ we obtain $m = 0$. Then $\mu_0 = g^2 n^2 \neq 0$ and this leads to the canonical form $9b$) from Table 2.

2) *Systems with zero-cycle* $\mathcal{D}_S(P,Q) = 0$. In this case systems have no finite singular points (i.e. all finite singularities have gone to infinity) and hence the total multiplicity equals zero. According to Lemma 2.3 in this case the conditions $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ and $\mu_4 \neq 0$ have to be satisfied. We note that according to (2.12) the polynomial μ_0 is the resultant of the binary forms $p_2(x,y)$ and $q_2(x,y)$. So the condition $\mu_0 = 0$ implies that these two binary forms has a non-constant common factor of degree at least one.

On the other hand denoting $K = \text{Jacob}\left(p_2, q_2\right)$ we conclude that this factor is of degree exactly one if $K \neq 0$ and it is of degree two (i.e. $p_2(x,y)$ and $q_2(x,y)$ are proportional) if $K = 0$. We shall examine both these cases.

***a*)** Assume first $K \neq 0$, i.e. the common factor of $p_2(x,y)$ and $q_2(x,y)$ is of degree one. Then via a linear transformation we can assume that this common factor is x, i.e. in systems (2.1) we have $a_{02} = b_{02} = 0$. Therefore these systems become

$$\dot{x} = a + cx + dy + gx^2 + 2hxy, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy \tag{2.14}$$

for which $K=(gm-hl)x^2\neq 0$. Then we can assume $c=e=0$ via the translation

$$x\to x+\frac{eh-cm}{2(gm-hl)},\quad y\to y+\frac{cl-eg}{2(gm-hl)}$$

and for systems (2.14) calculation yields: $\mu_0=0$ and $\mu_1=4(fh-dm)(hl-gm)x$. As $K\neq 0$ the condition $\mu_1=0$ yields $fh-dm=0$. Due to the condition $h^2+m^2\neq 0$ (as $K\neq 0$) without loss of generality we may set a new parameter u as follows: $f=um$ and $d=uh$. Then we have

$$\mu_2=(hl-gm)\big[4(bh-am)+u^2(hl-gm)\big]x^2,\quad \mu_3=2u(hl-gm)(al-bg)x^3$$

and due to $K\neq 0$ the condition $\mu_3=0$ yields $u(al-bg)=0$.

We claim that in order to satisfy the conditions $\mu_2=\mu_3=0$ and $\mu_4\neq 0$ it is necessary $u=0$. Indeed, supposing that $u\neq 0$ we obtain $al=bg$ and since $g^2+l^2\neq 0$ (as $K\neq 0$) we may set a new parameter v as follows: $a=gv$, $b=lv$. Then calculations yield

$$\mu_2=(hl-gm)^2(u^2+4v)x^2,\quad \mu_4=v(hl-gm)^2(u^2+4v)x^2y^2$$

and evidently the condition $\mu_2=0$ yields $\mu_4=0$. This proves our claim.

So $u=0$ and we have $\mu_3=0$. At the same time the condition $\mu_2=0$ yields $bh-am=0$. So we set again the parameter u as follows: $a=hu$, $b=mu$ and this leads to the canonical form 18*a*) from Table 2.

***b*)** Suppose now $K=0$, i.e. the polynomials $p_2(x,y)$ and $q_2(x,y)$ are proportional and via a linear transformation we obtain the system

$$\dot{x}=a+cx+dy,\quad \dot{y}=b+ex+fy+lx^2+2mxy+ny^2. \tag{2.15}$$

We shall consider two subcases: $d\neq 0$ and $d=0$.

***b*$_1$)** If $d\neq 0$ then via the affine transformation $x_1=x$, $y_1=cx+dy+a$ systems (2.15) will be brought to the systems:

$$\dot{x}=y,\quad \dot{y}=b+ex+fy+lx^2+2mxy+ny^2,$$

for which calculations yield: $\mu_0=\mu_1=0$, $\mu_2=l(lx^2+2mxy+ny^2)$. Therefore, the condition $\mu_2=0$ implies $l=0$ and then we have: $\mu_3=-e(2mx+ny)y^2$, $\mu_4=b(2mx+ny)y^3$. So due to $\mu_4\neq 0$ the condition $\mu_3=0$ yields $e=0$.

It remains to note that as $b\neq 0$ we may assume $b=1$ due to the rescaling $(x,y)\mapsto(bx,by)$ and this leads to the systems 18*b*) (see Table 2).

***b*$_2$)** Assume now $d=0$. Then for the systems (2.15) we calculate $\mu_2=c^2n(lx^2+2mxy+ny^2)=0$.

If $c\neq 0$ then $n=0$ and we may consider $c=1$ and $a=0$ via a time rescaling and a translation. This leads to the systems

$$\dot{x}=x,\quad \dot{y}=b+ex+fy+lx^2+2mxy,$$

for which calculations yield: $\mu_3=f(lx+2my)x^2$, $\mu_4=b(lx+2my)x^3$. Therefore from $\mu_3=0$ it follows $f=0$ and assuming $b=1$ (by the same reasons as above) we get the canonical form 18*c*) from Table 2.

Assume now $c=0$. Then for systems (2.15) with $d=0$ we calculate: $\mu_2=\mu_3=0$ and $\mu_4=a^2(clx^2+2mxy+ny^2)^2\neq 0$. So we could consider $a=1$ and $b=0$ due to the transformation $x_1=x$, $y_1=ay-bx$ and $t_1=at$ and this leads to the canonical form 18*d*) from Table 2.

As all the needed cases are examined Proposition 2.2 is proved ∎

2.3 The trace function

We denote

$$\sigma(\tilde{a},x,y)=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=\sigma_0(\tilde{a})+\sigma_1(\tilde{a},x,y)\ (\equiv D_1(\tilde{a})+D_2(\tilde{a},x,y))$$

and observe that the polynomial $\sigma(\tilde{a},x,y)$ is an affine comitant of systems (2.1). It is known, that if (x_i,y_i) is a singular point of a system (2.1) then for the trace of its respective linear matrix we have $\rho_i=\sigma(x_i,y_i)$.

Applying the differential operators $\mathcal{L}$ and $(*,*)^{(k)}$ (i.e. transvectant of index k) we shall define the following polynomial function which governs the values of the traces for finite singularities of systems (2.1).

Definition 2.2. We call the *trace function* $\mathfrak{T}(w)$ the function defined as follows:

$$\mathfrak{T}(w)=\sum_{i=0}^{4}\frac{1}{(i!)^2}\left(\sigma_1^i,\ \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0)\right)^{(i)}w^{4-i}=\sum_{i=0}^{4}\mathcal{G}_i\,w^{4-i} \tag{2.16}$$

where the coefficients $\mathcal{G}_i(\tilde{a})=\dfrac{1}{(i!)^2}(\sigma_1^i,\ \mu_i)^{(i)},\ i=0,1,2,3,4\ \big(\mathcal{G}_0(\tilde{a})\equiv\mu_0(\tilde{a})\big)$ are GL–invariants.

Using the function $\mathfrak{T}(w)$ we could construct the following four affine invariants $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}_4$:

$$\mathcal{T}_{i+1}(\tilde{a})=\frac{1}{i!}\,\frac{d^i\mathfrak{T}}{dw^i}\Big|_{w=\sigma_0},\ i=0,1,2,3\quad\big(\mathcal{T}_1\equiv\mathfrak{T}(\sigma_0)\big).$$

The geometric meaning of these invariants is revealed by the next theorem.

Theorem 2.1. *Consider a non-degenerate system* (2.1) *and let* $\boldsymbol{a}\in\mathbb{R}^{12}$ *be its 12-tuple of coefficients. Denote by* ρ_s *the trace of the linear matrix corresponding to a singular point* M_s*,* $1\le s\le 4$ *(real or complex, simple or multiple). Then the following relations hold, respectively:*
1) For $\mu_0(\boldsymbol{a})\neq 0$ *(total multiplicity 4):*

$$\begin{aligned}
\mathcal{T}_1(\boldsymbol{a})&=\mathcal{G}_0(\boldsymbol{a})\rho_1\rho_2\rho_3\rho_4,\\
\mathcal{T}_2(\boldsymbol{a})&=\mathcal{G}_0(\boldsymbol{a})(\rho_1\rho_2\rho_3+\rho_1\rho_2\rho_4+\rho_1\rho_3\rho_4+\rho_2\rho_3\rho_4),\\
\mathcal{T}_3(\boldsymbol{a})&=\mathcal{G}_0(\boldsymbol{a})(\rho_1\rho_2+\rho_1\rho_3+\rho_1\rho_4+\rho_2\rho_3+\rho_2\rho_4+\rho_3\rho_4),\\
\mathcal{T}_4(\boldsymbol{a})&=\mathcal{G}_0(\boldsymbol{a})(\rho_1+\rho_2+\rho_3+\rho_4);
\end{aligned} \tag{2.17}$$

2) For $\mu_0(\boldsymbol{a})=0,\ \mu_1(\boldsymbol{a},x,y)\neq 0$ *(total multiplicity 3):*

$$\begin{aligned}
&\mathcal{T}_1(\boldsymbol{a})=\mathcal{G}_1(\boldsymbol{a})\rho_1\rho_2\rho_3, && \mathcal{T}_2(\boldsymbol{a})=\mathcal{G}_1(\boldsymbol{a})(\rho_1\rho_2+\rho_1\rho_3+\rho_2\rho_3),\\
&\mathcal{T}_3(\boldsymbol{a})=\mathcal{G}_1(\boldsymbol{a})(\rho_1+\rho_2+\rho_3), && \mathcal{T}_4(\boldsymbol{a})=\mathcal{G}_1(\boldsymbol{a});
\end{aligned} \tag{2.18}$$

3) For $\mu_0(\boldsymbol{a})=\mu_1(\boldsymbol{a},x,y)=0,\ \mu_2(\boldsymbol{a},x,y)\neq 0$ *(total multiplicity 2):*

$$\begin{aligned}
&\mathcal{T}_1(\boldsymbol{a})=\mathcal{G}_2(\boldsymbol{a})\rho_1\rho_2, && \mathcal{T}_2(\boldsymbol{a})=\mathcal{G}_2(\boldsymbol{a})(\rho_1+\rho_2),\\
&\mathcal{T}_3(\boldsymbol{a})=\mathcal{G}_2(\boldsymbol{a}), && \mathcal{T}_4(\boldsymbol{a})=0;
\end{aligned} \tag{2.19}$$

4) For $\mu_0(\boldsymbol{a})=\mu_1(\boldsymbol{a},x,y)=\mu_2(\boldsymbol{a},x,y)=0,\ \mu_3(\boldsymbol{a},x,y)\neq 0$ *(one singularity):*

$$\mathcal{T}_1(\boldsymbol{a})=\mathcal{G}_3(\boldsymbol{a})\rho_1,\quad \mathcal{T}_2(\boldsymbol{a})=\mathcal{G}_3(\boldsymbol{a}),\quad \mathcal{T}_3(\boldsymbol{a})=\mathcal{T}_4(\boldsymbol{a})=0. \tag{2.20}$$

Proof: To prove Theorem 2.1 it is sufficient to calculate the respective affine invariant polynomials for each of the canonical systems given be Table 2 and to check the formulas provided by the statement of this theorem. ∎

2.4 Affine invariant polynomials attached to Poincaré-Lyapunov constants

In order to be able to calculate the values of the needed invariant polynomials directly for every canonical system we shall define here a family of T–comitants (see [27] for detailed definitions) expressed through C_i $(i = 0, 1, 2)$ and D_j $(j = 1, 2)$:

$$
\begin{aligned}
\tilde{A} =& \left(C_1, T_8 - 2T_9 + D_2^2\right)^{(2)}/144,\\
\widetilde{D} =& \Big[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6 - (C_1, T_5)^{(1)} + 6D_1(C_1D_2 - T_5) - 9D_1^2C_2\Big]/36,\\
\widetilde{E} =& \Big[D_1(2T_9 - T_8) - 3\,(C_1, T_9)^{(1)} - D_2(3T_7 + D_1D_2)\Big]/72,\\
\widetilde{F} =& \big[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0\,(D_2, T_9)^{(1)} - 9D_2^2T_4 + 288D_1\widetilde{E}\\
& - 24\left(C_2, \widetilde{D}\right)^{(2)} + 120\left(D_2, \widetilde{D}\right)^{(1)} - 36C_1\,(D_2, T_7)^{(1)} + 8D_1\,(D_2, T_5)^{(1)}\Big]/144,\\
\widetilde{B} =& \Big\{16D_1\,(D_2, T_8)^{(1)}\,(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0\,(D_2, T_9)^{(1)}\,(3D_1D_2 - 5T_6 + 9T_7)\\
& + 2\,(D_2, T_9)^{(1)}\left(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32\,(C_0, T_5)^{(1)}\right)\\
& + 6\,(D_2, T_7)^{(1)}\,[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5) + C_2(9T_4 + 96T_3)]\\
& + 6\,(D_2, T_6)^{(1)}\,[32C_0T_9 - C_1(12T_7 + 52D_1D_2) - 32C_2D_1^2] + 48D_2\,(D_2, T_1)^{(1)}\left(2D_2^2 - T_8\right)\\
& - 32D_1T_8\,(D_2, T_2)^{(1)} + 9D_2^2T_4\,(T_6 - 2T_7) - 16D_1\,(C_2, T_8)^{(1)}\left(D_1^2 + 4T_3\right)\\
& + 12D_1\,(C_1, T_8)^{(2)}\,(C_1D_2 - 2C_2D_1) + 6D_1D_2T_4\left(T_8 - 7D_2^2 - 42T_9\right)\\
& + 12D_1\,(C_1, T_8)^{(1)}\,(T_7 + 2D_1D_2) + 96D_2^2\Big[D_1\,(C_1, T_6)^{(1)} + D_2\,(C_0, T_6)^{(1)}\Big]\\
& - 16D_1D_2T_3\left(2D_2^2 + 3T_8\right) - 4D_1^3D_2\left(D_2^2 + 3T_8 + 6T_9\right) + 6D_1^2D_2^2\,(7T_6 + 2T_7)\\
& - 252D_1D_2T_4T_9\big\}/(2^83^3).\\
\widetilde{K} =& (T_8 + 4T_9 + 4D_2^2)/72 \quad \equiv \Big(p_2(x, y), q_2(x, y)\Big)^{(1)}/4,\\
\widetilde{H} =& (-T_8 + 8T_9 + 2D_2^2)/72.
\end{aligned}
$$

These polynomials in addition with (2.2) and (2.3) will serve as bricks in constructing affine algebraic invariants for systems (2.1). Using these bricks the minimal polynomial basis of affine invariants up to degree 12, containing 42 elements $A_1 - A_{42}$, was constructed in [16]. We shall apply here the following elements of this basis (keeping the notation from [16]):

$$
\begin{aligned}
&A_1 = \tilde{A}, && A_2 = (C_2, D)^{(3)}/12,\\
&A_3 = \left[C_2, D_2)^{(1)}, D_2\right)^{(1)}, D_2\big)^{(1)}/48, && A_4 = (\widetilde{H}, \widetilde{H})^{(2)},\\
&A_5 = (\widetilde{H}, \widetilde{K})^{(2)}/2, && A_6 = (\widetilde{E}, \widetilde{H})^{(2)}/2,\\
&A_7 = \left[C_2, \widetilde{E})^{(2)}, D_2\right)^{(1)}/8, && A_8 = \left[\widetilde{D}, \widetilde{H})^{(2)}, D_2\right)^{(1)}/8,\\
&A_9 = \left[\widetilde{D}, D_2)^{(1)}, D_2\right)^{(1)}, D_2\big)^{(1)}/48, && A_{10} = \left[\widetilde{D}, \widetilde{K})^{(2)}, D_2\right)^{(1)}/8,\\
&A_{11} = (\widetilde{F}, \widetilde{K})^{(2)}/4, && A_{12} = (\widetilde{F}, \widetilde{H})^{(2)}/4,\\
&A_{14} = (\widetilde{B}, C_2)^{(3)}/36, && A_{15} = (\widetilde{E}, \widetilde{F})^{(2)}/4,\\
&A_{33} = \left[\widetilde{D}, D_2)^{(1)}, \widetilde{F}\right)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/128, && A_{34} = \left[\widetilde{D}, \widetilde{D})^{(2)}, D_2\right)^{(1)}, \widetilde{K}\big)^{(1)}, D_2\big)^{(1)}/64,
\end{aligned}
$$

Finally we construct the needed affine invariants

$$
\begin{aligned}
\mathcal{F}_1 =& A_2,\\
\mathcal{F}_2 =& -2A_1^2A_3+2A_5(5A_8+3A_9)+A_3(A_8-3A_{10}+3A_{11}+A_{12})-\\
& -A_4(10A_8-3A_9+5A_{10}+5A_{11}+5A_{12}),\\
\mathcal{F}_3 =& -10A_1^2A_3+2A_5(A_8-A_9)-A_4(2A_8+A_9+A_{10}+A_{11}+A_{12})+\\
& +A_3(5A_8+A_{10}-A_{11}+5A_{12}),\\
\mathcal{F}_4 =& \,20A_1^2A_2-A_2(7A_8-4A_9+A_{10}+A_{11}+7A_{12})+A_1(6A_{14}-22A_{15})-4A_{33}+4A_{34},\\
\mathcal{F} =& A_7,\\
\mathcal{E} =& -(3A_8+2A_9+A_{10}+A_{11}+A_{12}),\\
\mathcal{H} =& -(A_4+2A_5),
\end{aligned}
\tag{2.21}
$$

as well as the needed additional CT-comitants:

$$
\begin{aligned}
\mathcal{E}_1 =& \Big\{(T_7,D_2)^{(1)}\big[12D_1T_3+2D_1^3+9D_1T_4+36(T_1,D_2)^{(1)}\big]\\
& -2D_1(T_6,D_2)^{(1)}\big[D_1^2+12T_3\big]+D_1^2\big[D_1(T_8,C_1)^{(2)}+6\big((T_6,C_1)^{(1)},D_2\big)^{(1)}\big]\Big\}/144,\\
\mathcal{E}_2 =& \Big\{(T_7,D_2)^{(1)}\big[8T_3(T_6,D_2)^{(1)}-D_1^2(T_8,C_1)^{(2)}-4D_1\big((T_6,C_1)^{(1)},D_2\big)^{(1)}\big],\\
& +\Big[(T_7,D_2)^{(1)}\Big]^2(8T_3-3T_4+2D_1^2)\Big\}/384.
\end{aligned}
\tag{2.22}
$$

We note that the CT-comitants above actually are GL-invariants (see [27] for detailed definitions) and not affine invariant polynomials. So everywhere in this article when we need to calculate the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$ we shall apply the algorithm described by the next remark.

Remark 2.3. *Assume that we need to evaluate the invariant polynomials constructed above for a quadratic system $S(\boldsymbol{a},x,y)$ of the form (2.1) corresponding to a point $\boldsymbol{a}\in\mathbb{R}^{12}$. Then all affine invariants (i.e. $\mathcal{T}_i$, $\mathcal{F}_i$ $(i=1,2,3,4)$, $\mathcal{F}$, $\mathcal{E}$ and $\mathcal{H}$) will be calculated for this system directly, except the polynomials $\mathcal{E}_1$ and $\mathcal{E}_2$. The last two polynomials will be calculated for any system in the orbit under the translation group action (say, $\tau(x_0,y_0): x=X+x_0,\ y=Y+y_0$) of the system $S(\boldsymbol{a},x,y)$, i.e. for the family of systems $S(\boldsymbol{a}(x_0,y_0),X,Y)$.*

The indicated above algorithm is needed in order to prove that the conditions given by $\mathcal{E}_1$ and $\mathcal{E}_2$ do not depend of the coordinates of an arbitrary translation vector $\tau(x_0,y_0)$, in other words that these conditions are affine invariant ones.

3 Proof of Main Theorem

The *Proof* of Main Theorem is organized in two steps:

1) providing that a non-degenerate quadratic system (2.1) has at least one simple real finite singular point we prove that the respective conditions given by Main Theorem are necessary and sufficient for the existence of a weak focus of the respective order or a center arbitrarily located on its phase plan;

2) in order to complete proof we show the incompatibility of the conditions given by Main Theorem for a non-degenerate quadratic system (2.1) which has not any simple finite real singular point.

3.1 Step 1: The necessity and sufficiency of the conditions for systems with at least one simple real finite singular point

The *statement* (a) follows immediately from Theorem 2.1 as a quadratic system in the case $\mathcal{T}_1 \neq 0$ cannot possess a singular point with zero trace.

Statement (b). Assume $\mathcal{T}_1 = 0$ and $\mathcal{T}_2 \neq 0$. Considering Theorem 2.1 in this case a quadratic system could possess only one real simple singular point with zero trace. Due to a translation we replace this point at the origin of coordinates and this leads to the family of systems:

$$\dot{x} = cx + dy + gx^2 + 2hxy + ky^2, \quad \dot{y} = ex - cy + lx^2 + 2mxy + ny^2, \tag{3.1}$$

with $c^2 + de \neq 0$ (as $(0,0)$ is a simple point). Then via a straightforward computation we obtain:

$$\begin{aligned} &\mathcal{T}_1 = 0 = G_0,\ \mathcal{F}_1 = -G_1/72,\ \mathcal{F}_2 = -2^{-7}3^{-3}G_2\mathcal{F},\ \mathcal{F}_3 = -2^{-7}3^{-3}G_3\mathcal{F},\\ &\mathcal{F}_4 = 2^{-6}3^{-2}G_4\mathcal{F},\ \mathcal{T}_2 = -G_5\mathcal{F}/8,\ \operatorname{sign}(\mathcal{T}_2\mathcal{F}) = -\operatorname{sign}(\det \mathcal{A}). \end{aligned} \tag{3.2}$$

where $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{F}$, $\mathcal{F}_j$, $j = 1, \ldots, 4$ and are the affine invariants from (2.3) and (2.21), respectively and G_j ($j = 0, \ldots, 5$) are the GL-invariants from (2.4).

Since for systems (3.1) the condition $\mathcal{T}_2 \neq 0$ holds, from (3.2) it follows $\mathcal{F} \neq 0$. Therefore considering Lemma 2.1 the statement (b) of the Main Theorem follows obviously.

Next we shall consider step by step each family of the systems 1), 2), 4), 8), 10), 11), 12), 14a), 14b), 17a) and 17b) (see Table 2) which possesses at least one finite real simple singular point.

Remark 3.1. *In what follows for each of the families of systems mentioned above we shall assume that the conditions $\mathcal{T}_1 = \mathcal{T}_2 = 0$ are fulfilled.*

3.1.1 Family of systems with the zero-cycle $\mathcal{D}_S(P,Q) = p + q + r + s$

According to Table 2 we consider the family of systems 1). For the finite singularities of this family we obtain:

$$\begin{aligned} &\rho_1 = c + f, \quad \rho_2 = -c + f + 2m, \quad \rho_3 = c - f + 2h,\\ &\rho_4 = c + f + 2(m - c)\tfrac{\beta(\beta-2\gamma)}{\beta^2-4\alpha\gamma} + 2(h - f)\tfrac{\beta(\beta-2\alpha)}{\beta^2-4\alpha\gamma}, \end{aligned} \tag{3.3}$$

where $\alpha = eh - cm$, $\beta = cf - de$, $\gamma = dm - fh$ and $\beta^2 - 4\alpha\gamma \neq 0$.

On the other hand according to Theorem 2.1 the relations (2.17) are fulfilled, where $\mathcal{G}_0 = \mu_0 = \beta^2 - 4\alpha\gamma \neq 0$. Clearly the conditions $\mathcal{T}_1 = \mathcal{T}_2 = 0$ (see the remark above) imply the vanishing of two traces. Due to an affine transformation without loss of generality we can assume $\rho_1 = \rho_2 = 0$ that yields $f = -c$ and $m = c$. Then we obtain the systems

$$\dot{x} = cx + dy - cx^2 + 2hxy - dy^2, \quad \dot{y} = ex - cy - ex^2 + 2cxy + cy^2, \tag{3.4}$$

and for the singular points $M_1(0,0)$ and $M_2(1,0)$ we have $\Delta_1 = -c^2 - de$ and $\Delta_2 = -c^2 + de + 2eh$, respectively. For systems (3.4) calculations yield:

$$\begin{aligned} &\mathcal{T}_1 = \mathcal{T}_2 = 0, \quad \mathcal{T}_3 = -4(c+h)^2\Delta_1\Delta_2, \quad \mathcal{F}_1 = 2(c+h)(2c^3 + c^2e + de^2 - 2ceh),\\ &\mathcal{E} = -2e^2(c+h)^4, \quad \mathcal{H} = -2(c+h)^2(\Delta_1 + \Delta_2), \quad \sigma = 2(c+h)y. \end{aligned} \tag{3.5}$$

3.1.1.1 The case $\mathcal{T}_3 \neq 0$. As for systems (3.4) we have $\rho_1 = \rho_2 = 0$ then considering (2.17) we obtain $\mathcal{T}_3 = \mathcal{G}_0\rho_3\rho_4 \neq 0$. Hence the remaining points could not have zero traces. So the point M_1 (respectively M_2) is either a weak focus or a center if and only if $\Delta_1 > 0$ (respectively $\Delta_2 > 0$)

We observe that $\operatorname{sign}\mathcal{T}_3 = -\operatorname{sign}(\Delta_1\Delta_2)$. Then if $\mathcal{T}_3 > 0$ we have $\Delta_1\Delta_2 < 0$ and this means that only one point is either a weak focus or a center (other one being a saddle). If $\mathcal{T}_3 < 0$ we obtain $\Delta_1\Delta_2 > 0$ and since in this case $\operatorname{sign}(\Delta_1 + \Delta_2) = -\operatorname{sign}(\mathcal{H})$ we conclude that systems (3.4) possess two weak foci or two centers if and only if $\mathcal{H} < 0$.

It remains to note that according to Lemma 2.2 if $\Delta_1 > 0$ (respectively $\Delta_2 > 0$) the point M_1 (respectively M_2) is a weak focus if $\mathcal{F}_1 \neq 0$ and it is a center if $\mathcal{F}_1 = 0$.

We claim that for the existence of at least one either weak focus or a center the condition $\mathcal{E} < 0$ is necessary. Indeed, according to (3.5) we have $\mathcal{E} \leq 0$ and since $\mathcal{T}_3 \neq 0$ the condition $\mathcal{E} = 0$ implies $e = 0$ and then $\Delta_1 < 0$ and $\Delta_2 < 0$. Hence our claim is proved. Thus we arrive to the conditions provided by the Main Theorem in the considered case.

3.1.1.2 The case $\mathcal{T}_3 = 0$. Since the singular points $M_1(0,0)$ and $M_2(1,0)$ of systems (3.4) are simple(i.e. $\Delta_1\Delta_2 \neq 0$), according to (3.5) the condition $\mathcal{T}_3 = 0$ yields $c + h = 0$. Therefore $\sigma(a, x, y) = 0$ and systems (3.4) become Hamiltonian:

$$\dot{x} = cx + dy - cx^2 - 2cxy - dy^2, \quad \dot{y} = ex - cy - ex^2 + 2cxy + cy^2. \tag{3.6}$$

As it is known for Hamiltonian systems a simple point could be either a center (if it is an anti-saddle) or an integrable saddle.

Thus, for the points M_i $(i = 1, \ldots, 4)$ of systems (3.6) we have $\rho_i = 0$ for every $i = 1, \ldots, 4$ and calculations yield:

$$\begin{aligned} &\Delta_1 = -(c^2 + de), \quad \Delta_2 = -c^2 - 2ce + de, \quad \Delta_3 = -c^2 + 2cd + de, \\ &\Delta_4 = \Delta_1\Delta_2\Delta_3/\mu_0, \quad \mu_0 = \Delta_1^2 + (\Delta_1 + \Delta_2)(\Delta_1 + \Delta_3). \end{aligned}$$

Therefore we obtain that $\operatorname{sign}(\mu_0) = \operatorname{sign}(\Delta_1\Delta_2\Delta_3\Delta_4)$. Taking into account that at least one determinant Δ_i $(i = 1, \ldots, 4)$ is positive and at least one is negative (as we could have neither four saddles nor four anti-saddles (sf. [11]) we conclude, that for $\mu_0 > 0$ systems (3.6) possess two centers.

Assume $\mu_0 < 0$, i.e. there exists an odd number of negative quantities among Δ_i $(i = 1, \ldots, 4)$. As a quadratic systems cannot possess three centers we obtain that for $\mu_0 < 0$ systems (3.6) possess only one center.

Thus considering Table 1 the conditions (f_1) and (f_2) of Main Theorem are verified, respectively.

3.1.2 Systems with the zero–cycle $\mathcal{D}_S(P, Q) = p + q + r^c + s^c$

Considering Table 2 for the finite singularities $M_{1,2}(0, \pm i)$, $M_3(1, 0)$ and $M_4(x_4, y_4)$ of systems 2) we obtain, respectively:

$$\begin{aligned} &\rho_{1,2} = -(a + g) \pm 2i(b + h), \quad \rho_3 = g - a + 2m, \\ &\rho_4 = -(a + g) + 2(g + m)\frac{\beta^2 + 4\gamma^2}{\mu_0} - 2(b + h)\frac{2\beta(\alpha + \gamma)}{\mu_0}, \end{aligned} \tag{3.7}$$

where $\alpha = gm - hl$, $\beta = bg - al$, $\gamma = bh - am$ and $\mu_0 = \beta^2 - 4\alpha\gamma \neq 0$.

According to Theorem 2.1 the relations (2.17) (where $\mathcal{G}_0 = \mu_0 \neq 0$) are fulfilled and hence, the condition $\mathcal{T}_1 = \mathcal{T}_2 = 0$ (see Remark 3.1) implies the vanishing of two traces.

3.1.2.1 The case $\mathcal{T}_3 \neq 0$. We claim that in this case the zero traces correspond to real singularities due to the necessary condition $\mathcal{E} < 0$ provided by statement (c) of Main Theorem. Indeed supposing $\rho_1 = \rho_2 = 0$ we obtain $g = -a$, $h = -b$ and then calculation yields $\mathcal{E} = 8a^2(a-m)^4 \geq 0$.

Thus our claim is proved and hence the condition $\mathcal{T}_1 = \mathcal{T}_2 = 0$ yields $\rho_3 = \rho_4 = 0$. Considering (3.7) the condition $\rho_3 = 0$ yields $g = a - 2m$. This leads to the systems

$$\begin{aligned} \dot{x} &= a - 2(a-m)x + (a-2m)x^2 + 2hxy + ay^2, \\ \dot{y} &= b - (b+l)x + lx^2 + 2mxy + by^2, \end{aligned} \tag{3.8}$$

for which we calculate

$$\rho_4 = 2\big[al(b+h) - ab^2 + abh - 2a^2m + 2b^2m + 2am^2\big]\Delta_3/\mu_0 \equiv 2W\Delta_3/\mu_0. \tag{3.9}$$

Hence the condition $\rho_4 = 0$ yields $W = 0$ and as W is linear with respect to the parameter l we shall consider three cases: $\alpha_1)$ $a(b+h) \neq 0$; $\alpha_2)$ $a \neq 0,\ b+h = 0$ and $\alpha_3)$ $a = 0$.

$\alpha_1)$ For $a(b+h) \neq 0$ the condition $W = 0$ yields

$$l = \frac{ab(b-h) + 2am(a-m) - 2b^2m}{a(b+h)}$$

and then for systems (3.8) we calculate:

$$\begin{aligned} &\Delta_3 = \frac{4(bh-am)(ah+bm)}{a(b+h)}, \quad \Delta_4 = -\frac{4\Delta_3(bh-am)^2\big[(a-m)^2+(b+h)^2\big]}{(b+h)^2\mu_0}, \\ &\mathcal{F}_1 = -4(ah+bm)\Delta_3, \quad \mathcal{T}_3 = -\Delta_3\Delta_4\frac{\mu_0^2a^2(b+h)^4}{16(bh-am)^4(ah+bm)^2}, \\ &\mathcal{H} = -(\Delta_3+\Delta_4)\frac{9\mu_0^2a^2(b+h)^4}{288(bh-am)^4(ah+bm)^2}, \quad \mathcal{E} = -8(a^2+b^2+bh-am)^2(ah+bm)^2/a^2. \end{aligned} \tag{3.10}$$

Thus $\operatorname{sign}(\Delta_3\Delta_4) = -\operatorname{sign}(\mathcal{T}_3)$ and since $\Delta_3 \neq 0$ we have $\mathcal{F}_1 \neq 0$. According to Lemma 2.2 each of the points M_3 and M_4 could only be a weak focus of the first order. Therefore, if $\mathcal{T}_3 > 0$ then $\Delta_3\Delta_4 < 0$ and this means that only one point is a weak focus.

In the case $\mathcal{T}_3 < 0$ we obtain $\Delta_3\Delta_4 > 0$ and then $\operatorname{sign}(\Delta_3+\Delta_4) = -\operatorname{sign}(\mathcal{H})$. Therefore systems (3.8) possess two weak foci of the first order if and only if $\mathcal{H} < 0$. It remains to note that considering (3.10) we have $\mathcal{E} \leq 0$ and we claim that the condition $\mathcal{E} < 0$ is necessary for the existence of at least one focus. Indeed suppose that at least one point is a weak focus and $\mathcal{E} = 0$. As $\Delta_3 \neq 0$ (i.e. $ah+bm \neq 0$) and $a \neq 0$ we obtain $m = (a^2+b^2+bh)/a$. However in this case we get $\Delta_3 = 4(a^2+b^2)^2/a^2 < 0$ and $\Delta_3 = \Delta_4 = 4(a^2+b^2)^2/a^2 < 0$, i.e. both points are saddles and our claim is proved. So $\mathcal{E} < 0$ and this leads to the respective conditions of statement (c) of Main Theorem.

$\alpha_2)$ Assume now that $a \neq 0$ and $b = -h$. Then we may consider $a = 1$ via a time rescaling and considering (3.9) we obtain:

$$W = 2(1-m)(h^2+m), \quad \mathcal{T}_3 = (1-m)^2\widetilde{W}(h,l,m),$$

where $\widetilde{W}(h,l,m)$ is a polynomial. Since $\mathcal{T}_3 \neq 0$ the condition $W = 0$ yields $m = -h^2$ and then for systems (3.8) calculations yield:

$$\Delta_3 = -2h(2h^3 + h + l) = -\Delta_4 \neq 0, \quad \mathcal{T}_3 = -\Delta_3\Delta_4(h^2+1)^2/h^2, \quad \mathcal{F}_1 = -4h(h^2+1)\Delta_3.$$

We observe that in this case $\Delta_3\Delta_4 < 0$ and according to Lemma 2.2 only one point is a weak focus of the first order. On the other hand we have $\mathcal{T}_3 > 0$ and $\mathcal{F}_1 \neq 0$, i.e. the set of conditions (c_1) of Main Theorem is verified.

α_3) If $a = 0$ then, considering (3.9), the condition $W = 0$ yields $bm = 0$. Since $b \neq 0$ (otherwise systems (3.8) become degenerate) we obtain $m = 0$. Then for systems (3.8) calculations yield:

$$\Delta_3 = 2h(b-l), \quad \Delta_4 = -2bh(b-l)/l, \quad \mu_0 = 4blh^2 \neq 0,$$
$$\mathcal{T}_3 = -\Delta_3\Delta_4\frac{4l^2(b+h)^2}{(b-l)^2}, \quad \mathcal{H} = -(\Delta_3+\Delta_4)\frac{2l^2(b+h)^2}{(b-l)^2}, \quad \mathcal{F}_1 = 0.$$

So, the condition $\mathcal{F}_1 = 0$ holds and by Lemma 2.2 the point M_3 (respectively M_4) will be a center if $\Delta_3 > 0$ (respectively $\Delta_4 > 0$). If $\mathcal{T}_3 > 0$ then $\Delta_3\Delta_4 < 0$ and therefore only one point is a center and this leads to conditions (c_3) of Main Theorem.

Assume now $\mathcal{T}_3 < 0$. Then $\operatorname{sign}(\Delta_3 + \Delta_4) = -\operatorname{sign}(\mathcal{H})$ and we conclude that systems (3.8) possess two centers if and only if $\mathcal{H} < 0$. In this case we get the conditions (c_4) of Main Theorem.

3.1.2.2 The case $\mathcal{T}_3 = 0$. According to formulas (2.17) the conditions $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = 0$ imply the vanishing of three traces. We claim that for a non-degenerate system having two real distinct and two complex finite singularities the traces corresponding to all four points have to vanish.

Indeed, if a singular point has a respective nonzero trace then necessary it must be real. So the relations $\rho_1 = \rho_2 = 0$ yield $g = -a$, $h = -b$ and then we calculate

$$\mathcal{T}_1 = \mathcal{T}_2 = 0,\ \mathcal{T}_3 = 4(a-m)^2\big[4(b^2+am)^2 + a^2(b+l)^2\big],\ \mathcal{T}_4 = 4(m-a)(b+l)\big[2b(b^2+am) + a^2(b+l)\big].$$

Evidently the condition $\mathcal{T}_3 = 0$ implies $\mathcal{T}_4 = 0$ and this proves our claim.

Thus for systems (3.8) the conditions $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0$ hold and considering (3.7) we obtain $m = a$, $h = -b$ and $g = -a$. This leads to the family of Hamiltonian systems

$$\dot{x} = a - ax^2 - 2bxy + ay^2, \quad \dot{y} = b - (b+l)x + lx^2 + 2axy + by^2 \tag{3.11}$$

for the real singular points of which we calculate

$$\Delta_3 = -2(2a^2 + b^2 - bl), \quad \Delta_4 = -\Delta_3\big[4(a^2+b^2)^2 + a^2(b+l)^2\big]/\mu_0,$$
$$\mu_0 = a^2(b+l)^2 - 4(a^2+b^2)(a^2-bl).$$

Therefore we have $\operatorname{sign}(\Delta_3\Delta_4) = -\operatorname{sign}(\mu_0)$. Moreover calculations yield

$$\mu_0(\Delta_3+\Delta_4) = 2(a^2+b^2)\Delta_3^2 > 0$$

and this implies $\operatorname{sign}(\Delta_3+\Delta_4) = \operatorname{sign}(\mu_0)$. Hence, the condition $\mu_0 > 0$ yields $\Delta_3\Delta_4 < 0$ and we obtain only one center. If $\mu_0 < 0$ then we have $\Delta_3 < 0$, $\Delta_4 < 0$ and hence systems (3.11) could not have any center. So considering Table 1 the conditions (f_3) of Main Theorem hold.

3.1.3 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 2p + q + r$

Considering Table 2 for the finite singularities $M_{1,2}(0,0)$ (double), $M_3(1,0)$ and $M_4(0,1)$ of systems 4) we obtain, respectively:

$$\rho_{1,2} = c + eu, \quad \rho_3 = -c + 2m + eu, \quad \rho_4 = c + 2h - eu. \tag{3.12}$$

According to Theorem 2.1 the relations (2.17) (where $\mathcal{G}_0 = \mu_0 \neq 0$) are fulfilled and hence, the condition $\mathcal{T}_1 = \mathcal{T}_2 = 0$ (see Remark 3.1 implies the vanishing of at least two traces among ρ_i, $i \in \{1,2,3,4\}$.

3.1.3.1 The case $\mathcal{T}_3 \neq 0$. Then only two traces vanish and we have either $\rho_{1,2} = 0$ or $\rho_3 = \rho_4 = 0$. We claim that due to the conditions $\mathcal{F} = 0$ and $\mathcal{E} < 0$ the case $\rho_{1,2} = 0$ could not occur.

Indeed, supposing $\rho_{1,2} = 0$ from (3.12) we have $c = -eu$ and then calculation yields:

$$\begin{aligned} &\mu_0 = 4e^2u(h+mu)^2 \neq 0, \quad \mathcal{E} = -384e^2(eu^2 + eu - h + mu)^4, \\ &\mathcal{T}_3 = \mu_0\rho_3\rho_4, \quad \mathcal{F} = 4\rho_3\rho_4 e(h+mu)(eu^2 + eu - h + mu). \end{aligned}$$

Therefore due to the condition $\mu_0\mathcal{T}_3 \neq 0$ the relation $\mathcal{F} = 0$ implies $eu^2 + eu - h + mu = 0$ and then $\mathcal{E} = 0$. This proves our claim.

Thus the conditions $\rho_3 = \rho_4 = 0$ hold and considering (3.12) we obtain: $c = eu - 2h$, $m = -h$ and this leads to the systems:

$$\begin{aligned} &\dot{x} = (eu-2h)x + u(eu-2h)y - (eu-2h)x^2 + 2hxy - u(eu-2h)y^2, \\ &\dot{y} = ex + euy - ex^2 - 2hxy - euy^2, \end{aligned} \tag{3.13}$$

for which calculations yield:

$$\begin{aligned} &\Delta_4 = 2h(e - 2h + eu), \quad \Delta_3 = -u\Delta_4, \quad \mu_0 = u\Delta_4^2 \neq 0, \\ &\mathcal{T}_3 = \mu_0(eu-h)^2, \quad \mathcal{E} = -2(1+u)^2(e-2h+eu)^2(-h+eu)^4, \\ &\mathcal{F}_1 = 8hu(e-2h+eu)^2(eu-h), \quad \mathcal{H} = -2(\Delta_3 + \Delta_4)(eu-h)^2. \end{aligned}$$

As $\operatorname{sign}(\mu_0) = \operatorname{sign}(u)$ we obtain $\operatorname{sign}(\Delta_3\Delta_4) = -\operatorname{sign}(\mu_0) = -\operatorname{sign}(\mathcal{T}_3)$. Moreover, since $\mathcal{T}_3\Delta_3 \neq 0$ we have $\mathcal{F}_1 \neq 0$ and according to Lemma 2.2 the point M_3 as well as the point M_4 could only be weak foci of the first order. Therefore, if $\mathcal{T}_3 > 0$ then $\Delta_3\Delta_4 < 0$ and this means that only one point is a weak focus.

In the case $\mathcal{T}_3 < 0$ we obtain $\Delta_3\Delta_4 > 0$ and since $\operatorname{sign}(\Delta_3 + \Delta_4) = -\operatorname{sign}(\mathcal{H})$ systems (3.13) possess two weak foci of the first order if and only if $\mathcal{H} < 0$.

It remains to note that we have $\mathcal{E} \leq 0$ and $\mathcal{E} = 0$ only if $u = -1$. However in this case we obtain $\Delta_3 = \Delta_4 = -4h^2 < 0$, i.e. both points are saddles. So, the condition $\mathcal{E} < 0$ is necessary and this leads to the respective conditions provided by statement (c) of Main Theorem.

3.1.3.2 The case $\mathcal{T}_3 = 0$. By formulas (2.17) the conditions $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = 0$ imply the vanishing of three traces. Then clearly we necessarily have $\rho_{1,2} = 0$ (i.e. $c = -eu$) and $\rho_3\rho_4 = 0$ and we may assume $\rho_4 = 0$ do to the change $(x, y, h, m, e, u) \mapsto (y, x, m, h, -eu^{-2}, u^{-1})$ (if necessary). So considering (3.12) we obtain $\rho_4 = 2(h - eu) = 0$, i.e. $h = eu$. Then we get the following family of systems

$$\dot{x} = -eux - eu^2y + eux^2 + 2euxy + euy^2, \quad \dot{y} = ex + euy - ex^2 + 2mxy - euy^2, \tag{3.14}$$

for which we have $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = 0$ and

$$\Delta_3 = 2e(e+m)u, \quad \Delta_4 = -u\Delta_3, \quad \rho_3 = 2(m+eu), \quad \rho_4 = 0, \quad \mu_0 = u\Delta_3^2 \neq 0,$$
$$\mathcal{T}_4 = \mu_0\rho_3, \quad \mathcal{H} = -\Delta_4\rho_3^2/2, \quad \mathcal{F}_1 = eu\Delta_4\rho_3, \quad \mathcal{E} = -e^2u^4\rho_3^4/8, \quad \mathcal{F} = 0.$$

3.1.3.2.1 If $\mathcal{T}_4 \neq 0$ then $\rho_3 \neq 0$ and only the point $M_4(0,1)$ could be either a weak focus or a center for systems (3.14). We observe that $\text{sign}\,(\Delta_4) = -\text{sign}\,(\mathcal{H})$ and $\mathcal{F}_1 \neq 0$ due to $\mathcal{T}_4 \neq 0$. According to Lemma 2.2 and Remark 2.1 (the second point with zero trace is the double point $M_{1,2}(0,0)$) we conclude that the point M_4 is a weak focus of the first order if and only if $\mathcal{H} < 0$. It remains to note that $\mathcal{F} = 0$ and due to $\Delta_3 \neq 0$ (i.e. $eu \neq 0$) the condition $\mathcal{E} < 0$ holds in this case. This leads to the conditions (d_1) of Main Theorem.

3.1.3.2.2 Assume $\mathcal{T}_4 = 0$. Then $\rho_3 = 0$ (i.e. $m = -eu$) and systems (3.14) become Hamiltonian. For the singular points M_3 and M_4 we calculate

$$\Delta_3 = -2e^2u(u-1), \quad \Delta_4 = -u\Delta_3, \qquad \mu_0 = u\Delta_3^2.$$

Therefore we have

$$\text{sign}\,(\Delta_3\Delta_4) = -\text{sign}\,(u) = -\text{sign}\,(\mu_0), \quad \mu_0(\Delta_3+\Delta_4) = 8e^6(1-u)^3u^4.$$

If $\mu_0 > 0$ we have $\Delta_3\Delta_4 < 0$ and we obtain only one center. If $\mu_0 < 0$ (i.e. $u < 0$) then we have $\Delta_3\Delta_4 > 0$ and $(\Delta_3+\Delta_4) < 0$. Hence $\Delta_3 < 0$, $\Delta_4 < 0$ and systems (3.14) could not have any center. So considering Table 1 the conditions (f_4) of Main Theorem are verified.

3.1.4 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 3p + q$

Considering Table 2 for the finite singularities $M_{1,2,3}(0,0)$ (triple) and $M_4(1,0)$ of systems 8) we obtain, respectively:

$$\rho_{1,2,3} = c + eu, \quad \rho_4 = -c + 2m + eu. \tag{3.15}$$

Clearly the condition $\mathcal{T}_1 = \mathcal{T}_2 = 0$ (see Remark 3.1) implies the vanishing of at least two traces and hence $\rho_{1,2,3} = 0$ and then $\mathcal{T}_3 = 0$. So $c = -eu$ and we calculate:

$$\Delta_4 = 2e(h+mu), \quad \rho_4 = 2(m+eu), \quad \mu_0 = (u^2-v)\Delta_4^2 \neq 0, \quad \mathcal{T}_4 = \mu_0\rho_4.$$

Evidently the simple point $M_4(1,0)$ could be either a weak focus or a center only if $\mathcal{T}_4 = 0$. The last relation yields $m = -eu$ and this leads to the following family of systems:

$$\begin{aligned} \dot{x} &= -eux - eu^2y + eux^2 + 2hxy + u(2h-ev)y^2, \\ \dot{y} &= ex + euy - ex^2 - 2euxy - e(2u^2-v)y^2, \end{aligned} \tag{3.16}$$

for which we have $\mathcal{T}_i = 0$ $(i = 1,2,3,4)$ and

$$\Delta_4 = 2e(eu^2-h), \quad \rho_4 = 0, \quad \sigma = 2(h-2eu^2+ev)y, \quad \mu_0 = (u^2-v)\Delta_4^2 \neq 0,$$
$$\mathcal{H} = -2\Delta_4(h-2eu^2+ev)^2, \quad \mathcal{E} = -2e^2(h-2eu^2+ev)^4, \quad \mathcal{F}_1 = \mathcal{F} = 0.$$

3.1.4.1 The case $\sigma \neq 0$. Then systems (3.16) are not Hamiltonian and we have $\mathcal{H} \neq 0$ and $\mathrm{sign}\,(\Delta_4) = -\mathrm{sign}\,(\mathcal{H})$. As $\mathcal{F}_1 = 0$ according to Lemma 2.2 and Remark 2.1 (the second point with zero trace is the triple point $M_{1,2,3}(0,0)$) we conclude that the point M_4 is a center if and only if $\mathcal{H} < 0$. It remains to note that due to $\sigma\Delta_3 \neq 0$ the condition $\mathcal{E} < 0$ is fulfilled in this case. This leads to the conditions (d_2) (second part) of Main Theorem.

3.1.4.2 The case $\sigma = 0$. In this case systems (3.16) become Hamiltonian and considering the relation $h = e(2u^2 - v)$ for the simple singular point $M_4(1,0)$ we calculate:

$$\Delta_4 = 2e^2(u^2 - v), \quad \rho_4 = 0, \quad \Delta_4^3 = 2e^2\mu_0.$$

Therefore $\mathrm{sign}\,(\Delta_4) = \mathrm{sign}\,(\mu_0)$ and then the simple point of systems (3.16) will be a center if and only if $\mu_0 > 0$. So considering Table 1 the conditions (f_5) of Main Theorem are verified.

3.1.5 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = p + q + r$

Considering Table 2 for the simple finite singularities $M_1(0,0)$, $M_2(1,0)$ and $M_3(\alpha,\beta)$ of systems 10) we obtain, respectively:

$$\begin{array}{ll} \rho_1 = c + f, \quad \rho_3 = c + f + 2(m-c)\frac{\beta}{2\alpha} - 2h\frac{\beta(\beta-2\alpha)}{4\alpha\gamma}, \\ \rho_2 = -c + f + 2m, \quad \mathcal{T}_4 = \mathcal{G}_1 = -8h\alpha\gamma, \quad \mu_0 = 0, \quad \mu_1 = 4\alpha\gamma x, \end{array} \tag{3.17}$$

where $\alpha = eh - cm$, $\beta = cf - de$, $\gamma = dm - fh$ and $\alpha\beta\gamma(\beta - 2\alpha) \neq 0$.

As for this family of systems the conditions $\mu_0 = 0$ and $\mu_1 \neq 0$ according to Theorem 2.1 (see formulas (2.18)) we shall consider two subcases: $\mathcal{T}_4 \neq 0$ and $\mathcal{T}_4 = 0$.

3.1.5.1 The case $\mathcal{T}_4 \neq 0$. Then by (3.17) we have $\mathcal{G}_1 \neq 0$ and hence the condition $\mathcal{T}_1 = \mathcal{T}_2 = 0$ (see Remark 3.1) implies the vanishing of at least two traces. Clearly that due to an affine transformation (if necessary) we may assume $\rho_1 = \rho_2 = 0$. So considering (3.17) we obtain $f = -c$ and $m = c$. This leads to the following family of systems:

$$\dot{x} = cx + dy - cx^2 + 2hxy, \quad \dot{y} = ex - cy - ex^2 + 2cxy, \tag{3.18}$$

for which we have $\mathcal{T}_1 = \mathcal{T}_2 = 0$ and

$$\begin{array}{l} \Delta_1 = -(c^2 + de), \quad \Delta_2 = -c^2 + de + 2eh, \quad \mathcal{T}_4 = -4ch(d+h)(\Delta_1 + \Delta_2), \quad \mathcal{F} = 0, \\ \mathcal{T}_3 = -4h^2\Delta_1\Delta_2, \quad \mathcal{F}_1 = -12ch(\Delta_1 + \Delta_2), \quad \mathcal{E} = -2e^2h^4, \quad \mathcal{H} = -2h^2(\Delta_1 + \Delta_2). \end{array} \tag{3.19}$$

We observe that the condition $\Delta_1\Delta_2\mathcal{T}_4 \neq 0$ implies $\mathcal{F}_1\mathcal{H}\mathcal{T}_3 \neq 0$ and then $\mathrm{sign}\,(\Delta_3\Delta_4) = -\mathrm{sign}\,(\mathcal{T}_3)$ and $\mathrm{sign}\,(\Delta_3 + \Delta_4) = -\mathrm{sign}\,(\mathcal{H})$. Moreover, since $\mathcal{F}_1 \neq 0$ according to Lemma 2.2 the point M_3 as well as the point M_4 could only be a weak focus of the first order. Therefore, if $\mathcal{T}_3 > 0$ then $\Delta_3\Delta_4 < 0$ and this means that only one point is a weak focus. In the case $\mathcal{T}_3 < 0$ we obtain $\Delta_3\Delta_4 > 0$ and systems (3.18) possess two weak foci of the first order if and only if $\mathcal{H} < 0$.

It remains to note that we have $\mathcal{E} \leq 0$ and $\mathcal{E} = 0$ only if $e = 0$. However in this case we obtain $\Delta_3 = \Delta_4 = -c^2 < 0$, i.e. both points are saddles. So, the condition $\mathcal{E} < 0$ is necessary. So the respective conditions provided by statement (c) of Main Theorem are verified.

3.1.5.2 The case $\mathcal{T}_4 = 0$. Since $\mu_1 \neq 0$ considering (3.17) we obtain $h = 0$ and then for systems (3.18) we have $\mathcal{T}_i = 0$ for each $i = 1, \ldots, 4$ and (considering Remark 2.3) we calculate

$$\mathcal{E}_1 = -2cdm\, \rho_1\rho_2\rho_3, \quad \mu_1 = -4cdm^2 x \neq 0.$$

Due to the condition $\mu_1 \neq 0$ in order to have at least one zero trace it is necessary and sufficient that $\mathcal{E}_1 = 0$. In this case by Remark 2.2 we may assume $\rho_1 = 0$, i.e. $f = -c$ and then we obtain the systems

$$\dot{x} = cx + dy - cx^2, \quad \dot{y} = ex - cy - ex^2 + 2mxy. \tag{3.20}$$

For these systems calculations yield:

$$\begin{aligned}
&\Delta_1 = -(c^2 + de), \quad \rho_2 = 2(m - c), \quad \rho_3 = (c - m)\Delta_1/(cm), \quad \sigma = 2(m - c)x,\\
&\mathcal{E}_2 = -4d^2(c - m)^4\Delta_1, \quad \mathcal{F}_1 = 24d(m - c)(c^2 + de + 2cm) = G_1/6, \quad \mathcal{E}_1 = \mathcal{H} = 0,
\end{aligned}$$

where G_1 is one of the GL-invariants associated to the origin (see Lemma 2.1). So if $\mathcal{F}_1 \neq 0$ then $\operatorname{sign}(\Delta_1) = -\operatorname{sign}(\mathcal{E}_2)$ and $G_1 \neq 0$. As for systems (3.20) we have $G_0 = 0$ then by Lemma 2.1 in the case $\mathcal{F}_1 \neq 0$ the singular point $M_1(0,0)$ is a weak focus of the first order if and only if $\mathcal{E}_2 < 0$. As $\mathcal{F}_1 \neq 0$ implies $\sigma \neq 0$ and for these systems $\mathcal{E}_1 = \mathcal{H} = 0$ we get the conditions (e_1) of Main Theorem.

Assuming $\mathcal{F}_1 = 0$ we shall consider two subcases: $\sigma \neq 0$ and $\sigma = 0$.

3.1.5.2.1 If $\sigma \neq 0$ then $m - c \neq 0$ and this implies $\rho_2\rho_3 \neq 0$, i.e. none of the points M_2 and M_3 could be a weak focus. We shall examine the point $M_1(0,0)$. Since for systems (3.20) we have $G_0 = G_1 = G_2 = G_3 = 0$ then by Lemma 2.1 this point is a center if and only if $\mathcal{E}_2 < 0$. Thus the conditions (e_2) (first part) of Main Theorem are satisfied.

3.1.5.2.2 Suppose now $\sigma = 0$. This leads to the Hamiltonian systems

$$\dot{x} = cx + dy - cx^2, \quad \dot{y} = ex - cy - ex^2 + 2cxy, \tag{3.21}$$

for which calculations yield:

$$\Delta_1 = -c^2 - de, \quad \Delta_2 = -c^2 + de, \quad \Delta_3 = \frac{1}{2c^2}\Delta_1\Delta_2.$$

We claim that among the determinants Δ_i $(i = 1, 2, 3)$ one and only one is positive. Indeed, we could not have $\Delta_1 > 0$ and $\Delta_2 > 0$ (otherwise we obtain the contradictory relation: $-2c^2 > 0$). Therefore we have either $\Delta_1\Delta_2 < 0$ (and then $\Delta_3 < 0$) or $\Delta_1 < 0$ and $\Delta_2 < 0$ (and then $\Delta_3 > 0$). So our claim is proved and this means that only one singular point of systems (3.20) is a center.

On the other hand considering Table 1 we observe that the conditions (f_6) of Main Theorem are verified.

3.1.6 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = p + q^c + r^c$

Considering Table 2 for the simple finite singularities $M_1(0,0)$ and $M_{2,3}(u \pm i, 1)$ of systems 11) we obtain, respectively:

$$\begin{aligned}
&\rho_1 = l(u^2 + 1) + 2(h - gu), \quad \rho_{2,3} = l(u^2 + 1) - 2mu \pm 2i(g - m),\\
&\mathcal{T}_4 = \mathcal{G}_1 = 8h(u^2 + 1)(hl - gm)^2, \quad \mu_0 = 0, \quad \mu_1 = 4(u^2 + 1)(hl - gm)^2 x.
\end{aligned} \tag{3.22}$$

As for this family of systems the conditions $\mu_0 = 0$ and $\mu_1 \neq 0$ hold then according to Theorem 2.1 (see formulas (2.18)) we shall consider two subcases: $\mathcal{T}_4 \neq 0$ and $\mathcal{T}_4 = 0$.

3.1.6.1 The case $\mathcal{T}_4 \neq 0$. Then by (3.22) we have $\mathcal{G}_1 \neq 0$ and hence the condition $\mathcal{T}_1 = \mathcal{T}_2 = 0$ (see Remark 3.1 implies the vanishing of at least two traces. Since we have two complex traces clearly the equalities $\rho_2 = \rho_3 = 0$ have to be fulfilled. Considering (3.22) we obtain $g = m$, $l = 2mu/(u^2+1)$ and then we obtain $\rho_1 = 2h \neq 0$ due to $\mathcal{T}_4 \neq 0$. So if $\mathcal{T}_1 = \mathcal{T}_2 = 0$ and $\mathcal{T}_4 \neq 0$ then the real singular point could not be a weak focus.

It remains to note that in this case we have

$$\mathcal{T}_3 = 16h^2m^2(2hu - m - mu^2)^2/(u^2+1) = 2h\mathcal{T}_4 \neq 0, \quad \mathcal{E} = 32h^4m^2u^2/(u^2+1)^2 \geq 0.$$

and hence the conditions provided by statement (c) are not verified.

3.1.6.2 The case $\mathcal{T}_4 = 0$. Since $\mu_1 \neq 0$ considering (3.22) we obtain $h = 0$ and then for systems 11) we have $\mathcal{T}_i = 0$ for each $i = 1, \ldots, 4$. On the other hand for these systems with $h = 0$ we calculate:

$$\mathcal{E}_1 = -2g^2m\,\rho_1\rho_2\rho_3, \quad \mu_1 = 4g^2m^2(1+u^2)x \neq 0, \quad \sigma = l(u^2+1) - 2gu + 2(g-m)x.$$

Since $\mu_1 \neq 0$ the condition $\mathcal{E}_1 = 0$ is equivalent to $\rho_1\rho_2\rho_3 = 0$, where ρ_i are given in (3.22) (setting $h = 0$). So assuming $\mathcal{E}_1 = 0$ we shall consider two subcases: $\sigma \neq 0$ and $\sigma = 0$.

3.1.6.2.1 Suppose first $\sigma \neq 0$. We claim that in this case the condition $\mathcal{E}_1 = 0$ is equivalent to $\rho_1 = 0$ (i.e. only the trace corresponding to real point vanishes). Indeed, admit the contrary, that $\rho_2 = 0$ (then $\rho_3 = 0$). Considering (3.22) we get $g - m = l(u^2+1) - 2mu = 0$ and this evidently implies $\sigma = 0$, i.e. our claim is proved.

Thus we have $\rho_1 = 0$ and this yields $l = 2gu/(u^2+1)$. So we arrive to the following family of systems:

$$\begin{aligned} \dot{x} &= -2gu\,x + g(u^2+1)y + gx^2, \\ \dot{y} &= \frac{2}{u^2+1}(m - 2gu^2 + mu^2)x + 2gu\,y + \frac{2gu}{u^2+1}x^2 - 2mxy, \end{aligned} \tag{3.23}$$

for which calculations yield:

$$\begin{aligned} &\Delta_1 = -2gm(1+u^2), \quad \sigma = 2(g-m)x, \quad G_0 = G_2 = G_3 = 0, \\ &\mathcal{E}_2 = -4g^2(1+u^2)^2(g-m)^4\Delta_1, \quad \mathcal{F}_1 = 48g(m-g)u\Delta_1 = G_1/6. \end{aligned}$$

We observe that the condition $\sigma\Delta_1 \neq 0$ implies $\mathcal{E}_2 \neq 0$ and then $\mathrm{sign}\,(\Delta_1) = -\mathrm{sign}\,(\mathcal{E}_2)$.

If $\mathcal{F}_1 \neq 0$ we get $G_1 \neq 0$ and since for systems (3.23) we have $G_0 = 0$, by Lemma 2.1 in this case the singular point $M_1(0,0)$ is a weak focus of the first order if and only if $\mathcal{E}_2 < 0$.

If $\mathcal{F}_1 = 0$ then for systems (3.23) we have $G_0 = G_1 = G_2 = G_3 = 0$ and by Lemma 2.1 this point is a center if and only if $\mathcal{E}_2 < 0$. So the respective conditions provided by statement (e) of Main Theorem are verified.

3.1.6.2.2 Assume now $\sigma = 0$. Then $m = g \neq 0$ (otherwise systems (3.23) become degenerate) and after the rescaling $t \to t/g$ we get the following family of Hamiltonian systems:

$$\dot{x} = -2u\,x + (u^2+1)y + x^2, \quad \dot{y} = \frac{2(1-u^2)}{1+u^2}\,x + 2u\,y + \frac{2u}{u^2+1}x^2 - 2xy. \tag{3.24}$$

According to Table 1 for this family of systems the conditions $\mu_0 = 0$, $\mathbf{D} > 0$ and $\mathbf{R} \neq 0$ are verified. So this contradicts to the statement (f) of Main Theorem.

3.1.7 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 2p + q$

Considering Table 2 for the finite singularities $M_{1,2}(0,0)$ (double) and $M_3(1,0)$ of systems 12) we obtain, respectively:

$$\begin{aligned} &\rho_{1,2} = c + eu, \quad \rho_3 = -c + 2m + eu, \quad \mu_0 = 0, \\ &\mathcal{T}_4 = \mathcal{G}_1 = 8h(eh - cm)^2 u, \quad \mu_1 = -4(eh - cm)^2 ux \neq 0. \end{aligned} \tag{3.25}$$

As for this family of systems the conditions $\mu_0 = 0$ and $\mu_1 \neq 0$ hold then in accordance with Theorem 2.1 (see formulas (2.18)) we shall consider two subcases: $\mathcal{T}_4 \neq 0$ and $\mathcal{T}_4 = 0$.

3.1.7.1 The case $\mathcal{T}_4 \neq 0$.

From (3.25) it follows $\mathcal{G}_1 \neq 0$ and hence the condition $\mathcal{T}_1 = \mathcal{T}_2 = 0$ (see Remark 3.1) implies the vanishing of at least two traces. Therefore the equality $\rho_{1,2} = 0$ has to be fulfilled and considering (3.25) we obtain $c = -eu$. So we get the family of systems

$$\dot{x} = -eux - eu^2 y + eux^2 + 2hxy, \quad \dot{y} = ex + euy - ex^2 + 2mxy, \tag{3.26}$$

for which we calculate

$$\begin{aligned} &\Delta_3 = 2e(h + mu), \ \ \rho_3 = 2(m + eu), \ \ \mathcal{T}_3 = 2hu\rho_3\Delta_3^2, \ \ \mathcal{E} = -2e^2(mu - h + eu^2)^4, \\ &\mathcal{T}_4 = 2hu\Delta_3^2, \ \ \mathcal{F}_1 = -2eu(mu - h + eu^2)\Delta_3, \ \ \mathcal{F} = -h(mu - h + eu^2)\rho_3\Delta_3/2. \end{aligned}$$

We observe that if $\mathcal{T}_3 \neq 0$ the simple point $M_3(0,0)$ could not be a weak focus (or center). On the other hand the conditions provided by statement (b) of Main Theorem could not be satisfied. Indeed, the necessary condition $\mathcal{F} = 0$ due to $\mathcal{T}_3 \neq 0$ implies $mu - h + eu^2 = 0$ and then $\mathcal{E} = 0$ that contradicts to the necessary condition $\mathcal{E} < 0$.

Assume $\mathcal{T}_3 = 0$. Since $\mathcal{T}_4 \neq 0$ we get $\rho_3 = 0$ (i.e. $m = -eu$) and then we have

$$\Delta_3 = 2e(h - eu^2), \ \ \mathcal{T}_4 = 2hu\Delta_3^2, \ \ \mathcal{F}_1 = 2ehu\Delta_3, \ \ \mathcal{H} = -2h^2\Delta_3, \ \ \mathcal{E} = -2e^2h^4, \ \ \mathcal{F} = 0.$$

As $\mathcal{T}_4 \neq 0$ we obtain $\operatorname{sign}(\Delta_3) = -\operatorname{sign}(\mathcal{H})$. We observe that as $m = -eu$ systems (3.26) possess two real points with zero traces (one of them being double). So according to Lemma 2.2 and Remark 2.1 we conclude that the point M_3 is a weak focus of the first order (respectively a center) if and only if $\mathcal{H} < 0$ and $\mathcal{F}_1 \neq 0$ (respectively $\mathcal{F}_1 = 0$). It remains to note that the condition $\mathcal{H}\mathcal{T}_4 \neq 0$ implies $\mathcal{E} < 0$ and this leads to the respective conditions provided by statement (d) of Main Theorem.

3.1.7.2 The case $\mathcal{T}_4 = 0$.

Since $\mu_1 \neq 0$ considering (3.25) we obtain $h = 0$ and then for systems 12) we have $\mathcal{T}_i = 0$ for each $i = 1, \dots, 4$. On the other hand for these systems with $h = 0$ we calculate:

$$\begin{aligned} &\mathcal{E}_1 = 2cu(c + eu)^2\rho_3\Delta_3, \quad \mu_1 = -4c^2m^2ux \neq 0, \\ &\mathcal{E}_2 = c^2u^2(c - m)^2(c - 4m - 3eu)(c + eu)\Delta_3. \end{aligned} \tag{3.27}$$

It is clear that in order to have a weak focus the condition $\mathcal{E}_1 = 0$ is necessary.

3.1.7.2.1 If $\mathcal{E}_2 \neq 0$ then the condition $\mathcal{E}_1 = 0$ yields $\rho_3 = 0$, i.e. according to (3.25) we obtain $c = 2m + eu$. Then after translation of the point M_3 to the origin of coordinates we get the family of systems

$$\dot{x} = (2m + eu)(-x - x^2 + uy), \quad \dot{y} = -ex + (2m + eu)y - ex^2 + 2mxy, \tag{3.28}$$

for which we calculate

$$\Delta_3 = -2m(2m+eu), \quad \rho_3 = 0, \quad G_0 = G_2 = G_3 = 0, \quad \mathcal{H} = 0, \ \sigma = 2(m+eu)(1-x),$$
$$\mathcal{E}_2 = -4u^2(m+eu)^4(2m+eu)^2\Delta_3, \quad \mathcal{F}_1 = 96mu(m+eu)(2m+eu)^2 = G_1/6.$$

We observe that the condition $\Delta_3\mathcal{E}_2 \neq 0$ implies $\sigma\mathcal{F}_1 \neq 0$ (then $G_1 \neq 0$) and $\operatorname{sign}(\Delta_3) = -\operatorname{sign}(\mathcal{E}_2)$.

Therefore as for systems (3.28) we have $G_0 = 0$ then by Lemma 2.1 the singular point $M_1(0,0)$ is a weak focus of the first order if and only if $\mathcal{E}_2 < 0$. We observe that in this case the conditions (e_1) of Main Theorem hold.

3.1.7.2.2 Assume now $\mathcal{E}_2 = 0$. Since $\mu_1 \neq 0$ considering (3.27) the conditions $\mathcal{E}_1 = \mathcal{E}_2 = 0$ implies $c + eu = 0$. We claim that in the case $\sigma \neq 0$ the simple singular point could not be a weak focus (or a center). Indeed setting $c = -eu$ we obtain systems (3.26) with $h = 0$ and we calculate: $\sigma = 2(m+eu)x = \rho_3 x \neq 0$ and this proves our claim. On the other hand the condition $\mathcal{E}_2 = 0$ contradicts to statement (e) of Main Theorem.

Assuming $\sigma = 0$ we have $h = 0$, $m = c = -eu \neq 0$ and setting $e = 1$ (due to a rescaling) this leads to the family of Hamiltonian systems

$$\dot{x} = -ux - u^2y + ux^2, \quad \dot{y} = x + uy - x^2 - 2uxy. \tag{3.29}$$

For these systems we obtain $\Delta_3 = -2u^2 < 0$ and hence the simple point of systems (3.29) could not be a focus (or center). On the other hand according to Table 1 for this family of systems the conditions $\mu_0 = \mathbf{D} = 0$ and $\mathbf{PR} \neq 0$ are verified. So this contradicts to the statement (f) of Main Theorem.

3.1.8 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = p + q$

According to Table 2 for the systems in this class there exist two canonical forms: $14a$) (when quadratic parts $p_2(x,y)$ and $q_2(x,y)$ are not proportional, i.e. $K = \operatorname{Jacob}(p_2,q_2) \neq 0$) and $14b$) (when p_2 and q_2 are proportional, i.e. $K = 0$) and we shall examine each of them.

3.1.8.1 Canonical systems $14a$)**.** Considering Table 2 for the simple finite singularities $M_1(0,0)$ and $M_2(1,0)$ of these systems we obtain, respectively:

$$\begin{aligned} &\rho_1 = c+f, \quad \rho_2 = -c+f+2fu, \quad \Delta_1 = cf - de, \quad \Delta_2 = -(2u+1)\Delta_1, \\ &\mathcal{T}_3 = \mathcal{G}_2 = -4d^2u^2\Delta_1\Delta_2, \quad \mu_0 = \mu_1 = 0, \quad \mu_2 = -\Delta_1\Delta_2x^2 \neq 0, \quad K = -u\Delta_1x^2. \end{aligned} \tag{3.30}$$

As for this family of systems the conditions $\mu_0 = \mu_1 = 0$ and $\mu_2 \neq 0$ hold then in accordance with Theorem 2.1 (see formulas (2.19)) we shall consider two subcases: $\mathcal{T}_3 \neq 0$ and $\mathcal{T}_3 = 0$.

3.1.8.1.1 The case $\mathcal{T}_3 \neq 0$. From (3.30) it follows $\mathcal{G}_2 \neq 0$ and hence considering (2.19) the condition $\mathcal{T}_1 = \mathcal{T}_2 = 0$ (see Remark 3.1) implies the vanishing of both traces, i.e. $\rho_1 = \rho_2 = 0$. Considering (3.30) the relation $\rho_1 = 0$ implies $f = -c$ and then $\rho_2 = -2c(u+1) = 0$. Since for systems $14a$) in this case we have $\mathcal{F}_1 = -2cd(u-1)\Delta_2$, we shall consider two possibilities: $\mathcal{F}_1 \neq 0$ and $\mathcal{F}_1 = 0$.

1) *The subcase* $\mathcal{F}_1 \neq 0$. Then $c \neq 0$ and the condition $\rho_2 = 0$ yields $u = -1$. This leads to the family of systems

$$\dot{x} = cx + dy - cx^2 - 2dxy, \quad \dot{y} = ex - cy - ex^2 + 2cxy, \tag{3.31}$$

for which calculations yield:

$$\Delta_1 = \Delta_2 = -(c^2 + de), \quad \mathcal{F} = 0, \quad \mathcal{H} = -4d^2\Delta_1,$$
$$\mathcal{T}_3 = -4d^2(c^2 + de)^2 < 0, \quad \mathcal{E} = -2d^4e^2 \leq 0.$$

We observe that $\text{sign}(\Delta_1) = \text{sign}(\Delta_2) = -\text{sign}(\mathcal{H})$. Since $\mathcal{F}_1 \neq 0$, according to Lemma 2.2 we conclude, that systems (3.31) possess two weak foci of the first order if and only if $\mathcal{H} < 0$. It remains to note that the condition $\mathcal{E} < 0$ holds for systems (3.31) in the case $\mathcal{H} < 0$ (otherwise the condition $\mathcal{E} = 0$ implies $e = 0$ and then $\Delta_1 < 0$). This leads to the conditions (c_2) of Main Theorem.

2) *The subcase* $\mathcal{F}_1 = 0$. Due to the condition $\mathcal{T}_3 \neq 0$ (i.e. $d \neq 0$) we get $c(u-1) = 0$ and then the condition $\rho_2 = -2c(u+1) = 0$ yields $c = 0$. Therefore we obtain the following family of systems (assuming $d = 1$ due to a time rescaling):

$$\dot{x} = y + 2uxy, \quad \dot{y} = ex - ex^2. \tag{3.32}$$

For these systems we we have $\rho_1 = \rho_2 = 0$ and

$$\Delta_1 = -e, \quad \Delta_2 = e(1 + 2u), \quad \mathcal{T}_3 = -4u^2\Delta_1\Delta_2, \quad \mathcal{F} = 0,$$
$$\mathcal{H} = -2u^2(\Delta_1 + \Delta_2), \quad \mathcal{E} = -384e^2u^4 < 0.$$

We observe that $\text{sign}(\Delta_1\Delta_2) = -\text{sign}(\mathcal{T}_3)$. If $\mathcal{T}_3 > 0$ then $\Delta_1\Delta_2 < 0$ and since $\mathcal{F}_1 = 0$ according to Lemma 2.2 only one point of systems (3.32) is a center. If $\mathcal{T}_3 < 0$ we obtain $\Delta_1\Delta_2 > 0$ and since $\text{sign}(\Delta_1 + \Delta_2) = -\text{sign}(\mathcal{H})$ we conclude, that systems (3.32) possess two centers if and only if $\mathcal{H} < 0$. As $\mathcal{F} = 0$ and $\mathcal{E} < 0$ this leads to the respective conditions provided by statement (c) of Main Theorem.

3.1.8.1.2 The case $\mathcal{T}_3 = 0$. We claim that in this case systems $14a)$ could not possess a weak focus (or a center). Indeed, as $K \neq 0$ from (3.30) we obtain $d = 0$ and this leads to the family of systems

$$\dot{x} = cx(1 - x), \quad \dot{y} = ex + fy - ex^2 + 2fuxy.$$

So both finite singularities are located on the invariant lines $x = 0$ and $x = 1$ and this proves our claim. On the other hand for these systems calculations yield:

$$\mathcal{T}_i = 0, \ (i = 1, 2, 3, 4), \quad \sigma = c + f - 2(c - fu)x, \quad \mathcal{H} = \mathcal{E}_2 = 0$$

and in the case $\sigma \neq 0$ this contradicts to the statement (e) of Main Theorem.

If $\sigma = 0$ then considering Table 1 and the condition $K \neq 0$ we get a contradiction with statement (f) of Main Theorem.

3.1.8.2 Canonical systems $14b)$. From Table 2 we observe that if $u = 0$ then we get the family of systems

$$\dot{x} = (x - 1)(gx + 2hy) + ky^2, \quad \dot{y} = y, \tag{3.33}$$

possessing two simple singularities $M_1(0, 0)$ and $M_2(1, 0)$ located on the invariant line $y = 0$. Clearly in this case we could not have a focus or a center.

We claim that for these systems could not be satisfied any set of the conditions provided by Main Theorem for the existence of a weak focus or a center. Indeed for systems (3.33) calculations yield:

$$\begin{aligned} &\rho_1 = 1-g, \quad \rho_2 = 1+g, \quad \Delta_1 = -g, \quad \Delta_2 = g,\\ &\mathcal{T}_2 = 8g^2(gk-h^2) = 2\mathcal{T}_3 = \mathcal{G}_2, \quad \mathcal{T}_1 = \mathcal{G}_2\rho_1\rho_2. \end{aligned} \tag{3.34}$$

The conditions $\mathcal{T}_1 = 0$ and $\mathcal{T}_2 \neq 0$ yields $\rho_1\rho_2 = 0$ and by Remark 2.2 we may assume $\rho_1 = 0$ (i.e. $g = 1$). Then $\mathcal{T}_1 = 8(k-h^2)^2 = 8\mathcal{F}$ and hence we obtain $\mathcal{T}_2\mathcal{F} = 8(k-h^2)^4 \geq 0$ and this contradicts to statement (b) of Main Theorem.

Assume $\mathcal{T}_1 = \mathcal{T}_2 = 0$. Then $g = 1$ and $k = h^2$ and for systems (3.33) in this case we have

$$\mathcal{T}_i = 0, \ (i = 1,2,3,4), \quad \sigma = 2(x+hy) \neq 0, \quad \mathcal{H} = \mathcal{E}_2 = 0$$

and this contradicts to the statement (e) of Main Theorem. So our claim is proved.

Assume $u \neq 0$. Then via the transformation $(x,y) \mapsto (y-x, \ -uy)$ we get the following family of systems

$$\dot{x} = cx - ky + gx^2 + 2hxy + ky^2, \quad \dot{y} = x, \tag{3.35}$$

possessing two simple singularities $M_1(0,0)$ and $M_2(0,1)$. Calculations yield:

$$\begin{aligned} &\rho_1 = c, \quad \rho_2 = c+2h, \quad \Delta_1 = k = -\Delta_2,\\ &\mathcal{T}_3 = \mathcal{G}_2 = 4g(h^2-gk)\Delta_1, \quad \mu_0 = \mu_1 = 0, \quad \mu_2 = \Delta_1(gx^2+2hxy+ky^2). \end{aligned} \tag{3.36}$$

As for this family of systems $\mu_2 \neq 0$ then in accordance with Theorem 2.1 (see formulas (2.19)) we shall consider two subcases: $\mathcal{T}_3 \neq 0$ and $\mathcal{T}_3 = 0$.

3.1.8.2.1 The case $\mathcal{T}_3 \neq 0$. From (3.36) it follows $\mathcal{G}_2 \neq 0$ and hence the condition $\mathcal{T}_1 = \mathcal{T}_2 = 0$ (see Remark 3.1 implies the vanishing of both traces, i.e. $\rho_1 = \rho_2 = 0$. Considering (3.36) we obtain $c = h = 0$ and then for systems (3.35) we have:

$$\mathcal{F} = \mathcal{F}_1 = 0, \quad \mathcal{T}_3 = 4g^2k^2 > 0, \quad \mathcal{E} = -2g^4k^2 < 0.$$

Since $\Delta_1\Delta_2 < 0$ and $\mathcal{F}_1 = 0$ according to Lemma 2.2 only one point is a center. It remains to note that $\mathcal{T}_3 > 0$, $\mathcal{F} = 0$ and $\mathcal{E} = -2h^4 < 0$, i.e. the conditions (c_3) of Main Theorem are satisfied.

3.1.8.2.2 The case $\mathcal{T}_3 = 0$. Considering (3.36) we obtain $g(h^2 - gk) = 0$ and we have to consider two subcases: $g = 0$ and $g = h^2/k \neq 0$.

1) *The subcase* $g = 0$. Then we get the systems

$$\dot{x} = cx - ky + 2hxy + ky^2, \quad \dot{y} = x, \tag{3.37}$$

for which we have:

$$\mathcal{T}_i = 0, \ (i = 1,2,3,4), \quad \mathcal{E}_1 = -2h\rho_1\rho_2\Delta_1, \quad \mathcal{F}_1 = -2h\Delta_1, \quad \mathcal{E}_2 = 2h^3(\rho_1+\rho_2)\Delta_1. \tag{3.38}$$

It is clear that in order to have a weak focus the condition $\mathcal{E}_1 = 0$ is necessary.

a) Assume first that $\mathcal{F}_1 \neq 0$. Then the condition $\mathcal{E}_1 = 0$ yields $\rho_1\rho_2 = 0$ and due to Remark 2.2 we may assume $\rho_1 = 0$. Hence $c = 0$ and for systems (3.37) we have:

$$\Delta_1 = k, \quad \rho_2 = 2h \neq 0, \quad \mathcal{E}_2 = -4h^4\Delta_1, \quad G_0 = G_2 = G_3 = 0 = \mathcal{H}, \quad G_1 = 6\mathcal{F}_1 \neq 0, \ \sigma = 2hy$$

and according to Lemma 2.1 the singular point $M_1(0,0)$ is a weak focus of the first order if and only if $\mathcal{E}_2 < 0$. As the conditions $\mathcal{H} = 0$ and $\sigma \neq 0$ hold this leads to the conditions (e_1) of Main Theorem.

***b*)** Suppose now $\mathcal{F}_1 = 0$. Considering (3.38) we obtain $h = 0$ and for systems (3.37) we have:

$$\rho_1 = \rho_2 = c = \sigma, \quad \Delta_1 = -\Delta_2 = k.$$

So evidently in order to have a weak point the condition $\sigma = 0$ is necessary and then we arrive to the Hamiltonian systems possessing only one center. Considering Table 1 the conditions (f_7) of Main Theorem hold in this case.

2) *The subcase* $g = h^2/k \neq 0$. Then we get the systems

$$\dot{x} = cx - ky + h^2x^2/k + 2hxy + ky^2, \quad \dot{y} = x, \tag{3.39}$$

for which considering Remark 2.3 we have:

$$\begin{aligned} &\mathcal{E}_1 = -2h(ch + h^2 + k)\rho_1\rho_2, \quad \mathcal{F}_1 = -2h(ch + h^2 + k),\\ &\mathcal{T}_i = 0, \ (i = 1, 2, 3, 4), \quad \mathcal{H} = 0, \quad \sigma = c + 2h^2x/k + 2hy \neq 0. \end{aligned} \tag{3.40}$$

It is clear that in order to have a weak focus the condition $\mathcal{E}_1 = 0$ is necessary.

***a*)** Assume first $\mathcal{F}_1 \neq 0$. Then the condition $\mathcal{E}_1 = 0$ yields $\rho_1\rho_2 = 0$ and due to Remark 2.2 we may assume $\rho_1 = 0$. Hence $c = 0$ and for systems (3.39) we have:

$$\begin{aligned} &\Delta_1 = k, \quad \rho_2 = 2h \neq 0, \quad G_1 = 6\mathcal{F}_1 \neq 0, \quad \mathcal{F}_1 = -2h(h^2 + k),\\ &G_0 = G_2 = G_3 = 0, \quad \mathcal{E}_2 = -(4h^4(h^2 + k)^2)/k. \end{aligned}$$

So the condition $\mathcal{F}_1 \neq 0$ implies $\mathcal{E}_2 \neq 0$ and then $\operatorname{sign}(\mathcal{E}_2) = \operatorname{sign}(\Delta_1)$. Hence according to Lemma 2.1 the singular point $M_1(0,0)$ is a weak focus of the first order if and only if $\mathcal{E}_2 < 0$ and this focus is unique due to $\rho_2 = 2h \neq 0$. So we get the conditions (e_1) of Main Theorem.

***b*)** Suppose now $\mathcal{F}_1 = 0$. Considering (3.40) and $h \neq 0$ we obtain $k = -h(c + h) \neq 0$.

We claim that in this case the considered family of systems could not have a weak focus or a center. Indeed calculations yield:

$$\rho_1 = c, \ \rho_2 = c + 2h, \quad \Delta_2 = h(c + h) = -\Delta_1, \ \sigma = c + 2hx/(c + h) + 2hy \neq 0.$$

We observe that if $\rho_1 = 0$ (respectively $\rho_2 = 0$) we have $c = 0$ (respectively $c = -2h$) and then $\Delta_1 = -h^2 < 0$ (respectively $\Delta_2 = -h^2 < 0$) and this proves our claim.

On the other hand we have $\sigma \neq 0$ and $\mathcal{H} = \mathcal{E}_2 = 0$ and this is in contradiction with statement (e) of Main Theorem.

3.1.9 Systems with the zero–cycle $\mathcal{D}_S(P, Q) = p$

According to Table 2 for the systems in this class having the simple finite singularity $M_1(0,0)$ there exist two canonical forms: $17a)$ (in the case $K = \operatorname{Jacob}(p_2, q_2) \neq 0$) and $17b)$ (when $K = 0$).

Remark 3.2. *We note that according to Table 1 for both families of systems the conditions* $\mu_0 = \mathbf{R} = \mathbf{P} = 0$ *and* $\mathbf{U} \neq 0$ *hold.*

3.1.9.1 Canonical systems $17a$**).** For these systems (see Table 2) we calculate:

$$\begin{aligned}&\rho_1 = c+f, \quad \Delta_1 = cf-de, \quad \mathcal{T}_1 = -8d^3u\rho_1\Delta_1^2, \ \mathcal{F}_1 = -6d^2u\Delta_1,\\ &\mathcal{T}_2 = -8d^3u\Delta_1^2, \quad \mu_0=\mu_1=\mu_2=0, \quad \mu_3 = u\Delta_1^2x^3 \neq 0.\end{aligned} \tag{3.41}$$

Considering Remark 3.1 and setting $\mathcal{T}_1 = \mathcal{T}_2 = 0$ due to $\mu_3 \neq 0$ we get $d = 0$ and then $\Delta_1 = cf$. So, the condition $\rho_1 = 0$ implies $\Delta_1 = -c^2 < 0$ and systems $17a$) could not possess a weak focus in this case.

On the other hand for these systems we have $\mathcal{T}_i = 0,\ (i = 1,2,3,4)$ and $\mathcal{H} = \mathcal{E}_2 = 0$. Therefore as $\sigma = c+f+2(2c+f)x \neq 0$ (otherwise we obtain degenerate systems) we get a contradiction with statement (e) of Main Theorem.

3.1.9.2 Canonical systems $17b$**).** Considering Table 2 for these systems we calculate:

$$\begin{aligned}&\rho_1 = 1+f, \quad \Delta_1 = f-de, \quad \mathcal{T}_1 = 8d^2(ld-2m)(ld-m)^3\rho_1\Delta_1,\\ &\mathcal{F}_1 = -2d(ld-2m)(ld-m)\Delta_1, \ \mathcal{T}_2 = 8d^2(ld-2m)(ld-m)^3\Delta_1.\end{aligned} \tag{3.42}$$

So the conditions $\mathcal{T}_1 = \mathcal{T}_2 = 0$ (see Remark 3.1) imply $d(ld-m)(ld-2m) = 0$ and then we have

$$\mathcal{T}_i = 0, \ (i=1,2,3,4) \quad \mathcal{H} = \mathcal{F}_1 = \mathcal{E}_2 = 0.$$

We shall consider three cases.

3.1.9.2.1 The case $d = 0$**.** Then by (3.42) we obtain that the relation $\rho_1 = 0$ implies $\Delta_1 = -1 < 0$, i.e. in this case systems $17b$) could not possess a weak focus. On the other hand if $\sigma = 1+f+2mx \neq 0$ then due to $\mathcal{H} = \mathcal{E}_2 = 0$ we get a contradiction with statement (e) of Main Theorem. If $\sigma = 0$ (i.e. $f+1 = m = 0$) then we get a contradiction with statement (f) due to Remark 3.2.

3.1.9.2.2 The case $m = dl$**.** This leads to the family of systems

$$\dot{x} = x+dy, \quad \dot{y} = ex+fy+l(x+dy)^2, \tag{3.43}$$

for which we have

$$\mathcal{E}_1 = 4d^3l^2\rho_1\Delta_1^2, \quad \mathcal{E}_2 = -4d^6l^2\Delta_1^3, \quad \mu_3 = l\Delta_1(x+dy)^3 \neq 0. \tag{3.44}$$

It is clear that in order to have a weak focus the condition $\mathcal{E}_1 = 0$ is necessary.

If $\mathcal{E}_2 \neq 0$ the last condition yields $\rho_1 = 0$ and we conclude that the point $M_1(0,0)$ is either a weak focus or a center if and only if $\mathcal{E}_2 < 0$. We claim that this point is a center. Indeed, assume $\rho_1 = 0$ (i.e. $f = -1$). Then for systems (3.43) we have $G_0 = G_1 = G_2 = G_3 = 0$ and $G_5 = -\Delta$. So according to Lemma 2.1 this point is a center and our claim is proved. It remains to note that in this case $\mathcal{H} = \mathcal{F}_1 = 0$ and hence the conditions (e_2) of Main Theorem are verified.

If $\mathcal{E}_2 = 0$ then due to $\mu_3 \neq 0$ we get $d = 0$ and this leads to the previous case.

3.1.9.2.3 The case $m = dl/2$**.** This leads to the family of systems

$$\dot{x} = x+dy, \quad \dot{y} = ex+fy+lx(x+dy), \tag{3.45}$$

for which we have

$$\mathcal{E}_1 = d^3l^2\rho_1\Delta_1^2, \quad \mathcal{E}_2 = -d^6l^2\Delta_1/4, \quad \mu_3 = l\Delta_1 x(x+dy)^2 \neq 0. \tag{3.46}$$

The necessary condition $\mathcal{E}_1 = 0$ yields $\rho_1 = 0$ (i.e. $f = -1$) if $\mathcal{E}_2 \neq 0$. In this case for systems (3.45) we have $G_0 = G_1 = G_2 = G_3 = 0$ and $G_5 = -\Delta$. So according to Lemma 2.1 the singular point $M_1(0,0)$ is is a center if and only if $\mathcal{E}_2 < 0$. So taking into consideration $\mathcal{H} = \mathcal{F}_1 = 0$ the conditions (e_2) of Main Theorem hold.

If $\mathcal{E}_2 = 0$ then due to $\mu_3 \neq 0$ we get $d = 0$ and this leads to the case above.

As all the systems possessing at least one simple finite real singularity are examined, this completes the proof of the Step 1.

3.2 Step 2: incompatibility of the conditions for systems without simple real singularities

According to Table 2 we shall consider step by step the family of systems corresponding to the following values of the zero–cycle $\mathcal{D}_S(P,Q)$:

$$p^c + q^c + r^c + s^c;\ 2p + q^c + r^c;\ 2p^c + 2q^c;\ 4p;\ 3p;\ p^c + q^c;\ 2p;\ 0.$$

3.2.1 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = p^c + q^c + r^c + s^c$

The systems in this class have four complex singular points and according to Table 1 the conditions $\mu_0 \neq 0$, $\mathbf{D} < 0$ and either $\mathbf{R} \leq 0$, or $\mathbf{S} \leq 0$ are verified. Hence in the case $\sigma = 0$ (i.e. when these systems are Hamiltonian) we get a contradiction with the statement (f) of Main Theorem.

In Table 2 this class of systems is presented by two canonical forms: $3a$) and $3b$). We shall examine each of them assuming $\sigma \neq 0$.

3.2.1.1 Canonical systems $3a$)**.** Considering Table 2 for the complex singular points $M_{1,2}(0,\pm i)$ and $M_{3,4}(u \pm i, v)$ of these systems we obtain, respectively:

$$\begin{aligned} \rho_{1,2} &= -2(gu+hv) \pm i\Big[2h + \frac{2l(u^2+1)}{v^2+1}\Big] \equiv U_1 \pm iV_1, \\ \rho_{3,4} &= 2mu + \frac{2lv(u^2+1)}{v^2+1} \pm 2i(g+m) \equiv U_2 \pm iV_2, \end{aligned} \tag{3.47}$$

Clearly the conditions $\mathcal{T}_1 = 0$ and $\mathcal{T}_2 \neq 0$ (see formulas (2.17)) could not hold (i.e. at least two traces must be zero).

Assume $\mathcal{T}_1 = \mathcal{T}_2 = 0$ and $\mathcal{T}_3 \neq 0$, i.e. only two traces vanish. Due to an affine transformation we may assume $\rho_{1,2} = 0$ and considering (3.47) we get $U_1 = V_1 = 0$. Since $g^2 + h^2 \neq 0$ (otherwise we obtain degenerate systems) we may set a new parameter w as follows: $u = hw$, $v = -gw$. This implies $U_1 = 0$ and from $V_1 = 0$ we obtain: $l = -h(1+g^2w^2)/(1+h^2w^2)$. Then for the canonical systems 3a) we obtain:

$$\mathcal{E} = 8g^2(g+m)^2(1+h^2w^2)^2(1+g^2w^2)^{-2}, \quad \rho_{3,4} = 2(g+m)(hw \pm i), \quad \sigma = 2(g+m)x.$$

Hence $\mathcal{E} \geq 0$ and for the systems under the examination cannot be fulfilled any of the set of conditions from the statement (c) of Main Theorem.

If $\mathcal{T}_3 = 0$ then according to formulas (2.17) at least one more trace vanishes, but this implies $g + m = 0$ and then we get $\sigma = 0$. However this is in contradiction with our assumption: $\sigma \neq 0$.

3.2.1.2 Canonical systems $3b$**).** Considering Table 2 for the complex singular points $M_{1,2}(0, \pm i)$ and $M_{3,4}(u \pm i, v)$ of these systems we obtain, respectively:

$$\rho_{1,2} = a(u^2 - 1) - g \pm 2ib \quad \rho_{3,4} = a(u^2 - 1) + g \pm 2ibu.$$

So we again could not have only one zero trace and the conditions $\mathcal{T}_1 = \mathcal{T}_2 = 0$ and $\mathcal{T}_3 \neq 0$ implies the vanishing of two traces. By the same reasons as above we may consider $\rho_{1,2} = 0$ and we obtain $b = 0$ and $g = a(u^2 - 1)$. Then calculations yield:

$$\mathcal{E} = 8a^6(u^2 - 1)^4, \quad \rho_3 = \rho_4 = 2a(u^2 - 1), \quad \sigma = 2a(u^2 - 1)x.$$

As $\mathcal{E} \geq 0$ for these systems cannot be fulfilled any of the set of conditions from the statement (c) of Main Theorem. If we assume in addition $\rho_3 = 0$ (then $\rho_4 = 0$) in order to have $\mathcal{T}_3 = 0$, we get again the contradiction $\sigma = 0$.

3.2.2 Systems with the zero–cycle $\mathcal{D}_S(P, Q) = 2p + q^c + r^c$

Taking into account Table 2 for the family 5) having one double singular point $M_{1,2}(0, 0)$ and two complex points $M_{3,4}(1, \pm i)$ we calculate:

$$\rho_{1,2} = cm + 2en \quad \rho_{3,4} = 2g + cm \pm i(2l + 2em - 2cn).$$

So we again could not have only one zero trace and the conditions $\mathcal{T}_1 = \mathcal{T}_2 = 0$ and $\mathcal{T}_3 \neq 0$ implies the vanishing of two traces (which could coincide).

3.2.2.1 The case $\rho_{1,2} = 0$**.** Then we obtain $cm + 2en = 0$. Since $m^2 + n^2 \neq 0$ (see Table 2) we could set a new parameter u as follows: $e = mu$ and $c = -2nu$. Then for the family of systems 4) we calculate

$$\begin{aligned} \mathcal{T}_3 =& 4u^2(gm + 2cln)^2(m^2 + 4n^2)\Big[(cl + 2n^2u)^2 + m^2(m^2 + 5n^2)u^2 + 2m(clm - gn)u + g^2\Big], \\ \mathcal{F} =& 8u(gm + 2cln)\Big[(cl + 2n^2u)^2 + m^2(m^2 + 5n^2)u^2 + 2m(clm - gn)u + g^2\Big] \times \\ & (2gn - clm - m^3u - 4mn^2u), \quad \mathcal{E} = -384u^2(2gn - clm - m^3u - 4mn^2u)^4. \end{aligned}$$

As it can be observed the necessary conditions $\mathcal{T}_3 \neq 0$ and $\mathcal{F} = 0$ from the statement (c) of Main theorem implies $\mathcal{E} = 0$ that contradicts $\mathcal{E} < 0$.

3.2.2.2 The case $\rho_3 = \rho_4 = 0$**.** In this case we get $g = -cm/2$, $l = cn - em$ and then we calculate:

$$\rho_{1,2} = cm + 2en, \quad \sigma = (cm + 2en)(1 - x), \quad \mathcal{E} = 24c^2m^2(cm + 2en)^4.$$

So we obtain $\mathcal{E} \geq 0$ and this again contradicts to $\mathcal{E} < 0$.

Assuming that all the traces vanish we get $\sigma = 0$ that contradicts to our assumption.

3.2.3 Systems with the zero–cycle $\mathcal{D}_S(P, Q) = 2p + 2q$

Considering Table 2 for the family 6) having two double singular point $M_{1,2}(0, 0)$ and $M_{3,4}(1, 0)$ we calculate:

$$\rho_{1,2} = c + eu \quad \rho_{3,4} = -c + eu + 2ev.$$

By formulas (2.17) (as $\rho_1 = \rho_2$ and $\rho_3 = \rho_4$) clearly the identity $\mathcal{T}_1 = 0$ implies $\mathcal{T}_2 = 0$. Due to an affine transformation we may assume $\rho_1 = \rho_2 = 0$, i.e. $c = -u$ and then we obtain:

$$\begin{aligned}\mathcal{T}_3 =& 4e^4(k+nu)^2(u+v)^2, \quad \mathcal{E} = -384e^2(-n+eu^2+2euv)^4\\ \mathcal{F} =& 8e^3(k+nu)(u+v)^2(-n+eu^2+2euv), \quad \rho_{3,4} = 2e(u+v),\\ \mathcal{T}_4 =& 4e^3(k+nu)^2(u+v), \quad \mu_0 = e^2(k+nu)^2.\end{aligned}$$

So if $\mathcal{T}_3 \neq 0$ then the condition $\mathcal{F} = 0$ (see statement (c) of Main theorem) implies $\mathcal{E} = 0$ that contradicts $\mathcal{E} < 0$.

Assume $\mathcal{T}_3 = 0$. Then evidently $\mathcal{T}_4 = 0$ and since $\mu_0 \neq 0$ we get $v = -u$ (i.e. $\rho_3 = \rho_4 = 0$) and this leads to the following family of systems:

$$\begin{aligned}\dot{x} &= -eux - eu^2y + eux^2 + 2eu^2xy + ky^2 \equiv p(x,y),\\ \dot{y} &= ex + euy - ex^2 - 2euxy + ny^2 \equiv q(x,y).\end{aligned} \tag{3.48}$$

For these systems calculations yield:

$$\mathcal{T}_i = 0 \ (i = 1,2,3,4), \quad \mathcal{H} = 0, \quad \sigma = 2(n+eu^2)y$$

and we shall consider two cases: $\sigma \neq 0$ and $\sigma = 0$.

3.2.3.1 The case $\sigma \neq 0$.

Then systems (3.48) are not Hamiltonian and the conditions provided by the statement (e) of Main Theorem could be satisfied. However we claim, that for this family the conditions $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 < 0$ could not be fulfilled simultaneously.

Indeed, considering Remark 2.3 calculations yield:

$$\begin{aligned}\mathcal{E}_1(\boldsymbol{a}) &= 16e(k+nu)(n+eu^2)^3y_0^2\big[2q(x_0,y_0) - (eu^2+n)y_0^2\big],\\ \mathcal{E}_2(\boldsymbol{a}) &= 8e^2(k+nu)(n+eu^2)^4y_0(1-2x_0-2uy_0)\big[2q(x_0,y_0) - (eu^2+n)y_0^2\big],\end{aligned}$$

As $\mathcal{H} = 0$ according to the statement (e) of Main Theorem the conditions $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 < 0$ must be satisfied. However evidently for any translation vector $\tau(x_0, y_0)$ the condition $\mathcal{E}_1 = 0$ implies $\mathcal{E}_2 = 0$ and this proves our claim.

3.2.3.2 The case $\sigma = 0$.

Then systems (3.48) become Hamiltonian and according to Table 1 for this family of systems the conditions $\mu_0 \neq 0$, $\mathbf{D} = \mathbf{T} = 0$ and $\mathbf{PR} > 0$ are verified. So this contradicts to the statement (f) of Main Theorem.

3.2.4 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 2p^c + 2q^c$

Considering Table 2 for the family 7) having two double complex singular points $M_{1,2}(0,i)$ and $M_{3,4}(0,-i)$ we calculate:

$$\begin{aligned}\rho_{1,2} &= au + 2i(b+av) \quad \rho_{3,4} = au - 2i(b+av),\\ \mathcal{T}_1 &= (al-bg)^2\big[4(b+av)^2 + a^2u^2\big], \quad \mu_0 = (al-bg)^2.\end{aligned}$$

As $\mu_0 \neq 0$ the condition $\mathcal{T}_1 = 0$ implies $au = 0$ and $b + av = 0$ (i.e. all traces vanish). Since $a^2 + b^2 \neq 0$ we obtain $u = 0$ and $b = -av$. Then we get the systems

$$\begin{aligned}\dot{x} &= a + gx^2 + 2avxy + ay^2 \equiv p(x,y),\\ \dot{y} &= -av + lex^2 - 2av^2xy - avy^2 \equiv q(x,y),\end{aligned} \tag{3.49}$$

for which calculations yield:

$$\mathcal{T}_i = 0 \ (i = 1, 2, 3, 4), \quad \mathcal{H} = 0, \quad \sigma = 2(g - av^2)x.$$

If $\sigma \neq 0$ then systems (3.49) are not Hamiltonian and the conditions provided by the statement (e) of Main Theorem could be satisfied. Taking into consideration Remark 2.3 we calculate:

$$\mathcal{E}_1(\boldsymbol{a}) = -16a(cl + gv)(-g + av^2)^3 x_0^2 \big[2p(x_0, y_0) + (av^2 - g)x_0^2\big],$$
$$\mathcal{E}_2(\boldsymbol{a}) = -16a^2(cl + gv)(-g + av^2)^4 x_0(vx_0 + x_0)\big[2p(x_0, y_0) + (av^2 - g)x_0^2\big],$$

As $\mathcal{H} = 0$ then according to the statement (e) of Main Theorem the conditions $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 < 0$ must be satisfied. However evidently for any translation vector $\tau(x_0, y_0)$ the condition $\mathcal{E}_1 = 0$ implies $\mathcal{E}_2 = 0$ and this contradicts to the statement (e).

In the case $\sigma = 0$ systems (3.49) become Hamiltonian and we have to consider the statement (f) of Main Theorem. According to Table 1 for this family of systems the conditions $\mu_0 \neq 0$, $\mathbf{D} = \mathbf{T} = 0$ and $\mathbf{PR} < 0$ are verified and this contradicts to the statement (f).

3.2.5 Systems with the zero–cycle $\mathcal{D}_S(P, Q) = 4p$

The systems in this class have one singularity $M_0(0, 0)$ of multiplicity four and according to Table 1 the conditions $\mu_0 \neq 0$ and $\mathbf{D} = \mathbf{T} = \mathbf{P} = \mathbf{R} = 0$ are fulfilled. Hence in the case $\sigma = 0$ (i.e. when these systems are Hamiltonian) we get a contradiction with the statement (f) of Main Theorem.

In Table 2 this class of systems is presented by three canonical forms: $9a)$ (semi-elementary point), $9b)$ (non-elementary point) and $9c)$ (homogeneous systems). We shall examine each of them assuming $\sigma \neq 0$.

3.2.5.1 Canonical systems $9a)$.

Considering Table 2 for this family of systems we obtain:

$$\rho_0 = f, \quad \mathcal{T}_1 = f^4 k^2 l^2, \quad \mu_0 = k^2 l^2. \tag{3.50}$$

As for this family the condition $fkl \neq 0$ holds (see Table 2) then $\mathcal{T}_1 \neq 0$ and only the condition of the statement (a) of Main Theorem could be satisfied.

3.2.5.2 Canonical systems $9b)$.

For this family of systems we calculate

$$\mathcal{T}_i = 0 \ (i = 1, 2, 3, 4), \quad \mathcal{H} = 288 g^2 n^2 = 288 \mu_0 \neq 0, \quad \sigma = 2gx + 2(h + n)y.$$

As $\sigma \neq 0$ and $\mathcal{H} > 0$ this contradicts to the statement (e) of Main Theorem.

3.2.5.3 Canonical systems $9c)$.

For this family of homogeneous systems we have

$$\mathcal{T}_i = 0 \ (i = 1, 2, 3, 4), \quad \sigma = 2(g + m)x + 2(h + n)y, \quad \mu_0 = n(4clh^2 - 4ghm + g^2 n)$$
$$\mathcal{H} = 288\big[clh(h + n)^2 - ghm(h + 3n) - n(hm^2 - g^2 n)\big], \quad \mathcal{E} = 0.$$

As by assumption $\sigma \neq 0$ the statement (e) of Main Theorem has to be considered and since $\mathcal{E} = 0$ the condition $\mathcal{H} \neq 0$ contradicts to this statement.

So we assume $\mathcal{H} = 0$ and we claim that in this case the condition $h(h + n) \neq 0$ has to be fulfilled. Indeed if $h = 0$ then $\mathcal{H} = 288 g^2 n^2 = 288 \mu_0 \neq 0$, whereas in the case $h = -n \neq 0$ we have $\mathcal{H} = 288(g + m)^2 n^2 \neq 0$ because of $\sigma = 2(g + m)x \neq 0$.

So our claim is proved and we assume $h(h+n)\neq 0$. Then the condition $\mathcal{H}=0$ yields $l=\big[ghm(h+3n)+n(hm^2-g^2n)\big]/\big[h(h+n)^2\big]$. In this case considering Remark 2.3 we calculate the invariant polynomials $\mathcal{E}_i$ $(i=1,2)$ in the examined case:

$$\mathcal{E}_1(\boldsymbol{a})=\frac{16n}{h(h+n)^2}(gh+2hm-gn)(gx_0+mx_0+hy_0+ny_0)^2\Phi(g,h,m,n,x_0,y_0),$$
$$\mathcal{E}_2(\boldsymbol{a})=\frac{16n}{h^2(h+n)^2}(gh+2hm-gn)^2(gx_0+mx_0+hy_0+ny_0)\Phi(g,h,m,n,x_0,y_0).$$

Here $\Phi(g,h,m,n,x_0,y_0)$ is a homogeneous polynomial of degree two in x_0, y_0 and homogeneous of degree four with respect to the parameters of the systems. Evidently for any translation vector $\tau(x_0,y_0$ the condition $\mathcal{E}_1=0$ implies $\mathcal{E}_2=0$ and this again contradicts to the statement (e).

3.2.6 Systems with the zero–cycle $\mathcal{D}_S(P,Q)=3p$

Considering Table 2 for the family 13) having the triple singular point $M_{1,2,3}(0,0)$ we calculate:

$$\rho_{1,2,3}=lu,\quad \mu_0=0,\ \mu_1=4(lh-gm)^2ux,\quad \mathcal{T}_1=-8l^3u^4h(lh-gm)^2.$$

As $\mu_1\neq 0$ (see Lemma 2.3) the condition $\mathcal{T}_1=0$ implies $lh=0$.

Assume first $h\neq 0$. Then $l=0$ (i.e. $\rho_{1,2,3}=0$) and for systems 13) we obtain:

$$\mathcal{T}_1=\mathcal{T}_2=\mathcal{T}_3=0,\quad \mathcal{T}_4=-8g^2hm^2u,\quad \mu_1=4g^2m^2ux,$$
$$\mathcal{F}=16ghmu(g+m)^2,\quad \mathcal{H}=-4gh^2m.$$

As $h\mu_1\neq 0$ we obtain $\mathcal{T}_4\neq 0$ and we have to consider the statement (d) of Main Theorem. Therefore the conditions $\mathcal{F}=0$ and $\mathcal{H}<0$ are necessary for the existence of a weak focus or of a center. However the condition $\mathcal{F}=0$ implies $\mathcal{H}=4h^2m^2>0$ that contradicts to this statement.

Suppose now $h=0$. Then for these systems we calculate

$$\mathcal{T}_i=0\ (i=1,2,3,4),\quad \sigma=lu+2(g+m)x,\quad \mu_1=4g^2m^2ux,$$
$$\mathcal{H}=0,\quad \mathcal{E}_1=2l^3g^2mu^4,\quad \mathcal{E}_2=-6l^2g^3m(g+m)^2u^4.$$

If $\sigma\neq 0$ then according to the statement (e) of Main Theorem the conditions $\mathcal{E}_1=0$ and $\mathcal{E}_2<0$ have to be satisfied. However clearly the equality $\mathcal{E}_1=0$ implies $\mathcal{E}_2=0$, i.e. we get a contradiction.

Suppose finally $\sigma=0$, i.e. systems 13) (see Table 2) become Hamiltonian. According to Table 1 for this family of systems the conditions $\mu_0=\mathbf{D}=\mathbf{P}=0$ and $\mathbf{R}\neq 0$ are fulfilled and this leads to a contradiction with the statement (f) of Main Theorem.

3.2.7 Systems with the zero–cycle $\mathcal{D}_S(P,Q)=p^c+q^c$

We note that the systems in this class have two complex singular points $M_{1,2}(0,\pm i)$ and according to Table 1 the conditions $\mu_0=\mathbf{R}=0$, $\mathbf{P}\neq 0$ and $\mathbf{U}<0$ hold. Hence in the case $\sigma=0$ (i.e. when these systems are Hamiltonian) we get a contradiction with the statement (f) of Main Theorem.

In Table 2 this class of systems is presented by two canonical forms: 15a) and 15b). We shall examine each of them assuming $\sigma\neq 0$.

3.2.7.1 Canonical systems $15a$**).** Considering Table 2 for this family of systems we calculate:

$$\begin{aligned}&\rho_{1,2} = hu \pm 2i(b+h), \quad \mu_0 = \mu_1 = 0, \quad \mu_2 = (bh-am)^2(4+u^2)y^2,\\ &\mathcal{T}_1 = 4m^2(4+u^2)(bh-am)^2\big[h^2u^2 + 4(b+h)^2\big].\end{aligned} \tag{3.51}$$

Assume first $m \neq 0$. As $\mu_2 \neq 0$ the condition $\mathcal{T}_1 = 0$ implies $hu = b + h = 0$ (i.e. $\rho_{1,2} = 0$). Setting $b = -h$ and considering the relation $hu = 0$ for systems $15a$) we obtain:

$$\mathcal{T}_1 = \mathcal{T}_2 = 0, \quad \mathcal{T}_3 = 4m^2(h^2+am)^2(4+u^2), \quad \mu_2 = (h^2+am)^2(4+u^2)y^2, \quad \mathcal{E} = 1536a^2m^4.$$

Therefore the condition $m\mu_2 \neq 0$ implies $\mathcal{T}_3 \neq 0$ and since $\mathcal{E} \geq 0$ we get a contradiction with the statement (c) of Main Theorem.

Suppose now $m = 0$. Then we calculate:

$$\mathcal{T}_i = 0 \ (i = 1,2,3,4), \quad \sigma = hu + 2(b+h)y, \quad \mathcal{H} = \mathcal{E}_2 = 0.$$

As $\sigma \neq 0$ then due to the conditions $\mathcal{H} = \mathcal{E}_2 = 0$ we get a contradiction with the statement (e) of Main Theorem.

3.2.7.2 Canonical systems $15b$**).** For this family of systems (see Table 2) calculations yield:

$$\rho_{1,2} = c \pm 2ih, \quad \mathcal{T}_1 = 4ae^2g(g-h^2)(c^2+4h^2), \quad \mathcal{T}_3 = 4ae^2g(g-h^2). \tag{3.52}$$

3.2.7.2.1 Assume first $\mathcal{T}_3 \neq 0$. Then the condition $\mathcal{T}_1 = 0$ yields $c = h = 0$ (i.e. $\rho_{1,2} = 0$) and then we have

$$\mathcal{T}_1 = \mathcal{T}_2 = 0, \quad \mathcal{T}_3 = 4a^2e^2g^2, \quad \mathcal{E} = 1536a^2g^4.$$

Therefore the condition $\mathcal{T}_3 \neq 0$ implies $\mathcal{E} > 0$ and we get a contradiction with the statement (c) of Main Theorem.

3.2.7.2.2 Suppose now $\mathcal{T}_3 = 0$. Since $ae \neq 0$ (otherwise systems $15b$) become degenerate) we obtain $g(ag - h^2) = 0$. As $a \neq 0$ we may assume $a = 1$ (due to a rescaling) and we shall consider two subcases: $g = 0$ and $g = h^2$.

If $g = 0$ then we obtain

$$\mathcal{T}_i = 0 \ (i = 1,2,3,4), \quad \sigma = c + 2hy, \quad \mathcal{H} = 0, \quad \mathcal{E}_1 = -2e^2h(c^2+4h^2), \quad \mathcal{E}_2 = -4ce^3h^3,$$

and in the case $g = h^2$ ($a = 1$) for systems systems $15b$) we calculate:

$$\begin{aligned}&\mathcal{T}_i = 0 \ (i = 1,2,3,4), \quad \sigma = c + 2h^2x + 2hy, \quad \mathcal{H} = 0,\\ &\mathcal{E}_1 = -2eh(e+ch)(c^2+4h^2), \quad \mathcal{E}_2 = -4ceh^3(e+ch)^2\end{aligned}$$

As $\sigma \neq 0$ (i.e. $c^2 + h^2 \neq 0$) in both cases the condition $\mathcal{E}_1 = 0$ implies $\mathcal{E}_2 = 0$ and we get a contradiction with the statement (e) of Main Theorem.

3.2.8 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 2p$

The systems of this class have one real double finite singularity and according to Table 1 the conditions $\mu_0 = \mathbf{R} = U = 0$ and $\mathbf{P} \neq 0$ are verified. Hence in the case $\sigma = 0$ (i.e. when these systems are Hamiltonian) we get a contradiction with the statement (f) of Main Theorem.

In Table 2 this class of systems is presented by two canonical forms: $16a$) and $16b$). We shall examine each of them assuming $\sigma \neq 0$.

3.2.8.1 Canonical systems $16a$**).** Considering Table 2 for this family of systems we calculate:

$$\mu_0 = \mu_1 = 0,\ \mu_2 = (dl - fg)^2x^2,\ \mathcal{T}_1 = 4d^2f^2(cld - fg)^2u^2,\ \mathcal{T}_3 = 4d^2(cld - fg)^2u^2.$$

3.2.8.1.1 Assume first $\mathcal{T}_3 \neq 0$. Then the condition $\mathcal{T}_1 = 0$ yields $f = 0$ and we calculate

$$\mathcal{T}_1 = \mathcal{T}_2 = 0, \quad \mathcal{T}_3 = 4d^4l^2u^2, \quad \mathcal{F} = -8d^3glu^2, \quad \mathcal{E} = -384d^2g^4.$$

So as $\mathcal{T}_3 \neq 0$ then the necessary condition $\mathcal{F} = 0$ (see statement (c) of Main theorem) implies $\mathcal{E} = 0$ that contradicts to this statement.

3.2.8.1.2 Suppose $\mathcal{T}_3 = 0$. Since $\mu_2 \neq 0$ we get $du = 0$ and then calculations yield:

$$\mathcal{T}_i = 0\ (i = 1, 2, 3, 4), \quad \sigma = f + 2(g + fu)x, \quad \mathcal{H} = 0$$

and either $\mathcal{E}_1 = \mathcal{E}_2 = 0$ if $d = 0$, or $\mathcal{E}_1 = 2df^2g(cld - fg)$ and $\mathcal{E}_2 = -4d^2fg^3(cld - fg)$ if $u = 0$. So as in the second case the condition $\mathcal{E}_1 = 0$ implies $\mathcal{E}_2 = 0$ then considering the condition $\sigma \neq 0$ we get a contradiction with the statement (e) of Main Theorem.

3.2.8.2 Canonical systems $16b$**).** Taking into account Table 2 for this family of systems we calculate:

$$\mu_0 = \mu_1 = 0, \quad \mu_2 = (d^2l - 2cdm + c^2n)(lx^2 + 2mxy + ny^2),$$
$$\mathcal{T}_1 = 4n(d^2l - 2cdm + c^2n)(ln - m^2)(c + du)^2,\ \ \mathcal{T}_3 = 4n(d^2l - 2cdm + c^2n)(ln - m^2).$$

3.2.8.2.1 Assume first $\mathcal{T}_3 \neq 0$. Then the condition $\mathcal{T}_1 = 0$ yields $c = -du$ and calculations yield:

$$\mathcal{T}_1 = \mathcal{T}_2 = 0, \quad \mathcal{T}_3 = 4d^2n(ln - m^2)(l + 2mu + nu^2),$$
$$\mathcal{F} = 8dn(m^2 - ln)(m + nu), \quad \mathcal{E} = -384d^2(m + nu)^4.$$

Therefore due to $\mathcal{T}_3 \neq 0$ the condition $\mathcal{F} = 0$ implies $\mathcal{E} = 0$ and we get a contradiction with the statement (c) of Main Theorem.

3.2.8.2.2 Suppose now $\mathcal{T}_3 = 0$. Since $\mu_2 \neq 0$ we get $n(ln - m^2) = 0$ and we shall examine two subcases: $n = 0$ and $n \neq 0$ (then $n = 1$ due to a time rescaling) and $l = m^2$.

If $n = 0$ we have

$$\mathcal{T}_i = 0\ (i = 1, 2, 3, 4), \quad \sigma = c + du + 2mx, \quad \mathcal{H} = 0,$$
$$\mathcal{E}_1 = 2dm(2cm - dl)(c + du)^2, \quad \mathcal{E}_2 = 4d^2m^3(2cm - dl)(c + du)$$

and if $n = 1$ and $l = m^2$ we obtain:

$$\mathcal{T}_i = 0\ (i = 1, 2, 3, 4), \quad \sigma = c + du + 2mx + 2y, \quad \mathcal{H} = 0,$$
$$\mathcal{E}_1 = 2(c - dm)^2(m + u)(c + du)^2, \quad \mathcal{E}_2 = 4(c - dm)^3(m + u)^2(c + du).$$

Evidently in both cases the condition $\mathcal{E}_1 = 0$ implies $\mathcal{E}_2 = 0$ and since $\sigma \neq 0$ we get a contradiction with the statement (e) of Main Theorem.

3.2.9 Systems with the zero–cycle $\mathcal{D}_S(P,Q)=0$

The systems of this class have not any finite singularity (real or complex) and according to Table 1 the conditions $\mu_0=\mathbf{R}=\mathbf{P}=\mathbf{U}=0$ and $\mathbf{V}\neq 0$ hold. Hence in the case $\sigma=0$ (i.e. when these systems are Hamiltonian) we get a contradiction with the statement (f) of Main Theorem.

We observe that in Table 2 this class of systems is presented by four canonical forms: $18a)$–$18d)$, for which the conditions $\mu_0=\mu_1=\mu_2=\mu_3=0$ and $\mu_4\neq 0$ hold. Considering Table 2 we shall examine each of them assuming $\sigma\neq 0$. For systems $18a)$– $18d)$ calculations yield, respectively:

$18a) \Rightarrow \quad \mathcal{T}_1=16h^4(hl-gm)^2u^2, \quad \mu_4=(hl-gm)^2u^2x^4;$
$18b) \Rightarrow \quad \mathcal{T}_1=-16m^4n, \quad \mu_4=y^3(2mx+ny);$
$18c) \Rightarrow \quad \mathcal{T}_1=0, \quad \mu_4=x^3(clx+2my);$
$18d) \Rightarrow \quad \mathcal{T}_1=16n^2(m^2-ln)^2, \quad \mu_4=(lx^2+2mxy+ny^2)^2.$

As $\mu_4\neq 0$ for all these systems the condition $\mathcal{T}_1=0$ leads to the systems (we note that $18c)$ are already such systems) for which we have: $\mathcal{T}_i=0$ $(i=1,2,3,4)$ and $\mathcal{H}=\mathcal{E}_2=0$. As $\sigma\neq 0$ this is in contradiction with the statement (e) of Main Theorem.

Thus all possible cases were examined and hence Main Theorem is proved. ∎

4 Reflections on weak saddles

According to Lemma 2.1 the GL-invariants G_i $(i=0,1,\dots,5)$ constructed in Subsection 2.1 are equivalent to the Poincaré-Lyapunov constants associated to the singular point $M_0(0,0)$ and hence characterize the order of the weak focus at the origin. In this section we shall prove that in terms of the same invariants could be characterized also the order of weakness of the saddle $M_0(0,0)$, i.e. simultaneously they are equivalent to the dual Poincaré-Lyapunov constants [15] (see also [19]) associated to the saddle $M_0(0,0)$.

Lemma 4.1. *A quadratic system*

$$\begin{aligned}\dot{x}&=a_{10}x+a_{01}y+a_{20}x^2+2a_{11}xy+a_{02}y^2,\\ \dot{y}&=b_{10}x+b_{01}y+b_{20}x^2+2b_{11}xy+b_{02}y^2,\end{aligned}\tag{4.1}$$

has at the origin of coordinates a weak saddle of the indicated order or an integrable saddle (see Remark 4.1) if and only if $G_0=0$, $G_5<0$ *and the following respective set of conditions holds:*

1st order $\Leftrightarrow$ $G_1\neq 0;$
2nd order $\Leftrightarrow$ $G_1=0,\ G_2\neq 0;$
3rd order $\Leftrightarrow$ $G_1=G_2=0,\ G_3G_4\neq 0;$
int. saddle $\Leftrightarrow$ $G_1=G_2=G_3G_4=0.$

Proof: It is known that the simple singular point $M_0(0,0)$ of systems (4.1) is a saddle with zero trace if and only if $\det(\mathcal{A})<0$ (i.e. $G_5<0$) and $\operatorname{tr}(\mathcal{A})=0$ (i.e. $G_0=0$). In this case according to [15] (see also, [19]) via a linear transformation this system can be brought to the form

$$\dot{x}=x+Ax^2+Bxy+Cy^2,\quad \dot{y}=-y-Kx^2-Lxy-My^2,\tag{4.2}$$

for which Cai Sui Lin [15] has calculated the first three dual Poincaré-Lyapunov constants:

$$
\begin{aligned}
L_1^* =& LM - AB,\\
L_2^* =& KB(2M-B)(M+2B) - CL(2A-L)(A+2L),\\
L_3^* =& (CK-LB)\big[ACL(2A-L) - BKM(2M-B)\big].
\end{aligned}
$$

Remark 4.1. According to [15] the weak saddle $(0,0)$ is of the first (respectively second; third) order if $L_1^* \neq 0$ (respectively $L_1^* = 0, L_2^* \neq 0$; $L_1^* = L_2^* = 0, L_3^* \neq 0$) and it is *integrable saddle* if $L_1^* = L_2^* = L_3^* = 0$.

Considering (2.4) for systems (4.2) calculations yield $G_0 = 0$, $G_6 = -1 < 0$ and

$$
\begin{aligned}
G_1 &= 864(AB - LM) = -864\, L_1^*,\\
G_2 &= 2^6 3^3 \big[L_2^* - L_1^*(5CK + 2BL - 3AM)\big],\\
G_3G_4 &= -2^{14} 3^5 \big[L_3^* + L_1^*(CK - BL)(CK + AM)\big].
\end{aligned}
\tag{4.3}
$$

So, the conditions $L_1^* = 0$ (respectively $L_1^* = L_2^* = 0$; $L_1^* = L_2^* = L_3^* = 0$) are equivalent to $G_1 = 0$ (respectively $G_1 = G_2 = 0$; $G_1 = G_2 = G_3G_4 = 0$) and this completes the proof of the lemma. ∎

Considering lemma above and the proof of Main Theorem (see statement (b) on page 16) we get the next result.

Theorem 4.1. *Assume that for a system* (2.1) *the conditions* $\mathcal{T}_1 = 0$ *and* $\mathcal{T}_2 \neq 0$ *hold (i.e. it possesses only one singular point with zero trace and the respective traces of all other finite singularities do not vanish). Then this system has either one weak focus (respectively saddle) of the indicated order if and only if* $\mathcal{T}_2\mathcal{F} < 0$ *(respectively* $\mathcal{T}_2\mathcal{F} > 0$*) and the following respective set of conditions holds:*

$$
\begin{aligned}
&\textit{1st order} &\Leftrightarrow\quad & \mathcal{F}_1 \neq 0;\\
&\textit{2nd order} &\Leftrightarrow\quad & \mathcal{F}_1 = 0,\ \mathcal{F}_2 \neq 0;\\
&\textit{3rd order} &\Leftrightarrow\quad & \mathcal{F}_1 = \mathcal{F}_2 = 0,\ \mathcal{F}_3\mathcal{F}_4 \neq 0;
\end{aligned}
$$

and it has one center (respectively integrable saddle) if and only if $\mathcal{T}_2\mathcal{F} < 0$ *(respectively* $\mathcal{T}_2\mathcal{F} > 0$*) and*

$$
\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3\mathcal{F}_4 = 0.
$$

Proof: Follows immediately taking into consideration factorizations (3.2) and Lemmas 2.1 and 4.1. ∎

Comparing Lemmas 4.1 and 2.1 we conclude that Lemma 2.2 could be generalized as follows:

Lemma 4.2. *Assume that a quadratic system* (2.1) *possesses two real simple singular points* M_1 *and* M_2 *with zero traces, i.e.* $\operatorname{tr}(\mathcal{A}^{(1)}) = \operatorname{tr}(\mathcal{A}^{(2)}) = 0$. *Then these points are of the types indicated below if and only if the respective conditions hold:*

$$
\begin{aligned}
S^{(1)},\ S^{(1)} &\quad\Leftrightarrow\quad \mathcal{F}_1 \neq 0,\ \det(\mathcal{A}^{(1)}) < 0,\ \det(\mathcal{A}^{(2)}) < 0;\\
S^{(1)},\ F^{(1)} &\quad\Leftrightarrow\quad \mathcal{F}_1 \neq 0,\ \det(\mathcal{A}^{(1)})\det(\mathcal{A}^{(2)}) < 0;\\
F^{(1)},\ F^{(1)} &\quad\Leftrightarrow\quad \mathcal{F}_1 \neq 0,\ \det(\mathcal{A}^{(1)}) > 0,\ \det(\mathcal{A}^{(2)}) > 0;\\
S^{(I)},\ S^{(I)} &\quad\Leftrightarrow\quad \mathcal{F}_1 = 0,\ \det(\mathcal{A}^{(1)}) < 0,\ \det(\mathcal{A}^{(2)}) < 0;\\
S^{(I)},\ \ C &\quad\Leftrightarrow\quad \mathcal{F}_1 = 0,\ \det(\mathcal{A}^{(1)})\det(\mathcal{A}^{(2)}) < 0;\\
C,\ \ C &\quad\Leftrightarrow\quad \mathcal{F}_1 = 0,\ \det(\mathcal{A}^{(1)}) > 0,\ \det(\mathcal{A}^{(2)}) > 0;
\end{aligned}
$$

where $F^{(1)}$ *(respectively* C; $S^{(1)}$; $S^{(I)}$;*) denotes a focus of first order (respectively a center; a saddle of first order; an integrable saddle) and* $\mathcal{F}_1$ *is the affine invariant defined in* (2.21).

Proof: Follows directly from the proof of Lemma 2.2. ∎

We note that on the base of the above lemma we could not generalize the statement (c) of Main Theorem in the same manner as we did for statement (b) (see Theorem 4.1). One of the obstacles is the condition $\mathcal{E} < 0$, that is necessary only for weak focus and not for a weak saddle. For example, the system

$$\dot{x} = x - x^2 - y + 2xy, \quad \dot{y} = -y + 2xy$$

has two weak saddles $M_1(0,0)$ and $M_0(1,0)$. At the same time for this system we calculate

$$\mathcal{T}_1 = \mathcal{T}_2 = 0, \quad \mathcal{T}_3 = -4 < 0, \quad \mathcal{H} = 4 > 0, \quad \mathcal{F} = \mathcal{E} = 0.$$

On the other hand from the proof of Main Theorem it follows that in the case $\mathcal{T}_1 = \mathcal{T}_2 = 0$ and $\mathcal{T}_3 \neq 0$ for the existence of weak saddles (obviously obtained from weak foci by changing the sign of the respective determinants), the condition $\mathcal{E} \leq 0$ is necessary.

Thus if the conditions in statement (c) of Main Theorem will be weakened as follows, then we get the next result (we keep the notations used in the lemma above).

Theorem 4.2. *Assume that for a system* (2.1) *the conditions* $\mathcal{T}_1 = \mathcal{T}_2 = 0$, $\mathcal{T}_3 \neq 0$ *and* $\mathcal{E} < 0$ *hold. Then these singular points are of the types indicated below if and only if the respective conditions hold:*

$$\begin{array}{ccccl} S^{(1)}, & S^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \mathcal{T}_3 < 0,\ \mathcal{H} > 0; \\ S^{(1)}, & F^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \mathcal{T}_3 > 0; \\ F^{(1)}, & F^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \mathcal{T}_3 < 0,\ \mathcal{H} < 0; \\ S^{(I)}, & S^{(I)} & \Leftrightarrow & \mathcal{F}_1 = 0,\ \mathcal{T}_3 < 0,\ \mathcal{H} > 0; \\ S^{(I)}, & C & \Leftrightarrow & \mathcal{F}_1 = 0,\ \mathcal{T}_3 > 0; \\ C, & C & \Leftrightarrow & \mathcal{F}_1 = 0,\ \mathcal{T}_3 < 0,\ \mathcal{H} < 0; \end{array}$$

Proof: Follows directly from the proof of Main Theorem, taking into attention only the cases when the conditions $\mathcal{T}_1 = \mathcal{T}_2 = 0$ and $\mathcal{T}_3 \neq 0$ hold. ∎

Remark 4.2. *We note that in order to complete the characterization of weak saddles for the class of quadratic systems, it remains to examine the subcase* $\mathcal{T}_1 = \mathcal{T}_2 = 0$, $\mathcal{T}_3 \neq 0$ *and* $\mathcal{E} = 0$ *as well as the case* $\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = 0$. *And in the remaining cases there could be only either weak saddles of the first order, or integrable saddles.*

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