Global Classical Solutions Close to Equilibrium to the Vlasov-Euler-Fokker-Planck System

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Abstract

We are concerned with the global well-posedness of a two-phase flow system arising in the modelling of fluid-particle interactions. This system consists of the Vlasov-Fokker-Planck equation for the dispersed phase (particles) coupled to the incompressible Euler equations modelling a dense phase (fluid) through the friction forcing. Global existence of classical solutions to the Cauchy problem in the whole space is established when initial data is a small smooth perturbation of a constant equilibrium state, and moreover an algebraic rate of convergence of solutions toward equilibrium is obtained under additional conditions on initial data. The proof is based on the macro-micro decomposition and Kawashima's hyperbolic-parabolic dissipation argument. This result is generalized to the periodic case, when particles are in the torus, improving the rate of convergence to exponential.

1 Introduction

Fluid-particle interaction systems have been proposed to describe the behavior of sprays, aerosols or more generically two phase flows where one phase (disperse) can be considered as a suspension of particles onto the other one (dense) thought as a fluid. In many of these applications, the assumption that particles are solid non-deformable spheres suspended on the fluid leads to simplified but meaningful models [2]. This kind of systems have been used in sedimentation of solid grain by external forces [1], for fuel-droplets in combustion theory [28] and biosprays in medicine [3, 22] for instance.

The particles behavior is obtained *via* the evolution of the statistical distribution of particles in phase space, where the only forces taken into account on particles are the friction forces due to the fluid and an stochastic term of fluctuations around the fluid velocity. This kinetic modelling of the particle phase, as in [6, 4, 7], leads to the Vlasov-Fokker-Planck equation coupled with some fluid equation with a friction term due to the action/reaction principle. Here, we will model the fluid by the incompressible Euler system. The resulting system reads as

$$\partial_t F + \xi \cdot \nabla_x F = \nabla_\xi \cdot ((\xi - u)F + \nabla_\xi F), \tag{1.1}$$

$$\nabla_x \cdot u = 0, \tag{1.2}$$

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \int_{\mathbb{R}^3} (\xi - u) F \, d\xi, \tag{1.3}$$

with

$$F(0, x, \xi) = F_0(x, \xi), \quad u(0, x) = u_0(x).$$

Here, the unknowns are $F = F(t, x, \xi) \geq 0$ for $t > 0, x \in \mathbb{R}^3, \xi \in \mathbb{R}^3$, denoting the density distribution function of particles in the phase space, and $u = u(t, x) \in \mathbb{R}^3$ and $p = p(t, x) \in \mathbb{R}$ for $t > 0, x \in \mathbb{R}^3$, denoting respectively the velocity field and pressure of the fluid. Initial data $F_0 = F_0(x, \xi)$ and $u_0 = u_0(x)$ for $x \in \mathbb{R}^3$, $\xi \in \mathbb{R}^3$ are given with the compatible condition

$$\nabla_x \cdot u_0(x) = 0. \tag{1.4}$$

The above system describes the motion of the interactive particle and fluid subject to a mutual friction forcing proportional to the relative velocity $\xi - u$; see [28, 6]. In what follows, we shall call (1.1), (1.2) and (1.3) by the Vlasov-Euler-Fokker-Planck system (VEFP for simplicity).

Previous work related to the mathematical analysis of coupled kinetic-fluid systems in the sense above can be traced back to [17] where global existence and large time behavior of solutions to the Vlasov-Stokes system was obtained. In [17], the fluid is assumed to be viscous and incompressible and its velocity satisfies the Stokes equations with the same friction forcing as in (1.3). When the motion of the fluid is described by the incompressible Navier-Stokes equations, [14, 15] considered hydrodynamic limits of the Vlasov-Navier-Stokes system in different regimes, [9, 12, 8] dealt with similar singular perturbation problems, and [5] recently gave a proof of global existence of weak solutions on the periodic domain. [21, 20] provided a detailed study of the global existence and asymptotic analysis for the coupled system of the Vlasov-Fokker-Planck equation with the compressible Navier-Stokes equations in $\mathbb{R}^3$, and [13] also proved global existence of classical solutions near equilibrium for the incompressible model. In the framework of the inviscid compressible flow under friction forces, existence of smooth solutions for short time was proved in [4] when there is no Brownian effect in the kinetic equation, and stability and asymptotic analysis were discussed in [7] when the velocity diffusion is considered.

The goal of this paper is to prove the global existence of classical solutions to the Cauchy problem of the VEFP system for initial data which is a small perturbation around the following spatially homogeneous steady states $F \equiv M,\ u \equiv 0,\ P \equiv 0$, where

$$M = M(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(-|\xi|^2/2\right),$$

which has been normalized to have zero bulk velocity and unit density and temperature. Compared with some existing results, although the fluid is inviscid, solutions close to equilibrium are shown to be asymptotically stable under smooth perturbations. From the later proof, this essentially results from the coupling term friction forcing through which the dissipation of the momentum component of the kinetic distribution can be transferred to the damped fluid velocity field. For the above purpose, let us reformulate the Cauchy problem in the framework of perturbations. Set

$$F = M + M^{1/2} f,$$

so that the reformulated Cauchy problem reads

$$\partial_t f + \xi\cdot\nabla_x f + u\cdot\nabla_\xi f - \frac{1}{2} u\cdot\xi f - u\cdot\xi M^{1/2} = \mathbf{L} f, \tag{1.5}$$

$$\nabla_x\cdot u = 0, \tag{1.6}$$

$$\partial_t u + u\cdot\nabla_x u + \nabla_x p + u(1+a) = b, \tag{1.7}$$

with initial data

$$f(0,x,\xi) = f_0(x,\xi) \equiv M^{-1/2}(F_0 - M), \quad u(0,x) = u_0(x), \tag{1.8}$$

where $\mathbf{L}$ is the linearized Fokker-Planck operator defined by

$$\mathbf{L} f = \frac{1}{M^{1/2}} \nabla_\xi \cdot \left[M \nabla_\xi \left(\frac{f}{M^{1/2}} \right) \right],$$

and $a = a^f, b = b^f$ depending on f are the moments of f defined by

$$a^f(t,x) = \int_{\mathbb{R}^3} M^{1/2} f(t,x,\xi)\, d\xi, \quad b^f(t,x) = \int_{\mathbb{R}^3} \xi M^{1/2} f(t,x,\xi)\, d\xi. \tag{1.9}$$

Theorem 1.1. *Let (1.4) and $F_0 = M + M^{1/2} f_0 \geq 0$ hold. Suppose that $\|f_0\|_{\mathrm{L}^2_\xi(\mathrm{H}^3_x)} + \|u_0\|_{\mathrm{H}^3}$ is small enough. Then, the Cauchy problem (1.5)-(1.7) and (1.8) admits a unique global classical solution $(f(t,x,\xi), u(t,x))$ satisfying*

$$f \in \mathscr{C}^0([0,\infty); \mathrm{L}^2_\xi(\mathrm{H}^3_x)), \quad u \in \mathscr{C}^0([0,\infty); \mathrm{H}^3),$$

$$F = M + M^{1/2} f \geq 0,$$

$$\sup_{t\geq 0}(\|f(t)\|_{\mathrm{L}^2_\xi(\mathrm{H}^3_x)} + \|u(t)\|_{\mathrm{H}^3}) \leq C(\|f_0\|_{\mathrm{L}^2_\xi(\mathrm{H}^3_x)} + \|u_0\|_{\mathrm{H}^3}).$$

Moreover, for any given $\varepsilon > 0$ which is close to zero, if $\|f_0\|_{\mathrm{L}^2_\xi(\mathrm{H}^3_x\cap \mathrm{L}^1_x)} + \|u_0\|_{\mathrm{H}^3\cap\mathrm{L}^1}$ is sufficiently small, the solution (f,u) enjoys the time-decay:

$$\|f(t)\|_{\mathrm{L}^2_\xi(\mathrm{H}^3_x)} + \|u(t)\|_{\mathrm{H}^3} \leq C_\varepsilon (1+t)^{-\frac{3}{4}+\varepsilon} (\|f_0\|_{\mathrm{L}^2_\xi(\mathrm{H}^3_x\cap\mathrm{L}^1_x)} + \|u_0\|_{\mathrm{H}^3\cap\mathrm{L}^1}) \tag{1.10}$$

for any $t \geq 0$, where C_ε depends only on ε and C_ε may blow up as ε tends to zero.

The main novelty we develop in this paper is the use of a refined energy method to take advantage of the damping of the velocity field in the Euler equation due to the kinetic part. This is the reason why we do not need a viscosity term in the fluid equation compared to [13], see remark 3.1 at the end of Section 3. Energy methods have been used for collisional kinetic equations [16, 10] and for other nonlinearly coupled Fokker-Planck equations and systems as in [19, 11]. The rest of this paper is organized as follows. In the next section, we start by giving an elementary observation for computing the dissipation of the linearized Fokker-Planck operator on the basis of a macro-micro decomposition, and also we introduce some notations used in the later proof. The global existence and rate of convergence of solutions are respectively obtained in the end of Section 2 and Section 3. We adapt our proof to the periodic in space case in Section 4 and eventually give a few technical lemmas in Section 5.

2 Global existence

In what follows, our analysis is based on the reformulated Cauchy problem (1.5)-(1.8). To obtain the global existence, the most important point is to obtain the uniform-in-time *a priori* estimates. Then, we will construct an approximation scheme allowing use to show short-time existence of the smooth solutions for which the *a priori* estimates become real estimates, and finally we will show that due to the uniform-in-time estimates, we can extend the solutions to all times. We now first introduce some notations and set some basic properties of the operators involved.

2.1 Preliminaries

Let $\nu(\xi) = 1 + |\xi|^2$ and denote $|\cdot|_\nu$ by

$$|g|_\nu^2 = \int_{\mathbb{R}^3} \left[|\nabla_\xi g(\xi)|^2 + \nu(\xi)|g(\xi)|^2\right] d\xi, \quad g = g(\xi).$$

The operator $\mathbf{L}$ satisfies that there is a positive constant $\lambda_0 > 0$ such that

$$-\int_{\mathbb{R}^3} g\mathbf{L}g \, d\xi \geq \lambda_0 |\{\mathbf{I} - \mathbf{P}_0\}g|_\nu^2,$$

for any $g = g(\xi)$, where $\mathbf{P}_0 g = a^g M^{1/2}$. Generally, u or b are not integrable in time-space. To control them in a smart way, an idea from the recent paper [11] can be employed. Let us define the velocity orthogonal projection $\mathbf{P}:\ \mathrm{L}^2_\xi \to \mathrm{Span}\{M^{1/2}, \xi_1 M^{1/2}, \xi_2 M^{1/2}, \xi_3 M^{1/2}\}$ by

$$\begin{aligned} \mathbf{P} &:= \mathbf{P}_0 + \mathbf{P}_1, \\ \mathbf{P}_0 g &:= a^g M^{1/2}, \\ \mathbf{P}_1 g &:= b^g \cdot \xi M^{1/2}. \end{aligned}$$

Decompose $\mathbf{L}g$ as

$$\mathbf{L}g = \mathbf{L}\{\mathbf{I} - \mathbf{P}\}g + \mathbf{L}\mathbf{P}g = \mathbf{L}\{\mathbf{I} - \mathbf{P}\}g - \mathbf{P}_1 g. \tag{2.1}$$

Notice that since $\mathbf{L}$ is self-adjoint,

$$\begin{aligned}\langle -\mathbf{L}\{\mathbf{I}-\mathbf{P}\}g, g\rangle &= \langle -\{\mathbf{I}-\mathbf{P}\}g, \mathbf{L}g\rangle = \langle -\{\mathbf{I}-\mathbf{P}\}g, \mathbf{L}\{\mathbf{I}-\mathbf{P}\}g - \mathbf{P}_1 g\rangle \\ &= \langle -\mathbf{L}\{\mathbf{I}-\mathbf{P}\}g, \{\mathbf{I}-\mathbf{P}\}g\rangle \geq \lambda_0 |\{\mathbf{I}-\mathbf{P}\}g|_\nu^2. \end{aligned} \tag{2.2}$$

Therefore,

$$\langle -\mathbf{L}g, g\rangle \geq \lambda_0 |\{\mathbf{I}-\mathbf{P}\}g|_\nu^2 + |b^g|^2. \tag{2.3}$$

We introduce some conventions for later use. C denotes some positive (generally large) constant and λ denotes some positive (generally small) constant, where both C and λ may take different values in different places. In addition, $A \sim B$ means $\lambda_1 A \leq B \leq \frac{1}{\lambda_1} A$ for a generic constant $\lambda_1 > 0$. For an integrable function $g : \mathbb{R}^3 \to \mathbb{R}$, its Fourier transform $\widehat{g} = \mathcal{F}g$ is defined by

$$\widehat{g}(k) = \mathcal{F}g(k) = \int_{\mathbb{R}^3} e^{-\mathrm{i}x\cdot k} g(x)\, dx, \quad x \cdot k =: \sum_{j=1}^{3} x_j k_j,$$

for $k \in \mathbb{R}^3$. For simplicity, we use $\|\cdot\|$ to denote L^2 norm over L^2_x or $\mathrm{L}^2_{x,\xi}$ if no confusion arises. We use $\langle \cdot, \cdot \rangle$ to denote the inner product over the Hilbert space L^2_ξ, i.e.

$$\langle g, h\rangle = \int_{\mathbb{R}^3} g(\xi) h(\xi)\, d\xi, \quad g, h \in \mathrm{L}^2_\xi.$$

Define

$$\|g\|_\nu^2 = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left[|\nabla_\xi g(x,\xi)|^2 + \nu(\xi)|g(x,\xi)|^2 \right] d\xi dx, \ g = g(x,\xi).$$

For $q \geq 1$, we also define

$$Z_q = \mathrm{L}^2_\xi(\mathrm{L}^q_x) = \mathrm{L}^2(\mathbb{R}^3_\xi; \mathrm{L}^q(\mathbb{R}^3_x)), \quad \|g\|_{Z_q} = \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |g(x,\xi)|^q\, dx \right)^{2/q} d\xi \right)^{1/2}.$$

For brevity, we introduce norms $\|(\cdot,\cdot)\|_{\mathcal{H}^m}$, $\|(\cdot,\cdot)\|_{\mathcal{Z}_q}$ with the integer $m \geq 0$ and $q \geq 1$ by

$$\|(f,u)\|_{\mathcal{H}^m}^2 = \|f\|^2_{\mathrm{L}^2_\xi(\mathrm{H}^m_x)} + \|u\|^2_{\mathrm{H}^m}, \quad \|(f,u)\|_{\mathcal{Z}_q} = \|f\|_{Z_q} + \|u\|_{\mathrm{L}^1},$$

for $f = f(x,\xi)$ and $u = u(x)$, and we set $\mathcal{L}^2 = \mathcal{H}^0$ as usual. Finally, for a multiple index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we denote $\partial^\alpha = \partial^\alpha_x = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}$. The length of α is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For simplicity, we also use ∂_i to denote ∂_{x_i} for each $i = 1, 2, 3$.

2.2 Uniform-in-time *a priori* estimates

In this subsection, let us assume that (1.5)-(1.8) admits a solution (f,u) with enough regularity and fast decaying at infinity over $[0,T]$ with $T > 0$. We begin with a technical lemma useful in the subsequent estimates.

Lemma 2.1. *There exist positive constants C, such as for any $f, g \in \mathrm{H}^3(\mathbb{R}^3)$ and any multi-index $\gamma \in \mathbb{N}^3$ verifying $1 \le |\gamma| \le 3$ we have*

$$\|f\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \le C\|\nabla_x f\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3)}\|\nabla_x^2 f\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3)}, \tag{2.4}$$

$$\|fg\|_{\mathrm{H}^2(\mathbb{R}^3)} \le C\|f\|_{\mathrm{H}^2(\mathbb{R}^3)}\|\nabla_x g\|_{\mathrm{H}^2(\mathbb{R}^3)}, \tag{2.5}$$

$$\|\partial_x^\gamma (fg)\|_{\mathrm{L}^2(\mathbb{R}^3)} \le C\|\nabla_x f\|_{\mathrm{H}^2(\mathbb{R}^3)}\|\nabla_x g\|_{\mathrm{H}^2(\mathbb{R}^3)}. \tag{2.6}$$

Proof. We start with (2.4). For all $R > 0$, decomposing the frequency space in $\{|k \le R\}$ and $\{|k| > R\}$ we have by Hölder's inequality

$$\int_{\mathbb{R}^3} |\hat{f}(k)|\, dk \le 4\pi \left[R\|k\hat{f}\|_{\mathrm{L}^2(\mathbb{R}^3)} + \frac{1}{R}\|k^2\hat{f}\|_{\mathrm{L}^2(\mathbb{R}^3)} \right].$$

Choosing $R = \|k^2\hat{f}\|^{1/2}_{\mathrm{L}^2(\mathbb{R}^3)}\|k\hat{f}\|^{-1/2}_{\mathrm{L}^2(\mathbb{R}^3)}$ and applying then the inverse Fourier transform concludes the proof of (2.4).

Now for (2.5), using Leibnitz formula, it is enough to bound the following terms $\partial^{\alpha-\beta} f \partial^\beta g$, where α and β are multi-indices verifying $|\alpha| \le 2$ and $\beta \le \alpha$, (meaning $\beta_i \le \alpha_i$ for all $i \in \{1, 2, 3\}$). We use (2.4) and Sobolev's embeddings, in the two following cases

- $|\beta| \le 1$ and
$$\|\partial^{\alpha-\beta} f \partial^\beta g\|_{\mathrm{L}^2(\mathbb{R}^3)} \le \|\partial^{\alpha-\beta} f\|_{\mathrm{L}^2(\mathbb{R}^3)}\|\partial^\beta g\|_{\mathrm{L}^\infty(\mathbb{R}^3)} \le C\|f\|_{\mathrm{H}^2(\mathbb{R}^3)}\|\nabla_x g\|_{\mathrm{H}^2(\mathbb{R}^3)},$$
- $\beta = \alpha$ and $\|f\partial^\alpha g\|_{\mathrm{L}^2(\mathbb{R}^3)} \le \|f\|_{\mathrm{L}^\infty(\mathbb{R}^3)}\|\partial^\alpha g\|_{\mathrm{L}^2(\mathbb{R}^3)} \le C\|f\|_{\mathrm{H}^2(\mathbb{R}^3)}\|\nabla_x g\|_{\mathrm{H}^2(\mathbb{R}^3)}$.

For the last inequality, let us work in Fourier variable again. We denote by $|\gamma| = \gamma_1+\gamma_2+\gamma_3$ and for all $k = (k_1, k_2, k_3) \in \mathbb{R}^3$, $k^\gamma := k_1^{\gamma_1}k_2^{\gamma_2}k_3^{\gamma_3}$. We have by Young's inequality ($|\gamma| \neq 0$):

$$|k^\gamma| \le \frac{\gamma_1}{|\gamma|}|k_1|^{|\gamma|} + \frac{\gamma_2}{|\gamma|}|k_2|^{|\gamma|} + \frac{\gamma_3}{|\gamma|}|k_3|^{|\gamma|}.$$

Hence, we have for all k and η in $\mathbb{R}^3$

$$|k^\gamma| \le C\sum_{i=1}^3 \left\{ |k_i - \eta_i|^{|\gamma|} + |\eta_i|^{|\gamma|} \right\},$$

for some constant C. Hence :

$$\begin{aligned} \frac{1}{C}\left|\hat{f} \star \hat{g}(k)k^\gamma\right| &\le \sum_{i=1}^3 \left\{ \int_{\mathbb{R}^3} |\hat{f}(k-\eta)\hat{g}(\eta)(k_i-\eta_i)^{|\gamma|}|\, d\eta + \int_{\mathbb{R}^3} |\hat{f}(k-\eta)\hat{g}(\eta)\eta_i^{|\gamma|}|\, d\eta \right\} \\ &= \sum_{i=1}^3 \left\{ \Phi_i(f,g)(k) + \Phi_i(g,f)(k) \right\}, \end{aligned}$$

where the last obvious notation is justified by a change of variable. We now observe that up to a constant $\|\hat{f}\|_{L^1(\mathbb{R}^3)}$ and $\|f\|_{L^\infty(\mathbb{R}^3)}$ are equal. In our case they are both finite by (2.4), hence by Hölder's inequality

$$|\Phi_i(g,f)|^2(k) \le \|\hat{f}\|_{\mathrm{L}^1} \int_{\mathbb{R}^3} |\hat{f}(k-\eta)||\hat{g}(\eta)|^2|\eta_i|^{2|\gamma|}\, d\eta,$$

which implies integrating on k and by symmetry that,

$$\|\Phi_i(g,f)\|_{\mathrm{L}^2(\mathbb{R}^3)} + \|\Phi(g,f)\|_{\mathrm{L}^2(\mathbb{R}^3)} \le \|\hat{f}\|_{\mathrm{L}^1(\mathbb{R}^3)}\|k_i^{|\gamma|}\hat{g}\|_{\mathrm{L}^2(\mathbb{R}^3)} + \|\hat{g}\|_{\mathrm{L}^1(\mathbb{R}^3)}\|k_i^{|\gamma|}\hat{f}\|_{\mathrm{L}^2(\mathbb{R}^3)},$$

to conclude

$$\|k^\gamma \hat{f}\star\hat{g}\|_{\mathrm{L}^2(\mathbb{R}^3)} \le D\sum_{i=1}^{3}\Big[\|\hat{g}\|_{\mathrm{L}^1(\mathbb{R}^3)}\|k_i^{|\gamma|}\hat{f}\|_{\mathrm{L}^2(\mathbb{R}^3)} + \|\hat{f}\|_{\mathrm{L}^1(\mathbb{R}^3)}\|k_i^{|\gamma|}\hat{g}\|_{\mathrm{L}^2(\mathbb{R}^3)}\Big].$$

After applying inverse Fourier transform and using (2.4), (2.6) is proved. □

The first step is to produce energy estimates for obtaining the dissipation of the kinetic equation on the basis of the coercivity property (2.3) of $-\mathbf{L}$.

Lemma 2.2. *For smooth solutions of the system* (1.5)-(1.7)*, we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|f\|^2+\|u\|^2)+\lambda_0\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|u-b\|^2\\ &\le C\|u\|_{\mathrm{H}^2}\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2 + C\|u\|_{\mathrm{H}^1}\|u-b\|^2 + C\|u\|_{\mathrm{H}^1}\|\nabla_x(a,b)\|^2\end{aligned} \tag{2.7}$$

for any $0\le t\le T$ *and any* $T>0$ *with* C *and* λ_0 *not depending on* T*.*

Proof. By using (1.6) and (2.1), the direct energy integration of (1.5) and (1.7) and then their summation give

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|f\|^2+\|u\|^2)+\int_{\mathbb{R}^3}\langle -\mathbf{L}\{\mathbf{I}-\mathbf{P}\}f,f\rangle\,dx+\|u-b\|^2\\ &\qquad=\int_{\mathbb{R}^3}\frac{1}{2}u\cdot\langle\xi f,f\rangle\,dx-\int_{\mathbb{R}^3}a|u|^2\,dx.\end{aligned}\tag{2.8}$$

Using the macro-micro decomposition $f=\mathbf{P}f+\{\mathbf{I}-\mathbf{P}\}f$, one can compute

$$\langle\xi f,f\rangle=\langle\xi,|\mathbf{P}f|^2\rangle+2\langle\xi\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f\rangle+\langle\xi,|\{\mathbf{I}-\mathbf{P}\}f|^2\rangle,$$

and further $\langle\xi,|\mathbf{P}f|^2\rangle=2ab$. We deduce

$$\begin{aligned}&\int_{\mathbb{R}^3}\frac{1}{2}u\cdot\langle\xi f,f\rangle\,dx-\int_{\mathbb{R}^3}a|u|^2\,dx\\ &\quad=-\int_{\mathbb{R}^3}au\cdot(u-b)\,dx+\int_{\mathbb{R}^3}u\cdot\langle\xi\mathbf{P}f,\{\mathbf{I}-\mathbf{P}\}f\rangle\,dx\\ &\qquad+\frac{1}{2}\int_{\mathbb{R}^3}u\cdot\langle\xi,|\{\mathbf{I}-\mathbf{P}\}f|^2\rangle\,dx.\end{aligned}$$

Then, using Young's, Sobolev's and Cauchy-Schwarz's inequalities,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{2} u \cdot \langle \xi f, f \rangle \, dx - \int_{\mathbb{R}^3} a|u|^2 \, dx \\
&\quad \leq \|a\|_{\mathrm{L}^6} \|u\|_{\mathrm{L}^3} \|u-b\|_{\mathrm{L}^2} + C\|u\|_{\mathrm{L}^3} \|(a,b)\|_{\mathrm{L}^6} \|\{\mathbf{I}-\mathbf{P}\}f\| \\
&\qquad + C\|u\|_{\mathrm{L}^\infty} \|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2 \\
&\quad \leq C(\|\nabla u\|_{\mathrm{H}^1} + \|u\|_{\mathrm{H}^1}) \|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2 + C\|u\|_{\mathrm{H}^1} \|u-b\|_{\mathrm{L}^2}^2 \\
&\qquad + C\|u\|_{\mathrm{H}^1} \|\nabla_x (a,b)\|_{\mathrm{L}^2}^2 .
\end{aligned}$$

By plugging the last inequality into (2.8), then (2.7) follows due to (2.2). □

Lemma 2.3. *For smooth solutions of the system (1.5)-(1.7), we have*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \sum_{1\leq|\alpha|\leq 3} (\|\partial^\alpha f\|^2 + \|\partial^\alpha u\|^2) + \lambda_0 \sum_{1\leq|\alpha|\leq 3} (\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_\nu^2 + \|\partial^\alpha (u-b)\|^2) \\
&\qquad \leq C\|\nabla_x u\|_{\mathrm{H}^2} \Big(\sum_{1\leq|\alpha|\leq 3} \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_\nu^2 + \|\nabla_x (a,b,u-b)\|_{\mathrm{H}^2}^2 \Big) ,
\end{aligned} \tag{2.9}$$

for any $0 \leq t \leq T$ and any $T > 0$ with C and λ_0 not depending on T.

Proof. Take α with $1 \leq |\alpha| \leq 3$. The energy integration of α-order for (1.5) and (1.7) and their summation give

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\partial^\alpha f\|^2 + \|\partial^\alpha u\|^2) + \int_{\mathbb{R}^3} \langle -\mathbf{L}\partial^\alpha \{\mathbf{I}-\mathbf{P}\} f, \partial^\alpha f \rangle \, dx + \|\partial^\alpha (u-b)\|^2 \\
&\quad = -\int_{\mathbb{R}^3} \langle [\partial^\alpha, u\cdot\nabla_\xi] f, \partial^\alpha f\rangle \, dx + \frac{1}{2}\int_{\mathbb{R}^3} \langle \partial^\alpha (u\cdot \xi f), \partial^\alpha f \rangle \, dx \\
&\qquad - \int_{\mathbb{R}^3} \partial^\alpha (u \cdot \nabla_x u) \cdot \partial^\alpha u \, dx - \int_{\mathbb{R}^3} \partial^\alpha (ua)\cdot \partial^\alpha u \, dx = \sum_{i=1}^4 I_i,
\end{aligned} \tag{2.10}$$

where $[A, B]$ means the commutator $AB - BA$ for two operators, and I_i $(1 \leq i \leq 4)$ denote the corresponding terms on the r.h.s. of the above equation. We easily get the bounds

$$\begin{aligned}
I_1 &= -\int_{\mathbb{R}^3} \langle \partial^\alpha [u\cdot\nabla_\xi f], \partial^\alpha f \rangle \, dx = \int_{\mathbb{R}^3} \langle \partial^\alpha [uf], \nabla_\xi \partial^\alpha f\rangle \, dx \leq \|\partial^\alpha (uf)\| \cdot \|\nabla_\xi \partial^\alpha f\|, \\
I_2 &= \frac{1}{2}\int_{\mathbb{R}^3} \langle \partial^\alpha (u\cdot\xi f), \partial^\alpha f\rangle \, dx = \frac{1}{2}\int_{\mathbb{R}^3} \langle \partial^\alpha [uf], \xi \partial^\alpha f \rangle \, dx \leq \frac{1}{2}\|\partial^\alpha (uf)\| \cdot \|\xi \partial^\alpha f\|, \\
I_4 &= -\int_{\mathbb{R}^3} \partial^\alpha (ua)\cdot \partial^\alpha u \, dx \leq \|\partial^\alpha (ua)\| \cdot \|\partial^\alpha u\|.
\end{aligned}$$

Since $1 \leq |\alpha| \leq 3$ we may use inequality (2.6) of Lemma 2.1 successively to obtain :

$$\begin{aligned}
I_1 &\leq C\|\nabla_x u\|_{\mathrm{H}^2} \|\nabla_x f\|_{\mathrm{L}^2_\xi(\mathrm{H}^2_x)} \|\nabla_\xi \partial^\alpha f\|, \\
I_2 &\leq C\|\nabla_x u\|_{\mathrm{H}^2} \|\nabla_x f\|_{\mathrm{L}^2_\xi(\mathrm{H}^2_x)} \|\xi \partial^\alpha f\|, \\
I_4 &\leq C\|\nabla_x u\|_{\mathrm{H}^2} \|\nabla_x a\|_{\mathrm{H}^2} \|\partial^\alpha u\|.
\end{aligned}$$

Due to (1.6), I_3 is computed by

$$I_3 = -\sum_{\beta<\alpha} C_\beta^\alpha \int_{\mathbb{R}^3} (\partial^{\alpha-\beta} u \cdot \nabla_x \partial^\beta u) \cdot \partial^\alpha u \, dx \leq C \|\nabla_x u\|_{\mathrm{H}^2}^2 \|\partial^\alpha u\|, \tag{2.11}$$

where the final inequality follows that for $\beta < \alpha$,

$$\int_{\mathbb{R}^3} (\partial^{\alpha-\beta} u \cdot \nabla_x \partial^\beta u) \cdot \partial^\alpha u \, dx \leq \begin{cases} \|\partial^\alpha u\|_{\mathrm{L}^2} \|\nabla_x u\|_{\mathrm{L}^\infty} \|\partial^\alpha u\|_{\mathrm{L}^2} & (|\beta| = 0) \\ \|\partial^{\alpha-\beta} u\|_{\mathrm{L}^3} \|\nabla_x \partial^\beta u\|_{\mathrm{L}^6} \|\partial^\alpha u\|_{\mathrm{L}^2} & (|\beta| = 1) \\ \|\partial^{\alpha-\beta}\|_{\mathrm{L}^\infty} \|\nabla_x \partial^\beta u\|_{\mathrm{L}^2} \|\partial^\alpha u\|_{\mathrm{L}^2} & (|\beta| \geq 2) \end{cases}$$

and Sobolev inequalities were further used. Putting estimates on I_i $(1 \leq i \leq 4)$ into (2.10) and taking summation over $1 \leq |\alpha| \leq 3$, then (2.9) follows. □

Corollary 2.1. *For smooth solutions of the system* (1.5)-(1.7)*, it holds that*

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \sum_{|\alpha| \leq 3} (\|\partial^\alpha f\|^2 + \|\partial^\alpha u\|^2) + \lambda_0 \sum_{|\alpha| \leq 3} (\|\{\mathbf{I} - \mathbf{P}\} \partial^\alpha f\|_\nu^2 + \|\partial^\alpha (u - b)\|^2) \\ &\quad \leq C \|u\|_{\mathrm{H}^3} \left[\sum_{|\alpha| \leq 3} (\|\{\mathbf{I} - \mathbf{P}\} \partial^\alpha f\|_\nu^2 + \|\partial^\alpha (u - b)\|^2) + \|\nabla_x (a, b)\|_{\mathrm{H}^2}^2 \right] \end{aligned} \tag{2.12}$$

for any $0 \leq t \leq T$ *and any* $T > 0$ *with* C *and* λ_0 *not depending on* T*.*

The goal of the second step in the energy estimates is to obtain the energy dissipation rate $\|\nabla_x (a, b)\|_{\mathrm{H}^2}^2$. For this purpose, we shall firstly derive the following coupled hyperbolic-parabolic system satisfied by a and b which are coefficient functions of $\mathbf{P} f$:

$$\partial_t a + \nabla_x \cdot b = 0, \tag{2.13}$$

$$\partial_t b_i + \partial_i a + \sum_j \partial_j \Gamma_{ij}(\{\mathbf{I} - \mathbf{P}\} f) = -b_i + u_i(1 + a), \tag{2.14}$$

$$\partial_i b_j + \partial_j b_i - (u_i b_j + u_j b_i) = -\partial_t \Gamma_{ij}(\{\mathbf{I} - \mathbf{P}\} f) + \Gamma_{ij}(\ell + r), \tag{2.15}$$

for $1 \leq i, j \leq 3$, where Γ_{ij} is the moment functional defined by $\Gamma_{ij}(g) = \langle (\xi_i \xi_j - 1) M^{1/2}, g \rangle$, for any $g = g(\xi)$, and ℓ, r denote

$$\begin{aligned} \ell &= -\xi \cdot \nabla_x \{\mathbf{I} - \mathbf{P}\} g + \mathbf{L} \{\mathbf{I} - \mathbf{P}\} g, \\ r &= -u \cdot \nabla_\xi \{\mathbf{I} - \mathbf{P}\} g + \frac{1}{2} u \cdot \xi \{\mathbf{I} - \mathbf{P}\} g. \end{aligned}$$

In fact, it is straightforward to get (2.13) and (2.14) by multiplying (1.5) by $M^{1/2}$ and $\xi_i M^{1/2}$ $(1 \leq i \leq 3)$ and then taking velocity integration over $\mathbb{R}^3$. To obtain (2.15), let us rewrite (1.5) as

$$\partial_t \mathbf{P} f + \xi \cdot \nabla_x \mathbf{P} f + u \cdot \nabla_\xi \mathbf{P} f - \frac{1}{2} u \cdot \xi \mathbf{P} f - u \cdot \xi M^{1/2} + \mathbf{P}_1 f = -\partial_t \{\mathbf{I} - \mathbf{P}\} f + \ell + r,$$

and then apply Γ_{ij} to it so that (2.15) follows, where (2.13) was also used. Define a temporal functional $\mathcal{E}_0(f(t))$ by

$$\mathcal{E}_0(f(t)) = \sum_{|\alpha|\leq 2}\sum_{ij}\int_{\mathbb{R}^3}\partial^\alpha(\partial_i b_j+\partial_j b_i)\partial^\alpha\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f)\,dx - \sum_{|\alpha|\leq 2}\int_{\mathbb{R}^3}\partial^\alpha a\partial^\alpha\nabla_x\cdot b\,dx, \tag{2.16}$$

The following lemma is in the same spirit of Kawashima's hyperbolic-parabolic dissipation estimates [18]. At the level of linearization, it corresponds to the estimate (3.7) which we shall prove later.

Lemma 2.4. *For smooth solutions of the system (1.5)-(1.7), it holds that*

$$\frac{d}{dt}\mathcal{E}_0(f(t))+\lambda\|\nabla_x(a,b)\|_{\mathrm{H}^2}^2 \leq C(\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{\mathrm{L}^2_\xi(\mathrm{H}^3_x)}+\|u-b\|^2_{\mathrm{H}^2}) +C\|u\|^2_{\mathrm{H}^2}\Big[\|\nabla_x(a,b)\|^2_{\mathrm{H}^2}+\|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|^2_{\mathrm{L}^2_\xi(\mathrm{H}^2_x)}\Big], \tag{2.17}$$

for any $0\leq t\leq T$ and any $T>0$ with C and λ not depending on T.

Proof. Take α with $|\alpha|\leq 2$. Notice

$$\sum_{ij}\|\partial^\alpha(\partial_i b_j+\partial_j b_i)\|^2 = 2\|\nabla_x\partial^\alpha b\|^2+2\|\nabla_x\cdot\partial^\alpha b\|^2. \tag{2.18}$$

On the other hand, it follows from (2.15) that

$$\begin{aligned}
&\sum_{ij}\|\partial^\alpha(\partial_i b_j+\partial_j b_i)\|^2\\
&\quad=\sum_{ij}\int_{\mathbb{R}^3}\partial^\alpha(\partial_i b_j+\partial_j b_i)\partial^\alpha[(u_ib_j+u_jb_i)-\partial_t\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f)+\Gamma_{ij}(\ell+r)]\,dx\\
&\quad=-\frac{d}{dt}\sum_{ij}\int_{\mathbb{R}^3}\partial^\alpha(\partial_i b_j+\partial_j b_i)\partial^\alpha\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f)\,dx\\
&\qquad+\sum_{ij}\int_{\mathbb{R}^3}\partial^\alpha(\partial_i\partial_t b_j+\partial_j\partial_t b_i)\partial^\alpha\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f)\,dx\\
&\qquad+\sum_{ij}\int_{\mathbb{R}^3}\partial^\alpha(\partial_i b_j+\partial_j b_i)\partial^\alpha[(u_ib_j+u_jb_i)+\Gamma_{ij}(\ell+r)]\,dx.
\end{aligned}\tag{2.19}$$

Using (2.14) to replace the time derivative of b, one has

$$\begin{aligned}
&\sum_{ij}\int_{\mathbb{R}^3}\partial^\alpha(\partial_i\partial_t b_j+\partial_j\partial_t b_i)\partial^\alpha\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f)\,dx\\
&\quad=-2\sum_{ij}\int_{\mathbb{R}^3}\partial^\alpha\partial_t b_i\partial^\alpha\partial_j\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f)\,dx\\
&\quad=2\sum_{ij}\int_{\mathbb{R}^3}\partial^\alpha[\partial_i a+\sum_m\partial_m\Gamma_{im}(\{\mathbf{I}-\mathbf{P}\}f)-(u_i-b_i)-u_ia]\partial^\alpha\partial_j\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f)\,dx.
\end{aligned}$$

Hence, using Young's inequality and inequality (2.5) of Lemma 2.1 one gets :

$$\sum_{ij}\int_{\mathbb{R}^3}\partial^\alpha(\partial_i\partial_t b_j+\partial_j\partial_t b_i)\partial^\alpha\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f)\,dx$$
$$\leq \epsilon\|\nabla_x a\|^2_{\mathrm{H}^2}+C_\epsilon\|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|^2_{\mathrm{L}^2_\xi(\mathrm{H}^2_x)}+C(\|u-b\|^2_{\mathrm{H}^2}+\|u\|^2_{\mathrm{H}^2}\|\nabla_x a\|^2_{\mathrm{H}^2}),$$

where $0<\epsilon\leq 1$ is arbitrary to be chosen later. The final term on the r.h.s. of (2.19) is estimated by

$$\sum_{ij}\int_{\mathbb{R}^3}\partial^\alpha(\partial_i b_j+\partial_j b_i)\partial^\alpha[(u_ib_j+u_jb_i)+\Gamma_{ij}(\ell+r)]\,dx\leq\frac{1}{2}\sum_{ij}\|\partial^\alpha(\partial_i b_j+\partial_j b_i)\|^2$$
$$+C\sum_{ij}(\|\partial^\alpha(u_ib_j+u_jb_i)\|^2+\|\partial^\alpha\Gamma_{ij}(\ell)\|^2+\|\partial^\alpha\Gamma_{ij}(r)\|^2).$$

Using again (2.5) it further holds that

$$\sum_{ij}\|\partial^\alpha(u_ib_j+u_jb_i)\|^2\leq C\|u\otimes b\|^2_{\mathrm{H}^2}\leq C\|u\|^2_{\mathrm{H}^2}\|\nabla_x b\|^2_{\mathrm{H}^2},$$

and since the moment functional appearing in Γ_{ij} can absorb any velocity derivative and any velocity weight we have also

$$\sum_{ij}\|\partial^\alpha\Gamma_{ij}(\ell)\|^2\leq C\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{\mathrm{L}^2_\xi(\mathrm{H}^3_x)},$$
$$\sum_{ij}\|\partial^\alpha\Gamma_{ij}(r)\|^2\leq C\|u\|^2_{\mathrm{H}^2}\|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|^2_{\mathrm{L}^2_\xi(\mathrm{H}^2_x)}.$$

Putting all the above estimates into (2.19) and then taking summation over $|\alpha|\leq 2$, due to (2.18), one has

$$\frac{d}{dt}\sum_{|\alpha|\leq 2}\sum_{ij}\int_{\mathbb{R}^3}\partial^\alpha(\partial_i b_j+\partial_j b_i)\partial^\alpha\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f)\,dx+2\|\nabla_x b\|^2_{\mathrm{H}^2}+2\|\nabla_x\cdot b\|^2_{\mathrm{H}^2}$$
$$\leq\epsilon\|\nabla_x a\|^2_{\mathrm{H}^2}+C_\epsilon\|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|^2_{\mathrm{L}^2_\xi(\mathrm{H}^2_x)}+C\|u-b\|^2_{\mathrm{H}^2}+\|\{\mathbf{I}-\mathbf{P}\}f\|^2_{\mathrm{L}^2_\xi(\mathrm{H}^3_x)}$$
$$+C\|u\|^2_{\mathrm{H}^2}(\|\nabla_x(a,b)\|^2_{\mathrm{H}^2}+\|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|^2_{\mathrm{L}^2_\xi(\mathrm{H}^2_x)}).\tag{2.20}$$

On the other hand, take again α with $|\alpha|\leq 2$ and calculate

$$\|\partial^\alpha\nabla_x a\|^2=\sum_i\int_{\mathbb{R}^3}\partial^\alpha\partial_i a\partial^\alpha\partial_i a\,dx$$
$$=\sum_i\int_{\mathbb{R}^3}\partial^\alpha\partial_i a\partial^\alpha[-\partial_t b_i+(u_i-b_i)-\sum_j\partial_j\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f)+u_ia]\,dx$$
$$=-\frac{d}{dt}\sum_i\int_{\mathbb{R}^3}\partial^\alpha\partial_i a\partial^\alpha b_i\,dx+\sum_i\int_{\mathbb{R}^3}\partial^\alpha\partial_i\partial_t a\partial^\alpha b_i\,dx$$
$$+\sum_i\int_{\mathbb{R}^3}\partial^\alpha\partial_i a\partial^\alpha[(u_i-b_i)-\sum_j\partial_j\Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f)+u_ia]\,dx.$$

So finally, we get

$$\|\partial^\alpha \nabla_x a\|^2 = \frac{d}{dt}\sum_i \int_{\mathbb{R}^3} \partial^\alpha a \partial^\alpha \nabla_x \cdot b\, dx + \sum_i \int_{\mathbb{R}^3} \partial^\alpha \partial_i \partial_t a \partial^\alpha b_i\, dx \\ + \sum_i \int_{\mathbb{R}^3} \partial^\alpha \partial_i a \partial^\alpha [(u_i - b_i) - \sum_j \partial_j \Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f) + u_i a]\, dx. \quad (2.21)$$

Here, $\partial_t a$ can be replaced by the mass conservation law (2.13) so as to obtain

$$\sum_i \int_{\mathbb{R}^3} \partial^\alpha \partial_i \partial_t a \partial^\alpha b_i\, dx = -\int_{\mathbb{R}^3} \partial^\alpha \partial_t a \partial^\alpha \nabla_x \cdot b\, dx = \|\partial^\alpha \nabla_x \cdot b\|^2.$$

The last term is estimated by

$$\sum_i \int_{\mathbb{R}^3} \partial^\alpha \partial_i a \partial^\alpha \Big[(u_i - b_i) - \sum_j \partial_j \Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f) + u_i a??\Big]\, dx \\ \leq \frac{1}{2}\|\nabla_x \partial^\alpha a\|^2 + C(\|u-b\|_{\mathrm{H}^2}^2 + \|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|_{\mathrm{L}^2_\xi(\mathrm{H}^2_x)}^2 + \|u\|_{\mathrm{H}^2}^2\|\nabla_x a\|_{\mathrm{H}^2}^2),$$

where we used (2.5) another time. Then, after taking summation over $|\alpha| \leq 2$, it follows from (2.21) that

$$-\frac{d}{dt}\sum_{|\alpha|\leq 2} \int_{\mathbb{R}^3} \partial^\alpha a \partial^\alpha \nabla_x \cdot b\, dx + \frac{1}{2}\|\nabla_x a\|_{\mathrm{H}^2}^2 \\ \leq \|\nabla_x \cdot b\|_{\mathrm{H}^2}^2 + C(\|u-b\|_{\mathrm{H}^2}^2 + \|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|_{\mathrm{L}^2_\xi(\mathrm{H}^2_x)}^2 + \|u\|_{\mathrm{H}^2}^2\|\nabla_x a\|_{\mathrm{H}^2}^2). \quad (2.22)$$

Now adding (2.20) to (2.22), we get :

$$\frac{d}{dt}\mathcal{E}_0(f(t)) + 2\|\nabla_x b\|_{\mathrm{H}^2}^2 + \|\nabla_x \cdot b\|_{\mathrm{H}^2}^2 + \left[\frac{1}{2} - \epsilon\right]\|\nabla_x a\|_{\mathrm{H}^2}^2 \\ \leq C(\|\{\mathbf{I}-\mathbf{P}\}f\|_{\mathrm{L}^2_\xi(\mathrm{H}^3_x)}^2 + \|u-b\|_{\mathrm{H}^2}^2) \\ + C\|u\|_{\mathrm{H}^2}^2\Big[\|\nabla_x(a,b)\|_{\mathrm{H}^2}^2 + \|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|_{\mathrm{L}^2_\xi(\mathrm{H}^2_x)}^2\Big].$$

Hence, (2.17) follows after taking $\epsilon = \frac{1}{4}$ for instance. □

Combining estimates obtained in the above two steps, one can finish the proof of uniform-in-time *a priori* estimates as follows. Define a total temporal energy functional $\mathcal{E}(f,u)$ and corresponding dissipation rate $\mathcal{D}(f,u)$ by

$$\mathcal{E}(f(t),u(t)) = \|(f(t),u(t))\|_{\mathcal{H}^3}^2 + \kappa_1 \mathcal{E}_0(f(t)), \quad (2.23)$$

$$\mathcal{D}(f(t),u(t)) = \sum_{|\alpha|\leq 3}(\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2 + \|\partial^\alpha(u-b)\|^2) + \|\nabla_x(a,b,u)\|_{\mathrm{H}^2}^2, \quad (2.24)$$

where $\kappa_1 > 0$ is a small constant to be chosen later. For the sake of clarity, let us introduce further notation:

$$\mathsf{K}(f,u) := \sum_{|\alpha|\le 3} \Big\{ \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_\nu^2 + \|\partial^\alpha(u-b)\|^2 \Big\} + \|\nabla_x(a,b)\|_{\mathrm{H}^2}^2 ,$$

so that we can rewrite (2.12) as

$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{\mathcal{H}^3}^2 + \lambda_0 \left[\mathsf{K}(f,u) - \|\nabla_x(a,b)\|_{\mathrm{H}^2}^2\right] \le C\|u\|_{\mathrm{H}^3}\mathsf{K}(f,u) \le C\|u\|_{\mathrm{H}^3}^2\mathsf{K}(f,u) + \kappa_2\mathsf{K}(f,u), \tag{2.25}$$

for a small constant κ_2 to be fixed later. Obviously we have $\|\cdot\|_{\mathrm{L}^2_\xi(\mathrm{L}^2_x)} \le \|\cdot\|_\nu$ and $\|\cdot\|_{\mathrm{H}^2_x} \le \|\cdot\|_{\mathrm{H}^3_x}$ hence (2.17) implies

$$\frac{d}{dt}\mathcal{E}_0(f(t)) + \lambda\|\nabla_x(a,b)\|_{\mathrm{H}^2}^2 \le C\left[\mathsf{K}(f,u) - \|\nabla_x(a,b)\|_{\mathrm{H}^2}^2\right] + C\|u\|_{\mathrm{H}^3}^2\mathsf{K}(f,u). \tag{2.26}$$

Finally, by adding $2\times$(2.25) to $\kappa_1\times$(2.26) we obtain

$$\frac{d}{dt}\mathcal{E}(f(t),u(t)) + \min(2\lambda_0 - C\kappa_1, \kappa_1\lambda)\mathsf{K}(f,u) \le 2\kappa_2\mathsf{K}(f,u) + C\|u\|_{\mathrm{H}^3}^2\mathsf{K}(f,u),$$

so picking first κ_1 and then κ_2 small enough we have $\mathcal{E}(f(t),u(t)) \sim \|(f(t),u(t))\|_{\mathcal{H}^3}^2$ and

$$\frac{d}{dt}\mathcal{E}(f(t),u(t)) + \lambda\mathsf{K}(f,u) \le C\|u\|_{\mathrm{H}^3}^2\mathsf{K}(f,u). \tag{2.27}$$

Now notice

$$\|\nabla_x u\|_{\mathrm{H}^2}^2 \le 2\|\nabla_x(u-b)\|_{\mathrm{H}^2}^2 + 2\|\nabla_x b\|_{\mathrm{H}^2}^2 \le 2\mathsf{K}(f,u),$$

and $\mathcal{D}(f,u) = \mathsf{K}(f,u) + \|\nabla_x u\|_{\mathrm{H}^2}^2$, so that by adding $\kappa_3\|\nabla_x u\|_{\mathrm{H}^2}^2$ to both sides in (2.27) with κ_3 small enough we have

$$\frac{d}{dt}\mathcal{E}(f(t),u(t)) + \kappa_3\mathcal{D}(f(t),u(t)) \le C\|u(t)\|_{\mathrm{H}^3}^2\mathcal{D}(f(t),u(t)), \tag{2.28}$$

for any $0 \le t \le T$ and any $T > 0$ with C and κ_3 not depending on T, which is the desired uniform-in-time estimate for the global existence.

2.3 Approximating Scheme & Global Existence

We are now going to prove first the existence of local regular solutions for which all the previous computations will be rigorous. These estimates will allow us to prove that the solution is in fact global at the end of this subsection. Let us introduce $\mathbb{X} = \mathrm{W}^{1,\infty}\big([0,T];\mathrm{H}^2(\mathbb{R}^3)\big) \cap \mathrm{L}^\infty\big([0,T];\mathrm{H}^3(\mathbb{R}^3)\big)$ and

$$S_T = \left\{ (f,u,p) \,\middle|\, \begin{array}{c} f \in \mathrm{L}^\infty\big([0,T];\mathrm{L}^2_\xi(\mathrm{H}^3_x)\big) \cap \mathscr{C}^1_{x,t} \cap \mathscr{C}^2_\xi, a^f, b^f \in \mathbb{X} \\ u \in \mathbb{X} \cap \mathscr{C}^1([0,T]\times\mathbb{R}^3) \\ \nabla_x \cdot u = 0, f \ge 0,\ p \in \mathscr{C}^0\big([0,T];\mathrm{H}^3(\mathbb{R}^3)\big) \end{array} \right\}.$$

Let us consider the iterative Cauchy problem, initialized by $(u^0, F^0) = (u_0, M + M^{1/2} f_0)$:

$$
\begin{aligned}
&\partial_t F^{n+1} + \xi \cdot \nabla_x F^{n+1} = \nabla_\xi \cdot (\xi F^{n+1} + \nabla_\xi F^{n+1}) - \gamma^n [u^n \star \tau^n] \cdot \nabla_\xi \widetilde{F^n},\\
&\widetilde{F^n} := M + M^{1/2}\big[\rho^n \star (M^{-1/2}(F^n - M))\big],\\
&\nabla_x \cdot u^{n+1} = 0,\\
&\partial_t u^{n+1} + u^n \cdot \nabla_x u^{n+1} + \nabla_x p^{n+1} = \int_{\mathbb{R}^3} (\xi - u^{n+1}) F^n \, d\xi,\\
&u^{n+1}(0) = u_0, \quad F^{n+1}(0) = M + M^{1/2} (\rho_n \star f_0)\gamma^n,
\end{aligned}
$$

where γ^n is some tensorial cut-off function $\gamma^n(x,\xi) := \theta\left(\frac{x}{n}\right)\theta\left(\frac{\xi}{n}\right) = \sigma^n \mu^n$, θ being a smooth function with support in the unit ball and $\tau^n(x)$ and $\rho^n(x,\xi)$ are standard mollifiers. In particular, $(\gamma^n)_n$, $(\sigma^n)_n$, $(\mu^n)_n$ are bounded in the Schwartz class $\mathscr{S}(\mathbb{R}^3)$. Considering $u^n_\star := u^n \star \tau^n$ and $f^n_\star := f^n \star \rho^n$, it reads in terms of perturbations as:

$$
\partial_t f^{n+1} + \xi \cdot \nabla_x f^{n+1} + \gamma^n u^n_\star \cdot \nabla_\xi f^n_\star - \frac{1}{2} \gamma^n u^n_\star \cdot \xi f^n_\star - \gamma^n u^n_\star \cdot \xi M^{1/2} = \mathbf{L} f^{n+1}, \tag{2.29}
$$

$$
\nabla_x \cdot u^{n+1} = 0, \tag{2.30}
$$

$$
\partial_t u^{n+1} + u^n \cdot \nabla_x u^{n+1} + \nabla_x p^{n+1} + u^{n+1}(1 + a^n) = b^n, \tag{2.31}
$$

$$
u^{n+1}(0) = u_0, \quad f^{n+1}(0) = (\rho_n \star f_0)\gamma^n, \tag{2.32}
$$

Lemma 2.5. *There exists $\delta_0 > 0$ and $T_0 > 0$ such as for $\|(f_0, u_0)\|_{\mathcal{H}^3} \le \delta_0$, the previous sequence (f^n, u^n) is well-defined and bounded in $\mathrm{L}^\infty([0,T_0];\mathcal{H}^3)$. (u^n) is furthermore bounded in $\mathrm{W}^{1,\infty}\Big([0,T_0];\mathrm{H}^2(\mathbb{R}^3)\Big)$.*

Proof. Some parts of the proof have been placed in the Appendix for the sake of clarity. Let us verify by induction that there exist, for all n, global classical solutions to the system (2.29)-(2.32), all belonging to S_T (but the kinetic part is not necessarily positive at this stage). Take $(f^n, u^n, p^n) \in S_T$ (except the positivity condition). The existence of $f^{n+1} \in \mathrm{L}^2_\xi(\mathrm{H}^3_x)$ is a direct consequence of Lemma 5.5 in Appendix and then $a^{f^{n+1}}, b^{f^{n+1}} \in \mathbb{X}$ is straightforward. As for u^{n+1} and p^{n+1}, Lemma 5.3 of the Appendix applies directly. Since all the equations are verified strongly and given the regularity of the solution, we are now able to compute several estimates in a rigorous way. As done in (2.10) but including this time the case $\alpha = 0$, the energy integration of α-order (2.29), (2.30) and (2.31), and their summation give

$$\frac{1}{2}\frac{d}{dt}(\|\partial^\alpha f^{n+1}\|^2 + \|\partial^\alpha u^{n+1}\|^2) + \int_{\mathbb{R}^3} \langle -\mathbf{L}\partial^\alpha f^{n+1}, \partial^\alpha f^{n+1}\rangle\, dx + \|\partial^\alpha u^{n+1}\|^2$$
$$= \overbrace{-\int_{\mathbb{R}^3} \langle \partial^\alpha \Big[\sigma^n u^n_\star \cdot \big[\nabla_\xi (f^n_\star \mu^n) - f^n_\star \nabla_\xi \mu^n\big]\Big], \partial^\alpha f^{n+1}\rangle\, dx}^{I_{1,n}} \overbrace{-\int_{\mathbb{R}^3} \partial^\alpha (u^{n+1} a^n)\cdot \partial^\alpha u^{n+1}\, dx}^{I_{2,n}}$$
$$\overbrace{-\int_{\mathbb{R}^3} \partial^\alpha (u^n\cdot \nabla_x u^{n+1})\cdot \partial^\alpha u^{n+1}\, dx}^{I_{3,n}} + \overbrace{\frac{1}{2}\int_{\mathbb{R}^3} \langle \partial^\alpha (\gamma^n u^n_\star \cdot \xi f^n_\star), \partial^\alpha f^{n+1}\rangle\, dx}^{I_{4,n}}$$
$$+ \int_{\mathbb{R}^3} \partial^\alpha b^n \cdot \partial^\alpha u^{n+1}\, dx + \int_{\mathbb{R}^3\times\mathbb{R}^3} \partial^\alpha f^{n+1} \mu^n M^{1/2} \xi \cdot \partial^\alpha (\sigma^n u^n_\star(x))\, d\xi\, dx.$$

Using (2.3) we hence have

$$\frac{1}{2}\frac{d}{dt}(\|\partial^\alpha f^{n+1}\|^2 + \|\partial^\alpha u^{n+1}\|^2) + \lambda_0 \|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f^{n+1}\|^2_\nu + \|\partial^\alpha u^{n+1}\|^2 + \|\partial^\alpha b^{n+1}\|^2$$
$$\le \sum_{i=1}^4 I_{i,n} + \int_{\mathbb{R}^3} \partial^\alpha b^n \cdot \partial^\alpha u^{n+1}\, dx + \int_{\mathbb{R}^3} |\partial^\alpha b^{n+1}||\partial^\alpha [\sigma^n u^n_\star]|\, dx,$$

and since for any function $g(x,\xi)$, $\|\mathbf{P}g\|_\nu$, $\|b^g\|$ and $\|a^g\|$ are all smaller than $C\|g\|$ for some constant C, σ^n is bounded in $\mathscr{S}(\mathbb{R}^3)$ and $\|\partial^\beta u^n_\star\| \le \|\partial^\alpha u^n\|$ for all $\beta \in \mathbb{N}^3$, we deduce

$$\frac{1}{2}\frac{d}{dt}(\|\partial^\alpha f^{n+1}\|^2 + \|\partial^\alpha u^{n+1}\|^2) + \lambda_0 \|\partial^\alpha f^{n+1}\|^2_\nu + \|\partial^\alpha u^{n+1}\|^2$$
$$\le \sum_{i=1}^4 I_{i,n} + C(\|\partial^\alpha u^{n+1}\|^2 + \|u^n\|^2_{\mathrm{H}^3} + \|\partial^\alpha f^n\|^2 + \|\partial^\alpha f^{n+1}\|^2).$$

Now for $I_{1,n}$, $I_{4,n}$ put (if necessary use integration by parts) operators ∇_ξ and $\xi\cdot$ on f^{n+1} (except for the term $f^n_\star \nabla_\xi \mu^n$) and apply Young to obtain

$$|I_{1,n}| + |I_{2,n}| + |I_{4,n}| \le C\|\partial^\alpha(\gamma^n u^n_\star f^n_\star)\|^2 + C\|\partial^\alpha(\sigma^n u^n_\star \cdot f^n_\star \nabla_\xi \mu^n)\|^2 + \frac{\lambda_0}{2}\|\partial^\alpha f^{n+1}\|^2_\nu$$
$$+ C\|\partial^\alpha(u^{n+1}a^n)\|^2 + \|\partial^\alpha u^{n+1}\|^2.$$

Since $(\gamma^n)_n$, $(\sigma^n)_n$ and $(\mu^n)_n$ are all bounded in $\mathscr{S}(\mathbb{R}^3)$, and $\|\partial^\beta g_\star\| \le \|\partial^\beta g\|$ for any convolution operation and any β, we have, using Lemma 2.1,

$$|I_{1,n}| + |I_{2,n}| + |I_{4,n}| \le C\|u^n\|^2_{\mathrm{H}^3}\|f^n\|^2_{\mathrm{L}^2_\xi(\mathrm{H}^3_x))} + \frac{\lambda_0}{2}\|\partial^\alpha f^{n+1}\|_\nu$$
$$+ C\big\{1 + \|f^n\|^2_{\mathrm{L}^2_\xi(\mathrm{H}^3_x)}\big\}\|u^{n+1}\|^2_{\mathrm{H}^3}.$$

Finally $I_{3,n}$ is equal to zero whenever $\alpha = 0$, so that we can use directly what we have done in (2.11) (notice that the term $[u^n\cdot\nabla_x \partial^\alpha u^{n+1}]\cdot \partial^\alpha u^{n+1}$ in the expansion is still zero) and get

$$|I_{3,n}| \le C\|u^n\|_{\mathrm{H}^3}\|u^{n+1}\|^2_{\mathrm{H}^3}.$$

We have finally, summing over α

$$\frac{d}{dt}\|(f^{n+1},u^{n+1})\|^2_{\mathcal{H}^3}+\min(\lambda_0,2)\sum_{|\alpha|\le 3}\Big\{\|\partial^\alpha f^{n+1}\|^2_\nu+\|\partial^\alpha u^{n+1}\|^2\Big\}$$
$$\le C_1(1+\|(f^n,u^n)\|^2_{\mathcal{H}^3})\|(f^{n+1},u^{n+1})\|^2_{\mathcal{H}^3}+C_2(1+\|(f^n,u^n)\|^2_{\mathcal{H}^3})\|(f^n,u^n)\|^2_{\mathcal{H}^3}.$$

Denote

$$A_n(t):=\|(f^n,u^n)\|^2_{\mathcal{H}^3},$$
$$B_n(t):=\sum_{|\alpha|\le 3}\Big\{\|\partial^\alpha f^{n+1}\|^2_\nu+\|\partial^\alpha u^{n+1}\|^2\Big\}.$$

We have obviously $A_n(t)\le B_n(t)$ and furthermore from the previous inequality

$$A_{n+1}(t)+\lambda\int_0^t B_{n+1}(s)ds\le A_{n+1}(0)+C_1\int_0^t(1+A_n(s))A_{n+1}(s)ds$$
$$+C_2\int_0^t(1+A_n(s))A_n(s)ds, \tag{2.33}$$

where $A_{n+1}(0)=\|(f_0,u_0)\|^2_{\mathcal{H}^3}$ is independent of n. Define successively

$$K:=C_1/\lambda,\qquad \delta_0:=K/3,$$
$$T_0:=\min\left\{\frac{1}{2C_1},\frac{K-2\delta_0}{2C_2K(1+K)}\right\}>0.$$

Let us prove the following property by induction on n: if $\|(f_0,u_0)\|_{\mathcal{H}^3}\le\delta_0$, then:

$$\Big[P(p):\qquad M_p:=\sup_{t\in[0,T_0]}A_p(t)\le K\Big]$$

holds for all $p\in\mathbb{N}$. $P(0)$ is obviously satisfied, let us suppose $P(n)$ is true. We have then by (2.33), for $t\in[0,T_0]$:

$$A_{n+1}(t)+\lambda\int_0^t B_{n+1}(s)ds\le\delta_0+C_1T_0M_{n+1}+C_1T_0M_nM_{n+1}+C_2T_0M_n(1+M_n)$$
$$\le\delta_0+C_1T_0M_{n+1}+C_1T_0KM_{n+1}+C_2T_0K(1+K)$$

since $C_1K=\lambda$, for $t\in[0,T_0]$, we have also

$$A_{n+1}(t)+\lambda\int_0^t\left[B_{n+1}(s)-A_{n+1}(s)\right]ds\le\delta_0+CT_0M_{n+1}+C(M_n+M_n^2)T_0\,,$$

and noting that $B_{n+1}(s)-A_{n+1}(s)\ge 0$, we conclude

$$M_{n+1}\le\delta_0+C_1T_0M_{n+1}+C_2K(1+K)T_0,$$

and since $2C_1T_0 \le 1$, we have:

$$M_{n+1} \le 2\delta_0 + 2C_2K(1+K)T_0 \le K,$$

because of the definition of T_0.

Finally the bound of (u^n) in $\mathrm{W}^{1,\infty}\Big([0,T_0];\mathrm{H}^2(\mathbb{R}^3)\Big)$ is a direct consequence of the previous bound, using the fluid equation, after applying the Leray projection (see the appendix section for the precise definition) to get rid of the pressure function. □

Lemma 2.6. *There exists a subsequence $(u^{\sigma(n)})_n$ strongly converging in $\mathrm{L}^\infty\big([0,T_0];\mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^3)\big)$ to an element of $\mathscr{C}^0\big([0,T_0];\mathrm{H}^2(\mathbb{R}^3)\big)$.*

Proof. Consider, an exhaustive family of increasing balls centered in 0, of radius $m \in \mathbb{N}^*$: $(\mathsf{B}_m)_m$. Given m, the injection $\mathrm{H}^3(\mathsf{B}_m) \hookrightarrow \mathrm{H}^2(\mathsf{B}_m)$ is compact. We know from Lemma 2.5 that $(u^n)_n$ is bounded in $\mathrm{W}^{1,\infty}\Big([0,T_0];\mathrm{H}^2(\mathsf{B}_m)\Big)$ and $\mathrm{L}^\infty\big([0,T_0];\mathrm{H}^3(\mathsf{B}_m)\big)$, hence the family $(u^n)_n$ is equicontinuous in $\mathscr{C}^0\Big([0,T_0];\mathrm{H}^2(\mathsf{B}_m)\Big)$ and pointwisely (in time) relatively compact. We may hence apply Ascoli's theorem to obtain the convergence of a subsequence in $\mathscr{C}^0\Big([0,T_0];\mathrm{H}^2(\mathsf{B}_m)\Big)$. We also extract to have almost everywhere convergence. Since the previous extractions are countable, we may extract diagonally a subsequence following the exhaustive sequence of balls. In such a way, for a fixed ball B_m the sequence (still) denoted $(u^n)_n$ converges to some u^m in $\mathscr{C}^0\Big([0,T_0];\mathrm{H}^2(\mathsf{B}_m)\Big)$. All the functions are continuous hence all the u^m are equal in their domain of definition and we may define pointwisely a limit function u on B_m as the common value of all the $(u^k)_{k\ge m}$ on this ball, for all $m \in \mathbb{N}^*$. By construction $u \in \mathscr{C}^0\big([0,T];\mathrm{H}^2(\mathsf{B}_m)\big)$ for all m and the sequence of corresponding norms is bounded: $u \in \mathscr{C}^0\big([0,T_0];\mathrm{H}^2(\mathbb{R}^3)\big)$. Since the sequence is exhaustive, we have strong convergence in $\mathrm{L}^\infty\big([0,T_0];\mathrm{H}^2_{\mathrm{loc}}(\mathbb{R}^3)\big)$. □

Lemma 2.7. *There exists $\delta_1 > 0$ such as if $\|(f_0,u_0)\|_{\mathcal{H}^3} \le \delta_1$ then the Cauchy problem (1.5)-(1.7) with initial data verifying (1.8) has a unique global and classical solution defined on $\mathbb{R}_+$, belonging to $(f,u,p) \in S_t$ for all $t > 0$.*

Proof. With $\|(f_0,u_0)\|_{\mathcal{H}^3} \le \delta_0$, by Lemma 2.5 we now that (f^n,u^n) is (up to a subsequence) weakly-$\star$ convergent in $\mathrm{L}^\infty([0,T_0];\mathcal{H}^3)$ and so is (u^n) in $\mathrm{W}^{1,\infty}\Big([0,T_0];\mathrm{H}^2(\mathbb{R}^3)\Big)$. Let us note (f,u) the limit. Together with the strong convergence given by Lemma 2.6 we can pass to the limit and prove that (f,u) is a weak solution of (1.5)-(1.7) and (1.8). Using Lemma 5.2 of the appendix we obtain that u is in fact a strong regular solution. Using then the uniqueness of solutions for the Vlasov-Fokker-Planck [23], we finally have that f is also regular and strong solution of the equation. Until now we have just obtained local solution on some interval $[0,T_0]$ but since it is regular we may use all the *a priori* estimates that we performed earlier. In particular, we have (2.28), i.e.

$$\frac{d}{dt}\mathcal{E}(f(t),u(t)) + \kappa_3\mathcal{D}(f(t),u(t)) \le C\|u(t)\|^2_{\mathrm{H}^3}\mathcal{D}(f(t),u(t)),$$

with C and κ_3 independent of T_0. Now, define

$$\epsilon = \min\{\delta_0, \sqrt{\kappa_3/(2C)}\},$$

and let initial data (f_0, u_0) be small enough such that

$$\mathcal{E}(f_0, u_0) \le \delta_1 := \frac{\epsilon}{2(1+C)}.$$

Denote

$$T_\star = \sup\left\{ t \in \mathbb{R}_+^* \;\middle|\; \begin{array}{c} \exists (f,u,p) \in S_t \text{ verifying } (1.5) - (1.7) \text{ and } (1.8) \text{ on } [0,t] \\ \text{and } \sup_{0\le s\le t} \mathcal{E}(f(s),u(s)) \le \varepsilon \end{array} \right\}.$$

Note first that the uniqueness on any interval is a direct consequence of the previous estimates since the solutions are regular, therefore in what follow $(f(t), u(t))$ is well-defined. Since $\|(f_0, u_0)\|_{\mathcal{H}^3} \le \epsilon/2 \le \delta_0$, it follows from the previous study that $T_\star$ is well-defined (since $\mathcal{E}(\cdot)$ and $\|\cdot\|_{\mathcal{H}^3}$ are equivalent). Furthermore by definition of ε, $\mathcal{E}(f(t), u(t))$ is a decreasing function of time on $[0, T_\star[$: the right hand side term of (2.28) can be absorbed due to

$$C\|u(t)\|_{\mathrm{H}^3}^2 \le C\|(f(t), u(t))\|_{\mathcal{H}^3}^2 \le C\epsilon^2 \le \frac{\kappa_3}{2},$$

hence, for $t \in [0, T_\star[$, $\mathcal{E}(f(t), u(t)) \le \varepsilon/2$. Let us check that if $T_\star < \infty$, then by local existence our solution is in fact defined at least on some interval $[0, T_\star + \eta]$. Indeed, the time of existence T_0 is granted as soon as the initial data is in the ball $\mathsf{B}(0, \delta_0)$ of $\mathcal{H}^3$, which is the case for $(f(t), u(t))$ for $t < T_\star$. Then, one could take the couple $(f(t), u(t))$ (t sufficiently close to $T_\star$) as admissible initial condition and obtain local existence from t to some $t + T_0 > T_\star$. Pasting this extension with the previous solution defined on $[0, T_\star]$, we get some (at least) weak solution on the whole interval $[0, T_\star + \eta]$. We may then use the positive fundamental solution of the Fokker-Planck equation with a field given in [23] to have all the regularity that we need. We are in position to use the previous result since the only assumption required in [23] is to have a field in $\mathrm{L}^\infty\big([0,T]; \mathrm{W}^{1,\infty}(\mathbb{R}^3)\big)$, and this is still the case here, even after the "pasting operation". In that way we can prove first that the kinetic part f is regular in time and positive, then we treat the fluid equation as before to finally extend the global strong solution on $[0, T_\star + \eta]$ by uniqueness.

By continuity $\mathcal{E}(f(T_\star), u(T_\star)) \le \varepsilon/2$ so that $\mathcal{E}(f(t), u(t))$ remains strictly less than ε on some interval $[T_\star, T_\star + \eta']$. This contradicts the definition of $T_\star$ which is therefore infinite. This shows the global-in-time existence in our main Theorem 1.1. □

Corollary 2.2. *Take δ_1 as in Lemma 2.7 and (f_0, u_0) verifying $\|(f_0, u_0)\|_{\mathcal{H}^3} \le \delta_1$. Then the unique solution to the system (1.5)-(1.8) verifies*

$$\frac{d}{dt}\mathcal{E}(f(t), u(t)) + \lambda \mathcal{D}(f(t), u(t)) \le 0,$$

for some fixed constant $\lambda > 0$. In particular the energy $\mathcal{E}(f(t), u(t))$ is a non-increasing function.

3 Rate of convergence

In order to study the time-decay of solutions obtained in last section, we firstly consider the linearized Cauchy problem

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f - u \cdot \xi M^{1/2} = \mathbf{L} f + S_f, \\ \nabla_x \cdot u = 0, \\ \partial_t u + \nabla_x p + (u - b) = 0, \\ f|_{t=0} = f_0, \quad u|_{t=0} = u_0. \end{cases} \tag{3.1}$$

Here, the non-homogeneous source S_f takes the form

$$S_f = \nabla_\xi \cdot G - \frac{1}{2}\xi \cdot G + h,$$

for $G = (G_i)$, $G_i = G_i(t, x, \xi) \in \mathbb{R}$, $(1 \le i \le 3)$ and $h = h(t, x, \xi) \in \mathbb{R}$, where it is supposed that

$$\mathbf{P}_0 G_i \equiv 0 \ (1 \le i \le 3), \quad \mathbf{P} h \equiv 0$$

for all $t \ge 0$ and $x \in \mathbb{R}^3$. We first note that this coupled linearized problem is well-posed in $\mathcal{L}^2$.

Proposition 3.1. *There is a well-defined linear semigroup $\mathbf{E}_t : \mathcal{L}^2 \longrightarrow \mathcal{L}^2$, $t \ge 0$, such that for any given $(f_0, u_0) \in \mathcal{L}^2$ with $\nabla_x \cdot u_0 = 0$, then $\mathbf{E}_t(f_0, u_0)$ is the unique distributional solution to (3.1) with $S_f = 0$. Moreover, for any given $(f_0, u_0) \in \mathcal{L}^2$ with $\nabla_x \cdot u_0 = 0$, the unique distributional solution to (3.1) satisfies*

$$(f(t), u(t)) = \mathbf{E}_t(f_0, u_0) + \int_0^t \mathbf{E}_{t-s}(S_f(s), 0)\, ds. \tag{3.2}$$

Proof. The well-posedness part follows the same guidelines as in the local existence theorem in the previous section but simpler due to its linearity. We do not repeat here the whole argument, but one needs to define smooth approximations by regularizing the initial data and splitting the coupling term as in the system (2.29)-(2.32). The passing to the limit is trivial in this case, moreover the solutions are strong for initial smooth compactly supported data. Due to standard regularization procedures, it is enough to show the propagation of the $\mathcal{L}^2$-norm for smooth compactly supported initial data. Taking $(f^\epsilon(t), u^\epsilon(t)) = \mathbf{E}_t(f_0^\epsilon, u_0^\epsilon)$, a direct computation shows

$$\frac{1}{2}\frac{d}{dt}(\|f^\epsilon\|^2 + \|u^\epsilon\|^2) + \int_{\mathbb{R}^3} \langle -\mathbf{L}\{\mathbf{I} - \mathbf{P}\} f^\epsilon, f^\epsilon \rangle\, dx + \|u^\epsilon - b^\epsilon\|^2 = 0\,,$$

implying

$$\|\mathbf{E}_t(f_0^\epsilon, u_0^\epsilon)\|_{\mathcal{L}^2} \le \|(f_0^\epsilon, u_0^\epsilon)\|_{\mathcal{L}^2}$$

for all $t \ge 0$ and all $\epsilon > 0$. Passing to the limit in the regularization parameter ϵ leads trivially to the stated propagation. Finally, the variation-of-constants formula (3.2) for problem (3.1) is again direct by approximation procedures. □

We can now show the following uniform estimates on solutions of (3.1).

Theorem 3.1. *Let $1 \le q \le 2$ and $(f_0, u_0) \in \mathcal{L}^2$ with $\nabla_x \cdot u_0 = 0$. For any α, α' with $\alpha' \le \alpha$ and $m = |\alpha - \alpha'|$,*

$$\|\partial^\alpha \mathbf{E}_t(f_0, u_0)\|_{\mathcal{L}^2} \le C(1+t)^{-\sigma_{q,m}}(\|\partial^{\alpha'}(f_0, u_0)\|_{\mathcal{Z}_q} + \|\partial^\alpha (f_0, u_0)\|_{\mathcal{L}^2}), \tag{3.3}$$

and

$$\left\|\partial^\alpha \int_0^t \mathbf{E}_{t-s}(S_f(s), 0)ds\right\|_{\mathcal{L}^2}^2 \le C\int_0^t (1+t-s)^{-2\sigma_{q,m}} \times (\|\partial^{\alpha'}(G(s), \nu^{-1/2}h(s))\|_{Z_q}^2 + \|\partial^\alpha (G(s), \nu^{-1/2}h(s))\|_{\mathrm{L}^2}^2)ds, \tag{3.4}$$

hold for $t \ge 0$, where C is a positive constant depending only on m, q and

$$\sigma_{q,m} = \frac{3}{2}\left(\frac{1}{q} - \frac{1}{2}\right) + \frac{m}{2}.$$

Proof. By Fourier transforming (3.1) in x, one has

$$\begin{cases} \partial_t \widehat{f} + \mathrm{i}\xi \cdot k \widehat{f} - \widehat{u} \cdot \xi M^{1/2} = \mathbf{L}\widehat{f} + \nabla_\xi \cdot \widehat{G} - \frac{1}{2}\xi \cdot \widehat{G} + \widehat{h}, \\ \mathrm{i}k \cdot \widehat{u} = 0, \\ \partial_t \widehat{u} + \mathrm{i}k\widehat{p} + \widehat{u} - \widehat{b} = 0. \end{cases} \tag{3.5}$$

By taking the inner product of the first equation in (3.5) with the conjugate of $\widehat{f}$ and integrating in ξ, its real part gives

$$\frac{1}{2}\partial_t \|\widehat{f}\|_{L^2_\xi}^2 + \mathrm{Re}\int_{\mathbb{R}^3} (-\mathbf{L}\{\mathbf{I} - \mathbf{P}\}\widehat{f} | \{\mathbf{I} - \mathbf{P}\}\widehat{f})\, d\xi + |\widehat{b}|^2 - \mathrm{Re}\,(\widehat{u}|\widehat{b})$$
$$= \mathrm{Re}\int_{\mathbb{R}^3} (\nabla_\xi \cdot \widehat{G} - \frac{1}{2}\xi \cdot \widehat{G} | \{\mathbf{I} - \mathbf{P}\}\widehat{f})\, d\xi + \mathrm{Re}\int_{\mathbb{R}^3} (\widehat{h} | \{\mathbf{I} - \mathbf{P}\}\widehat{f})\, d\xi,$$

where we used the observation

$$\nabla_\xi \cdot G - \frac{1}{2}\xi \cdot G \perp \mathrm{Rang}\,\mathbf{P}$$

due to $\mathbf{P}_0 G = 0$. Then, the coercivity of $-\mathbf{L}$, integration by parts in ξ and Cauchy-Schwarz inequality further imply

$$\frac{1}{2}\partial_t \|\widehat{f}\|_{L^2_\xi}^2 + \lambda_0 |\{\mathbf{I} - \mathbf{P}\}\widehat{f}|_\nu^2 + |\widehat{b}|^2 - \mathrm{Re}(\widehat{u}|\widehat{b}) \le C(\|\widehat{G}\|^2 + \|\nu^{-1/2}\widehat{h}\|^2).$$

Similarly, from the last two equations in (3.5) we infer $\frac{1}{2}\partial_t |\widehat{u}|^2 + |\widehat{u}|^2 - \mathrm{Re}(\widehat{b}|\widehat{u}) = 0$, that together in the previous estimate shows

$$\frac{1}{2}\partial_t (\|\widehat{f}\|_{L^2_\xi}^2 + |\widehat{u}|^2) + \lambda_0 |\{\mathbf{I} - \mathbf{P}\}\widehat{f}|_\nu^2 + |\widehat{u} - \widehat{b}|^2 \le C(\|\widehat{G}\|^2 + \|\nu^{-1/2}\widehat{h}\|^2). \tag{3.6}$$

Next, we consider the estimates on a, b. Since $\mathbf{P}S_f = 0$, then similar to get (2.13)-(2.15), corresponding to the system (3.1), one has

$$\begin{aligned}
&\partial_t a + \nabla_x \cdot b = 0,\\
&\partial_t b_i + \partial_i a + \sum_j \partial_j \Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f) = u_i - b_i,\\
&\partial_i b_j + \partial_j b_i = -\partial_t \Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f) + \Gamma_{ij}(\ell + S_f),
\end{aligned}$$

where ℓ still denotes

$$\ell = -\xi \cdot \nabla_x \{\mathbf{I}-\mathbf{P}\}f + \mathbf{L}\{\mathbf{I}-\mathbf{P}\}f.$$

Taking the Fourier transform in x gives

$$\begin{cases}
\partial_t \widehat{a} + \mathrm{i}k \cdot \widehat{b} = 0,\\
\partial_t \widehat{b}_i + \mathrm{i}k_i \widehat{a} + \sum_j \mathrm{i}k_j \Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}f) = \widehat{u}_i - \widehat{b}_i,\\
\mathrm{i}k_i \widehat{b}_j + \mathrm{i}k_j \widehat{b}_i = -\partial_t \Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}\widehat{f}) + \Gamma_{ij}(\widehat{\ell} + \widehat{S}_f).
\end{cases}$$

By using the same proof as in Lemma 2.4, we conclude that there exist $\lambda > 0$ and $C > 0$ such that

$$\begin{aligned}
&\partial_t \mathrm{Re}\{\sum_{ij}(\mathrm{i}k_i \widehat{b}_j + \mathrm{i}k_j \widehat{b}_i | \Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}\widehat{f})) - (\widehat{a}|\mathrm{i}k\cdot\widehat{b})\} + \lambda|k|^2(|\widehat{a}|^2 + |\widehat{b}|^2)\\
&\qquad \le C((1+|k|^2)\|\{\mathbf{I}-\mathbf{P}\}\widehat{f}\|^2_{L^2_\xi} + |\widehat{u}-\widehat{b}|^2) + C(\|\widehat{G}\|^2 + \|\nu^{-1/2}\widehat{h}\|^2).
\end{aligned}$$

By setting

$$\mathcal{E}_1(\widehat{f}) = \frac{1}{1+|k|^2}\sum_{ij}(\mathrm{i}k_i \widehat{b}_j + \mathrm{i}k_j \widehat{b}_i | \Gamma_{ij}(\{\mathbf{I}-\mathbf{P}\}\widehat{f})) - \frac{1}{1+|k|^2}(\widehat{a}|\mathrm{i}k\cdot\widehat{b}),$$

it follows that

$$\partial_t \,\mathrm{Re}\mathcal{E}_1(\widehat{f}) + \frac{\lambda|k|^2}{1+|k|^2}|(\widehat{a},\widehat{b},\widehat{u})|^2 \le C(\|\{\mathbf{I}-\mathbf{P}\}\widehat{f}\|^2_{L^2_\xi} + |\widehat{u}-\widehat{b}|^2) + C(\|\widehat{G}\|^2 + \|\nu^{-1/2}\widehat{h}\|^2). \tag{3.7}$$

Now, given $t \ge 0$ and $k \in \mathbb{R}^3$, define the functional $\mathcal{E}_{\mathcal{F}}(\widehat{f},\widehat{u})$ by

$$\mathcal{E}_{\mathcal{F}}(\widehat{f},\widehat{u}) = (\|\widehat{f}\|^2_{L^2_\xi} + |\widehat{u}|^2) + \kappa_4 \mathrm{Re}\,\mathcal{E}_1(\widehat{f})$$

for a small constant $\kappa_4 > 0$ to be chosen later. Firstly, let $\kappa_4 > 0$ be small enough such that $\mathcal{E}_{\mathcal{F}}(\widehat{f},\widehat{u}) \sim \|\widehat{f}\|^2_{L^2_\xi} + |\widehat{u}|^2$ since

$$|\mathcal{E}_1(\widehat{f})| \le C(\|\widehat{f}\|^2_{L^2_\xi} + |\widehat{u}|^2)$$

holds for all $t \ge 0$ and $k \in \mathbb{R}^3$. By making $\kappa_4 > 0$ further small enough, the linear combination $2\times$(3.6) + $\kappa_4\times$(3.7) gives

$$\partial_t \,\mathcal{E}_{\mathcal{F}}(\widehat{f},\widehat{u}) + \lambda(|\{\mathbf{I}-\mathbf{P}\}\widehat{f}|^2_\nu + |\widehat{u}-\widehat{b}|^2) + \frac{\lambda|k|^2}{1+|k|^2}|(\widehat{a},\widehat{b},\widehat{u})|^2 \le C(\|\widehat{G}\|^2 + \|\nu^{-1/2}\widehat{h}\|^2),$$

which also implies

$$\partial_t \mathcal{E}_{\mathcal{F}}(\widehat{f},\widehat{u}) + \frac{\lambda|k|^2}{1+|k|^2}\mathcal{E}_{\mathcal{F}}(\widehat{f},\widehat{u}) \le C(\|\widehat{G}\|^2 + \|\nu^{-1/2}\widehat{h}\|^2).$$

It follows from Gronwall's inequality that

$$\mathcal{E}_{\mathcal{F}}(\widehat{f},\widehat{u}) \le e^{-\frac{\lambda|k|^2}{1+|k|^2}t}\mathcal{E}_{\mathcal{F}}(\widehat{f}_0,\widehat{u}_0) + \int_0^t e^{-\frac{\lambda|k|^2}{1+|k|^2}(t-s)}(\|\widehat{G}(s)\|^2 + \|\nu^{-1/2}\widehat{h}(s)\|^2)\,ds.$$

As in [18] or [11, Theorem 3.1], the further k-integration yields the desired time-decay estimates (3.3) and (3.4) by setting homogeneous source $S_f = 0$ and zero initial data $(f_0, u_0) = 0$, respectively. $\square$

We now need two technical lemmas for the later proof.

Lemma 3.1. *Given any* $0 < \beta_1 \neq 1$ *and* $\beta_2 > 1$,

$$\int_0^t (1+t-s)^{-\beta_1}(1+s)^{-\beta_2}\,ds \le C(1+t)^{-\min\{\beta_1,\beta_2\}}$$

for all $t \ge 0$.

Proof. Denote

$$J(t) = \int_0^t (1+t-s)^{-\beta_1}(1+s)^{-\beta_2}\,ds.$$

Let us divide the time integral into two parts $s \in (0, t/2)$ and $s \in (t/2, t)$, then it is easy to check that

$$\begin{aligned} J(t) &\le \left(1+\frac{t}{2}\right)^{-\beta_1}\int_0^{t/2}(1+s)^{-\beta_2}\,ds + \left(1+\frac{t}{2}\right)^{-\beta_2}\int_{t/2}^{t}(1+t-s)^{-\beta_1}\,ds \\ &= \left(1+\frac{t}{2}\right)^{-\beta_1}\int_0^{t/2}(1+s)^{-\beta_2}\,ds + \left(1+\frac{t}{2}\right)^{-\beta_2}\int_0^{t/2}(1+s)^{-\beta_1}\,ds\,. \end{aligned}$$

If $\beta_1 > 1$, we use that $(1+s)^{-\beta_i}$, $i = 1,2$, are integrable on $(0,\infty)$ to conclude. If $0 < \beta_1 < 1$, we again use that $(1+s)^{-\beta_2}$ is integrable in $(0,\infty)$ to infer

$$\int_0^t (1+t-s)^{-\beta_1}(1+s)^{-\beta_2}\,ds \le C\left[(1+t)^{-\beta_1} + (1+t)^{-\beta_1+1-\beta_2}\right]$$

yielding the desired estimate. $\square$

Lemma 3.2. *Let* $\gamma > 1$ *and* $g_1, g_2 \in \mathscr{C}^0(\mathbb{R}_+,\mathbb{R}_+)$ *with* $g_1(0) = 0$. *For* $A \in \mathbb{R}_+$, *define* $\mathscr{B}_A := \{y \in \mathscr{C}^0(\mathbb{R}_+,\mathbb{R}_+) \,|\, y \le A + g_1(A)y + g_2(A)y^\gamma,\ y(0) \le A\}$. *Then, there exists a constant* $A_0 \in (0, \min\{A_1, A_2\})$ *such that for any* $0 < A \le A_0$,

$$y \in \mathscr{B}_A \Longrightarrow \sup_{t\ge 0} y(t) \le 2A.$$

Proof. One can fix $A_0 \in (0, \min\{A_1, A_2\})$ such that

$$\sup_{0\le A\le A_0} \left[g_1(A) + g_2(A)(2A)^{\gamma-1}\right] \le \frac{1}{3}$$

due to $\gamma > 1$ and assumptions on functions $g_i(\cdot)$, $i = 1, 2$. Take $0 < A \le A_0$. Define

$$t_\star = \sup\left\{t \ge 0 \text{ such that } \sup_{0\le s\le t} y(t) \le 2A\right\}.$$

Notice $t_\star > 0$ since $y(0) \le A$ and $y(t)$ is continuous. We claim $t_\star = \infty$. Otherwise, $t_\star > 0$ is finite. Thus, from the definition of $t_\star$, $y(t_\star) = 2A$ and $y(t) \le 2A$ for any $0 \le t \le t_\star$. The latter implies that for $0 \le t \le t_\star$,

$$y(t) \le A + g_1(A)y(t) + g_2(A)[y(t)]^\gamma \le A + \sup_{0\le A\le A_0} \left[g_1(A) + g_2(A)(2A)^{\gamma-1}\right] y(t)$$

which by the choice of A_0, further gives

$$\sup_{0\le t\le t_\star} y(t) \le \frac{1}{1 - \sup\limits_{0\le A\le A_0} \left[g_1(A) + g_2(A)(2A)^{\gamma-1}\right]} A \le \frac{3}{2} A < 2A.$$

This is a contradiction to $y(t_\star) = 2A$. Therefore, $t_\star = \infty$ follows. □

Proof of the rate of convergence part in the main Theorem 1.1: By definition of $\mathcal{E}_0$ in (2.16), we have

$$\mathcal{E}_0(f(t)) \le C(\|\nabla_x b\|^2_{\mathrm{H}^2} + \|\{\mathbf{I} - \mathbf{P}\}f\|^2_{\mathrm{H}^2} + \|a\|^2_{\mathrm{H}^2}),$$

which implies the following by the definitions of $\mathcal{E}(f(t), u(t))$ and $\mathcal{D}(f(t), u(t))$ in (2.23) and (2.24)

$$\begin{aligned}\mathcal{E}(f(t), u(t)) &\le C(\|\{\mathbf{I} - \mathbf{P}\}f\|^2_{\mathrm{H}^3} + \|a\|^2_{\mathrm{H}^3} + \|b\|^2_{\mathrm{H}^3} + \|u\|^2_{\mathrm{H}^2}) \\ &\le C(\mathcal{D}(f(t), u(t)) + \|(f(t), u(t))\|^2_{\mathcal{L}^2}).\end{aligned}$$

From Corollary 2.2, we have the existence of λ such as

$$\frac{d}{dt}\mathcal{E}(f(t), u(t)) + \lambda \mathcal{D}(f(t), u(t)) \le 0,$$

so that we finally infer the existence of λ and C such as

$$\frac{d}{dt}\mathcal{E}(f(t), u(t)) + \lambda \mathcal{E}(f(t), u(t)) \le C\|(f(t), u(t))\|^2_{\mathcal{L}^2}.$$

Gronwall's inequality gives

$$\mathcal{E}(f(t), u(t)) \le e^{-\lambda t}\mathcal{E}(f_0, u_0) + C\int_0^t e^{-\lambda(t-s)}\|(f(s), u(s))\|^2_{\mathcal{L}^2} ds. \tag{3.8}$$

Next, we use the mild form of the system (1.5)-(1.7) to estimate the above zero-order energy. Indeed, the system (1.5)-(1.7) can written as

$$(f(t),u(t))=\mathbf{E}_t(f_0,u_0)+\int_0^t \mathbf{E}_{t-s}(S_f(s),S_u(s))ds,$$

where

$$S_f=-u\cdot\nabla_\xi f-\frac{1}{2}u\cdot\xi f,\quad S_u=-\mathbb{P}\left\{u\cdot\nabla_x u+au\right\}.$$

Here, $\mathbb{P}$ is the Leray projector given in (5.1) in the Appendix. One can decompose S_f as

$$\begin{aligned}S_f&=-u\cdot\nabla_\xi\{\mathbf{I}-\mathbf{P}_0\}f-\frac{1}{2}u\cdot\xi\{\mathbf{I}-\mathbf{P}_0\}f-u\cdot\nabla_\xi\mathbf{P}_0 f-\frac{1}{2}u\cdot\xi\mathbf{P}_0 f\\&=\nabla_\xi\cdot G-\frac{1}{2}\xi\cdot G+u\cdot a\xi M^{1/2},\end{aligned}$$

with $G=:-u\{\mathbf{I}-\mathbf{P}_0\}f$. So, $(f(t),u(t))$ can be rewritten as the sum of three terms

$$\begin{aligned}(f(t),u(t))&=\mathbf{E}_t(f_0,u_0)+\int_0^t \mathbf{E}_{t-s}(\nabla_\xi\cdot G(s)-\frac{1}{2}\xi\cdot G(s),0)\,ds\\&\quad+\int_0^t \mathbf{E}_{t-s}(u\cdot a\xi M^{1/2},0)\,ds+\int_0^t \mathbf{E}_{t-s}(0,-\mathbb{P}\left\{u\cdot\nabla_x u+au\right\})\,ds\\&=I_1(t)+I_2(t)+I_3(t)+I_4(t).\end{aligned}$$

By applying (3.3) to $I_1(t)$ and $I_3(t)$, one has

$$\|I_1(t)\|_{\mathcal{L}^2}\le C(1+t)^{-\frac{3}{4}}\|(f_0,u_0)\|_{\mathcal{Z}_1\cap\mathcal{L}^2},$$

and

$$\|I_3(t)\|_{\mathcal{L}^2}\le C\int_0^t(1+t-s)^{-\frac{3}{4}}\|u\cdot a\xi M^{1/2}\|_{\mathcal{Z}_1\cap\mathcal{L}^2}\,ds\le C\int_0^t(1+t-s)^{-\frac{3}{4}}\mathcal{E}(f(s),u(s))\,ds,$$

where Hölder and Sobolev inequalities were used. For $I_2(t)$, since $\mathbf{P}_0 G=0$, one can apply (3.4) to it to compute

$$\begin{aligned}\|I_2(t)\|_{\mathcal{L}^2}^2&\le C\int_0^t(1+t-s)^{-\frac{3}{2}}\|u\{\mathbf{I}-\mathbf{P}_0\}f\|_{\mathcal{Z}_1\cap\mathcal{L}^2}^2 ds\\&\le C\int_0^t(1+t-s)^{-\frac{3}{2}}[\mathcal{E}(f(s),u(s))]^2 ds,\end{aligned}$$

where again Hölder and Sobolev inequalities were used. Now, for any given $0<\varepsilon<\frac{1}{4}$ using (3.3) with $q=3/(3-2\varepsilon)$, we deduce

$$\|I_4(t)\|_{\mathcal{L}^2}\le C\int_0^t(1+t-s)^{-\frac{3}{4}+\varepsilon}\|\mathbb{P}\left\{u\cdot\nabla_x u+au\right\}\|_{\mathrm{L}^q\cap\mathrm{L}^2}\,ds\,.$$

Using the Calderon-Zygmund Theorem for Riesz transforms [26], the Leray projection operator in (5.1) is continuous on L^q for all $1 < q < \infty$, and thus there exists C_ε such that

$$\|\mathbb{P}\{u\cdot\nabla_x u + au\}\|_{L^q\cap L^2} \le C_\varepsilon\|u\cdot\nabla_x u + au\|_{L^q\cap L^2} \le C_\varepsilon\|u\cdot\nabla_x u + au\|_{L^1\cap L^2}$$

where interpolation inequality was used. We remark finally that

$$\|u\cdot\nabla_x u + au\|_{L^1\cap L^2} \le C\mathcal{E}(f,u),$$

and thus

$$\|I_4(t)\|_{\mathcal{L}^2} \le C_\varepsilon\int_0^t (1+t-s)^{-\frac{3}{4}+\varepsilon}\mathcal{E}(f(s),u(s))\,ds\,.$$

Therefore, it holds

$$\begin{aligned}\|(f(t),u(t))\|_{\mathcal{L}^2}^2 &\le 2\sum_{i=1}^4\|I_i(t)\|_{\mathcal{L}^2}^2\\ &\le C(1+t)^{-\frac{3}{2}}\|(f_0,u_0)\|_{\mathcal{Z}_1\cap\mathcal{L}^2}^2 + C\int_0^t (1+t-s)^{-\frac{3}{2}}[\mathcal{E}(f(s),u(s))]^2 ds\\ &\quad + C\left[\int_0^t (1+t-s)^{-\frac{3}{4}}\mathcal{E}(f(s),u(s))ds\right]^2\\ &\quad + C_\varepsilon\left[\int_0^t (1+t-s)^{-\frac{3}{4}+\varepsilon}\mathcal{E}(f(s),u(s))\,ds\right]^2.\end{aligned} \tag{3.9}$$

Define

$$\mathcal{E}_\infty(t) = \sup_{0\le s\le t}(1+s)^{\frac{3}{2}-2\varepsilon}\mathcal{E}(f(s),u(s)). \tag{3.10}$$

Fix a constant $\delta < 1/3$ close enough to $1/3$. Using (3.10) and that $\mathcal{E}(f(t),u(t))$ and $\mathcal{E}_\infty(t)$ are non-increasing in time, we get

$$\begin{aligned}&\int_0^t (1+t-s)^{-\frac{3}{4}+\varepsilon}\mathcal{E}(f(s),u(s))\,ds\\ &\qquad = \int_0^t (1+t-s)^{-\frac{3}{4}+\varepsilon}[\mathcal{E}(f(s),u(s))]^{\frac{2}{3}+\delta}[\mathcal{E}(f(s),u(s))]^{\frac{1}{3}-\delta}\,ds\\ &\qquad \le [\mathcal{E}_\infty(t)]^{(\frac{2}{3}+\delta)}[\mathcal{E}(f_0,u_0)]^{(\frac{1}{3}-\delta)}\int_0^t (1+t-s)^{-\frac{3}{4}+\varepsilon}(1+s)^{-(\frac{3}{2}-2\varepsilon)(\frac{2}{3}+\delta)}\,ds.\end{aligned}$$

Since $\delta < 1/3$ is close to $1/3$, we deduce from Lemma 3.1 that

$$\int_0^t (1+t-s)^{-\frac{3}{4}+\varepsilon}\mathcal{E}(f(s),u(s))\,ds \le C(1+t)^{-\frac{3}{4}+\varepsilon}[\mathcal{E}_\infty(t)]^{(\frac{2}{3}+\delta)}[\mathcal{E}(f_0,u_0)]^{(\frac{1}{3}-\delta)}.$$

Let us also remark that the third term in the r.h.s. of (3.9) is also trivially estimated by the above quantity. Similarly, using Lemma 3.1 it holds that

$$\begin{aligned}\int_0^t (1+t-s)^{-\frac{3}{2}}[\mathcal{E}(f(s),u(s))]^2 ds &\le \mathcal{E}_\infty(t)\mathcal{E}(f_0,u_0)\int_0^t (1+t-s)^{-\frac{3}{2}}(1+s)^{-\frac{3}{2}+2\varepsilon}ds\\ &\le C\mathcal{E}_\infty(t)\mathcal{E}(f_0,u_0)(1+t)^{-\frac{3}{2}+2\varepsilon}.\end{aligned}$$

Thus, one has from (3.9) that

$$\|(f(t),u(t))\|_{\mathcal{L}^2}^2 \leq C_\varepsilon(1+t)^{-\frac{3}{2}+2\varepsilon}\Big\{\|(f_0,u_0)\|_{\mathcal{Z}_1\cap\mathcal{L}^2}^2 + \mathcal{E}(f_0,u_0)\mathcal{E}_\infty(t) + [\mathcal{E}(f_0,u_0)]^{2(\frac{1}{3}-\delta)}[\mathcal{E}_\infty(t)]^{2(\frac{2}{3}+\delta)}\Big\}.$$

Plugging this into the r.h.s. of (3.8), multiplying the resulting inequality by $(1+t)^{\frac{3}{2}-2\varepsilon}$ and using again Lemma 3.1 replacing $e^{-\lambda(t-s)}$ by $(1+t-s)^{-\beta_1}$ with $\beta_1 > \frac{3}{2}-2\varepsilon$ and the fact that $\mathcal{E}_\infty(t)$ is non-increasing in time, we conclude

$$(1+t)^{\frac{3}{2}-2\varepsilon}\mathcal{E}(f(t),u(t)) \leq C_\varepsilon\Big\{\|(f_0,u_0)\|_{\mathcal{Z}_1\cap\mathcal{H}^3}^2 + \mathcal{E}(f_0,u_0)\mathcal{E}_\infty(t) + [\mathcal{E}(f_0,u_0)]^{2(\frac{1}{3}-\delta)}[\mathcal{E}_\infty(t)]^{2(\frac{2}{3}+\delta)}\Big\}$$

for any $t \geq 0$, which implies that

$$\mathcal{E}_\infty(t) \leq C_\varepsilon\{\|(f_0,u_0)\|_{\mathcal{Z}_1\cap\mathcal{H}^3}^2 + \mathcal{E}(f_0,u_0)\mathcal{E}_\infty(t) + [\mathcal{E}(f_0,u_0)]^{2(\frac{1}{3}-\delta)}[\mathcal{E}_\infty(t)]^{2(\frac{2}{3}+\delta)}\}.$$

Since $\|(f_0,u_0)\|_{\mathcal{Z}_1\cap\mathcal{H}^3}^2$ and $\mathcal{E}(f_0,u_0) \sim \|(f_0,u_0)\|_{\mathcal{H}^3}^2$ are small enough and $1 < 2(\frac{2}{3}+\delta) < 2$, then

$$y(t) \leq A\,[1+y(t)] + C_\varepsilon^{1-2(\frac{1}{3}-\delta)}A^{2(\frac{1}{3}-\delta)}y(t)^2$$

for all $t \geq 0$, with $y(t) = \mathcal{E}_\infty(t)$ and $A = C_\varepsilon\|(f_0,u_0)\|_{\mathcal{Z}_1\cap\mathcal{H}^3}^2$. A direct application of Lemma 3.2 implies

$$\mathcal{E}_\infty(t) \leq 2A = 2C_\varepsilon\|(f_0,u_0)\|_{\mathcal{Z}_1\cap\mathcal{H}^3}^2$$

holds uniformly in time. Recalling the definition (3.10) of $\mathcal{E}_\infty(t)$, (1.10) follows and the proof of Theorem 1.1 is completed. □

Remark 3.1. *[Adding Viscosity] The same theorem applies directly by adding viscosity to the system (1.1)-(1.3). More precisely, if we consider the Navier-Stokes-Vlasov-Fokker-Planck system as in [13]:*

$$\begin{aligned}
&\partial_t F + \xi\cdot\nabla_x F = \nabla_\xi\cdot((\xi-u)F + \nabla_\xi F),\\
&\nabla_x\cdot u = 0,\\
&\partial_t u + u\cdot\nabla_x u + \nabla_x p = \mu\Delta_x u + \int_{\mathbb{R}^3}(\xi-u)F\,d\xi,
\end{aligned}$$

where the constant $\mu > 0$ is the viscosity of the fluid, then all estimates in Sections 2 and 3 can be made independently on $\mu > 0$. Therefore, Theorem 1.1 holds for the Navier-Stokes-Vlasov-Fokker-Planck system for any $\mu > 0$. Moreover, the constructed solutions in Theorem 1.1 are their weak limits as $\mu \to 0$.

4 The periodic case

In this section, we are concerned with the case when the spatial domain is periodic. Precisely, consider the Cauchy problem over the Torus $\mathbb{T}^3$:

$$\begin{cases} \partial_t F + \xi \cdot \nabla_x F = \nabla_\xi \cdot ((\xi - u)F + \nabla_\xi F), \\ \nabla_x \cdot u = 0, \\ \partial_t u + u \cdot \nabla_x u + \nabla_x p = \int_{\mathbb{R}^3} (\xi - u)F \, d\xi, \quad t > 0, x \in \mathbb{T}^3, \xi \in \mathbb{R}^3 \end{cases}$$

with $F(0,x,\xi) = F_0(x,\xi)$, $u(0,x) = u_0(x)$, $x \in \mathbb{T}^3, \xi \in \mathbb{R}^3$. Here, $\nabla_x \cdot u_0 = 0$. Similarly as before, set $F(t,x,\xi) = M + M^{1/2} f(t,x,\xi)$ to obtain

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f + u \cdot \nabla_\xi f - \frac{1}{2} u \cdot \xi f - u \cdot \xi M^{1/2} = \mathbf{L} f, \\ \nabla_x \cdot u = 0, \\ \partial_t u + u \cdot \nabla_x u + \nabla_x p + u(1+a) = b, \end{cases} \tag{4.1}$$

with initial data

$$f(0,x,\xi) = f_0(x,\xi) \equiv M^{-1/2}(F_0 - M), \quad u(0,x) = u_0(x). \tag{4.2}$$

Here, the moment functions a, b are defined in (1.9).

Theorem 4.1. *Let $\nabla_x \cdot u_0 = 0$ and $F_0 \equiv M + M^{1/2} f_0 \geq 0$. Assume that $\|f_0\|_{\mathrm{L}^2_\xi(\mathbb{R}^3; \mathrm{H}^3_x(\mathbb{T}^3))} + \|u_0\|_{\mathrm{H}^3(\mathbb{T}^3)}$ is small enough and*

$$\int_{\mathbb{T}^3} a_0 dx = 0, \quad \int_{\mathbb{T}^3} (u_0 + b_0) \, dx = 0.$$

Then, the Cauchy problem (4.1)-(4.2) admits a unique global solution $(f(t,x,\xi), u(t,x))$ satisfying

$$f \in \mathscr{C}^0([0,\infty); \mathrm{L}^2_\xi(\mathbb{R}^3; \mathrm{H}^3_x(\mathbb{T}^3))), \quad u \in \mathscr{C}^0([0,\infty); \mathrm{H}^3(\mathbb{T}^3)),$$
$$F = M + M^{1/2} f \geq 0,$$
$$\|f(t)\|_{\mathrm{L}^2_\xi(\mathbb{R}^3; \mathrm{H}^3_x(\mathbb{T}^3))} + \|u(t)\|_{\mathrm{H}^3(\mathbb{T}^3)} \leq C e^{-\lambda t} (\|f_0\|_{\mathrm{L}^2_\xi(\mathbb{R}^3; \mathrm{H}^3_x(\mathbb{T}^3))} + \|u_0\|_{\mathrm{H}^3(\mathbb{T}^3)})$$

for any $t \geq 0$, where $C > 0$ and $\lambda > 0$ are some constants.

Proof. Let us only sketch the proof of uniform-in-time *a priori* estimates as follows. First of all, the estimate (2.12) in Corollary 2.1 can be modified for the periodic case as

$$\frac{1}{2} \frac{d}{dt} \sum_{|\alpha| \leq 3} (\|\partial^\alpha f\|^2 + \|\partial^\alpha u\|^2) + \lambda_0 \sum_{|\alpha| \leq 3} (\|\{\mathbf{I} - \mathbf{P}\} \partial^\alpha f\|_\nu^2 + \|\partial^\alpha (u - b)\|^2)$$
$$\leq C \|u\|_{\mathrm{H}^3} \left[\sum_{|\alpha| \leq 3} (\|\{\mathbf{I} - \mathbf{P}\} \partial^\alpha f\|_\nu^2 + \|\partial^\alpha (u - b)\|^2) + \|(a,b)\|_{\mathrm{H}^3}^2 \right]. \tag{4.3}$$

This is obtained by changing the estimates whenever pure Sobolev inequalities were used by adding the zero-order terms in the energy. For instance, in the end of proof of Lemma 2.2, we rewrite that bound by

$$\begin{aligned}\int_{\mathbb{T}^3} \frac{1}{2} u \cdot \langle \xi f, f\rangle \, dx - \int_{\mathbb{T}^3} a|u|^2 \, dx \\ &\leq \|a\|_{\mathrm{L}^6}\|u\|_{\mathrm{L}^3}\|u-b\|_{\mathrm{L}^2} + C\|u\|_{\mathrm{L}^3}\|(a,b)\|_{\mathrm{L}^6}\|\{\mathbf{I}-\mathbf{P}\}f\| \\ &\quad + \|u\|_{\mathrm{L}^\infty}\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2 \\ &\leq C\|u\|_{\mathrm{H}^2}\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2 + C\|u\|_{\mathrm{H}^1}\|u-b\|_{\mathrm{L}^2}^2 + C\|u\|_{\mathrm{H}^1}\|(a,b)\|_{\mathrm{H}^1}^2.\end{aligned}$$

Analogously, it is done for the proof of Lemma 2.3. In a similar way, the estimate (2.17) in Lemma 2.4 can be modified for the torus case as

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_0(f(t)) + \lambda\|\nabla_x(a,b)\|_{\mathrm{H}^2}^2 \leq & C(\|\{\mathbf{I}-\mathbf{P}\}f\|_{\mathrm{L}^2_\xi(\mathrm{H}^3_x)}^2 + \|u-b\|_{\mathrm{H}^3}^2) \\ & + C\|u\|_{\mathrm{H}^3}^2\Big[\|(a,b)\|_{\mathrm{H}^3}^2 + \|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|_{\mathrm{L}^2_\xi(\mathrm{H}^3_x)}^2\Big],\end{aligned} \tag{4.4}$$

where $\mathcal{E}_0(f(t))$ is the same as in (2.16), with $\mathbb{R}^3$ replaced by $\mathbb{T}^3$. Now, we also define $\mathcal{E}(f(t),u(t))$ in the same way as in (2.23). Therefore, for properly chosen $\kappa_1 > 0$ in (2.23), from (4.3) and (4.4), one has $\mathcal{E}(f(t),u(t)) \sim \|(f,u)\|_{\mathcal{H}^3}^2$ and

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}(f(t),u(t)) + \lambda\mathcal{D}(f(t),u(t)) \\ &\quad \leq C(\|u\|_{\mathrm{H}^3} + \|u\|_{\mathrm{H}^3}^2)\left[\sum_{|\alpha|\leq 3}\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_\nu^2 + \|u-b\|_{\mathrm{H}^3}^2 + \|(a,b)\|_{\mathrm{H}^3}^2\right],\end{aligned} \tag{4.5}$$

where $\mathcal{D}(f(t),u(t))$ is the same as in (2.24). Using the conservation laws in the case of Torus

$$\frac{d}{dt}\int_{\mathbb{T}^3} a dx = 0, \quad \frac{d}{dt}\int_{\mathbb{T}^3}(u+b)\, dx = 0$$

due to the system (4.1),

$$\int_{\mathbb{T}^3} a dx = 0, \quad \int_{\mathbb{T}^3}(u+b)\, dx = 0$$

hold for any $t > 0$ since they hold initially by the assumptions in Theorem 4.1. Thus, from the Poincaré inequality, one has

$$\|a\|_{\mathrm{L}^2} \leq C\|\nabla_x a\|_{\mathrm{L}^2}, \quad \|u+b\|_{\mathrm{L}^2} \leq C\|\nabla_x(u+b)\|_{\mathrm{L}^2}.$$

It further holds that

$$\begin{aligned}\|u\|_{\mathrm{L}^2} + \|b\|_{\mathrm{L}^2} &\leq C(\|u+b\|_{\mathrm{L}^2} + \|u-b\|_{\mathrm{L}^2}) \\ &\leq C(\|\nabla_x(u+b)\|_{\mathrm{L}^2} + \|u-b\|_{\mathrm{L}^2}) \\ &\leq C(\|\nabla_x(b,u)\|_{\mathrm{L}^2} + \|u-b\|_{\mathrm{L}^2}).\end{aligned}$$

Applying the above estimates to the inequality (4.5), one has

$$\frac{d}{dt}\mathcal{E}(f(t),u(t)) + \lambda\mathcal{D}_{\mathbb{T}}(f(t),u(t)) \le C(\|u\|_{\mathrm{H}^3} + \|u\|_{\mathrm{H}^3}^2)\mathcal{D}_{\mathbb{T}}(f(t),u(t)) \tag{4.6}$$

with the definition

$$\mathcal{D}_{\mathbb{T}}(f(t),u(t)) = \mathcal{D}(f(t),u(t)) + \|(a,b,u)\|_{\mathrm{L}^2}^2.$$

Therefore, using the similar proof as in the case of $\mathbb{R}^3$, the global existence of solutions to the Cauchy problem (4.1)-(4.2) stated as in Theorem 4.1 follows. Finally, since $\mathcal{E}(f(t),u(t))$ is small enough uniformly in time, (4.6) gives

$$\frac{d}{dt}\mathcal{E}(f(t),u(t)) + \lambda\mathcal{D}_{\mathbb{T}}(f(t),u(t)) \le 0.$$

Noticing $\mathcal{E}(f(t),u(t)) \le C\mathcal{D}_{\mathbb{T}}(f(t),u(t))$ in the torus case, it follows

$$\frac{d}{dt}\mathcal{E}(f(t),u(t)) + \lambda\mathcal{E}(f(t),u(t)) \le 0.$$

This implies the exponential decay of $\mathcal{E}(f(t),u(t)) \sim \|(f(t),u(t))\|_{\mathcal{H}^3}^2$ in time. $\square$

5 Appendix

In the sequel $\langle\cdot,\cdot\rangle$ will denote the distribution bracket of $\mathscr{D}'(\mathbb{R}^3)^3$ over $\mathscr{D}(\mathbb{R}^3)^3$. Let us introduce the sets

$$\begin{aligned}\mathrm{V} &:= \left\{\varphi \in \mathscr{D}(]0,T[\times\mathbb{R}^3)^3 \,|\, \mathrm{div}_x(\varphi) = 0\right\},\\ \Lambda &:= \left\{h \in \mathscr{D}'(]0,T[\times\mathbb{R}^3)^3 \,|\, \varphi \in \mathrm{V} \Rightarrow \langle h,\varphi\rangle = 0\right\}.\end{aligned}$$

Here and in the sequel, for clearness, we add a superscript in vector fields to show the dimension of their ranges. Consider $\mathbb{X} = \mathrm{W}^{1,\infty}\big([0,T];\mathrm{H}^2(\mathbb{R}^3)^3\big) \cap \mathrm{L}^\infty\big([0,T];\mathrm{H}^3(\mathbb{R}^3)^3\big)$ and $\mathbb{Y} =:= \mathrm{W}^{1,\infty}\big([0,T];\mathrm{H}^2(\mathbb{R}^3)\big) \cap \mathrm{L}^\infty\big([0,T];\mathrm{H}^3(\mathbb{R}^3)\big)$. Then,

$$\begin{aligned}\Pi : \mathbb{X}^3 \times \mathbb{Y} &\longrightarrow \mathrm{L}^\infty\big([0,T];\mathrm{H}^2(\mathbb{R}^3)\big)\\ (v,w,s,r) &\longmapsto \partial_t v + w\cdot\nabla_x v + v(r+1) - s,\end{aligned}$$

is a well-defined map since:

1. by Rademacher's theorem v is a.e. differentiable, $\partial_t v \in \mathrm{L}^\infty\big([0,T];\mathrm{H}^2(\mathbb{R}^3)^3\big)$ and we have the corresponding integration formula.
2. $\mathrm{H}^2(\mathbb{R}^3)$ is stable by product, hence all the other terms belong to $\mathrm{L}^\infty\big([0,T];\mathrm{H}^2(\mathbb{R}^3)^3\big)$.

In the sequel $\mathbb{P}$ is the Leray projector on the closed space of $\mathrm{L}^2(\mathbb{R}^3)^3$ formed by the divergence-free functions. Recall its definition *via* Fourier transform

$$\mathbb{P}(v) := \mathcal{F}^{-1}\left(\hat{v} - \frac{k\cdot\hat{v}}{|k|^2}k\right). \tag{5.1}$$

From now on we will only consider elements (v, w, s, r) of $\mathbb{X}^3 \times \mathbb{Y}$, with v and w divergence-free vector flows. In such conditions we will say that v is a *weak* solution of the system

$$\partial_t v + \mathbb{P}\{w \cdot \nabla_x v + v(r+1) - s\} = 0 \tag{5.2}$$

$$\operatorname{div}_x(v) = 0, \tag{5.3}$$

if and only if $\Pi(v, w, s, r) \in \Lambda$, which is equivalent to satisfy (5.2)–(5.3) a.e. as elements of $\mathrm{L}^2(\mathbb{R}^3)^3$. We will say that v is a *strong* solution of (5.2)–(5.3) when v is $\mathscr{C}^1([0, T_0] \times \mathbb{R}^3)^3$ and verifies the equalities for all $(t, x) \in]0, T[\times \mathbb{R}^3$. We may now state the first result

Lemma 5.1. *1. If $v \in \mathscr{C}^0\big([0, T]; \mathrm{H}^3(\mathbb{R}^3)^3\big)$ is a weak solution of (5.2)–(5.3) then it is a strong solution.*

2. For every strong solution there exists a real valued function $q \in \mathscr{C}^0\big([0, T]; \mathrm{H}^3(\mathbb{R}^3_{\mathrm{loc}})^3\big)$ such that the following equation holds pointwisely in $]0, T[\times \mathbb{R}^3$:

$$\partial_t v + w \cdot \nabla_x v + v(r+1) + \nabla_x q = s,$$

Proof.

1. $\mathbb{P}$ clearly maps continuously $\mathrm{H}^2(\mathbb{R}^3)^3$ into itself. Since $\mathrm{H}^3(\mathbb{R}^3)$ is a topological algebra, the assumptions made on v, w, s and r insure us that $w \cdot \nabla_x v + rv - s$ lies in $\mathscr{C}^0\big([0, T]; \mathrm{H}^2(\mathbb{R}^3)^3\big)$. Thus, $\partial_t v$ is continuous in time with value in $\mathrm{H}^2(\mathbb{R}^3)^3 \hookrightarrow \mathscr{C}^0(\mathbb{R}^3)^3$. Furthermore, v is continuous in time with values in $\mathrm{H}^3(\mathbb{R}^3) \hookrightarrow \mathscr{C}^1(\mathbb{R}^3)^3$. Eventually $v \in \mathscr{C}^1([0, T] \times \mathbb{R}^3)^3$ and since it verifies the weak form of the equation, it verifies it pointwisely.

2. In this case $\Pi(v, w, s, r) \in \mathscr{C}^0\big([0, T]; \mathrm{H}^2(\mathbb{R}^3)^3\big) \cap \Lambda$, so we may directly use Lemma 5.4 below to get the existence of $q \in \mathscr{C}^0\big(]0, T[; \mathrm{H}^3(\mathbb{R}^3_{\mathrm{loc}})^3\big)$ such as $\Pi(v, w, s, r) = \nabla_x q$, first as distributions and then as functions since both are continuous. So we have for $(t, x) \in]0, T[\times \mathbb{R}^3$ that $\partial_t v + w \cdot \nabla_x v + v(r+1) + \nabla_x q = s$.

□

Weak and strong solutions finally becomes one by the following lemma.

Lemma 5.2. *Any weak solution v of (5.2)–(5.3) is in $\mathscr{C}^0\big([0, T]; \mathrm{H}^3(\mathbb{R}^3)^3\big)$.*

Proof. We refer to [27], p.364, Proposition 1.4 for a complete proof (in a more general framework), we will only give the main steps here. By interpolation $v \in \mathscr{C}^0_{\mathrm{w}}\big(I; \mathrm{H}^3(\mathbb{R}^3)^3\big)$ (weak topology). So one just need to prove that $\|v(t)\|_{\mathrm{H}^3(\mathbb{R}^3)^3}$ is continuous. For that apply a mollifier ρ_n on the equation and use the usual estimates (using the divergence-free property of w) to obtain

$$\Big| \|\rho_n \star v(t_2)\|^2_{\mathrm{H}^3(\mathbb{R}^3)^3} - \|\rho_n \star v(t_1)\|^2_{\mathrm{H}^3(\mathbb{R}^3)^3} \Big| \leq C \int_{t_1}^{t_2} \|v(\tau)\|^2_{\mathrm{H}^3(\mathbb{R}^3)^3} d\tau + C(t_2 - t_1),$$

for some constant C independent of n. $\|v(t)\|^2_{\mathrm{H}^3}$ is hence Lipschitz continuous and the lemma is proved. □

Lemma 5.3. *Given $v_0 \in \mathrm{H}^3_{\mathrm{div}}(\mathbb{R}^3)^3$, there exists a strong global solution to (5.2)–(5.3) verifying $v(0) = v_0$.*

Proof. By the two previous lemmas we just need to build a weak solution. Consider the Banach space $\mathrm{H}^3_{\mathrm{div}}(\mathbb{R}^3)^3$, closure of V in $\mathrm{H}^3(\mathbb{R}^3)^3$. The Leray projector is again a continuous endomorphism of $\mathrm{H}^3_{\mathrm{div}}(\mathbb{R}^3)^3$. Consider a sequence $(\rho_n)_n$ of mollifiers (in space). Then, the Cauchy problem

$$\partial_t v_n + \mathbb{P}\Big\{\rho_n \star \big[w \cdot \nabla_x[\rho_n \star v_n]\big] + v_n(r+1) - s\Big\} = 0, \tag{5.4}$$

$$v_n(0) = v_0, \tag{5.5}$$

has clearly a unique, global, strong solution on $[0,T]$. Indeed, since we have

$$\begin{aligned}\|\rho_n \star \big[w \cdot \nabla_x[\rho_n \star v_n]\big]\|_{\mathrm{H}^3(\mathbb{R}^3)^3} &\leq \|\rho_n\|_{\mathrm{L}^1(\mathbb{R}^3)^3}\|w\|_{\mathrm{H}^3(\mathbb{R}^3)^3}\|\nabla_x\rho_n \star v_n\|_{\mathrm{H}^3(\mathbb{R}^3)^3} \\ &\leq \|\rho_n\|_{\mathrm{L}^1(\mathbb{R}^3)^3}\|w\|_{\mathrm{H}^3(\mathbb{R}^3)^3}\|\nabla_x\rho_n\|_{\mathrm{L}^1(\mathbb{R}^3)^3}\|v_n\|_{\mathrm{H}^3(\mathbb{R}^3)^3},\end{aligned}$$

we see that the problem is just solving an affine ordinary differential equation

$$\dot{v}_n(t) = A_n(t)v_n(t) + \mathbb{P}(s),$$

for some continuous linear map of continuous operators $A_n(t)$ which is known to have global solutions. The rest of the proof is close to what is done in [27] (p.360-363, theorem 1.2) but a bit simpler (linear), let us sketch briefly what will happen :

- *The sequence $(v^n)_n$ is bounded in $\mathrm{L}^\infty\big([0,T_0];\mathrm{H}^3_{\mathrm{div}}(\mathbb{R}^3)^3\big)$.*
  Since the weak derivatives ∂^α commutes with the strong one ∂_t and with the operator $\mathbb{P}$ which is continuous from $\mathrm{H}^3_{\mathrm{div}}(\mathbb{R}^3)^3$ to itself and self-adjoint, usual estimates (using the divergence-free property of w) infer

  $$\|v^n(t)\|^2_{\mathrm{H}^3_{\mathrm{div}}(\mathbb{R}^3)^3} \leq \|v_0\|^2_{\mathrm{H}^3_{\mathrm{div}}(\mathbb{R}^3)^3} + A_0\int_0^t \|v^n(\tau)\|^2_{\mathrm{H}^3_{\mathrm{div}}(\mathbb{R}^3)^3}d\tau,$$

  for some constant A_0 independent of n. Grönwall's lemma then insures that $(v_n)_n$ is bounded in the desired space and the associated weak$-\star$ convergence (of a subsequence) follows from Banach-Alaoglu's theorem.

- *The sequence $(v^n)_n$ is bounded in $\mathrm{W}^{1,\infty}\big([0,T_0];\mathrm{H}^2(\mathbb{R}^3)^3\big)$.*
  The previous item and the equation itself give us an estimate in $\mathscr{C}^1\big([0,T_0];\mathrm{H}^2_{\mathrm{div}}(\mathbb{R}^3)^3\big)$, which as previously gives weak$-\star$ compactness in $\mathrm{W}^{1,\infty}\big([0,T_0];\mathrm{H}^2(\mathbb{R}^3)^3\big)$.

- *The limit is solution.*
  The strong equations (5.4)–(5.5) implies clearly for all $n \in \mathbb{N}^*$, $\Pi(v_n, w, r, s) \in \Lambda$, which is kept at the limit $n \to \infty$.

□

The following lemma proves the classical existence of the pressure in our simple case, see [24] for related results.

Lemma 5.4. *Let $h \in \mathscr{C}^0\big([0,T];\mathrm{H}^2(\mathbb{R}^3)^3\big) \cap \Lambda$. Then there exists $q \in \mathscr{C}^0\big(]0,T[;\mathrm{H}^3(\mathbb{R}^3_{\mathrm{loc}})\big)$ such as $h = \nabla_x q$.*

Proof. Consider first the case where $g \in \mathrm{L}^2(\mathbb{R}^3)^3$ verifies $\langle g, \varphi\rangle = 0$ for all divergence-free function φ, and let us prove $g = \nabla_x q$ for some $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ scalar function. A straightforward density argument implies $\mathbb{P}(g) = 0$ and hence by the definition of $\mathbb{P}$ (5.1)

$$\hat{g} = \frac{k\cdot\hat{g}}{|k|^2}k,$$

so one just need to prove that $\dfrac{k\cdot\hat{g}}{|k|^2} \in \mathcal{F}(\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3))$ (in the tempered sense). But we have

$$\frac{\hat{g}\cdot k}{|k|^2} = \frac{\hat{g}\cdot k}{|k|^2}\mathbb{1}_{|k|\le 1} + \frac{\hat{g}\cdot k}{|k|^2}\mathbb{1}_{|k|>1} \in \mathrm{L}^{11/10}(\mathbb{R}^3) + \mathrm{L}^2(\mathbb{R}^3),$$

where we used the Hölder inequality and the integrability of $|k| \mapsto |k|^{-22/9}\mathbb{1}_{|k|\le 1}$ in $\mathbb{R}^3$. Now $\mathrm{L}^2(\mathbb{R}^3) = \mathcal{F}(\mathrm{L}^2(\mathbb{R}^3))$ by isometry and $\mathrm{L}^{11/10}(\mathbb{R}^3) \subset \mathcal{F}(\mathrm{L}^{11}(\mathbb{R}^3))$ by Riesz-Thorin's interpolation. So we indeed get the existence of $q \in \mathrm{L}^{11}(\mathbb{R}^3) \subset \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ such as $g = \nabla_x q$ and since the previous inclusions are continuous we have the estimate, for some constant C

$$\|q\|_{\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)} \le C\|g\|_{\mathrm{L}^2(\mathbb{R}^3)^3}.$$

Now take h as in the lemma and $\varphi \in \mathscr{D}(\mathbb{R}^3)^3$, with $\operatorname{div}(\varphi) = 0$. φ is not really an admissible test function for $h \in \Lambda$ since it does not depend on time. But,

$$\Psi(t) := \int_{\mathbb{R}^3} h(t,x)\cdot\varphi(x)\,dx$$

is a continuous function of $t \in [0,T]$, since $h \in \mathscr{C}^0\big([0,T];\mathrm{L}^2(\mathbb{R}^3)^3\big)$. And since $h \in \Lambda$, we get by Fubini's theorem that $\Psi = 0$ in $\mathscr{D}'(]0,T[)$, hence $\Psi(t) = 0$ for all $t \in [0,T]$.

We can hence apply the previous study with $g := h(t)$ to obtain the existence of $q(t) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$, for all $t \in [0,T]$, such as $h(t) = \nabla_x q(t)$. Furthermore q depends linearly on h, so for $s,t \in [0,T]$ we have

$$\|q(t) - q(s)\|_{\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)} \le C\|h(t) - h(s)\|_{\mathrm{L}^2(\mathbb{R}^3)^3},$$

and since $h \in \mathscr{C}^0\big([0,T];\mathrm{H}^2(\mathbb{R}^3)^3\big)$ we eventually have $q \in \mathscr{C}^0\big([0,T];\mathrm{H}^3(\mathbb{R}^3_{\mathrm{loc}})\big)$. □

Lemma 5.5. *Consider the Fokker-Planck equation*

$$\partial_t F + \xi\cdot\nabla_x F - \nabla_\xi\cdot(\xi F) - \Delta_\xi F = A, \tag{5.6}$$

$$F(0,x,\xi) = F_0(x,\xi), \tag{5.7}$$

where $A \in \mathrm{L}^\infty\big([0,T];\mathscr{C}^\infty(\mathbb{R}^3\times\mathbb{R}^3)\big) \cap \mathscr{C}^0_{t,x,\xi}$ and $F_0 - M \in \mathscr{C}^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ are both compactly supported in space and velocity. Then the Cauchy problem (5.6)–(5.7) has a unique classical solution belonging to $\mathscr{C}^1\big([0,T];\mathscr{C}^0(\mathbb{R}^3\times\mathbb{R}^3)\big)$. For each $t_0 > 0$, $M^{-1/2}(F-M) \in \mathscr{C}^0\big([t_0,T];\mathscr{S}(\mathbb{R}^3\times\mathbb{R}^3)\big)$, and furthermore $M^{-1/2}(F-M)$ belongs to $\mathrm{L}^\infty\big([0,T];\mathrm{L}^2_\xi(\mathrm{H}^3_x)\big)$.

Proof. By following directly the computations made in [25] (same notations), we first describe the characteristics

$$\Phi_t(x,\xi) := [X_t(x,\xi), \dot{X}_t(x,\xi)] = [x + \xi(1-e^{-t}), \xi e^{-t}].$$

We have then (see [25]) an explicit formula describing the solution F of (5.6)–(5.7)

$$F(t,\Phi_t(x,\xi)) = e^{3t}H(t)\star F_0(x,\xi) + \int_0^t e^{3(t-s)}[H(t-s)\star A(s)]\big(\Phi_s(x,\xi)\big)ds,$$

where the convolutions are acting on both variables x and ξ. Of course since M solves the homogenous equation, we have also :

$$\begin{aligned}(F-M)(t,\Phi_t(x,\xi)) &= e^{3t}H(t)\star(F_0-M)(x,\xi)\\ &\quad+\int_0^t e^{3(t-s)}[H(t-s)\star A(s)]\big(\Phi_s(x,\xi)\big)ds,\end{aligned} \tag{5.8}$$

The function H is the fundamental solution of some partial differential equations with constant (in space and velocity) coefficients and in our case it takes the form (directly taken from [25] again)

$$H(t,x,\xi) = \frac{\exp\big(-\frac{\nu(t)|x|^2+\lambda(t)|\xi|^2+\mu(t)(x\cdot\xi)}{4\lambda(t)\nu(t)-\mu^2(t)}\big)}{(2\pi)^3(4\lambda(t)\nu(t)-\mu^2(t))^{3/2}},$$

where

$$\lambda(t) = t + 2(1-e^t) + \frac{1}{2}(e^{2t}-1), \quad \nu(t) = \frac{1}{2}(e^{2t}-1), \quad \mu(t) = (1-e^t)^2.$$

A straightforward computation gives the ellipticity condition

$$4\nu(t)\lambda(t) - \mu^2(t) = (e^t-1)[(e^t+1)2t + 4(1-e^t)] \geq 0,$$

so that (5.8) is clearly defined. For $t_0 > 0$, F belongs to $\mathscr{C}^1\big([t_0,T];\mathscr{S}(\mathbb{R}^3\times\mathbb{R}^3)\big)$ is a consequence of (5.8), as noticed in [25] (our source term is smooth and compactly supported). To prove that it is still the case for $M^{-1/2}F$, given (5.8), it is clearly sufficient to prove that

- $G : (t,x,\xi) \mapsto \exp\Big[\frac{|\xi|^2}{4}e^{-2t}\Big]\big[H(t)\star(F_0-M)\big](x,\xi) \in \mathscr{C}^0\big([t_0,T];\mathscr{S}(\mathbb{R}^3\times\mathbb{R}^3)\big)$,
- $B_s : (t,x,\xi) \mapsto \exp\Big[\frac{|\xi|^2}{4}e^{-2t}\Big][H(t-s)\star A(s)]\big(\Phi_s(x,\xi)\big) \in \mathscr{C}^0\big([t_0,T];\mathscr{S}(\mathbb{R}^3\times\mathbb{R}^3)\big)$, for $t > s$.

The fundamental solution can also be written in a simpler form

$$H(t,x,\xi) = K(t)e^{-\alpha(t)|x|^2}e^{-\beta(t)|\xi|^2}e^{-\gamma(t)x\cdot\xi},$$

so that for all $t > 0$, $x, y, \xi, \zeta \in \mathbb{R}^3$

$$H(t, x-y, \xi-\zeta) = K(t) e^{-\alpha(t)|x|^2 - \beta(t)|\xi|^2 - \alpha(t)|y|^2 - \beta(t)|\zeta|^2} e^{2\alpha(t) x\cdot y} e^{2\beta(t)\xi\cdot\zeta} e^{-\gamma(t)(x-y)\cdot(\xi-\zeta)},$$

and hence for any multi-indices $\ell, p \in \mathbb{N}^3$, we deduce

$$\partial_x^\ell \partial_\xi^p H(t, x-y, \xi-\zeta) = P_{\ell,p}^t(x, \xi, y, \zeta) H(t, x-y, \xi-\zeta),$$

for some polynomial function $P_{\ell,p}^t$. Now, by the basic property of the fundamental solution, and since F_0 is compactly supported in space and velocity, the fact that $G(t)$ is $\mathscr{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ for any fixed time is obvious, and

$$\partial_x^\ell \partial_\xi^p G(t) = \sum_{0\le k\le p} Q_k^t(|\xi|) \exp\Big[\frac{|\xi|^2}{4} e^{-2t}\Big] [\partial_x^\ell \partial_\xi^k H(t)] \star (F_0 - M)(x, \xi),$$

where Q_k^t is a polynomial. And since F_0 has its support in space and velocity included in, say, $\mathsf{B}_x(0, R) \times \mathsf{B}_\xi(0, R)$, we have

$$\begin{aligned} &|[\partial_x^\ell \partial_\xi^k H(t)] \star (F_0 - M)(x, \xi)| \\ &\qquad \le C_{F_0} R_{\ell,p}^t(x, \xi) K(t) e^{-\alpha(t)|x|^2 - \beta(t)|\xi|^2} e^{(2\alpha(t)+\gamma(t))|x|R} e^{(2\beta(t)+\gamma(t))|\xi|R} e^{-\gamma(t) x\cdot\xi}, \end{aligned}$$

where $R_{\ell,p}^t(x, \xi)$ is another polynomial (actually the one obtained by letting $x = \xi = (1, 1, 1)$ in $P_{\ell,p}^t$). Eventually we see that a sufficient condition to have $G(t)$ in $\mathscr{S}(\mathbb{R}^3 \times \mathbb{R}^3)$ is that the quadratic form

$$q_t(x, \xi) := \frac{\nu(t)|x|^2 + \lambda(t)|\xi|^2 + \mu(t)(x\cdot\xi)}{4\lambda(t)\nu(t) - \mu^2(t)} - \frac{1}{4}|\xi|^2 e^{-2t}$$

is positive definite. This is equivalent to

$$\lambda(t)(e^{2t} - \nu(t)) > -\frac{1}{4}\mu^2(t),$$

which is obviously true with the definition of $\nu(t)$, so the eigenvalue keep its sign. Furthermore, the determinant is still strictly positive because

$$\frac{1}{4} e^{-2t} < \frac{1}{\nu(t)}.$$

The regularity in time for $t > 0$ is obvious. As for B_s, the same study applies since $B_s(t) \in \mathscr{S}(\mathbb{R}^3 \times \mathbb{R}^3)$ is equivalent to $B_s(t) \circ \Phi_{-s} \in \mathscr{S}(\mathbb{R}^3 \times \mathbb{R}^3)$ and all the previous inequalities are true if we substitue t by $t - s$. The end of the lemma is proved in the following way : since F_0 is smooth, $\partial_t M^{-1/2}(F - M)(t, x, \xi)$ has a limit as t goes to 0, and it is equal to $M^{-1/2}[-\xi\cdot\nabla_x F_0 + \nabla_\xi\cdot(\xi F_0) + \Delta_\xi F_0 + A(0)] \in \mathrm{L}^2_\xi(\mathrm{H}^3_x)$ and hence $M^{-1/2} F \in \mathrm{L}^\infty\big([0, \varepsilon]; \mathrm{L}^2_\xi(\mathrm{H}^3_x))\big)$, and hence for the whole interval $[0, T]$. $\square$

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