Sharp logarithmic Sobolev inequalities on gradient solitons and applications

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Abstract

We show that gradient solitons, expanding, shrinking or steady, for the Ricci flow have potentials leading to suitable reference probability measures on the manifold. Under suitable conditions these reference measures satisfy sharp logarithmic Sobolev inequalities with lower bounds charaterized by the geometry of the manifold. In the proof various useful volume growth estimates are also established for gradient shrinking and expanding solitons.

1 Introduction

A complete Riemannian manifold (M, g) is called a gradient shrinking soliton (shrinker) if there exists a (smooth) function f such that its Hessian f_{ij} satisfies

$$R_{ij} + f_{ij} - \frac{1}{2}g_{ij} = 0. \tag{1.1}$$

Here R_{ij} denotes the Ricci curvature. As shown in [CLN], Theorem 4.1, associated to the metric and the *potential function* f, there exists a family of metrics $g(\eta)$, a solution to Ricci flow $\frac{\partial}{\partial\eta}g(\eta) = -2\,\mathrm{Ric}(g(\eta))$, with the property that $g(0) = g$, the original metric, and a family of diffeomorphisms $\phi(\eta)$, which is generated by the vector field $X = \frac{1}{\tau}\nabla f$, such that $\phi(0) = \mathrm{id}$ and $g(\eta) = \tau(\eta)\phi^*(\eta)g$ with $\tau(\eta) = 1-\eta$, as well as $f(x, \eta) = \phi^*(\eta)f(x)$. Namely it gives a self-similar (shrinking) family of metrics which is a solution to the Ricci flow. The metric $g(\eta)$ and $f(\eta)$ (sometimes also written as g^τ and f^τ, or simply g and f when the meaning is clear) satisfy that

$$R_{ij} + f_{ij} - \frac{1}{2\tau}g_{ij} = 0.$$

Here S denotes the scalar curvature.

Gradient shrinking solitons arise as the singularity models of Ricci flow. The more interesting cases are the noncompact ones. Trivial examples includes the Euclidean space $\mathbb{R}^n$ and the cylinders $\mathbb{S}^k \times \mathbb{R}^{n-k}$ for $k \geq 2$. Non-trivial noncompact examples can be found in, for example [FIK]. There is also a more recent systematical way of constructing solitons with symmetry in [DW]. The main purpose of this paper is to prove the following theorem, which generalizes the sharp logarithmic Sobolev inequality (referred as Stam-Gross logarithmic Sobolev inequality in [Vi1, Vi2], where one can also find detailed historic accounts and more complete references) of the Euclidean space $\mathbb{R}^n$ [Gr].

Theorem 1.1 *Assume that (M, g, f) be a gradient shrinking soliton so that either $|Ric|$ is bounded or* $\mathrm{Ric} \geq 0$*. Then there exists a geometric invariant (under the isometry) μ_s which depends only on the value of f and S at the minimum point of f under the normalization $\frac{1}{(4\pi\tau)^{n/2}} \int_M e^{-f} \, d\Gamma_\tau = 1$, and is independent of τ, such that for $\tau > 0$ and any compact supported smooth function $\rho = \frac{e^{-\psi}}{(4\pi\tau)^{n/2}}$ with $\int_M \rho \, d\Gamma_\tau = 1$, where $d\Gamma_\tau$ is the volume element of g^τ, we have that*

$$\int_M \left(\tau(|\nabla \psi|^2_\tau + S(\cdot, \tau)) + \psi - n\right) \rho \, d\Gamma_\tau \geq -\mu_s.$$

Moreover, for this geometric constant μ_s the above inequality is sharp. In the case that $\mathrm{Ric} \geq 0$ *and bounded, $\mu_s \geq 0$.*

The interested reader should consult Section 2 for the notations and the discussions on how the invariant μ_s is defined. The proof of the inequality uses the Bakry-Emery criterion for logarithmic Sobolev inequalities as obtain from the so-called HWI inequalities derived by Villani and coauthors in different settings, see Section 4 and references therein. Here, the main difficulty resides in showing the necessary bounds on the potential to be able to apply these inequalities, which is done in Section 2.

An immediate consequence of the theorem is the (strong) non-collapsing result for the gradient shrinking solitons. In the case that M has bounded nonnegative Ricci curvature the sharp logarithmic Sobolev inequality of Theorem 1.1 implies the logarithmic Sobolev inequality for all scales (may not necessarily with the sharp constant). In our analysis of the gradient shrinking solitons we also proves a volume estimate which implies the following statement on gradient shrinking solitons:

Corollary 1.1 *Any non-flat gradient shrinking soliton with nonnegative Ricci curvature must have zero asymptotic volume ratio.*

The above result is also proved in Section 2, where one can find a quantified estimate on the volume growth. This, in the case of gradient shrinking solitons, already generalizes a previous important result of Perelman [P] on ancient solutions with bounded nonnegative curvature operator. The result of Perelman, Proposition 11.4 of [P], draws the same conclusion for any ancient solutions with bounded nonnegative curvature operator. (Perelman also assumes the non-collapsing condition, which is not needed in his proof.) Gradient steady solitons are special ancient solutions. However our result is more general than that of Perelman in two ways. First one can not derive the above result from the result of Perelman's since we do not assume that the curvature operator has any sign, nor put any bound on the curvature. On the other hand, the result of Perelman can be derived out of the result above on gradient steady solitons as a consequence via the limiting to the *asymptotic solitons*.

Gradient steady/expanding solitons (expanders) arise also in the singularity analysis of Ricci flow [H1]. The technique employed here yields some sharp geometric inequalities for steady/expanding solitons as well. The following is the sharp logarithmic Sobolev inequality for the expanders.

Theorem 1.2 *Assume that (M, g, f) is gradient expanding soliton with* $\mathrm{Ric} \geq 0$*. Then for* *any* $\rho(x) = \frac{e^{-\psi(x)}}{(4\pi)^{n/2}}$ *with* $\int_M \rho(x)\, d\Gamma(x) = 1$*, we have that*

$$\int_M \left(|\nabla\psi|^2 - 3S + \psi - n\right) \rho \, d\Gamma \geq -\mu_e. \tag{1.2}$$

Here μ_e *is a geometric constant only depending on the value of* f *and* S *at the minimum point of* f*. Moreover, the inequality is sharp for such* μ_e*. Moreover* $\mu_e \geq 0$ *with equality if and only if* (M, g) *is isometric to* $\mathbb{R}^n$*.*

We refer the readers to Section 5 for the notaions involved. An equivalent expression of the integrand also appeared in a recent interesting preprint of Cao and Hamilton [CH] on pointwise differential estimates of Li-Yau-Hamilton type.

For the expanding solitons, we also obtain a volume estimate, which generalizes a recent result of Hamilton [H2](see also [CLN]) asserting that *the asymptotic volume ratio of gradient expanding solitons with bounded positive Ricci curvature must be positive.* The following is one of our statements.

Corollary 1.2 *Assume that* (M, g, f) *is a gradient expanding soltion with* $S(x) \geq -\beta$ *for some constant* $\beta \geq 0$*. Then for any* $o \in M$ *and* $r \geq r_0$

$$V(o, r) \geq V(o, r_0) \left(\frac{r + a}{r_0 + a}\right)^{n-2\beta}$$

with $a = 2\sqrt{f(o) + \mu_e + \beta}$*.*

The above mentioned Hamilton's result follows from the above statement applying to the case $\beta = 0$. For general β, the growth rate in our estimate is sharp as shown by easy examples. Detailed discussions can also be found in Section 5. A similar result can also be found in a recent preprint [CT].

For gradient steady soliton we obtain a sharp weighted Poincaré inequality instead. The proof is relatively easy without appealing the above mentioned theory involving the Bakry-Emery criterion.

The part of the conclusion that $\mu_s \geq 0$ in the main theorem is finally proved in Section 7. This is motivated by the c-theorem on the renormalization group flow. In view of the entropy monotonicity formula of Perelman, its connection with the logarithmic Sobolev inequality, as well as the fact that gradient shrinking solitons arise as the singularity models (at least for the cases that the blow-up has nonnegative curvature), this result can be viewed as an analogue of Zamolodchikov's c-theorem [Z] for the re-normalization group flow. The proof makes use a Li-Yau-Hamilton type inequality of Perelman [P] and the entropy formula/monotonicity for the linear heat equation of [N2].

2 Geometric estimates on gradient shrinking solitons

We shall follow the notations from the previous section. The following equations are simple consequences of the soliton equation (see for example [H1] or [CLN] for a proof):

$$
\begin{aligned}
S + \Delta f - \frac{n}{2\tau} &= 0, && (2.1)\\
S + |\nabla f|^2 - \frac{f}{\tau} &= \frac{\mu_s(\tau)}{\tau}. && (2.2)
\end{aligned}
$$

where μ is a constant so chosen that

$$\frac{1}{(4\pi\tau)^{\frac{n}{2}}} \int_M e^{-f}\, d\Gamma_\tau = 1. \tag{2.3}$$

Here S is the scalar curvature. The equations (2.1) and (2.2) imply that

$$2\Delta f - |\nabla f|^2 + S + \frac{f-n}{\tau} = -\frac{\mu_s(\tau)}{\tau}. \tag{2.4}$$

The integral $\int_M e^{-f}\, d\Gamma_\tau$ is well-defined if we assume that $|\operatorname{Ric}|$ is bounded or $\operatorname{Ric} \geq 0$. This is a consequence of the following lemma, together with the Bishop volume comparison theorem which implies that for a fixed $o \in M$ the volume of $B(o, r)$ is bounded from above by $C \exp((n-1)Kr)$ for some $C > 0$ if $\operatorname{Ric} \geq -(n-1)K^2$.

Lemma 2.1 *Assume that either $|\operatorname{Ric}| \leq C_1$ for some constant $C_1 > 0$ or $\operatorname{Ric} \geq 0$. Let $r(x)$ be the distance function to a fixed point $o \in M$ with respect to $g(\eta)$ metric. Then there exists $\delta_0 = \delta_0(M, f, \tau)$ and positive constants C_2, C_3 depending on M, f, τ such that for any $\delta \leq \delta_0$,*

$$f(x) \geq \delta r^2(x) \tag{2.5}$$

for $r(x) \geq C_2$ and

$$f(x) \leq C_3 r^2(x), \qquad |\nabla f|(x) \leq C_3(r(x)+1) \tag{2.6}$$

for $r(x) \geq C_2$.

Proof. If we assume that $|\operatorname{Ric}| \leq C_1$, by Hamilton [H1], Theorem 17.4, there exists $C_4 = C_4(M, C_1)$ such that any minimizing geodesic γ from $o \in M$ to $x = \gamma(s_0)$ with $s_0 \geq 2$,

$$\int_0^{s_0} \operatorname{Ric}(\gamma', \gamma')\, ds \leq C_4(M).$$

Hence as in [P, N3] we have that

$$\frac{d}{ds} f(\gamma(s)) = \langle \nabla f, \nabla r \rangle(\gamma(s)) \geq \frac{1}{2\tau} s - C_4 - |\nabla f|(o).$$

for $s \geq 2$, which implies the first claimed estimate. In fact one has that

$$f(x) \geq \frac{1}{4\tau} r^2(x) - (C_4 - |\nabla f|(o)) r(x) - f(o).$$

This implies (2.5) easily. From this we know that the minimum of f can be achieved at some point, say $o \in M$ and which is invariant for different η. The estimates of (2.6) follow from the fact $f_{ij} \le (C_1+1)g_{ij}$ and (2.2).

In the case $\mathrm{Ric} \ge 0$, The first estimate of (2.6) on the upper bound of f again follows easily from $f_{ij} \le \frac{1}{2\tau} g_{ij}$ and the second one of (2.6) follows from the upper bound on f and (2.2). In fact we have that for $r(x) \ge B$, some positive constant $B = B(M)$,

$$\left(|\nabla f|^2 + f\right)(x) \le \left(\frac{1}{2\tau} + 1\right)^2 r(x)^2. \tag{2.7}$$

By the proof of Proposition 1.1 in [N3], for any minimizing geodesic joining o to $x = \gamma(s_0)$ with $s_0 \ge 2$ and $r_0 \ge 1$, we have that

$$\int_0^{s_0 - r_0} \mathrm{Ric}(\gamma', \gamma')\, ds \le C_4(M) + \frac{n-1}{r_0}. \tag{2.8}$$

Now assume that the claimed estimate (2.5) fails at $x = \gamma(s_0)$. Namely we assume that

$$f(\gamma(s_0)) \le \delta_0 s_0^2.$$

Then for s_0 sufficiently large, (2.2) implies that

$$S(\gamma(s_0)) \le \frac{\delta_0}{\tau} s_0^2 + \frac{\mu_s(\tau)}{\tau} \le \frac{2\delta_0}{\tau} s_0^2.$$

Now we employ the trick from [N3] to obtain upper bound of $S(\gamma(s))$ for s near s_0. Using the equation

$$\nabla_i S = 2R_{ij} f_j$$

as in [N3] we have that for s_0 and s_1 satisfying $s_0 \ge s_1 \ge B$, for some $B = B(M) > 0$,

$$|\nabla \log S|(\gamma(s)) \le 2|\nabla f|(\gamma(s)) \le 2\left(\frac{1}{2\tau} + 1\right) s$$

which implies that

$$\log \frac{S(\gamma(s_1))}{S(\gamma(s_0))} \le 2\left(\frac{1}{2\tau} + 1\right) s_0 (s_0 - s_1)$$

for any $s_1 \le s_0$. Now choosing $r_0 = \frac{n-1}{\epsilon s_0}$ (with $\epsilon > 0$ to be decided later) as in [N3] we have that

$$S(\gamma(s)) \le \frac{2\delta_0}{\tau} s_0^2 \exp\left(\frac{2\left(\frac{1}{2\tau} + 1\right)(n-1)}{\epsilon}\right)$$

for all $s \ge s_0 - r_0$. Then

$$\begin{aligned}
\int_{s_0 - r_0}^{s_0} \mathrm{Ric}(\gamma', \gamma')\, ds &\le \int_{s_0 - r_0}^{s_0} S(\gamma(s))\, ds \\
&\le r_0 \frac{2\delta_0}{\tau} s_0^2 \exp\left(\frac{2\left(\frac{1}{2\tau} + 1\right)(n-1)}{\epsilon}\right) \\
&= \frac{2\delta_0 (n-1)}{\tau\epsilon} \exp\left(\frac{2\left(\frac{1}{2\tau} + 1\right)(n-1)}{\epsilon}\right) s_0.
\end{aligned}$$

Together with (2.8), taking $\epsilon = \frac{1}{4\tau}$, we have that

$$\int_0^{s_0} \mathrm{Ric}(\gamma', \gamma')\, ds \le \frac{1}{4\tau} s_0 + 8\delta_0 (n-1) \exp\left(\frac{8}{\tau}\left(\frac{1}{2\tau}+1\right)(n-1)\right) s_0 + C_4(M).$$

But this would implies that, if δ_0 is chosen to be sufficiently small such that

$$8\delta_0(n-1)\exp\left(\frac{8}{\tau}\left(\frac{1}{2\tau}+1\right)(n-1)\right) \le \frac{1}{100\tau},$$

$$\begin{aligned} \frac{d}{ds} f(\gamma(s)) &= \int_0^s \frac{d^2}{ds^2} f(\gamma(s)) + \frac{d}{ds} f(\gamma(0)) \\ &\ge \frac{s}{2\tau} - \int_0^{s_0} \mathrm{Ric}(\gamma', \gamma')\, ds - |\nabla f|(o) \\ &\ge \frac{1}{5\tau} s - C_5 \end{aligned}$$

for any $s_0 - 1 \le s \le s_0$ with s_0 sufficiently large and $C_5 = C(C_4, \delta_0, |\nabla f|(o))$. For $s \le s_0 - 1$, applying (2.8) with $r_0 = 1$, we have that

$$\begin{aligned} \frac{d}{ds} f(\gamma(s)) &= \int_0^s \frac{d^2}{ds^2} f(\gamma(s)) + \frac{d}{ds} f(\gamma(0)) \\ &= \frac{s}{2\tau} - \int_0^{s} \mathrm{Ric}(\gamma', \gamma')\, ds - |\nabla f|(o) \\ &\ge \frac{s}{2\tau} - C_4 - (n-1) - |\nabla f|(o). \end{aligned} \tag{2.9}$$

Hence

$$f(\gamma(s_0)) \ge \frac{1}{8\tau} s_0^2 - C_5.$$

This is a contradiction to the assumption that $f(\gamma(s_0)) \le \delta_0 s_0^2$, for s_0 sufficiently large and $\delta_0 \le \frac{1}{10\tau}$. $\square$

Corollary 2.1 *Under the same assumptions as in the lemma, the minimum of f can be achieved somewhere, say $o \in M$. Hence $f(o)$ and $S(o)$ are fixed for different η and the constant $\mu_s(\tau)$ in (2.2) is a constant independent of τ. Therefore $\mu_s \doteqdot \mu_s(\tau)$ is an invariant of the soliton. Moreover, both S and Δf either are bounded (in the case $|\mathrm{Ric}|$ is bounded) or grow at most quadratically (in the case $\mathrm{Ric} \ge 0$).*

Proof. The first part of the corollary is evident since o is the fixed point of $\phi(\eta)$. The change of S is compensated by the factor τ. The second part follows from the equations (2.1) and (2.2). $\square$

Note that by Theorem 4.1 of [CLN], $\frac{e^{-f}}{(4\pi\tau)^{n/2}}$ satisfies the conjugate heat equation

$$\left(\frac{\partial}{\partial \tau} - \Delta + S\right)\left(\frac{e^{-f}}{(4\pi\tau)^{n/2}}\right) = 0.$$

Hence the total mass of e^{-f} (namely the equation (2.3)) is preserved. In other words, if it holds at $\tau = 1$ (which corresponds to $\eta = 0$), it holds for all $\tau > 0$. Also note that $\mu_s(M, g) = \mu_s(M', g')$ if (M, g) is isometric to (M', g') by the virtue of Lemma 1.2 of [Na].

The next result concerns the behavior of the volume $V(o,r)$ of balls $B(o,r)$, especially as $r \to \infty$. We start with an easier version with Ricci curvature being bounded.

Corollary 2.2 *Let (M,g) be a nonflat gradient shrinking soliton with* $\mathrm{Ric} \geq 0$. *Assume further that* $\mathrm{Ric} \leq C_1$. *Then there exists a constant $\delta = \delta(M,f) > 0$ with the property that for any $o \in M$, there exists $a = a(M,f,C_1)$ such that for any $r \geq r_0 \geq a$*

$$V(o,r) \leq V(o,r_0)\left(\frac{r-a}{r_0-a}\right)^{n-\delta}. \tag{2.10}$$

Proof. Without the loss of generality we may assume that $\tau = 1$. By Proposition 1.1 of [N3], we have $\delta = \delta(M,f) > 0$ such that $S \geq \frac{\delta}{2}$. On the other hand, the argument from the proof of the previous lemma, more precisely (2.9), implies that

$$\frac{\partial f}{\partial r}(x) \geq \frac{r(x)}{2} - C_6(M,f,o).$$

Now integration by part (on equation (2.1)) over $B(o,r)$ yields that

$$\begin{aligned} \frac{n-\delta}{2}V(o,r) &\geq \int_{B(o,r)}\left(\frac{n}{2}-S\right)d\Gamma \\ &= \int_{\partial B(o,r)}\frac{\partial f}{\partial r}\,dA \\ &\geq A(o,r)\left(\frac{r}{2}-C_6\right). \end{aligned}$$

Here $A(o,r)$ is the surface area of $\partial B(o,r)$. The result follows from integrating the above estimate on $[r_0, r]$. □

Note that the above result, if can be shown under no upper bound on the Ricci curvature, can be used to prove Proposition 11.4 of [P], which concludes that *any nonflat ancient solution of Ricci flow with bounded nonnegative curvature operator must have the asymptotic volume ratio* $\lim_{r\to\infty}\frac{V(o,r)}{r^n} = 0$. In fact one can prove it by the contradiction. Assuming that the claim is not true, one can obtain an *asymptotic soliton* by Proposition 11.2 of [P], which is nonflat, also have maximum volume growth since it is easy to show that the asymptotic volume ratio for an ancient solution with bounded nonnegative curvature is monotone non-increasing in t, hence the asymptotic soliton must has positive asymptotic volume ratio. This is a contradiction with the estimate (2.10). With some extra effort, we can indeed prove a such desired volume estimate without assuming the Ricci curvature upper bound.

Proposition 2.1 *Let (M,g) be a nonflat gradient shrinking soliton with* $\mathrm{Ric} \geq 0$. *Then there exists $\delta = \delta(M,f)$ with the property that for any $o \in M$, there exists $a = a(M,f,o) > 1$ and $C = C(n,\delta)$ such that for any $R \geq R_0 \geq a$,*

$$V(o,R+1) \leq V(o,R_0+1)e^{\frac{C(n,\delta)}{R_0}}\left(\frac{R-a}{R_0-a}\right)^{n-\delta}. \tag{2.11}$$

Proof. First we introduce some notations. Let $\mathcal{C}(r)$ be the subset of the unit tangent sphere at o such that for all $\theta \in \mathcal{C}(r)$ the geodesic $\gamma(s) = \exp_o(s\theta)$ is minimizing up to $s = r$ (and not beyond). Clearly $\mathcal{C}(r_2) \subset \mathcal{C}(r_1)$ if $r_1 \le r_2$. Let $J(r,\theta)dr \wedge d\theta$ be the volume element with respect to this polar coordinate (r,θ). Then the surface area of the sphere $\partial B(o,r)$, $A(r) = \int_{\mathcal{C}(r)} J(r,\theta)\, d\theta$. Also we assume that $\tau = 1$ without the loss of the generality. As before

$$\begin{aligned} V(o,r)\left(\frac{n-\delta}{2}\right) &\ge \int_{B(o,r)} \left(\frac{n}{2} - S\right) d\Gamma \\ &= \int_{\partial B(o,r)} \frac{\partial f}{\partial r}\, dA. \end{aligned}$$

Now observe that from the proof of the previous lemma, precisely (2.9), it is evident that for any $\theta \in \mathcal{C}(r+1)$, and at $s=r$, on the geodesic $\gamma(s) = \exp_o(s\theta)$, $\frac{\partial f}{\partial r} \ge \frac{r}{2} - C_6$. Hence

$$\begin{aligned} \int_{\partial B(o,r)} \frac{\partial f}{\partial r}(r)\, dA &= \int_{\mathcal{C}(r)} \frac{\partial f}{\partial r}(r) J(r,\theta)\, d\theta \\ &\ge \int_{\mathcal{C}(r+1)} \left(\frac{r}{2} - C_6\right) J(r,\theta)\, d\theta \\ &\ge \left(\frac{r}{2} - C_6\right)\left(\frac{r}{r+1}\right)^{n-1} \int_{\mathcal{C}(r+1)} J(r+1,\theta)\, d\theta \\ &= \left(\frac{r}{2} - C_6\right)\left(\frac{r}{r+1}\right)^{n-1} A(o,r+1). \end{aligned}$$

In the second inequality we have used the volume comparison fact that $\frac{J(r,\theta)}{r^{n-1}}$ is monotone non-increasing. Putting together we have that

$$V(o,r+1)\left(\frac{n-\delta}{2}\right) \ge \left(\frac{r}{2} - C_6\right)\left(\frac{r}{r+1}\right)^{n-1} A(o,r+1).$$

For $r \ge 2$,

$$\begin{aligned} \frac{A(o,r+1)}{V(o,r+1)} &\le \frac{n-\delta}{2}\frac{1}{r/2 - C_6}\left(1 + \frac{1}{r}\right)^{n-1} \\ &\le (n-\delta)\frac{1}{r - 2C_6}\left(1 + \frac{C(n)}{r}\right). \end{aligned}$$

Let $2a = 4C_6$, then for $r \ge 2a$, $r - 2C_6 \ge \frac{r}{2}$. Thus

$$\frac{A(o,r+1)}{V(o,r+1)} \le \frac{n-\delta}{r-a} + \frac{C(n,\delta)}{r^2}.$$

Finally the claimed result follows from integrating the above on $[R_0, R]$. □

Corollary 2.3 *Let (M,g) be a nonflat gradient shrinking soliton with* Ric ≥ 0. *Then $\mathcal{V}(M) = 0$, where $\mathcal{V}(M) \doteqdot \lim_{r\to\infty} \frac{V(o,r)}{r^n}$.*

Note that the above result generalizes Proposition 11.4 of [P] in the case of the gradient shrinking solitons by replacing the nonnegativity of the curvature operator with nonnegativity of Ricci. Also the assumption on the curvature bound has been removed. It is interesting to find out if the same estimate as in Proposition 2.1 can be shown for any ancient solutions with nonnegative Ricci curvature.

3 Optimal Transport and LSIs

In this section, we will work with Riemannian manifolds (M, g) endowed with a reference probability measure $e^{-V} d\Gamma$ where the potential $V \in C^2(M)$ verifies a curvature-dimension bound of the type $C(K, \infty)$ with $K \in \mathbb{R}$, i.e.,

$$R_{ij} + V_{ij} \geq K g_{ij}.$$

Here $d\Gamma$ is the volume measure associated to (M, g). This section is devoted to collect several results present in the literature in the present context [Vi2]. A Riemannian manifold in this section refers to a smooth, complete connected finite-dimensional Riemannian manifold distinct from a point, equipped with a smooth metric tensor. Let us assume that the reference measure is normalized by

$$\int_M e^{-V} d\Gamma = 1.$$

Consider the positive solution ρ to the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} - \operatorname{div}\left(\rho \nabla (\log \rho + V)\right) = 0. \tag{3.1}$$

Let $\xi = \log \rho + V$. It is easy to see that

$$\left(\frac{\partial}{\partial t} - \Delta\right) \xi = \langle \nabla \xi, \nabla \log \rho \rangle.$$

Let us define the Boltzmann relative entropy functional, called also Nash entropy, as

$$H_V(\rho) \doteqdot \int_M \rho \xi \, d\Gamma.$$

We have immediately the following dissipation of the Boltzmann relative entropy functional,

$$\frac{d}{dt} H_V(\rho(t)) = -\int_M |\nabla \xi|^2 \rho \, d\Gamma \doteqdot -I_V(\rho(t)), \tag{3.2}$$

where computations have been made for smooth fast-decaying at infinity solutions on the manifold M. This computation show us that these two quantities, the relative Boltzmann entropy $H_V(\rho)$ and the relative Fisher information $I_V(\rho)$ are intimately related at least for solutions of (3.1). However, as it was discovered in the case of $\mathbb{R}^n$ or in the case of a manifold in [BE, AMTU] for linear diffussions or in [CT, O, DoPi] for nonlinear diffusions, this relation is really through functional inequalities, see also [OV].

Related to these functionals, there is another quantity that is involved in these inequalities: the Euclidean Wasserstein distance between any two probability measures ν_0, ν_1 on the manifold M, i.e.,

$$W_2(\nu_0, \nu_1) = \inf \left\{ \int_{M \times M} d(x, y)^2 \, d\theta(x, y); \; \theta \in \Theta(\nu_0, \nu_1) \right\}^{1/2}; \tag{3.3}$$

where $\Theta(\nu_0, \nu_1)$ is the set of probability measures on $M \times M$ having marginals ν_0 and ν_1. This distance is well defined for probability measures ν_0 and ν_1 with second moment bounded, $\mathcal{P}_2(M)$, and metrizes the weak convergence of measures in the sense of [Vi2, Definition 6.7, Theorem 6.8]. The expression "second moment bounded" refers to the fact that $r(x)^2$ is integrable against the measures ν_0 and ν_1. It is worthy to mention that the curvature-dimension bound $C(K,\infty)$ with $K>0$ implies that the second moment of the reference measure (actually, all moments) e^{-V} are bounded, see [Vi2, Theorem 18.11].

Recently, several authors [S, LV] based on early works [Mc, CMS], see [Vi2, Chapter 17] for a whole account of the history, have characterized curvature-dimension bounds in terms of the displacement convexity of the Boltzmann relative entropy functional. The notion of displacement convexity refers to convexity along pathes of minimal transport distance W_2 in the set of probability measures $\mathcal{P}_2(M)$. An expression of the convexity of these functionals are the so called HWI inequalities, named in this way since they involved the three functionals $H_V(\rho)$, $I_V(\rho)$ and W_2. In the following, we will work with measures absolutely continuous against volume measure and we identify the measures with their densities for notational convenience. The main results we need are the following:

Theorem 3.1 [Vi2, Corollary 20.13] and [BE] *Let M be a Riemannian manifold equipped with a reference measure $e^{-V}d\Gamma$ where the potential $V\in C^2(M)$ verifies a curvature-dimension bound of the type $C(K,\infty)$ with $K\in\mathbb{R}$. Then, for any given $\nu\in\mathcal{P}_2(M)$ absolutely continuous with respect to volume measure $d\Gamma$ with density ρ, it holds the HWI inequality:*

$$H_V(\rho)\le W_2(\rho,e^{-V})\sqrt{I_V(\rho)}-\frac{K}{2}W_2(\rho,e^{-V})^2.$$

As a consequence, we have that whenever $K>0$, the following Logarithmic Sobolev Inequality (LSI) follows

$$H_V(\rho)\le\frac{1}{2K}I_V(\rho).$$

The HWI inequalities were originally introduced in [OV] and used in other models in nonlinear PDEs in [CMV]. Later, they were generalized to compact manifolds in [LV] and in this generality in [Vi2]. To see that the LSI inequality follows from the HWI inequality it suffices to consider the right-hand side of the HWI inequality as a function of W_2 and maximize that function.

Let us remark that some proofs of the LSI inequality use the Fokker-Planck dynamics (3.1), called the Bakry-Emery stragegy, but the referred functional proof through the HWI inequalities allows to overcome discussions on integrability issues and decay to infinity of solutions to (3.1). In fact, a direct application of the LSI on (3.2) gives the exponential decay of the Boltzmann relative entropy functional for solutions of (3.1) with initial density in $\mathcal{P}_2(M)$ in case $C(K,\infty)$ with $K>0$ holds, i.e., given a solution $\rho(t)$ of (3.1) then

$$H_V(\rho(t))\le H_V(\rho(0))\,e^{-2Kt}\qquad\text{for all } t\ge 0.$$

Nevertheless, let us remind the reader that assuming all integrability and behavior at ∞ are met for the integration by parts below, we can obtain the evolution of the relative Fisher information, see [BE, AMTU, Vi1, Vi2] for these computations. To take the time derivative of $I_V(\rho(t))$ note the Bochner type formula

$$\left(\frac{\partial}{\partial t}-\Delta\right)|\nabla\xi|^2=-2\xi_{ij}^2+2\langle\nabla(\langle\nabla\xi,\nabla\log\rho\rangle),\nabla\xi\rangle-2R_{ij}\xi_i\xi_j. \tag{3.4}$$

Using the above formula we have that

$$\begin{aligned}\frac{d}{dt} I_V(\rho(t)) &= \int_M (\Delta|\nabla\xi|^2)\rho + |\nabla\xi|^2 \operatorname{div}(\nabla\rho + \rho\nabla V)\, d\Gamma \\ &\quad + \int_M \left(-2\xi_{ij}^2 + 2\langle\nabla(\langle\nabla\xi, \nabla\log\rho\rangle), \nabla\xi\rangle - 2R_{ij}\xi_i\xi_j\right)\rho\, d\Gamma.\end{aligned}$$

Since

$$\begin{aligned}\int_M \langle\nabla(\langle\nabla\xi, \nabla\log\rho\rangle), \nabla\xi\rangle\rho\, d\Gamma &= \int_M \langle\nabla(|\nabla\xi|^2 - \langle\nabla V, \nabla\xi\rangle), \nabla\xi\rangle\rho\, d\Gamma \\ &= \int_M \langle\langle\nabla|\nabla\xi|^2, \nabla\rho\rangle + \langle\nabla|\nabla\xi|^2, \nabla V\rangle\rho\, d\Gamma \\ &\quad - \int_M \langle\nabla\langle\nabla V, \nabla\xi\rangle, \nabla\xi\rangle\rho\, d\Gamma\end{aligned}$$

we arrive at

$$\begin{aligned}\frac{d}{dt} I_V(\rho(t)) &= \int_M \left(-2\xi_{ij}^2 - 2R_{ij}\xi_i\xi_j\right)\rho\, d\Gamma \\ &\quad + \int_M \langle\nabla|\nabla\xi|^2, \nabla V\rangle\rho - 2\langle\nabla\langle\nabla V, \nabla\xi\rangle, \nabla\xi\rangle\rho\, d\Gamma \\ &= \int_M \left(-2\xi_{ij}^2 - 2(R_{ij} + V_{ij})\xi_i\xi_j\right)\rho\, d\Gamma. \qquad (3.5)\end{aligned}$$

As a consequence, due to the curvature dimension bound $C(K, \infty)$, we have

$$\frac{d}{dt} I_V(\rho(t)) \leq -2K \int_M |\xi|^2 \rho\, d\Gamma,$$

and thus,

$$I_V(\rho(t)) \leq I_V(\rho(0))\, e^{-2Kt} \qquad \text{for all } t \geq 0.$$

4 Main Result and Applications

Now, let us come back to the precise situation we have, the case of a shriking soliton and prove the main Theorem 1.1. Let us define the potential $V = f - \frac{n}{2}\log(2\pi\tau)$ for the fixed time slice of the shrinking Riemannian manifold soliton (M, g) at time τ. Lemma 2.1 and Corollary 2.1 implies that e^{-V} is a well defined probability measure. Moreover, we deduce from the soliton definition (1.1) that this reference measure verifies the $C(\frac{1}{2\tau}, \infty)$ condition. Therefore, Theorem 3.1 implies that for any probability density of the form

$$\rho(x) = \frac{e^{-\psi(x)}}{(4\pi\tau)^{\frac{n}{2}}}$$

with second moment bounded then

$$H_V(\rho) \leq \tau I_V(\rho).$$

Using now the soliton equation (2.4), we deduce:

$$\begin{aligned} I_V(\rho) &= \int_M \left(|\nabla\psi|^2\rho + 2\langle\nabla f, \nabla\rho\rangle + |\nabla f|^2\rho\right)\, d\Gamma \\ &= \int_M \left(|\nabla\psi|^2\rho + (-2\Delta f + |\nabla f|^2)\rho\right)\, d\Gamma \\ &= \int_M (|\nabla\psi|^2 + S + \frac{f + \mu_s - n}{\tau})\rho\, d\Gamma. \end{aligned}$$

Thus, the LSI inequality is equivalent to

$$\int_M \left(\tau(|\nabla\psi|^2 + S) + \psi - n\right)\rho\, d\Gamma \geq -\mu_s, \tag{4.1}$$

for all densities ρ with bounded second moment for the shriking soliton, with μ_s characterized by Corollary 2.1. This finishes the proof of Theorem 1.1.

Finally, recall Perelman's entropy functional

$$\mathcal{W}(g^\tau, u, \tau) \doteqdot \int_M \left(\tau(|\nabla\psi|^2 + S) + \psi - n\right) u\, d\Gamma_\tau$$

is defined for $u = \frac{e^{-\psi}}{(4\pi\tau)^{n/2}}$ with $\int_M u\, d\Gamma_\tau = 1$. Theorem 1.1 implies that for (M, g^τ), $W(g^\tau, u, \tau) \geq -\mu_s$. Namely Perelman's μ-invariant

$$\mu(g^\tau, \tau) \doteqdot \inf_{\int_M u=1} \mathcal{W}(g^\tau, u, \tau)$$

is bounded from below by $-\mu_s$. From (2.4) it is easy to see that

$$\tau(2\Delta f - |\nabla f|^2 + S) + f - n = -\mu_s.$$

Hence $u = \frac{e^{-f}}{(4\pi\tau)^{n/2}}$ is the minimizer for Perelman's $\mu(g, \tau)$ (cf. Remark 3.2 of [P]). This shows that the inequality of Theorem 1.1 is sharp. Summarizing above we have that

Corollary 4.1 *Let (M, g, f) be a gradient shrinking soliton satisfying* (1.1) *with either* $|\operatorname{Ric}|$ *being bounded or* $\operatorname{Ric} \geq 0$. *Then*

$$\mu(g, 1) = -\mu_s.$$

When $f = $ constant, (M, g) is a Einstein manifold with $\operatorname{Ric}_M = \frac{1}{2} g_M$. In this case we obtain a log-Sobolev inequality for $S = \frac{n}{2}$ and

$$\mu_s = \frac{n}{2} - \log(V(M)) + \frac{n}{2}\log(4\pi)$$

where $V(M)$ is the volume of (M, g_M). The μ-invariant was computed in [CHI] for many examples of four manifolds.

When $M = \mathbb{R}^n$ with $f = \frac{1}{4}|x|^2$, direct calculation shows that $\mu_s = 0$. Hence we have the classical logarithmic Sobolev inequality of Sham-Gross.

Recall here that a solution of Ricci flow is called κ non-collapsed, if for any (x_0, t_0) and $r \geq 0$, such that on $P(x_0, t_0, r) = B_{g(t_0)}(x_0, r) \times [t_0 - r^2, t_0]$, $|Rm|(x, t) \leq r^{-2}$, then $V_{g(t_0)}(x_0, r) \geq \kappa r^n$. Here $V_{g(t_0)}(x_0, r)$ is the volume of $B_{g(t_0)}(x_0, r)$ with respect to $g(t_0)$. By the virtue of Perelman [P], Theorem 1.1 implies the following volume non-collapsing result for gradient shrinking solitons.

Corollary 4.2 *Let (M, g, f) be a gradient shrinking soliton satisfying* (1.1) *with either* $|\operatorname{Ric}|$ *being bounded or* $\operatorname{Ric} \geq 0$. *Then there exists a $\kappa = \kappa(\mu_s) > 0$ such that if in a ball $B(x_0, 1)$, $|\operatorname{Ric}| \leq 1$, then $V(x_0, 1) \geq \kappa$.*

Proof. Follows from Theorem 1.1 and Section 4 of [P]. See also [CLN, To]. □

In [Na] there *stated* a related result asserting the κ-noncollapsing of gradient shrinking solitons with bounded curvature, in the sense defined right above the corollary. The conclusion in above corollary is stronger.

When $\operatorname{Ric}(M, g_{\tau=1}) \geq 0$ with bounded curvature, one can derive the logarithmic Sobolev inequality for all scales. This is done in the following two propositions.

Proposition 4.1 (Scale ≥ 1) *Let (M, g) be a gradient shrinking soliton satisfying* (1.1). *Assume that* $\operatorname{Ric} \geq 0$. *Then, there exists positive $\delta = \delta(M) < 1$ such that for any $\sigma \geq 1$,*

$$\int_M \left(\sigma(|\nabla\tilde{\psi}|^2 + S) + \tilde{\psi} - n\right) \frac{e^{-\tilde{\psi}}}{(4\pi\sigma)^{\frac{n}{2}}}\, d\Gamma \geq -\mu_s + \frac{n}{2} - \delta - \frac{n}{2}\log(\frac{n}{2\delta})$$

for any $\tilde{\psi}$ satisfying that $\int_M \frac{e^{-\tilde{\psi}}}{(4\pi\sigma)^{\frac{n}{2}}} d\Gamma = 1$.

Proof. Clearly only the nonflat case worths the proof (since the flat one is isometric to $\mathbb{R}^n$). By Proposition 1.1 of [N3], for a nonflat gradient shrinking soliton, there exists $\delta = \delta(M, f) > 0$ such that $S(x) \geq \delta$ for any $x \in M$. Let $\psi = \tilde{\psi} + \frac{n}{2}\log\sigma$. Then it is easy to see that

$$\begin{aligned}
\int_M \left(\sigma(|\nabla\tilde{\psi}|^2 + S) + \tilde{\psi} - n\right) \frac{e^{-\tilde{\psi}}}{(4\pi\sigma)^{\frac{n}{2}}}\, d\Gamma \quad &= \quad \int_M \left(|\nabla\psi|^2 + S) + \psi - n\right) \frac{e^{-\psi}}{(4\pi)^{\frac{n}{2}}}\, d\Gamma \\
&\quad +(\sigma - 1)\int_M (|\nabla\psi|^2 + S)\frac{e^{-\psi}}{(4\pi)^{\frac{n}{2}}}\, d\Gamma - \frac{n}{2}\log\sigma \\
&\geq \quad -\mu_s + \delta(\sigma - 1) - \frac{n}{2}\log\sigma.
\end{aligned}$$

Here we have used Theorem 1.1 in the last estimate. Since $-\delta(\sigma - 1) - \frac{n}{2}\log\sigma \geq \frac{n}{2} - \delta - \frac{n}{2}\log(\frac{n}{2\delta})$, the claimed result follows. □

Proposition 4.2 (Scale ≤ 1) *Assume that $0 \leq \operatorname{Ric} \leq A$. Then for any $0 \leq \sigma \leq 1$,*

$$\int_M \left(\sigma(|\nabla\tilde{\psi}|^2 + S) + \tilde{\psi} - n\right) \frac{e^{-\tilde{\psi}}}{(4\pi\sigma)^{\frac{n}{2}}}\, d\Gamma \geq -\mu_s - nA$$

for any $\tilde{\psi}$ satisfying that $\int_M \frac{e^{-\tilde{\psi}}}{(4\pi\sigma)^{\frac{n}{2}}} d\Gamma = 1$.

Proof. Define

$$\mu_0(g, \sigma) \doteqdot \inf_{\int_M u_0 = 1} \int_M \left(\sigma|\nabla\tilde{\psi}|^2 + \tilde{\psi} - n\right) u_0$$

with $u_0 = \frac{e^{-\tilde{\psi}}}{(4\pi\sigma)^{\frac{n}{2}}}$. Theorem 1.1 implies that $\mu_0(g, 1) \geq -\mu_s - nA$. Now for any u_0 which is compactly supported, let $u(x, t)$ be the heat equation solution with $u(x, 0) = u_0$. Then by the entropy monotonicity result in [N2], for $\sigma \le 1$,

$$\begin{aligned}\int_M \left(\sigma|\nabla\tilde{\psi}|^2 + \tilde{\psi} - n\right) \frac{e^{-\tilde{\psi}}}{(4\pi\sigma)^{\frac{n}{2}}} &\geq \int_M \left(|\nabla\varphi|^2 + \varphi - n\right) u(y, 1-\sigma)\, d\Gamma(y) \\ &\geq \mu_0(g, 1)\end{aligned}$$

where $u(y, 1-\sigma) = \frac{e^{-\varphi(y)}}{(4\pi(1-\sigma))^{\frac{n}{2}}}$. This implies the claimed result. $\square$

The above two proposition implies $\nu(M, g) > \infty$, hence a the strong κ-non-collapsing result for gradient shrinking solitons with bounded and nonnegative Ricci curvature as in [P] (see also [To] [Chow, et al1]).

5 Expanding solitons

Let us proceed in a similar way with expanding solitons. Recall that (M, g) is called a gradient expanding soliton if there exists f such that

$$R_{ij} + \frac{1}{2} g_{ij} = f_{ij}. \tag{5.1}$$

It is easy to show that

$$\Delta f = S + \frac{n}{2} \tag{5.2}$$

$$S + |\nabla f|^2 - f = \mu_e \tag{5.3}$$

for some constant μ_e. As before one can normalize μ_e so that $\int_M \frac{e^{-f}}{(4\pi)^{n/2}}\, d\Gamma = 1$. This will make μ_e a geometric invariant of (M, g).

Our first concern is about the behavior of the volume of balls $B(o, r)$ in M for any given $o \in M$. Along this direction, Hamilton [H2] proved the following result:

Theorem 5.1 (Hamilton) *Let (M, g) be a gradient expanding soliton has bounded nonnegative Ricci curvature. Then (M, g) has maximum volume growth. Namely*

$$\liminf_{r\to\infty} \frac{V(o, r)}{r^n} > 0.$$

For the exposition of this result please see [CLN], Proposition 9.46. (The uniform bound of the Ricci curvature is used in the proof to bound $\int_\gamma \mathrm{Ric}(\gamma'\gamma')$ as in Section 2.) Here the limit always exists due to the Bishop-Gromov volume comparison. This limit is called *asymptotic volume ratio*. This compares sharply with the gradient shrinking solitons (cf. Corollary 2.3) and a result of Perelman [P] asserting that *any non-flat ancient solution with bounded nonnegative curvature operator its asymptotic volume ratio must be zero*. The result below is a generalization of the above result of Hamilton.

Proposition 5.1 *Let (M, g, f) be an gradient expanding soliton.*

(1) If $S(x) \ge 0$ for any $x \in M$, without assuming any curvature bound, then for any $o \in M$, $r \ge r_0$.

$$V(o, r) \ge V(o, r_0) \left(\frac{r + a}{r_0 + a}\right)^n$$

with $a = 2\sqrt{f(o) + \mu_e}$.

(2) *Assume that* $S(x) \geq -\beta$ *for some constant* $\beta > 0$. *Then for any* $o \in M$ *and* $r \geq r_0$

$$V(o, r) \geq V(o, r_0) \left(\frac{r + a}{r_0 + a}\right)^{n - 2\beta}$$

with $a = 2\sqrt{f(o) + \mu_e + \beta}$.

Proof. In the case (1), from the assumption and (5.3) we have that $f + \mu_e \geq 0$. Consider any minimizing geodesic $\gamma(s)$ from $o \in M$ a fixed point of M. Then (5.3) implies that for any s

$$\left|\frac{d}{ds} f(\gamma(s))\right|^2 \leq f + \mu_e.$$

This implies, by the ODE comparison, that

$$\left(2\sqrt{f + \mu_e}\right)(\gamma(s)) \leq s + a$$

where $a = 2\sqrt{f(o) + \mu_e}$, which then implies that

$$\left|\frac{\partial f}{\partial r}\right|(\gamma(s)) \leq \frac{s}{2} + \frac{a}{2}. \tag{5.4}$$

Now we integrate (5.2) on $B(o, r)$ and have that

$$\begin{aligned} \frac{n}{2} V(o, r) &\leq \frac{n}{2} V(o, r) + \int_{B(o,r)} S \, d\Gamma \\ &= \int_{B(o,r)} \Delta f \, d\Gamma \\ &\leq \int_{\partial B(o,r)} \left|\frac{\partial f}{\partial r}\right| (y) \, dA(y). \end{aligned}$$

Using (5.4) we have that

$$\frac{n}{2} V(o, r) \leq A(o, r)(\frac{r}{2} + \frac{a}{2}).$$

The result follows by dividing the both side of the above by $V(o, r)$ and then integrating the resulting estimate on the interval $[r_0, r]$. The proof for the case (2) is similar. $\square$

Remark 5.2 *The estimates in both cases have the sharp power. To see this consider* $M = N^k \times \mathbb{R}^{n-k}$ *where* N *is a compact Einstein manifold with* $\mathrm{Ric}_N = -\frac{1}{2} g_N$, $\mathbb{R}^{n-k}$ *is the Gaussian expanding soliton.*

Now we derive the logarithmic Sobolev inequality for the expanders. To make sure that the integral $\int_M e^{-f} \, d\Gamma$ is finite we have to make an assumption that there exists some $\epsilon > 0$,

$$f_{ij} = \frac{1}{2} g_{ij} + R_{ij} \geq \epsilon g_{ij}. \tag{5.5}$$

Under this assumption, it is easy to see that

$$f(x) \geq \frac{\epsilon}{4} r^2(x) - C$$

for some $C = C(M, f)$. Since $R_{ij} \geq -\frac{1}{2} g_{ij}$, the volume $V(o, r) \leq \exp(A(r+1))$ for some $A = A(n)$. This together with the lower estimate above ensures that the integral $\int_M \frac{e^{-f}}{(4\pi)^{n/2}} \, d\Gamma$ is finite. (For this fact, one may also use the result of [WW].) Notice that under our assumption (5.5), as in the proof of Proposition 5.1 we have that

$$f(x) \leq \left(\frac{r(x)}{2} + b \right)^2$$

for some $b = b(M, f)$. This ensures the finiteness of the integral

$$\int_M \left(|\nabla f|^2 + |\Delta f| + |S| \right) \frac{e^{-f}}{(4\pi)^{n/2}} \, d\Gamma.$$

Note that (5.2) and (5.3) implies that

$$2\Delta f - |\nabla f|^2 - 3S + f - n = -\mu_e. \tag{5.6}$$

Integrating (5.6), we have that

$$\int_M \left(|\nabla f|^2 - 3S + f - n \right) \frac{e^{-f}}{(4\pi)^{n/2}} \, d\Gamma = -\mu_e.$$

It is clear that assumption (5.5) is trivially satisfied for non-negative Ricci tensor.

Assume in the rest of this section that $\mathrm{Ric} \geq 0$, let us define the potential $V = f - \frac{n}{2} \log(4\pi)$. Previous arguments imply that the reference measure e^{-V} is a well defined probability measure. Moreover, we deduce from the soliton definition (5.1) and being $\mathrm{Ric} \geq 0$ that this reference measure verifies the $C(\frac{1}{2}, \infty)$ condition. Therefore, Theorem 3.1, together with a similar calculation as before, implies the following LSI inequality.

Theorem 5.3 *Assume that (M, g, f) is gradient expanding soliton with $\mathrm{Ric} \geq 0$. Then for any $\rho(x) = \frac{e^{-\psi(x)}}{(4\pi)^{n/2}}$ with $\int_M \rho(x) \, d\Gamma(x) = 1$, we have that*

$$\int_M \left(|\nabla \psi|^2 - 3S + \psi - n \right) \rho \, d\Gamma \geq -\mu_e. \tag{5.7}$$

Here μ_e, as before, is a geometric invariant (in the sense of Section 2), which is the same for two isometric metrics. One can write in the dynamic form by considering the family of metrics $g(\tau)$ (in this case with $g(1)$ being the original metric, and $0 < \tau < \infty$) generated by the diffeomorphisms as for the shrinking case. Since it is the same inequality by re-scaling we omit it. Note that in the left hand side of (5.7) an equivalent integrand is

$$\tau \left(2\Delta\psi - |\nabla \psi|^2 - 3S \right) + \psi - n.$$

This expression also showed itself up in a differential Harnack (or Li-Yau-Hamilton) type calculation, in a recent preprint of Cao and Hamilton [CH] (where however the nonnegativity of the curvature operator is required). It is certainly interesting to explore the connections between the log-Sobolev inequality here and the Li-Yau-Hamilton type estimate for Ricci flow solution.

6 Gradient steady solitons

Now we consider the gradient steady solitons. Recall that a steady gradient soliton (M, g) has a potential function f satisfying that

$$R_{ij} = f_{ij}. \tag{6.1}$$

It was shown in [H1] that

$$|\nabla f|^2 + S = \lambda \tag{6.2}$$

for some λ. Similar as before there is a solution to Ricci flow $g(\tau)$ associated with the gradient steady soliton (M, g, f) [CLN]. We first need the following lemma to ensure the finiteness of $\int_M e^{-f}\, d\Gamma$ and other integrals later involved, under some geometric assumptions.

Lemma 6.1 *Let (M, g, f) be a gradient steady soliton. Assume that there exists a point $o \in M$ such that $S(o) = \max_M S$ and either $\operatorname{Ric}(x) > 0$ for all $x \in M$, or $\operatorname{Ric} \geq 0$ and*

$$\limsup_{x\to\infty} S(x) < \max_M S.$$

Then o is a minimum of f and there exists $\delta > 0$ and $C = C(M, f)$ so that

$$f(x) \geq \delta r(x) - C. \tag{6.3}$$

Here $r(x)$ is the distance function to o.

Proof. For the first case, it was shown in [H1], Theorem 20.1 (the argument there actually requires $\operatorname{Ric} > 0$ even though it was not stated; it is also necessary shown by easy examples) that o is the unique minimum of f. Note that for any geodesic $\gamma(s)$ from o, we have that

$$\frac{d^2}{ds^2}(f(\gamma(s)) = \operatorname{Ric}(\gamma', \gamma') > 0.$$

Hence we have for any $s_0 > 0$, $\frac{d}{ds}(f(\gamma(s_0))) > 0$. Then $f(\gamma(s)) \geq \frac{d}{ds}(f(\gamma(s_0)))(s - s_0) + f(\gamma(s_0))$, which implies the desired lower estimate.

For the second case, the assumption already excludes the Ricci flat situation, on which clearly (6.3) fails for f being a constant. We first claim that under the assumption on the behavior of S at the infinity, $S(o) = \lambda$. Suppose it is not true, then $\max_M S < \lambda$ and $|\nabla f|^2 \geq \lambda - \max_M S$. Let $\sigma(u)$ be an integral curve of ∇f passing o with $\sigma(0) = o$. Direct calculation shows that $\frac{d}{du}\left(|\nabla f|^2(\sigma(u))\right) = 2\operatorname{Ric}(\nabla f, \nabla f)(\sigma(u)) \geq 0$. This shows that $|\nabla f|^2(\sigma(u)) = |\nabla f|^2(\sigma(0))$ for $u \leq 0$ since $|\nabla f|^2$ has its minimum at o. Hence we have that $S(\sigma(u)) = \max_M S$ for all $u \leq 0$. However since $-f(\sigma(u)) = -f(\sigma(0)) + \int_u^0 |\nabla f|^2\, du = -f(\sigma(0)) - u|\nabla f|^2(\sigma(0)) \to +\infty$ as $u \to -\infty$ we can conclude that $\sigma(u) \to \infty$. This is a contradiction with the assumption that $\limsup_{x\to\infty} S(x) < \max_M S$. Hence we have that $\lambda = \max_M S$ which implies $\nabla f = 0$ at o and

$$\liminf_{x\to\infty} |\nabla f|^2 \geq 2\eta^2 \doteqdot \lambda - \limsup_{x\to\infty} S(x) > 0.$$

By considering any minimizing geodesic $\gamma(s)$ emitting from o and the fact $\frac{d}{ds}(f(\gamma(0))) = 0$ and $\frac{d^2}{ds^2}(f(\gamma(s))) \geq 0$, it is clear that o is the minimal point of f and $\langle \nabla f, \nabla r\rangle(x) \geq 0$ for any $x \in M \setminus \{o\}$. Let R_0 be such that $|\nabla f|^2(x) \geq \eta^2$ for all $x \in M \setminus B(o, R_0)$. Consider again a integral curve $\sigma(u)$ passing x. Since $|\nabla f|$ is bounded and M is complete, the curve is defined for all $-\infty < u < +\infty$. Notice that $\sigma(u) \in B(o, r(x))$ for all $u \leq 0$ and

$$f(\sigma(0)) - f(\sigma(u)) = \int_u^0 |\nabla f|^2 \, du \geq (-u)\eta^2$$

as along as $\sigma(u) \in M \setminus B(o, R_0)$. From this we infer that there exist some u_0 such that $\sigma(u_0) \in B(o, R_0)$. On the other hand

$$\begin{aligned} f(x) &= f(\sigma(u_0)) + \int_{u_0}^0 |\nabla f|^2 \, du \\ &\geq f(\sigma(u_0)) + \eta \int_{u_0}^0 |\sigma'(u)| \, du \\ &\geq f(\sigma(u_0)) + \eta d(x, \sigma(u_0)). \end{aligned}$$

This implies the desired lower estimate. □

Remark 6.1 *If the sectional curvature of (M, g) is nonnegative, one can show that the claim of the lemma holds under the assumption that $S(o) = \max_M S$, as far as M does not admit any flat factor $\mathbb{R}^k$. The reason is the following. First if the claimed result fails, one can conclude that f_{ij} has an eigenvector corresponding to the zero eigenvalue somewhere. Note that for the associated Ricci flow, the function $f(x, \tau)$, defined as the pull back via the diffeomorphism generated by ∇f, satisfies the heat equation (cf.* [CLN] *for details). Then the result follows from the strong tensor maximum principle and splitting theorem on noncompact manifolds proved in* [N1].

In the both cases o is a minimum point of f and λ is a geometric invariant, namely $\max_{x \in M} S(x)$. Also we have seen that both $|\nabla f|$ and $|\Delta f|$ are bounded. We normalize f so that $\int_M e^{-f} \, d\Gamma = 1$. Integration by parts gives the following weighted Poincaré inequality.

Proposition 6.1 *Let (M, g, f) be a gradient steady soliton. Then for any compact supported smooth function $u = e^{-\psi}$ with $\int_M u \, d\Gamma = 1$, we have that*

$$\int_M \left(|\nabla \psi|^2 - 3S \right) u \, d\Gamma \geq -\lambda.$$

Proof. The proof follows from the following simple calculation:

$$\begin{aligned} \int_M \left(|\nabla \psi|^2 - 3S \right) u &= \int_M \left(|\nabla \psi|^2 - 2\langle \nabla \psi, \nabla f \rangle + |\nabla f|^2 + 2\Delta f - |\nabla f|^2 - 3S \right) u \\ &\geq \int_M \left(2\Delta f - |\nabla f|^2 - 3S \right) u = -\lambda. \end{aligned}$$

□

This is a sharp inequality, at least it is so under the assumption of Lemma 6.1, since for this case the equality holds when $u = e^{-f}$. An equivalent form is that

$$\int_M \left(4|\nabla \varphi|^2 - 3S\varphi^2 \right) \, d\Gamma \geq -\lambda \int_M \varphi^2 \, d\Gamma$$

for any $\varphi \in L^2(M)$.

The weighted Poincaré inequality and its geometric meanings have been studied recently by, for example Li and Wang [LW].

7 An analogue of the c-theorem

For the re-normalization group flow, there exists the so-called central charge $c(t)$ invariant [Z] for the flow such that it is monotone non-increasing along the flow. Moreover it is always nonnegative. For Ricci flow, there are Perelman's monotonic quantities such as the $\mathcal{W}(g, \sigma, f)$-entropy, defined as

$$\mathcal{W}(g, \sigma, \varphi) \doteqdot \int_M \left(\sigma(|\nabla \varphi|^2 + S) + \varphi - n\right) u \, d\Gamma$$

for any $u = \frac{e^{-\varphi}}{(4\pi\sigma)^{\frac{n}{2}}}$ with $\int_M u = 1$, and associated $\mu(g, \sigma) \doteqdot \inf_{\int_M u=1} \mathcal{W}(g, \sigma, \varphi)$, $\nu(g) \doteqdot \inf_{\sigma>0} \mu(g, \sigma)$ invariants, as well as the so-called reduced volume. (The quantity $\mathcal{W}(g, \sigma, \varphi)$, $\mu(g, \sigma)$ and $\nu(g)$ may not be finite when M is not compact. Proposition 4.1 and Proposition 4.2 ensures that is the case for the shrinkers with bounded nonnegative Ricci curvature.) The reduced volume is always nonnegative by the definition. However, it is monotone non-decreasing in stead of non-increasing along the flow. Utilizing the sharp logarithmic Sobolev inequalities proved for the shrinkers and expanders we shall show in this section that the logarithmic Sobolev constants μ_s and μ_e are nonnegative, at least for the gradient shrinking/expanding solitons with bounded nonnegative Ricci curvature (this amounts to say that the reduced volume is always less than 1). In view of the monotonicity of the entropy, and the fact that the gradient shrinking solitons almost always arises at the singularity, one can view the monotonicity of the entropy together with the result proved here as an analogue of the c-theorem. Namely, for the solution to the Ricci flow, one can view $-\mu(g, \tau)$ as the analogue of the $c(t)$-invariant. Perelman's entropy formula concludes that it is monotone non-increasing. Our result concludes that $-\mu(g, 1) = \mu_s$ and it is nonnegative for the shrinkers with bounded curvature tensor and nonnegative Ricci.

We shall show two results on the sign of the invariants μ_s and μ_e. The case of μ_e is an easy application of an early rigidity result in [N2].

Proposition 7.1 *Let (M, g, f) be a gradient expanding soliton with* $\mathrm{Ric} \geq 0$. *Then $\mu_e \geq 0$. If $\mu_e = 0$ then (M, g) must be isometric to $\mathbb{R}^n$.*

Proof. Assuming that $\mu_e \leq 0$, Theorem 5.3 then implies that

$$\int_M \left(|\nabla \psi|^2 + \psi - n\right) \rho \, d\Gamma \geq 0.$$

Then by the proof of Theorem 1.4 of [N2] (one can see a detailed account in [Chow, et al1], pages 314–333), we can conclude that (M, g) is isometric to $\mathbb{R}^n$, on which $\mu_e = 0$. $\square$

Similar result holds for gradient shrinking solitons. For that we have to assume that the curvature tensor of (M, g) is uniformly bounded.

Theorem 7.1 *Let (M, g) be a gradient shrinking soliton with bounded curvature and nonnegative Ricci curvature. Let f be the normalized potential function as above. Then $\mu_s \geq 0$.*

Proof. Recall from the introduction that there is an associated solution $g(t)$ (with $-\infty < t < 0$, $t = \eta - 1$) to Ricci flow generated by pulling back the metric via the diffeomorphisms generated by the vector field ∇f. The original metric g corresponds to the one $g(-1)$ (meaning $t = -1$). Proposition 4.1 and Proposition 4.2 imply that $\mu(g(-1), \sigma)$ and $\nu(g(-1))$ are finite. Since $g(t)$ is just the re-scale of $g(-1)$, we have that for any $-\infty < t < 0$, $\mu(g(t), \sigma)$, $\nu(g(t), \sigma)$ are also finite. Now let $H(y, t; x, t_0)$ (with $t < t_0 < 0$) be the (minimal) positive fundamental solution to the conjugate heat equation:

$$\left(-\frac{\partial}{\partial t} - \Delta_y + S(y,t)\right) H(y, t; x, t_0) = 0$$

being the $\delta_x(y)$ at $t = t_0$. By a result of Perelman, Corollary 9.3 of [P] (see also [CTY] and [N4]), we know that

$$v_H(y,t) \doteqdot (t_0 - t)\left(2\Delta\varphi - |\nabla\varphi|^2 + S\right) + \varphi - n \le 0$$

with $H(y, t; x, t_0) = \frac{e^{-\varphi(y,t)}}{(4\pi(t_0-t))^{\frac{n}{2}}}$. This would implies in particular

$$\mu(g(-1), t_0 + 1) \le \int_M v_H(y, -1) H(y, -1)\, d\Gamma_{g(-1)} \le 0.$$

On the other hand Theorem 1.1 asserts that $\mu(g(-1), 1) \ge -\mu_s$. The result would follow if we show that $\mu(g(-1), t_0 + 1) \to \mu(g(-1), 1)$ as $t_0 \to 0$. For $t_{0,i} \to 0$, consider the the minimizer φ_i for $\mathcal{W}(g(-1), 1 + t_{0,i}, \varphi)$ (for simplicity we write $g(-1)$ back to g from now on). Let $\sigma_i = 1 + t_{0,i} \to 1$. We assume that $\frac{1}{2} \le \sigma_i \le 1$. By Proposition 4.2 and the above we have that

$$0 \ge \mu(g, \sigma_i) \ge -\mu_s - nA.$$

Write $w_i = e^{-\varphi_i/2}$. $w_i \in W^{1,2}(M)$. The Euler-Lagrangian equation is

$$-4\sigma_i \Delta w_i + \sigma_i S w_i - n w_i - 2 w_i \log w_i = \mu(g, \sigma_i) w_i \tag{7.1}$$

for $\int_M w_i^2 = (4\pi\sigma_i)^{\frac{n}{2}} \le (4\pi)^{\frac{n}{2}}$. Integrating over M we have that

$$4\sigma_i \int_M |\nabla w_i|^2 = \mu(g, \sigma_i)(4\pi\sigma_i)^{\frac{n}{2}} + \int_M \left(w_i^2 \log w_i^2 + n w_i^2 - \sigma_i S w_i^2\right)$$

which implies

$$4\sigma_i \int_M |\nabla w_i|^2 \le \int_M w_i^2 \log w_i^2 + n(4\pi)^{\frac{n}{2}} \tag{7.2}$$

On the other hand, writing $\frac{w_i^2}{(4\pi\sigma_0)^{\frac{n}{2}}} = \frac{e^{-\tilde{\psi}}}{\pi^{\frac{n}{2}}}$ and using that $\mathcal{W}(g, \frac{1}{4}, \tilde{\psi}) \ge \mu(g, \frac{1}{4})$,

$$\int_M |\nabla w_i|^2 \ge (4\pi\sigma_i)^{\frac{n}{2}} \mu(g, \frac{1}{4}) - \frac{nA}{4}(4\pi\sigma_i)^{\frac{n}{2}} + \int_M w_i^2 \log w_i^2.$$

Combining with (7.2), one can find $C = C(A, n)$ such that

$$\int_M |\nabla w_i|^2 \le C(A, n)$$

which implies that $\|w_i\|_{W^{1,2}(M)}$ is uniformly bounded, which implies that $w_i \to w_\infty$ in the the dual norm of $W^{1,2}(M)$ and strongly in $L^2(M)$, for some $w_i \in W^{1,2}(M)$. Due to the bound $\mu(g,\sigma_i)$ we may also assume that $\mu(g,\sigma_i) \to \mu_\infty(g)$. Clearly $\mu_\infty(g) \le 0$. It is evident that $\int_M w_\infty^2 = (4\pi)^{\frac{n}{2}}$. We shall show that on every compact subset K, after passing to subsequences, w_i converges to some w_∞, say in C^0-fashion. This will imply that w_∞ satisfies the equation

$$-4\Delta w_\infty + Sw_\infty - nw_\infty - 2w_\infty \log w_\infty = \mu_\infty(g) w_\infty.$$

Integration by parts, one can then have that

$$\int_M \left(4|\nabla w_\infty|^2 + Sw_\infty^2 - 2w_\infty^2 \log w_\infty - nw_\infty^2\right) = \mu_\infty(g)(4\pi)^{\frac{n}{2}}.$$

This implies that $\mu_\infty(g) \ge \mu(g,1)$. This however is enough to conclude that $\mu_s \ge 0$ since $0 \ge \mu_\infty(g) \ge \mu(g,1) \ge -\mu_s$. The claim that $w_i \to w_\infty$ in C^0 norm can be proved using Sobolev embedding theorem (over compact region K), interior L^p-estimates, and the compactness of the Sobolev embedding. We leave the details to the interested reader. One can also find this in the forthcoming book [Chow, et al2]. □

When f = constant, (M,g) is a compact Einstein manifold with $\mathrm{Ric}_M = \frac{1}{2} g_M$. The theorem concludes that

$$\mu_s = \frac{n}{2} - \log(V(M)) + \frac{n}{2}\log(4\pi) \ge 0$$

where $V(M)$ is the volume of (M, g_M). Among all such manifolds the sphere $\mathbb{S}^n$ has the smallest μ_s. In this case μ_s has the limit $\frac{1}{2}\log\frac{e}{2}$ as $n \to \infty$, at least for the case that n is even.

Remark 7.2 *We believe that the similar result holds without assuming that Ricci curvature is nonnegative and shall return to this in a future study. One should also consult [T] for the definition of an 'entropy' for the renormalization group flow motivated by Perelman's entropy and its formula for Ricci flow. For the steady gradient soliton, it is clear that $\lambda \ge 0$ for any steady solitons with $S \ge 0$.*

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