AVERAGE OF THE MEAN CURVATURE INTEGRAL IN COMPLEX SPACE FORMS

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Abstract. In real space forms, given a regular hypersurface S, it is known that the integral over the space of s-planes of the mean curvature integral of intersections with S is a multiple of the mean curvature integral of S. We study the corresponding expression in a complex space form. The role of the s-planes is played by the complex s-planes. We prove that the same property does not hold but another term appears. We express this term as the integral of the normal curvature in the direction obtained from applying the complex structure to the normal direction. As an application, we give the measure of the set of complex lines meeting a compact domain in a complex space form, and we characterize the reproductive continuous invariant valuations of degree $2n-2$ in the standard Hermitian space.

1. Introduction

Let $\mathbb{M}^n(k)$ be the n-dimensional simply connected Riemannian manifold of constant sectional curvature k. The space of s-dimensional totally geodesic submanifolds of $\mathbb{M}^n(k)$, $\mathcal{L}_s^{\mathbb{R}}$, is a homogeneous space with a unique (up to a constant factor) measure dL_s invariant under the isometry group of $\mathbb{M}^n(k)$. If S is a compact oriented hypersurface of class $\mathcal{C}^2$, mean curvature integrals are defined as

$$M_r(S) = \binom{n-1}{r}^{-1} \int_S \sigma_r(\mathrm{II}) dx$$

where $\sigma_r(\mathrm{II})$ denotes the r-th symmetric elementary function of the eigenvalues of the second fundamental form with respect to the normal field giving the orientation. Santaló [San04] proved that on $\mathbb{M}^n(k)$ the mean curvature integrals satisfy the so-called reproductive property. That is,

$$\int_{\mathcal{L}_s^{\mathbb{R}}} M_r^{(s)}(S \cap L_s) dL_s = c M_r(S)$$

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where c is known and depends only on n, r and s and $M_r^{(s)}(S \cap L_s)$ denotes the r-th mean curvature integral of $S \cap L_s$ as a hypersurface in L_s. The normal vector taken on $S \cap L_s$ is the one making an acute angle with the normal vector chosen on S.

In this paper we prove that this reproductive property is not satisfied in complex space forms when $r = 1$. We denote (simply connected) complex space forms with constant holomorphic sectional curvature 4ϵ by $\mathbb{CK}^n(\epsilon)$. A complex space form is isometric to the standard Hermitian space, $\mathbb{C}^n$, if $\epsilon = 0$, to a complex projective space, $\mathbb{CP}^n$, if $\epsilon > 0$ or to a complex hyperbolic space, $\mathbb{CH}^n$, if $\epsilon < 0$. (As usual, we will denote by J the complex structure of the manifold.)

On $\mathbb{CK}^n(\epsilon)$, we denote by $\mathcal{L}_s$ the space of totally geodesic complex submanifolds of complex dimension s. Each $L_s \in \mathcal{L}_s$ is isometric to $\mathbb{CK}^s(\epsilon)$. The space $\mathcal{L}_s$ is a homogeneous space and admits a unique (up to a constant factor) invariant measure dL_s.

Notation 1.1. We denote the volume of the n-dimensional Euclidean ball of radius 1 by ω_n and the volume of the n-dimensional Euclidean sphere of radius 1 by O_n.

The main result of this paper gives an expression for the integral over $\mathcal{L}_s$ of the mean curvature integral of the intersection with a compact oriented hypersurface of class $\mathcal{C}^2$.

Theorem 1.2. *Let $S \subset \mathbb{CK}^n(\epsilon)$ be a compact oriented hypersurface of class $\mathcal{C}^2$ (possibly with boundary) and let $s \in \{1, \dots, n-1\}$. Then*

$$\int_{\mathcal{L}_s} M_1^{(s)}(S \cap L_s) dL_s = \frac{\omega_{2n-2}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{2s(2s-1)} \binom{n}{s}^{-1} \cdot \\ \cdot \left((2n-1)\frac{2ns - n - s}{n-s} M_1(S) + \int_S k_n(JN) \right)$$

where N is a normal vector field to S and $k_n(JN)$ denotes the normal curvature in the direction $JN \in TS$.

The main tool in the proof is the use of moving frames adapted to $S \cap L_s$, L_s or S.

Theorem 1.2 shows that the reproductive property of the mean curvature integral is not satisfied in complex space forms. The non-reproduction can be explained by the theory of valuations (see Section 2.2 for definitions). Let us denote by $\mathcal{K}(V)$ the family of non-empty compact convex subsets of a finite n-dimensional real vector space V.

Definition 1.3. A *valuation on V* is a scalar real valued functional $\phi : \mathcal{K}(V) \to \mathbb{R}$ which satisfies

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$$

whenever $A, B, A \cup B \in \mathcal{K}(V)$.

In 1957 Hadwiger proved the following result concerning valuations.

Theorem ([Had57])**.** *A basis of the space of continuous translation and $O(n)$ invariant valuations of $\mathbb{R}^n$ is*

$$\chi, M_1, M_2, \dots, M_{n-2}, \text{area}, \text{vol}$$

where M_i denotes the continuous extension to $\mathcal{K}(\mathbb{R}^n)$ of the i-th mean curvature integral of the boundary.

But, on the standard Hermitian space $\mathbb{C}^n$ with its isometry group $IU(n) = \mathbb{C}^n \rtimes U(n)$ Alesker [Ale03] proved that there are more linearly independent continuous invariant valuations than on $\mathbb{R}^{2n}$.

Theorem (Theorem 2.1.1 [Ale03])**.** *Let $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$ be the space of continuous translation invariant valuations on $\mathbb{C}^n$ invariant also under $U(n)$. Then*

$$\dim \mathrm{Val}^{U(n)}(\mathbb{C}^n) = \binom{n+2}{2}$$

and the dimension of the subspace of $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$ of homogeneous valuations of degree k is $\left\lfloor \frac{\min\{k, 2n-k\}}{2} \right\rfloor + 1$.

In [BF08], Bernig and Fu study further the space of continuous valuations on $\mathbb{C}^n$ and give some new bases.

In the following, when we need a compact domain Ω and not just its boundary, for simplicity we will restrict to compact domains with smooth boundary and call them *regular domains*.

On $\mathbb{C}^n$ the function $\int_{\mathcal{L}_s} M_1^{(s)}(\partial\Omega \cap L_s) dL_s$ is a homogeneous valuation of degree $2n-2$. By the last theorem the dimension of the space of homogeneous invariant valuations of degree $2n-2$ is 2. On the other hand, valuations $M_1(\partial\Omega)$ and $\int_{\partial\Omega} k_n(JN)$ are linearly independent. Hence, the expression in Theorem 1.2 in $\mathbb{C}^n$ has all the possible terms.

As $\int_{\partial\Omega} k_n(JN)$ is an invariant valuation on $\mathbb{CK}^n(\epsilon)$ linearly independent from $M_1(\partial\Omega)$, it is also natural to study the integral over $\mathcal{L}_s$ of this valuation instead of the mean curvature integral.

Theorem 1.4. *Let $S \subset \mathbb{CK}^n(\epsilon)$ be a compact oriented hypersurface of class $\mathcal{C}^2$ (possibly with boundary) and let $s \in \{1, \dots, n-1\}$. Then*

$$\int_{\mathcal{L}_s} \int_{S \cap L_s} \tilde{k}_n(J\tilde{N}) dp dL_s = \frac{\omega_{2n-2}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{2s} \binom{n}{s}^{-1} \cdot \left(\frac{2sn - s - n}{n-s} \int_S k_n(JN) + (2n-1) M_1(S) \right).$$

where $\tilde{N}$ denotes a normal vector field to $S \cap L_s \subset L_s$, N a normal vector field to $S \subset \mathbb{CK}^n(\epsilon)$, $\tilde{k}_n(J\tilde{N})$ the normal curvature of $J\tilde{N}$ in L_s and $k_n(JN)$ the normal curvature of JN in $\mathbb{CK}^n(\epsilon)$.

In this situation arise two natural questions: which are the invariant valuations $\mu(\Omega)$ of degree $2n-2$ satisfying the reproductive property? Which valuation do we have to integrate to obtain the mean curvature integral of the whole domain?

In this paper we prove the following result in $\mathbb{CK}^n(\epsilon)$.

Theorem 1.5. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain, let $s \in \{1, \dots, n-1\}$ and let*

$$\varphi_1(\Omega) = M_1(\partial\Omega) - \int_{\partial\Omega} k_n(JN)$$

and

$$\varphi_2(\Omega) = (2s-1)(2n-1)M_1(\partial\Omega) + \int_{\partial\Omega} k_n(JN).$$

Then,

$$\int_{\mathcal{L}_s} \varphi_1(\Omega \cap L_s) dL_s = \frac{\omega_{2n-2}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})(s-1)(2n-1)}{(2s-1)(n-s)} \binom{n}{s}^{-1} \varphi_1(\Omega)$$

and

$$\int_{\mathcal{L}_s} \varphi_2(\Omega \cap L_s) dL_s = \frac{\omega_{2n-2}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{(2s-1)} \binom{n-2}{s-1}^{-1} \varphi_2(\Omega)$$

Thus, each of $\varphi_1(\Omega)$, $\varphi_2(\Omega)$ expands a 1-dimensional subspace of reproductive valuations when we integrate over $\mathcal{L}_s$. In $\mathbb{C}^n$, φ_1 and φ_2 are all the reproductive continuous invariant valuations of degree $2n-2$.

Moreover, if

$$\varphi(\Omega) = (2ns - n - s)M_1(\partial\Omega) - \frac{n-s}{2s-1}\int_{\partial\Omega} k_n(JN),$$

then

$$\int_{\mathcal{L}_s} \varphi(\partial\Omega \cap L_s) dL_s = \frac{O_{2n-3}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})(2n-1)(s-1)}{(2s-1)} \binom{n-1}{s}^{-1} M_1(\partial\Omega).$$

In [Ale03], Alesker defined some bases of valuations, one of them is given by the valuations $\{U_{k,p}\}$ which we define on Section 2.2. Alesker also stated the following theorem:

Theorem 1.6 (Theorem 3.1.2 [Ale03])**.** *Let Ω be a regular domain in $\mathbb{C}^n$. Let $0 < q < n$, $0 < 2p < k < 2q$. Then*

$$\int_{L_q \in \mathcal{L}_q} U_{k,p}(\Omega \cap L_q) = \sum_{p=0}^{[k/2]+n-q} \gamma_p \cdot U_{k+2(n-q),p}(\Omega),$$

where the constants γ_p depend only on n, q, and p.

The value of the constants γ_p remains unknown. Using the results in this paper we shall give the constants for all $q \in \{1, \dots, n-1\}$ and $k = 2q-2$.

As another application of Theorems 1.2 and 1.4 we shall give the measure of the set of complex lines meeting a regular domain in $\mathbb{CK}^n(\epsilon)$ in terms of $M_1(\partial\Omega)$, $\int_{\partial\Omega} k_n(JN)$ and the volume (volume is necessary if $\epsilon \neq 0$).

Proposition 1.7. *Let Ω be a regular domain in $\mathbb{CK}^n(\epsilon)$. Then*

$$\int_{\mathcal{L}_1} \chi(\Omega \cap L_1) dL_1 = \frac{\omega_{2n-2}}{2O_1 n} \left((2n-1) M_1(\partial\Omega) + \int_{\partial\Omega} k_n(JN) + 8n\epsilon \mathrm{vol}(\Omega) \right).$$

This proposition is a particular case of a general result, obtained using a different approach, for the measure of the set of complex s-planes meeting a regular domain given in [AGS09].

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2. Preliminaries

2.1. **Moving frames.** Let U be manifold and $p \in U$. In a complex space form $\mathbb{CK}^n(\epsilon)$, a local orthonormal frame defined on U is a map $\{q; e_1, \ldots, e_{2n}\} : U \to (T\mathbb{CK}^n(\epsilon))^{2n}$, such that $\{e_1, \ldots, e_{2n}\}$ is an orthonormal basis of $T_{q(p)}\mathbb{CK}^n(\epsilon)$.

Let $U \subset \mathbb{CK}^n(\epsilon)$ be an open set containing $p \in \mathbb{CK}^n(\epsilon)$. We consider a local orthonormal frame $\{q; e_1, \ldots, e_{2n}\}$ in U with $q = \mathrm{id}$. We define the 1-forms $\{\omega_i\}$ as the dual forms of $\{e_i\}$ and the 2-forms $\{\omega_{ij}\}$ as the connection forms of $\mathbb{CK}^n(\epsilon)$ with respect to the Levi-Civita connection.

Now, let U be an open set of $\mathcal{L}_s$, the space of totally geodesic complex submanifolds of complex dimension s. We shall use an expression for the measure dL_s of $\mathcal{L}_s$, in terms of a moving frame defined on $U \subset \mathcal{L}_s$. We take a local orthonormal frame $\{q; e_1, \ldots, e_{2n}\}$ in U such that $q(L_s) = p$ and $\{e_1, \ldots, e_{2s}\}$ are an orthonormal basis of $T_p L_s$ with $Je_{2k-1} = e_{2k}$, $k \in \{1, \ldots, s\}$. Then, the measure of $\mathcal{L}_s$ is given by (cf. [San52])

$$dL_s = \omega_{2s+1} \wedge \omega_{2s+2} \wedge \cdots \wedge \omega_{2n-1} \wedge \omega_{2n} \wedge \bigwedge_{\substack{k=1,2,\ldots,s \\ j=2s+1,\ldots,2n}} \omega_{2k-1,j}.$$

The chosen normalization of the measure dL_s satisfies

$$\int_{\mathcal{L}_s} dL_s = \mathrm{vol}(G^{\mathbb{C}}_{n+1,s+1}). \tag{1}$$

We will also consider the space $\mathcal{L}_{s[p]}$ of totally geodesic complex submanifolds of complex dimension s containing a point $p \in \mathbb{CK}^n(\epsilon)$. If

we define a local orthonormal frame in $U \subset \mathcal{L}_{s[p]}$ in the same way as in $\mathcal{L}_s$, then the measure of $\mathcal{L}_{s[p]}$ is given by (cf. [San52])

$$dL_{s[p]} = \bigwedge_{\substack{k=1,2,\dots,s \\ j=2s+1,\dots,2n}} \omega_{2k-1,j}.$$

2.2. **Valuations.** See, for example, [KR97] or [Ale07] for a detailed treatment and references.

A valuation ϕ is said to be *homogeneous of degree* $k \in \mathbb{R}$ if

$$\phi(\lambda K) = \lambda^k \phi(K) \text{ for any } \lambda > 0,\ K \in \mathcal{K}(V).$$

Let us denote by $\mathrm{Val}(V)$ the space of *translation invariant continuous* (with respect to the Hausdorff topology) valuations. The Euler characteristic and the volume are translation invariant continuous valuations on any real vector space. Mean curvature integrals are also translation invariant continuous valuations on $\mathbb{R}^n$ and $\mathbb{C}^n$.

In [Ale03], Alesker gives two different bases of $\mathrm{Val}^{U(n)}(\mathbb{C}^n)$, the space of translation invariant continuous valuations invariant under $U(n)$. One of these two bases is defined as follows. Let k, p be two integers such that $0 \leq 2p \leq k \leq 2n$. Then $\{U_{k,p}\}_{\{k,p\}}$ is a basis if

$$U_{k,p}(\Omega) = \frac{1}{2(n-p)\omega_{2n-k}} \int_{\mathcal{L}_{n-p}} M_{2n-k-1}(\partial\Omega \cap L_{n-p}) dL_{n-p}. \tag{2}$$

The subscript k in $U_{k,p}$ denotes the homogeneous degree of the valuation.

3. Average of the mean curvature integral over complex s-planes

3.1. **Preliminary lemmas.** In this section we state the lemmas we shall use to prove Theorems 1.2 and 1.4.

Lemma 3.1. *Let E be a complex vector space of dimension 2 endowed with an inner product $\langle\,,\rangle$ and let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of E. Then $\langle e_a, Je_b\rangle^2 = \langle e_c, Je_d\rangle^2$ with $\{a,b,c,d\} = \{1,2,3,4\}$.*

Proof. We express Je_b and Je_d in terms of the orthonormal basis and we use that $\langle Je_b, Je_b\rangle = \langle Je_d, Je_d\rangle = 1$, $\langle Je_b, Je_d\rangle = 0$ and $\langle Je_b, e_d\rangle = -\langle Je_d, e_b\rangle$. □

Lemma 3.2. *If $u \in S^{2n-3}$, then*

$$\int_{S^{2n-3}} \langle u, v\rangle dv = 0$$

and

$$\int_{S^{2n-3}} \langle u, v\rangle^2 dv = \frac{O_{2n-3}}{2n-2}.$$

Proof. The first equality follows since the integral is over an odd function. For the second one, we decompose $u = \cos\theta v + \sin\theta w$ with $w \in \langle v\rangle^\perp$, then using polar coordinates with u the first vector of the polar coordinates system, we have

$$\int_{S^{2n-3}} \langle u, v\rangle^2 dS_{2n-3} = O_{2n-4}\int_0^\pi \cos^2\theta \sin^{2n-4}\theta d\theta = O_{2n-4}\frac{O_{2n-4+2+1}}{O_2 O_{2n-4}}$$
$$= \omega_{2n-2} = \frac{O_{2n-3}}{2n-2}.$$

□

Lemma 3.3. *Suppose S is a hypersurface of class $\mathcal{C}^2$ in a Riemannian manifold M and $L \subset T_pM$ is a subspace such that $\exp_p L$ is a $(r+1)$-dimensional submanifold of M intersecting S at p, then $\sigma_i(II|_u)$, the i-th symmetric elementary function of S restricted to $u = T_pS \cap L$, is related to $\tilde{\sigma}_i(II|_C)$, the i-th symmetric elementary function of the hypersurface $C = S \cap \exp_p L \subset \exp_p L$ by*

$$\sigma_i(II|_u) = \cos^i\theta\tilde{\sigma}_i(II|_C)$$

where θ denotes the angle between a normal vector of S and a normal vector of u at p .

Proof. If $A \subset B \subset M$ are submanifolds, then we denote the second fundamental form of A as a submanifold of B by $h_A^B : T_pA \times T_pA \to (T_pA)^\perp$. If $B = M$, we just put h_A instead of h_A^M.

Let N be a normal vector to S. Then for all $X, Y \in T_pC$

$$h_C(X,Y) = h_C^L(X,Y) + h_L(X,Y) = h_C^L(X,Y)$$

since L is a totally geodesic submanifold of M, but also

$$h_C(X,Y) = h_C^S(X,Y) + h_S(X,Y).$$

Note that $h_C^S(X,Y)$ is a multiple of a normal vector to C in S, so $\langle h_C^S(X,Y), N\rangle = 0$ (for $X, Y \in T_pC$).

If $X, Y \in T_pC$, then

$$\mathrm{II}_S(X,Y) := \langle h_S(X,Y), N\rangle = \langle h_C(X,Y) - h_C^S(X,Y), N\rangle$$
$$= \langle h_C(X,Y), N\rangle = \langle h_C^L(X,Y), N\rangle = \langle \mathrm{II}_C^L(X,Y)n, N\rangle$$

where n denotes a normal vector of C in L. So,

$$\mathrm{II}_C^L(X,Y) = \frac{1}{\langle N, n\rangle}\mathrm{II}_S(X,Y). \tag{3}$$

Since $\tilde{\sigma}_i$ is the sum of the minors of order i of II_C^L, by replacing by (3) each entry of the second fundamental form, we obtain the result. □

The following lemma generalizes in complex space forms a result given by Langevin and Shifrin [LS82] in real space forms.

Lemma 3.4. *Let E be a complex vector space of complex dimension n and let II be a bilinear form defined on E. We denote by $G^{\mathbb{C}}_{n,s}$ the Grassmanian of s-dimensional complex planes on E. Then,*

$$\int_{G^{\mathbb{C}}_{n,s}} \mathrm{tr}(II|_V)dV = \frac{s\,\mathrm{vol}(G^{\mathbb{C}}_{n,s})}{n}\mathrm{tr}(II|_E).$$

Proof. First, recall that

$$U(n-s)\times U(s) \longrightarrow U(n) \longrightarrow G^{\mathbb{C}}_{n,s} \tag{4}$$

is a fibration for each $s \in \{1,\dots,n-1\}$.

We prove the case $\dim_{\mathbb{C}} V \leq \frac{n}{2}$ by induction on the complex dimension of V. The case $\dim_{\mathbb{C}} V > \frac{n}{2}$ can be proved using similar arguments.

Suppose $\dim_{\mathbb{C}} V = 1$, that is, $s = 1$. Then,

$$\int_{G^{\mathbb{C}}_{n,1}} \mathrm{tr}(\mathrm{II}|_V)dV = \frac{1}{\mathrm{vol}(U(n-1))\mathrm{vol}(U(1))}\int_{U(n)} \mathrm{tr}(\mathrm{II}|_{V_1^1})dU$$

since $\mathrm{tr}(\mathrm{II}|_{V_1^1})$ is constant along the fibers. We denote by V_1^1 the complex vector subspace generated by the first column of the matrix $U \in U(n)$. In general, for $U \in U(n)$, we will denote by V_a^b the complex vector subspace generated by the columns b to $b+a-1$. The subscript a denotes the dimension of V_a^b, or equivalently, the number of columns we consider and the upperscript b denotes from which column we start to consider them. Then

$$\begin{aligned}\int_{U(n)} \mathrm{tr}(\mathrm{II}|_{V_1^1})dU &= \frac{1}{n}\int_{U(n)} (\mathrm{tr}(\mathrm{II}|_{V_1^1}) + \mathrm{tr}(\mathrm{II}|_{V_1^2}) + \cdots + \mathrm{tr}(\mathrm{II}|_{V_1^n}))dU \\ &= \frac{1}{n}\int_{U(n)} \mathrm{tr}(\mathrm{II}|_E)dU = \frac{\mathrm{vol}(U(n))}{n}\mathrm{tr}(\mathrm{II}|_E).\end{aligned}$$

Thus,

$$\int_{G^{\mathbb{C}}_{n,1}} \mathrm{tr}(\mathrm{II}|_V)dV = \frac{\mathrm{vol}(U(n))}{n\mathrm{vol}(U(n-1))\mathrm{vol}(U(1))}\mathrm{tr}(\mathrm{II}|_E) = \frac{\mathrm{vol}(G^{\mathbb{C}}_{n,1})}{n}\mathrm{tr}(\mathrm{II}|_E).$$

Suppose now that the result is true till $\dim_{\mathbb{C}} V = r-1$. We shall prove it for $\dim_{\mathbb{C}} V = r \leq \frac{n}{2}$. If R denotes the remainder of $\frac{n}{r}$, then $R < r$ and we can apply the induction hypothesis in R. Thus, using

similar arguments as before, we obtain

$$\begin{aligned}
\int_{G^{\mathbb{C}}_{n,r}} \operatorname{tr}(\mathrm{II}|_V) &= \frac{1}{\operatorname{vol}(U(n-r))\operatorname{vol}(U(r))}\int_{U(n)} \operatorname{tr}(\mathrm{II}|_{V_r^1}) \\
&= \frac{1}{\operatorname{vol}(U(n-r))\operatorname{vol}(U(r))\lfloor \frac{n}{r}\rfloor}\int_{U(n)} (\operatorname{tr}(\mathrm{II}|_{V_r^1})+\cdots+\operatorname{tr}(\mathrm{II}|_{V_r^{n-R-r+1}})) \\
&= \frac{1}{\operatorname{vol}(U(n-r))\operatorname{vol}(U(r))\lfloor \frac{n}{r}\rfloor}\left(\int_{U(n)} \operatorname{tr}(\mathrm{II}|_E) - \int_{U(n)} \operatorname{tr}(\mathrm{II}|_{V_R^{n-R+1}})\right) \\
&= \frac{\operatorname{vol}(U(n))\operatorname{tr}(\mathrm{II}|_E) - \operatorname{vol}(U(n-R))\operatorname{vol}(U(R))\int_{G^{\mathbb{C}}_{n,R}} \operatorname{tr}(\mathrm{II}|_{V_R})}{\operatorname{vol}(U(n-r))\operatorname{vol}(U(r))\lfloor \frac{n}{r}\rfloor} \\
&= \frac{\left(\operatorname{vol}(U(n)) - \operatorname{vol}(U(n-R))\operatorname{vol}(U(R))\operatorname{vol}(G^{\mathbb{C}}_{n,R})\frac{R}{n}\right)\operatorname{tr}(\mathrm{II}|_E)}{\operatorname{vol}(U(n-r))\operatorname{vol}(U(r))\lfloor \frac{n}{r}\rfloor} \\
&= \operatorname{vol}(G^{\mathbb{C}}_{n,r})\frac{r}{n-R}\left(1-\frac{R}{n}\right)\operatorname{tr}(\mathrm{II}|_E) \\
&= \operatorname{vol}(G^{\mathbb{C}}_{n,r})\frac{r}{n}\operatorname{tr}(\mathrm{II}|_E)
\end{aligned}$$

and the result follows when $2s \leq n$. □

3.2. **Integral of the r-th mean curvature integral over complex s-planes.** The following proposition shall be essential to prove the main results, since it gives a first expression of the integral we would like to study in terms of an integral on the boundary of the domain.

Proposition 3.5. *Let $S \subset \mathbb{CK}^n(\epsilon)$ be a compact oriented hypersurface of class $\mathcal{C}^2$ (possibly with boundary) and let r, s be integers such that $1 \leq s < n$ and $1 \leq r \leq 2s-1$. Then*

$$\int_{\mathcal{L}_s} M_r^{(s)}(S \cap L_s)dL_s = \binom{2s-1}{r}^{-1}\int_S \int_{\mathbb{RP}^{2n-2}} \int_{G^{\mathbb{C}}_{n-2,s-1}} \frac{\langle JN, e_s\rangle^{2s-r}}{(1-\langle JN, e_s\rangle^2)^{s-1}}\sigma_r(p; e_s \oplus V)dV\, de_s dp,$$

where N denotes a normal vector field at T_pS, $e_s \in T_pS$, V denotes a complex $(s-1)$-plane by p contained at $\{N, JN, e_s, Je_s\}^\perp$, and $\sigma_r(p; e_s \oplus V)$ the r-th symmetric elementary function of the second fundamental form of S restricted to the real subspace $e_s \oplus V$.

Proof. Let L_s be a complex s-plane such that $S \cap L_s \neq \emptyset$ and let $p \in S \cap L_s$. We denote by $\tilde{\sigma}_r$ the r-th symmetric elementary function of the second fundamental form of $S \cap L_s$ as a hypersurface of L_s. Then, by definition

$$\int_{\mathcal{L}_s} M_r^{(s)}(S \cap L_s)dL_s = \binom{2s-1}{r}^{-1}\int_{S\cap L_s \neq \emptyset}\int_{S\cap L_s} \tilde{\sigma}_r(s)ds\, dL_s.$$

We shall prove the result using moving frames adapted to $S \cap L_s$, L_s or S.

Let $g = \{e_1, Je_1, e_2, Je_2, \ldots, e_s, w_s, e_{s+1}, Je_{s+1} \ldots, e_n, N\}$ be a moving frame adapted to $S \cap L_s$ and S. That is, $\{e_1, Je_1, \ldots, e_s\}$ is an orthonormal basis of $T_p(S \cap L_s)$, $\{e_{s+1}, Je_{s+1}, \ldots, e_n\}$ is an orthonormal basis of $T_p(S \cap L_s^{\perp})$, N is a normal vector field to TS and w_s completes to an orthonormal basis of $T_p\mathbb{CK}^n(\epsilon)$. We denote by

$$\{\omega_1, \omega_{\overline{1}}, \ldots, \omega_{s-1}, \omega_{\overline{s-1}}, \omega_s, \omega_{\tilde{s}}, \omega_{s+1}, \omega_{\overline{s+1}}, \ldots, \omega_n, \omega_{\tilde{n}}\}$$

the dual basis of the vectors in g and by $\{\omega_{ij}\}$ the connection forms.

Let $g' = \{e'_1 = e_1, e'_2 = Je_1, \ldots, e'_{2s-1} = e_s, e'_{2s} = Je_s, e'_{2s+1} = e_{s+1}, e'_{2s+2} = Je_{s+1}, \ldots, e'_{2n-1} = e_n, e'_{2n} = Je_n\}$ be a moving frame adapted to $S \cap L_s$ and L_s. That is, $\{e_1, Je_1, \ldots, e_s, Je_s\}$ is an orthonormal basis of T_pL_s. We denote by

$$\{\omega'_1, \omega'_{\overline{1}}, \ldots, \omega'_n, \omega'_{\overline{n}}\}$$

the dual basis of the vectors in g' and by $\{\omega'_{ij}\}$ the connection forms.

The relation between g' and g is given by

$$\begin{aligned} e'_{2s} &= Je_s = \langle Je_s, w_s\rangle w_s + \langle Je_s, N\rangle N, \\ e'_{2n} &= Je_n = \langle Je_n, w_s\rangle w_s + \langle Je_n, N\rangle N \end{aligned}$$

and $e'_{2j+1} = e_j$, $e'_{2j} = Je_j$ if $j \in \{1, \ldots, s-1, s+1, \ldots, n-1\}$.

Then

$$\left\{ \begin{array}{ll} \omega'_j & = \omega_j, \text{ if } j \neq \overline{s}, \overline{n}, \\ \omega'_{\overline{n}} & = \langle Je_n, w_s\rangle \omega_{\tilde{s}} + \langle Je_n, N\rangle \omega_{\tilde{n}} \end{array} \right. \tag{5}$$

and

$$\left\{ \begin{array}{lcl} \omega'_{i\overline{s}} & = & \langle Je_s, w_s\rangle \omega_{i\tilde{s}} + \langle Je_s, N\rangle \omega_{i\tilde{n}} \\ \omega'_{i\overline{n}} & = & \langle Je_n, w_s\rangle \omega_{i\tilde{s}} + \langle Je_n, N\rangle \omega_{i\tilde{n}}, \\ \omega'_{ij} & = & \omega_{ij}, \quad \text{otherwise.} \end{array} \right. \tag{6}$$

The expression of ds (the density of $S \cap L_s$), dL_s and $dL_{s[p]}$ in terms of ω' is

$$\begin{aligned} ds &= \omega'_1 \wedge \cdots \wedge \omega'_s, \\ dL_s &= \omega'_{s+1} \wedge \omega'_{\overline{s+1}} \wedge \cdots \wedge \omega'_n \wedge \omega'_{\overline{n}} \wedge \bigwedge_{\substack{i=1,2,\ldots,s \\ j=s+1,\overline{s+1},\ldots,n,\overline{n}}} \omega'_{ij}, \\ dL_{s[p]} &= \bigwedge_{\substack{i=1,2,\ldots,s \\ j=s+1,\overline{s+1},\ldots,n,\overline{n}}} \omega'_{ij} \end{aligned}$$

and the expression of dp (the density of S) in terms of ω is $dp = \omega_1 \wedge \omega_{\overline{1}} \wedge \cdots \wedge \omega_n$.

On the other hand, by Lemma 3.1 it is satisfied

$$|\langle Je_n, w_s\rangle| = |\langle Je_s, N\rangle|. \tag{7}$$

Indeed, vectors $\{e_s, w_s, e_n, n\}$ are an orthonormal basis of a 2-dimensional complex plane, the orthogonal complement of the space generated by $\{e_1, Je_1, \ldots, e_{s-1}, Je_{s-1}, e_{s+1}, Je_{s+1}, \ldots, e_{n-1}, Je_{n-1}\}$, so we can apply Lemma 3.1.

By relations (5) and (7) we get

$$|ds \wedge dL_s| = |\langle Je_n, w_s\rangle dL_{s[p]} \wedge dp| = |\langle JN, e_s\rangle dL_{s[p]} \wedge dp|$$

since $\omega_{\tilde{n}}$ vanishes on TS.

Then, by Lemma 3.3,

$$\begin{aligned}\int_{S\cap L_s\neq\emptyset} M_r^{(s)}(S\cap L_s)dL_s &= \binom{2s-1}{r}^{-1}\int_S\int_{L_{s[p]}} |\langle JN, e_s\rangle|\tilde{\sigma}_r(p)dL_{s[p]}dp \\ &= \binom{2s-1}{r}^{-1}\int_S\int_{L_{s[p]}} \frac{|\langle JN, e_s\rangle|}{\langle N, Je_s\rangle^r}\sigma_r(p)dL_{s[p]}dp.\end{aligned}$$

Now, we shall express $dL_{s[p]}$ in terms of $dG^{\mathbb{C}}_{n-2,s-1}\wedge dS_{2n-2}$. For every generic complex s-plane L_s containing $p\in S$, the submanifold $S\cap L_s$ is a hypersurface of L_s. If $\tilde{N}$ is a normal vector field of $S\cap L_s$ as a hypersurface in L_s, then $J\tilde{N}\in T(S\cap L_s)$. Thus, $J\tilde{N}$ is a well defined vector in TS for every generic L_s and we can define the map

$$\begin{array}{rccc}\phi: & \mathcal{L}_{s[p]} & \longrightarrow & S^{2n-2}\times G^{\mathbb{C}}_{n-2,s-1} \\ & L_{s[p]} & \mapsto & (J\tilde{N}, \{\tilde{N}, J\tilde{N}\}^{\perp}\cap L_{s[p]})\end{array}$$

which has inverse

$$\begin{array}{rccc}\phi^{-1}: & S^{2n-2}\times G^{\mathbb{C}}_{n-2,s-1} & \longrightarrow & \mathcal{L}_{s[p]} \\ & (v, L_{(s-1)[p]}) & \mapsto & \exp_p\{v, Jv, L_{(s-1)[p]}\}\end{array}.$$

The expression of dS_{2n-2} in terms of ω and the expression of $dG^{\mathbb{C}}_{n-2,s-1}$ in terms of ω' are

$$\begin{aligned} dS_{2n-2} &= \bigwedge_{j=1,\overline{1},\ldots,s-1,\overline{s-1},\tilde{s},s+1,\overline{s+1},\ldots,n} \omega_{sj} \\ dG^{\mathbb{C}}_{n-2,s-1} &= \bigwedge_{\substack{i=1,\ldots,s-1 \\ j=s+1,\overline{s+1},\ldots,n-1,\overline{n-1}}} \omega'_{ij} \end{aligned} \tag{8}$$

By (6) and (8) we have

$$\begin{aligned} dL_{s[p]} &= \bigwedge_{\substack{i=1,\dots,s \\ j=s+1,\overline{s+1},\dots,n,\overline{n}}} \omega'_{ij} \\ &= dG^{\mathbb{C}}_{n-2,s-1} \wedge \bigwedge_{i=1,..,s-1} \omega_{in} \wedge \bigwedge_{i=1,\dots,s} \omega'_{i\overline{n}} \wedge \bigwedge_{j=s+1,\overline{s+1},\dots,\overline{n-1},n} \omega_{sj}, \\ dS_{2n-2} &= \bigwedge_{j=s+1,\overline{s+1},\dots,\overline{n-1},n} \omega_{sj} \wedge \bigwedge_{j=1,\overline{1},\dots,\overline{s-1},\tilde{s}} \omega_{sj}. \end{aligned}$$

Next, we relate $\bigwedge \omega_{in} \bigwedge \omega'_{i\overline{n}}$ with $\bigwedge \omega_{sj}$. From (6) follows

$$|\bigwedge_{i=1,\dots,s} \omega'_{i\overline{n}}| = \langle Je_n, w_s\rangle^s |\bigwedge_{i=1,\dots,s} \omega_{i\tilde{s}}|$$

and also using that $\omega_{in} = \omega_{\bar{i}\overline{n}}$ since $\omega_{in} = \langle de_i, e_n\rangle = \langle dJe_i, Je_n\rangle = \omega_{\bar{i},\overline{n}}$ we obtain

$$\begin{aligned} |\bigwedge_{i=1,\dots,s-1} \omega_{in}| = |\bigwedge_{i=1,\dots,s-1} \omega_{\bar{i}\overline{n}}| &= |\bigwedge_{i=1,\dots,s-1} \langle Je_n, w_s\rangle \omega_{\bar{i}\tilde{s}}| \\ &= |\langle Je_n, w_s\rangle|^{s-1} |\bigwedge_{i=1,\dots,s-1} \omega_{\bar{i}\tilde{s}}|. \end{aligned}$$

In order to study

$$|\bigwedge_{i=1,\overline{1},\dots,s-1,\overline{s-1}} \omega_{i\tilde{s}}|,$$

we use $e_{\bar{s}} = Je_s = \langle Je_s, w_s\rangle w_s + \langle Je_s, N\rangle \omega_{\tilde{n}}$ and we obtain

$$\begin{aligned} |\bigwedge_{i=1,\overline{1},\dots,s-1,\overline{s-1}} \omega_{is}| = |\bigwedge_{i=\overline{1},1,\dots,\overline{s-1},s-1} \omega_{\bar{i}s}| &= |\bigwedge_{i=1,\overline{1},\dots,s-1,\overline{s-1}} \omega_{i\bar{s}}| \\ &= \langle Je_s, w_s\rangle^{2(s-1)} |\bigwedge_{i=1,\overline{1},\dots,s-1,\overline{s-1}} \omega_{i\tilde{s}}|. \end{aligned}$$

Thus,

$$|\bigwedge_{i=1,\overline{1},\dots,s-1,\overline{s-1}} \omega_{i\tilde{s}}| = \langle Je_s, w_s\rangle^{-2(s-1)} |\bigwedge_{i=1,\overline{1},\dots,s-1,\overline{s-1}} \omega_{is}|$$

and

$$dL_{s[p]} = \frac{\langle Je_n, w_s\rangle^{2s-1}}{\langle Je_s, w_s\rangle^{2(s-1)}} dG^{\mathbb{C}}_{n-2,s-1} \wedge dS_{2n-2}.$$

Using $|\langle Je_n, w_s\rangle| = |\langle JN, e_s\rangle|$ and $\langle Je_s, w_s\rangle^2 = 1 - \langle JN, e_s\rangle^2$ we get the result. □

3.3. **Proof of the main results.**

Proof of Theorem 1.4. If $\{e_1, Je_1, \dots, e_s, Je_s\}$ is a moving frame adapted to $T_pS \cap L_s$ and L_s, then Je_s is a normal vector of $S \cap L_s$ in L_s, thus we can take $J\tilde{N} = e_s$.

By Proposition 3.5 and the relation $\tilde{k}_n(J\tilde{N}) = \dfrac{k_n(e_s)}{\langle JN, e_s\rangle}$ (cf. (3)) we obtain

$$\begin{aligned} I &= \int_{\mathcal{L}_s}\int_{S\cap L_s} \tilde{k_n}(J\tilde{N}) \\ &= \int_S \int_{\mathbb{RP}^{2n-2}} \int_{G^{\mathbb{C}}_{n-2,s-1}} \frac{\langle JN, e_s\rangle^{2s}}{(1-\langle JN, e_s\rangle^2)^{s-1}} \frac{k_n(e_s)}{\langle JN, e_s\rangle} dV dS dp \\ &= \mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1}) \int_S \int_{\mathbb{RP}^{2n-2}} \frac{\langle JN, e_s\rangle^{2s-1}}{(1-\langle JN, e_s\rangle^2)^{s-1}} k_n(e_s) dS dp. \end{aligned} \tag{9}$$

In order to compute the integral over $\mathbb{RP}^{2n-2}$, we use polar coordinates and express the normal curvature of e_s in terms of the principal curvatures of T_pS.

That is, if $\{f_1, \dots, f_{2n-1}\}$ is an orthonormal basis of principal directions of T_pS then $e_s = \sum_{j=1}^{2n-1} \langle e_s, f_j\rangle f_j$, and

$$k_n(e_s) = \sum_{j=1}^{2n-1} \langle e_s, f_j\rangle^2 k_n(f_j) = \sum_{j=1}^{2n-1} \langle e_s, f_j\rangle^2 k_j.$$

On the other hand, we consider JN to be the first vector of a polar coordinates system and we denote

$$\langle JN, e_s\rangle = \cos\theta_1.$$

Using spherical trigonometry we have

$$\begin{aligned} \langle e_s, f_j\rangle &= -\cos\theta_1 \cos(JN, f_j) + \sin\theta_1 \sin(JN, f_j)\cos(e_s, JN, f_j) \\ &= -\cos\theta_1 \cos\alpha_j + \sin\theta_1 \sin\alpha_j \cos\theta_2 \end{aligned}$$

where $\cos(e_s, JN, f_j)$ denotes the cosine of the spherical angle with vertex JN. Note that α_j are constants when the point is fixed. Then,

$$\begin{aligned} &\int_{\mathbb{RP}^{2n-2}} \frac{\langle JN, e_s\rangle^{2s-1}}{(1-\langle JN, e_s\rangle^2)^{s-1}} k_n(e_s) dS \\ &= \sum_{j=1}^{2n-1} k_j \Big(\int_{S^{2n-3}} \int_0^{\pi/2} \frac{\cos^{2s-1}\theta_1}{\sin^{2s-2}\theta_1} \cos^2\theta_1 \cos^2\alpha_j \sin^{2n-3}\theta_1 d\theta_1 dS_{2n-3} + \\ &+ \int_{S^{2n-4}} \int_0^{\pi} \cos^2\theta_2 \sin^{2n-4}\theta_2 d\theta_2 \int_0^{\pi/2} \frac{\cos^{2s-1}\theta_1}{\sin^{2s-2}\theta_1} \sin^2\theta_1 \sin^2\alpha_j \sin^{2n-3}\theta_1 d\theta_1 + 0 \Big) \\ &= \frac{\omega_{2n-2}}{2s} \binom{n}{s}^{-1} \sum_{j=1}^{2n-1} k_j \left(\frac{2sn-n-s}{n-s} \cos^2\alpha_j + 1 \right). \end{aligned}$$

Integrating over S and using

$$k_n(JN) = \sum_{j=1}^{2n-1} k_j \langle JN, f_j\rangle^2$$

we obtain the stated result. □

Proof of Theorem 1.2. By Proposition 3.5 and Lemma 3.4 we have

$$\begin{aligned}
&\int_{\mathcal{L}_s} M_1^{(s)}(S\cap L_s)dL_s\\
&= \frac{1}{2s-1}\int_S\int_{\mathbb{RP}^{2n-2}}\int_{G^{\mathbb{C}}_{n-2,s-1}} \frac{\langle JN,e_s\rangle^{2s-1}}{(1-\langle JN,e_s\rangle^2)^{s-1}}\sigma_1(p;e_s\oplus V)dV de_s dp\\
&= \frac{1}{2s-1}\int_S\int_{\mathbb{RP}^{2n-2}} \frac{\langle JN,e_s\rangle^{2s-1}}{(1-\langle JN,e_s\rangle^2)^{s-1}}\int_{G^{\mathbb{C}}_{n-2,s-1}} (\mathrm{trII}|_V+\mathrm{II}(e_s,e_s))dV de_s dp\\
&= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{2s-1}\int_S\int_{\mathbb{RP}^{2n-2}} \frac{\langle JN,e_s\rangle^{2s-1}}{(1-\langle JN,e_s\rangle^2)^{s-1}}\left(\frac{s-1}{n-2}\mathrm{trII}|_E+k_n(e_s)\right)de_s dp,
\end{aligned}$$

where $E=\langle N, JN, e_s, Je_s\rangle^{\perp}$.

Note that if $s=1$, then $\dim V=0$. Although the integral $\int_{G^{\mathbb{C}}_{n-2,0}}\mathrm{tr}(\mathrm{II}|_V)dV$ has no sense, last equality above remains true since $\frac{s-1}{n-2}\mathrm{trII}|_E=0$.

If $s=n-1$, then $\dim V=n-2$ and $e_s\oplus V=\{N,JN,Je_s\}^{\perp}$. As $\int_{G^{\mathbb{C}}_{n-2,n-2}}\mathrm{tr}(\mathrm{II}|_V)dV=\mathrm{tr}(\mathrm{II}|_E)$, the above equality also remains true.

We shall study the following integrals:

$$J_E = \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{2s-1}\frac{s-1}{n-2}\int_S\int_{\mathbb{RP}^{2n-2}}\frac{\langle JN,e_s\rangle^{2s-1}}{(1-\langle JN,e_s\rangle^2)^{s-1}}\mathrm{trII}|_E de_s dp,$$

$$J_s = \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{2s-1}\int_S\int_{\mathbb{RP}^{2n-2}}\frac{\langle JN,e_s\rangle^{2s-1}}{(1-\langle JN,e_s\rangle^2)^{s-1}}k_n(e_s)de_s dp.$$

The second integral is the same as the integral (9). Thus, we already know its expression.

In order to study the integral J_E, we shall use polar coordinates in the same way and with the same notation as in the proof of Theorem 1.4. Let $\{e_1, Je_1, \dots, e_{s-1}, Je_{s-1}\}$ be vectors in $T_pS\cap L_s$ and let $\{e_{s+1}, Je_{s+1}, \dots, e_{n-1}, Je_{n-1}\}$ be vectors in $T_pS\cap L_s^{\perp}$. If $a\in\{1,\dots,s-1,s+1,\dots,n-1\}$, then $\langle e_a, JN\rangle = \langle Je_a, N\rangle = 0$ and also $\langle e_a, N\rangle = 0$.

We denote, following the notation in the proof of Theorem 1.4, $\cos\phi_{ij} = \cos(e_i, JN, f_j)$ and $\cos\phi_{\bar{i}j} = \cos(Je_i, JN, f_j)$. Then,

$$\int_{S^{2n-3}}\int_0^{\pi/2} \frac{\cos^{2s-1}\theta_1}{\sin^{2s-2}\theta_1}\sin^2\alpha_j \sin^{2n-3}\theta_1 \cdot$$
$$\cdot(\cos^2\phi_{1j} + \cdots + \cos^2\phi_{\overline{s-1},j} + \cos^2\phi_{s+1,j} + \cdots + \cos^2\phi_{\overline{n-2},j})d\theta_1 dS_{2n-3}$$
$$= \sin^2\alpha_j \frac{\Gamma(s)\Gamma(n-s)}{2\Gamma(n)}\cdot$$
$$\cdot\int_{S^{2n-3}}(\cos^2\phi_{1j} + \cdots + \cos^2\phi_{\overline{s-1},j} + \cos^2\phi_{s+1,j} + \cdots + \cos^2\phi_{\overline{n-2},j})dS_{2n-3}.$$

Now, denote by $\tilde{S}^{2n-1}$ the subset of S^{2n-1} consisting of all points not in $\mathrm{span}\{N, JN\}$. Then, the map

$$\begin{array}{rccc} \Pi: & \tilde{S}^{2n-1} & \longrightarrow & \{N, JN\}^\perp \\ & v & \mapsto & \frac{\mathrm{proj}_{\{N,JN\}^\perp}(v)}{||\mathrm{proj}_{\{N,JN\}^\perp}(v)||} \end{array}$$

is well-defined.

We also denote by f_j a principal direction at p as in the proof of Theorem 1.4.

As $\cos(\phi_{aj}) = \cos(e_a, JN, f_j)$ denotes the cosine of the spherical angle with vertex JN and points in e_a and f_j, by definition, it coincides with $\langle\Pi(e_a), \Pi(f_j)\rangle$.

On the other hand, vectors $\{\Pi(e_1), \ldots, J\Pi(e_{n-1})\}$ constitute an orthonormal basis of $\{N, JN\}^\perp$. Since $\Pi(f_j)$ is a unitary vector in $\{N, JN\}^\perp$, we have $\sum_{a=1}^{n-1}(\langle\Pi(e_a), \Pi(f_j)\rangle^2 + \langle J\Pi(e_a), \Pi(f_j)\rangle^2) = 1$ and we obtain

$$\int_{S^{2n-3}}(\cos^2\phi_{1j} + \cdots + \cos^2\phi_{J(s-1),j} + \cos^2\phi_{s+1,j} + \cdots + \cos^2\phi_{J(n-2),j})dS_{2n-3}$$
$$= \int_{S^{2n-3}}(1 - \langle\Pi(e_s), \Pi(f_j)\rangle^2 - \langle J\Pi(e_s), \Pi(f_j)\rangle^2)dS_{2n-3}.$$

Now, using polar coordinates such that

$$\langle\Pi(e_s), \Pi(f_j)\rangle = \cos\theta_2, \;\; \theta_2 \in (0, \pi),$$

we obtain

$$\begin{aligned}\langle J\Pi(e_s), \Pi(f_j)\rangle &= \sin(\Pi(e_s), \Pi(f_j))\cos(\Pi(e_s), \Pi(f_j), J\Pi(f_j)) \\ &= \sin\theta_2\cos\theta_3, \;\; \theta_3 \in (0, \pi)\end{aligned}$$

By Lemma 3.2 and the relation

$$O_{2n-3} = \frac{\pi}{n-2}O_{2n-5},$$

we have

$$\int_{S^{2n-5}}\int_0^\pi\int_0^\pi (1-\cos^2\theta_2-\sin^2\theta_2\cos^2\theta_3)\sin^{2n-4}\theta_2\sin^{2n-5}\theta_3 d\theta_3 d\theta_2 dS_1$$
$$= O_{2n-3} - \int_{S^{2n-4}}\int_0^\pi \cos^2\theta_2\sin^{2n-4}\theta_2 d\theta_2 dS_{2n-4}$$
$$- \int_{S^{2n-5}}\int_0^\pi \cos^2\theta_3\sin^{2n-5}\theta_3 d\theta_3 \int_0^\pi \sin^{2n-2}\theta_2 d\theta_2$$
$$= O_{2n-3} - \frac{O_{2n-3}}{2n-2} - O_{2n-5}2\frac{\sqrt{\pi}\Gamma(n-2)}{4\Gamma(n-\frac{1}{2})}\frac{\sqrt{\pi}\Gamma(n-\frac{1}{2})}{\Gamma(n)}$$
$$= \frac{O_{2n-3}}{2(n-1)}(n-2) = \omega_{2n-2}(n-2).$$

Thus,

$$J_E = \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})\omega_{2n-2}n(s-1)}{2(2s-1)(n-s)s}\binom{n}{s}^{-1}\int_S\sum_{j=1}^{2n-1}\sin^2\alpha_j k_j(p)dp$$
$$= \frac{\omega_{2n-2}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})n(s-1)}{2s(2s-1)(n-s)}\binom{n}{s}^{-1}\left((2n-1)M_1(S)-\int_S k_n(JN)\right)$$

and adding the expression of both J_E and J_s we get the result. □

4. Applications

4.1. **Reproductive valuations.**

Proof of Theorem 1.5. It was proved in [Ale03] that every continuous invariant valuation of degree $2n-2$ in $\mathbb{C}^n$ has the form

$$\nu(\Omega) = aM_1(\partial\Omega) + b\int_{\partial\Omega} k_n(JN).$$

We look for relations between a and b to be ν a reproductive valuation, that is,

$$\int_{\mathcal{L}_s}\nu(\Omega\cap L_s)dL_s = \lambda\nu(\Omega).$$

By Theorems 1.2 and 1.4 we have

$$\int_{\mathcal{L}_s}\nu(\Omega\cap L_s)dL_s = \int_{\mathcal{L}_s}\left(aM_1(\partial\Omega\cap L_s) + b\int_{\partial\Omega\cap L_s}k_n(J\tilde{N})\right)$$
$$= \frac{\omega_{2n-2}\mathrm{vol}(G^{\mathbb{C}}_{n-2,s-1})}{2s(2s-1)}\binom{n}{s}^{-1}\Big(\Big(a+\frac{b(2s-1)(2sn-n-s)}{n-s}\Big)\int_{\partial\Omega}k_n(JN)+$$
$$+\big(\frac{a(2ns-n-s)}{n-s}+b(2s-1)\big)(2n-1)M_1(\partial\Omega)\Big).$$

Thus, ν is reproductive if and only if for some $\lambda \in \mathbb{R}$

$$(2n-1)\left(\frac{a(2ns-n-s)}{n-s}+b(2s-1)\right)=\lambda a,$$
$$a+\frac{b(2s-1)(2sn-n-s)}{n-s}=\lambda b.$$

Solving this system we get two solutions, $a=-b$, $\lambda=\dfrac{2s(s-1)(2n-1)}{n-s}$, and $a=b(2s-1)(2n-1)$, $\lambda=\dfrac{2n(n-1)}{n-s}$.

In $\mathbb{CK}^n(\epsilon)$ the results follow straightforward using Theorems 1.2 and 1.4. □

4.2. **Computation of some constants of Theorem 3.1.2 in** [Ale03]**.** First, we express $\int_{\partial\Omega} k_n(Jn)$ (a translation invariant continuous valuation) in terms of valuations $\{U_{k,p}\}$ defined in (2).

Proposition 4.1. *Let Ω be a regular domain in $\mathbb{C}^n$. Then*

$$\int_{\partial\Omega} k_n(Jn)=\frac{(2n-3)(2n-2)^2 n\omega_2}{\omega_{2n-2}}U_{2n-2,1}(\Omega)$$
$$-2n(2n-1)(2n^2-4n+1)\omega_2 U_{2n-2,0}(\Omega).$$

Proof. The homogeneous degree of $\int_{\partial\Omega} k_n(Jn)$ is $2n-2$. By Theorem 2.1.1 in [Ale03], we know that $U_{2n-2,0}$ and $U_{2n-2,1}$ constitute a basis for the continuous invariant valuations of degree $2n-2$. Since

$$U_{2n-2,0}(\Omega)=\frac{1}{2n\omega_2}M_1(\partial\Omega) \tag{10}$$

and

$$U_{2n-2,1}(\Omega)=\frac{1}{(2n-2)\omega_2}\int_{\mathcal{L}_{n-1}} M_1(\partial\Omega\cap L_{n-1})dL_{n-1},$$

Theorem 1.2 with $s=n-1$ gives the result. □

In the following theorem we give the constants for all $q\in\{1,\dots,n-1\}$ and $k=2q-2$ for Theorem 3.1.2 in [Ale03] (Theorem 1.6 in the introduction).

Theorem 4.2. *Let Ω be a regular domain in $\mathbb{C}^n$. Let $0<q<n$, $0<2p<k=2q-2$. Then*

$$\int_{\mathcal{L}_q} U_{2q-2,p}(\Omega\cap L_q)dL_q=\frac{\omega_{2q-2}\omega_{2n-2}\mathrm{vol}(G^{\mathbb{C}}_{n-2,q-1})\mathrm{vol}(G^{\mathbb{C}}_{q-2,q-p-1})}{(q-p)(2q-2p-1)\binom{n-2}{q-1}\binom{q-2}{q-p-1}}\cdot$$
$$\cdot\left(\frac{(2n-3)(n-1)(n-q+p)}{\omega_{2n-2}}U_{2n-2,1}(\Omega)-(2n-1)n(n-q+p-1)U_{2n-2,0}(\Omega)\right).$$

Proof. From equation (2) and Theorem 1.2 we get the following relation:

$$\begin{aligned} U_{2q-2,p}(\Omega \cap L_q) &= \frac{1}{2(q-p)\omega_2} \int_{\mathcal{L}_{q-p}} M_1((\partial\Omega \cap L_q) \cap L_{q-p}) dL_{q-p} \\ &= \frac{O_{2q-3}\mathrm{vol}(G^{\mathbb{C}}_{q-2,q-p-1})}{8(q-p)^2(q-1)(2q-2p-1)\omega_2} \binom{q}{q-p}^{-1} \cdot \\ &\cdot \left((2q-1)\frac{2q(q-p)-2q+p}{p} \int_{\mathcal{L}_q} M_1(\partial\Omega \cap L_q) + \int_{\mathcal{L}_q} k_n(J\tilde{N}) \right) \end{aligned}$$

where $\tilde{N}$ denotes a normal vector to $\partial\Omega \cap L_q$ as a hypersurface in L_q. Using again Theorems 1.2 and 1.4, we express the integrals over $\mathcal{L}_q$ as an integral over $\partial\Omega$. Finally, from the relation in Proposition 4.1 and (10) we get the result. □

4.3. **Measure of the set of complex lines meeting a compact domain in $\mathbb{C}^n$.**

Proposition 4.3. *Let Ω be a regular domain in $\mathbb{C}^n$. Then*

$$\int_{\mathcal{L}_1} \chi(\Omega \cap L_1) dL_1 = \frac{\omega_{2n-2}}{2O_1 n} \left((2n-1)M_1(\partial\Omega) + \int_{\partial\Omega} k_n(Jn) \right).$$

Proof. As a complex line in $\mathbb{C}^n$ is isometric to $\mathbb{R}^2$, from Gauss-Bonnet-Chern formula in $\mathbb{R}^2$ and Theorem 1.2 with $s = 1$ we obtain

$$\begin{aligned} \int_{\mathcal{L}_1} \chi(\Omega \cap L_1) dL_1 &= \frac{1}{O_1} \int_{\mathcal{L}_1} \int_{\partial\Omega \cap L_1} k_g dp dL_1 \\ &= \frac{1}{O_1} \frac{\omega_{2n-2}}{2n} \left((2n-1)M_1(\partial\Omega) + \int_{\partial\Omega} k_n(Jn) \right). \end{aligned}$$

□

Remark 4.4. The method of the previous proposition cannot be used to study the integral $\int_{\mathcal{L}_s} \chi(\Omega \cap L_s) dL_s$ since it is not known an expression of $\int_{\mathcal{L}_s} M_{2s-1}(\partial\Omega \cap L_s) dL_s$ in terms of a basis of degree $2n - 2s$.

4.4. **Measure of the set of complex lines meeting a compact domain in $\mathbb{CP}^n$ and $\mathbb{CH}^n$.** A complex line in $\mathbb{CP}^n$ (or $\mathbb{CH}^n$) is isometric to $\mathbb{CP}^1$ (or $\mathbb{CH}^1 \cong \mathbb{H}^2(-4)$). Gauss-Bonnet-Chern theorem is also known in these spaces but it involves the volume of the domain. We study first the integral $\int_{\mathcal{L}_s} \mathrm{vol}_{2s}(\Omega \cap L_s) dL_s$, although we just need it when $s = 1$.

The following result is stated in [APF07].

Proposition (Theorem 6.4 [APF07])**.** *Let Ω be a regular domain in $\mathbb{CK}^n(\epsilon)$. Then*

$$\int_{\mathcal{L}_s} \mathrm{vol}_{2s}(\Omega \cap L_s) dL_s = C_{n,s}\mathrm{vol}(\Omega) \tag{11}$$

with $C_{n,s}$ a constant depending only on dimensions n, s (and the normalization of the measure dL_s).

The constant $C_{n,s}$, which is not computed explicitly in [APF07], is independent of the holomorphic curvature of the manifold. We are interested in its value to obtain the measure of the set of complex lines meeting a domain in $\mathbb{CP}^n$ or $\mathbb{CH}^n$.

In order to obtain the value of C we shall use the so-called template method. That is, we apply both sides of the equality to $\mathbb{CP}^n$. As $C_{n,s}$ is independent of the holomorphic curvature we get this constant for all ϵ.

Proposition 4.5. *Let $\mathbb{CK}^n(\epsilon)$ be a complex space form with holomorphic sectional curvature $4\epsilon \neq 0$ and let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain. Then*

$$\int_{\mathcal{L}_s} \mathrm{vol}_{2s}(\Omega \cap L_s) dL_s = \mathrm{vol}(G^{\mathbb{C}}_{n,n-s})\mathrm{vol}(\Omega).$$

Proof. Let us take $\Omega = \mathbb{CP}^n$. Using the normalization of the measure dL_s given in (1), we obtain, for the left hand side of the equality (11)

$$\int_{\mathcal{L}_s} \mathrm{vol}_{2s}(\mathbb{CP}^n \cap L_s) dL_s = \int_{\mathcal{L}_s} \mathrm{vol}_{2s}(\mathbb{CP}^s) dL_s = \mathrm{vol}(\mathbb{CP}^s)\mathrm{vol}(G^{\mathbb{C}}_{n,n-s}).$$

Using that

$$\mathrm{vol}(G^{\mathbb{C}}_{n,n-s}) = \frac{\mathrm{vol}(U(n))}{\mathrm{vol}(U(n-s))\mathrm{vol}(U(s))}$$

and that $\mathbb{CP}^n \cong G^{\mathbb{C}}_{n,1}$, we get the value of the constant $C_{n,s}$. □

Proof of Proposition 1.7. By Gauss-Bonnet-Chern formula in $\mathbb{M}^2(4\epsilon)$ we have

$$\int_{\mathcal{L}_1} \chi(\Omega \cap L_1) dL_1 = \frac{1}{2\pi} \int_{\mathcal{L}_1} M_1(\partial\Omega \cap L_1) dL_1 + \frac{4\epsilon}{2\pi} \int_{\mathcal{L}_1} \mathrm{vol}_2(\Omega \cap L_1) dL_1.$$

The result follows from Proposition 4.5 with $s = 1$ and Theorem 1.2. □

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