COHOMOLOGICAL UNIQUENESS OF SOME p-GROUPS

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Abstract. In this paper we consider classifying spaces of a family of p-groups and we prove that mod p cohomology enriched with Bockstein spectral sequences determines their homotopy type among p-completed CW-complexes. We end with some applications to group theory.

1. Introduction

Let p be a prime number. A naive way of describing Bousfield-Kan p-completion functor [1] is to say that it does transform mod p cohomology isomorphisms into actual homotopy equivalences. It is then therefore natural to think that the homotopy type of a p-complete space X should be characterized in some sense by its mod p cohomology ring H^*X. Classifying spaces of finite p-groups provide nice examples of p-complete spaces. Then the following question arises: given a finite p-group P, and a p-complete space X such that $H^*X \cong H^*BP$, is then $X \simeq BP$?

One would like to give a positive answer to the question above, but the very first step towards that positive answer is to understand, or to give the appropriate meaning to, the isomorphism $H^*X \cong H^*BP$.

It is well known that there are infinitely many examples of non isomorphic finite p-groups (hence infinitely many examples of non homotopic p-complete spaces) having isomorphic mod p cohomology rings, even as unstable algebras (see [3] for a general proof of this fact in the case of $p = 2$). This is not surprising since p-completion does not invert abstract mod p cohomology isomorphisms, but just those which are induced by continuous maps, and these compare unstable algebras plus secondary operations.

In this direction, Broto-Levi [2] suggested that mod p cohomology rings of finite p-groups should be considered objects in the category $\mathcal{K}_\beta$ of unstable algebras endowed with Bockstein spectral sequences (see Section 2 for precise definitions). Here we follow that line and consider the family of groups studied by I. Leary in [6], proving:

Theorem 1.1. *Let p be an odd prime and define the finite p-group*

$$P(p,n) = \langle A, B, C \mid A^p = B^p = C^{p^{n-2}} = [A,C] = [B,C] = 1, [A,B] = C^{p^{n-3}} \rangle.$$

Given X a p-complete CW-complex:

(a) *If $n = 3, 4$ and $H^*(X) \cong H^*(BP(p,n))$ as unstable algebras, then $X \simeq BP(p,n)$.*

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(b) *If* $n \geq 5$ *and* $H^*_\beta(X) \cong H^*_\beta(BP(p,n))$ *as objects in* $\mathcal{K}_\beta$*, then* $X \simeq BP(p,n)$*.*

Proof. Statement (a) is proved in Corollary 4.6, $n = 3$, and Corollary 4.7.(a), $n = 4$. Statement (b) is proved in Corollary 4.7.(b). □

Besides of its own topological interest, the result above and the techniques developed in its proof may be appealing from a group theoretical point of view. First, since the classifying space of a finite p-group is a p-complete CW-complex, Theorem 1.1 provides a cohomological characterization of $P(p,n)$:

Theorem 1.2. *Let* p *be an odd prime and* G *be a finite* p*-group. Then* $G \cong P(p,n)$ *if and only if* $H^*_\beta(BG) \cong H^*_\beta(BP(p,n))$*.*

Second, the ideas in the proof of Theorem 1.1 can be used to obtain a cohomological characterization of $P(p,n)$ as a complement for some $N \trianglelefteq G$. This characterization can be seen as a generalization of Tate's cohomological criteria of p-nilpotency [7]:

Theorem 1.3. *Let* p *be an odd prime and* $P(p,n) \leq G$ *be a group. Then* $P(p,n)$ *is a complement for some* $N \trianglelefteq G$ *if and only if one of the following holds:*

(a) $n = 3, 4$ *and there exists* $\psi\colon H^*(BP(p,n)) \to H^*(BG)$ *as unstable algebras such that* $(\mathrm{res} \circ \psi)|_{H^1_\beta(BP(p,n))}$ *is the identity.*
(b) $n \geq 5$ *there exists* $\psi\colon H^*_\beta(BP(p,n)) \to H^*_\beta(BG)$ *in* $\mathcal{K}_\beta$ *such that* $(\mathrm{res} \circ \psi)|_{H^1_\beta(BP(p,n))}$ *is the identity.*

Proof. If $P(p,n)$ is a complement for some $N \trianglelefteq G$, then the induced projection $G \xrightarrow{\pi} G/N \cong P(p,n)$ gives rise to a map between classifying spaces $BG \xrightarrow{B\pi} BP(p,n)$ that provides the desired cohomological morphism $\psi = B\pi^*$.

The converse in proved in in Proposition 5.1, in case $n = 3$, and Proposition 5.2 in case $n > 3$. □

Organization of the paper: In Section 2 we introduce the notation we use along paper. In Section 3 the group $P(p,n)$ is defined and the mod p cohomology ring of its classifying space is described. In Section 4, we explore endomorphisms of the mod p cohomology ring of $BP(p,n)$, and we conclude that mod p cohomology determines the homotopy type of $BP(p,n)$. Finally, in Section 5 we apply the ideas developed in the previous section to the group theoretical framework.

2. Definitions and notation

We follow the notation and conventions in [2, Section 2]. As our study is done for a fixed odd prime p, we just recall the definitions in this case.

All the spaces considered here have the homotopy type of a p-complete CW-complex. Unless otherwise stated $H^*(X)$ refers to the cohomology of the space X with trivial coefficients in $\mathbb{F}_p$.

Definition 2.1. Let p be an odd prime and K be an unstable algebra. A *Bss (Bockstein spectral sequence) for* K is a spectral sequence of differential graded algebras $\{E_i(K), \beta_i\}_{i=1}^\infty$ where the differentials have degree one and such that

(a) $E_1(K) = K$ and $\beta_1 = \beta$ is the primary Bockstein operator.
(b) If $x \in E_i(K)^{\text{even}}$ and $x^p \neq 0$ in $E_{i+1}(K)$, $i \geq 1$, then $\beta_{i+1}(x^p) = x^{p-1}\beta_i(x)$.

We work in the category $\mathcal{K}_\beta$, whose objects are pairs $(K; \{E_i(K), \beta_i\}_{i=1}^\infty)$, where K is an unstable algebra and $\{E_i(K); \beta_i\}_{i=1}^\infty$ is a Bss for K. A morphism $f\colon K \to K'$ in $\mathcal{K}_\beta$ is a family of morphisms $\{f_i\}_{i=1}^\infty$, where $f_1\colon K \to K'$ is a morphism of $\mathcal{A}_p$-algebras and for each $i \geq 2$, $f_i\colon E_i(K) \to E_i(K')$ is a morphism of differential graded algebras, which as morphism of graded algebras, is induced by f_{i-1}.

The mod p cohomology of a space X is an object of $\mathcal{K}_\beta$ that is denoted by $H^*_\beta(X)$.

Definition 2.2. We say that two spaces X and Y are *comparable* if $H^*_\beta(X)$ and $H^*_\beta(Y)$ are isomorphic objects in the category $\mathcal{K}_\beta$. We say that X is *determined by cohomology* if given a space Y comparable to X, there is a homotopy equivalence $X \simeq Y$.

Definition 2.3. Let K_β be an object in $\mathcal{K}_\beta$. Let K be the underlying unstable algebra over $\mathcal{A}_p$. We say that K_β is *weakly generated by* $x_1, \dots, x_n$ if any endomorphism f of K_β such that the restriction of f to the vector subspace of K generated by $x_1, \dots, x_n$ is an isomorphism, is an isomorphism in $\mathcal{K}_\beta$.

3. The cohomology of some p-groups

In this section, the p-group $P(p,n)$, p an odd prime, and $n \geq 3$, is introduced, and in what follows the notation in [6] is used.

The group

$$P(p,n) = \langle A, B, C \mid A^p = B^p = C^{p^{n-2}} = [A,C] = [B,C] = 1, [A,B] = C^{p^{n-3}} \rangle \tag{1}$$

have order order p^n and fits in a central extension:

$$0 \longrightarrow \mathbb{Z}/p^{n-2} \longrightarrow P(p,n) \longrightarrow \mathbb{Z}/p \times \mathbb{Z}/p \longrightarrow 0\,. \tag{2}$$

The cohomology of $P(p,n)$ is calculated in [6]:

Theorem 3.1 ([6, Proposition 3, Theorem 7, Proposition 8])**.** *$H^*(BP(3,3))$ is generated by elements y, y', x, x', Y, Y', X, X', z with*

$$\deg(y) = \deg(y') = 1, \deg(x) = \deg(x') = \deg(Y) = \deg(Y') = 2,$$
$$\deg(X) = \deg(X') = 3 \text{ and } \deg(z) = 6$$

subject to the following relations:

$$yy' = 0, xy' = x'y, yY = y'Y' = xy', yY' = y'Y,$$
$$YY' = xx', Y^2 = xY', Y'^2 = x'Y,$$
$$yX = xY - xx', y'X' = x'Y' - xx',$$
$$Xy' = x'Y - xY', X'y = xY' - x'Y,$$
$$XY = x'X, X'Y' = xX', XY' = -X'Y, xX' = -x'X,$$
$$XX' = 0, x(xY' + x'Y) = -xx'^2, x'(xY' + x'Y) = -x'x^2,$$
$$x^3y' - x'^3y = 0, x^3x' - x'^3x = 0,$$
$$x^3Y' + x'^3Y = -x^2x'^2 \text{ and } x^3X' + x'^3X = 0.$$

Moreover, the action of the mod 3 Steenrod algebra is determined by:

$$\beta(y) = x, \beta(y') = x', \beta(Y) = X, \beta(Y') = X',$$

$$\mathcal{P}^1(X) = x^2X + zy,\ \mathcal{P}^1(X') = x'^2X' - zy',\ \text{ and } \mathcal{P}^1(z) = zc_2,$$

where $c_2 = xY' - x'Y - x^2 - x'^2$.

Theorem 3.2 ([6, Proposition 3, Theorem 6, Proposition 8])**.** *For an odd prime $p \geq 5$, the cohomology $H^*(BP(p,3))$ is generated by elements y, y', x, x', Y, Y', X, X', d_4, …, d_p, c_4, …, c_{p-1} and z with*

$$\deg(y) = \deg(y') = 1, \deg(x) = \deg(x') = \deg(Y) = \deg(Y') = 2,$$

$$\deg(X) = \deg(X') = 3, \deg(d_i) = 2i - 1, \deg(c_i) = 2i,\ \text{ and } \deg(z) = 2p$$

subject to the following relations:

$$yy' = 0,\ xy' = x'y,\ yY = y'Y' = 0,\ yY' = y'Y,$$

$$Y^2 = Y'^2 = YY' = 0,\ yX = xY,\ y'X' = x'Y',$$

$$Xy' = 2xY' + x'Y,\ X'y = 2x'Y + xY'$$

$$XY = X'Y' = 0,\ XY' = -X'Y,\ xX' = -x'X,$$

$$x(xY' + x'Y) = x'(xY' + x'Y) = 0,$$

$$x^py' - x'^py = 0,\ x^px' = x'^px = 0,$$

$$x^pY' + x'^pY = 0,\ x^pX' + x'^pX = 0,$$

$$c_iy = \begin{cases} 0 \\ -x^{p-1}y \end{cases} \quad c_iy' = \begin{cases} 0 & \text{for } i < p-1 \\ -x'^{p-1}y' & \text{for } i = p-1, \end{cases}$$

$$c_ix = \begin{cases} 0 \\ -x^{p} \end{cases} \quad c_ix' = \begin{cases} 0 & \text{for } i < p-1 \\ -x'^{p} & \text{for } i = p-1, \end{cases}$$

$$c_iY = \begin{cases} 0 \\ -x^{p-1}Y \end{cases} \quad c_iY' = \begin{cases} 0 & \text{for } i < p-1 \\ -x'^{p-1}Y' & \text{for } i = p-1, \end{cases}$$

$$c_iX = \begin{cases} 0 \\ -x^{p-1}X \end{cases} \quad c_iX' = \begin{cases} 0 & \text{for } i < p-1 \\ -x'^{p-1}X & \text{for } i = p-1, \end{cases}$$

$$c_ic_j = \begin{cases} 0 & \text{for } i+j < 2p-2 \\ x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1} & \text{for } i = j = p-1, \end{cases}$$

$$d_iy = \begin{cases} 0 \\ -x^{p-1}Y \end{cases} \quad d_iy' = \begin{cases} 0 & \text{for } i < p \\ -x'^{p-1}Y' & \text{for } i = p, \end{cases}$$

$$d_ix = \begin{cases} 0 \\ -x^{p-1}y \\ x^{p-1}X \end{cases} \quad d_ix' = \begin{cases} 0 & \text{for } i < p-1 \\ -x'^{p-1}y' & \text{for } i = p-1 \\ -x'^{p-1}X' & \text{for } i = p, \end{cases}$$

$$d_iY = 0,\ d_iY' = 0,$$

$$d_iX = \begin{cases} 0 \\ -x^{p-1}Y \end{cases} \quad d_iX' = \begin{cases} 0 & \text{for } i \neq p-1 \\ -x'^{p-1}Y' & \text{for } i = p-1, \end{cases}$$

$$d_id_j = \begin{cases} 0 & \text{for } i < p \text{ or } j < p-1 \\ x^{2p-3}Y + x'^{2p-3}Y' + x^{p-1}x'^{p-2}Y' & \text{for } i = p \text{ and } j = p-1, \end{cases}$$

$$d_ic_j = \begin{cases} 0 & \text{for } i < p-1 \text{ or } j < p-1 \\ -x^{2p-3}y + x'^{2p-3}y' - x^{p-1}x'^{p-2}y' & \text{for } i = j = p-1 \\ -x^{2p-3}X + x'^{2p-3}X' - x^{p-1}x'^{p-2}X' & \text{for } i = p,\ j = p-1. \end{cases}$$

Moreover, the action of the mod p Steenrod algebra is determined by:

$$\beta(y) = x,\ \beta(y') = x',\ \beta(Y) = X,\ \beta(Y') = X',$$

$$\beta(d_i) = \begin{cases} c_i & \text{for } i < p \\ 0 & \text{for } i = p. \end{cases}$$

$$\mathcal{P}^1(X) = x^{p-1}X + zy,\ \mathcal{P}^1(X') = x'^{p-1}X' - zy',$$

$$\mathcal{P}^1(c_i) = \begin{cases} izc_{i-1} & \text{if } 2 \leq i < p-1 \\ -zc_{p-2} + x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1} & \text{if } i = p-1, \end{cases}$$

$$\mathcal{P}^1(z) = zc_{p-1},$$

where c_2 and c_3 are non zero multiples of $xY' + x'Y$ and XX' respectively.

Remark 3.3. As stated in [6, p. 71] one can verify that in the cohomology ring $H^*(BP(p,3))$, $p \geq 5$, any product of the generators y, y', x, x', Y, Y', X, X' in degree greater than 6 may be expressed in the form

$$\begin{array}{ll} f_1 + f_2Y + f_3Y' & \text{for even total degree} \\ f_1y + f_2y' + f_3X + f_4X' & \text{for odd total degree,} \end{array}$$

where each f_i is a polynomial in x and x'. So, for $n \leq p$, any element $u \in H^{2n-1}(BP(p,3))$ can be expressed as:

$$u = ad_n + f_1y + f_2y' + f_3X + f_4X',$$

where $a \in \mathbb{F}_p$ and each f_i is a polynomial in x and x'.

Theorem 3.4 ([6, Theorem 4])**.** *For $n \geq 4$, $H^*(BP(p,n))$ is generated by elements u, y, y', x, x', c_2, c_3,..., c_{p-1}, z with*

$$\deg(u) = \deg(y) = \deg(y') = 1,\ \deg(x) = \deg(x') = 2,\ \deg(c_i) = 2i,\ \deg(z) = 2p,$$

subject to the following relations:

$$xy' = x'y,\ x^py' = x'^py,\ x^px' = x'^px,$$

$$c_iy = \begin{cases} 0 \\ -x^{p-1}y \end{cases} \quad c_iy' = \begin{cases} 0 & \text{for } i < p-1 \\ -x'^{p-1}y' & \text{for } i = p-1, \end{cases}$$

$$c_ix = \begin{cases} 0 \\ -x^p \end{cases} \quad c_ix' = \begin{cases} 0 & \text{for } i < p-1 \\ -x'^p & \text{for } i = p-1, \end{cases}$$

$$c_i c_j = \begin{cases} 0 & \text{for } i+j < 2p-2 \\ x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1} & \text{for } i = j = p-1. \end{cases}$$

Moreover, we have the following operations of the mod p Steenrod algebra:

$$\beta(y) = x, \beta(y') = x', \beta(u) = \begin{cases} 0 & \text{for } n > 4 \\ y'y & \text{for } n = 4, \end{cases}$$

and

$$\mathcal{P}^1(z) = zc_{p-1}, \mathcal{P}^1(c_i) = \begin{cases} izc_{i-1} & \text{for } i < p-1 \\ -zc_{p-2} + x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1} & \text{for } i = p-1, \end{cases}$$

where $c_1 = y'y$.

Remark 3.5. If we look at $H^*_\beta(BP(p,n))$ for $n \geq 4$ as an object in $\mathcal{K}_\beta$ we have that there is a Bockstein operator $\beta_{n-3}(u) = yy'$ [6, p. 66]. From this and the cohomology of $BP(p,n)$ we can deduce that for $n = 3$ and $n = 4$ the study of the cohomological uniqueness of these spaces can be done in $\mathcal{K}$, the category of unstable algebras, instead of working in $\mathcal{K}_\beta$.

Remark 3.6. Consider the groups $B\mathbb{Z}/p^i \times B\mathbb{Z}/p \times B\mathbb{Z}/p$ and fix the following notation for the cohomology:

$$H^*(B\mathbb{Z}/p^i \times B\mathbb{Z}/p \times B\mathbb{Z}/p; \mathbb{F}_p) = E(u_i, y, y') \otimes \mathbb{F}_p[v_i, x, x']$$

where generators are sorted as components.

There is a tower of extensions:

$$BP(p,n) \xrightarrow{\pi_{n-3}} B\mathbb{Z}/p^{n-3} \times B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{\pi_{n-4}}$$
$$B\mathbb{Z}/p^{n-4} \times B\mathbb{Z}/p \times B\mathbb{Z}/p \longrightarrow \cdots \qquad \cdots \xrightarrow{\pi_1} B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p$$

where each extension π_i for $i < n-3$:

$$0 \longrightarrow B\mathbb{Z}/p \longrightarrow B\mathbb{Z}/p^i \times B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{\pi_i} B\mathbb{Z}/p^{i-1} \times B\mathbb{Z}/p \times B\mathbb{Z}/p \longrightarrow 0$$

is classified by $\beta_{i-1}(u_i)$ and

$$0 \longrightarrow B\mathbb{Z}/p \to BP(p,n) \xrightarrow{\pi_{n-3}} B\mathbb{Z}/p^{n-3} \times B\mathbb{Z}/p \times B\mathbb{Z}/p \longrightarrow 0$$

is classified by $\beta_{n-3}(u_{n-3}) - yy'$.

4. Cohomological uniqueness

Let p be an odd prime, $n \geq 3$ and $P(p,n)$ be the group defined in Equation (1). In this section we prove that the homotopy type of the classifying space of $P(p,n)$ is determined by its cohomology (Definition 2.2). The very first step toward that result is to study the behaviour of some endomorphisms of the mod p cohomology ring of $BP(p,n)$.

First we consider the case $n \leq 4$. In this case we do not need to use higher Bocksteins and it is enough to consider the structure of unstable algebra.

Theorem 4.1. *Let $\varphi\colon H^*(BP(3,3)) \to H^*(BP(3,3))$ be a homomorphism of $\mathcal{A}_3$-algebras which restricts to the identity in H^1. Then φ is an isomorphism.*

Proof. In this proof we follow the notation in Theorem 3.1 for generators and relations in cohomology.

By hypothesis $\varphi(y) = y$ and $\varphi(y') = y'$. Now, since $\beta(y) = x$ and $\beta(y') = x'$, then $\varphi(x) = \varphi(\beta(y)) = \beta(\varphi(y)) = \beta(y) = x$ and analogously $\varphi(x') = x'$. Moreover, by dimensional reasons,

$$\varphi(Y) = aY + bY' + cx + dx'$$

for some $a, b, c, d \in \mathbb{F}_3$. Because $yY = xy'$ we obtain

$$xy' = \varphi(xy') = \varphi(yY) = y\varphi(Y) = ayY + byY' + cyx + dyx'$$

and regrouping terms

$$xy' = (a + d)xy' + byY' + cyx.$$

From here we obtain $a + d = 1$ and $b = c = 0$, and $\varphi(Y) = aY + dx'$ with $a + d = 1$. Analogously $\varphi(Y') = bY' + cx$ with $b, c \in \mathbb{F}_3$ and $b + c = 1$. Now, as $Y^2 = xY'$, we have

$$\begin{aligned} \varphi(Y)^2 &= x\varphi(Y') \\ a^2Y^2 + d^2x'^2 + 2adYx' &= bxY' + cx^2. \end{aligned}$$

This implies that $c = d = 0$ and $a^2 = a = b = 1$. So $\varphi(Y) = Y$ and $\varphi(Y') = Y'$, and applying Bockstein again $\varphi(X) = X$ and $\varphi(X') = X'$ too. So φ is the identity up to dimension five and it remains to check where does it map z.

Using the first Steenrod power of X

$$\begin{aligned} \varphi(\mathcal{P}^1(X)) &= \mathcal{P}^1(\varphi(X)) \\ \varphi(x^2X + zy) &= \mathcal{P}^1(X) \\ x^2X + \varphi(z)y &= x^2X + zy \\ \varphi(z)y &= zy. \end{aligned}$$

Thus $\varphi(z) = z + \alpha$ where $\alpha y = 0$ and $\alpha \in \langle y, y', x, x', Y, Y', X, X'\rangle$. So $\varphi(\alpha) = \alpha$, $z = \varphi(z - \alpha)$ and φ is an epimorphism. In fact, because $H^*(BP(3,3))$ is a finite dimensional $\mathbb{F}_3$-vector space in each dimension, φ is an isomorphism dimension-wise, and thus φ is an isomorphism. □

Theorem 4.2. *Let $p \geq 5$ be a prime. If $\varphi\colon H^*(BP(p,3)) \to H^*(BP(p,3))$ is a homomorphism in $\mathcal{K}$ that restricts to the identity in H^1, then φ is an isomorphism.*

Proof. Consider the notation of generators and relations in $H^*_\beta(BP(p,3))$ given in Theorem 3.2. We calculate the image under φ of every generator in $H^*(BP(p,3))$.

As φ is the identity on y and y', applying Bockstein operations we get that $\varphi(x) = x$ and $\varphi(x') = x'$.

As Y is in degree 2, there exist coefficients a, b, c, d such that

$$\varphi(Y) = ax + bx' + cY + dY'.$$

Using the relation $Y^2 = 0$, we get $\varphi(Y)^2 = 0$, which implies that $a = b = 0$, and so $\varphi(Y) = cY + dY'$. The relation $yY = 0$ implies $0 = y\varphi(Y) = dyY'$, so $d = 0$, getting that there is c such that $\varphi(Y) = cY$. Using the same arguments, there is d such that $\varphi(Y') = dY'$.

According to Remark 3.3, there are $a_n \in \mathbb{F}_p$ and $f_{n,i}$ polynomials in x and x' such that for $4 \le n \le p$

$$\varphi(d_n) = a_n d_n + f_{n,1}y + f_{n,2}y' + f_{n,3}X + f_{n,4}X',$$

and applying the Bockstein operation, we get that for $4 \le n \le p-1$:

$$\varphi(c_n) = a_n c_n + f_{n,1}x + f_{n,2}x'.$$

The relation $c_{p-1}x = -x^p$ gives rise to the following equalities:

$$\begin{aligned} -x^p &= \varphi(-x^p) = \varphi(c_{p-1}x) = \varphi(c_{p-1})\varphi(x) = \varphi(c_{p-1})x = \\ &= a_{p-1}c_{p-1}x + f_{p-1,1}x^2 + f_{p-1,2}xx' = -a_{p-1}x^p + f_{p-1,1}x^2 + f_{p-1,2}xx', \end{aligned}$$

so $(a_{p-1} - 1)x^p = f_{p-1,1}x^2 + f_{p-1,2}xx'$, and as there are no relations involving x and x' till degree $2p+2$, we can simplify:

$$f_{p-1,1}x + f_{p-1,2}x' = (a_{p-1} - 1)x^{p-1}. \tag{3}$$

Doing the same computations using the relation $c_{p-1}x' = -x'^p$ we get

$$f_{p-1,1}x + f_{p-1,2}x' = (a_{p-1} - 1)x'^{p-1}. \tag{4}$$

Comparing now (3) and (4), and using again that there is no relation between x and x' till degree $2p+2$, we get $a_{p-1} = 1$, $\varphi(c_{p-1}) = c_{p-1}$.

Now we see that $\varphi(c_n) = a_n c_n$, for $4 \le n < p-1$: using the relation $c_n x = 0$ and applying φ we get $f_{n,1}x + f_{n,2}x' = 0$, so

$$\varphi(c_n) = a_n c_n. \tag{5}$$

In order to calculate $\varphi(z)$, we apply φ to the equality:

$$\mathcal{P}^1(c_{p-1}) = -zc_{p-2} + x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1}.$$

Since $\varphi(c_{p-1}) = c_{p-1}$, $\varphi(x) = x$, and $\varphi(x') = x'$, we get

$$zc_{p-2} = \varphi(z)a_{p-2}c_{p-2}. \tag{6}$$

As the generator z does not appear in any relation, Equation (6) implies that $a_{p-2} \neq 0$ and $\varphi(z) = a_{p-2}^{-1}z + g$, where g is an expression not involving z, and such that $gc_{p-2} = 0$.

We use that we know that $a_{p-2} \neq 0$ to check that $a_n \neq 0$ for $4 \le n < p-2$ with an induction argument: assume $\varphi(c_n) = a_n c_n$ with $a_n \neq 0$ and $5 \le n \le p-2$, and compute $\varphi(c_{n-1})$:

$$nzc_{n-1} = \mathcal{P}^1(c_n) = \mathcal{P}^1(\varphi(a_n^{-1}c_n)) = \varphi(a_n^{-1}\mathcal{P}^1(c_n)) = a_n^{-1}n\varphi(z)a_{n-1}c_{n-1}.$$

This implies $zc_{n-1} = a_n^{-1}a_{n-1}\varphi(z)c_{n-1}$, and this can only happens if $a_{n-1} \neq 0$ and $\varphi(z) = a_n a_{n-1}^{-1}z + g$ (g not involving z).

From the expression $c_3 = \mu XX'$ we deduce that $\varphi(c_3) = a_3c_3$ with $a_3 = cd$, where c and d were introduced at the beginning of the proof and are such that $\varphi(Y) = cY$ and $\varphi(Y') = dY'$. The argument above for $\mathcal{P}^1(c_4)$ shows that a_3 is non-zero neither. Hence c, d and a_n for all $n \in \{3, \dots, p-1\}$ are non-zero.

Let us check now that the coefficients c and d are equal: recall that c_2 was defined as $\lambda(xY' + x'Y)$ with λ non-zero. Then, applying $\mathcal{P}^1$ to c_3 we get:

$$\begin{aligned} 3zc_2 &= \mathcal{P}^1(c_3) = \mathcal{P}^1(\varphi(a_3^{-1}c_3)) = a_3^{-1}\varphi(\mathcal{P}^1(c_3)) = a_3^{-1}(\varphi(3zc_2)) = \\ &= a_3^{-1}3\varphi(z)\varphi(c_2) = a_3^{-1}3\varphi(z)\lambda(dxY' + cx'Y), \end{aligned}$$

which implies $\lambda z(xY' + x'Y) = a_3^{-1}\lambda\varphi(z)(dxY' + cx'Y)$ and can be simplified to:

$$zxY' + zx'Y = da_3^{-1}\varphi(z)xY' + ca_3^{-1}\varphi(z)x'Y. \tag{7}$$

Again, as z does not appear in any relation, Equation (7) can be true only if $c = d$. In particular, $\varphi(c_2) = a_2c_2$ with $a_2 = c\lambda \neq 0$.

Now we can assume that all the coefficients a_n for $2 \leq n \leq p-1$ and c and d are equal to 1: as all are different to zero, and $r^{p-1} = 1$ if $r \in \mathbb{F}_p \setminus \{0\}$, φ^{p-1} is the identity in Y, Y' and c_n. Use now that φ is an isomorphism if and only if φ^{p-1} is so. Therefore at this point we have that:

$$\varphi(y) = y, \varphi(y') = y',\ \varphi(x) = x,\ \varphi(x') = x',$$
$$\varphi(Y) = Y,\ \varphi(Y') = Y',\ \varphi(X) = X,\ \varphi(X') = X',$$
$$\varphi(c_i) = c_i \text{ for } 2 \leq i \leq p-1\ ,\ \varphi(d_i) = d_i + g_i \text{ for } 4 \leq i \leq p-1, \text{ and } f(z) = z + g$$

where g and all g_i are expressions in x, x', y, y', X, X', Y and Y'. This implies that all generators but d_p are in the image of φ.

The image of d_p, as it is in odd degree greater than 6, must be:

$$\varphi(d_p) = a_pd_p + f_{p,1}y + f_{p,2}y' + f_{p,3}X + f_{p,4}X'$$

with $a_p \in \mathbb{F}_p$, and $f_{p,i}$ polynomials in x and x'. As $\beta(d_p) = 0$, the Bockstein operation on $\varphi(d_p)$ must vanish, and this means:

$$0 = \beta(\varphi(d_p)) = f_{p,1}x + f_{p,2}x'.$$

So this is a polynomial in x, x' which must be zero. As there are not relations involving just x and x' in this degree, we deduce that there exist f_p a polynomial in x and x' such that $f_{p,1} = f_px'$ and $f_{p,2} = f_px$. This implies that (recall $xy' = x'y$),

$$f_{p,1}y + f_{p,2}y' = f_p(x'y - xy') = 0,$$

and then

$$\varphi(d_p) = a_pd_p + f_{p,3}X + f_{p,4}X'.$$

As any expression on x, x', X and X' is in the image, we have only to check that $a_p \neq 0$. To do that we assume that $a_p = 0$, getting a contradiction.

If $\varphi(d_p) = f_{p,3}X + f_{p,4}X'$, using the relation $d_pY = 0$ we get $(f_{p,3}X + f_{p,4}X')Y = 0$. Applying that $XY = 0$ this means that $f_{p,4}X'Y = 0$. Recall now that $f_{p,4}$ is a polynomial on x, x' of degree $2p-4$. Looking at the relations under this degree, $f_{p,4}X'Y = 0$ means that $f_{p,4}$ must be zero. The same argument can be applied to the relation $d_pY' = 0$, obtaining that $f_{p,3} = 0$. So at this point we have that $\varphi(d_p) = 0$.

Use now that $d_px = x^{p-1}X \neq 0$, and applying φ to both sides of this equality we get the following contradiction:

$$0 = \varphi(d_px) = \varphi(x^{p-1}X) = x^{p-1}X \neq 0.$$

Therefore $a_p \neq 0$ and φ is an isomorphism. □

We now consider the case of $n > 3$. Here the use of Bockstein operators is needed.

Theorem 4.3. *Let p be an odd prime and consider the notation of the generators and relations in $H^*_\beta(BP(p,n))$ as in Theorem 3.4.*

(a) *If $\varphi\colon H^*(BP(p,4)) \to H^*(BP(p,4))$ is a homomorphism of unstable algebras that fixes y and y', then φ is an isomorphism.*
(b) *If $n \geq 5$ and $\varphi\colon H^*_\beta(BP(p,n)) \to H^*_\beta(BP(p,n))$ is a homomorphism in $\mathcal{K}_\beta$ which fixes y and y'. Then φ is an isomorphism.*

Proof. We prove both results at the same time. Just observe that the Bockstein used in the proof is β_{n-3}, which is part of the mod p Steenrod algebra when $n = 4$.

Starting from $\varphi(y) = y$ and $\varphi(y') = y'$ and using the Bockstein operator we reach $\varphi(x) = x$ and $\varphi(x') = x'$. On the other hand there exist $a, b, c \in \mathbb{F}_p$ such that $\varphi(u) = au + by + cy'$. From Remark 3.5 we deduce that $\beta_{n-3}(u) = y'y$, so, as the morphism is in $\mathcal{K}_\beta$,

$$\varphi(\beta_{n-3}(u)) = \beta_{n-3}(\varphi(u)) \Rightarrow y'y = \begin{cases} ay'y + bx + cx' & \text{for } n = 4 \\ ay'y & \text{for } n > 4. \end{cases}$$

We obtain then that $a = 1$ and $b = c = 0$ for $n = 4$, and that $a = 1$ for $n > 4$. Hence, $u = \varphi(u) - by - cy' = \varphi(u - by - cy')$ and $\langle u, y, y', x, x' \rangle \leq \operatorname{Im} \varphi$.

Now consider the generator c_{p-1}. We can write $\varphi(c_{p-1}) = a_{p-1}c_{p-1} + bx^{p-1} + g_{p-1}$ with $a_{p-1}, b \in \mathbb{F}_p$ and g_{p-1} not containing multiples of the monomials c_{p-1} and x^{p-1}. Applying φ to the equation $c_{p-1}x' = -x'^p$ we obtain $-x'^p = a_{p-1}c_{p-1}x' + bx^{p-1}x' + g_{p-1}x' = -a_{p-1}x'^p + bx^{p-1}x' + g_{p-1}x'$. The only equation at degree p involving x'^p is $c_{p-1}x' = -x'^p$ (notice that a multiple of the equation $xy' = x'y$ in degree 3 does not involve x'^p and that this may occur just for $p = 3$). As g_{p-1} does not contain neither c_{p-1} nor x^{p-1} we deduce that $a_{p-1} = 1$, $b = 0$. Hence $\varphi(c_{p-1}) = c_{p-1} + g_{p-1}$.

Next we deal with c_{p-2} of degree $2(p-2)$ and z of degree $2p$. Their images are $\varphi(c_{p-2}) = a_{p-2}c_{p-2} + g_{p-2}$ and $\varphi(z) = a_z z + g_z$, with $a_{p-2}, a_z \in \mathbb{F}_p$, and g_{p-2} and g_z not involving the monomials c_{p-2} and z respectively. Write the Steenrod power $\mathcal{P}^1(c_{p-1}) = -zc_{p-2} + x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1}$ as $\mathcal{P}^1(c_{p-1}) = -zc_{p-2} + f$, with $f = x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1}$. Applying φ we get:

$$\begin{aligned} \varphi(\mathcal{P}^1(c_{p-1})) &= \mathcal{P}^1(\varphi(c_{p-1})) \\ \varphi(-zc_{p-2}+) &= \mathcal{P}^1(c_{p-1} + g_{p-1}) \\ -(a_z z + g_z)(a_{p-2}c_{p-2} + g_{p-2}) + f &= -zc_{p-2} + f + \mathcal{P}^1(g_{p-1}) \\ -a_z a_{p-2} z c_{p-2} - a_z z g_{p-2} - a_{p-2} g_z c_{p-2} - g_z g_{p-2} &= -zc_{p-2} + \mathcal{P}^1(g_{p-1}). \end{aligned}$$

Notice that there is no relation involving the generator z and the equations involving c_{p-2} are $c_{p-2}y = c_{p-2}y' = c_{p-2}x = c_{p-2}x' = c_{p-2}c_j = 0$ for $j < p$. Also, the monomial zc_{p-2} cannot appear in zg_{p-2}, $g_z c_{p-2}$, and $g_z g_{p-2}$. Finally, $\mathcal{P}^1(g_{p-1})$ does not involve zc_{p-2} as g_{p-1} does not involve c_{p-1} and the action of $\mathcal{P}^1$ on u, y, y', x, x' is determined by the axioms. Hence, $a_z a_{p-2} = 1$ and both a_z and a_{p-2} are non-zero.

For the rest of the generators c_i for $i = 2, 3, \ldots, p-3$ we can write $\varphi(c_i) = a_i c_i + g_i$, with $a_i \in \mathbb{F}_p$ and g_i not involving c_i. The Steenrod power $\mathcal{P}^1(c_{i+1}) = (i+1)zc_i$ provides then

$$\begin{aligned}\varphi(\mathcal{P}^1(c_{i+1})) &= \mathcal{P}^1(\varphi(c_{i+1}))\\ \varphi((i+1)zc_i) &= \mathcal{P}^1(\alpha_{i+1}c_{i+1} + g_{i+1})\\ (i+1)(a_z z + g_z)(a_i c_i + g_i) &= (i+1)a_{i+1}zc_i + \mathcal{P}^1(g_{i+1})\\ (i+1)(a_z a_i z c_i + a_z z g_i + a_i a_z c_i + a_z g_i) &= (i+1)a_{i+1}zc_i + \mathcal{P}^1(g_{i+1}).\end{aligned}$$

Notice again that there is no relation involving the generator z and the relations involving c_i are $c_i y = c_i y' = c_i x = c_i x' = c_i c_j = 0$ for $j < 2p-2-i$. Also, the monomial zc_i cannot appear in zg_i, $g_z c_i$, and $g_z g_i$. Moreover, $\mathcal{P}^1(g_{i+1})$ does not involve zc_i as g_{i+1} does not involve c_{i+1}. We deduce that $(i+1)a_z a_i = (i+1)a_{i+1}$. As $a_z \neq 0$ and $a_{p-2} \neq 0$, an inductive argument shows that $a_i \neq 0$ for for $i = 2, 3, \ldots, p-3$, and hence for all $i = 2, 3, \ldots, p-1$.

To finish we show that all the generators $c_2, c_3, \ldots, c_{p-1}, z$ are in the image of φ. We start with $c_2 = \frac{1}{\alpha_2}(\varphi(c_2) - g_2)$. As $g_2 \in \langle u, x, x, y, y' \rangle \leq \operatorname{Im}\varphi$ then c_2 is also in the image of φ. An inductive argument shows that $c_i = \frac{1}{\alpha_i}(\varphi(c_i) - g_i)$ is in the image of φ as g_i belongs to $\langle u, x, x', y, y', c_2, c_3, \ldots, c_{i-1} \rangle$. This argument also applies to show that $z \in \operatorname{Im}\varphi$.

Hence, φ is an epimorphism. Because $H^*_\beta(BP(p,n))$ is finite in each dimension φ is an isomorphism. ☐

Then, the following corollary is straightforward:

Corollary 4.4. *$H^*_\beta(BP(p,n))$ for odd p and $n \geq 3$ is weakly generated (Definition* 2.3) by y, and y'.

Proof. Let φ be and endomorphism of $H^*_\beta(BP(p,n))$ which is an isomorphism on $\langle y, y' \rangle$. Using the outer automorphism group of $P(p,n)$ which is described in [4, Lemma A.5] there is a morphism $f\colon BP(p,n) \to BP(p,n)$ such that the composition $f^* \circ \varphi$ fixes y and y'. Use now Theorems 4.1, 4.2 and 4.3 to get the result. ☐

Notice that for any finite p-group there is a natural isomorphism $H^1P \cong P/\Phi(P)$, where $\Phi(P)$ stands for the Frattini subgroup of P [5, p. 173]. Therefore, Theorems 4.1, 4.2 and 4.3 can be seen as cohomological counterpart of the following group theoretical result.

Proposition 4.5. *Let P be a finite p-group and $f\colon P \to P$ a group morphism such that the induced morphism at the level of Frattini quotients $\tilde{f}\colon P/\Phi(P) \to P/\Phi(P)$ is an isomorphism. Then f is an isomorphism.*

Proof. Let n be such that $P/\Phi(P) = (\mathbb{Z}/p)^n$ [5, Theorem 5.1.3]. Assume f is not an isomorphism, then $f(P) \leq H < P$ for some maximal subgroup $H < P$, and therefore $\tilde{f}(P/\Phi(P)) < H/\Phi(P) = (\mathbb{Z}/p)^{n-1} < P/\Phi(P)$, that is, $\tilde{f}$ is not an isomorphism. ☐

Now, we apply the results above to obtain the cohomology uniqueness of the classifying space $BP(p,n)$. We split this result into two corollaries because the structure of $P(p,3)$ is essentially different from that of $P(p,n)$, $n > 4$.

Corollary 4.6. *Let p be an odd prime and X be a p-complete space such that $H^*(X) \cong H^*(BP(p,3))$ as unstable algebras. Then $X \simeq BP(p,3)$.*

Proof. We begin with the central extension

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow P(p,3) \xrightarrow{\pi} \mathbb{Z}/p \times \mathbb{Z}/p \longrightarrow 0$$

classified by $yy' \in H^2(B\mathbb{Z}/p \times B\mathbb{Z}/p)$, which gives raise to the principal fibration

$$BP(p,3) \xrightarrow{B\pi} B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{yy'} B^2\mathbb{Z}/p\,.$$

Consider the map $\pi_X\colon X \to B\mathbb{Z}/p \times B\mathbb{Z}/p$ that classifies the classes $y, y' \in H^1(X)$. Then the composite

$$X \xrightarrow{\pi_X} B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{yy'} B^2\mathbb{Z}/p$$

is null-homotopic because of Theorems 3.1 and 3.2, and so π_X lifts to $\varphi\colon X \to BP(p,3)$, giving the commutative diagram

$$\begin{array}{ccc} & & BP(p,3) \\ & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle B\pi} \\ X & \xrightarrow{\pi_X} & B\mathbb{Z}/p \times B\mathbb{Z}/p \end{array}$$

which implies that φ^* fixes y and y'. Now apply Theorems 4.1 and 4.2 to φ^*. ☐

Corollary 4.7. *Let p be an odd prime and X be a p-complete space.*
(a) *If $H^*(X) \cong H^*(BP(p,4))$ as unstable algebras then $X \simeq BP(p,4)$.*
(b) *If $n \geq 5$ and $H^*_\beta(X) \cong H^*_\beta(BP(p,n))$ as objects in $\mathcal{K}_\beta$. Then $X \simeq BP(p,n)$.*

Proof. Consider the central extension and notations as in Remark 3.6.

If $n = 4$ we have

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow P(p,4) \xrightarrow{\pi_1} \mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p \longrightarrow 0$$

Let now $\pi_{1,X}$ be the map $\pi_{1,X}\colon X \to B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p$ that classifies the classes $u, y, y' \in H^1(X)$.

The composite

$$X \xrightarrow{\pi_{1,X}} B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{yy'-\beta(u)} B^2\mathbb{Z}/p$$

is null-homotopic because of Remark 3.6, and so $\pi_{1,X}$ lifts to $\varphi\colon X \to BP(p,4)$, giving the commutative diagram

$$\begin{array}{ccc} & & BP(p,4) \\ & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle B\pi_1} \\ X & \xrightarrow{\pi_{1,X}} & B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p \end{array}$$

This implies that φ^* fixes y and y' and the result is a consequence of Theorem 4.3.

If $n > 4$, we must consider before the extensions:

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^i \times \mathbb{Z}/p \times \mathbb{Z}/p \xrightarrow{\pi_{i-1}} \mathbb{Z}/p^{i-1} \times \mathbb{Z}/p \times \mathbb{Z}/p \longrightarrow 0,$$

classified by $\beta_{i-1}(u) \in H^*_\beta(B\mathbb{Z}/p^{i-1} \times B\mathbb{Z}/p \times B\mathbb{Z}/p)$.

Let $\pi_{1,X}$ be now the map $\pi_{1,X}\colon X \to B\mathbb{Z}/p \times B\mathbb{Z}/p \times \mathbb{Z}/p$ that classifies the classes $u, y, y' \in H^1(X)$. As $\beta_1(u) = 0 \in H^*_\beta(X)$ the map $\pi_{1,X}$ extends to a map $\pi_{2,X}$ having the following commutative diagram:

$$\begin{array}{ccc} & & B\mathbb{Z}/p^2 \times B\mathbb{Z}/p \times B\mathbb{Z}/p \\ & \overset{\pi_{2,X}}{\nearrow} & \downarrow B\pi_1 \\ X & \xrightarrow{\pi_{1,X}} & B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p \end{array}$$

Using the same argument we can proceed extending the map till $B\pi_{n-3}\colon B\mathbb{Z}/p^{n-3} \times B\mathbb{Z}/p \times B\mathbb{Z}/p$. To do the last step we use again Remark 3.6 and that $\beta_{n-3}(u) = yy' \in H^*_\beta(X)$, and obtain a map φ which gives the commutative diagram:

$$\begin{array}{ccc} & & BP(p,n) \\ & \overset{\varphi}{\nearrow} & \downarrow B\pi_{n-2}\circ\cdots\circ B\pi_1 \\ X & \xrightarrow{\pi_{1,X}} & B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p \end{array}$$

which implies again that φ^* fixes y and y', so apply Theorem 4.3. □

5. Some applications to Group Theory

The techniques used along the proof of the last two corollaries in the previous section can be used to obtain a cohomological characterization of $P(p,n)$ as a complement for some $N \trianglelefteq G$, for a super group $P(p,n) \leq G$. Recall that given a group G, and a normal subgroup $N \trianglelefteq G$, $K \leq G$ is a complement for N if $G = NK$ and $N \cap K = 1$, that is, if $G = N \rtimes K$.

Again, we consider the case $n = 3$ separately.

Proposition 5.1. *Let p be an odd prime and G be a finite group such that $P(p,3) \leq G$. Assume also that there exists $\psi\colon H^*(BP(p,3)) \to H^*(BG)$ as unstable algebras such that $(\mathrm{res} \circ \psi)|_{H^1_\beta(BP(p,3))}$ is the identity. Then $P(p,3)$ is a complement for some $N \trianglelefteq G$.*

Proof. As it was announced, we work along the lines in the proof of Corollary 4.6. We begin by considering the map $B\pi_G\colon BG \to B\mathbb{Z}/p \times B\mathbb{Z}/p$ that classifies the classes $\psi(y), \psi(y') \in H^1(BG)$. Then $B\pi_G^*(yy') = B\pi_G^*(y)B\pi_G^*(y') = \psi(y)\psi(y') = \psi(yy') = \psi(0) = 0$ (Theorems 3.1 and 3.2), and the composite

$$BG \xrightarrow{B\pi_G} B\mathbb{Z}/p \times B\mathbb{Z}/p \xrightarrow{yy'} B^2\mathbb{Z}/p$$

is null-homotopic. Therefore $B\pi_G$ lifts to $B\phi\colon BG \to BP(p,3)$, giving the commutative diagram

$$\begin{array}{ccccc} & & & & BP(p,3) \\ & & & \overset{B\phi}{\nearrow} & \downarrow B\pi \\ BP(p,3) & \xrightarrow{\mathrm{res}} & BG & \xrightarrow{B\pi_G} & B\mathbb{Z}/p \times B\mathbb{Z}/p \end{array}$$

which implies that $B\phi^*(y) = \psi(y)$ and $B\phi^*(y') = \psi(y')$, and

$$(\mathrm{res} \circ B\phi)(y) = (\mathrm{res}^* \circ \psi)(y) = y \text{ and } (\mathrm{res} \circ B\phi)^*(y') = (\mathrm{res}^* \circ \psi)(y) = y'.$$

Now, applying Theorems 4.1 and 4.2, or Proposition 4.5 we obtain that $\phi|_{P(p,3)}$ is an automorphism of $P(p,3)$, that is, $P(p,3)$ is a complement for $N = \operatorname{Ker}\phi \trianglelefteq G$. □

We now proceed with the case $n > 3$.

Proposition 5.2. *Let p be an odd prime and G be a finite group such that $P(p,n) \leq G$.*

(a) *If $n = 4$ and there exists $\psi\colon H^*(BP(p,4)) \to H^*(BG)$ as unstable algebras such that $(\mathrm{res}\circ\psi)|_{H^1_\beta(BP(p,4n))}$ is the identity, then $P(p,4)$ is a complement for some $N \trianglelefteq G$*

(b) *If $n \geq 5$ and there exists $\psi\colon H^*_\beta(BP(p,n)) \to H^*_\beta(BG)$ in $\mathcal{K}_\beta$ such that $(\mathrm{res}\circ\psi)|_{H^1_\beta(BP(p,n))}$ is the identity, then $P(p,n)$ is a complement for some $N \trianglelefteq G$*

Proof. We now follow along the lines of the proof of Corollary 4.7.

If $n = 4$ we have

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow P(p,4) \xrightarrow{\pi_1} \mathbb{Z}/p\times\mathbb{Z}/p\times\mathbb{Z}/p \longrightarrow 0$$

Let now $B\pi_{1,G}$ be the map $B\pi_{1,G}\colon BG \to B\mathbb{Z}/p\times B\mathbb{Z}/p\times B\mathbb{Z}/p$ that classifies the classes $\psi(u), \psi(y), \psi(y') \in H^1(BG)$.

The composite

$$BG \xrightarrow{\pi_{1,G}} B\mathbb{Z}/p\times B\mathbb{Z}/p\times B\mathbb{Z}/p \xrightarrow{yy'-\beta(u)} B^2\mathbb{Z}/p$$

is null-homotopic because of Remark 3.6, and

$$\begin{aligned} B\pi^*_{1,G}(yy'-\beta(u)) &= B\pi^*_{1,G}(y)B\pi^*_{1,G}(y') - B\pi^*_{1,G}(\beta(u)) = \\ &= \psi(y)\psi(y') - \psi(\beta(u)) = \psi(yy'-\beta(u)) = \psi(0) = 0. \end{aligned} \quad (8)$$

Therefore $B\pi_{1,G}$ lifts to $B\phi\colon BG \to BP(p,4)$, giving the commutative diagram

$$\begin{array}{ccccc} & & & & BP(p,4) \\ & & & \overset{B\phi}{\nearrow} & \downarrow B\pi_1 \\ BP(p,3) & \xrightarrow{\mathrm{res}} & BG & \xrightarrow{B\pi_{1,G}} & B\mathbb{Z}/p\times B\mathbb{Z}/p\times B\mathbb{Z}/p. \end{array}$$

This implies that $B\phi^*(y) = \psi(y)$, $B\phi^*(y') = \psi(y')$, and $B\phi^*(u) = \psi(u)$, and

$$(\mathrm{res}\circ B\phi)(y) = (\mathrm{res}^*\circ\psi)(y) = y,$$

$$(\mathrm{res}\circ B\phi)^*(y') = (\mathrm{res}^*\circ\psi)(y) = y', \text{ and}$$

$$(\mathrm{res}\circ B\phi)(u) = (\mathrm{res}^*\circ\psi)(u) = u.$$

Now, according to either Proposition 4.5 or Theorem 4.3, $\phi|_{P(p,4)}$ is an automorphism of $P(p,4)$, that is, $P(p,4)$ is a complement for $N = \operatorname{Ker}\phi \trianglelefteq G$.

If $n > 4$, we must consider before the extensions:

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^i\times\mathbb{Z}/p\times\mathbb{Z}/p \xrightarrow{\pi_{i-1}} \mathbb{Z}/p^{i-1}\times\mathbb{Z}/p\times\mathbb{Z}/p \longrightarrow 0,$$

classified by $\beta_{i-1}(u) \in H^*_\beta(\mathbb{Z}/p^{i-1}\times\mathbb{Z}/p\times\mathbb{Z}/p)$.

Let $B\pi_{1,G}\colon BG \to B\mathbb{Z}/p \times B\mathbb{Z}/p \times \mathbb{Z}/p$ be the map that classifies the classes $\psi(u)$, $\psi(y)$, $\psi(y') \in H^1(BG)$. As $\beta_1(\psi(u)) = 0 \in H^*_\beta(BG)$ the map $B\pi_{1,G}$ extends to a map $B\pi_{2,G}$ having the following commutative diagram:

$$\begin{array}{ccc} & & B\mathbb{Z}/p^2 \times B\mathbb{Z}/p \times B\mathbb{Z}/p \\ & \overset{B\pi_{2,G}}{\nearrow} & \downarrow B\pi_1 \\ BG & \xrightarrow{B\pi_{1,G}} & B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p \end{array}$$

Using the same argument we can proceed extending the map till $B\pi_{n-3}\colon B\mathbb{Z}/p^{n-3} \times B\mathbb{Z}/p \times B\mathbb{Z}/p$. To do the last step we use again Remark 3.6 and that $\beta_{n-3}(\psi(u)) = \psi(y)\psi(y') \in H^*_\beta(BG)$, and obtain a map $B\phi$ which gives the commutative diagram:

$$\begin{array}{ccc} & & BP(p,n) \\ & \overset{B\phi}{\nearrow} & \downarrow B\pi_{n-2}\circ\cdots\circ B\pi_1 \\ BG & \xrightarrow{B\pi_{1,G}} & B\mathbb{Z}/p \times B\mathbb{Z}/p \times B\mathbb{Z}/p \end{array}$$

which implies again that $B\phi^*(y) = \psi(y)$, $B\phi^*(y') = \psi(y')$, and $B\phi^*(u) = \psi(u)$, and

$$(\mathrm{res} \circ B\phi)(y) = (\mathrm{res}^* \circ \psi)(y) = y,$$

$$(\mathrm{res} \circ B\phi)^*(y') = (\mathrm{res}^* \circ \psi)(y) = y', \text{ and}$$

$$(\mathrm{res} \circ B\phi)(u) = (\mathrm{res}^* \circ \psi)(u) = u.$$

Again, applying Proposition 4.5 or Theorem 4.3, we obtain that $\phi|_{P(p,n)}$ is an automorphism of $P(p,n)$, that is, $P(p,n)$ is a complement for $N = \mathrm{Ker}\,\phi \trianglelefteq G$. □

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