QUADRATIC SYSTEMS WITH A POLYNOMIAL FIRST INTEGRAL: A COMPLETE CLASSIFICATION IN THE COEFFICIENT SPACE $\mathbb{R}^{12}$

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Abstract. In this paper we are going to apply the invariant theory to give invariant conditions on the coefficients of any non degenerate quadratic system in order to determine if it has or not a polynomial first integral without using any normal form. We obtain that the existence of polynomial first integral is directly related with the fact that all the roots of a convenient cubic polynomial are rational and negative. The coefficients of this cubic polynomial are invariants related with some geometric properties of the system.

1. Introduction and the statement of the main result

Let $\mathbb{R}[x, y]$ be the ring of all polynomials in the variables x and y with coefficients in $\mathbb{R}$. In this paper we deal with *quadratic polynomial differential systems* in $\mathbb{R}^2$ of the form

$$\frac{dx}{dt} = x' = P(x, y), \qquad \frac{dy}{dt} = y' = Q(x, y), \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$ and $\max\{\deg P, \deg Q\} = 2$. In what follows such differential systems will be called simply *quadratic systems*.

Quadratic systems have been investigated intensively, and more than one thousand papers have been published about these systems (see for instance [18], [30] and [31]). But the problem of classifying the integrable quadratic systems remains open. For additional information on integrable differential systems in dimension 2, see [5].

The search of first integrals is a classical tool for classifying all trajectories of a polynomial system. Polynomial first integrals are a particular case of the Darbouxian first integrals. In 1878 Darboux [10] showed how the first integrals of planar polynomial systems possessing sufficient invariant algebraic curves can be constructed. The best improvements to Darboux's results for planar polynomial systems are due to Poincaré [16] in 1897, Jouanolou [12] in 1979, to Prelle and Singer [17] in 1983, and to Singer [27] in 1992. Some recent interesting results related with Darboux theory of integrability have been made by several authors, see for instance [7, 8, 9, 19], etc.

We say that $\mathcal{H} \in \mathbb{R}[x, y] \setminus \mathbb{R}$ is a *polynomial first integral* of system (1) on $\mathbb{R}^2$ if $\mathcal{H}(x(t), y(t)) =$ is constant for all values of t such that $(x(t), y(t))$ is defined on $\mathbb{R}^2$. Obviously, $\mathcal{H}$ is a first integral of system (1) if and only if

$$P\frac{\partial \mathcal{H}}{\partial x} + Q\frac{\partial \mathcal{H}}{\partial y} = 0 \tag{2}$$

on $\mathbb{R}^2$.

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Llibre and Xiang Zhang obtained in [14] all quadratic systems that have polynomial first integrals of degree less than or equal to 4 and did the topological classification of the quadratic phase portraits having such first integrals. But the complete study of all quadratic systems having polynomial first integral was done by Chavarriga, Garcia, Llibre, Pérez and Rodríguez in [6]. They provide the necessary and sufficient conditions in order that a quadratic system has a polynomial first integral. They also give the explicit polynomial first integral for all them in terms of the coefficients of a set of different normal forms. A remarkable fact of such study is that quadratic systems may have polynomial first integrals of any arbitrary degree. Finally, some of these authors characterize the phase portraits of all quadratic systems having a polynomial first integral (see [11]). Another remarkable fact is that no new phase portraits are possible beyond those which already are realizable by the subclass of all Hamiltonian quadratic systems (see [1]).

On the other hand, the use of the invariant theory applied to the study of differential polynomial systems (mainly quadratic ones) has allowed to extend the conditions for many families of quadratic systems from the used normal forms to the general system in the parameter space of 12 coefficients. In this direction, the works of the Sibirskii school have provided the necessary tools for determining the algebraic conditions on general systems to achieve most of the geometric properties of the problem. As example we can mention the classification of quadratic systems having a center [28], the global classification of all infinite singular points [23] and also finite [3], the classification of quadratic systems having more than 3 invariant straight lines [24, 20, 21, 22, 25], or the classification of quadratic systems having a rational first integral of degree up to 2 [4]. The invariant theory has also been very useful when applied to the study of some complicated bifurcation diagrams like the study of all the quadratic systems having a weak focus of second order [2].

In this paper we are going to apply the invariant theory to the work [6], that is we are going to give invariant conditions on the coefficients of any non degenerate quadratic system in order to determine if it has or not a polynomial first integral without using any normal form. Our main conclusion is that the existence of a polynomial first integral is directly related to the existence of negative rational roots of a convenient cubic polynomial $\Phi(z)$, whose coefficients are invariants related with some geometric properties of the system.

It is important to note that up to now the invariant theory had been applied in terms that some invariants must be zero, positive, negative or certain constants, but as far as we know this is the first time in which a quotient of invariants must belong to a non continuous set of numbers as the rationals.

Along the paper it becomes clear that the quadratic systems for which the roots of the polynomial $\Phi(z)$ are rational but not negative, have rational first integrals which can be of any degree like the polynomial ones. This will be stated as a corollary at the end of the paper. Unfortunately this does not complete the study of the rational first integrals for quadratic systems since some families of rational first integrals can be provided and proved that they cannot come from this possibility.

In the following, we denote by $\mathbb{N}$ and $\mathbb{Q}^-$ ($\mathbb{Q}^+$) the set of positive integers and the set of negative (positive) rational numbers, respectively.

Our main result is the following one.

Main Theorem. *A non-degenerate quadratic system has a polynomial first integral if and only if either*

$\mathcal{A}$) *it is Hamiltonian, or*
it is not Hamiltonian but one of the following sets of conditions holds:
$\mathcal{B}$) *If* $\theta = \mu_0 = 0$, $KM \neq 0$ *then* $R_1 = 0$ *and, either*
$\mathcal{B}_1$) $R_2 < 0$, $R_3 = 0$, $B_3 \neq 0$, $\mathcal{F}_1 = -2k$, $k \in \mathbb{N}$, *or*
$\mathcal{B}_2$) $R_2 < 0$, $R_3 = 0$, $B_3 = 0$, $\mathcal{F}_1 \in \mathbb{Q}^-$, *or*

$\mathcal{B}_3$) $R_2 > 0,\ B_3 \neq 0,\ \mathcal{F}_1 = -\dfrac{p}{q} \in \mathbb{Q}^-,\ \mathcal{F}_2 = \dfrac{r^2}{q^2} \in \mathbb{Q}^+,\ \dfrac{p+r}{2q} \in \mathbb{N},\ \mathcal{F}_2 < \mathcal{F}_1^2$, *or*

$\mathcal{B}_4$) $R_2 > 0,\ B_3 = 0,\ \mathcal{F}_1 = -\dfrac{p}{q} \in \mathbb{Q}^-,\ \mathcal{F}_2 = \dfrac{r^2}{s^2} \in \mathbb{Q}^+,\ \mathcal{F}_2 < \mathcal{F}_1^2$, *or*

$\mathcal{B}_5$) $R_2 = 0,\ R_3 = 0,\ B_3 \neq 0,\ \mathcal{F}_1 \in \mathbb{Q}^-,\ \mathcal{F}_1 < -1$.

$\mathcal{C}$) *If* $\theta = \mu_0 = K = M = 0$ *then* $B_3 = 0,\ R_9 \neq 0,\ \mathcal{F}_3 \in \mathbb{Q}^-$.

$\mathcal{D}$) *If* $\theta\mu_0 \neq 0$ *then* $B_1 = R_4 = 0$, $\Phi(z)$ *has three roots in* $\mathbb{Q}^-$ *and, either*

$\mathcal{D}_1$) $R_5 \neq 0,\ B_3 = 0$, *or*

$\mathcal{D}_2$) $R_5 \neq 0,\ B_3 \neq 0,\ R_6 = 0,\ R_7 = 0$, *or*

$\mathcal{D}_3$) $R_5 = 0,\ R_8 = 0$.

Here the invariant polynomials are defined in (5) *and* (6).

In Section 2 we introduce the necessary invariants and we recall the main theorem from [6].

In Section 3 we show the proof of our main result. We split it in several lemmas and divide it according to some subsets to make it clearer.

In Section 4 we give some corollaries related with polynomial first integrals and also with rational first integrals.

2. Preliminary

Consider real quadratic systems of the form:

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y) \end{aligned} \tag{3}$$

with homogeneous polynomials p_i and q_i $(i = 0,1,2)$ of degree i in x, y:

$$\begin{aligned} p_0 &= a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 &= b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2. \end{aligned}$$

We say that a quadratic system (3) is *non-degenerate* if $\gcd(P,Q) = 1$.

Let $a = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ be the 12-tuple of the coefficients of system (3) and denote $\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, x, y]$.

In order to find affine invariant conditions for determining the class of quadratic systems possessing a polynomial first integral we shall construct the necessary affine invariant polynomials as follows.

We consider the polynomials

$$\begin{aligned} C_i(a,x,y) &= yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y],\ i = 0,1,2, \\ D_i(a,x,y) &= \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y],\ i = 1,2. \end{aligned} \tag{4}$$

Using the so-called *transvectant of index* k (see [15]) of two polynomials f, $g \in \mathbb{R}[a,x,y]$

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}},$$

we construct the following GL—comitants of the second degree with the coefficients of the initial system

$$\begin{aligned} T_1 &= (C_0, C_1)^{(1)}, \quad T_2 = (C_0, C_2)^{(1)}, \quad T_3 = (C_0, D_2)^{(1)}, \\ T_4 &= (C_1, C_1)^{(2)}, \quad T_5 = (C_1, C_2)^{(1)}, \quad T_6 = (C_1, C_2)^{(2)}, \\ T_7 &= (C_1, D_2)^{(1)}, \quad T_8 = (C_2, C_2)^{(2)}, \quad T_9 = (C_2, D_2)^{(1)}. \end{aligned}$$

In order to be able to calculate the values of the needed invariant polynomials directly for every canonical system we shall define here a family of T—comitants (see [23] for detailed definitions) expressed through C_i $(i=0,1,2)$ and D_j $(j=1,2)$:

$$
\begin{aligned}
\tilde{A} &= \big(C_1, T_8 - 2T_9 + D_2^2\big)^{(2)}/144,\\
\widetilde{D} &= \Big[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6 - (C_1,T_5)^{(1)} + 6D_1(C_1D_2 - T_5) - 9D_1^2C_2\Big]/36,\\
\widetilde{E} &= \Big[D_1(2T_9 - T_8) - 3\,(C_1,T_9)^{(1)} - D_2(3T_7 + D_1D_2)\Big]/72,\\
\widetilde{F} &= \big[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0\,(D_2,T_9)^{(1)} - 9D_2^2T_4 + 288D_1\widetilde{E}\\
&\quad - 24\left(C_2,\widetilde{D}\right)^{(2)} + 120\left(D_2,\widetilde{D}\right)^{(1)} - 36C_1\,(D_2,T_7)^{(1)} + 8D_1\,(D_2,T_5)^{(1)}\Big]/144,\\
\widetilde{K} &= (T_8 + 4T_9 + 4D_2^2)/72,\\
\widetilde{H} &= (-T_8 + 8T_9 + 2D_2^2)/72.
\end{aligned}
$$

Finally we construct the invariant polynomials:

$$
\begin{aligned}
&M(a,x,y) = 2\,\mathrm{Hess}\big(C_2(x,y)\big) = (C_2,C_2)^{(2)},\\
&\eta(a) = \mathrm{Discriminant}\big(C_2(x,y)\big) = (M,M)^{(2)}/384,\\
&D(a,x,y) = \widetilde{D}(a,x,y),\\
&K(a,x,y) = 4\widetilde{K}(a,x,y) \equiv \mathrm{Jacob}\big(p_2(x,y), q_2(x,y)\big),\\
&H(a,x,y) = -4\widetilde{H}(a,x,y),\\
&\mu_0(a) = \mathrm{Discriminant}\big(K(x,y)\big) \equiv \mathrm{Discriminant}\big(H(x,y)\big) = -(K,K)^{(2)}/32,\\
&N(a,x,y) = K(a,x,y) + H(a,x,y),\\
&\theta(a) = \mathrm{Discriminant}\big(N(a,x,y)\big) = -(N,N)^{(2)}/2,\\
&B_3(a,x,y) = (C_2,\tilde{D})^{(1)} = Jacob\left(C_2,\tilde{D}\right),\\
&B_2(a,x,y) = (B_3,B_3)^{(2)} - 6B_3(C_2,\tilde{D})^{(3)},\\
&B_1(a) = \mathrm{Res}_x\left(C_2,\tilde{D}\right)/y^9 = -2^{-9}3^{-8}\,(B_2,B_3)^{(4)},\\
&R_1(a) = -\big((C_2,C_2)^{(2)},C_2)^{(1)},\tilde{D}\big)^{(3)},\\
&R_2(a,x,y) = 8H\big[(C_2,\tilde{D})^{(2)} + 8(\tilde{D},D_2)^{(1)}\big] + 3\big[(C_1,\,2H-N)^{(1)} - 2D_1N\big]^2,\\
&R_3(a,x,y) = (C_1,\ 2H-N)^{(1)} - 2D_1N,\\
&R_4(a) = (H,\tilde{E})^{(2)},\\
&R_5(a) = \big(((C_2,H)^{(1)},H)^{(2)},D_2\big)^{(1)},\\
&R_6(a,x,y) = \widetilde{E}(a,x,y),\\
&R_7(a) = 5\big[(C_2,\tilde{D})^{(3)}\big]^2 - 9\big(((\tilde{D},\tilde{D})^{(2)},D_2)^{(1)},D_2\big)^{(1)} - 63((\tilde{D},\tilde{F})^{(2)},D_2)^{(1)} +\\
&\qquad + 27((\tilde{D},\tilde{D})^{(2)},\tilde{H} - 4\tilde{K})^{(2)} - 18((C_2,\tilde{D})^{(2)},\tilde{F})^{(2)},\\
&R_8(a) = 2(C_2,\tilde{D})^{(3)}(\tilde{H},\tilde{H}-\tilde{K})^{(2)} - 144\tilde{A}(\tilde{E},\tilde{H})^{(2)} - 40\tilde{A}((C_2,\tilde{E})^{(2)},D_2)^{(1)} -\\
&\qquad - 18((\tilde{F},\tilde{H})^{(1)},\tilde{K})^{(2)} + \big(((C_2,\tilde{D})^{(2)},\tilde{K})^{(1)},\tilde{H}\big)^{(2)},\\
&R_9(a,x,y) = \big(2D_1^2 + (C_1,C_1)^{(2)}\big)/8,
\end{aligned}
\tag{5}
$$

as well as the following rational functions

$$
\begin{aligned}
&\mathcal{F}_1(a,x,y) = -2H/K, \quad \mathcal{F}_2(a,x,y) = 3\mathcal{F}_1^2 R_3^2/R_2, \quad \mathcal{F}_3(a,x,y) = -\tilde{D}/(C_2R_9),\\
&\Phi(a,z) = z^3 + \frac{2}{\theta}(4\mu_0 - 4\eta - \theta)z^2 + \frac{1}{\theta}(\theta - 16\mu_0)z + \frac{8\mu_0}{\theta},
\end{aligned}
\tag{6}
$$

which will be responsible for the polynomial integrability of quadratic systems.

Some geometrical meaning of the T-comitants C_2, M, η, μ_0, K, H, B_1, B_2, B_3, D, N, θ can be revealed (see [20]):

(i) The number and the type (complex or/and real) of infinite singularities of a system (3) are completely determined by the invariant polynomials C_2, M and η. More precisely, if $C_2 \neq 0$ (i.e. the infinite line is not fulfilled with singularities) then there exist three real distinct singularities (respectively one real and two complex; one double and one simple; one triple) if and only if $\eta > 0$ (respectively $\eta < 0$; $\eta = 0$ and $M \neq 0$; $\eta = M = 0$). Here by double (triple) point we means only the number of infinite singular points which can bifurcate from it.

(ii) $\gcd(p_2(x,y), q_2(x,y)) =$ constant (respectively $bx+cy$; $(bx+cy)(dx+ey)$; $(bx+cy)^2$) if and only if $\mu_0 \neq 0$ (respectively $\mu_0 = 0$ and $K \neq 0$; $\mu_0 = K = 0$ and $H \neq 0$; $\mu_0 = K = H = 0$), where $bx + cy, dx + ey \in \mathbb{C}[x,y]$ are some linear forms with $be - cd \neq 0$.

(iii) A necessary condition for the existence of one (respectively 2; 3) invariant straight line(s) in one (respectively two; three distinct) direction(s) in the affine plane of a system (3) is $B_1 = 0$ (respectively $B_2 = 0$; $B_3 = 0$).

(iv) If $ux + vy + w = 0$ is an invariant affine line of a system (3), then $D(-v,u) = 0 = C_2(-v,u)$, i.e. the T-comitants $D(a,x,y)$ and $C_2(a,x,y)$ capture the directions of invariant affine straight lines of systems (3).

(v) A necessary condition for the existence of one pair (respectively, two pairs) of parallel invariant straight lines of a system (3) is $\theta = 0$ (respectively, $\theta = N = 0$).

(vi) If $ux + vy + w_i = 0$ $(i = 1,2)$ are two parallel invariant affine lines of a system (3) then $H(-v,u) = 0$, i.e. the T-comitant $H(a,x,y)$ captures the directions of parallel invariant affine straight lines of systems (3).

Remark 1. We say that a comitant $U(a,x,y)$ is of *the type* (r,g,d) if it is a homogeneous polynomial of degree r in x and y, as well as homogeneous of degree d in the coefficients of the systems and if its weight equals g (for detailed definitions of the polynomials which are invariant under the action of the group of affine transformations see [26], [23]).

The proof of the Main Theorem is based on the classification of all non-degenerate quadratic systems having a polynomial first integral [6], using the constructed explicit normal form. We present here this result in the form as it is stated in [11].

Theorem PFI [*Theorem 4* [11]]. *A quadratic vector field without common factors has a polynomial first integral $H(x,y)$ if and only if there exists a change of the variables and of the time such that the new vector field $(P(x,y),\ a + bx + cy + dx^2 + exy + fy^2)$ is one of the followings:*

(i) $P(x,y) = 1+xy$, $f = -\frac{1}{2} \in \mathbb{Q}^-$, $c = e = 0$. *Then* $H(x,y) = ax+\frac{b}{2}x^2+\frac{d}{3}x^3-\frac{xy^2}{2}-y$.

(ii) $P(x,y) = xy$, $f \neq 0$ *and furthermore one has one of the following conditions:*

(a) $f = -\frac{p}{q} \in \mathbb{Q}^-$, $c = e = 0$ *and* $a^2 + b^2 + d^2 \neq 0$. *Then*

$$
H(x,y) = x^{2p}\left(\frac{a}{2f} + \frac{b}{2f-1}x + \frac{d}{2f-2}x^2 + \frac{y^2}{2}\right)^q.
$$

(b) $f=-\dfrac{s}{p+2k}$, *with* $s,p,k\in\mathbb{N}$, $e\neq 0$, $a=\dfrac{b^2f(f-1)}{d(1-2f)^2}$, $c=\dfrac{ebf}{d(2f-1)}$, *and* $d=-\dfrac{k(p+k)e^2}{p^2(f-1)}$. *Then* $H=x^sA^kB^{p+k}$, *where*

$$A=-b(f-1)^2p^2+e(2f-1)(k+p)(ekx-(f-1)py),$$
$$B=-b(f-1)^2p^2+e(2f-1)k(e(k+p)x+(f-1)py).$$

(c) $f=-1$, $d=\dfrac{k(p+k)e^2}{2p^2}$, $e\neq 0$ *and* $b=c=0$. *Then* $H=A^{k+p}B^k$, *where* $A=2p^2a+ekx(e(k+p)x-2py)$ *and* $B=2p^2a+e(k+p)x(ekx+2py)$.

(iii) $P(x,y)=y$, $f=c=e=0$ *and* $d\neq 0$. *Then* $H(x,y)=6ax+3bx^2-3y^2+2dx^3$.

(iv) $P(x,y)=1+x^2$ *and one of the following conditions holds:*

(a) $f=c=b=d=0$, $e=-2k$ *with* $k\in\mathbb{N}$ *and* $a=1$, *then*

$$H(x,y)=y(1+x^2)^k-\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{x^{2j+1}}{2j+1}.$$

(b) $f=a=c=b=d=0$, $e=-\frac{p}{q}\in\mathbb{Q}^-$, *then*

$$H(x,y)=(1+x^2)^py^{2q}.$$

(v) $P(x,y)=-1+x^2$ *and one of the following conditions holds:*

(a) $f=a=d=b=0$, $e=-\frac{p}{q}\in\mathbb{Q}^-$, $c=\frac{r}{s}\in(e,-e)\cap\mathbb{Q}$, *then*

$$H(x,y)=(x+1)^{sp+rq}(x-1)^{sp-rq}y^{2sq};$$

(b) $f=d=b=0$, $e=-\dfrac{p}{q}\in\mathbb{Q}^-$, $c=\dfrac{r}{q}$, *with* $r\in\mathbb{Z}$, $-p<r<p$ *and* $p\equiv r$ *(mod 2) and furthermore* $\alpha=\dfrac{p-r}{2q}\in\mathbb{N}$ *and* $a=1$. *Then*

$$H(x,y)=(x+1)^{p+r}\left(y(x-1)^{\alpha}-\left[\sum_{j=0}^{\alpha-1}(-2)^{\alpha-1-j}\binom{\alpha-1}{j}\frac{(x+1)^j}{\beta+j}\right]\right)^{2q},$$

where $\beta=(p+r)/2q$.

(c) $f=d=b=0$, $e=-\frac{p}{q}\in\mathbb{Q}^-$, $c=\dfrac{r}{q}$*with* $r\in\mathbb{Z}$, $-p<r<p$ *and* $p\equiv r$ *(mod 2) and furthermore* $\beta=\dfrac{p+r}{2q}\in\mathbb{N}$ *and* $a=1$. *Then*

$$H(x,y)=(x-1)^{p-r}\left(y(x+1)^{\beta}-\left[\sum_{j=0}^{\beta-1}2^{\beta-1-j}\binom{\beta-1}{j}\frac{(x-1)^j}{\alpha+j}\right]\right)^{2q},$$

where $\alpha=\dfrac{p-r}{2q}$.

(vi) $P(x,y)=x^2$, $f=c=b=d=0$, $a=1$ *and* $e=-\frac{p}{q}\in\mathbb{Q}^-\cap(-\infty,-1)$. *Then* $H(x,y)=x^{p-q}\left(xy+\dfrac{q}{q-p}\right)^q$.

(vii) $P(x,y)=x$, $f=e=a=b=0$ *and* $c=-\frac{p}{q}\in\mathbb{Q}^-$. *Then* $H(x,y)=x^p\left(y-\frac{d}{2-c}x^2\right)^q$.

(viii) $P(x,y)=1$, $f=c=e=a=0$ *and* $d\neq 0$. *Then* $H(x,y)=y-\frac{1}{2}bx^2-\frac{1}{3}dx^3$.

Remark 2. *Note that statement* (vi) *sets that the first integral is* $H(x,y)=x^{p-q}\left(xy+\dfrac{q}{q-p}\right)^q$ *instead of* $H(x,y)=x^{p-q}\left(xy+\dfrac{q-p}{q}\right)^q$ *as it is given in* [6] *and* [11]. *Sure this is a typeset* *error in the original paper since it is clear that the formula does not fit and the repaired one does.*

3. Proof of the Main Theorem

We split the proof of the Main Theorem in four theorems respective to the cases $\mathcal{A}$), $\mathcal{B}$), $\mathcal{C}$) and $\mathcal{D}$).

According to Theorem PFI the following proposition holds.

Proposition 3. *If a quadratic system* (3) *possesses a polynomial first integral then either it is* $\mathcal{A}$) *Hamiltonian, or via a linear transformation its quadratic part* (p_2, q_2) *could be brought to one of the form:*

$$\mathcal{B})\ (x^2,\ exy); \qquad \mathcal{C})\ (0,\ x^2); \qquad \mathcal{D})\ (xy,\ dx^2+exy+fy^2).$$

Lemma 4. *The homogeneous parts* (p_2, q_2) *of a quadratic system* (3) *could be brought via a linear transformation to the form* $\mathcal{B}$) *with* $e(e-1)\neq 0$ *(respectively, to the form* $\mathcal{C}$) *if and only if* $\theta=\mu_0=0$, $KM\neq 0$ *(respectively,* $K=M=0$ *and this implies* $\theta=\mu_0=0$*).*

Proof. Necessity. Assume that a system (3) has the quadratic parts of the form $(p_2, q_2) = (x^2, exy)$. Then calculations yield

$$\theta=\mu_0=0, \quad K=2ex^2, \quad M=-8(e-1)^2x^2,$$

and clearly $KM\neq 0$ due to the condition $e(e-1)\neq 0$.

If $(p_2,q_2)=(0,x^2)$ then we have $\theta=\mu_0=K=M=0$.

Sufficiency. We consider the homogeneous quadratic systems

$$\dot{x}=gx^2+2hxy+ky^2, \quad \dot{y}=lx^2+2mxy+ny^2. \tag{7}$$

Without loss of generality we may assume $k=0$ doing a rotation (if necessary). Then for these systems we calculate

$$\begin{gathered}\theta=-64h\big[l(n-h)^2+gm(n-h)-m^2n\big],\\ \mu_0=n\big[4h(hl-gm)+g^2n\big],\ K=4\big[(gm-hl)x^2+gnxy+hny^2\big],\\ M=-8\big[(g-2m)^2+3l(2h-n)\big]x^2-8(g-2m)(2h-n)xy-8(2h-n)^2y^2\big].\end{gathered} \tag{8}$$

1) Assume first that for systems (7) the conditions $\theta=\mu_0=0$ and $KM\neq 0$ hold.

a) If $h=0$ then the conditions $\mu_0=g^2n^2=0$ and $K=4gx(mx+ny)\neq 0$ imply $g\neq 0$, $n=0$ and we get the family of systems

$$\dot{x}=gx^2, \quad \dot{y}=lx^2+2mxy,$$

for which $M=-8(g-2m)^2x^2\neq 0$. Then applying the transformation $(x,y)\mapsto\big(x/g,\ (lx+gy)/(g(g-2m))\big)$ we get the systems $\dot{x}=x^2,\ \dot{y}=exy$, where $e=2m/g$.

b) If $h\neq 0$ then $n\neq 0$, otherwise we get a contradiction. Indeed supposing $n=0$ we obtain $\theta=64h^2(gm-hl)=0$ and $K=4(gm-hl)x^2\neq 0$. So $hn\neq 0$ and considering (8), the conditions $\theta=\mu_0=0$ yield $m=g(n-h)/(2h)$ and $l=g^2(n-2h)/(4h^2)$. In this case applying the transformation $(x,y)\mapsto\big(y,\ (2hx-gny)/(2hn)\big)$ we obtain the systems $\dot{x}=x^2,\ \dot{y}=exy$, where $e=2h/n$.

2) Assume now that for systems (7) the conditions $K=M=0$ hold. Then from (8) it obviously follows $n=h=g=m=0$ and then $\theta=\mu_0=0$. Since the system must be quadratic (i.e. $l\neq 0$) then due to the rescalling $y\to ly$ we get the system $\dot{x}=0,\ \dot{y}=x^2$ and this completes the proof of Lemma 4. ∎

Remark 5. *According to Theorem PFI (see subcases* (ii),(a)-(c)*) in the case* $\mathcal{D}$) *for the existence of a polynomial first integral it is necessary* $f<0$. *Then the conditions* $\theta=\mu_0=0$ *yield* $d=e=0$, *and then we have* $KM=-16f(f-1)^2y^4\neq 0$, *i.e. we come to the case* $\mathcal{B}$)

(with $e = 1/f$). So to obtain from $\mathcal{D}$) the cases different from $\mathcal{B}$), the condition $\theta^2 + \mu_0^2 \neq 0$ must hold for this family.

We observe that for these four families of systems (i.e. systems $\mathcal{A}$) and systems with quadratic parts $\mathcal{B}$), $\mathcal{C}$) and $\mathcal{D}$)) we have the following distribution of the subfamilies from Theorem PFI:

$\mathcal{A}$): $\Leftrightarrow$ the cases: (i); (iii); (iv), (a) in the case $k = 1$; (viii);
$\mathcal{B}$): $\Leftrightarrow$ the cases: (iv), (a),(b); (v), (a)-(c); (vi); (ii), (a) in the case $d = 0$;
$\mathcal{C}$): $\Leftrightarrow$ the case (vii);
$\mathcal{D}$): $\Leftrightarrow$ the cases: (ii), (a)-(c).

3.1. **Case $\mathcal{A}$).** In the case of Hamiltonian systems a more generic assertion can be done for polynomial systems of degree n

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) + ... + p_n(x,y) \equiv P(x,y),\\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) + ... + q_n(x,y) \equiv Q(x,y). \end{aligned} \tag{9}$$

More exactly we have the next result.

Theorem $\mathcal{A}$. *If a polynomial system* (9) *verifies the conditions*

$$D_i(x,y) = 0 \ (i = 1, 2, ..., n) \ \Leftrightarrow \frac{\partial}{\partial x}P(x,y) + \frac{\partial}{\partial y}Q(x,y) \equiv 0, \tag{10}$$

i.e. it is Hamiltonian, then this system possesses the following polynomial first integral

$$H(x,y) = \sum_{j=0}^{n} \frac{1}{j+1} C_j(x,y), \tag{11}$$

where

$$\begin{aligned} C_i(x,y) &= yp_i(x,y) - xq_i(x,y), \ i = 0, 1, ..., n,\\ D_i(x,y) &= \frac{\partial}{\partial x}p_i(x,y) + \frac{\partial}{\partial y}q_i(x,y), \ i = 1, 2, ...n \end{aligned}$$

are the GL-comitants of systems (9).

Proof. Assuming that the conditions (10) hold we shall show that polynomial $H(x,y)$ of the form (11) is a first integral of systems (9). Indeed considering (11) and applying the Euler's formula for homogeneous polynomials we have

$$\begin{aligned} \frac{\partial H}{\partial x} &= \sum_{j=0}^{n} \frac{1}{j+1}\left(y\frac{\partial p_j}{\partial x} - q_j - x\frac{\partial q_j}{\partial x}\right) \overset{(10)}{=} \sum_{j=0}^{n} \frac{1}{j+1}(-jq_j - q_j) = -\sum_{j=0}^{n} q_j = -Q(x,y);\\ \frac{\partial H}{\partial y} &= \sum_{j=0}^{n} \frac{1}{j+1}\left(p_j + y\frac{\partial p_j}{\partial y} - x\frac{\partial q_j}{\partial y}\right) \overset{(10)}{=} \sum_{j=0}^{n} \frac{1}{j+1}(p_j + jp_j) = \sum_{j=0}^{n} p_j = P(x,y). \end{aligned}$$

Therefore

$$H_x\frac{dx}{dt} + H_y\frac{dy}{dt} = -Q(x,y)P(x,y) + P(x,y)Q(x,y) = 0$$

and this proves the lemma. $\square$

3.2. **Case $\mathcal{B}$).** In what follows we shall use the notation $\mathcal{R} = \dfrac{\partial}{\partial x}P(x,y) + \dfrac{\partial}{\partial y}Q(x,y)$ for the divergence of system (3). Therefore in order that a quadratic system be non-Hamiltonian it must satisfy the condition $\mathcal{R} \neq 0$.

Theorem $\mathcal{B}$. *Assume that for a non-hamiltonian and non-degenerate quadratic system (3) the conditions $\theta = \mu_0 = 0$ and $KM \neq 0$ are fulfilled. Then this system possesses a polynomial first integral if and only if $R_1 = 0$ and one of the following sets of conditions holds.*

$\mathcal{B}_1$) $R_2 < 0$, $R_3 = 0$, $B_3 \neq 0$, $\mathcal{F}_1 = -2k$, $k \in \mathbb{N}$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x} = 1 + x^2, \quad \dot{y} = 1 - 2kxy, \tag{12}$$

possessing the polynomial first integral

$$\mathcal{H}(x,y) = y(1+x^2)^k - \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{x^{2j+1}}{2j+1}. \tag{13}$$

$\mathcal{B}_2$) $R_2 < 0$, $R_3 = 0$, $B_3 = 0$, $\mathcal{F}_1 = -\dfrac{p}{q} \in \mathbb{Q}^-$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x} = 1 + x^2, \quad \dot{y} = -\frac{p}{q}\, xy, \tag{14}$$

possessing the polynomial first integral

$$\mathcal{H}(x,y) = y^{2q}\left(1+x^2\right)^p; \tag{15}$$

$\mathcal{B}_3$) $R_2 > 0$, $B_3 \neq 0$, $\mathcal{F}_1 = -\dfrac{p}{q} \in \mathbb{Q}^-$, $\mathcal{F}_2 = \dfrac{r^2}{q^2} \in \mathbb{Q}^+$, $\mathcal{F}_2 < \mathcal{F}_1^2$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x} = -1 + x^2, \quad \dot{y} = 1 + \frac{r}{q}\, y - \frac{p}{q}\, xy, \quad p,q,r \in \mathbb{N},\ p > r, \tag{16}$$

and if the additional condition $\dfrac{p+r}{2q} = \beta \in \mathbb{N}$ *holds, then this system possesses the polynomial first integral (with* $\alpha = \dfrac{p-r}{2q}$*)*

$$\mathcal{H}(x,y) = (x-1)^{p-r}\left(y(x+1)^\beta - \left[\sum_{j=0}^{\beta-1} 2^{\beta-1-j}\binom{\beta-1}{j}\frac{(x-1)^j}{\alpha+j}\right]\right)^{2q}. \tag{17}$$

$\mathcal{B}_4$) $R_2 > 0$, $B_3 = 0$, $\mathcal{F}_1 = -\dfrac{p}{q} \in \mathbb{Q}^-$, $\mathcal{F}_2 = \dfrac{r^2}{s^2} \in \mathbb{Q}^+$, $\mathcal{F}_2 < \mathcal{F}_1^2$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x} = -1 + x^2, \quad \dot{y} = \frac{r}{s}\, y - \frac{p}{q}\, xy, \quad p,q,r,s \in \mathbb{N},\ ps - qr > 0, \tag{18}$$

possessing the polynomial first integral

$$\mathcal{H}(x,y) = (x+1)^{ps+qr}(x-1)^{ps-qr}y^{2qs}; \tag{19}$$

$\mathcal{B}_5$) $R_2 = 0$, $R_3 = 0$, $B_3 \neq 0$, $\mathcal{F}_1 = -\dfrac{p}{q} \in \mathbb{Q}^-$, $\mathcal{F}_1 < -1$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x} = x^2, \quad \dot{y} = 1 - \frac{p}{q}\, xy, \quad p > q, \tag{20}$$

possessing the polynomial first integral

$$\mathcal{H}(x,y) = x^{p-q}\left(xy + \frac{q}{q-p}\right)^q. \tag{21}$$

Proof. As for a quadratic system the conditions $\theta = \mu_0 = 0$ and $KM \neq 0$ hold, considering Lemma 4 and applying an additional translation (excluding the linear terms in the variable x from both polynomials $P(x,y)$ and $Q(x,y)$ we shall consider the following family of systems

$$\dot{x} = a_1 + c_1 y + x^2, \quad \dot{y} = a + cy + exy. \tag{22}$$

We calculate

$$\begin{gathered} \mathcal{R} = c + (e+2)x,\ K = 2ex^2,\ H = -e^2x^2, \\ M = -8(e-1)^2x^2,\ R_1 = 576c_1^2(e-1)^3. \end{gathered} \tag{23}$$

Since this family is not Hamiltonian the condition $\mathcal{R} \neq 0$ is verified.

Remark 6. *Since the polynomials K and H are both of the same type* $(2,0,2)$ *(see Remark 1), we conclude that the rational function $F_1 = -\dfrac{2H}{K} = e$ is an absolute invariant for the family* (22) *with respect to the group Aff*$(2,\mathbb{R})$ *of affine transformations as well as with respect to the time rescalling.*

According to Theorem PFI if a quadratic of system could be brought via an affine transformation and time rescaling to the form (22) (i.e. its quadratic part $(p_2, q_2) = (x^2,\ exy)$) then this system could possess a polynomial first integral only if $c_1 = 0$. As $M \neq 0$ (i.e. $e - 1 \neq 0$) the last condition is equivalent to $R_1 = 0$ in a continues way. Thus we get the family of systems

$$\dot{x} = a_1 + x^2, \quad \dot{y} = a + cy + exy, \tag{24}$$

for which

$$R_2 = -192a_1e^4x^4, \quad R_3 = -8cex^2 \neq 0, \quad B_3 = 3ae^2(e-1)x^4. \tag{25}$$

Following Theorem PFI we shall consider the cases when $a_1 < 0$, $a_1 > 0$ and $a_1 = 0$. Taking into account (25) and $K \neq 0$ these conditions are equivalent with $R_2 > 0$, $R_2 < 0$ and $R_2 = 0$, respectively.

1) Assume first $R_2 < 0$. Then $a_1 > 0$ and we may consider $a_1 = 1$ due to the transformation $(x,y,t) \mapsto (a^{1/2}x, a^{1/2}y, a^{-1/2}t)$. So we arrive to case (iv) of Theorem PFI and from this theorem it follows, that systems (24) with $a_1 = 1$ possess a polynomial first integral only if $c = 0$. According to (25) the last condition is equivalent to $R_3 = 0$. Thus we obtain the systems

$$\dot{x} = 1 + x^2, \quad \dot{y} = a + exy \tag{26}$$

and following Theorem PFI we shall consider two subcases: $a \neq 0$ and $a = 0$. Considering (25) these conditions are governed by the invariant polynomial B_3.

a) If $B_3 \neq 0$ then $a \neq 0$ and we may consider $a = 1$ due to the change $y \mapsto ay$. According to Theorem PFI (see (iv),(a)) systems (26) (with $a = 1$) possess a polynomial first integral if and only if $e = -2k$ with $k \in \mathbb{N}$. Since for these systems $\mathcal{F}_1 = e$ we conclude that for $\mathcal{F}_1 = -2k$ we get the family of systems (12) possessing the polynomial first integral (13). So we obtain the case $\mathcal{B}_1$) of the theorem.

b) Admit now $B_3 = 0$, i.e. $a = 0$. According to Theorem PFI (see (iv),(b)) systems (26) (with $a = 0$) possess a polynomial first integral if and only if $e = -p/q \in \mathbb{Q}^-$. Since for these systems $\mathcal{F}_1 = e$ we conclude that for $\mathcal{F}_1 = -p/q \in \mathbb{Q}^+$ we get the family of systems (14) possessing the polynomial first integral (15). Thus we arrive to the case $\mathcal{B}_2$) of the theorem.

2) Assuming $R_2 > 0$ from (25) we obtain $a_1 < 0$ and we may consider $a_1 = -1$ due to the transformation $(x, y, t) \mapsto ((-a)^{1/2}x, (-a)^{1/2}y, (-a)^{-1/2}t)$. So we arrive to the case (v) of Theorem PFI and we shall consider the family of systems (24) with $a_1 = -1$. We need again to distinguish the subcases $a \neq 0$ and $a = 0$.

a) If $B_3 \neq 0$ by (25) we have $a \neq 0$ and we may consider $a = 1$ due to the change $y \mapsto ay$. Therefore we obtain the family of systems

$$\dot{x} = -1 + x^2, \quad \dot{y} = 1 + cy + exy, \tag{27}$$

which according to Theorem PFI (see (v),(c)) possess a polynomial first integral if and only if $e = -p/q \in \mathbb{Q}^-$ and $c = r/q$ with $r \in \mathbb{Z}$, $-p < r < p$ and $\dfrac{p+r}{2q} \in \mathbb{N}$.

On the other hand for systems (27) calculations yield

$$\mathcal{F}_1 = e, \quad \mathcal{F}_2 = c^2. \tag{28}$$

Remark 7. *Since the polynomial R_3 (respectively R_2) is of type $(2, 0, 3)$ (respectively $(4, 0, 6)$) we obtain that the T-comitants R_3^2 and R_2 are both of the same type $(4, 0, 6)$ (see Remark 1). Therefore considering Remark 6 we conclude that the rational function $\mathcal{F}_2(a, x, y) = 3\mathcal{F}_1^2 \dfrac{R_3^2}{R_2}$ is an absolute invariant for the family (27) with respect to the group $Aff(2, \mathbb{R})$ as well as with respect to the time rescalling.*

In short we need to have $\mathcal{F}_1 = -\dfrac{p}{q}$ and $\mathcal{F}_2 = \dfrac{r^2}{q^2}$ with $r \in \mathbb{Z}$ and $\dfrac{p+r}{2q} \in \mathbb{N}$. Moreover if the previous conditions hold then we have $\mathcal{F}_2 - \mathcal{F}_1^2 = \dfrac{r^2 - p^2}{q^2}$ and clearly, the condition $-p < r < p$ is equivalent to $\mathcal{F}_2 < \mathcal{F}_1^2$. In other words we get the family of systems (16) possessing the polynomial first integral (17). In such a way we obtain the case $\mathcal{B}_3$) of the theorem.

b) Assume $B_3 = 0$, i.e. $a = 0$. According to Theorem PFI (see (v),(a)) systems (27) (with $a = 0$) possess a polynomial first integral if and only if $e = -p/q \in \mathbb{Q}^-$ and $c = r/s \in (e, -e) \bigcap \mathbb{Q}$. As for systems (27) the relations (28) hold, we need to have $\mathcal{F}_1 = -\dfrac{p}{q}$ and $\mathcal{F}_2 = \dfrac{r^2}{s^2}$. Moreover in this case we obtain $\mathcal{F}_2 - \mathcal{F}_1^2 = \left(\dfrac{r}{s}\right)^2 - \left(\dfrac{p}{q}\right)^2$ and clearly, the condition $\mathcal{F}_2 < \mathcal{F}_1^2$ is equivalent to $e = -p/q < r/s < p/q = -e$. So we get the family of systems (18) possessing the polynomial first integral (19). This leads to the case $\mathcal{B}_4$) of the theorem.

3) Assume finally $R_2 = 0$. Then $a_1 = 0$ and systems (24) become

$$\dot{x} = x^2, \quad \dot{y} = a + cy + exy, \tag{29}$$

and according to Theorem PFI (see (vi)) these systems possess a polynomial first integral only if $c = 0$ and $a \neq 0$ (then we may assume $a = 1$ due to the change $y \mapsto ay$). Considering (25) and the relation $KM \neq 0$, the last conditions are equivalent to $R_3 = 0$ and $B_3 \neq 0$, respectively. Moreover according to Theorem PFI (see (vi)) systems (29) (with $c = 0$ and $a = 1$) possess a polynomial first integral if and only if $e = -p/q \in \mathbb{Q}^- \bigcap (-\infty, -1)$. As for these systems we have $\mathcal{F}_1 = e$, we need to have $\mathcal{F}_1 = -\dfrac{p}{q} < -1$. In in this case we arrive to the family of systems (20) possessing the polynomial first integral (21). This leads to the case $\mathcal{B}_5$) of the theorem. ☐

Remark 8. *We note that the condition $\dfrac{p+r}{2q} \in \mathbb{N}$ (see case $\mathcal{B}_3$) of Theorem $\mathcal{B}$) implies $p \equiv r (\text{mod } 2)$ (see Theorem PFI,* (v), (b) and (c)*). We remark also that in the case* (v) *of Theorem PFI the subcase* (b) *could be reduced to the subcase* (c) *via the transformation* $(x, y, t) \mapsto (-x, -y, -t)$.

3.3. **Case $\mathcal{C}$). Theorem $\mathcal{C}$.** *Assume that for a non-Hamiltonian and non-degenerate quadratic system* (3) *the conditions* $K = M = 0$ *(then* $\theta = \mu_0 = 0$*) are fulfilled. Then this system possesses a polynomial first integral if and only if the following conditions hold*

$$B_3 = 0, \quad R_9 \neq 0, \quad \mathcal{F}_3 = -\frac{p}{q} \in \mathbb{Q}^-.$$

Moreover in this case the system can be brought via an affine transformation and time rescalling to the system

$$\dot{x} = x, \quad \dot{y} = -\frac{p}{q}\, y + x^2, \tag{30}$$

possessing the polynomial first integral

$$\mathcal{H}(x, y) = x^p \left(y - \frac{q}{p + 2q} x^2 \right)^q. \tag{31}$$

Proof. Assume that for a quadratic system the conditions $K = 0 = M$ are satisfied (by Lemma 4 this implies $\theta = \mu_0 = 0$). Taking into account Lemma 4 and applying an additional translation (excluding the linear terms with variable x from the polynomial $Q(x, y)$, we shall consider the following family of systems

$$\dot{x} = a_1 + b_1 x + c_1 y, \quad \dot{y} = a + cy + x^2. \tag{32}$$

For these systems we calculate

$$\mathcal{R} = b_1 + c,\ B_3 = 6c_1 x^3 (cx - c_1 y),\ R_9 = b_1 c. \tag{33}$$

Since this family of systems is not Hamiltonian (i.e. $\mathcal{R} \neq 0$), according to Theorem PFI (see subcase (vii)) systems (33) could have a polynomial first integral only if $c_1 = 0$ (i.e. $B_3 = 0$) and $b_1 c \neq 0$.

We note that in the case $c_1 = 0$ the GL-invariant R_9 keeps the value, indicated in (33), after any translation $(x, y) \mapsto (\tilde{x} + x_0, \tilde{y} + y_0)$ with arbitrary $(x_0, y_0) \in \mathbb{R}^2$. Indeed, for any system located in the orbit under the translation group action of a system (32) we have $R_9 = b_1 c - 2c_1 x_0$. This means that the polynomial R_9 is a CT-comitant [20] for the family of systems (33) with $c_1 = 0$ (i.e. $B_3 = 0$), and it forces the condition $b_1 c \neq 0$. So in what follows we assume $B_3 = 0$ and $R_9 \neq 0$. Then $c_1 = 0$ and we may assume $b_1 = 1$ and $a = 0$ due to the transformation $(x, y, t) \mapsto (x, y/b_1 - a/c, t/b_1)$. Thus we get the following family of systems

$$\dot{x} = a_1 + x, \quad \dot{y} = cy + x^2, \tag{34}$$

for which we calculate: $\mathcal{F}_3 = c$.

Remark 9. *We note that the invariant polynomials D and C_2 are T-comitants of quadratic systems* (3)*, whereas R_9 (as it was mentioned above) is a CT-comitant for the canonical systems* (34)*. Therefore since D and $R_9 C_2$ are both invariant polynomials of the same type* $(3, -1, 3)$ *(see Remark 1), we conclude that the rational function* $\mathcal{F}_3 = -\dfrac{D}{R_9 C_2} = c$ *is an absolute invariant for the family* (34) *with respect to the group* $Aff(2, \mathbb{R})$ *as well as with respect to the time rescalling.*

On the other hand in the case $a_1 = 0$ the family (34) corresponds to the family (vii) of the Theorem PFI. According to this theorem we conclude, that a system (34) will possess a polynomial first integral only if $c = -\dfrac{p}{q} \in \mathbb{Q}^-$ (i.e. if $\mathcal{F}_3 = -\dfrac{p}{q}$) and this implies $c(c-1) \neq 0$.

Then without loss of generality we may assume $a_1 = 0$ due to the transformation $(x, y) \mapsto (x - a_1, 2a_1x/(c-1) + y - a_1^2/c)$.

Thus, we arrive to the family of systems (30) possessing the polynomial first integral (31) and this completes the proof of the theorem. $\square$

3.4. **Case $\mathcal{D}$).** We first shall prove some results concerning the homogenous parts of quadratic systems.

We consider the homogeneous quadratic systems

$$\dot{x} = gx^2 + hxy + ky^2 = p_2(x, y), \quad \dot{y} = lx^2 + mxy + ny^2 = q_2(x, y), \tag{35}$$

and the following equation

$$\Phi(z) = z^3 + \varphi_1 z^2 + \varphi_2 z + \varphi_3 = 0, \tag{36}$$

where according to (6) we have

$$\varphi_1 = \frac{8\mu_0 - 8\eta - 2\theta}{\theta}, \quad \varphi_2 = \frac{\theta - 16\mu_0}{\theta}, \quad \varphi_3 = \frac{8\mu_0}{\theta}.$$

As the polynomials μ_0, η and θ are of the same type $(0, 2, 4)$ (see Remark 1), we conclude that in the case $\theta \neq 0$ the quotients φ_i $(i = 1, 2, 3)$ are absolute rational affine invariants of these systems.

Considering Theorem PFI the next result could be obtained immediately.

Lemma 10. *Assume that for a non-Hamiltonian and non-degenerate quadratic system* (3) *the condition $\theta^2 + \mu_0^2 \neq 0$ holds. Then this system possesses a polynomial first integral only if $\theta\mu_0 \neq 0$.*

Proof. According to Theorem PFI and considering Theorems $\mathcal{B}$ and $\mathcal{C}$ we conclude that for $\theta^2 + \mu_0^2 \neq 0$ a non-Hamiltonian and non-degenerate quadratic system could possess a polynomial first integral only if via a linear transformation it could be brought to the system with quadratic parts $(p_2, q_2) = (xy, dx^2 + exy + fy^2)$, i.e. to the families *(ii), (a), (b), (c)* of Theorem PFI. Moreover for these families of systems we must have $f < 0$.

On the other hand for the class of systems with such quadratic parts we calculate

$$\mu_0 = df, \quad \theta = 8\big[e^2 f - d(2f - 1)^2\big].$$

Suppose the contrary that $\theta\mu_0 = 0$.

1) If $\mu_0 = 0$ then due to $f < 0$ we get $d = 0$, and then the condition $\theta = 8e^2 f \neq 0$ yields $e \neq 0$ and this leads to the families *(ii), (b)* and *(ii), (c)*. However for these families according to Theorem PFI the condition $d = -\dfrac{k(p+k)}{p^2(f-1)}$ holds, where $f < 0$ (in the case *(ii), (c)* we have $f = -1$). So we get the contradiction $d = 0 = k(p + k)$ and $p, k \in \mathbb{N}$.

2) Assume now $\theta = 0$ and $\mu_0 \neq 0$. Then $d = \dfrac{e^2 f}{(2f-1)^2}$, and as $f < 0$ we obtain $d < 0$. On the other hand the condition $d = -\dfrac{k(p+k)}{p^2(f-1)}$ implies $d > 0$ (due to $f < 0$), and the obtained contradiction completes the proof of the lemma. $\square$

According to Theorem PFI and considering Theorems $\mathcal{B}$, $\mathcal{C}$ and Lemma 10 we conclude, that for $\theta\mu_0 \neq 0$ a non-Hamiltonian and non-degenerate quadratic system could possess a polynomial first integral only if via a linear transformation it could be brought to the form $(\dot{x}, \dot{y}) = (xy, Q)$, i.e. if this system belongs to the families *(ii), (a), (b), (c)* of Theorem PFI. The next assertion gives the respective affine invariant conditions.

Lemma 11. *Assume that for a non-Hamiltonian and non-degenerate quadratic system* (3) *the conditions* $\mu_0\theta \neq 0$ *and* $R_4 = 0$ *hold. Then this system could be brought via an affine transformation to the system*

$$\dot{x} = xy, \quad \dot{y} = a + bx + cy + dx^2 + exy + fy^2, \tag{37}$$

if and only if the condition $B_1 = 0$ *is verified.*

Proof. Consider a quadratic system such that its homogeneous part coincides with the one of systems (35). Clearly we may assume $k = 0$ doing a rotation (if necessary), and for this family of systems we calculate

$$\theta = -8h\big[l(2n-h)^2 + gm(2n-h) - m^2 n\big].$$

Hence the condition $\theta \neq 0$ implies $h \neq 0$, and therefore via the linear transformation $x_1 = x$, $y_1 = gx + hy$ we get a quadratic system with homogeneous parts of the form (37). Obviously applying an additional translation we can remove the linear terms from the first equation and this leads to the family of systems

$$\dot{x} = a_1 + xy, \quad \dot{y} = a + bx + cy + dx^2 + exy + fy^2. \tag{38}$$

For these systems we calculate

$$\theta = -8[d(1-2f)^2 - e^2 f], \quad \mu_0 = df, \quad R_4 = 2[cd(2f-1) - bef], \quad B_1 = a_1\tilde{B}_1,$$

where $\tilde{B}_1(a_1, a, b, c, d, e, f)$ is a polynomial. So the *necessity* of the condition $B_1 = 0$ in order to have $a_1 = 0$ is evident.

Sufficiency. Assume now $B_1 = 0$. As the condition $R_4 = 0$ holds we shall consider two cases $f \neq 1/2$ and $f = 1/2$.

1) The case $f \neq 1/2$**.** Then $R_4 = 0$ yields $c = \dfrac{bef}{d(2f-1)}$ ($d \neq 0$ as $\mu_0 \neq 0$), and calculations yield

$$B_1 = \frac{a_1\theta^2}{64d^2(2f-1)^4}\big[a_1^2 d^3(2f-1)^4 + a_1 de(2f-1)^2\varkappa + (f-1)\varkappa^2\big] \equiv \frac{a_1\theta^2\bar{B}_1}{64d^2(2f-1)^4}, \tag{39}$$

where $\varkappa = ad(2f-1)^2 - b^2 f(f-1)$ and we obtain

$$D_B = \text{Discrim}\,[\bar{B}_1, a_1] = d^2(2f-1)^4\big[4d(1-f) + e^2\big]\varkappa^2. \tag{40}$$

So since $B_1 = 0$ and $\theta \neq 0$ we have either $a_1 = 0$, or $\bar{B}_1 = 0$, and this implies $D_B \geq 0$. Supposing $a_1 \neq 0$ we get the second relation and we consider two subcases $4d(1-f)+e^2 \geq 0$ and $4d(1-f)+e^2 < 0$.

a) Assume first $4d(1-f)+e^2 \geq 0$. As for systems (38) we have $C_2 = -x\big[dx^2 + exy + (f-1)y^2\big]$ and this leads to the following factorization over $\mathbb{R}$ of the binary form $dx^2 + exy + (f-1)y^2 = (px+qy)(rx+sy)$. Thus we obtain the relations

$$d = pr, \quad e = qr + ps, \quad f = 1 + qs$$

and in this case for system (38) we calculate

$$\bar{B}_1 = Z_1 Z_2, \quad \mu_0 = pr(1+qs), \quad \theta = -8(p + pqs - q^2 r)(r + qrs - ps^2), \tag{41}$$

where

$$Z_1 = (a_1 p + aq)pr(1+2qs)^2 - b^2 q^2 s(1+qs),$$
$$Z_2 = (a_1 r + as)pr(1+2qs)^2 - b^2 qs^2(1+qs),$$

Therefore the condition $B_1 = 0$ yields $Z_1Z_2 = 0$, and without loss of generality we may assume $Z_1 = 0$ due to the change $p \leftrightarrow r$, $q \leftrightarrow s$. Therefore since $\mu_0 \neq 0$ and $2f - 1 =$

$1+2qs \neq 0$ the relation $Z_1 = 0$ yields

$$a_1 = \frac{b^2q^2s(1+qs)}{p^2r(1+2qs)^2} - aq/p.$$

Then via the affine transformation

$$\chi: \; x_1 = px + qy + \frac{bqs}{r(1+2qs)}, \quad y_1 = qrx + (1+qs)y + \frac{bq(1+qs)}{p(1+2qs)}$$

with $\det(\chi) = (p+pqs-q^2r) \neq 0$ (due to $\theta \neq 0$) systems (38) will be written as systems (37).

b) Assume now $4d(1-f)+e^2 < 0$. Then the condition $D_B \geq 0$ implies $\varkappa = 0$, and this yields $a = \dfrac{b^2(f-1)}{d(2f-1)^2}$. Then for systems (38) we obtain $B_1 = a_1^3 d\theta^2/64 = 0$, and this yields $a_1 = 0$, and hence in the case $f \neq 1/2$ lemma is proved.

2) The case $f = 1/2$**.** Then $R_4 = -be = 0$ and $\theta = 4e^2 \neq 0$ and this implies $b = 0$. So we get the family of systems

$$\dot{x} = a_1 + xy, \quad \dot{y} = a + cy + dx^2 + exy + y^2/2, \tag{42}$$

for which calculations yield

$$B_1 = \frac{a_1}{8}\big[2a_1^2de^4 + 2a_1e^3(c^2d+ae^2) - (c^2d+ae^2)^2\big] \equiv \frac{a_1}{8}B_1^*.$$

As $a_1 \neq 0$ we get $B_1^* = 0$ and this implies

$$\text{Discrim}\,[B_1^*, a_1] = 4e^4(c^2d+ae^2)^2(2d+e^2) \geq 0.$$

a) If $2d+e^2 \geq 0$ then we may assume $2d+e^2 = u^2 \geq 0$ and hence, we get the relation $d = (u^2-e^2)/2$. So in this case we obtain

$$B_1^* = -\frac{1}{4}\big[2e^2(e-u)a_1 - 2ae^2 + c^2(e^2-u^2)\big]\big[2e^2(e+u)a_1 - 2ae^2 + c^2(e^2-u^2)\big] \equiv -\frac{1}{4}\tilde{Z}_1\tilde{Z}_2,$$
$$\mu_0 = (u-e)(u+e)/4, \quad \theta = 4e^2,$$

and the condition $B_1^* = 0$ yields $\tilde{Z}_1\tilde{Z}_2 = 0$. Then without loss of generality we may assume $Z_1 = 0$ due to the change $u \mapsto -u$, and since $\mu_0\theta \neq 0$ the relation $Z_1 = 0$ gives

$$a_1 = \frac{1}{2e^2(e-u)}\big[2ae^2 - c^2(e^2-u^2)\big].$$

Then via the affine transformation

$$\chi_1: \; x_1 = (e-u)x - y + \frac{c(u-e)}{e}, \quad y_1 = \frac{(e+u)}{2}x + \frac{1}{2}y + \frac{c(u+e)}{2e}, \; \det(\chi_1) = e \neq 0,$$

systems (42) will be written as systems (37).

b) Assume finally $2d+e^2 < 0$. Then Discrim $[B_1^*, a_1] \geq 0$ due to $e \neq 0$ (because $\theta \neq 0$). This implies $c^2d+ae^2 = 0$, i.e. $a = -c^2d/e^2$. Therefore for (42) we obtain $B_1 = a_1^3de^4/4 = 0$ and due to $\mu_0\theta \neq 0$ (i.e. $de \neq 0$) we get again $a_1 = 0$. This completes the proof of the lemma. $\square$

Lemma 12. *Assume that for a non-Hamiltonian and non-degenerate quadratic system* (3) *the conditions* $\mu_0\theta \neq 0$ *and* $R_4 = R_5 = 0 = B_3$ *hold. Then this system could be written via an affine transformation as the system*

$$\dot{x} = xy, \quad \dot{y} = a + bx + dx^2 + fy^2. \tag{43}$$

Proof. Since the condition $B_3 = 0$ implies $B_1 = 0$ (see (5)), taking into consideration Lemma 11 due to the conditions $\mu_0\theta \neq 0$ and $R_4 = 0$ we may consider the family of systems (37), for which calculations yield

$$R_5 = 48e\big[e^2 f(f+1) - d(f-1)(2f+1)^2\big] \equiv eU, \quad R_4 = 2[cd(2f-1) - bef].$$

Then the condition $R_5 = 0$ implies either $e = 0$, or $U = 0$. If $e = 0$ then the conditions $R_4 = 2cd(2f-1)0$ and $\theta\mu_0 \neq 0$ (i.e. $d(2f-1) \neq 0$) yield $c = 0$, and the lemma follows.

We assume $e \neq 0$. In this case the condition $U = 0$ implies $(f-1)(2f+1) \neq 0$, and then $d = \dfrac{e^2 f(f+1)}{(f-1)(2f+1)^2}$. Therefore for systems (37) the condition $R_4 = 2[cd(2f-1) - bef] = 0$ yields $b = ce(1+f)(2f-1)/\big[(f-1)(2f+1)^2\big]$, and then

$$B_3 = \frac{-3e\big[a(2f+1)^2 - c^2(f+1)\big]}{(f-1)(2f+1)^4}\big[ex + (2f-1)y\big]\big[e(2f^2+f+1)x + (f-1)(2f+1)^2 y\big]x^2.$$

So the condition $B_3 = 0$ yields $a = \dfrac{c^2(f+1)}{(2f+1)^2}$, and then via the affine transformation

$$\chi: \; x_1 = e(f+1)x + (f-1)(2f+1)y + \frac{c(f^2-1)}{f}, \quad y_1 = \frac{ef}{2f+1}x + fy + \frac{cf}{2f+1},$$

with $\det(\chi) = 2ef \neq 0$ (due to $\theta \neq 0$) we arrive to systems (43). □

Theorem $\mathcal{D}$. *Assume that for a non-Hamiltonian and non-degenerate quadratic system* (3) *the condition $\theta\mu_0 \neq 0$ holds. Then this system possesses a polynomial first integral if and only if $B_1 = 0 = R_4$, $\Phi(z)$ has three roots in $\mathbb{Q}^-$ and one of the following sets of conditions holds.*

$\mathcal{D}_1$) $R_5 \neq 0$, $B_3 = 0$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x} = xy, \quad \dot{y} = a + bx + cy + dx^2 + xy + fy^2, \tag{44}$$

where $f = -\dfrac{s}{p+2k} \in \mathbb{Q}^-$, $\Phi(f) = 0$, $c = \dfrac{bf}{d(2f-1)}$, $a = \dfrac{b^2 f(f-1)}{d(2f-1)^2}$, $d = \dfrac{k(p+k)}{p^2(1-f)} \in \mathbb{Q}^+$, *and it possesses the polynomial first integral*

$$\begin{aligned} &\mathcal{H}(x,y) = x^s \mathcal{A}^k \mathcal{B}^{k+p}, \;\; s, k, p+k \in \mathbb{N}, \quad \textit{where} \\ &\mathcal{A} = -b(f-1)^2 + (2f-1)(k+p)\big[kx - (f-1)py)\big], \\ &\mathcal{B} = -b(f-1)^2 + (2f-1)k\big[(k+p)x + (f-1)py)\big]. \end{aligned} \tag{45}$$

$\mathcal{D}_2$) $R_5 \neq 0$, $B_3 \neq 0$, $R_6 = 0$, $R_7 = 0$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x} = xy, \quad \dot{y} = a + dx^2 + xy - y^2, \tag{46}$$

with $d = \dfrac{k(p+k)}{2p^2} \in \mathbb{Q}^+$, *and it possesses the polynomial first integral*

$$\begin{aligned} &\mathcal{H}(x,y) = \mathcal{A}^{k+p}\mathcal{B}^k, \quad k, p+k \in \mathbb{N}, \quad \textit{where} \\ &\mathcal{A} = 2ap^2 + kx\big[(k+p)x - 2py)\big], \quad \mathcal{B} = 2ap^2 + (k+p)x(kx + 2py). \end{aligned} \tag{47}$$

$\mathcal{D}_3$) $R_5 = 0$, $R_8 = 0$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x} = xy, \quad \dot{y} = a + bx + dx^2 + fy^2, \tag{48}$$

$f = -\dfrac{p}{q} \in \mathbb{Q}^-$ $p, q \in \mathbb{N}$, $\Phi(f) = 0$, *and it possesses the polynomial first integral*

$$\mathcal{H}(x, y) = x^{2p}\left(\frac{a}{2f} + \frac{b}{2f-1}x + \frac{d}{2f-2}x^2 + \frac{y^2}{2}\right)^q. \tag{49}$$

Proof. We consider a non-Hamiltonian and non-degenerate quadratic system (3) for which the conditions $\theta\mu_0 \neq 0$ and $B_1 = 0 R_4 =$ hold. According to Lemma 11 via an affine transformation this system could be written in the form

$$\dot{x} = xy, \quad \dot{y} = a + bx + cy + dx^2 + exy + fy^2, \tag{50}$$

for which we calculate

$$\begin{gathered} R_4 = 2[cd(2f-1) - bef] = 0, \quad \theta = -8[d(1-2f)^2 - e^2 f], \quad \mu_0 = df, \\ \eta = (f-1)^2(4d + e^2 - 4df), \quad R_5 = 48e\big[e^2 f(f+1) - d(f-1)(2f+1)^2\big], \\ \Phi(z) = (z-f)\Big[z^2 - \frac{8}{\theta}(2d + e^2 - 4df)z - \frac{8d}{\theta}\Big] \equiv (z-f)\psi(z). \end{gathered} \tag{51}$$

As in the families of systems (ii) (a),(b) and (c) from Theorem PFI the parameter f is negative and, on the other hand, f is the root of the function $\Phi(z)$, then during the proof of this theorem without loss of generality we can consider $f < 0$ in both directions of the proof.

The case $\mathcal{D}_1$) Assume that for systems (50) the following conditions are fulfilled

$$\mu_0\theta R_5 \neq 0, \quad R_4 = B_3 = 0. \tag{52}$$

Since $d \neq 0$ (as $\mu_0 \neq 0$) and $2f - 1 \neq 0$ (as $f < 0$) the condition $R_4 = 0$ yields $c = \dfrac{bef}{d(2f-1)}$. In this case for systems (50) we calculate

$$B_3 = \frac{-3}{d(2f-1)^2}\big[ad(2f-1)^2 - b^2 f(f-1)\big]x^2\big[(4df - 2d - e^2)x + ey\big]\big[ex + (2f-1)y\big].$$

Obviously the condition $B_3 = 0$ implies $a = \dfrac{b^2 f(f-1)}{d(2f-1)^2}$ and we arrive to the subfamily *(ii) (b)* of Theorem PFI, which possesses a polynomial first integral if and only if the following additional conditions hold

$$f = -\frac{s}{p+2k} \in \mathbb{Q}^-, \quad d = -\frac{k(p+k)e^2}{p^2(f-1)} \in \mathbb{Q}^+, \quad s, p, k \in \mathbb{N}. \tag{53}$$

Now we claim that given (52) the set (53) is equivalent to $\Phi(z) = 0$ having three roots in $\mathbb{Q}^-$.

Indeed assume first that $z_i \in \mathbb{Q}^-$, $\Phi(z_i) = 0$, $i = 1, 2, 3$. Then considering (51) we have $\text{Discrim}\,(\psi(z)) = 64e^2(4d + e^2 - 4df)/\theta^2 \geq 0$ and we could not have the equality (i.e. $4d + e^2 - 4df = 0$), because the double root will be $z_2 = z_3 = 1 > 0$. So the condition $\eta > 0$ holds, and as $e \neq 0$ we may set a new parameter u as follows: $4d + e^2 - 4df = u^2 e^2$. Thus we obtain $d = e^2(u^2 - 1)/(4(1-f))$ and then for systems (50) we calculate

$$F(z) = (z-f)\big(z - \frac{1-u}{1+u-2fu}\big)\big(z - \frac{1+u}{1-u+2fu}\big) = (z-z_1)(z-z_2)(z-z_3).$$

Assuming $z_1 = -p/q \in \mathbb{Q}^-$ and $z_2 = -r/s \in \mathbb{Q}^-$ we get $u = -q(r+s)/(2pr + qr - qs)$, then $z_3 = (pr - qs)/(2pr + qr + ps)$, and hence the condition $pr - qs < 0$ has to be satisfied in order to have $z_3 \in \mathbb{Q}^-$. On the other hand for this value of the parameter u we obtain $d = -\dfrac{e^2 r(p+q)(qs - pr)}{(2pr + qr - qs)^2(f-1)}$. Then setting $\tilde{s} = p(r+s)$ and either $\tilde{k} = r(p+q)$,

$\tilde{p} = qs - qr - 2pr$ if $q(s-r) > 2pr$, or $\tilde{k} = qs - pr$, $\tilde{p} = qr - qs + 2pr$ if $q(s-r) < 2pr$ we obtain $f = -\dfrac{\tilde{s}}{\tilde{p} + 2\tilde{k}}$ and $d = -\dfrac{\tilde{k}(\tilde{p}+\tilde{k})e^2}{\tilde{p}^2(f-1)}$ with $\tilde{s}, \tilde{k}, \tilde{p} \in \mathbb{N}$.

Conversely assuming now the conditions (53) we shall show that the function $\Phi(z)$ has three roots in $\mathbb{Q}^-$.

Indeed for these values of the parameters f and d taking into consideration the above expressions for c and a, we determine the following three roots of the function $\Phi(z)$

$$z_1 = -\frac{s}{p+2k}, \quad z_2 = -\frac{k+p}{k+s}, \quad z_3 = -\frac{k}{k+p+s}.$$

So all three roots are in $\mathbb{Q}^-$. We observe that since $e \neq 0$ we can assume $e = 1$ due to the change $x \mapsto x/e$ and then we get the family of systems (44) possessing the first integral (45). This completes the proof of the theorem in the case $\mathcal{D}_1$).

The case $\mathcal{D}_2$) For systems (50) calculations yield

$$\mathcal{R} = c + ex + (2f+1)y, \quad R_6 = \big[(2cd - be)x^2 + (b + 2bf - ce)xy + cy^2\big]/4.$$

Since $\mathcal{R} \neq 0$ (these systems are not Hamiltonian) the condition $R_6 = 0$ implies $b = c = 0$. Then for systems (50) we have

$$B_3 = -3a\big[(4df - 2d - e^2)x + ey\big]\big[ex + (2f-1)y\big]x^2, \quad R_7 = 162a^2\theta(f+1),$$

and as $B_3 \neq 0$ (i.e. $a \neq 0$) the condition $R_7 = 0$ yields $f = -1$. So we obtain the family of systems

$$\dot{x} = xy, \quad \dot{y} = a + dx^2 + exy - y^2. \tag{54}$$

According to Theorem PFI (see subfamily ii) (c)) systems (54) posses a polynomial first integral if and only if the condition $d = \dfrac{k(p+k)e^2}{2p^2}$ $(k, p \in \mathbb{N})$ holds. We claim that this condition is equivalent to the existence of three roots in $\mathbb{Q}^-$ of the function $\Phi(z)$.

Indeed assume first that all three roots of the function $\Phi(z)$ are in $\mathbb{Q}^-$. Then repeating the same arguments as in the case $\mathcal{D}_1$, we conclude that the condition $\eta > 0$ must hold. Hence considering (51) and $f = -1$ we get $8d + e^2 > 0$, and as $e \neq 0$ we may set a new parameter u as follows: $8d + e^2 = u^2e^2$. Thus we obtain $d = e^2(u^2-1)/8$ and then for systems (54) we calculate the roots of the function $\Phi(z)$

$$z_1 = -1, \quad z_2 = \frac{1+u}{1-3u}, \quad z_3 = \frac{1-u}{1+3u}.$$

Assuming $z_2 = -r/s \in \mathbb{Q}^-$ we get $u = (r+s)/(3r-s)$, and then $z_3 = (r-s)/(3r+s)$. Hence the condition $r < s$ has to be satisfied. On the other hand for this value of the parameter u we obtain $d = \dfrac{2e^2r(s-r)}{2(3r-s)^2}$ and setting either $k = 2r$, $p = s - 3r$ if $s > 3r$, or $k = s - r$, $p = 3r - s$ if $s < 3r$ we obtain $d = \dfrac{k(p+k)e^2}{2p^2}$ with $k, p \in \mathbb{N}$.

Conversely assume that for systems (54) the condition $d = \dfrac{k(p+k)e^2}{2p^2}$ is fulfilled. Then considering (51) we calculate the roots of the function $\Phi(z)$

$$z_1 = -1, \quad z_2 = -\frac{k+p}{3k+p}, \quad z_3 = -\frac{k}{3k+2p}$$

and as $k, p \in \mathbb{N}$ we get all three roots are in $\mathbb{Q}^-$. So our claim is proved.

As $e \neq 0$ we can assume $e = 1$ due to the change $x \mapsto x/e$ and we obtain the family of systems (46) possessing the first integral (47).

The case $\mathcal{D}_3$) Now the conditions $R_5 = R_8 = 0$ hold and we claim, that in this case any such system (50) could be written doing an affine transformation and a time rescaling in the form

$$\dot{x} = xy, \quad \dot{y} = a + bx + dx^2 + fy^2. \tag{55}$$

Indeed if $B_3 = 0$ then, since the conditions $\theta\mu_0 \neq 0$ and $R_4 = 0$ hold, by Lemma 12 this follows immediately, and we observe that for the systems (55) we have $R_8 = 0$.

Assume now that for systems (50) the condition $B_3 \neq 0$ holds and we calculate

$$\begin{aligned} R_8 =& 10e\Big[b^2(f^2-1)(2f+1) - 2cef(f+1)b + e^2(f+1)\left(c^2 - 2af\right) + \\ &+ d(2f+1)\left(-fc^2 + c^2 + 4af^2 + 2a - 2af\right). \end{aligned}$$

We shall show that when $R_5 = R_4 = 0$ and $B_3 \neq 0$ the condition $R_8 = 0$ is equivalent to $e = 0$. Indeed suppose the contrary that $e \neq 0$. Then considering (51) the conditions $R_5 = R_4 = 0$ yield

$$d = \frac{e^2 f(f+1)}{(f-1)(2f+1)^2}, \qquad b = \frac{ce(1+f)(2f-1)}{(f-1)(2f+1)^2},$$

and therefore we obtain

$$R_8 = \frac{40e^3 f(f+1)}{(f-1)(2f+1)^3}\big[a(2f+1)^2 - c^2(f+1)\big], \quad \mu_0 = \frac{e^2 f^2(f+1)}{(f-1)(2f+1)^2},$$

$$B_3 = \frac{-3e}{(f-1)(2f+1)^4}\big[a(2f+1)^2 - c^2(f+1)\big]x^2$$
$$\big[ex + (2f-1)y\big]\big[e(f+1+2f^2)x + (f-1)(2f+1)^2 y\big].$$

Since $\mu_0 B_3 \neq 0$ we get $R_8 \neq 0$ and hence, the contradiction obtained says that the condition $R_8 = 0$ is equivalent to $e = 0$. So for $R_8 = 0$ we get $e = 0$, and then the conditions $R_4 = 2cd(2f-1)$ and $\theta\mu_0 \neq 0$ (i.e. $d(2f-1) \neq 0$) yield $c = 0$.

Thus our claim is proved and in what follows we shall consider the family of systems (55) which contains the subfamily (ii), (a) of Theorem PFI. According to this theorem systems (55) possess a polynomial first integral if and only if $f \in \mathbb{Q}^-$. We shall show that this condition is equivalent to the existence of three roots in $\mathbb{Q}^-$ of the function $\Phi(z)$.

Indeed for systems (54) we calculate

$$\Phi(z) = (z-f)\Big(z - \frac{1}{2f-1}\Big)^2.$$

It obviously can be seen that $f = -p/q \in \mathbb{Q}^-$ $(p, q \in \mathbb{N})$ if and only if all three roots of the function $\Phi(z)$ (two of them coincide) are in $\mathbb{Q}^-$. It remains to observe that in the canonical form (54) f is a simple root of $\Phi(z)$. Therefore we get the family of systems (48) possessing the first integral (49). This completes the proof of the theorem. □

4. Some corollaries from Theorems $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$

First we shall prove a result on the rational first integrals in invariant form for a class of quadratic systems.

Lemma 13. *Assume that for a non-degenerate quadratic system* (3) *the conditions* $\mu_0\theta \neq 0$, $R_4 = 0 = B_3$ *and* $\eta > 0$ *hold. Then this system has three invariant straight lines* $L_i(x,y) = u_i x + v_i y + w_i = 0$ *in three distinct directions and a first integral of the Darboux form*

$$\mathcal{H}(x,y) = L_1(x,y)^{\frac{z_1}{z_1-1}}\, L_2(x,y)^{\frac{z_2}{z_2-1}}\, L_3(x,y)^{\frac{z_3}{z_3-1}}, \tag{56}$$

where z_i *are the real roots of the function* $\Phi(z)$, *satisfying*

$$\frac{z_1}{z_1-1} + \frac{z_2}{z_2-1} + \frac{z_3}{z_3-1} = 1. \tag{57}$$

Proof. As $\eta > 0$ is the discriminant of the cubic form $C_2(x,y)$, the quadratic system possesses three distinct real infinite singular points. So due to a linear transformation we may consider without loss of generality that these three points (in the projective plane) are $N_1(1,0,0)$, $N_2(1,1,0)$ and $N_3(0,1,0)$. Then clearly we get the following family of systems

$$\dot{x} = a_1 + b_1x + c_1y + ex^2 + (f-1)xy, \quad \dot{y} = a + bx + cy + (e-1)xy + fy^2, \tag{58}$$

for which $C_2(x,y) = xy(x-y)$, $\theta = -8(e-1)(f-1)(e+f)$ and since $\theta \neq 0$ we can consider $c_1 = b = 0$ doing a translation (if necessary). Considering these conditions for systems (58) we calculate: Coefficient$[B_3, x^4] = -3a(e-1)^2$, Coefficient$[B_3, y^4] = 3a_1(f-1)^2$ and since $\theta \neq 0$ (i.e. $(e-1)(f-1) \neq 0$) the condition $B_3 = 0$ implies $a = a_1 = 0$. Then we obtain

$$B_3 = 3(b_1 - c)(b_1f + ce)x^2y^2, \quad R_4 = -2(e-1)(f-1)(b_1f + ce), \quad \mu_0 = ef(e+f-1),$$

and therefore the condition $b_1f + ce = 0$ has to be fulfilled. As $\mu_0 \neq 0$ (i.e. $e \neq 0$) we may assume $b_1 = ue$ and then we obtain $c = -uf$. Thus we arrive to the family of systems

$$\dot{x} = eux + ex^2 + (f-1)xy, \quad \dot{y} = -fuy + (e-1)xy + fy^2, \tag{59}$$

possessing three invariant lines $x = 0$, $y = 0$ and $x - y + u = 0$. Considering (36) for these systems calculations yield

$$\Phi(z) = \Big(z - \frac{e}{e-1}\Big)\Big(z - \frac{f}{f-1}\Big)\Big(z - \frac{e+f-1}{e+f}\Big) = (z-z_1)(z-z_2)(z-z_3),$$

and hence

$$\frac{z_1}{z_1-1} = e, \ \frac{z_2}{z_2-1} = f, \ \frac{z_3}{z_3-1} = 1 - e - f.$$

On the other hand a straightforward calculation gives that the function

$$\mathcal{H}(x,y) = x^f y^e (x - y + u)^{1-e-f}$$

is a first integral of systems (59). As the values z_i (and as a consequence, the values $\frac{z_i}{z_i-1}$), $i = 1,2,3$ are absolute invariant constants with respect to the group $Aff(2,\mathbb{R})$ of affine transformations as well as with respect to time rescalling.

We note that the identity (57) follows immediately from the the form of the first integral expressed through parameters of systems (59). However it could be proved directly considering the form of the function $\Phi(z)$, as the identity (57) is equivalent to

$$1 = z_1z_2 + z_1z_3 + z_2z_3 - 2z_1z_2z_3 = \varphi_2 + 2\varphi_3 = \frac{\theta - 16\mu_0}{\theta} + 2\frac{8\mu_0}{\theta} = 1$$

according to Viète's relations between the roots of a polynomial and its coefficients. □

Corollary $\mathcal{B}^*$. *Assume that for a non-degenerate quadratic system* (3) *the conditions* $\theta = \mu_0 = 0$ *and* $KM \neq 0$ *are fulfilled. Then this system possesses a rational first integral if* $R_1 = 0$ *and one of the following sets of conditions holds.*

$\mathcal{B}_2^*$) $R_2 < 0$, $R_3 = 0$, $B_3 = 0$, $\mathcal{F}_1 = \dfrac{p}{q} \in \mathbb{Q}^+$, $p, q \in \mathbb{N}$; *in this case the system can be brought via an affine transformation and a time rescalling to the system* $\dot{x} = 1 + x^2$, $\dot{y} = \dfrac{p}{q}xy$, *possessing the rational first integral* $\mathcal{H}(x,y) = y^{2q}\left(1 + x^2\right)^{-p}$.

$\mathcal{B}_4^*a$) $R_2 > 0$, $B_3 = 0$, $\mathcal{F}_1 = -\dfrac{p}{q} \in \mathbb{Q}^-$, $\mathcal{F}_2 = \dfrac{r^2}{s^2} \in \mathbb{Q}^+$, $\mathcal{F}_2 > \mathcal{F}_1^2$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x} = -1 + x^2, \quad \dot{y} = \frac{r}{s}y - \frac{p}{q}xy, \quad p,q,r,s \in \mathbb{N}, \ ps - qr < 0,$$

possessing the rational first integral $\mathcal{H}(x,y) = (x+1)^{ps+qr}(x-1)^{ps-qr}y^{2qs}$.

$\mathcal{B}_4^*b)$ $R_2>0,\ B_3=0,\ \mathcal{F}_1=\dfrac{p}{q}\in\mathbb{Q}^+,\ \mathcal{F}_2=\dfrac{r^2}{s^2}\in\mathbb{Q}^+$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x}=-1+x^2,\quad \dot{y}=\frac{r}{s}\,y+\frac{p}{q}\,xy,\quad p,q,r,s\in\mathbb{N},$$

possessing the rational first integral $\mathcal{H}(x,y)=(x+1)^{ps+qr}(x-1)^{ps-qr}y^{-2qs}$.

$\mathcal{B}_5^*a)$ $R_2=0,\ R_3=0,\ B_3\neq 0,\ \mathcal{F}_1=-\dfrac{p}{q}\in\mathbb{Q}^-,\ p,q\in\mathbb{N},\ \mathcal{F}_1>-1$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x}=x^2,\quad \dot{y}=1-\frac{p}{q}\,xy,\quad p<q$$

possessing the rational first integral $\mathcal{H}(x,y)=x^{p-q}\left(xy+\frac{q}{q-p}\right)^q$.

$\mathcal{B}_5^*b)$ $R_2=0,\ R_3=0,\ B_3\neq 0,\ \mathcal{F}_1=\dfrac{p}{q}\in\mathbb{Q}^+,\ \ p,q\in\mathbb{N}$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x}=x^2,\quad \dot{y}=1+\frac{p}{q}\,xy,$$

possessing the rational first integral $\mathcal{H}(x,y)=x^{p+q}\left(xy-\frac{q}{q+p}\right)^{-q}$.

Corollary $\mathcal{C}^*$. *Assume that for a non-Hamiltonian and non-degenerate quadratic system* (3) *the conditions* $K=M=0$ *(then* $\theta=\mu_0=0$*) are fulfilled. Then this system possesses a rational first integral if the following conditions hold*

$$B_3=0,\quad R_9\neq 0,\quad \mathcal{F}_3=\frac{p}{q}\in\mathbb{Q}^+,\quad p,q\in\mathbb{N},\quad \mathcal{F}_3\neq 2.$$

Moreover in this case the system can be brought via an affine transformation and a time rescalling to the system $\dot{x}=x,\quad \dot{y}=\dfrac{p}{q}\,y+x^2,\quad p-2q\neq 0$, *possessing the rational first integral* $\mathcal{H}(x,y)=x^p\left(y+\dfrac{q}{p-2q}x^2\right)^{-q}$.

Corollary $\mathcal{D}^*$. *Assume that for a non-Hamiltonian and non-degenerate quadratic system* (3) *the condition* $\theta\mu_0\neq 0$ *holds. Then this system possesses a rational first integral if* $B_1=0=R_4$, $\Phi(z)$ *has three rational roots, among which at least one is in* $\mathbb{Q}^+$, *and one of the following sets of conditions holds.*

$\mathcal{D}_1^*)$ $R_5\neq 0,\ B_3=0$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x}=eux+c_1y+ex^2+(f-1)xy,\quad \dot{y}=fuy+(e-1)xy+fy^2,$$

possessing the rational first integral $\mathcal{H}(x,y)=x^fy^e(x-y+u)^{1-e-f}$, *where* $e,f,1-e-f\in\mathbb{Z}$ *not all have the same sign.*

$\mathcal{D}_2^*)$ $R_5\neq 0,\ B_3\neq 0,\ R_6=0,\ R_7=0$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x}=xy,\quad \dot{y}=a+\frac{k(p+k)}{2p^2}x^2+xy-y^2,$$

possessing the rational first integral $\mathcal{H}(x,y)=\mathcal{A}^{k+p}\mathcal{B}^k$, *where* $k,p+k\in\mathbb{Z},k(p+k)<0$ *and*

$$\mathcal{A}=2ap^2+kx\big[(k+p)x-2py)\big],\quad \mathcal{B}=2ap^2+(k+p)x\big[kx+2py)\big].$$

$\mathcal{D}_3$) $R_5 = 0$, $R_8 = 0$; *in this case the system can be brought via an affine transformation and a time rescalling to the system*

$$\dot{x} = xy, \quad \dot{y} = a + bx + dx^2 + \frac{p}{q}y^2, \quad p, q \in \mathbb{N},$$

possessing the rational first integral

$$\mathcal{H}(x, y) = x^{2p}\left(\frac{a}{2f} + \frac{b}{2f-1}x + \frac{d}{2f-2}x^2 + \frac{y^2}{2}\right)^{-q}.$$

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