# THE CORONA FACTORIZATION PROPERTY, STABILITY, AND THE CUNTZ SEMIGROUP OF A C\*-ALGEBRA

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ABSTRACT. The Corona Factorization Property, originally invented to study extensions of  $C^*$ -algebras, appears to convey essential information about the intrinsic structure of the  $C^*$ -algebra. We show that the Corona Factorization Property of a  $\sigma$ -unital  $C^*$ -algebra A is completely captured by its Cuntz semigroup W(A) of equivalence classes of positive elements in matrix algebras over A. The corresponding condition in W(A) is a (weak) comparability property that is termed the Corona Factorization Property (for the semigroup). Using this result one can for example show that all unital  $C^*$ -algebras with finite decomposition rank have the Corona Factorization Property.

Applying similar techniques we study the related question of when  $C^*$ -algebras are stable. We give an intrinsic characterization, that we term property (S), of  $C^*$ -algebras with the absence of non-zero unital quotients and non-zero bounded 2-quasitraces. We then show that property (S) is equivalent to stability provided that the Cuntz semigroup of the  $C^*$ -algebra satisfies another (also very weak) comparability property, that we call the  $\omega$ -comparison property.

## 1. INTRODUCTION

The Corona Factorization Property was defined and studied by Kucerovsky and Ng in [10] building up on work by Elliott and Kucerovsky, [4], in which *purely large*  $C^*$ -algebras were studied. Both concepts relate to the theory of extensions and in particular to the important question on when extensions are automatically absorbing.

A  $C^*$ -algebra satisfies the Corona Factorization Property if every full projection in the multiplier algebra of its stabilization is properly infinite (and hence equivalent to the unit). The existence of non-properly infinite full projections in the multiplier algebra of a stable  $C^*$ -algebra was noted (implicitly) in [17], and more explicitly in [18], in connection with the construction of non-stable  $C^*$ -algebras that become stable when being tensored with a matrix algebra. The existence of finite full projections in the multiplier algebra of a stable  $C^*$ -algebra was also essential in the construction in [19] of a simple  $C^*$ -algebra with a finite and an infinite projection. In the language of Kucerovsky and Ng it is shown in [19] that the  $C^*$ -algebra  $C(\prod_{n=1}^{\infty} S^2)$  does not have the Corona Factorization Property.

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Zhang proved a (partial) converse of these results, that a simple  $C^*$ -algebra of real rank zero with the Corona Factorization Property is either stably finite or purely infinite.

It thus appears that failure to have the Corona Factorization Property is an "infinite dimensional" property, and conversely that all  $C^*$ -algebras with "finite dimensional behavior" should have the Corona Factorization Property. (By finite and infinite dimensinality we are, of course, not referring to the vector space dimension of the  $C^*$ -algebra, but rather to its non-commutative dimension—perhaps best defined through Kirchberg and Winter's notion of decomposition rank.) Pimsner, Popa, and Voiculescu studied in [14] extensions of  $C(X) \otimes \mathcal{K}$ , where X is a finite-dimensional compact metric space, and developed an  $\operatorname{Ext}(X, -)$  theory. It follows in particular from their work that  $C(X) \otimes \mathcal{K}$ has the Corona Factorization Property when X has finite dimension. The assumption that X is finite dimensional is crucial.

Using Kirchberg and Winter notion of decomposition rank, [9], mentioned above, Kucerovsky and Ng, [11], studied extensions of type I  $C^*$ -algebras with finite decomposition.

In this paper we show that a  $\sigma$ -unital  $C^*$ -algebra (simple or not) satisfies the Corona Factorization Property if, and only if, its Cuntz semigroup W(A) satisfies a (weak) comparison property that we call the Corona Factorization Property for semigroups (also considered in [12]). We also introduce stronger notions of comparison for ordered abelian semigroups, one of which is verified for the Cuntz semigroup of a unital  $C^*$ algebras with finite decomposition rank, whencefore the Corona Factorization Property also holds for these algebras.

This parallels the property that the authors introduced and examined in the article [12]. There it was shown, using entirely different techniques than those used here, that a  $\sigma$ -unital  $C^*$ -algebras of *real rank zero* has the Corona Factorization Property if and only if its monoid V(A) of Murray-von Neumann equivalence classes of projections in the stabilization of a  $C^*$ -algebra A has the Corona Factorization Property (for monoids).

In outline the paper is as follows. In Section 2, we define a number of comparability properties for ordered abelian semigroups, including *n*-comparison and  $\omega$ -comparison (and their weak counterparts), and the Corona Factorization Property for semigroups. These properties can be viewed as weakened forms of the almost unperforation property for semigroups (considered in [20]). In fact, an ordered abelian semigroup has the 0comparison property if and only if it is almost unperforated. It follows from a result of Toms and Winter, [21], that the Cuntz semigroup of any unital simple  $C^*$ -algebra with decomposition rank *n* has the *n*-comparison property (and hence also  $\omega$ -comparison and the Corona Factorization Property). It was this result by Toms and Winter that led us to consider *n*-comparison.

In Section 3 we establish (using the before mentioned result of Toms and Winter) that the Cuntz semigroup of (non-simple) unital  $C^*$ -algebras with finite decomposition rank has the weak *n*-comparison property, and hence also the Corona Factorization property.

In Section 4 we consider an intrinsic property (that we call property (S)) of a  $C^*$ -algebra defined in terms of Cuntz' comparison theory for  $C^*$ -algebras, and we show

that it is equivalent to the absence of unital quotients and bounded 2-quasitraces. It is further shown that property (S) is equivalent to stability of a  $C^*$ -algebra if its Cuntz semigroup has the  $\omega$ -comparison property, thus generalizing a result from [6].

In Section 5 we prove our main result, that the Corona Factorization Property for a  $\sigma$ unital  $C^*$ -algebra can be read off from a comparability property of its Cuntz semigroup; and we show that every ideal in a  $\sigma$ -unital  $C^*$ -algebra has the Corona Factorization Property if and only if its Cuntz semigroup satisfies a comparability property, that we call the strong Corona Factorization Property.

#### 2. Comparability in ordered abelian semigroups

In this section we shall discuss a number of comparability properties of ordered abelian semigroups.

Consider an ordered abelian semigroup  $(W, +, \leq)$ . We shall exclusively be interested in *positive* semigroups, i.e., semigroups where  $x \leq x + z$  for all  $x, z \in W$ . (However, we do not assume that the order is the algebraic order, given by  $x \leq y$  if and *only if* y = x + z for some z in W.) We remind the reader of some commonly used terminology. A *state on* W *normalized at*  $x \in W$  is an additive order preserving map from W into  $\mathbb{R}^+ \cup \{\infty\}$  that maps x to 1. The set of all states normalized at x is denoted by S(W, x). Given two elements  $x, y \in W$ , one writes  $x \propto y$  if there exists  $n \in \mathbb{N}$  such that  $x \leq ny$ .

The result below has appeared already in several versions in the literature, perhaps first as the extension result of Goodearl and Handelman in [5, Lemma 4.1]. We wish to emphasize the following formulation that will be essential for our paper.

**Proposition 2.1.** Let  $(W, +, \leq)$  be an ordered abelian semigroup, and let  $x, y \in W$ . Then the following statements are equivalent:

- (i) There exists  $k \in \mathbb{N}$  such that  $(k+1)x \leq ky$ .
- (ii) There exists  $k_0 \in \mathbb{N}$  such that  $(k+1)x \leq ky$  for every  $k \geq k_0$ .
- (iii)  $x \propto y$  and f(x) < f(y) for every  $f \in S(W, y)$ .

*Proof.* (iii)  $\Rightarrow$  (i). This is the heart of the proof, and is an easy consequence of the extension result of Goodearl and Handelman from [5, Lemma 4.1] mentioned above, see also [16, Proposition 3.1] and [20, Proposition 3.2].

(ii)  $\Rightarrow$  (iii). This is contained in the proof of [20, Proposition 3.2], but here it comes again: First, if  $(k+1)x \leq ky$ , then  $x \leq ky$ , so  $x \propto y$ . Second,  $(k+1)x \leq ky$  implies that  $f(x) \leq k(k+1)^{-1} < 1 = f(y)$  for every  $f \in S(W, y)$ .

(i)  $\Rightarrow$  (ii). Suppose  $(m+1)x \leq my$  for some positive integer m. Put  $k_0 = (m+1)m$ . For each  $k \geq k_0$  write k = (m+1)r + s, where r and s are non-negative integers with  $s \leq m$ . Note that necessarily  $r \geq m$ . Therefore

$$(k+1)x = (m+1)rx + (s+1)x \le (m+1)rx + (m+1)x \le mry + my \le ky.$$

**Definition 2.2.** Given x, y in an ordered abelian semigroup W. Then we say that x is stably dominated by y, written  $x <_s y$ , if the equivalent conditions (i)–(iii) in Proposition 2.1 hold.

**Remark 2.3.** Notice that the relation  $\langle s \rangle$  is transitive. Indeed, let  $x, y, z \in W$  be such that  $x \langle s \rangle y$  and  $y \langle s \rangle z$ . Then by Proposition 2.1 (ii) there exists  $k \in \mathbb{N}$  such that  $(k+1)x \leq ky$  and  $(k+1)y \leq kz$ . But then  $(k+1)x \leq ky \leq (k+1)y \leq kz$ , so  $x \langle s \rangle z$ .

**Remark 2.4.** The notion of almost unperforation from [20, Definition 3.3] can in terms of stable domination be rephrased as follows. An ordered abelian semigroup W is almost unperforated if and only if for all x, y in W,  $x <_s y$  implies  $x \leq y$ .

Let us briefly remind the reader about the ordered Cuntz semigroup W(A) associated to a  $C^*$ -algebra. Let  $M_{\infty}(A)^+$  denote the disjoint union  $\bigcup_{n=1}^{\infty} M_n(A)^+$ . For  $a \in M_n(A)^+$ and  $b \in M_m(A)^+$  set  $a \oplus b = \text{diag}(a, b) \in M_{n+m}(A)^+$ , and write  $a \preceq b$  if there exists a sequence  $\{x_k\}$  in  $M_{m,n}(A)$  such that  $x_k^*bx_k \to a$ . Write say that a and b are *Cuntz* equivalent, in symbols  $a \approx b$ , if  $a \preceq b$  and  $b \preceq a$ . Put  $W(A) = M_{\infty}(A)^+ \approx$ , and let  $\langle a \rangle \in W(A)$  be the equivalence class containing a. The set W(A) becomes an ordered abelian semigroup when equipped with addition and order inherited from  $\oplus$  and  $\preceq$ . See [8, Section 2] for further properties of W(A).

Given  $a \in A^+$  and  $n \in \mathbb{N}$  we shall denote, as customary, by  $a \otimes 1_n$  the  $n \times n$  matrix with a's in the diagonal and zeroes elsewhere. Clearly,  $\langle a \otimes 1_n \rangle = n \langle a \rangle$  in W(A).

We shall occasionally also consider the following (stronger) equivalence relation. Given two positive elements  $a, b \in A^+$ , write  $a \sim b$  if there exists  $x \in A^+$  such that  $xx^* = a$  and  $x^*x = b$ . Observe that  $a \sim b$  implies  $a \approx b$ .

Given  $a, b \in A^+$  we will write  $a \prec_s b$  if  $\langle a \rangle <_s \langle b \rangle$  in W(A). About this relation we have the lemma below, which is similar to [16, Proposition 2.4].

**Lemma 2.5.** Let a and b be positive elements in a C<sup>\*</sup>-algebra A and suppose that  $a \prec_s b$ . Then, for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $(a - \varepsilon)_+ \prec_s (b - \delta)_+$ .

*Proof.* For  $c \in A^+$  note that  $((c \otimes 1_k) - \eta)_+ = (c - \eta)_+ \otimes 1_k$ .

There exists a positive integer k such that  $(k+1)\langle a \rangle \leq k\langle b \rangle$ . Hence  $a \otimes 1_{k+1} \preceq b \otimes 1_k$ (in  $M_{k+1}(A)$ ). Let  $\varepsilon > 0$ . It then follows from [16, Proposition 2.4] that there exists  $\delta > 0$  such that

$$(a-\delta)_+ \otimes 1_{k+1} = (a \otimes 1_{k+1} - \delta)_+ \precsim (b \otimes 1_k - \varepsilon)_+ = (b-\varepsilon)_+ \otimes 1_k.$$

This shows that  $(a - \delta)_+ \prec_s (b - \varepsilon)_+$ .

Our first comparability property, given in Definition 2.7 below, is prompted by a result of Toms and Winter, [21, Lemma 6.1]. Recall that if  $\tau$  is a positive trace (or a 2quasitrace), then one can associate to it a dimension function  $d_{\tau}: W(A) \to [0, \infty]$  given by

$$d_{\tau}(\langle a \rangle) = \lim_{n \to \infty} \tau(a^{1/n}),$$

where a is a positive element in A or in a matrix algebra over A (in the latter case we must extend  $\tau$  to the same matrix algebra over A). The trace property ensures that  $d_{\tau}$  is well-defined. We can also view  $d_{\tau}$  as being a function on the positive elements in A (and in matrix algebras over A). We shall not distinguish between the two situations.

**Proposition 2.6** (Toms and Winter). Let A be a simple, separable, and unital C<sup>\*</sup>algebra of decomposition rank  $n < \infty$ . Let  $a, d_0, d_1, \ldots, d_n$  be positive elements in A such that  $d_{\tau}(a) < d_{\tau}(d_j)$  for  $j = 0, 1, \ldots, n$  and for all tracial states  $\tau$  on A (where  $d_{\tau}$ is the dimension function on A associated with the trace  $\tau$ ). It follows that

$$\langle a \rangle \le \langle d_0 \rangle + \langle d_1 \rangle + \dots + \langle d_n \rangle,$$

in the Cuntz semigroup W(A).

**Definition 2.7** (The *n*-comparison property). Let  $(W, +, \leq)$  be an ordered abelian semigroup and let *n* be a natural number. Then *W* is said to have the *n*-comparison property if whenever  $x, y_0, \ldots, y_n$  are elements in *W* with  $x <_s y_j$  for all *j*, then  $x \leq y_0 + y_1 + \cdots + y_n$ .

Note that W has the 0-comparison property if and only if W is almost unperforated, cf. Remark 2.4. Note also, that if W has the *n*-comparison property for some n, then it has the *m*-comparison property for all  $m \ge n$ .

With Definition 2.7 at hand we can rephrase the proposition of Toms and Winter above as follows:

**Proposition 2.8.** Let A be a simple, separable and unital  $C^*$ -algebra with decomposition rank  $n < \infty$ . Then W(A) has n-comparison.

*Proof.* Let  $x, y_0, \ldots, y_n \in W(A)$  be such that  $x <_s y_j$  for every  $j = 0, \ldots, n$ . Upon replacing A by a matrix algebra over A (which does not change the decomposition rank) we may assume that there are positive elements  $a, d_0, d_1, \ldots, d_n$  in A such that  $x = \langle a \rangle$  and  $y_j = \langle d_j \rangle$ .

We know from Proposition 2.1 that  $f(x) < f(y_j)$  for every dimension function fon A normalized at  $y_j$ . As A is simple and unital, every such f is a multiple of a dimension function which is normalized at the unit:  $\langle 1_A \rangle$ . It follows that  $d_\tau(a) < d_\tau(d_j)$ for every tracial state  $\tau$  on A (because  $d_\tau$  then is a dimension function on A normalized at  $1_A$ ) Thus, by [21, Lemma 6.1] (which in fact is Proposition 2.6), we get that  $a \preceq d_0 \oplus d_1 \oplus \cdots \oplus d_n$ , which in turn implies that  $x \leq y_0 + y_1 + \cdots + y_n$ , as desired.  $\Box$ 

We do not know if the Cuntz semigroup of any (non-simple, non-unital)  $C^*$ -algebra with decomposition rank equal to  $n < \infty$  has the *n*-comparison property. It seems very likely that it should be the case. In Proposition 3.2 we show that a weaker version of *n*-comparison holds for all unital  $C^*$ -algebras with decomposition rank *n*. (That weaker form implies the Corona Factorization Property of the  $C^*$ -algebra.)

We proceed to define a comparison property which is weaker than any n-comparison property. First we remind the reader of the notion of compact containement formalized

[2]: Given two elements x, y in an abelian ordered semigroup W, x is compactly contained in y, or in other words, x is way below y, denoted by  $x \ll y$ , if whenever  $y_1 \leq y_2 \leq \cdots$ is an increasing sequence with supremum greater than or equal to y, eventually  $x \leq y_n$ .

If A is a  $C^*$ -algebra and if a is a positive element in (a matrix algebra over) A, then  $\langle (a-\varepsilon)_+ \rangle \ll \langle a \rangle$  for every  $\varepsilon > 0$ . Let us note some properties that can be deduced from [16, Proposition 2.4] and Lemma 2.5. If  $x, y \in W(A)$ , then

- (a)  $x \leq y$  if and only if  $x_0 \leq y$  for every  $x_0 \in W(A)$  with  $x_0 \ll x$ ;
- (b) if  $x \leq y$  and if  $x_0 \in W(A)$  is such that  $x_0 \ll x$ , then there is  $y_0 \in W(A)$  with  $y_0 \ll y$  and  $x_0 \leq y_0$ ;
- (c) if  $x \leq y$  and if  $x_0 \in W(A)$  is such that  $x_0 \ll x$ , then there is  $y_0 \in W(A)$  with  $y_0 \ll y$  and  $x_0 \leq y_0$ .

**Definition 2.9** (The  $\omega$ -comparison property). An ordered abelian semigroup  $(W, +, \leq)$  is said to have the  $\omega$ -comparison property if whenever  $x', x, y_0, y_1, y_2, \ldots$  are elements in W such that  $x <_s y_j$  for all j and  $x' \ll x$ , then  $x' \leq y_0 + y_1 + \cdots + y_n$  for some n (that may depend on the element x').

It is clear that if W has the n-comparison property for some n, then W also has the  $\omega$ -comparison property.

We shall now (re-)define two even weaker comparison properties for an ordered abelian semigroup, the strong Corona Factorization Property and the Corona Factorization Property. These were also defined in our paper, [12], written in parallel with this paper. In [12] we were only interested in the case where the ordering was the algebraic one, and where all elements were compactly contained in themselves. In our more general situation the definition below is more appropriate (and it extends the definition given in [12]), see Remark 2.13.

**Definition 2.10** (The strong Corona Factorization Property for semigroups). Let  $(W, +, \leq)$  be an ordered abelian semigroup. Then W is said to satisfy the *strong Corona Factorization Property* if given any x', x in W, any sequence  $\{y_n\}$  in W, and any positive integer m satisfying  $x' \ll x$  and  $x \leq my_n$  for all n, then there exists a positive integer k such that  $x' \leq y_1 + y_2 + \cdots + y_k$ .

Fullness, as defined below, was also considered in [12], and again we must extend this definition so that it applies to general ordered (positive) semigroups.

**Definition 2.11** (Full elements and sequences). Let  $(W, +, \leq)$  be an ordered abelian semigroup.

An element x in W is said to be *full* if for any  $y', y \in W$  with  $y' \ll y$ , one has  $y' \propto x$ . A sequence  $\{x_n\}$  in W is said to be *full* if it is increasing and for any  $y', y \in W$  with  $y' \ll y$ , one has  $y' \propto x_n$  for some (hence all sufficiently large) n.

Every order unit in W is full, but the reverse is not always true. The constant sequence  $\{x_n\}$ , with  $x_n = x$  for all n, is full if and only if x is full.

Suppose that A is a  $C^*$ -algebra and that  $\{a_n\}$  is a sequence of positive elements in A. Then  $\{\langle a_n \rangle\}$  is full in W(A) if and only if  $a_1 \preceq a_2 \preceq a_3 \preceq \cdots$  and  $\{a_n\}$  is full in A (in the sense of  $\{a_n\}$  not being contained in a proper closed two-sided ideal in A).

**Definition 2.12** (Corona Factorization Property for semigroups). Let  $(W, +, \leq)$  be an ordered abelian semigroup. Then W is said to satisfy the *Corona Factorization Property* if given any full sequence  $\{x_n\}$  of W, any sequence  $\{y_n\}$  of W, an element x' in W, and a positive integer m satisfying  $x' \ll x_1$  and  $x_n \leq my_n$  for all n, then there exists a positive integer k such that  $x' \leq y_1 + y_2 + \cdots + y_k$ .

It is clear that any semigroup that satisfies the strong Corona Factorization Property also satisfies the Corona Factorization Property. It was shown in [12] that a conical refinement monoid satisfies the strong Corona Factorization Property if and only if every ideal of the monoid satisfies the Corona Factorization Property. It is not clear if this remains true without assuming the refinement property, but we shall show (implicit in Theorem 5.14) that this also holds for semigroups arising as the Cuntz semigroup of a  $\sigma$ -unital  $C^*$ -algebra. We shall also see that, as in the case of  $C^*$ -algebras with real rank zero, the Corona Factorization Property defined for semigroups matches the corresponding property for the  $C^*$ -algebras (see Section 5).

It is also easily checked that, if W is an algebraically ordered semigroup where every element is compactly contained in itself, then the definitions given above for the (strong) Corona Factorization Property agree with the ones in [12]. The precise connection between the two notions is found in the remark below.

**Remark 2.13.** Recall that an *interval* in an ordered abelian monoid V is a non-empty subset I of V which is order-hereditary and upwards directed. An interval I is said to be *countably generated* if there is a sequence  $\{x_n\}$  of elements in I (that can be taken to be increasing) such that  $I = \{x \in V \mid x \leq x_n \text{ for some } n\}$ .

Given two intervals I and J in V, their addition is defined to be  $I + J = \{x \in V \mid x \leq y + z \text{ with } y \in I, z \in J\}$ . Denote by  $\Lambda(V)$  the monoid of all intervals in V, which is naturally ordered by set inclusion, and denote by  $\Lambda_{\sigma}(V)$  the submonoid of  $\Lambda(V)$  whose elements are the countably generated intervals in V.

Now assume that V is algebraically ordered and that all of its elements are compactly contained in themselves. Then V satisfies the (strong) Corona Factorization Property if, and only if,  $\Lambda_{\sigma}(V)$  (ordered by inclusion) satisfies the (strong) Corona Factorization Property. Let us sketch part of the arguments needed. Assume that V has the strong Corona Factorization Property, let  $X, Y_1, Y_2, \ldots$  be elements in  $\Lambda_{\sigma}(V)$ , and let  $m \in \mathbb{N}$ be such that  $X \subseteq mY_n$  for all n. Let  $X' \ll X$  be given. Since X is countably generated by a sequence, say  $\{x_n\}$ , that without loss of generality can be assumed to be increasing, there is i such that  $X' \subseteq [0, x_i]$ . It suffices to check that  $[0, x_i] \subseteq Y_1 + \cdots + Y_l$  for some l. As  $x_i$  belongs to  $mY_n$  for all n, and since each  $Y_n$  is upwards directed, there is  $y_n \in Y_n$  such that  $x_i \leq my_n$ . By the assumption that V satisfies the strong Corona Factorization Property, this implies that  $x_i \leq y_1 + y_2 + \cdots + y_n$  for some n. This again implies  $X' \subseteq Y_1 + \cdots + Y_n$ , and thus proves that  $\Lambda_{\sigma}(V)$  has the strong Corona Factorization Property.

In [12], the authors proved that a  $C^*$ -algebra with real rank zero has the (strong) Corona Factorization Property if and only if the projection semigroup V(A) has the corresponding property (for monoids). Since V(A) is algebraically ordered and every element is compactly contained in itself, the above applies to get that  $\Lambda_{\sigma}(V(A))$  satisfies the (strong) Corona Factorization Property if and only if V(A) does. In particular, since for a  $C^*$ -algebra with real rank zero A, one has that  $W(A \otimes \mathcal{K})$  is order-isomorphic to  $\Lambda_{\sigma}(V(A))$ —see [13]—, our observations yield that, within this class, V(A) has the (strong) Corona Factorization Property if and only if  $W(A \otimes \mathcal{K})$  does, if and only if A has the (strong) Corona Factorization Property if and only if  $W(A \otimes \mathcal{K})$  does, if and only if A has the (strong) Corona Factorization Property—the latter equivalence follows from the results in [12].

**Proposition 2.14.** Any abelian ordered semigroup, which satisfies the  $\omega$ -comparison property, also satisfies the Corona Factorization Property.

*Proof.* Let W be an abelian ordered semigroup with the  $\omega$ -comparison property. Let  $\{x_n\}$  be a full sequence in W, let  $\{y_n\}$  be another sequence in W, let  $x' \in W$ , and let m be a positive integer such that  $x' \ll x_1$  and  $x_n \leq my_n$  for all n. For each integer  $n \geq 0$  put

Then

$$(m+1)x_1 \leq x_{n(m+1)+1} + x_{n(m+1)+2} + \dots + x_{n(m+1)+m+1}$$

 $z_n = y_{n(m+1)+1} + y_{n(m+1)+2} + \dots + y_{n(m+1)+m+1}.$ 

$$\leq m(y_{n(m+1)+1} + y_{n(m+1)+2} + \dots + y_{n(m+1)+m+1}) = mz_n,$$

whence  $x_1 <_s z_n$  for all n. It follows that  $x' \leq z_0 + z_1 + \cdots + z_n$  for some n, which entails that  $x' \leq y_1 + y_2 + \cdots + y_{(n+1)(m+1)}$ .

We shall finally consider the following notion of *n*-comparison that only involves full elements of the semigroup. (This shall be appropriate for studying the Corona Factor-ization Property for non-simple  $C^*$ -algebras of finite decomposition rank.)

**Definition 2.15** (Weak *n*- and weak  $\omega$ -comparison property). Let  $(W, +, \leq)$  be an ordered abelian semigroup. Say that W has the *weak n-comparison* property if whenever  $y_0, y_1, \ldots, y_n$  are *full* elements in W and  $x \in W$  are such that  $x <_s y_i$  for all i, then  $x \leq y_0 + y_1 + \cdots + y_n$ .

We say that W has the weak  $\omega$ -comparison property if whenever  $x, x', y_0, y_1, \ldots$ , are elements in W such that  $y_0, y_1, \ldots$  are full elements,  $x' \ll x$ , and  $x <_s y_i$  for all i, then  $x' \leq y_0 + y_1 + \cdots + y_n$  for some positive integer n.

The weak n- and the weak  $\omega$ -comparison properties only makes sense for semigroups that contain a full element (if they don't, then they automatically posses this property). It is clear that if W satisfies the weak n-comparison property for some n, then it satisfies the weak m-comparison property for all  $m \geq n$  and it satisfies the weak  $\omega$ -comparison property.

**Proposition 2.16.** Any abelian ordered semigroup, which satisfies the weak  $\omega$ -comparison property and which contains a full element that is compactly contained in another (full) element, also satisfies the Corona Factorization Property.

*Proof.* Let W be an abelian ordered semigroup with the weak  $\omega$ -comparison property. By assumption there are elements  $v \ll w$  in W such that v is full.

Let  $\{x_k\}$  be a full sequence in W, let  $\{y_k\}$  be another sequence in W, let  $x' \in W$ , and let m be a positive integer such that  $x' \ll x$  and  $x_k \leq my_k$  for all k. By the definition of a full sequence (applied to  $v \ll w$ ) we have that  $v \propto x_k$  for all large enough k. Hence  $v \propto y_k$  for all large enough k, whence  $y_k$  is full whenever k is large enough. Upon removing the first finitely many elements from the sequences  $\{x_k\}$  and  $\{y_k\}$  we can assume that all  $y_k$  are full.

Let now  $z_0, z_1, z_2, \ldots$  be as in the proof of Proposition 2.14 above. Then  $z_0, z_1, z_2, \ldots$  are full and  $x_1 <_s z_j$  for all j. It follows that

$$x' \le z_0 + z_1 + \dots + z_n = y_1 + y_2 + \dots + y_{(n+1)(m+1)}$$

for some n, whence W has the Corona Factorization Property.

In conclusion, we have defined the following comparability properties of an ordered abelian semigroup, listed in decreasing strength: 0-comparison (which is the same as being almost unperforated), 1-comparison, 2-comparison, ...,  $\omega$ -comparison, the strong Corona Factorization Property, and the Corona Factorization Property for semigroups. Moreover, we have defined weak *n*- and the weak  $\omega$ -comparison properties. We show below that the comparison properties above are in fact strictly decreasing in strength, except that we do not have an example of an abelian ordered semigroup that has the strong Corona Factorization Property but not the  $\omega$ -comparison property. (That the strong Corona Factorization Property is strictly stronger than the Corona Factorization Property was already noted in [12].)

**Example 2.17.** Let *n* be a positive integer, let  $W_n$  be the subsemigroup of  $\mathbb{Z}^+$  generated by  $\{0, n + 1, n + 2\}$ , and equip  $W_n$  with the algebraic order. Notice that

$$(n+1)(n+2) - (n+1) - (n+2) = n^2 + n - 1$$

is the largest natural number that does not belong to  $W_n$ . Suppose that  $x, y_0, y_1, \ldots, y_n$  belong to  $W_n$ , that  $x <_s y_j$  for all j, and that x is non-zero. Then all  $y_j$ 's are non-zero, whence

$$y_0 + y_1 + \dots + y_n - x \ge y_1 + y_2 + \dots + y_n \ge n(n+1),$$

(where the ordering above is the usual one in  $\mathbb{Z}$ ), which shows that  $x \leq y_0 + y_1 + \cdots + y_n$ (with respect to the order given on  $W_n$ ). Hence  $W_n$  has the *n*-comparison property.

On the other hand, if we take x = n + 1 and  $y_0 = y_1 = \cdots = y_{n-1} = n + 2$ , then  $x <_s y_j$  for all j, but

$$y_0 + y_1 + \dots + y_{n-1} - x = n(n+2) - (n+1) = n^2 + n - 1,$$

which does not belong to  $W_n$ . Hence  $x \not\leq y_0 + y_1 + \cdots + y_{n-1}$  in  $W_n$ , whence  $W_n$  does not have the (n-1)-comparison property.

Next, put  $W_{\omega} = \bigoplus_{n=1}^{\infty} W_n$  (as an ordered abelian semigroup). Then  $W_{\omega}$  has the  $\omega$ -comparison property, but does not have the *n*-comparison property for any finite n. Indeed, suppose that  $x, y^0, y^1, \ldots$  in  $W_{\omega}$  are such that  $x <_s y^j$  for all j. Write  $x = (x_1, x_2, \ldots)$  and  $y^j = (y_1^j, y_2^j, \ldots)$ , with  $x_k$  and  $y_k^j$  in  $W_k$  for all k. Then  $x_k = 0$  for all k greater than some  $k_0 \in \mathbb{N}$ . Since  $W_k$  has  $k_0$ -comparison when  $k \leq k_0$ , we have

$$x_k \le y_k^0 + y_k^1 + \dots + y_k^{k_0}$$

(in  $W_k$ ) for all k, whence  $x \leq y^0 + y^1 + \dots + y^{k_0}$  (in  $W_{\omega}$ ).

Conversely, given a positive integer n, choose  $x', y'_0, y'_1, \ldots, y'_n$  in  $W_{n+1}$  such that  $x' \leq s y'_j$  for all j, and such that  $x' \nleq y'_0 + y'_1 + \cdots + y'_n$ . (This is possible because  $W_{n+1}$  does not have the *n*-comparison property.) Let  $x, y_0, y_1, \ldots, y_n$  in  $W_{\omega}$  be the elements whose coordinates in the (n + 1) position are, respectively,  $x', y'_0, y'_1, \ldots, y'_n$ , and whose other coordinates are zero. Then  $x <_s y_j$  for all j while  $x \nleq y_0 + y_1 + \cdots + y_n$ . Hence  $W_{\omega}$  does not have the *n*-comparison property.

# 3. The Corona Factorization Property for $C^*$ -algebras with finite decomposition rank

We have already mentioned the result, [21, Lemma 6.1], of Toms and Winter which implies that the Cuntz semigroup of a simple unital separable  $C^*$ -algebra has *n*-comparison property. We wish to extend this result to the non-simple case, and state for this purpose a lemma whose proof actually is contained in the proof of [21, Lemma 6.1] (follow that proof from Equation (10) to its end) and therefore is omitted.

**Lemma 3.1** (Toms and Winter). Let A be a separable  $C^*$ -algebra with finite decomposition rank n. Suppose that  $a, d_0, \ldots, d_n \in A^+$  and  $\alpha > 0$  satisfy

$$\forall \tau \in T(A) : \quad d_{\tau}(a) < d_{\tau}(d_i) - \alpha.$$

Then  $a \preceq d_0 \oplus d_1 \oplus \cdots \oplus d_n$ .

**Proposition 3.2.** Let A be a separable, unital C<sup>\*</sup>-algebra with decomposition rank  $n < \infty$ . Then W(A) has the weak n-comparison property.

*Proof.* Let  $x, y_0, \ldots, y_n \in W(A)$  with  $y_i$  full and  $x <_s y_i$  be given for every *i*. Then, by Proposition 2.1, there exists k such that  $(k+1)x \leq ky_i$  for all *i*.

As  $y_0, \ldots, y_n$  are full, there is a natural number N such that  $\langle 1_A \rangle \leq Ny_i$  for all *i*. Choose  $0 < \alpha' < (k+1)^{-1}$  and let  $\alpha = \alpha'/N$ . Let now  $f \in S(W(A), \langle 1_A \rangle)$ . We then have that  $1 \leq Nf(y_i)$  for all *i*, whence  $\alpha < f(y_i)/(k+1)$ . Therefore:

$$f(x) \le \frac{k}{k+1} f(y_i) = f(y_i) - \frac{f(y_i)}{k+1} < f(y_i) - \alpha$$

for all i.

In particular, with  $d_{\tau}$  denoting the (lower semicontinuous) dimension function associated to a tracial state  $\tau$  on A, we have  $d_{\tau}(x) < d_{\tau}(y_i) - \alpha$  for all i and for every tracial state  $\tau$  on A

Finite decomposition rank passes to matrices, so upon replacing A with a matrix algebra over A, we can suppose that there exist positive elements a and  $d_0, d_1, \ldots, d_n$  in A, with  $d_i$  full, such that  $x = \langle a \rangle$  and  $y_i = \langle d_i \rangle$  for all i. Then  $d_{\tau}(a) < d_{\tau}(d_i) - \alpha$  for all i and for all tracial states  $\tau$  on A. Lemma 3.1 then implies that  $a \preceq d_0 \oplus d_1 \oplus \cdots \oplus d_n$ , which again implies that  $x \leq y_0 + y_1 + \cdots + y_n$  as desired.

Combining Proposition 2.16 and Proposition 3.2 we get:

**Corollary 3.3.** Let A be a separable, unital  $C^*$ -algebra with finite decomposition rank. Then W(A) has the Corona Factorization Property.

# 4. Stability of $C^*$ -algebras

We show in this section a  $C^*$ -algebra whose Cuntz semigroup has the  $\omega$ -comparison property is stable if and only if it has no unital quotient and no bounded 2-quasitrace. We introduce a property (S) of a  $C^*$ -algebra that we show is equivalent to having no non-zero unital quotients and no bounded 2-quasitraces.

Recall from [7] that F(A) is the set of *compactly supported* positive elements, that is,

$$F(A) = \{a \in A^+ \mid \text{ there exists } e \in A^+ \text{ with } ea = a\}.$$

It was shown in [7] that a separable  $C^*$ -algebra A is stable if and only if to every  $a \in F(A)$  there exists  $b \in A^+$  such that  $a \perp b$  and  $a \preceq b$ . We shall here consider a weaker version of this condition, where we replace the relation  $a \preceq b$  with the relation  $a \prec_s b$  considered in Section 2.

**Definition 4.1.** A  $C^*$ -algebra A is said to have property (S) if for every  $a \in F(A)$  there exists  $b \in A^+$  such that  $a \perp b$  and  $a \prec_s b$ .

It follows immediately from the definition, the results from [7] quoted above, and from Remark 2.4, that if A is a separable  $C^*$ -algebra for which W(A) is almost unperforated, then A has property (S) if and only if A is stable. It is easy to see that every stable  $C^*$ -algebra has property (S).

**Lemma 4.2.** Let A be a separable  $C^*$ -algebra with property (S). Then A has no non-zero unital quotients.

Proof. Let I be an ideal of A such that A/I is unital. Let e + I be the unit of A/I, with  $e \in A^+$ . Upon replacing e with g(e), where  $g: \mathbb{R}^+ \to [0, 1]$  is a continuous function which vanishes on, say [0, 1/2], and with g(1) = 1, we can assume that  $e \in F(A)$ . By the assumption that A has property (S) there exists  $b \in A^+$  such that  $e \perp b$  and  $e \prec_s b$ . Now, 0 = eb + I = b + I, so b belongs to I. The relation  $e \prec_s b$  implies that e belongs to the closed two-sided ideal generated by b, and hence to I. Thus, e + I = 0 and A/I = 0. We say that d is strictly full if  $(d-\varepsilon)_+$  is full for some  $\varepsilon > 0$ , and hence for all sufficiently small  $\varepsilon > 0$ .

**Lemma 4.3.** Let A be a  $C^*$ -algebra such that  $A \otimes \mathcal{K}$  contains a full projection. Then every full positive element in A is strictly full.

Proof. We can view A as a full hereditary sub-C\*-algebra of its stabilization  $A \otimes \mathcal{K}$ . Let p be a full projection in  $A \otimes \mathcal{K}$ , and let d be a full positive element in A. For each  $\varepsilon > 0$  consider the closed two-sided ideal  $I_{\varepsilon}$  of  $A \otimes \mathcal{K}$  generated by  $(d - \varepsilon)_+$ . The closure of  $\bigcup_{\varepsilon > 0} I_{\varepsilon}$  is a closed two-sided ideal in  $A \otimes \mathcal{K}$  which contains d and hence is equal to  $A \otimes \mathcal{K}$ . It follows that  $\bigcup_{\varepsilon > 0} I_{\varepsilon}$  is a dense (algebraic) ideal in  $A \otimes \mathcal{K}$ . Being a dense ideal,  $\bigcup_{\varepsilon > 0} I_{\varepsilon}$  contains every projection of  $A \otimes \mathcal{K}$ . Hence p belongs to  $I_{\varepsilon}$  for some  $\varepsilon > 0$ , whence  $I_{\varepsilon} = A \otimes \mathcal{K}$ , which again implies that  $(d - \varepsilon)_+$  is full (in  $A \otimes \mathcal{K}$  and hence in A).

**Lemma 4.4.** Let A be a separable  $C^*$ -algebra with property (S). Then given any  $a \in F(A)$  there exists  $b \in F(A)$  such that

$$a \perp b, \qquad a \prec_s b, \qquad a+b \in F(A).$$

If, moreover,  $A \otimes \mathcal{K}$  is assumed to contain a full projection, then b above can be chosen to be strictly full in A.

*Proof.* Let  $a \in F(A)$ , and choose d in  $A^+$  with da = ad = a. Let  $g: \mathbb{R}^+ \to [0, 1]$  be a continuous function which is zero on [0, 1/2] and with g(1) = 1, and put e = g(d). Then

 $e \in F(A),$  ea = ae = a,  $a \preceq (e - 1/2)_+,$  ||e|| = 1.

Since A has property (S) there exists  $b_0 \in A^+$  such that  $e \perp b_0$  and  $e \prec_s b_0$ . It follows from Lemma 2.5 that there exists  $\delta > 0$  such that  $(e - 1/2)_+ \prec_s (b_0 - \delta)_+$ . Put  $b = (b_0 - \delta)_+ \in F(A)$ , and set  $f = h(b_0)$  where  $h \colon \mathbb{R}^+ \to [0, 1]$  is a continuous function such that h(0) = 0 and h(t) = 1 for  $t \geq \delta$ . Then  $a \perp b$ ,  $a \prec_s b$ , and

$$(e+f)(a+b) = ea + fb = a + b.$$

The latter shows that a + b belongs to F(A).

Assume now that there exists a full projection in  $A \otimes \mathcal{K}$ . Let *B* be the hereditary sub-*C*<sup>\*</sup>-algebra of *A* consisting of all elements which are orthogonal to *e*. Then *B* is full in *A*. Indeed, because *e* belongs to F(A) there exists a positive element *e'* in *A* such that e'e = ee' = e. Let *I* be the closed two-sided ideal in *A* generated by *B*, and assume, to reach a contradiction, that *I* were proper. Then e' + I would be a unit for A/I, thus contradicting Lemma 4.2.

It follows from Brown's theorem that  $B \otimes \mathcal{K}$  is isomorphic to  $A \otimes \mathcal{K}$ , and so  $B \otimes \mathcal{K}$  contains a full projection. Hence, by Lemma 4.3, any full element in B is strictly full. Upon adding onto  $b_0$  a positive full element in B we can assume that  $b_0$  is full. It follows that  $b = (b_0 - \delta)_+$  is full (and hence strictly full) if  $\delta_0 > 0$  is chosen sufficiently small. **Lemma 4.5.** Let A be a separable  $C^*$ -algebra with property (S). Then for every  $a \in F(A)$  there is a sequence  $b_0, b_1, b_2, \ldots$  of elements in F(A) such that the elements  $a, b_0, b_1, b_2, \ldots$  are pairwise orthogonal,  $a + b_0 + b_1 + \cdots + b_n$  belongs to F(A) for all n, and such that  $a \prec_s b_0 \prec_s b_1 \prec_s \cdots$ .

If, moreover,  $A \otimes \mathcal{K}$  is assumed to contain a full projection, then  $b_0, b_1, b_2, \ldots$  above can be chosen to be strictly full in A.

Proof. The existence of  $b_0$  such that  $a \perp b_0$ ,  $a \prec_s b_0$ , and  $a + b_0$  belongs to F(A) follows from Lemma 4.4. Suppose that  $n \geq 0$  and that  $b_0, b_1, \ldots, b_n$  have been found such that  $a, b_0, b_1, \ldots, b_n$  are pairwise orthogonal,  $a \prec_s b_0 \prec_s b_1 \prec_s \cdots \prec_s b_n$ , and  $a + b_0 + b_1 + \cdots + b_n$  belongs to F(A). Then, by Lemma 4.4, there is  $b_{n+1}$  in F(A) which is orthogonal to the sum  $a + b_0 + b_1 + \cdots + b_n$  (and hence to each of the summands), such that  $a + b_0 + b_1 + \cdots + b_n \prec_s b_{n+1}$  (and hence  $b_n \prec_s b_{n+1}$ ) and such that  $a + b_0 + b_1 + \cdots + b_{n+1}$  belongs to F(A).

Finally, use Lemma 4.4 to see that each of the positive elements  $b_j$  above can be chosen to be strictly full if  $A \otimes \mathcal{K}$  contains a full projection.

We will now give an algebraic characterization of property (S) for a  $C^*$ -algebra. The characterization is very similar to, but sharpens, [6, Theorem 3.6]. The reader is referred to [1] for the definition and properties of 2-quasitraces. Let us just here remind the reader than any 2-quasitrace on an exact  $C^*$ -algebra is a trace, and that the short-coming of a quasitrace (compared with a trace) is that it only is assumed to be additive on commuting elements.

**Proposition 4.6.** Let A be a separable  $C^*$ -algebra. Then A has property (S) if and only if A has no non-zero bounded 2-quasitrace and no non-zero unital quotient.

*Proof.* The "if" part is contained in the proof of [6, Theorem 3.6].

To prove the "only if" part, suppose that A has property (S). By Lemma 4.2, A has no non-zero unital quotients. Suppose, to reach a contradiction, that  $\tau$  is a non-zero bounded 2-quasitrace on A, and let  $d_{\tau}$  be the associated lower semicontinuous dimension function on W(A). Since  $\tau$  is non-zero there is a positive element a in A such that  $d_{\tau}(\langle a \rangle) > 0$ , and since  $d_{\tau}$  is lower semicontinuous,  $d_{\tau}(\langle (a - \varepsilon)_+ \rangle) > 0$  for some  $\varepsilon > 0$ . We can now use Lemma 4.5 to find a sequence  $b_0 = (a - \varepsilon)_+, b_1, b_2, \ldots$  of pairwise orthogonal elements in F(A) such that  $b_0 \prec_s b_1 \prec_s b_2 \prec_s \cdots$ . By Proposition 2.1 we have  $0 < d_{\tau}(\langle b_0 \rangle) < d_{\tau}(\langle b_1 \rangle) < d_{\tau}(\langle b_2 \rangle) < \cdots$ , and in particular

$$d_{\tau}(\langle b_0 + b_1 + b_2 + \dots + b_n \rangle) \ge (n+1)d_{\tau}(\langle (a-\varepsilon)_+ \rangle).$$

On the other hand, one has  $d_{\tau}(\langle c \rangle) \leq ||\tau||$  for all c in  $A^+$ , and so the inequality above is in contradiction with the assumed boundedness of  $\tau$ .

It is well-known that stability is not a stable property (see [17]). Property (S), however, is a stable property, as easily follows from Proposition 4.6 above:

**Corollary 4.7.** Let A be a separable  $C^*$ -algebra. Then the following conditions are equivalent:

- (i) A has property (S).
- (ii)  $M_n(A)$  has property (S) for some natural number n.
- (iii)  $M_n(A)$  has property (S) for all natural numbers n.

*Proof.* By Proposition 4.6 it suffices to check that each of the two properties: having a non-zero unital quotient, and having a non-zero bounded 2-quasitrace, passes to matrix algebras and back again. This is trivial for the first. It is a theorem (see [1]) that 2-quasitraces extend to all matrix algebras (and vice versa).

The result, [6, Theorem 3.6], that we have used extensively in the proof of Proposition 4.6 above, actually says that a separable  $C^*$ -algebra with almost unperforated Cuntz semigroup is stable if and only if it has no non-zero unital quotient and no nonzero bounded 2-quasitrace. Reminding the reader that almost unperforation is the same as the "0-comparison" property, we can extend [6, Theorem 3.6] as in the proposition below.

Recall that every element in  $W(A \otimes \mathcal{K})$  is represented by a positive element in  $A \otimes \mathcal{K}$ (we do not need to take matrix algebras), and every element in W(A) is represented by a positive element in  $M_{\infty}(A)$ . If a belongs to  $F(A \otimes \mathcal{K})$ , then a is equivalent (in the sense of Cuntz comparison) to an element in  $M_{\infty}(A)^+$ , whence  $\langle a \rangle$  belongs to W(A). Suppose that B is a hereditary sub-C<sup>\*</sup>-algebra of  $A \otimes \mathcal{K}$ . Then W(B) is a sub-semigroup of  $W(A \otimes \mathcal{K})$ ; and if a belongs to F(B), then  $\langle a \rangle$  belongs to W(A).

**Proposition 4.8.** Let A be a separable C<sup>\*</sup>-algebra such that W(A) satisfies the  $\omega$ comparison property (cf. Definition 2.9). Let B be a hereditary sub-C<sup>\*</sup>-algebra of  $A \otimes \mathcal{K}$ .
Then the following conditions are equivalent:

- (i) B is stable
- (ii) B has no non-zero unital quotient and no non-zero bounded 2-quasitrace.
- (iii) B has property (S).

*Proof.* Conditions (ii) and (iii) are equivalent for all separable  $C^*$ -algebras by Proposition 4.6, and (i) clearly implies (ii) (again for all  $C^*$ -algebras) (see eg. [7]).

(iii)  $\Rightarrow$  (i). By [7, Proposition 2.2] it is enough to show that for every  $a \in B^+$ and every  $\varepsilon > 0$  there exists  $b \in B^+$  such that  $(a - \varepsilon)_+ \preceq b$  and  $(a - \varepsilon)_+ \perp b$ . (Indeed, if such an element *b* exists, then  $(a - 2\varepsilon)_+ = x^*bx$  for some  $x \in B$ ; whence  $(a - 2\varepsilon)_+ \sim b^{1/2}xx^*b^{1/2} := b_0 \perp (a - 2\varepsilon)_+$ , and  $||a - (a - 2\varepsilon)_+|| \leq 2\varepsilon$ .)

Since  $(a - \varepsilon/2)_+$  belongs to F(B) we can apply Lemma 4.5 to get a sequence of positive elements  $b_0, b_1, b_2, \ldots$  in F(B) such that  $(a - \varepsilon/2)_+ \prec_s b_0 \prec_s b_1 \prec_s b_2 \prec_s \cdots$  and for which  $(a - \varepsilon/2)_+, b_0, b_1, b_2, \ldots$  are mutually orthogonal. By the remarks preceding this proposition, the elements  $\langle (a - \varepsilon/2)_+ \rangle, \langle b_j \rangle \in W(B)$  belong to W(A); so by the assumption that W(A) satisfies the  $\omega$ -comparison property there is a natural number nsuch that

$$\langle ((a - \varepsilon/2)_+ - \varepsilon/2)_+ \rangle = \langle (a - \varepsilon)_+ \rangle \leq \langle b_0 \rangle + \dots + \langle b_n \rangle = \langle b_0 \oplus \dots \oplus b_n \rangle = \langle b_0 + \dots + b_n \rangle$$

in W(A) (and hence in W(B)). Thus,  $(a-\varepsilon)_+ \preceq b_0 + \cdots + b_n$  and  $(a-\varepsilon)_+ \perp b_0 + \cdots + b_n$  as desired.

We have the following analog of Proposition 4.8, where the assumption on the comparison property of the Cuntz semigroup is weakened, but where we instead have to assume the existence of a full projection:

**Proposition 4.9.** Let A be a separable  $C^*$ -algebra such that W(A) satisfies the weak  $\omega$ -comparison property and such that  $A \otimes \mathcal{K}$  contains a full projection. Let B be a full hereditary sub- $C^*$ -algebra of  $A \otimes \mathcal{K}$ . Then the following conditions are equivalent:

- (i) B is stable
- (ii) B has no non-zero unital quotient and no non-zero bounded 2-quasitrace.
- (iii) B has property (S).

*Proof.* Proceeding as in the proof of Proposition 4.8, we only need to prove (iii)  $\Rightarrow$  (i); and to prove that (i) holds it suffices to show that for every  $a \in B^+$  and every  $\varepsilon > 0$  there exists  $b \in B^+$  such that  $(a - \varepsilon)_+ \preceq b$  and  $(a - \varepsilon)_+ \perp b$ .

Since  $(a - \varepsilon)_+$  belongs to F(B) we can apply Lemma 4.5 to get *full* positive elements  $b_0, b_1, b_2, \ldots$  in F(B) such that  $(a - \varepsilon)_+ \prec_s b_0 \prec_s b_1 \prec_s b_2 \prec_s \cdots$  and such that  $(a - \varepsilon)_+, b_0, b_1, b_2 \ldots$  are mutually orthogonal. As remarked above Proposition 4.8, the elements  $\langle (a - \varepsilon)_+ \rangle, \langle b_j \rangle \in W(B)$  actually belong to W(A). Moreover, each  $\langle b_j \rangle$  is full in W(A) (because B is full in  $A \otimes \mathcal{K}$ ), so by the assumption that W(A) has the weak  $\omega$ -comparison property it follows that

$$\langle (a-\varepsilon)_+ \rangle \le \langle b_0 \rangle + \dots + \langle b_n \rangle = \langle b_0 \oplus \dots \oplus b_n \rangle = \langle b_0 + \dots + b_n \rangle$$

for some natural number n (the order is relatively to W(A) and hence also in W(B)). We conclude that  $(a - \varepsilon)_+ \preceq b_0 + \cdots + b_n$  and  $(a - \varepsilon)_+ \perp b_0 + \cdots + b_n$ .

We end this section by describing when separable  $C^*$ -algebras with finite decomposition rank are stable (under the assumption that their stabilization contains a full projection):

**Corollary 4.10.** Let A be a separable C<sup>\*</sup>-algebra with finite decomposition rank, and assume that  $A \otimes \mathcal{K}$  contains a full projection. Then the following conditions are equivalent:

- (i) A is stable.
- (ii) A has no non-zero unital quotients and no non-zero bounded positive traces.
- (iii) A has property (S).

Proof. Let p be a full projection in  $A \otimes \mathcal{K}$  and put  $B = p(A \otimes \mathcal{K})p$ . Then B has the same decomposition rank as A, say n; and B is unital. It follows from Proposition 3.2 that W(B) has weak n-comparison and therefore also weak  $\omega$ -comparison property. As A is (isomorphic to) a full hereditary sub- $C^*$ -algebra of  $B \otimes \mathcal{K}$  the result now follows from Proposition 4.9. (In (ii) we have used that any 2-quasitrace on a nuclear  $C^*$ -algebra is a trace.)

## 5. The Corona Factorization Property and the Cuntz semigroup

Recall that a  $C^*$ -algebra A is said to have the *Corona Factorization Property* if every full projection in the multiplier algebra of  $A \otimes \mathcal{K}$  is properly infinite. The fact that the Corona Factorization Property is equivalent to a statement regarding stability of full hereditary sub- $C^*$ -algebras of the stabilized  $C^*$ -algebra was observed by Kucerovsky and Ng in [10]. Our aim here is to characterize the Corona Factorization Property for  $C^*$ -algebras in terms of a certain comparison property of the Cuntz semigroup.

For each  $\varepsilon > 0$  define the continuous function  $g_{\varepsilon} \colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$  by

$$g_{\varepsilon}(t) = \begin{cases} 0 & \text{if } 0 \le t \le \varepsilon \\ \varepsilon^{-1}t - 1 & \text{if } \varepsilon \le t \le 2\varepsilon \\ 1 & \text{if } 2\varepsilon \le t \end{cases}$$

**Lemma 5.1.** Let A be a  $\sigma$ -unital C<sup>\*</sup>-algebra and suppose that  $\{e_k\}$  is an increasing approximate unit for A consisting of positive contractions. Then:

- (i) For every positive a in A and for every  $\varepsilon > 0$  one has  $(a \varepsilon)_+ \preceq e_k$  for all large enough k.
- (ii)  $\{\langle e_k \rangle\}$  is a full sequence in W(A).

*Proof.* (i). We have  $||a^{1/2}e_ka^{1/2} - a|| < \varepsilon$  for k large enough, whence  $(a - \varepsilon)_+ \preceq a^{1/2}e_ka^{1/2} \preceq e_k$ .

(ii). The sequence  $\{\langle e_k \rangle\}$  is clearly increasing. The fullness property of this sequence follows from (i) and from the fact that  $\{e_k \otimes \mathbb{1}_n\}_{k=1}^{\infty}$  is an approximate unit for  $M_n(A)$ .  $\Box$ 

Recall from Section 4 the definition of the set F(A) of compactly supported elements in a  $C^*$ -algebra A. Suppose that A is  $\sigma$ -unital. Then, to any strictly positive element c in A one can associate the set

$$F_c(A) := \{ b \in A^+ \mid g_{\varepsilon}(c)b = b \text{ for some } \varepsilon > 0 \},\$$

cf. [7]. It is easy to see that  $F_c(A)$  is a dense subset of F(A), which—unlike F(A)—is closed under addition.

We shall use below that whenever  $c \in A$  is a strictly positive element of A, then  $c \otimes 1_n$  is a strictly positive element of  $M_n(A)$ .

**Lemma 5.2.** Let c be a strictly positive element of a C<sup>\*</sup>-algebra A, and let  $a = (a_{ij})$  be a positive element in  $M_n(A)^+$ . Let  $d = \sum_{j=1}^n a_{jj} \in A^+$  be the sum of the diagonal elements of a. Then  $\langle a \rangle \leq n \langle d \rangle$ ; and d belongs to  $F_c(A)$  if a belongs to  $F_{c\otimes 1_n}(M_n(A))$ .

*Proof.* Let  $\varepsilon > 0$ . For each i = 1, 2, ..., n, let  $\{e_k^{(i)}\}_{k=1}^{\infty}$  be an approximate unit for  $\overline{a_{ii}Aa_{ii}}$ , and put  $e_k = \operatorname{diag}(e_k^{(1)}, \ldots, e_k^{(n)})$ . Then  $e_k^{(i)} \preceq a_{ii} \preceq d$  for all k. Also,  $\{e_k\}$  is an approximate unit for  $\overline{aM_n(A)a}$ , whence  $(a - \varepsilon)_+ \preceq e_k$  for all large enough k, cf. Lemma 5.1. We therefore conclude that

$$\langle (a-\varepsilon)_+ \rangle \leq \langle e_k \rangle = \sum_{i=1}^n \langle e_k^{(i)} \rangle \leq n \langle d \rangle.$$

This proves the claim because  $\varepsilon > 0$  was arbitrary.

Suppose that a belongs to  $F_{c\otimes 1_n}(M_n(A))$ . Then there is  $\varepsilon > 0$  with  $g_{\varepsilon}(c \otimes 1_n)a = a$ . As  $g_{\varepsilon}(c \otimes 1_n) = g_{\varepsilon}(c) \otimes 1_n$ , this is easily seen to imply that  $g_{\varepsilon}(c)a_{ii} = a_{ii}$  for all *i*, hence  $a_{ii} \in F_c(A)$ .  $\Box$ 

**Lemma 5.3.** Let A be a  $\sigma$ -unital C<sup>\*</sup>-algebra, and fix a strictly positive element c in A. Suppose that  $M_n(A)$  is stable for some positive integer n. Then for all elements a, b in  $F_c(A)$  there exists an element d in  $F_c(A)$  with  $a \perp d$  and  $b \otimes 1_n \preceq d \otimes 1_n$ .

Proof. Let  $a, b \in F_c(A)$ . Then there exists  $\delta > 0$  such that  $a, b \in \overline{c_0Ac_0}$ , where  $c_0 = g_{\delta}(c)$ . Clearly,  $a, b \preceq c_0$  and also  $c_0 \otimes 1_n \in F_{c \otimes 1_n}(M_n(A))$ . Using that  $M_n(A)$  is stable, we find an element  $b' \in F_{c \otimes 1_n}(M_n(A))$  such that  $b' \perp c_0 \otimes 1_n$  and  $c_0 \otimes 1_n \preceq b'$ , cf. [7, Lemma 2.6 (i)]. Let  $d \in A$  be the sum of the diagonal elements in b'. By Lemma 5.2, we get that  $d \in F_c(A)$  and  $b' \preceq d \otimes 1_n$ . This shows that

$$b \otimes 1_n \precsim c_0 \otimes 1_n \precsim b' \precsim d \otimes 1_n$$
.

Since  $b' \perp c_0 \otimes 1_n$ , it follows that  $d \perp c_0$ , whence  $d \perp a$ .

The lemma below is a reformulation of the characterization of stability from [7].

**Lemma 5.4.** Let A be a  $\sigma$ -unital C<sup>\*</sup>-algebra with a strictly positive element c. Then A is stable if and only if for every  $\varepsilon > 0$  there exists  $b \in A^+$  such that  $b \perp (c - \varepsilon)_+$  and  $(c - \varepsilon)_+ \preceq b$ .

*Proof.* The "only if" part follows from [7, Theorem 2.1]. To prove the "if" part, we verify that condition (b) of [7, Proposition 2.2] is satisfied. To this end, let  $a \in F(A)$  and  $\varepsilon > 0$  be given. Choose  $\delta > 0$  such that  $||a - g_{\delta}(c)ag_{\delta}(c)|| < \varepsilon$ . Then find  $d \in A^+$  such that  $(c - \delta)_+ \perp d$  and  $(c - \delta)_+ \precsim d$ . Then

$$g_{\delta}(c)ag_{\delta}(c) \perp d, \qquad a \precsim g_{\delta}(c)ag_{\delta}(c) \precsim g_{\delta}(c) \precsim d.$$

Hence there exists t in A such that  $b' := (a - \varepsilon)_+ = t^* dt$ . Put  $c' = d^{1/2} t t^* d^{1/2}$ . Then  $||a - b'|| \le \varepsilon, b' \perp c'$ , and  $b' \sim c'$ .

**Proposition 5.5.** Let A be a  $\sigma$ -unital C<sup>\*</sup>-algebra whose Cuntz semigroup W(A) satisfies the Corona Factorization Property for monoids. Then A is stable if  $M_m(A)$  is stable for some  $m \in \mathbb{N}$ .

*Proof.* Suppose that  $M_m(A)$  is stable for some natural number m. Let c be a strictly positive element in  $A^+$ , and let  $\varepsilon > 0$  be given. Choose a decreasing sequence  $\{\varepsilon_n\}$  of strictly positive real numbers that converges to zero, and such that  $\varepsilon_1 < \varepsilon$ . Let  $a_n = (c - \varepsilon_n)_+$ . Since  $a_n \approx g_{\varepsilon_n}(c)$  and  $\{g_{\varepsilon_n}(c)\}$  is an increasing approximate unit for cAc = A, it follows from Lemma 5.1 that  $\{\langle a_n \rangle\}$  is a full sequence in W(A).

We use Lemma 5.3 to construct a sequence  $d_1, d_2, d_3...$  of positive elements in  $F_c(A)$  such that  $a_1, d_1, d_2, ...$  are pairwise orthogonal and  $a_n \preceq d_n \otimes 1_m$  for all n. Indeed, at stage n, since  $a_1, d_1, ..., d_{n-1}$  belong to  $F_c(A)$ , so does their sum, and so we can find  $d_n \in F_c(A)$  orthogonal to  $a_1 + d_1 + \cdots + d_{n-1}$  satisfying  $a_n \preceq d_n \otimes 1_m$ .

Now, apply the Corona Factorization Property for W(A) to  $\{\langle a_n \rangle\}$  and  $\{\langle d_n \rangle\}$  (that satisfies  $\langle a_n \rangle \leq m \langle d_n \rangle$  for all n). Thus, for our  $\varepsilon > 0$ , there is a natural number k such that

$$\langle (c-\varepsilon)_+ \rangle \leq \langle (c-\varepsilon_1)_+ \rangle = \langle a_1 \rangle \leq \langle d_1 \rangle + \langle d_2 \rangle + \dots + \langle d_k \rangle = \langle d_1 + d_2 + \dots + d_k \rangle,$$

which implies that A is stable (by virtue of Lemma 5.4).

If A is a non-unital C<sup>\*</sup>-algebra, then we shall denote the unit of its multiplier algebra  $\mathcal{M}(A)$  by 1.

**Lemma 5.6.** Let A be a  $\sigma$ -unital stable C<sup>\*</sup>-algebra and let a be a positive contraction in A. Then 1 - a is a properly infinite, full, positive element in  $\mathcal{M}(A)$ .

*Proof.* It follows from [7, Corollary 4.3] that (1-a)A(1-a) is stable. Hence 1-a is properly infinite, cf. [8, Proposition 3.7]. We proceed to prove that 1-a is full in  $\mathcal{M}(A)$ .

Take positive functions  $f, g: [0, 1] \rightarrow [0, 1]$  such that f is zero on [0, 1/2], f + g = 1, and g(1) = 0. Then g(a) belongs to (1-a)A(1-a). Since A is stable and  $\sigma$ -unital we can find a positive element b in A such that  $b \perp (a - 1/2)_+$  and  $(a - 1/2)_+ \preceq b$ . Then  $f(a) \perp b$ , whence

$$b = (f(a) + g(a))b(f(a) + g(a)) = g(a)bg(a) \in (1 - a)A(1 - a).$$

As  $f(a) \preceq (a - 1/2)_+ \preceq b$ , we see that f(a) belongs to the closed two-sided ideal in  $\mathcal{M}(A)$  generated by 1 - a. As g(a) belongs to  $\overline{(1 - a)\mathcal{M}(A)(1 - a)}$ , we conclude that the closed two-sided ideal generated by 1 - a contains 1 = f(a) + g(a), and hence is equal to  $\mathcal{M}(A)$ .

**Lemma 5.7.** Let A be a  $\sigma$ -unital stable C<sup>\*</sup>-algebra and let T be a positive element in  $\mathcal{M}(A)$  such that  $1 \preceq T$  (or, equivalently, such that T is full and properly infinite). Then  $\overline{TAT}$  is stable.

*Proof.* Put  $B = \overline{TAT}$ . Since A is  $\sigma$ -unital, then so is B.

There is  $\delta > 0$  such that  $1 \preceq (T - 2\delta)_+$ , whence  $1 = R^*(T - \delta)_+R$  for some element R in  $\mathcal{M}(A)$ . Put  $V = (T - \delta)_+^{1/2}R$  and put T' = g(T), where  $g: \mathbb{R}^+ \to [0, 1]$  is a continuous function such that g(0) = 0, g(t) = 1 for  $t \geq \delta$ , and g is linear on  $[0, \delta]$ . Then  $\overline{T'AT'} = \overline{TAT} = B$ , and V is an isometry whose range projection satisfies  $VV^*T' = VV^*$ .

To show that B is stable, we use [7], by which it suffices to show that for each  $a \in F(A)$  there is  $b \in A^+$  such that  $a \perp b$  and  $a \sim b$ . Take  $a \in F(B)$ , and let e be a positive contraction in B such that ae = ea = a. Put  $T_0 = (1 - e)T'(1 - e)$ , and note that  $\overline{T_0AT_0} \subseteq B$ . Now,

$$V^*T_0V = V^*T'V - V^*(eT' + T'e - eT'e)V = 1 - c,$$

with  $c = V^*(eT' + T'e - eT'e)V \in A$ . As  $V^*T_0V$  is a positive contraction, the element c is also a positive contraction. We can therefore use Lemma 5.6 to conclude that  $V^*T_0V$ 

is properly infinite and full. As  $V^*T_0V \preceq T_0$  we also have that  $T_0$  is properly infinite and full. This again entails that  $1 \preceq T_0$ , and so there is an isometry W in  $\mathcal{M}(A)$ whose range projection,  $WW^*$ , belongs to  $\overline{T_0\mathcal{M}(A)T_0}$ . In particular,  $WW^* \perp a$ . Put  $b = WaW^*$ . Then b is a positive element in  $B, b \perp a$ , and  $b \sim a$  as desired.  $\Box$ 

Let A be a stable C\*-algebra. Then there exists a sequence  $\{S_n\}$  of isometries in  $\mathcal{M}(A)$  with orthogonal range projections and such that  $\sum_{n=1}^{\infty} S_n S_n^* = 1$  (the sum being convergent in the strict topology). Let  $\{a_n\}$  be any bounded sequence of elements in A (or in  $\mathcal{M}(A)$ ). Then  $\sum_{n=1}^{\infty} S_n a_n S_n^*$  is strictly convergent to an element in  $\mathcal{M}(A)$ . We shall denote this element by  $\bigoplus_{n=1}^{\infty} a_n$ . If  $\{T_n\}$  is another sequence of isometries in  $\mathcal{M}(A)$  with range projections adding up to 1 in the strict topology, then  $\sum_{n=1}^{\infty} T_n S_n^*$  is strictly convergent to a unitary U in  $\mathcal{M}(A)$  and  $U\left(\sum_{n=1}^{\infty} S_n a_n S_n^*\right)U^* = \sum_{n=1}^{\infty} T_n a_n T_n^*$ . This shows that the element  $\bigoplus_{n=1}^{\infty} a_n$  is independent on the choice of the sequence  $\{S_n\}$  of isometries, up to unitary equivalence.

**Lemma 5.8.** Let A be a stable  $\sigma$ -unital C<sup>\*</sup>-algebra which satisfies the Corona Factorization Property. Let T be a full, positive element in  $\mathcal{M}(A)$ . Then  $a \preceq T$  for every positive element a in A.

*Proof.* Put  $B = \overline{TAT}$ . Then B is a full hereditary sub-C<sup>\*</sup>-algebra of A because T is full in  $\mathcal{M}(A)$ .

Again using that T is a full element in the properly infinite  $C^*$ -algebra  $\mathcal{M}(A)$ , there is a positive integer n such that  $T \otimes 1_n$  is properly infinite. As,

$$M_n(B) = \overline{(T \otimes 1_n)M_n(A)(T \otimes 1_n)},$$

we conclude from Lemma 5.7 that  $M_n(B)$  is stable. Because A is assumed to satisfy the Corona Factorization Property, we can now conclude from [10, Theorem 4.2] that B is stable.

Let *a* be a positive element in *A* and let  $\varepsilon > 0$  be given. As *B* is full in *A* we can find a positive integer *n*, positive elements  $b_1, \ldots, b_n$  in *B*, and elements  $x_1, \ldots, x_n$  in *A* such that  $(a - \varepsilon)_+ = \sum_{j=1}^n x_j^* b_j x_j$ . Because *B* is stable there are isometries  $S_1, \ldots, S_n$  in  $\mathcal{M}(B)$  with orthogonal range projections. We now get

$$(a-\varepsilon)_+ \preceq b_1 \oplus b_2 \oplus \cdots \oplus b_n \approx S_1 b_1 S_1^* + S_2 b_2 S_2^* + \cdots + S_n b_n S_n^* \preceq T.$$

As this holds for all  $\varepsilon > 0$ , we have  $a \preceq T$  as desired.

The lemma below is similar to [15, Corollary 2.7], but we do not assume below that A is unital. If a and b are positive elements in a  $C^*$ -algebra and if m is a positive integer, then we shall write  $a \preceq_m b$  to denote that  $a \preceq b \otimes 1_m$ .

**Lemma 5.9.** Let A be a  $\sigma$ -unital stable C<sup>\*</sup>-algebra, and let c be a strictly positive contraction in A.

Let  $\{a_n\}$  be a bounded sequence of positive elements in A. Then  $\bigoplus_{n=1}^{\infty} a_n$  defines a full element in  $\mathcal{M}(A)$  if there exist  $\delta > 0$  and a positive integer m such that for every

## $\varepsilon > 0$ and for every positive integer k there is an integer $\ell > k$ such that

$$(c-\varepsilon)_+ \preceq_m (a_k-\delta)_+ \oplus (a_{k+1}-\delta)_+ \oplus \cdots \oplus (a_\ell-\delta)_+$$

*Proof.* We show that  $1 \preceq_m \bigoplus_{n=1}^{\infty} a_n$ , which of course will imply that  $\bigoplus_{n=1}^{\infty} a_n$  is full. By assumption we can find integers  $1 = k_1 < k_2 < k_3 < \cdots$  such that

$$(c-\frac{1}{n})_+ \precsim_m (a_{k_n}-\delta)_+ \oplus (a_{k_n+1}-\delta)_+ \oplus \cdots \oplus (a_{k_{n+1}-1}-\delta)_+$$

for all n.

Choose isometries  $T_1, T_2, \ldots, T_m$  in  $\mathcal{M}(A)$  with range projections adding up to 1. Then we can identify  $(\bigoplus_{n=1}^{\infty} a_n) \otimes 1_m$  with

$$\sum_{j=1}^{m} T_j \Big( \bigoplus_{n=1}^{\infty} a_n \Big) T_j^* = \sum_{n=1}^{\infty} \sum_{j=1}^{m} T_j S_n a_n S_n^* T_j^* \sim \sum_{n=1}^{\infty} S_n \Big( \sum_{j=1}^{m} T_j a_n T_j^* \Big) S_n^*.$$

(We have here used that the range projections of the two families of isometries,  $\{S_nT_j\}$ and  $\{T_jS_n\}$ , sum to 1 in the strict topology.) Put

$$b_n = \sum_{k=k_n}^{k_{n+1}-1} S_k \Big( \sum_{j=1}^m T_j a_n T_j^* \Big) S_k^* \sim (a_{k_n} \oplus a_{k_n+1} \oplus \dots \oplus a_{k_{n+1}-1}) \otimes 1_m$$

Then  $\left(\bigoplus_{n=1}^{\infty} a_n\right) \otimes 1_m \sim \sum_{n=1}^{\infty} b_n$ , the latter sum is strictly convergent, and  $(c - \frac{1}{n})_+ \preceq (b_n - \delta)_+$  for all n. We must show that  $1 \preceq \sum_{n=1}^{\infty} b_n$ . Choose a strictly decreasing sequence  $\{\delta_n\}$  of positive real numbers such that  $\delta_2 = 1$ 

Choose a strictly decreasing sequence  $\{\delta_n\}$  of positive real numbers such that  $\delta_2 = 1$ and  $\delta_{n+2} > 1/n$  for all n. Define  $g_n: [0,1] \to [0,1]$  to be the continuous function which is zero on  $[0, \delta_{n+2}] \cup [\delta_n, 1]$  (note that  $[\delta_1, 1] = \emptyset$ ),  $g_n(\delta_{n+1}) = 1$ , and  $g_n$  is linear on  $[\delta_{n+2}, \delta_{n+1}]$  and on  $[\delta_{n+1}, \delta_n]$ . Then  $1 = \sum_{n=1}^{\infty} g_n(c)$  and the sum is strictly convergent. Moreover, since  $\delta_{n+2} > 1/n$ , we have  $g_n(c) = x_n^*(b_n - \delta)_+ x_n$  for some element  $x_n$  in A. Let  $h: [0,1] \to \mathbb{R}^+$  be the continuous function which satisfies h(0) = 0,  $h(t) = t^{-1/2}$ for  $t \ge \delta$ , and h is linear on  $[0,\delta]$ . Put  $y_n = h(b_n)(b_n - \delta)_+^{1/2}x_n$ . Then  $||y_n|| \le \delta^{-1/2}$ (because  $||(b_n - \delta)_+^{1/2}x_n|| = ||g_n(c)||^{1/2} = 1$  and  $||h(b_n)|| \le \delta^{-1/2}$ ), and  $y_n^*b_n y_n = g_n(c)$ . Notice that  $y_n$  belongs to the set  $\overline{b_n}Ag_n(c)$ . Put  $Y = \sum_{n=1}^{\infty} y_n \in \mathcal{M}(A)$  (the sum is strictly convergent). Then,

$$Y^* \Big(\sum_{n=1}^{\infty} b_n\Big) Y = \sum_{n=1}^{\infty} Y^* b_n Y = \sum_{n=1}^{\infty} y_n^* b_n y_n = \sum_{n=1}^{\infty} g_n(c) = 1,$$
  
hat  $1 \preceq \sum_{n=1}^{\infty} b_n$ .

which shows that  $1 \preceq \sum_{n=1}^{\infty} b_n$ .

**Lemma 5.10.** Let A be a stable  $\sigma$ -unital C\*-algebra which satisfies the Corona Factorization Property. Let  $a_1, a_2, \ldots, b_1, b_2, \ldots$  be positive elements in A, and let m be a positive integer such that  $a_1 \preceq a_2 \preceq a_3 \preceq \cdots$ , such that the set  $\{a_n\}$  is full in A, and such that  $a_n \preceq b_n \otimes 1_m$  for all n. It follows that for each  $\eta > 0$  there is a natural number k such that

$$(a_1 - \eta)_+ \precsim b_1 \oplus b_2 \oplus \cdots \oplus b_k.$$

*Proof.* We note first that we can choose  $\delta_n > 0$  such that

$$(a_1 - \delta_1)_+ \precsim (a_2 - \delta_2)_+ \precsim (a_3 - \delta_3) \precsim \cdots,$$

and such that  $\{(a_n - \delta_n)_+\}_{n=1}^{\infty}$  is full in A. (Let us prove this fact: As  $a_j \preceq a_n$ whenever  $1 \leq j < n$  there is  $\eta_n > 0$  such that  $(a_j - 1/n)_+ \preceq (a_n - \eta_n)_+$  for  $j = 1, 2, \ldots, n-1$ . We choose now  $\delta_n$  inductively such that  $0 < \delta_n \leq \eta_n$  and such that  $(a_{n-1} - \delta_{n-1})_+ \preceq (a_n - \delta_n)_+$ . For n = 1 we can take  $\delta_1 = \eta_1$ . For  $n \geq 2$ , since  $a_{n-1} \preceq a_n$ , there is  $\delta_n \in (0, \eta_n]$  such that  $(a_{n-1} - \delta_{n-1})_+ \preceq (a_n - \delta_n)_+$ . To see that the sequence  $\{(a_n - \delta_n)_+\}_{n=1}^{\infty}$  is full in A, let I be the closed two-sided ideal generated by this sequence. Since  $(a_j - 1/n)_+ \preceq (a_n - \eta_n)_+ \preceq (a_n - \delta_n)_+ \in I$  whenever  $1 \leq j < n$ , we see that  $(a_j - 1/n)_+$  belongs to I whenever n > j. It follows that  $a_j$  belongs to Ifor all j, whence I = A, because the sequence  $\{a_n\}$  was assumed to be full.)

Next we choose  $\delta'_n > 0$  such that  $(a_n - \delta_n)_+ \preceq_m (b_n - \delta'_n)_+$  for all n. Let  $g_n \colon [0, 1] \to [0, 1]$  be the continuous function given by  $g_n(0) = 0$ ,  $g_n(t) = 1$  for  $t \geq \delta'_n$ , and  $g_n$  is linear on  $[0, \delta'_n]$ . Put  $b'_n = g_n(b_n)$ . Then  $b_n$  is Cuntz equivalent to  $b'_n$ , and  $(b_n - \delta'_n)_+ \preceq (b'_n - 1/2)_+$ .

We claim that  $T := \bigoplus_{n=1}^{\infty} b'_n$  is full in  $\mathcal{M}(A)$ . To this end, take a strictly positive contraction c in A. Let  $k \in \mathbb{N}$  and  $\varepsilon > 0$  be given. Note that the tail  $\{(a_n - \delta_n)_+\}_{n=k}^{\infty}$ is full in A (because the sequence  $\{(a_n - \delta_n)_+\}_{n=1}^{\infty}$  is Cuntz increasing). It follows that c belongs to the closed two-sided ideal generated by  $\{(a_n - \delta_n)_+\}_{n=k}^{\infty}$ , whence  $(c - \varepsilon)_+$ belongs to the algebraic ideal generated by this sequence, and hence to the algebraic ideal generated by  $\{(a_n - \delta_n)_+\}_{n=k}^{k'}$  for some k' > k. This entails that

$$(c-\varepsilon)_+ \preceq_p (a_k - \delta_k)_+ \oplus (a_{k+1} - \delta_{k+1})_+ \oplus \cdots \oplus (a_{k'} - \delta_{k'})_+$$

for some positive integer p. Using again the sequence  $\{(a_n - \delta_n)_+\}_{n=1}^{\infty}$  is Cuntz increasing, we get that

$$(c-\varepsilon)_+ \stackrel{\checkmark}{\underset{m}{\sim}} (a_k-\delta_k)_+ \oplus (a_{k+1}-\delta_{k+1})_+ \oplus \cdots \oplus (a_\ell-\delta_\ell)_+$$

when  $\ell \ge k + p(k' - k + 1)$ . Lemma 5.9 now yields that T is full in  $\mathcal{M}(A)$ .

Since A is assumed to have the Corona Factorization Property we can use Lemma 5.8 to conclude that  $a_1 \preceq T$ . Hence  $(a_1 - \eta/2)_+ = R^*TR$  for some R in  $\mathcal{M}(A)$ . Take a positive contraction e in A such that  $e(a_1 - \eta/2)_+ = (a_1 - \eta/2)_+ = (a_1 - \eta/2)_+e$ . Put  $r = Re \in A$ . As  $\bigoplus_{n=1}^k b'_n \to T$  in the strict topology as  $k \to \infty$ , it follows that  $r^*(\bigoplus_{n=1}^k b'_n)r \to r^*Tr = (a_1 - \eta/2)_+$  in the norm topology (on A) as  $k \to \infty$ . Take k such that

$$||r^*(b'_1 \oplus b'_2 \oplus \cdots \oplus b'_k)r - (a_1 - \eta/2)_+|| < \eta/2.$$

Then

$$(a_1 - \eta)_+ \preceq r^* (b'_1 \oplus b'_2 \oplus \cdots \oplus b'_k) r \preceq b'_1 \oplus b'_2 \oplus \cdots \oplus b'_k \approx b_1 \oplus b_2 \oplus \cdots \oplus b_k$$

as desired.

We wish to apply the result above to the stabilization of a  $C^*$ -algebra A with the Corona Factorization Property. Let us for this purpose consider the Cuntz semigroup of A, and compare it with the Cuntz semigroup of its stabilization. Identifying A and matrix algebras over A with corners of  $A \otimes \mathcal{K}$  we can write  $A \subset M_{\infty}(A) \subset A \otimes \mathcal{K}$ . In this way we can view W(A) as a sub-semigroup (in fact an ideal) of  $W(A \otimes \mathcal{K})$ . Every element in  $W(A \otimes \mathcal{K})$  is represented by a positive element in  $A \otimes \mathcal{K}$  (we do not need to take matrix algebras), and every element in W(A) is represented by a positive element in  $M_{\infty}(A)$ . Every element in  $W(A \otimes \mathcal{K})$  which is compactly supported by another element in  $W(A \otimes \mathcal{K})$  belongs to W(A) (see the remarks above Proposition 4.8).

**Lemma 5.11.** Let A be a  $\sigma$ -unital C<sup>\*</sup>-algebra and let B be a full herditary sub-C<sup>\*</sup>algebra of A. Then W(A) has the Corona Factorization Property if and only if W(B)has this property.

*Proof.* By Browns theorem it suffices to consider the case where  $A = B \otimes \mathcal{K}$ . If  $W(B \otimes \mathcal{K})$  has the Corona Factorization Property, then so does W(B), because W(B) is a subsemigroup of  $W(B \otimes \mathcal{K})$ , and any full sequence in W(B) is also full in  $W(B \otimes \mathcal{K})$ .

Suppose that W(B) has the Corona Factorization Property. Let  $\{x_n\}$  be a full sequence in  $W(B \otimes \mathcal{K})$ , let  $\{y_n\}$  be a sequence in  $W(B \otimes \mathcal{K})$ , let  $x' \in W(B \otimes \mathcal{K})$ , and let m be a positive integer such that  $x' \ll x_1$  and  $x_n \leq my_n$  for all n. Arguing as in the proof of Lemma 5.10 we can find a full sequence  $\{x'_n\}$  in  $W(B \otimes \mathcal{K})$  such that  $x'_n \ll x_n$  for all n and such that  $x' \ll x'_1$ . Next, using item (b) above Definition 2.9, we find  $y'_n \ll y_n$  such that  $x'_n \leq my'_n$ . By the remark above, the elements  $x', x'_n, y'_n$  all belong to W(B), and  $\{x'_n\}$  is full in W(B). Hence

$$x' \le y_1' + y_2' + \dots + y_n' \le y_1 + y_2 + \dots + y_n$$

for some n. This shows that  $W(B \otimes \mathcal{K})$  has the Corona Factorization Property.

**Theorem 5.12.** Let A be a  $\sigma$ -unital C<sup>\*</sup>-algebra. Then A has the Corona Factorization Property if and only if its Cuntz semigroup, W(A), has the Corona Factorization Property (for monoids).

*Proof.* Assume first that A has the Corona Factorization Property. By Lemma 5.11 above we can assume that A is stable. Let  $\{x_n\}$  be a full sequence in W(A), let  $\{y_n\}$  be another sequence in W(A), let  $x' \in W(A)$ , and let  $m \in \mathbb{N}$  be such that  $x_n \leq my_n$  for all n and  $x' \ll x_1$ . Take positive elements  $a_n$  and  $b_n$  in A such that  $x_n = \langle a_n \rangle$  and  $y_n = \langle b_n \rangle$ , and take  $\eta > 0$  such that  $x' \leq \langle (a_1 - \eta)_+ \rangle$ . Then  $\{a_n\}$  is full in A,  $a_1 \preceq a_2 \preceq \cdots$ , and  $a_n \preceq_m b_n$  for all n. Hence, by Lemma 5.10, we get that

$$(a_1 - \eta)_+ \precsim b_1 \oplus b_2 \oplus \cdots \oplus b_k$$

for some k. Thus

$$x' \le \langle (a_1 - \eta)_+ \rangle \le \langle b_1 \rangle + \langle b_2 \rangle + \dots + \langle b_k \rangle = y_1 + y_2 + \dots + y_k.$$

This shows that W(A) has the Corona Factorization Property.

To prove the converse direction, let B be a full hereditary subalgebra of A. We are going to show that, if  $M_n(B)$  is stable for some n, then B is itself stable. Then A will have the Corona Factorization Property by virtue of [10, Theorem 4.2]

Since W(B) inherits the Corona Factorization Property (for monoids), it will suffice to show that if a  $C^*$ -algebra A is such that W(A) has the Corona Factorization Property and  $M_m(A)$  is stable for some m, then A is stable. But this follows from Proposition 5.5.

**Corollary 5.13.** Let A be a separable, unital  $C^*$ -algebra with finite decomposition rank. Then A has the Corona Factorization Property.

*Proof.* Combine Theorem 5.12 above with Corollary 3.3.

The corollary above extends the result of Pimsner, Popa and Voiculescu, [14], and Kucerovsky and Ng, [11], that the  $C^*$ -algebra  $C(X) \otimes \mathcal{K}$  is absorbing, or equivalently, that it satisfies the Corona Factorization Property, when X has finite covering dimension (as the decomposition rank of  $C(X) \otimes \mathcal{K}$  coincides with the covering dimension of the space X).

We end this paper by describing for which  $C^*$ -algebras the Cuntz semigroup has the strong Corona Factorization Property.

**Theorem 5.14.** Let A be a  $\sigma$ -unital C<sup>\*</sup>-algebra. Then W(A) has the strong Corona Factorization Property if and only if every ideal in A has the Corona Factorization Property.

*Proof.* Assume that W(A) has the strong Corona Factorization Property, and let I be a closed two-sided ideal in A. Then W(I) is an ideal in W(A). As the strong Corona Factorization Property trivially passes to ideals, we conclude that W(I) satisfies the (strong) Corona Factorization Property. It therefore follows from Theorem 5.12 that I has the Corona Factorization Property (for  $C^*$ -algebras).

Suppose now that all ideals in A have the Corona Factorization Property. To show that W(A) has the strong Corona Factorization Property, it suffices to show that whenever  $a, b_1, b_2, \ldots$  are positive elements in  $M_{\infty}(A)$ ,  $\varepsilon > 0$ , and m is a positive integer such that  $a \preceq b_n \otimes 1_m$ , then  $(a - \varepsilon)_+ \preceq b_1 \oplus b_2 \oplus \cdots \oplus b_k$  for some positive integer k. Upon replacing A by a matrix algebra over A, we can assume that a belongs to A. Each  $b_n$  belongs to some matrix algebra over A, say  $b_n \in M_{r_n}(A)$ . There are rectangular matrices  $t_n \in M_{mr_n,1}(A)$  such that  $t_n^*(b_n \otimes 1_m)t_n = (a - \varepsilon/3n)_+$ .

Let I be the closed two-sided ideal in A generated by a. Then a is full in I, and  $\langle a \rangle$  is full in W(I); however, the elements  $b_n$  may not belong to (a matrix algebra over) I. To fix this problem, take a quasi-central increasing approximate unit  $\{e_k\}_{k=1}^{\infty}$  for I consisting of positive contractions. For each n find k such that

$$\|t_n^*(e_k \otimes 1_{mr_n})(b_n \otimes 1_m)(e_k \otimes 1_{mr_n})t_n - (a - \varepsilon/3n)_+\| < \varepsilon/3n,$$

and put  $a_n = (a - 2\varepsilon/3n)_+$  and  $c_n = (e_k \otimes 1_{r_n})b_n(e_k \otimes 1_{r_n})$ . Then  $c_n$  belongs to  $M_{r_n}(I)$ ,  $c_n \preceq b_n, a_n \preceq c_n \otimes 1_m$  (relatively to A, and hence also relatively to I),  $(a - \varepsilon)_+ =$   $(a_1 - \varepsilon/3)_+$ , and  $\{\langle a_n \rangle\}$  is a full sequence in W(I). Since W(I) is assumed to satisfy the Corona Factorization Property we conclude that

$$(a - \varepsilon)_+ \precsim c_1 \oplus c_2 \oplus \cdots \oplus c_k \precsim b_1 \oplus b_2 \oplus \cdots \oplus b_k$$

for some k as desired.

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