HARDY-LITTLEWOOD-SOBOLEV INEQUALITIES VIA FAST DIFFUSION FLOWS

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Abstract

We give a simple proof of the $\lambda = d-2$ cases of sharp Hardy-Littlewood-Sobolev inequalities for $d \geq 3$, and also the sharp Log-HLS inequality for $d = 2$, via a monotone flow governed by the fast diffusion equation.

keywords: Hardy-Littlewood-Sobolev — Fast Diffusion — Gagliardo-Nirenberg-Sobolev.

1 Introduction

We explain an interesting relation between the sharp Hardy-Littlewood-Sobolev (HLS) inequality for the resolvent of the Laplacian, the sharp Gagliardo-Nirenberg-Sobolev (GNS) inequality, and the fast diffusion equation (FDE). As a consequence of this relation, we obtain a new identity expressing the HLS functional as an integral involving the fast diffusion flow and the GNS functional. From this identity we obtain a simple proof of the sharp HLS inequality in the cases that express the regularizing properties of the Green's function for the Laplacian in $\mathbb{R}^d$, for $d \geq 3$, and of the Logarithmic Hardy-Littlewood-Sobolev (Log-HLS) inequality, for $d = 2$. The proof also provides interesting information about the HLS functional that does not follow from previous proofs of the HLS inequality.

Throughout the paper, we shall use $\|f\|_p$ to denote the usual L^p norms with respect to Lebesgue measure:

$$\|f\|_p = \left(\int_{\mathbb{R}^d} |f|^p \mathrm{d}x \right)^{1/p} ,$$

for $1 \leq p < \infty$.

1.1 The sharp Hardy-Littlewood-Sobolev inequality

The sharp form of the *HLS inequality* is due to Lieb [12]. It states that for all non-negative measurable functions f on $\mathbb{R}^d$, and all $0 < \lambda < d$,

$$\frac{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \dfrac{f(x)f(y)}{|x-y|^\lambda} \, \mathrm{d}x \, \mathrm{d}y}{\|f\|_p^2} \leq \frac{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \dfrac{h(x)h(y)}{|x-y|^\lambda} \, \mathrm{d}x \, \mathrm{d}y}{\|h\|_p^2} \tag{1}$$

where

$$h(x) = \left(\frac{1}{1+|x|^2} \right)^{(2d-\lambda)/2} . \tag{2}$$

and $p = 2d/(2d - \lambda)$. Moreover, there is equality in(1) if and only if for some $x_0 \in \mathbb{R}^d$ and $s \in \mathbb{R}_+$, f is a non-zero multiple of $h(x/s - x_0)$.

The $\lambda = d - 2$ cases of the sharp HLS inequality are particularly interesting since they express the L^p smoothing properties of $(-\Delta)^{-1}$ on $\mathbb{R}^d$: for $d \geq 3$,

$$\int_{\mathbb{R}^d} f(x) \left[(-\Delta)^{-1} f \right] (x) \mathrm{d}x = \frac{1}{(d-2)|S^{d-1}|} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f(x)f(y)}{|x-y|^{d-2}} \, \mathrm{d}x \mathrm{d}y ,$$

where $|S^{d-1}|$ denotes the surface area of the $d-1$ dimensional unit sphere in $\mathbb{R}^d$. The integrals on the right hand side of (1) can be computed explicitly in terms of Γ-functions, and, after some computation with the constants, one sees that for $\lambda = d - 2$, (1) can be rewritten as $\mathcal{F}[f] \geq 0$ for all $f \in L^{2d/(d+2)}(\mathbb{R}^d)$ where

$$\mathcal{F}[f] := C_S \|f\|_{2d/(d+2)}^2 - \int_{\mathbb{R}^d} f(x) \left[(-\Delta)^{-1} f \right] (x) \mathrm{d}x , \tag{3}$$

with

$$C_S := \frac{4}{d(d-2)} |S^d|^{-2/d} . \tag{4}$$

We refer to this functional $\mathcal{F}$ on $L^{2d/(d+2)}(\mathbb{R}^d)$ as the *HLS functional*.

Let g be any smooth function of compact support, and let $\langle g, f \rangle$ denote $\int_{\mathbb{R}^d} g(x)f(x)\mathrm{d}x$. Then the positivity of $\mathcal{F}$ on $L^{2d/(d+2)}(\mathbb{R}^d)$ implies that for all $f \in L^{2d/(d+2)}(\mathbb{R}^d)$,

$$2\langle g, f \rangle - C_S \|f\|_{2d/(d+2)}^2 \leq 2\langle g, f \rangle - \int_{\mathbb{R}^d} f(x) \left[(-\Delta)^{-1} f \right] (x) \mathrm{d}x .$$

Taking the supremum over f on both sides; i.e., computing two Legendre transforms, one finds

$$\frac{1}{C_S} \|g\|_{2d/(d-2)}^2 \leq \|\nabla g\|_2^2 . \tag{5}$$

Notice that C_S is the least constant for which (5) can hold for all smooth compactly supported functions g, since the Legendre transform can be undone so that any improvement in the constant in (5) would yield an improvement in the constant in the HLS inequality, and this is impossible.

We can summarize the last paragraph by saying the that sharp HLS inequality for $\lambda = d - 2$, $d \geq 3$, is *dual* to the sharp Sobolev inequality (5), and because of this duality, once one knows the sharp constant to one of these inequalities, one knows the sharp constant to the other. A little thought shows that the same is true for optimizers: Once one knows the optimizers for one inequality, one knows the optimizers for the other.

In this paper, we shall explain another kind of "duality" involving the $\lambda = d - 2$ cases of the HLS inequality. This duality relation, which does not have any evident connection with the Legendre transform argument explained above, relates the $\lambda = d - 2$ cases of the sharp HLS inequality to certain sharp Gagliardo-Nirenberg-Sobolev (GNS) inequalities, again with an identification of sharp constants and optimizers. The GNS inequalities in question are due, in their sharp form, to Del Pino and Dolbeault [9]. They state that for all locally integrable functions f on $\mathbb{R}^d$, $d \geq 2$, with a square integrable distributional gradient, and p with $1 < p < d/(d-2)$

$$\frac{\|\nabla f\|_2^\theta \|f\|_{p+1}^{1-\theta}}{\|f\|_{2p}} \geq \frac{\|\nabla \tilde{h}\|_2^\theta \|\tilde{h}\|_{p+1}^{1-\theta}}{\|\tilde{h}\|_{2p}} \tag{6}$$

where

$$\tilde{h}(x) = \left(\frac{1}{1+|x|^2}\right)^{1/(p-1)} . \tag{7}$$

and $\theta = d(p-1)/(p(d+2-(d-2)p))$. Moreover, there is equality in (6) if and only if for some $x_0 \in \mathbb{R}^d$ and $s \in \mathbb{R}_+$, f is a non-zero multiple of $\tilde{h}(x/s - x_0)$.

Notice that GNS optimizers are certain powers of HLS optimizers, and vice-versa. We shall see that this is no accident. In fact, there is yet another context in which the HLS optimizers play an important role: they are the steady-state solutions of certain nonlinear evolution equations pertaining to *fast diffusion*.

1.2 The Fast Diffusion Equation

The equation

$$\frac{\partial u}{\partial t}(x,t) = \Delta u^m(x,t) \tag{8}$$

with $0 < m < 1$ describes fast diffusion. (For $m = 1$, it is the usual heat equation describing ordinary diffusion, and $m > 1$ corresponds to *slow diffusion*.) As in [7, 8], note that $u(x,t)$ solves (8) if and only if

$$v(x,t) := e^{td}u(e^t x, e^{\beta t}) \tag{9}$$

with $\beta = 2 - d(1-m)$ satisfies the equation

$$\frac{\partial v}{\partial t}(x,t) = \beta \Delta v^m(x,t) + \nabla \cdot [xv(x,t)] . \tag{10}$$

For $m = 1$, this is the Fokker-Planck equation, and (10) is a non-linear relative of the Fokker-Planck equation. For all $1 - 2/d < m < 1$, the equation (10) has integrable stationary solutions.

Computing them, one finds

$$v_{\infty,M}(x) := \left(D(M) + \frac{1-m}{2\beta m}|x|^2 \right)^{-1/(1-m)}. \tag{11}$$

The parameter $D(M)$ fixes the *mass* M of the steady state $v_{\infty,M}(x)$; i.e., the quantity

$$M := \int_{\mathbb{R}^d} v_{\infty,M}(x)\,\mathrm{d}x.$$

Computing the integral one finds that

$$D(M) = C(d,m,\beta)M^{2/(2-2(1-m))}$$

where $C(d,m,\beta)$ is a constant that may be expressed in terms of Γ-functions.

The self-similar solutions of (8) corresponding through the change of variables (9) to the $v_{\infty,M}$ are known as *Barenblatt solutions*, and $v_{\infty,M}$ is known as the *Barenblatt profile* for (8) with mass M. Notice in the limiting case $m = 1$, the Barenblatt profile approaches a Gaussian, as one would expect. The Barenblatt self-similar solutions are natural generalizations of the fundamental solutions of the heat equation. The Cauchy problem for the FDE (8) has been studied by many authors, we refer to [17] for a full account of the literature.

It is established in [11] that the range of mass conservation for the fast diffusion equation is $1-2/d < m < 1$. As noted above, this is exactly the range of $m < 1$ in which integrable self-similar solutions exist. Within this range, the flow associated to the fast diffusion equation is in many ways *even better* than the flow associated to the heat equation; see [4] and the references therein. The solutions of (8) with positive integrable initial data are C^∞ and strictly positive everywhere instantaneously, just as for the heat flow.

Moreover, for non-negative initial data f of mass M satisfying

$$\sup_{|x|>R} f(x)|x|^{2/(1-m)} < \infty \tag{12}$$

for some $R > 0$, which means that f is decaying at infinity at least as fast as the Barenblatt profile $v_{\infty,M}$, the solution $v(x,t)$ of (8) with initial data f satisfies the following remarkable bounds: For any $t_* > 0$, there exists a constant $C = C(t^*) > 0$ such that

$$\frac{1}{C} \leq \frac{v(x,t)}{v_{\infty,M}} \leq C, \tag{13}$$

for all $t \geq t_*$ and $x \in \mathbb{R}^d$. The lower bound is remarkable, as our assumption on the initial data is an upper bound. This shows "how fast" fast diffusion really is: It spreads mass out to infinity to produce the "right tails" instantly.

The proof of these bounds is based on the L^∞-error estimate obtained in [16] and improved to global Harnack inequalities in [4], see also [8]. Moreover, one can show global smoothness estimates on the quotient, that is, for any $t_* > 0$

$$\sup_{t \geq t_*} \left\| \frac{v(\cdot,t)}{v_{\infty,M}} \right\|_{C^k(\mathbb{R}^d)} < \infty, \tag{14}$$

for all $k \in \mathbb{N}$. Finally, it is well-known that

$$\lim_{t\to\infty} \|v(t) - v_{\infty,M}\|_{L^1(\mathbb{R}^d)} = 0. \tag{15}$$

For the best known rates of convergence see [2].

2 Monotonicity of $\mathcal{F}$ along fast diffusion

Since the HLS minimizers are the attracting steady states for a certain fast diffusion flow, one might hope that the HLS functional $\mathcal{F}$ would be monotone decreasing along this flow. This is indeed the case.

2.1 THEOREM. *Let $f \in L^{2d/(d+2)}(\mathbb{R}^d)$ be non-negative, and suppose that f satisfies (12) for some $R > 0$, and $m = d/(d+2)$, ensuring in particular that f is integrable. Let us further suppose that*

$$\int_{\mathbb{R}^d} f(x)\, \mathrm{d}x = \int_{\mathbb{R}^d} h(x)\, \mathrm{d}x = M_* \tag{16}$$

where h is given by (2) with $\lambda = d - 2$. Let $u(x,t)$ be the solution of (8) with $m = d/(d+2)$ and $u(x,1) = f(x)$. Then, for all $t > 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[u(\cdot,t)] = -2\mathcal{D}[u^{(d-1)/(d+2)}(\cdot,t)] \leq 0 \tag{17}$$

where

$$\mathcal{D}[g] := C_S \frac{d(d-2)}{(d-1)^2}\|g\|_{2d/(d-1)}^{4/(d-1)}\|\nabla g\|_2^2 - \|g\|_{2(d+1)/(d-1)}^{2(d+1)/(d-1)}\ . \tag{18}$$

Proof: There are two things to be proved, namely the identity on the left hand side of (17), and also the non-negativity of the functional $\mathcal{D}$ defined in (18).

We begin with the former. The uniform bounds on the regularity of the quotient (14) justify all of the integration-by-parts used in the following computation of the derivative of $\mathcal{F}$ along the FDE flow for $m = d/(d+2)$:

$$\frac{\partial}{\partial t}u(x,t) = \Delta u^{d/(d+2)}(x,t)\ . \tag{19}$$

Therefore, let $u(x,t)$ solve (19). We compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[u] = {} & 2\,C_S\left(\int_{\mathbb{R}^d} u^{2m}\mathrm{d}x\right)^{2/d}\int_{\mathbb{R}^d} u^{(d-2)/(d+2)}\Delta u^m\, \mathrm{d}x \\
& -2\int_{\mathbb{R}^d}(\Delta u^m)\left[(-\Delta)^{-1}u\right]\,\mathrm{d}x \\
= {} & -\frac{2C_S\, d(d-2)}{(d+2)^2}\left(\int_{\mathbb{R}^d} u^{2m}\mathrm{d}x\right)^{2/d}\int_{\mathbb{R}^d} u^{-6/(d+2)}|\nabla u|^2\mathrm{d}x \\
& +2\int_{\mathbb{R}^d} u^{(2d+2)/(d+2)}u\mathrm{d}x
\end{aligned} \tag{20}$$

Now define $g = u^{(d-1)/(d+2)}$. Then one computes

$$\int_{\mathbb{R}^d} u^{-6/(d+2)}|\nabla u|^2\mathrm{d}x = \left(\frac{d+2}{d-1}\right)^2\int_{\mathbb{R}^d}|\nabla g|^2\mathrm{d}x\ .$$

Rewriting the right hand side of (20) in terms of g, one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[u] = -2\,C_S\left(\int_{\mathbb{R}^d} g^{2d/(d-1)}\mathrm{d}x\right)^{2/d}\frac{d(d-2)}{(d-1)^2}\int_{\mathbb{R}^d}|\nabla g|^2\mathrm{d}x + 2\int_{\mathbb{R}^d} g^{(2d+2)/(d-1)}\mathrm{d}x\ .$$

Expressing this in terms of $\mathcal{D}[u]$, we have proved the left hand side of (17).

We shall now show that $\mathcal{D}[u]$ is non-negative as a consequence of the $p = (d+1)/(d-1)$ cases of the GNS inequality (6). These can be written in the form

$$C_{\text{GNS}}\|\nabla g\|_2^2 \|g\|_{2d/(d-1)}^{4/(d-1)} \geq \|g\|_{2(d+1)/(d-1)}^{2(d+1)/(d-1)} \, , \tag{21}$$

where, by definition, C_{GNS} is the best constant for which this inequality is valid for all smooth g with compact support. One could compute the right hand side of (6) to determine the explicit value of C_{GNS} and find that

$$C_{\text{GNS}} = C_S \frac{d(d-2)}{(d-1)^2} \, . \tag{22}$$

An easier way to see this is to go back to the first part of the proof, and consider the initial data $f = h$, so that $u(x,t)$ does not depend on t. Then by what we just proved, $\mathcal{D}(h^{(d-1)/(d+2)}) = 0$. Notice that $h^{(d-1)/(d+2)}$ is an optimizer for the $p = \frac{d+1}{d-1}$ case of (6). Hence, for the optimal g,

$$C_S \frac{d(d-2)}{(d-1)^2}\|\nabla g\|_2^2 \|g\|_{2d/(d-1)}^{4/(d-1)} = \|g\|_{2(d+1)/(d-1)}^{2(d+1)/(d-1)} \, ,$$

and this proves (22), and now the non-negativity of $\mathcal{D}$ follows from (21) and (22). ∎

As we show in the next section, all of the information that we have used about the sharp GNS inequality can also be proved by a fast diffusion flow argument without bringing anything else into the argument. Thus, while at present, our analysis may not look self-contained, this will be remedied shortly. For now though, let us finish with the HLS inequality.

As a direct consequence of the previous theorem, we deduce an identity for the HLS functional that manifestly displays its non-negativity.

2.2 THEOREM. *Let $f \in L^{2d/(d+2)}(\mathbb{R}^d)$, $d \geq 3$ be non-negative. Suppose also that f satisfies*

$$\sup_{|x|>R} f(x)|x|^{-(d+2)} < \infty \tag{23}$$

for some $R > 0$. Then

$$\mathcal{F}[f] = \frac{8}{d+2} \int_0^\infty e^{\beta t} \mathcal{D}[u^{(d-1)/(d+2)}(\cdot, e^{\beta t})] \, \mathrm{d}t \geq 0 \, . \tag{24}$$

Moreover, $\mathcal{F}[f] = 0$ if and only if f is a multiple of $h(x/s - x_0)$ for some $s > 0$ and $x_0 \in \mathbb{R}^d$, with h given by (2), $\lambda = d - 2$.

Proof: The assumption (23) together with the fact that $f \in L^{2d/(d+2)}(\mathbb{R}^d)$ implies the integrability of f. Since for all $\alpha > 0$, $\mathcal{F}[\alpha f] = \alpha^2 \mathcal{F}[f]$, it is harmless to assume (16), which we do. We may now apply the previous theorem. Let $v(x,t)$ be the solution of (10) with $v(x,0) = f(x)$. Let $u(x,t)$ be the solution of (8) with $u(x,1) = f(x)$. Because of the scaling relation (9), we have

$$\mathcal{F}[v(\cdot,t)] = e^{t(d-2)}\mathcal{F}[u(\cdot, e^{\beta t})]$$

with $\beta = 4/(d+2)$. Then Theorem 2.1 implies that, for all $t > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t(d-2)}\mathcal{F}[v(\cdot,t)]\right) = -\frac{8}{d+2} e^{\beta t}\mathcal{D}[u^{(d-1)/(d+2)}(\cdot, e^{\beta t})] \, . \tag{25}$$

We now claim that

$$\lim_{t\to 0} \mathcal{F}[v(\cdot,t)] = \mathcal{F}[f] \quad \text{and} \quad \lim_{t\to\infty} \mathcal{F}[v(\cdot,t)] = 0 \ . \tag{26}$$

Since $\mathcal{F}[h] = 0$, the latter fact is an easy consequence of the the global bounds in (13) due to the assumptions (12) and (16), and a dominated convergence argument. The former is slightly more subtle: First, it is easy to show, using known facts about the Cauchy problem for the FDE [17], that under our hypothesis, $\lim_{t\to 0} \|v(\cdot,t) - f\|_{2d/(d+2)} = 0$. By an argument using Fatou's Lemma, the potential integral term can only jump downwards in the limit. Thus, at least we have $\mathcal{F}[f] \geq \lim_{t\to 0} \mathcal{F}[v(\cdot,t)]$, and hence integrating (25) over $[0,\infty)$, we obtain

$$\mathcal{F}[f] \geq \frac{8}{d+2} \int_0^t e^{\beta s} \mathcal{D}[u^{(d-1)/(d+2)}(\cdot, e^{\beta s})] \, \mathrm{d}s \geq 0 \ .$$

In particular, we have shown that $\mathcal{F}[f] \geq 0$ under the hypotheses of the theorem. Then an obvious truncation and monotone convergence argument using the sequence of function $f_n = \min\{f, nh\}$ shows that $\mathcal{F}[f]$ is well defined, finite and non-negative for all non-negative $f \in L^{2d/(d+2)}(\mathbb{R}^d)$. This proves the $\lambda = d - 2$ HLS inequality, and then by a standard argument using the positive definite nature of the potential integral, shows that the potential integral is continuous on $L^{2d/(d+2)}(\mathbb{R}^d)$. Thus, $\mathcal{F}[f]$ is continuous on $L^{2d/(d+2)}(\mathbb{R}^d)$, and (26) now follows. Now integrating (25) over $[0,\infty)$ and using (26) yields the identity (24).

We now conclude from (24) that $\mathcal{F}[f] = 0$ if and only if $\mathcal{D}[u^{(d-1)/(d+2)}(\cdot, e^{\beta t})] = 0$ for all t. This is equivalent to the existence of a constant C and continuous functions $s(t)$ and $x_0(t)$, defined for $t > 1$ such that

$$u(x, e^{\beta t}) = C s^{-d}(t) \left[\tilde{h}\left(\frac{x}{s(t)} - x_0(t) \right) \right]^{(d+2)/(d-1)}$$

due to the characterization of the optimizers in the GNS inequality (6). Thus $u(x, e^{\beta t})$ is at each $t > 0$ a Barenblatt profile, thus by uniqueness of the Cauchy problem for the FDE (8), $u(x, e^{\beta t})$ is a self-similar Barenblatt solution of the FDE (8). Since $f(x) = \lim_{t\to 0} u(x, e^{\beta t})$ in $L^1(\mathbb{R}^d)$, hence f is itself a Barenblatt profile, meaning that f is a multiple of $h(x/s - x_0)$ for some $s > 0$ and some $x_0 \in \mathbb{R}^d$. ∎

The identity (24) has been derived under the hypothesis (23). However, it is easy to pass from Theorem 2.2 to to the following, which is simply a restatement of the $\lambda = d - 2$ cases of Lieb's Theorem:

2.3 THEOREM. *Let $f \in L^{2d/(d+2)}(\mathbb{R}^d)$, $d \geq 3$ be non-negative. Then $\mathcal{F}[f] \geq 0$, and $\mathcal{F}[f] = 0$ if and only if f is a multiple of $h(x/s - x_0)$ for some $s > 0$ and some $x_0 \in \mathbb{R}^d$, and where h is given by (2), $\lambda = d - 2$.*

Proof: We have already proved the inequality in the course of proving Theorem 2.2. The cases of equality are somewhat more subtle, and are dealt with in the next lemma. ∎

2.4 LEMMA. *If $f \in L^{2d/(d+2)}(\mathbb{R}^d)$ is non-negative and satisfies satisfies $\mathcal{F}[f] = 0$, then f satisfies (12) for some $R > 0$.*

To prove Lemma 2.4 we make use, for the first time, of rearrangement inequalities and the conformal invariance of the HLS functional. Recently, Frank and Lieb gave an interesting proof of certain cases of the HLS inequality [10] that uses *reflection positivity* in place of rearrangements.

Proof of Lemma 2.4: By a well–known theorem of Lieb [13] on the cases of equality in the Riesz rearrangement inequality, every optimizer f in $L^{2d/(d+2)}(\mathbb{R}^d)$ must be a translate of its spherically symmetric decreasing rearrangement, f^*. Making any necessary translation, we may assume $f = f^*$. Next, as also shown by Lieb, the HLS functional is invariant under the inversion mapping $f \mapsto \widehat{f}$ where $\widehat{f}(x) = |x|^{-(d+2)} f(x/|x|^2)$, which is an isometry on $L^{2d/(d+2)}(\mathbb{R}^d)$. Letting x_0 be any unit vector, f is uniformly bounded on the unit ball centered at $2x_0$. Thus $|x|^{-(d+2)} f(x/|x|^2 - 2x_0)$ is also an optimizer, and satisfies (12) for some $R = 1$. Now the previous Theorem applies, and this function must be a Barenblatt profile. It follows that the same is true of the original f. ∎

Note that we have only used the invariance of $\mathcal{F}$ under inversion, and hence under the full conformal group, to settle the final points regarding cases of equality. It is remarkable that neither the fast diffusion flow, nor the GNS inequalities possess this invariance, and yet for a dense class of functions functions f, (24) expresses $\mathcal{F}[f]$ in terms of the fast diffusion flow and the GNS functional.

3 The sharp GNS inequalities and fast diffusion

As we have seen, a calculation using fast diffusion reduces the proof of the $\lambda = d - 2$ cases of the HLS inequality to the proof of certain GNS inequalities. We now show, using results in [7], that another sort of calculation using a different fast diffusion reduces the proof these GNS inequalities to the Schwarz inequality.

The FDE (10) with $1 - 2/d < m < 1$ is a gradient flow of the functional

$$\mathcal{H}[v] = \int_{\mathbb{R}^d} \left[\frac{|x|^2}{2} v + \frac{\beta}{m-1} v^m \right] dx \,, \tag{27}$$

with respect to the Euclidean Wasserstein distance, see [15]. In particular, $\mathcal{H}[v]$ is a Liapunov functional for (10), being its dissipation given by

$$\frac{d}{dt} \mathcal{H}[v] = - \int_{\mathbb{R}^d} \left| x + \frac{m\beta}{m-1} \nabla v^{m-1} \right|^2 v \, dx := -\mathcal{I}[v] \tag{28}$$

for any solution $v(\cdot, t)$ to (10) and initial data $v(x, 0)$ satisfying the hypotheses of Theorem 2.2. Here, the regularity properties of the solution (13) and (14) that justified the computations in the previous section ensure that at least when the initial data satisfies (12), the dissipation of the entropy along the evolution is given by

$$\frac{d}{dt} \mathcal{I}[v] = -2\mathcal{I}[v] - 2(m-1) \int_{\mathbb{R}^d} v^m \, [\Delta \xi]^2 \, dx - 2 \int_{\mathbb{R}^d} v^m \left[\sum_{i,j=1}^{d} \left(\frac{\partial^2 \xi}{\partial x_i \partial x_j} \right)^2 \right] dx. \tag{29}$$

with

$$\xi = \frac{|x|^2}{2} + \frac{m\beta}{m-1} v^{m-1},$$

as shown in [7]. Define

$$\mathcal{R}[v] := \int_{\mathbb{R}^d} v^m \left[\Delta\xi\right]^2 \, dx \; . \tag{30}$$

By the Schwarz inequality for the Hilbert-Schmidt inner product,

$$\int_{\mathbb{R}^d} v^m \left[\sum_{i,j=1}^d \left(\frac{\partial^2\xi}{\partial x_i \partial x_j}\right)^2\right] dx \geq \frac{1}{d} \int_{\mathbb{R}^d} v^m \left[\Delta\xi\right]^2 \, dx \, ,$$

and thus from (29) and (30) we get

$$\frac{d}{dt}\mathcal{I}[v] \leq -2\mathcal{I}[v] - 2\left(m - 1 + \frac{1}{d}\right)\mathcal{R}[v] \; . \tag{31}$$

As long as $(d-1)/d < m < 1$, the factor in from of $\mathcal{R}[v]$ is strictly positive.

Now combine (28) and (31) to conclude

$$\frac{d}{dt}\mathcal{H}[v] \geq \frac{1}{2}\frac{d}{dt}\mathcal{I}[v] + \left(m - 1 + \frac{1}{d}\right)\mathcal{R}[v]. \tag{32}$$

Integrating this inequality in t from 0 to ∞, and using the fact that

$$\lim_{t\to\infty} \mathcal{H}[v(\cdot,t)] = \lim_{t\to\infty} \mathcal{I}[v(\cdot,t)] = 0,$$

one gets

$$\mathcal{H}[v(\cdot,0)] \leq \frac{1}{2}\mathcal{I}[v(\cdot,0)] + \int_0^\infty \left(m - 1 + \frac{1}{d}\right)\mathcal{R}[v(\cdot,t)]\mathrm{d}t \tag{33}$$

for all $v(\cdot,0)$ satisfying the assumptions of Theorem 2.2. Since $\left(m - 1 + \frac{1}{d}\right)\mathcal{R}[v] \geq 0$,

$$2\mathcal{H}[v(\cdot,0)] \leq \mathcal{I}[v(\cdot,0)] \; . \tag{34}$$

This inequality is equivalent to the sharp GNS inequalities (6). One see this by expanding the squares in this inequality, cancelling the second moment terms from both sides, and performing an integration-by-parts allowed by (13). Then with $m = (p+1)/(2p)$ and the change of dependent variable $v(x,0) =: f(x)^{2p}$, and finally using a standard scaling argument, one arrives at (6); see [9] for details. This finishes the summary of the relevant results in [7, 9].

The exponent m of the FDE (8) used to prove the particular cases of the GNS inequalities involved in the proof of the HLS inequality in previous sections is $m = d/(d+1)$.

On the other hand, the exponent m in the FDE along which the HLS functional is monotone is $m = d/(d+2)$, which corresponds to the critical exponent of FDE related to the boundedness of the second moment of the stationary states $v_{\infty,M}$, and it plays certain role in the large-time assymptotics of the FDE, see [8, 2].

We finally show how to extract from (32) the characterization of the optimizers of the GNS inequalities (6), at least under the conditions that are relevant for the application in the proof of Theorem 2.2.

3.1 THEOREM. *Let f be a positive measurable function on $\mathbb{R}^d$, $d \geq 2$, with a square integrable distributional gradient f, such that*

$$\sup_{x\in\mathbb{R}^d} \frac{f^{-(p-1)/2p}(x)}{1 + |x|^2} < \infty \, , \tag{35}$$

and f being an optimizer of the GNS inequality (6). Then, f is given by (7) up to translations and dilations.

Proof: Let us consider $v(x,0) = f^{2p}(x)$ as initial data for the FDE (10) with $m = (p+1)/2p$. Under these conditions, we have derived (33), and since (34) is equivalent to the sharp GNS inequality for f and f is indeed a stationary state of the FDE (10) due to (11), it must be the case that

$$\int_0^\infty \mathcal{R}[v(\cdot,t)]\,\mathrm{d}t = 0\,.$$

Due to the positivity of $v(\cdot,t)$ for all $t > 0$, we conclude that $\Delta\xi = 0$ for all $t > 0$. It is straightforward to infer from the global bounds (13) that for any $t_* > 0$, there exists $D_1 = D_1(t_*) > 0$ such that

$$\frac{1}{D_1} \le \frac{|x|^2}{2} + \frac{m\beta}{m-1}v^{m-1}(x,t) \le D_1$$

for all $t \ge t_*$, $x \in \mathbb{R}^d$. Thus, ξ is a globally bounded harmonic function, and then Liouville's theorem implies that ξ is constant. It follows that for each t, $v(\cdot,t)$ is a Barenblatt profile, which determines the form of f as in Theorem 2.2. ∎

4 Proof of the sharp Log-HLS inequality via fast diffusion

In this section, we prove the sharp Log-HLS inequality on $\mathbb{R}^2$ by a similar fast diffusion flow argument. The sharp Log-HLS inequality [1, 6] states that for all non-negative measurable functions f on $\mathbb{R}^2$ such that $f \ln f$ and $f \ln(e + |x|^2)$ belong to $L^1(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} f(x)\log f(x)\mathrm{d}x + \frac{2}{M}\int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)\log|x-y|f(y)\mathrm{d}x\mathrm{d}y \ge C\,, \tag{36}$$

where $M := \int_{\mathbb{R}^2} f\,\mathrm{d}x$ with $C(M) := M(1 + \log\pi - \log(M))$. Moreover, there is equality if and only if $f(x) = h_{\gamma,M}(x - x_0)$ for some $\gamma > 0$ and some $x_0 \in \mathbb{R}^2$, where

$$h_{\gamma,M}(x) := \frac{M}{\pi}\frac{\gamma}{(\gamma + |x|^2)^2}\,. \tag{37}$$

Note that all of integrals figuring in the logarithmic HLS inequality are at least well defined with no cancellation of infinities in their sum under the condition that $f \ln f$ and $f \ln(e + |x|^2)$ belong to $L^1(\mathbb{R}^2)$.

We therefore define the Log HLS functional $\mathcal{F}$ by

$$\mathcal{F}[f] := \int_{\mathbb{R}^2} f(x)\log f(x)\mathrm{d}x + 2\left(\int_{\mathbb{R}^2} f(x)\mathrm{d}x\right)^{-1}\iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)\log|x-y|f(y)\mathrm{d}x\mathrm{d}y$$

on the domain introduced above. The logarithmic HLS functional involves three distinct integral functionals of f while for $d \ge 3$, the HLS functional involves only two. A more significant difference is that the logarithmic HLS functional $\mathcal{F}$ is invariant under scale changes. That is, for $a > 0$ and f in the domain of $\mathcal{F}$, define $f_{(a)} := a^2 f(ax)$. One then computes that $\mathcal{F}[f_{(a)}] = \mathcal{F}[f]$ for all $a > 0$.

This simplifies our application of the fast diffusion equation, to which we now turn. For $d = 2$, $m = d/(d+2)$ reduces to $m = 1/2$. Thus, the relevant cases of the fast diffusion equation in $d = 2$ are

$$\frac{\partial u}{\partial t}(x,t) = \Delta\sqrt{u}(x,t) \ , \tag{38}$$

and

$$\frac{\partial v}{\partial t}(x,t) = \Delta\sqrt{v}(x,t) + \nabla \cdot [xv(x,t)] \ . \tag{39}$$

As before, it is easily checked that the stationary states are given by

$$v_{\infty,M}(x) = \left(D + \frac{1}{2}|x|^2\right)^{-2} \ ,$$

for any mass $M > 0$ with suitably chosen $D = D(M)$. Note that

$$h(x) := h_{1,4\pi}(x) = v_{\infty,M_*}(x) = \frac{4}{(1 + |x|^2)^2} \tag{40}$$

for a suitable M_*. For $d = 2$, the scaling relation between these two equations is that $u(x,t)$ solves (38) if and only if $v(x,t) := e^{2t}u(e^t x, e^t)$ solves (39). Notice that with u and v related in this way, the scale invariance of $\mathcal{F}$ implies that

$$\mathcal{F}[v(\cdot,t)] = \mathcal{F}[u(\cdot,e^t)] \ . \tag{41}$$

We now differentiate along the fast diffusion flow as before. For convenience, without loss of generality, we fix the initial mass. We also impose the appropriate version of (12).

4.1 THEOREM. *Let f be a non-negative measurable functions on $\mathbb{R}^2$ such that $f \ln f$ and $f \ln(e + |x|^2)$ belong to $L^1(\mathbb{R}^2)$. Suppose that $\int_{\mathbb{R}^2} f(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} h(x)\,\mathrm{d}x = M_*$, with h given by (40). Them $\mathcal{F}[f] \geq \mathcal{F}[h]$, and there is equality if and only if $f(x/s - x_0) = h(x)$ for some $s > 0$ and some $x_0 \in \mathbb{R}^2$.*

Suppose in addition that f satisfies (12) for some $R > 0$ and $m = 1/2$. Let $u(x,t)$ be the solution of (38) with $u(x,1) = f(x)$. Then we have the identity

$$\mathcal{F}[f] = \mathcal{F}[h] + \int_0^\infty \mathcal{D}[u^{1/4}(\cdot,e^t)]\,\mathrm{d}t \geq \mathcal{F}[h] \ , \tag{42}$$

where $\mathcal{D}[g] = \|\nabla g\|_2^2\|g\|_4^4 - \pi\|g\|_6^6$ is non-negative by the $d = 2$, $p = 3$ case of the sharp GNS inequality (6).

Proof: Let Let $v(x,t)$ be the solution of (39) with $v(x,0) = f(x)$. Suppose initially that $f \leq Ch$ for some finite C. Under this additional argument is it easy to prove (26), though the $t = 0$ limit is more subtle since in $d = 2$, the potential integral is neither pointwise positive, nor positive definite. However, our hypotheses ensure integrability of the positive and negative parts in the potential integral, and then monotone convergence may be used as before. A truncation argument, left to the reader, then removes the extra assumption $f \leq Ch$.

Differentiability of $\mathcal{F}[v(\cdot, t)]$ is justifies as before, and we have

$$\mathcal{F}[h] - \mathcal{F}[f] = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}[v(\cdot, t)] \, \mathrm{d}t \ .$$

But by (41), $\dfrac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}[v(\cdot, t)] = \dfrac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}[u(\cdot, e^t)]$. By the uniform regularity bounds on the quotient (14), we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}[u(\cdot, t)] = & -\frac{8\pi}{M_*} \int_{\mathbb{R}^2} \left[(-\Delta)^{-1} u \right] \Delta u^{1/2} \, \mathrm{d}x \\
& + \int_{\mathbb{R}^2} \log u \, \Delta u^{1/2} \mathrm{d}x \\
= & \frac{8\pi}{M_*} \int_{\mathbb{R}^2} u^{3/2} \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{u^{3/2}} \mathrm{d}x \ .
\end{aligned} \tag{43}$$

Making the change of variables $g = u^{1/4}$, $\|u\|_{3/2}^{3/2} = \|g\|_6^6$, $\|g\|_4^4 \mathrm{d}x = M_*$ and

$$\int_{\mathbb{R}^2} \frac{|\nabla u|^2}{u^{3/2}} \mathrm{d}x = 16 \|\nabla g\|_2^2 \ ,$$

leading together with (43) to (42). The proof of the statement about cases of equality proceed exactly as in Theorem 2.2. The proof that the condition (12) may be relaxed as far as the inequality itself is concerned, is similar, with the the integrability condition on $f \ln(e + |x|^2)$ being used to insure integrability of the positive part of the potential integral. ∎

5   Consequences of the monotonicity

The monotonicity of the HLS and Log-HLS functionals along fast diffusion flows has interesting consequences. One of these is the simplicity of the "landscape" of the Log-HLS functional: Let $\mathcal{C}$ be the (convex) set of non-negative functions on $\mathbb{R}^2$ satisfying all of the hypotheses of Theorem 4.1. Let $\mathcal{F}$ be Log-HLS functional on $\mathcal{C}$. Then there are no strict local minimizers of $\mathcal{F}$ in $\mathcal{C}$ other than the absolute minimizers.

Indeed, this is an immediate consequence of Theorems 2.1 and 4.1: One can go monotonically down to the absolute minimizers from any point in $\mathcal{C}$. Of course, a similar result holds for the HLS funcional, but in this case the Euler-Lagrange equation is thoroughly studied, and there are no non-negative critical points apart form global minimizers.

Next, as we have noted, the fast diffusion flow along which we have shown $\mathcal{F}$, corresponding to the Log-HLS inequality in $d = 2$, to be monotone decreasing is gradient flow in the 2-Wasserstein metric for the entropy functional $\mathcal{H}$ defined in (27). There is a kind of duality between the HLS functional $\mathcal{F}$ and the entropy functional $\mathcal{H}$, as we now explain.

We first recall an observation of Matthes, McCann and Savare [14] concerning pairs of gradient flow equations. Consider two smooth functions Φ and Ψ on $\mathbb{R}^d$, and consider the two ordinary differential equations describing gradient flow:

$$\dot{x}(t) = -\nabla \Phi[x(t)] \qquad \text{and} \qquad \dot{y}(t) = -\nabla \Psi[y(t)] \ .$$

Then of course $\Phi[(x(t)]$ and $\Psi[(t(t)]$ are monotone decreasing. Now differentiate each function along the others flow:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi[(y(t)] = -\langle\nabla\Phi[y(t)],\nabla\Psi[y(t)]\rangle$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi[(x(t)] = -\langle\nabla\Psi[x(t)],\nabla\Phi[x(t)]\rangle\ .$$

Thus, Φ is decreasing along the gradient flow of Ψ for any initial data if and only if Ψ is decreasing along the gradient flow of Φ for any initial data.

An analog of this holds for well-behaved gradient flows in the 2-Wasserstein sense, which is the result of [14]. In our case, we can apply it to the Log-HLS functional in $d = 2$. Thus, since $\mathcal{F}$ for the Log-HLS functional is decreasing along the 2-Wasserstein gradient flow for $\mathcal{H}$, one can expect that $\mathcal{H}$ is decreasing along the 2-Wasserstein gradient flow for $\mathcal{F}$, which turns out to be nothing other than the critical mass case of the Patlak-Keller-Segel equation. Actually, the $m = 1/2$, $d = 2$ version of $\mathcal{H}$ must be "renormalized" since in this case $v_{\infty,M}$ does not have finite second moments, nor an integrable square root. Still, this "second Lyapunov function" has been shown to be very useful in analyzing the critical mass Patlak-Keller-Segel equation; see [3].

Finally, the main new results here, namely, the integral identities (24) and (42), provide the starting point of an analysis of "remainder terms" and "stability" results for the the sharp HLS and Log-HLS inequalities. A quantitative stability theorem shall be developed elsewhere. Finally, we expect to be able to carry out a similar proof for the cases $d - 2 < \lambda < d$, which involves a non-local analog of the fast diffusion equation.

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