Characterization of the finite weak singularities of quadratic systems via invariant theory

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Abstract

This article is about weak singularities of quadratic differential systems, that is, non-degenerate singular points with traces of the corresponding linearized systems at such points equal to zero. These could be foci, centers or saddles. Necessary and sufficient conditions for a real quadratic system to possess a fixed number of weak singularities of a specific order are given. The conditions are stated in terms of affine invariant polynomials in the 12–dimensional space of the coefficients.

1 Introduction and the statement of the main result

We consider the real polynomial differential systems

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{1.1}$$

where P and Q are polynomials in the variables x and y with real coefficients, i.e. $P, Q \in \mathbb{R}[x, y]$. We shall say that systems (1.1) are *quadratic* if $\max(\deg(P), \deg(Q)) = 2$. We say that the quadratic system (1.1) is *non–degenerate* if the polynomials P and Q are relatively prime or coprime.

The notion of the center was introduced by Poincaré in his 1885 article [27]. Poincaré gave an algorithm for deciding when a polynomial differential system has a center. An algebraic version of this algorithm was stated by Shi Song Ling [35]. Placing the singular point at the origin, the necessary and sufficient conditions to have a center at the origin is the annihilation of an infinite number of polynomials in the coefficients of the system. In view of Hilbert's basis theorem, this amounts to the annihilation of only a finite number of them. These finite number of polynomials in the coefficients of the systems are called the Poincaré-Lyapunov constants as Lyapunov generalized Poincaré's results for analytic systems.

The next result on centers was Dulac's theorem [20] saying that a quadratic system possessing a center is integrable in finite terms. Dulac gave a finite number of conditions for such a system to have a center.

Dulac's notion of center is for complex systems: a non-degenerate singularity is a center if and only if the quotient λ of its eigenvalues is negative and rational and the system has a local nonconstant analytic first integral. Dulac had much insight in working with this notion. However, as his work was for complex systems, his canonical form for the case $\lambda = -1$ was for systems with a saddle and his conditions were not readily applicable to real systems. Kapteyn's work [23] dealt with real

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systems and he obtained conditions for the center in a more compact form for systems in the normal form which now bears his name.

In 1939, using Kapteyn's normal form, Bautin proved his now well known result [10] which says that a center in a quadratic system has cyclicity three . He gave necessary and sufficient conditions for a singular point of a quadratic system with a center for its linearization to have cyclicity one two and three.

In [36] (see also [38]) Sibirsky gave necessary and sufficient conditions for the origin to be a center (respectively a weak focus of given order i, where $i \in \{1, 2, 3\}$) in terms of algebraic invariants under the action of the linear group $GL(2, \mathbb{R})$.

The above mentioned set of conditions for the center was given for a system possessing a singular point placed at the origin. The question remained as to how to state conditions for a system to possess a center (respectively a weak focus of given order i, where $i \in \{1, 2, 3\}$) irrespectively of its position in the plane. In other words, how to formulate the conditions for existence of a center or weak focus independent of how system may be presented. This problem was solved for the case of the center in [14] and later the equivalent affine invariant conditions have been constructed in [40].

However in [14] the obtained conditions for the existence of one or two centers are associated to several canonical forms which were constructed using some invariant algebraic equalities and have not a relevant geometrical meaning. The conditions constructed in [40] are associated to the configurations of the finite singularities (real and/or complex, simple and/or multiple) and therefore there are needed additional conditions in order to determine such concrete configuration.

We remark that the class of quadratic systems with a weak focus at the origin of coordinates is examined by many authors. Thus in the paper [24] (respectively in [2], [3]) using the algebro-geometrical concepts of divisor and zero-cycle the topological classification of the class of quadratic systems possessing a weak focus of the third (respectively second) order is provided and the respective bifurcation diagrams are constructed. These classifications are based on the corresponding normal forms and in order to extend them in the whole space $\mathbb{R}^{12}$ of the coefficients of quadratic systems the affine invariant conditions for the existence of a weak focus of a given order are necessary.

A *weak saddle* is a hyperbolic saddle such that the trace of its linear part is zero. The notion of weak saddle was introduced by Cai Suilin [16] for the quadratic systems, and generalized to any polynomial differential system by Joyal and Rousseau [22], see these papers for additional information.

The class of quadratic systems with a weak saddle at the origin of coordinates is also examined by many authors (cf. [45, 44, 17, 18, 46, 39, 47, 43], including the examination of the co-existence of weak singularities in quadratic systems.

In [37] the GL-invariant conditions for the existence at the origin of coordinates of a *simple center in the sense of Dulac* (which in the case of real eigenvalues is an integrable saddle) are determined.

Applying the theory of algebraic invariants of differential equations developed by K. Sibirsky and his disciples (cf. [38], [42], [28], [9], [19]) we propose here a complete characterization of weak singularities (saddles and/or foci, integrable saddles and/or centers) for the class of non-degenerate quadratic systems. And this characterization is double global: the conditions are given in the whole coefficient space $\mathbb{R}^{12}$ and at the same time they describe the existence and the order of weak foci (respectively saddles) and centers (respectively integrable saddles) arbitrarily located on the whole phase plane.

In order to be able to give the statement of the Main Theorem we shall denote by $f^{(i)}$ (respectively $s^{(i)}$) a weak focus (respectively a weak saddle) of order i with $i \in \{1, 2, 3\}$ and by C (respectively $\$$) a singularity of center type (respectively an integrable saddle).

Our main result is the following one.

Main Theorem. *Consider a non-degenerate quadratic system.*

(a) *If* $\mathcal{T}_4 \neq 0$ *then this system has no weak singularity.*

(b) *If* $\mathcal{T}_4 = 0$ *and* $\mathcal{T}_3 \neq 0$ *then the system has exactly one weak singularity. Moreover this singularity is either a weak focus (respectively a weak saddle) of the indicated order below, or a center (respectively an integrable saddle) if and only if* $\mathcal{T}_3\mathcal{F} < 0$ *(respectively* $\mathcal{T}_3\mathcal{F} > 0$*) and the following corresponding condition holds:*

$$\begin{array}{lll}
(b_1) & f^{(1)} \ \big(\textit{respectively } s^{(1)}\big) & \Leftrightarrow \ \mathcal{F}_1 \neq 0;\\
(b_2) & f^{(2)} \ \big(\textit{respectively } s^{(2)}\big) & \Leftrightarrow \ \mathcal{F}_1 = 0,\ \mathcal{F}_2 \neq 0;\\
(b_3) & f^{(3)} \ \big(\textit{respectively } s^{(3)}\big) & \Leftrightarrow \ \mathcal{F}_1 = \mathcal{F}_2 = 0,\ \mathcal{F}_3\mathcal{F}_4 \neq 0;\\
(b_4) & c \ \ \big(\textit{respectively } \$\big) & \Leftrightarrow \ \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3\mathcal{F}_4 = 0.
\end{array}$$

(c) *If* $\mathcal{T}_4 = \mathcal{T}_3 = 0$ *and* $\mathcal{T}_2 \neq 0$ *then the system could possess two and only two weak singularities and none of them is of order 2 or 3. Moreover this system possesses two weak singularities, which are of the types indicated below, if and only if* $\mathcal{F} = 0$ *and one of the following conditions holds:*

$$\begin{array}{llll}
(c_1) & s^{(1)},\ s^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \mathcal{T}_2 < 0,\ \mathcal{B} \leq 0,\ \mathcal{H} > 0;\\
(c_2) & s^{(1)},\ f^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \mathcal{T}_2 > 0,\ \mathcal{B} < 0;\\
(c_3) & f^{(1)},\ f^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \mathcal{T}_2 < 0,\ \mathcal{B} < 0,\ \mathcal{H} < 0;\\
(c_4) & \$,\ \ \$ & \Leftrightarrow & \mathcal{F}_1 = 0,\ \mathcal{T}_2 < 0,\ \mathcal{B} < 0,\ \mathcal{H} > 0;\\
(c_5) & \$,\ \ c & \Leftrightarrow & \mathcal{F}_1 = 0,\ \mathcal{T}_2 > 0,\ \mathcal{B} < 0;\\
(c_6) & c,\ \ c & \Leftrightarrow & \mathcal{F}_1 = 0,\ \mathcal{T}_2 < 0,\ \mathcal{B} < 0,\ \mathcal{H} < 0.
\end{array}$$

(d) *If* $\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = 0$ *and* $\mathcal{T}_1 \neq 0$ *then the system could possess one and only one weak singularity (which is of order 1). Moreover this system has one weak singularity of the type indicated below if and only if* $\mathcal{F} = 0$ *and one of the following conditions holds:*

$$\begin{array}{llll}
(d_1) & s^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \mathcal{B} < 0,\ \mathcal{H} > 0;\\
(d_2) & f^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \mathcal{B} < 0,\ \mathcal{H} < 0.
\end{array}$$

(e) *If* $\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = \mathcal{T}_1 = 0$ *and* $\sigma(a,x,y) \neq 0$ *then the system could possess one and only only one weak singularity. Moreover this system has one weak singularity, which is of the type indicated below, if and only if one of the following conditions holds:*

$$\begin{array}{llll}
(e_1) & s^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \mathcal{H} = \mathcal{B}_1 = 0,\ \mathcal{B}_2 > 0;\\
(e_2) & f^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \mathcal{H} = \mathcal{B}_1 = 0,\ \mathcal{B}_2 < 0;\\
(e_3) & \$ & \Leftrightarrow & \begin{cases} [\alpha] \ \ \mathcal{F}_1 = 0,\ \mathcal{F} = 0,\ \mathcal{B} < 0,\ \mathcal{H} > 0,\ \textit{or}\\ [\beta] \ \ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B}_1 = 0,\ \mathcal{B}_2 > 0,\ \textit{or}\\ [\gamma] \ \ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = \mu_0 = 0,\ K(\mu_2^2 + \mu_3^2) \neq 0,\ \textit{or}\\ [\delta] \ \ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = K = 0,\ \mu_2\mathcal{G} \neq 0,\ \textit{or}\\ [\varepsilon] \ \ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = \mathcal{B}_4 = K = \mu_2 = 0,\ \mu_3 \neq 0; \end{cases}\\
(e_4) & c & \Leftrightarrow & \begin{cases} [\alpha] \ \ \mathcal{F}_1 = 0,\ \mathcal{F} = 0,\ \mathcal{B} < 0,\ \mathcal{H} < 0,\ \textit{or}\\ [\beta] \ \ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B}_1 = 0,\ \mathcal{B}_2 < 0. \end{cases}
\end{array}$$

(f) *If* $\sigma(a,x,y) = 0$ *then the system is Hamiltonian and it possesses* i $(1 \leq i \leq 4)$ *weak singular* *points of the types indicated below if and only if one of the following conditions holds:*

$$
\begin{array}{lccl}
(f_1) & \$,\$,\$,c & \Leftrightarrow & \mu_0<0,\ \mathbf{D}<0,\ \mathbf{R}>0,\ \mathbf{S}>0;\\
(f_2) & \$,\$,\ c,\ c & \Leftrightarrow & \mu_0>0,\ \mathbf{D}<0,\ \mathbf{R}>0,\ \mathbf{S}>0;\\
(f_3) & \$,\$,\ c & \Leftrightarrow & \mu_0=0,\ \mathbf{D}<0,\ \mathbf{R}\neq 0;\\
(f_4) & \$,\$ & \Leftrightarrow & \begin{cases}[\alpha]\ \ \mu_0<0,\ \mathbf{D}>0,\ \ or\\ [\beta]\ \ \mu_0<0,\ \mathbf{D}=0,\ \mathbf{T}<0,\ \ or\\ [\gamma]\ \ \mu_0=\mathbf{R}=0,\ \mathbf{P}\neq 0,\ \mathbf{U}>0,\ K\neq 0;\end{cases}\\
(f_5) & \$,\ c & \Leftrightarrow & \begin{cases}[\alpha]\ \ \mu_0>0,\ \mathbf{D}>0,\ \ or\\ [\beta]\ \ \mu_0>0,\ \mathbf{D}=0,\ \mathbf{T}<0,\ \ or\\ [\gamma]\ \ \mu_0=\mathbf{R}=0,\ \mathbf{P}\neq 0,\ \mathbf{U}>0,\ K=0;\end{cases}\\
(f_6) & \$ & \Leftrightarrow & \begin{cases}[\alpha]\ \ \mu_0<0,\ \mathbf{D}=\mathbf{T}=\mathbf{P}=0,\ \mathbf{R}\neq 0,\ \ or\\ [\beta]\ \ \mu_0=0,\ \mathbf{D}>0,\ \mathbf{R}\neq 0,\ \ or\\ [\gamma]\ \ \mu_0=0,\ \mathbf{D}=0,\ \mathbf{PR}\neq 0,\ \ or\\ [\delta]\ \ \mu_0=\mathbf{R}=\mathbf{P}=0,\ \mathbf{U}\neq 0;\end{cases}\\
(f_7) & c & \Leftrightarrow & \mu_0>0,\ \mathbf{D}=\mathbf{T}=\mathbf{P}=0,\ \mathbf{R}\neq 0;
\end{array}
$$

Here the invariant polynomials are defined in Subsections 2.3 and 2.4.

This article is organized as follows:

In Section 2 we construct the necessary invariant polynomials and functions and prove some needed auxiliary results. More precisely, in Subsection 2.1 we define some new GL-invariants which are responsible for the existence of a weak singularity (focus or saddle) and its order at the origin of coordinates. These invariant polynomials are more convenient (than the constructed respectively in [38] and [37]) by two reasons: *(i)* they have served as a base for the construction of the respective affine invariant conditions; *(ii)* they are applied to characterize also the degree of the weakness of the saddle at the origin of coordinates. So it is naturally that in invariant form the Poincaré-Lyapunov constants (i.e. focus values) and the "dual Poincaré-Lyapunov constants" (i.e. saddle quantities) coincide.

In Subsection 2.2 the whole class of non-degenerate quadratic systems (2.1) splits in invariant way in several subfamilies according to the number and multiplicities of the finite singularities (real and/or complex, simple and/or multiple). Actually the proof of the Main Theorem is based on the examination of these canonical forms.

In Subsection 2.3 we define the *trace function* $\mathfrak{T}(w)$ (see Definition 2.2) and the associated affine invariants $\mathcal{T}_1$–$\mathcal{T}_4$, which are responsible for the number of finite singularities having zero traces. These invariant polynomials served as a fundament for the partition of the coefficient space $\mathbb{R}^{12}$ of quadratic systems (2.1), which serves as a basic support for the Main Theorem.

Subsection 2.4 is dedicated to the construction of the affine invariant polynomials $\mathcal{F}_1$–$\mathcal{F}_4$ associated to Poincaré-Lyapunov constants. More precisely, when the singular point with zero trace is located at the origin of coordinate, the polynomials $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3\mathcal{F}_4$ become equivalent to Poincaré-Lyapunov constants (respectively dual Poincaré-Lyapunov constants) if this point is a focus (respectively a saddle).

In Section 3 we prove the Main Theorem. The proof proceeds in two steps:

1) In Subsection 3.1 providing that a non-degenerate quadratic system (2.1) has at least one simple real finite singular point, we prove that the conditions given by Main Theorem are necessary and sufficient for the existence of the respective weak singularities of the corresponding types arbitrarily located on the phase plan of the system;

2) In order to complete the proof, in Subsection 3.2 we prove the incompatibility of the conditions given by Main Theorem for a non-degenerate quadratic system (2.1) which does not have a simple finite real singular point.

2 Preliminary

Consider real quadratic systems of the form:

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + \ p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + \ q_2(x,y) \equiv Q(x,y) \end{aligned} \tag{2.1}$$

with homogeneous polynomials p_i and q_i $(i = 0, 1, 2)$ of degree i in x, y:

$$\begin{aligned} p_0 &= a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) + a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\ q_0 &= a_{00}, \quad q_1(x,y) = a_{10}x + a_{01}y, \quad q_2(x,y) + a_{20}x^2 + 2a_{11}xy + a_{02}y^2. \end{aligned}$$

Let $\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ be the 12-tuple of the coefficients of systems (2.1) and denote $\mathbb{R}[\tilde{a}, x, y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, \ b_{11}, b_{02}, x, y]$.

Notation 2.1. *We denote by* $\boldsymbol{a} = (\boldsymbol{a}_{00}, \boldsymbol{a}_{10} \ldots, \boldsymbol{b}_{02})$ *a specific point in* $\mathbb{R}^{12}$ *and we keep* a_{ij} *and* b_{ij} *as parameters. Each particular system* (2.1) *yields an ordered 12-tuple* $\boldsymbol{a}$ *of its coefficients.*

2.1 Local GL–invariant conditions

In [38] the necessary and sufficient GL-invariant conditions for the existence at the origin of coordinates of a weak focus of the given order or a center are constructed. However we shall construct here other GL-invariant conditions which are equivalent to them from [38, Theorem 34.3] but are more convenient by two reasons: *(i)* the defined GL-invariants could serve as a base for the construction of the respective affine invariant conditions; *(ii)* they could be also applied to characterize the degree of weakness of the saddle at the origin of coordinate.

We single out the following five polynomials, basic ingredients in constructing invariant polynomials for systems (2.1):

$$\begin{aligned} &C_i(\tilde{a}, x, y) = yp_i(x,y) - xq_i(x,y), \ (i = 0, 1, 2) \\ &D_i(\tilde{a}, x, y) = \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y}, \ (i = 1, 2). \end{aligned} \tag{2.2}$$

As it was shown in [38] these polynomials of degree one in the coefficients of systems (2.1) are GL–comitants of these systems. Let $f, \, g \in \mathbb{R}[\tilde{a}, x, y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

The polynomial $(f,g)^{(k)} \in \mathbb{R}[\tilde{a}, x, y]$ is called *the transvectant of index k of (f,g)* (cf. [21], [25])).

Theorem 2.1 (see [42])**.** *Any GL–comitant of systems* (2.1) *can be constructed from the elements* (2.2) *by using the operations: $+$, $-$, $\times$, and by applying the differential operation* $(*,*)^{(k)}$.

Remark 2.1. *We point out that the elements* (2.2) *generate the whole set of GL–comitants and hence also the set of affine comitants as well as the set of T-comitants.*

We construct the following GL–comitants of the second degree with respect to the coefficients of the initial systems

$$
\begin{array}{lll}
T_1 = (C_0, C_1)^{(1)}, & T_2 = (C_0, C_2)^{(1)}, & T_3 = (C_0, D_2)^{(1)}, \\
T_4 = (C_1, C_1)^{(2)}, & T_5 = (C_1, C_2)^{(1)}, & T_6 = (C_1, C_2)^{(2)}, \\
T_7 = (C_1, D_2)^{(1)}, & T_8 = (C_2, C_2)^{(2)}, & T_9 = (C_2, D_2)^{(1)}.
\end{array} \tag{2.3}
$$

Denoting $\mathcal{A} = \begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix}$ we define the following GL–invariants:

$$
\begin{aligned}
&G_0 = \frac{\partial p_1}{\partial x} + \frac{\partial q_1}{\partial y} \equiv D_1(\tilde{a}) = \operatorname{tr}(\mathcal{A}), \\
&G_1 = (C_1 T_7,\, C_2)^{(3)}, \\
&G_2 = \big(5(T_8,\, T_6)^{(1)} - 2\big(7T_8 + 8T_9,\, T_7\big)^{(1)},\, D_2\big)^{(1)}, \\
&G_3 = \big((T_8,\, T_6)^{(1)} + 2(T_8 + 8T_9,\, T_7)^{(1)},\, D_2\big)^{(1)}, \\
&G_4 = T_4\left[(T_8,\, C_1)^{(2)} + 8(T_7 - T_6,\, D_2)^{(1)}\right] - 3\big((T_6,\, C_1)^{(1)},\, T_6\big)^{(1)}, \\
&G_5 = (p_1,\, q_1)^{(1)} = \det(\mathcal{A}).
\end{aligned} \tag{2.4}
$$

In what follows we shall use the next two useful lemmas.

Lemma 2.1. *A quadratic system*

$$
\begin{aligned}
\dot{x} &= a_{10}x + a_{01}y + a_{20}x^2 + 2a_{11}xy + a_{02}y^2, \\
\dot{y} &= b_{10}x + b_{01}y + b_{20}x^2 + 2b_{11}xy + b_{02}y^2,
\end{aligned} \tag{2.5}
$$

has at the origin of coordinates either a weak focus (respectively a weak saddle) of the indicated order below, or a center (respectively an integrable saddle) if and only if $G_0 = 0$ and $G_5 > 0$ (respectively $G_5 < 0$) and the following condition holds:

$$
\begin{array}{lcl}
f^{(1)} \ (\textit{respectively } s^{(1)}) & \Leftrightarrow & G_1 \neq 0; \\
f^{(2)} \ (\textit{respectively } s^{(2)}) & \Leftrightarrow & G_1 = 0,\ G_2 \neq 0; \\
f^{(3)} \ (\textit{respectively } s^{(3)}) & \Leftrightarrow & G_1 = G_2 = 0,\ G_3G_4 \neq 0; \\
C \quad\ (\textit{respectively } \$) & \Leftrightarrow & G_1 = G_2 = G_3G_4 = 0.
\end{array}
$$

Proof: It is known that the matrix $\mathcal{A}$ via a linear transformation and a time rescaling can be brought to the form $a_{10} = 0$, $a_{01} = -1$, $b_{10} = 1$ and $b_{01} = 0$ (respectively, $a_{10} = 1$, $a_{01} = 0$, $b_{10} = 0$ and $b_{01} = -1$) if and only if $G_0 = \operatorname{tr}(\mathcal{A}) = 0$ and $G_5 = \det(\mathcal{A}) > 0$ (respectively, $G_5 < 0$). So, in what follows we assume that the condition $G_0 = 0$ holds.

Subcase $G_5 > 0$. Then the corresponding eigenvalues of the matrix $\mathcal{A}$ are purely imaginary and according to [10] systems (2.5) via a linear transformation and time rescaling can be brought to the systems

$$
\begin{aligned}
\dot{x} &= -y - mx^2 + (2a + l)xy + ny^2, \\
\dot{y} &= x + ax^2 + (2m + b)xy - ay^2,
\end{aligned} \tag{2.6}
$$

for which the Poincaré-Lyapunov quantities are as follows:

$$
\begin{aligned}
L_1 &= l(n - m), \\
L_2 &= ab(n - m)(5n - 5m - b), \\
L_3 &= ab(m - n)^2[a^2 + n(2n - m)].
\end{aligned}
$$

On the other hand for systems (2.6) calculations yield: $G_0 = 0$, $G_5 = 1 > 0$ and

$$\begin{aligned}
&G_1 = 72l(n-m) = 72L_1,\\
&G_2 = 864(n-m)\big[10l(a^2+m^2) + b^2(a+l) + 5ab(m-n) + l(4al + 7bm)\big],\\
&G_3 = 864(n-m)\big[b^2(a+l) + ab(m-n) + 2l(a^2+m^2) + 3blm\big],\\
&G_4 = 288\big[(4a+l)^2 + (b+5m-n)(b+m+3n)\big].
\end{aligned}$$

Evidently, the condition $G_1 \neq 0$ is equivalent to $L_1 \neq 0$. Assume $G_1 = 0$, i.e. $l(n-m) = 0$. The condition $m-n = 0$ yields $G_1 = G_2 = G_3G_4 = 0$ and simultaneously $L_1 = L_2 = L_3 = 0$, i.e. we have a center. If $m-n \neq 0$ then $l = 0$ and considering the expressions above we obtain

$$G_2 = -864L_2, \qquad G_3G_4 = 2^{10}3^5\big[64L_3 - L_2\big(16a^2 + (m-5n)^2 + b^2 + 2bm + 6bn\big)\big]$$

and this proves our assertion in the case $G_5 > 0$.

Subcase $G_5 < 0$. In this case the simple singular point $M_0(0,0)$ of systems (2.5) is a saddle with zero trace. So according to [16] (see also, [22]) via a linear transformation this system can be brought to the form

$$\dot{x} = x + Ax^2 + Bxy + Cy^2, \quad \dot{y} = -y - Kx^2 - Lxy - My^2, \tag{2.7}$$

for which Cai Sui Lin [16] has calculated the first three dual Poincaré-Lyapunov constants:

$$\begin{aligned}
&L_1^* = LM - AB,\\
&L_2^* = KB(2M-B)(M+2B) - CL(2A-L)(A+2L),\\
&L_3^* = (CK - LB)\big[ACL(2A-L) - BKM(2M-B)\big].
\end{aligned}$$

Remark 2.2. According to [16] the weak saddle $(0,0)$ is of the first (respectively second; third) order if and only if $L_1^* \neq 0$ (respectively $L_1^* = 0, L_2^* \neq 0$; $L_1^* = L_2^* = 0, L_3^* \neq 0$) and it is an *integrable saddle* if and only if $L_1^* = L_2^* = L_3^* = 0$.

Considering (2.4) for systems (2.7) calculations yield $G_0 = 0$, $G_5 = -1 < 0$ and

$$\begin{aligned}
&G_1 = 144(AB - LM) = -144\,L_1^*,\\
&G_2 = 2^6 3^3\big[L_2^* - L_1^*(5CK + 2BL - 3AM)\big],\\
&G_3G_4 = -2^{14}3^5\big[L_3^* + L_1^*(CK - BL)(CK + AM)\big].
\end{aligned}$$

So, the conditions $L_1^* = 0$ (respectively $L_1^* = L_2^* = 0$; $L_1^* = L_2^* = L_3^* = 0$) are equivalent to $G_1 = 0$ (respectively $G_1 = G_2 = 0$; $G_1 = G_2 = G_3G_4 = 0$) and this completes the proof of the lemma. ■

Lemma 2.2. *Assume that a quadratic system* (2.1) *possesses two simple real weak singularities* M_1 *and* M_2, *i.e.* $\mathrm{tr}\,(\mathcal{A}^{(1)}) = \mathrm{tr}\,(\mathcal{A}^{(2)}) = 0$ *and* $\det(\mathcal{A}^{(1)})\det(\mathcal{A}^{(2)}) \neq 0$. *Then these points are of the types indicated below if and only if the respective conditions hold:*

$$\begin{array}{rcl}
s^{(1)},\ s^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \det(\mathcal{A}^{(1)}) < 0,\ \det(\mathcal{A}^{(2)}) < 0;\\
s^{(1)},\ f^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \det(\mathcal{A}^{(1)})\det(\mathcal{A}^{(2)}) < 0;\\
f^{(1)},\ f^{(1)} & \Leftrightarrow & \mathcal{F}_1 \neq 0,\ \det(\mathcal{A}^{(1)}) > 0,\ \det(\mathcal{A}^{(2)}) > 0;\\
\$,\quad \$ & \Leftrightarrow & \mathcal{F}_1 = 0,\ \det(\mathcal{A}^{(1)}) < 0,\ \det(\mathcal{A}^{(2)}) < 0;\\
\$,\quad c & \Leftrightarrow & \mathcal{F}_1 = 0,\ \det(\mathcal{A}^{(1)})\det(\mathcal{A}^{(2)}) < 0;\\
c,\quad c & \Leftrightarrow & \mathcal{F}_1 = 0,\ \det(\mathcal{A}^{(1)}) > 0,\ \det(\mathcal{A}^{(2)}) > 0;
\end{array}$$

where $\mathcal{F}_1$ *is the affine invariant defined in* (2.17).

Proof: If a quadratic system (2.1) possesses two real singular points then via an affine transformation these points can be localized at $M_1(0,0)$ and $M_2(1,0)$, respectively. In this case we obtain the systems

$$\dot{x} = cx + dy - cx^2 + 2hxy + ky^2, \quad \dot{y} = ex + fy - ex^2 + 2mxy + ny^2, \tag{2.8}$$

and for the singular points $M_1(0,0)$ and $M_2(1,0)$ we have $\rho_1 = c + f$ and $\rho_2 = -c + f + 2m$. Hence the conditions $\rho_1 = \rho_2 = 0$ yield $f = -c$ and $m = c$.

In order to apply the local conditions provided by Lemma 2.1 we shall examine the systems above in two forms: one of them having at the origin the point M_1 and another one having at the origin the point M_2. The first form evidently will be obtained from (2.8) by setting $f = -c$ and $m = c$:

$$\dot{x} = cx + dy - cx^2 + 2hxy + ky^2, \quad \dot{y} = ex - cy - ex^2 + 2cxy + ny^2.$$

For these systems calculations yield:

$$\begin{aligned}
&G_0^{(1)} = 0, \quad G_1^{(1)} = -144(h+n)(2c^3 - 2ceh - e^2k + cen) = -72\mathcal{F}_1,\\
&G_2^{(1)} = 864(5d + 8h + 3n)\mathcal{F}_1, \quad G_3^{(1)} = 864(d-n)\mathcal{F}_1,\\
&G_5^{(1)} = -(c^2 + de) = \det\left(\mathcal{A}^{(1)}\right)
\end{aligned}$$

Replacing the point $M_2(1,0)$ at the origin due to a translation the systems above become:

$$\begin{aligned}
&\dot{x} = -cx + (d + 2h)y - cx^2 + 2hxy + ky^2,\\
&\dot{y} = -ex + cy - ex^2 + 2cxy + ny^2,
\end{aligned}$$

and for these systems we calculate again the values of G_i, attached to the point M_2 (which is located at the origin of coordinate now):

$$\begin{aligned}
&G_0^{(2)} = 0, \quad G_1^{(2)} = -144(h+n)(2c^3 - 2ceh - e^2k + cen) = -72\mathcal{F}_1,\\
&G_2^{(2)} = 864(5d + 2h - 3n)\mathcal{F}_1, \quad G_3^{(2)} = 864(d + 2h + n)\mathcal{F}_1,\\
&G_5^{(2)} = -c^2 + de + 2eh = \det\left(\mathcal{A}^{(2)}\right).
\end{aligned}$$

We observe that $\mathcal{F}_1 \neq 0$ implies $G_1^{(1)}G_1^{(2)} \neq 0$ whereas $\mathcal{F}_1 = 0$ implies $G_i^{(1)} = G_i^{(2)} = 0$, $i = 1,2,3$. So according to Lemma 2.1 if $\det\left(\mathcal{A}^{(i)}\right) > 0$ (respectively $\det\left(\mathcal{A}^{(i)}\right) < 0$) $(i = 1,2)$ then the singular point M_i is a weak focus (respectively a weak saddle) of the first order if $\mathcal{F}_1 \neq 0$ and it is a center (respectively an integrable saddle) if $\mathcal{F}_1 = 0$. ∎

Remark 2.3. *If one of the points either M_1 or M_2 is not a simple singularity (for example $\det\left(\mathcal{A}^{(1)}\right) = 0$), then the statement of Lemma 2.2 regarding the second point M_2 is still valid.*

In what follows the next remark will be useful.

Remark 2.4. *Assume that we have obtained a normal form of a family of quadratic systems moving a simple singular point to the origin of coordinates and fixing the position of all other singular points, even depending on some parameters. Then any geometrical propriety of any simple finite singular point can be considered to be hold by the origin point.*

2.2 Canonical forms associated to the finite singularities

We shall use the notion of zero–cycle in order to describe the number and multiplicity of singular points of a quadratic system. This notion as well as the notion of *divisor*, were used for classification purposes of planar quadratic differential systems by Pal and Schlomiuk [26], Llibre and Schlomiuk [24], Schlomiuk and Vulpe [30] and by Artes and Llibre and Schlomiuk [3].

Definition 2.1. *We consider formal expressions $\mathcal{D} = \sum n(w)w$ where $n(w)$ is an integer and only a finite number of $n(w)$ are nonzero. Such an expression is called a zero–cycle of $\mathbf{P}_2(\mathbb{C})$ if all w appearing in $\mathcal{D}$ are points of $\mathbf{P}_2(\mathbb{C})$. We call degree of the zero–cycle $\mathcal{D}$ the integer $\deg(\mathcal{D}) = \sum n(w)$. We call support of $\mathcal{D}$ the set $\mathrm{Supp}\,(\mathcal{D})$ of w's appearing in $\mathcal{D}$ such that $n(w) \neq 0$.*

We note that $\mathbf{P}_2(\mathbb{C})$ denotes the complex projective space of dimension 2. For a system (S) belonging to the family (2.1) we denote $\nu(P,Q) = \{w \in \mathbb{C}_2 \mid P(w) = Q(w) = 0\}$ and we define the following zero–cycle $\mathcal{D}_S(P,Q) = \sum_{w\in\nu(P,Q)} I_w(P,Q)w$, where $I_w(P,Q)$ is the intersection number or multiplicity of intersection at w. It is clear that for a non–degenerate quadratic system $\deg(\mathcal{D}_S) \leq 4$ as well as $\mathrm{Supp}\,(\mathcal{D}_S) \leq 4$. For a degenerate system the zero–cycle $\mathcal{D}_S(P,Q)$ is undefined.

Consider now the differential operator $\mathcal{L} = x\cdot\mathbf{L}_2 - y\cdot\mathbf{L}_1$ (see [7]) acting on $\mathbb{R}[\tilde{a},x,y]$, where

$$\begin{aligned}
\mathbf{L}_1 &= 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}};\\
\mathbf{L}_2 &= 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}.
\end{aligned}\tag{2.9}$$

Using this operator we construct the following important set of invariant polynomials:

$$\begin{aligned}
\mu_0(\tilde{a}) &= \mathrm{Res}_{\,x}\big(p_2(x,y), q_2(x,y)\big)/y^4,\\
\mu_i(\tilde{a},x,y) &= \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ \ i = 1,..,4,
\end{aligned}\tag{2.10}$$

where $\mathcal{L}^{(i)}(\mu_0) = \mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$. These polynomials are in fact invariant polynomials of systems (2.1) with respect to the group $GL(2,\mathbb{R})$ (see [7]). Their geometrical meaning is revealed in the following two lemmas:

Lemma 2.3. ([7]) *The total multiplicity of all finite singularities of a quadratic system* (2.1) *equals k if and only if for every $i \in \{0,1,\ldots,k-1\}$ we have $\mu_i(\tilde{a},x,y) = 0$ in $\mathbb{R}[x,y]$ and $\mu_k(\tilde{a},x,y) \neq 0$. Moreover a system* (2.1) *is degenerate (i.e.* $\gcd(P,Q) \neq \text{constant}$*) if and only if $\mu_i(\tilde{a},x,y) = 0$ in $\mathbb{R}[x,y]$ for every $i = 0,1,2,3,4$.*

Lemma 2.4. ([8]) The point $M_0(0,0)$ is a singular point of multiplicity k $(1 \leq k \leq 4)$ for a quadratic system (2.1) if and only if for every $i \in \{0,1,\ldots,k-1\}$ we have $\mu_{4-i}(\tilde{a},x,y) = 0$ in $\mathbb{R}[x,y]$ and $\mu_{4-k}(\tilde{a},x,y) \neq 0$.

Using the invariant polynomials μ_i $(i = 0,1,\ldots,4)$ we could construct the invariant polynomials $\mathbf{D}$, $\mathbf{P}$, $\mathbf{R}$, $\mathbf{S}$, $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$, which are responsible for the number and multiplicities of finite singularities of a non-degenerate quadratic system. We note that these polynomials were constructed (using another way) and applied in [6, 7] (see also [4]). Here they are constructed as follows:

$$\begin{aligned}
\mathbf{D} =& \Big[3\big((\mu_3,\mu_3)^{(2)},\ \mu_2\big)^{(2)} - \big(6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\ \mu_4\big)^{(4)}\Big]/48,\\
\mathbf{P} =& 12\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\\
\mathbf{R} =& 3\mu_1^2 - 8\mu_0\mu_2,\\
\mathbf{S} =& \mathbf{R}^2 - 16\mu_0^2\mathbf{P},\\
\mathbf{T} =& 18\mu_0^2(3\mu_3^2 - 8\mu_2\mu_4) + 2\mu_0(2\mu_2^3 - 9\mu_1\mu_2\mu_3 + 27\mu_1^2\mu_4) - \mathbf{PR},\\
\mathbf{U} =& \mu_3^2 - 4\mu_2\mu_4,\\
\mathbf{V} =& \mu_4.
\end{aligned}\tag{2.11}$$

The geometrical meaning of the invariant polynomials above is revealed in the next proposition:

Proposition 2.1. ([7]) *The form of the divisor $\mathcal{D}_S(P,Q)$ for non-degenerate quadratic systems* (2.1) *is determined by the corresponding conditions indicated in Table 1, where we write $p+q+r^c+s^c$ if two of the finite points, i.e. r^c, s^c, are complex but not real.*

Table 1

No.	Zero–cycle $\mathcal{D}_S(P,Q)$	Invariant criteria	No.	Zero–cycle $\mathcal{D}_S(P,Q)$	Invariant criteria
1	$p+q+r+s$	$\mu_0\neq 0, \mathbf{D}<0,$ $\mathbf{R}>0, \mathbf{S}>0$	10	$p+q+r$	$\mu_0=0, \mathbf{D}<0, \mathbf{R}\neq 0$
2	$p+q+r^c+s^c$	$\mu_0\neq 0, \mathbf{D}>0$	11	$p+q^c+r^c$	$\mu_0=0, \mathbf{D}>0, \mathbf{R}\neq 0$
3	$p^c+q^c+r^c+s^c$	$\mu_0\neq 0, \mathbf{D}<0, \mathbf{R}\leq 0$ $\mu_0\neq 0, \mathbf{D}<0, \mathbf{S}\leq 0$	12	$2p+q$	$\mu_0=\mathbf{D}=0, \mathbf{PR}\neq 0$
4	$2p+q+r$	$\mu_0\neq 0, \mathbf{D}=0, \mathbf{T}<0$	13	$3p$	$\mu_0=\mathbf{D}=\mathbf{P}=0, \mathbf{R}\neq 0$
5	$2p+q^c+r^c$	$\mu_0\neq 0, \mathbf{D}=0, \mathbf{T}>0$	14	$p+q$	$\mu_0=\mathbf{R}=0, \mathbf{P}\neq 0,$ $\mathbf{U}>0$
6	$2p+2q$	$\mu_0\neq 0, \mathbf{D}=\mathbf{T}=0,$ $\mathbf{PR}>0$	15	p^c+q^c	$\mu_0=\mathbf{R}=0, \mathbf{P}\neq 0,$ $\mathbf{U}<0$
7	$2p^c+2q^c$	$\mu_0\neq 0, \mathbf{D}=\mathbf{T}=0,$ $\mathbf{PR}<0$	16	$2p$	$\mu_0=\mathbf{R}=0, \mathbf{P}\neq 0,$ $\mathbf{U}=0$
8	$3p+q$	$\mu_0\neq 0, \mathbf{D}=\mathbf{T}=0,$ $\mathbf{P}=0, \mathbf{R}\neq 0$	17	p	$\mu_0=\mathbf{R}=0\mathbf{P}=0,$ $\mathbf{U}\neq 0$
9	$4p$	$\mu_0\neq 0, \mathbf{D}=\mathbf{T}=0,$ $\mathbf{P}=\mathbf{R}=0$	18	0	$\mu_0=\mathbf{R}=\mathbf{P}=0,$ $\mathbf{U}=0, \mathbf{V}\neq 0$

Considering the expressions (2.11) the next remark follows.

Remark 2.5. *If $\mu_0=0$ then the condition $\mathbf{R}=0$ (respectively $\mathbf{R}=\mathbf{P}=0$; $\mathbf{R}=\mathbf{P}=\mathbf{U}=0$; $\mathbf{R}=\mathbf{P}=\mathbf{U}=\mathbf{V}=0$) is equivalent to $\mu_1=0$ (respectively $\mu_1=\mu_2=0$; $\mu_1=\mu_2=\mu_3=0$; $\mu_1=\mu_2=\mu_3=\mu_4=0$).*

Using this partition of the coefficient space $\mathbb{R}^{12}$ of the family of non-degenerate quadratic systems (2.1) the respective canonical forms could be associated. We have the next result.

Proposition 2.2. *Any non-degenerate quadratic system in dependence of the number and multiplicities of finite singularities could be brought via an affine transformation and time rescaling to one of the canonical forms given by Table 2.*

Proof: All the canonical forms corresponding to the configurations of finite singularities given by Table 1, were constructed in [4] and [5] (up to some additional rescaling), except the configurations corresponding to the cases 9 ($\mathcal{D}_S(P,Q)=4p$) and 18 ($\mathcal{D}_S(P,Q)=0$). We note that some of canonical forms corresponding to these cases (and to other cases from Table 1) were constructed earlier in different papers (see for example, [11, 12, 13]). However we shall construct here the needed canonical forms which are more convenient for our propose.

1) *Systems with zero-cycle $\mathcal{D}_S(P,Q)=4d$.* In this case systems have one finite singular point of multiplicity 4 and via a translation we may locate this point at the origin of coordinates. Clearly we could have either a semi-elementary (with one non-zero eigenvalue), or a non-elementary (with two zero eigenvalues) singular point .

a) In the case of semi-elementary singular point it is known (cf. for example, [1]) that via a linear transformation and time rescaling a quadratic system in this case can be transformed to the

Table 2

Form of $D_S(P,Q)$	Canonical form	Finite singularities and conditions on parameters
$p+q+r+s$	1) $\dot x = cx+dy-cx^2+2hxy-dy^2,$ $\dot y = ex+fy-ex^2+2mxy-fy^2$	$M_1(0,0), M_2(1,0), M_3(0,1),$ $M_4\Big(\frac{\beta(\beta-2\gamma)}{\beta^2-4\alpha\gamma},\ \frac{\beta(\beta-2\alpha)}{\beta^2-4\alpha\gamma}\Big)$, where $\alpha = eh-cm,\quad \beta = cf-de,$ $\gamma = dm-fh;\ \ \beta^2-4\alpha\gamma\neq 0$
$p+q+r^c+s^c$	2) $\dot x = a-(a+g)x+gx^2+2hxy+ay^2,$ $\dot y = b-(b+l)x+lx^2+2mxy+by^2$	$M_1(0,i),\ M_2(0,-i),\ M_3(1,0),$ $M_4\Big(\frac{\beta^2+4\gamma^2}{\beta^2-4\alpha\gamma},\ \frac{2\beta(\alpha+\gamma)}{4\alpha\gamma-\beta^2}\Big)$, where $\alpha = gm-hl,\quad \beta = bg-al,$ $\gamma = bh-am;\ \ \beta^2-4\alpha\gamma\neq 0$
$p^c+q^c+r^c+s^c$	3a) $\dot x = \frac{g(u^2+1)}{v^2+1}(1+y^2)-2(gu+hv)x$ $+gx^2+2hxy,\ \dot y = \frac{l(u^2+1)}{v^2+1}(1+y^2)$ $-2(lu+mv)x+lx^2+2mxy$	$M_{1,2}(0,\pm i),\ \ M_{3,4}(u\pm i,v);$ $gm-lh\neq 0$
	3b) $\dot x = a+[a(u^2-1)-g]x+gx^2+ay^2,$ $\dot y = b+[b(u^2-1)-l]x+lx^2+by^2$	$M_{1,2}(0,\pm i),\ \ M_{3,4}(1,\pm iu);$ $al-bg\neq 0$
$2p+q+r$	4) $\dot x = cx+cuy-cx^2+2hxy-cuy^2,$ $\dot y = ex+euy-ex^2+2mxy-euy^2$	$M_{1,2}(0,0)-double, M_3(1,0), M_4(0,1);$ $u(cm-eh)\neq 0$
$2p+q^c+r^c$	5) $\dot x = cm\,x+2cn\,y+g\,x^2-2cn\,xy+$ $+(g+cm)\,y^2,\ \ \dot y = em\,x+2en\,y+$ $+l\,x^2-2en\,xy+(l+em)\,y^2$	$M_{1,2}(0,0)-double,\ \ M_{3,4}(1,\pm i);$ $(cl-eg)(m^2+n^2)\neq 0$
$2p+2q$	6) $\dot x = cx+cuy-cx^2+2cvxy+ky^2,$ $\dot y = ex+euy-ex^2+2evxy+ny^2$	$M_{1,2}(0,0),\ \ M_{3,4}(1,0);$ $cn-ek\neq 0$
$2p^c+2q^c$	7) $\dot x = a+aux+gx^2+2avxy+ay^2,$ $\dot y = b+bux+lx^2+2bvxy+by^2$	$M_{1,2}(0,i),\ M_{3,4}(0,-i);$ $al-bg\neq 0$
$3p+q$	8) $\dot x = cx+cuy-cx^2+2hxy+$ $+(2hu+cv)y^2,\quad \dot y = ex+euy-$ $-ex^2+2mxy+(2mu+ev)y^2$	$M_{1,2,3}(0,0)-triple,\ M_4(1,0);$ $(u^2-v)(cm-eh)\neq 0$
$4p$	9a) $\dot x = y^2,\ \ \dot y = y+x^2+2mxy+ny^2$	$M_0(0,0)-of\ multiplicity\ 4;$
	9b) $\dot x = y+x^2+ky^2,\ \ \dot y = y^2$	$M_0(0,0)-of\ multiplicity\ 4;$
	9c) $\dot x = gx^2+2hxy,$ $\dot y = lx^2+2mxy+ny^2$	$M_0(0,0)-of\ multiplicity\ 4;$ $g^2n^2-4hn(gm-hl)\neq 0$

canonical form

$$\dot x = gx^2+2hxy+ky^2,\quad \dot y = y+lx^2+2mxy+ny^2.$$

By Lemma 2.4 the singular point $M_0(0,0)$ has multiplicity 4 if and only if $\mu_4=\mu_3=\mu_2=\mu_1=0$ and $\mu_0\neq 0$. For the systems above we calculate $\mu_4=\mu_3=0,\ \ \mu_2 = 2g(gx^2+2hxy+ky^2)$ and the condition $\mu_2=0$ yields $g=0$. Then we have:

$$\mu_1 = 2hl(2hx+ky)=0,\quad \mu_0 = l(k^2l-4hkm+4h^2n)\neq 0$$

that implies $h=0$ and therefore we obtain $\mu_0 = k^2l^2\neq 0$. Then we may assume $k=l=1$ due to the rescaling $(x,y)\mapsto (k^{-\frac13}l^{-\frac23}x,\ k^{-\frac23}l^{-\frac13}y)$. This leads to the canonical form 9a) from Table 2.

b) Assume that the singular point $M_0(0,0)$ is non-elementary. If these systems are homogeneous then the canonical form is trivial and could be considered as 19c).

Table 2 *(continued)*

Form of $D_S(P,Q)$	Canonical form	Finite singularities and conditions on parameters
$p+q+r$	10) $\dot{x}=cx+dy-cx^2+2hxy,$ $\dot{y}=ex+fy-ex^2+2mxy$	$M_1(0,0), M_2(1,0),$ $M_3\Big(\frac{\beta}{2\alpha}, \frac{\beta(2\alpha-\beta)}{4\alpha\gamma}\Big)$, where $\alpha=eh-cm,\ \ \beta=cf-de,$ $\gamma=dm-fh; \alpha\beta\gamma(2\alpha-\beta)\neq 0$
$p+q^c+r^c$	11) $\dot{x}=2(h-gu)x+g(u^2+1)y+$ $+gx^2-2hxy,\ \ \dot{y}=2(m-lu)x+$ $+l(u^2+1)y+lx^2-2mxy$	$M_1(0,0),\quad M_{2,3}(u\pm i,1);$ $gm-hl\neq 0$
$2p+q$	12) $\dot{x}=cx+cy-cx^2+2hxy,$ $\dot{y}=ex+ey-ex^2+2mxy$	$M_{1,2}(0,0)-double, M_3(1,0);$ $cm-eh\neq 0$
$3p$	13) $\dot{x}=gy+gx^2+2hxy,$ $\dot{y}=ly+lx^2+2mxy$	$M_{1,2,3}(0,0)-triple;$ $gm-lh\neq 0$
$p+q$	14a) $\dot{x}=cx+dy-cx^2+2duxy,$ $\dot{y}=ex+fy-ex^2+2fuxy$	$M_1(0,0),\quad M_2(1,0);$ $(cf-de)(2u+1)u\neq 0$
	14b) $\dot{x}=-(g+ku^2)x-2hy+gx^2+$ $+2hxy+ky^2,\quad \dot{y}=ux+y$	$M_1(0,0),\quad M_2(1,-u);$ $g-2hu+ku^2\neq 0$
p^c+q^c	15a) $\dot{x}=a+hux+2hxy+ay^2,$ $\dot{y}=b+mux+2mxy+by^2$	$M_{1,2}(0,\pm i);$ $am-bh\neq 0$
	15b) $\dot{x}=a+cx+gx^2+2hxy+ay^2,$ $\dot{y}=x$	$M_{1,2}(0,\pm i);$ $a\neq 0$
$2p$	16a) $\dot{x}=dy+gx^2+2dxy,$ $\dot{y}=fy+lx^2+2fxy$	$M_{1,2}(0,0)-double;$ $fg-dl\neq 0$
	16b) $\dot{x}=cx+dy,$ $\dot{y}=lx^2+2mxy+ny^2$	$M_{1,2}(0,0)-double;$ $c^2n-2cdm+dl^2\neq 0$
p	17a) $\dot{x}=cx+dy+(2c+d)x^2+2dxy,$ $\dot{y}=ex+fy+(2e+f)x^2+2fxy$	$M_1(0,0);$ $cf-de\neq 0$
	17b) $\dot{x}=x+dy,\quad \dot{y}=ex+fy+$ $+lx^2+2mxy+(2dm-ld^2)y^2$	$M_1(0,0);$ $(de-f)(l^2+m^2)\neq 0$
0	18a) $\dot{x}=h+gx^2+2hxy,$ $\dot{y}=m+lx^2+2mxy$	$hl-gm\neq 0$
	18b) $\dot{x}=y,\ \ \dot{y}=1+ex+fy+2mxy+ny^2$	$m^2+n^2\neq 0$
	18c) $\dot{x}=x,\quad \dot{y}=1+lx^2+2mxy$	$l^2+m^2\neq 0$
	18d) $\dot{x}=1,\quad \dot{y}=ex+fy+$ $+lx^2+2mxy+ny^2$	$(e,f\in\{0,1\})$ $l^2+m^2+n^2\neq 0$

Suppose now that linear matrix is not zero identically. Then via a linear transformation and time rescaling these systems can be brought to the form

$$\dot{x}=y+gx^2+2hxy+ky^2,\quad \dot{y}=lx^2+2mxy+ny^2.$$

Considering Lemma 2.4 we calculate $\mu_4=\mu_3=0,\quad \mu_2=l(lx^2+2mxy+ny^2)$ and the condition $\mu_2=0$ yields $l=0$. Then we have:

$$\mu_1=2gm(2mx+ny)=0,\quad \mu_0=g(4km^2-4hmn+gn^2)$$

and due to $\mu_0\neq 0$ we obtain $m=0$. So $\mu_0=g^2n^2\neq 0$ and we may consider $g=n=1$ and $h=0$

due to the transformation $(x, y, t) \mapsto \big((nx - hy)/n^2,\, gy/n^2, nt/g\big)$. Thus we obtain the canonical form $9b$) from Table 2.

2) *Systems with zero-cycle* $\mathcal{D}_S(P,Q) = 0$. In this case systems (2.1) have no finite singular points (i.e. all finite singularities have gone to infinity) and hence the total multiplicity equals zero. According to Lemma 2.3 in this case the conditions $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ and $\mu_4 \neq 0$ have to be satisfied. We note that according to (2.10) the polynomial μ_0 is the resultant of the binary forms $p_2(x, y)$ and $q_2(x, y)$. So the condition $\mu_0 = 0$ implies that these two binary forms has a non-constant common factor of degree at least one.

On the other hand denoting $K = \text{Jacob}\big(p_2, q_2\big)$ we conclude that this factor is of degree exactly one if $K \neq 0$ and it is of degree two (i.e. $p_2(x, y)$ and $q_2(x, y)$ are proportional) if $K = 0$.

***a*)** Assume first $K \neq 0$, i.e. the common factor of $p_2(x, y)$ and $q_2(x, y)$ is of degree one. Then via a linear transformation we can consider that this common factor is x, i.e. in systems (2.1) we have $a_{02} = b_{02} = 0$. Therefore these systems become

$$\dot{x} = a + cx + dy + gx^2 + 2hxy, \quad \dot{y} \;= b + ex + fy + lx^2 + 2mxy,$$

for which $K = (gm - hl)x^2 \neq 0$. In this case we may assume $c = e = 0$ via the translation

$$x \to x + \frac{eh - cm}{2(gm - hl)}, \quad y \to y + \frac{cl - eg}{2(gm - hl)}$$

and then for the systems above calculation yields: $\mu_0 = 0$ and $\mu_1 = 4(fh - dm)(hl - gm)x$. As $K \neq 0$ the condition $\mu_1 = 0$ yields $fh - dm = 0$. Due to the condition $h^2 + m^2 \neq 0$ (as $K \neq 0$) without loss of generality we may set a new parameter u as follows: $f = um$ and $d = uh$. Then we have

$$\mu_2 = (hl - gm)\big[4(bh - am) + u^2(hl - gm)\big]x^2, \quad \mu_3 = 2u(hl - gm)(al - bg)x^3$$

and due to $K \neq 0$ the condition $\mu_3 = 0$ yields $u(al - bg) = 0$.

We claim that in order to satisfy the conditions $\mu_2 = \mu_3 = 0$ and $\mu_4 \neq 0$ it is necessary $u = 0$. Indeed, supposing that $u \neq 0$ we obtain $al = bg$ and since $g^2 + l^2 \neq 0$ (as $K \neq 0$) we may set a new parameter v as follows: $a = gv$, $b = lv$. Then calculations yield

$$\mu_2 = (hl - gm)^2(u^2 + 4v)x^2, \quad \mu_4 = v(hl - gm)^2(u^2 + 4v)x^2y^2$$

and evidently the condition $\mu_2 = 0$ yields $\mu_4 = 0$. This proves our claim.

So $u = 0$ and we have $\mu_3 = 0$. At the same time the condition $\mu_2 = 0$ yields $bh - am = 0$. So we set again the parameter u as follows: $a = hu$, $b = mu$, where $u \neq 0$ due to $\mu_4 = (hl - gm)^2u^2x^4 \neq 0$. therefore we may assume $u = 1$ due to the change $y \to uy$, $h \to h/u$ and $l \to lu$. This leads to the canonical form $18a$) from Table 2.

***b*)** Suppose now $K = 0$, i.e. the polynomials $p_2(x, y)$ and $q_2(x, y)$ are proportional. Then via a linear transformation we obtain the system

$$\dot{x} = a + cx + dy, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy + ny^2 \tag{2.12}$$

and we shall consider two subcases: $d \neq 0$ and $d = 0$.

***b*$_1$)** If $d \neq 0$ then via the affine transformation $x_1 = x$, $y_1 = cx + dy + a$ systems (2.12) will be brought to the systems:

$$\dot{x} = y, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy + ny^2,$$

for which calculations yield: $\mu_0 = \mu_1 = 0$, $\mu_2 = l(lx^2 + 2mxy + ny^2)$. Therefore, the condition $\mu_2 = 0$ implies $l = 0$ and then we have: $\mu_3 = -e(2mx + ny)y^2$, $\mu_4 = b(2mx + ny)y^3$. So due to $\mu_4 \neq 0$ the condition $\mu_3 = 0$ yields $e = 0$.

It remains to note that as $b \neq 0$ we may assume $b = 1$ due to the rescaling $(x, y) \mapsto (bx, by)$ and this leads to the systems 18b) (see Table 2).

***b*$_2$)** Assume now $d = 0$. Then for the systems (2.12) we calculate $\mu_2 = c^2n(lx^2+2mxy+ny^2)$=0. If $c \neq 0$ then $n = 0$ and we may consider $c = 1$ and $a = 0$ via a time rescaling and a translation. This leads to the systems

$$\dot{x} = x, \quad \dot{y} = b + ex + fy + lx^2 + 2mxy,$$

for which calculations yield: $\mu_3 = f(lx + 2my)x^2$, $\mu_4 = b(lx + 2my)x^3$. Therefore the conditions $\mu_3 = 0$ and $\mu_4 \neq 0$ imply $f = 0$ and assuming $b = 1$ (due to the rescaling $y \to by$) we get the canonical form 18c) from Table 2.

Assume now $c = 0$. Then for systems (2.12) with $d = 0$ we calculate: $\mu_2 = \mu_3 = 0$ and $\mu_4 = a^2(lx^2 + 2mxy + ny^2)^2 \neq 0$. So we could consider $a = 1$ and $b = 0$ due to the transformation $x_1 = x$, $y_1 = ay - bx$ and $t_1 = at$. Moreover we may assume $e, f \in \{0, 1\}$ due to a rescaling and this leads to the canonical form 18d) from Table 2.

As all the needed cases are examined Proposition 2.2 is proved ∎

2.3 The trace function

We denote

$$\sigma(\tilde{a}, x, y) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \sigma_0(\tilde{a}) + \sigma_1(\tilde{a}, x, y) \ (\equiv D_1(\tilde{a}) + D_2(\tilde{a}, x, y))$$

and remark that the polynomial $\sigma(\tilde{a}, x, y)$ is an affine comitant of systems (2.1). It is known, that if (x_i, y_i) is a singular point of a system (2.1) then $\rho_i = \sigma(x_i, y_i)$ is the trace of the linear part of the system at this point.

Applying the differential operators $(*, *)^{(k)}$ (i.e. transvectant of index k) and $\mathcal{L}$ (see (2.9)) we shall define the following polynomial function which governs the values of the traces for finite singularities of systems (2.1).

Definition 2.2. We call the *trace function* $\mathfrak{T}(w)$ the function defined as follows:

$$\mathfrak{T}(w) = \sum_{i=0}^{4} \frac{1}{(i!)^2} \left(\sigma_1^i, \ \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0) \right)^{(i)} w^{4-i} = \sum_{i=0}^{4} \mathcal{G}_i \, w^{4-i}$$

where the coefficients $\mathcal{G}_i(\tilde{a}) = \dfrac{1}{(i!)^2}(\sigma_1^i, \ \mu_i)^{(i)}$, $i = 0, 1, 2, 3, 4$ $\big(\mathcal{G}_0(\tilde{a}) \equiv \mu_0(\tilde{a})\big)$ are GL–invariants.

Using the function $\mathfrak{T}(w)$ we construct the following four affine invariants $\mathcal{T}_4$, $\mathcal{T}_3$, $\mathcal{T}_2$, $\mathcal{T}_1$:

$$\mathcal{T}_{4-1}(\tilde{a}) = \frac{1}{i!} \frac{d^i \mathfrak{T}}{dw^i}\Big|_{w=\sigma_0}, \ i = 0, 1, 2, 3 \quad \big(\mathcal{T}_4 \equiv \mathfrak{T}(\sigma_0)\big).$$

The geometric meaning of these invariants is revealed by the next theorem.

Theorem 2.2. *Consider a non-degenerate system* (2.1) *and let* $\boldsymbol{a} \in \mathbb{R}^{12}$ *be its 12-tuple of coefficients. Denote by* ρ_s *the trace of the linear part of this system at a finite singular point* M_s, $1 \leq s \leq 4$ *(real or complex, simple or multiple). Then the following relations hold, respectively:*
(i) For $\mu_0(\boldsymbol{a}) \neq 0$ *(total multiplicity 4):*

$$\begin{aligned}
\mathcal{T}_4(\boldsymbol{a}) &= \mathcal{G}_0(\boldsymbol{a})\rho_1\rho_2\rho_3\rho_4, \\
\mathcal{T}_3(\boldsymbol{a}) &= \mathcal{G}_0(\boldsymbol{a})(\rho_1\rho_2\rho_3 + \rho_1\rho_2\rho_4 + \rho_1\rho_3\rho_4 + \rho_2\rho_3\rho_4), \\
\mathcal{T}_2(\boldsymbol{a}) &= \mathcal{G}_0(\boldsymbol{a})(\rho_1\rho_2 + \rho_1\rho_3 + \rho_1\rho_4 + \rho_2\rho_3 + \rho_2\rho_4 + \rho_3\rho_4), \\
\mathcal{T}_1(\boldsymbol{a}) &= \mathcal{G}_0(\boldsymbol{a})(\rho_1 + \rho_2 + \rho_3 + \rho_4);
\end{aligned} \tag{2.13}$$

(ii) For $\mu_0(\boldsymbol{a}) = 0,\ \mu_1(\boldsymbol{a},x,y) \neq 0$ *(total multiplicity 3):*

$$\begin{array}{ll} \mathcal{T}_4(\boldsymbol{a}) = \mathcal{G}_1(\boldsymbol{a})\rho_1\rho_2\rho_3, & \mathcal{T}_3(\boldsymbol{a}) = \mathcal{G}_1(\boldsymbol{a})(\rho_1\rho_2 + \rho_1\rho_3 + \rho_2\rho_3), \\ \mathcal{T}_2(\boldsymbol{a}) = \mathcal{G}_1(\boldsymbol{a})(\rho_1 + \rho_2 + \rho_3), & \mathcal{T}_1(\boldsymbol{a}) = \mathcal{G}_1(\boldsymbol{a}); \end{array} \tag{2.14}$$

(iii) For $\mu_0(\boldsymbol{a}) = \mu_1(\boldsymbol{a},x,y) = 0,\ \mu_2(\boldsymbol{a},x,y) \neq 0$ *(total multiplicity 2):*

$$\begin{array}{ll} \mathcal{T}_4(\boldsymbol{a}) = \mathcal{G}_2(\boldsymbol{a})\rho_1\rho_2, & \mathcal{T}_3(\boldsymbol{a}) = \mathcal{G}_2(\boldsymbol{a})(\rho_1 + \rho_2), \\ \mathcal{T}_2(\boldsymbol{a}) = \mathcal{G}_2(\boldsymbol{a}), & \mathcal{T}_1(\boldsymbol{a}) = 0; \end{array} \tag{2.15}$$

(iv) For $\mu_0(\boldsymbol{a}) = \mu_1(\boldsymbol{a},x,y) = \mu_2(\boldsymbol{a},x,y) = 0,\ \mu_3(\boldsymbol{a},x,y) \neq 0$ *(one singularity):*

$$\mathcal{T}_4(\boldsymbol{a}) = \mathcal{G}_3(\boldsymbol{a})\rho_1, \quad \mathcal{T}_3(\boldsymbol{a}) = \mathcal{G}_3(\boldsymbol{a}), \quad \mathcal{T}_2(\boldsymbol{a}) = \mathcal{T}_1(\boldsymbol{a}) = 0. \tag{2.16}$$

Proof: To prove this theorem it is sufficient to evaluate the affine invariant polynomials $\mathcal{T}_4$, $\mathcal{T}_3$, $\mathcal{T}_2$ and $\mathcal{T}_1$ for each of the canonical forms given by Table 2. More exactly we have to consider for each case provided by Theorem 2.2 the following canonical forms (see Table 2 and Remark 2.5), respectively:

(i)	for $\mu_0 \neq 0$	$\Rightarrow$	systems 1), 2), 3a), 3b), 4) $-$ 8), 9a), 9b), 9c);
(ii)	for $\mu_0 = 0$, $\mu_1 \neq 0$	$\Rightarrow$	systems 10) $-$ 13);
(iii)	for $\mu_0 = \mu_1 = 0$, $\mu_2 \neq 0$	$\Rightarrow$	systems 14a), 14b), 15a), 15b), 16a), 16b);
(iv)	for $\mu_0 = \mu_1 = \mu_2 = 0$, $\mu_3 \neq 0$	$\Rightarrow$	systems 17a), 17b)

and to check the relations provided by the statement of this theorem. ∎

Corollary 2.1. *Assume that for a non-degenerate system* (2.1) *the conditions* $\mathcal{T}_4 = 0$ *and* $\mathcal{T}_3 \neq 0$ *are verified. Then this system possesses exactly one real weak singularity.*

Proof: Indeed, if this system has a multiple singularity (for example a double point) then we obtain the respective multiple trace ($\rho_1 = \rho_2$) and clearly $\rho_1 = 0$ implies $\rho_2 = 0$ and therefore $\mathcal{T}_4 = \mathcal{T}_3 = 0$.

In the case of an imaginary singular point with the respective trace ρ_1, evidently for the complex conjugate singularity we have $\rho_2 = \bar{\rho}_1$. So, again the condition $\rho_1 = 0$ gives $\rho_2 = 0$ and this leads to $\mathcal{T}_4 = \mathcal{T}_3 = 0$. ∎

2.4 Affine invariant polynomials associated to Poincaré-Lyapunov constants

In order to be able to calculate the values of the needed invariant polynomials directly for every canonical system we shall define here a family of T–comitants (see [30] for detailed definitions) expressed through C_i $(i = 0,1,2)$ and D_j $(j = 1,2)$ (see (2.2) and (2.3)):

$$\begin{aligned}
\tilde{A} = &\left(C_1, T_8 - 2T_9 + D_2^2\right)^{(2)}/144, \\
\widetilde{B} = &\Big\{16D_1\left(D_2, T_8\right)^{(1)}\left(3C_1D_1 - 2C_0D_2 + 4T_2\right) + 32C_0\left(D_2, T_9\right)^{(1)}\left(3D_1D_2 - 5T_6 + 9T_7\right) \\
&+ 2\left(D_2, T_9\right)^{(1)}\left(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32\left(C_0, T_5\right)^{(1)}\right) \\
&+ 6\left(D_2, T_7\right)^{(1)}\left[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5) + C_2(9T_4 + 96T_3)\right] \\
&+ 6\left(D_2, T_6\right)^{(1)}\left[32C_0T_9 - C_1(12T_7 + 52D_1D_2) - 32C_2D_1^2\right] + 48D_2\left(D_2, T_1\right)^{(1)}\left(2D_2^2 - T_8\right) \\
&- 32D_1T_8\left(D_2, T_2\right)^{(1)} + 9D_2^2T_4\left(T_6 - 2T_7\right) - 16D_1\left(C_2, T_8\right)^{(1)}\left(D_1^2 + 4T_3\right) \\
&+ 12D_1\left(C_1, T_8\right)^{(2)}\left(C_1D_2 - 2C_2D_1\right) + 6D_1D_2T_4\left(T_8 - 7D_2^2 - 42T_9\right) \\
&+ 12D_1\left(C_1, T_8\right)^{(1)}\left(T_7 + 2D_1D_2\right) + 96D_2^2\left[D_1\left(C_1, T_6\right)^{(1)} + D_2\left(C_0, T_6\right)^{(1)}\right] \\
&- 16D_1D_2T_3\left(2D_2^2 + 3T_8\right) - 4D_1^3D_2\left(D_2^2 + 3T_8 + 6T_9\right) + 6D_1^2D_2^2\left(7T_6 + 2T_7\right) \\
&- 252D_1D_2T_4T_9\Big\}/(2^8 3^3).
\end{aligned}$$

$$
\begin{aligned}
\widetilde{D} =&\Big[2C_0(T_8-8T_9-2D_2^2)+C_1(6T_7-T_6-(C_1,T_5)^{(1)}+6D_1(C_1D_2-T_5)-9D_1^2C_2\Big]/36,\\
\widetilde{E} =&\Big[D_1(2T_9-T_8)-3\,(C_1,T_9)^{(1)}-D_2(3T_7+D_1D_2)\Big]/72,\\
\widetilde{F} =&\big[6D_1^2(D_2^2-4T_9)+4D_1D_2(T_6+6T_7)\;+48C_0\,(D_2,T_9)^{(1)}-9D_2^2T_4+288D_1\widetilde{E}\\
&-24\left(C_2,\widetilde{D}\right)^{(2)}+120\left(D_2,\widetilde{D}\right)^{(1)}-36C_1\,(D_2,T_7)^{(1)}+8D_1\,(D_2,T_5)^{(1)}\Big]/144,\\
\widetilde{K} =&(T_8+4T_9+4D_2^2)/72 \quad\equiv\Big(p_2(x,y),q_2(x,y)\Big)^{(1)}/4,\\
\widetilde{H} =&(-T_8+8T_9+2D_2^2)/72,\\
\widetilde{M} =&T_8.
\end{aligned}
$$

These polynomials in addition with (2.2) and (2.3) will serve as bricks in constructing affine algebraic invariants for systems (2.1). Using these bricks the minimal polynomial basis of affine invariants up to degree 12, containing 42 elements A_1-A_{42}, was constructed in [15]. We shall apply here the following elements of this basis (keeping the notation from [15]):

$$
\begin{array}{ll}
A_1=\tilde{A}, & A_2=(C_2,D)^{(3)}/12,\\
A_3=\big[C_2,D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48, & A_4=(\widetilde{H},\widetilde{H})^{(2)},\\
A_5=(\widetilde{H},\widetilde{K})^{(2)}/2, & A_6=(\widetilde{E},\widetilde{H})^{(2)}/2,\\
A_7=\big[C_2,\widetilde{E})^{(2)},D_2\big)^{(1)}/8, & A_8=\big[\widetilde{D},\widetilde{H})^{(2)},D_2\big)^{(1)}/8,\\
A_9=\big[\widetilde{D},D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48, & A_{10}=\big[\widetilde{D},\widetilde{K})^{(2)},D_2\big)^{(1)}/8,\\
A_{11}=(\widetilde{F},\widetilde{K})^{(2)}/4, & A_{12}=(\widetilde{F},\widetilde{H})^{(2)}/4,\\
A_{14}=(\widetilde{B},C_2)^{(3)}/36, & A_{15}=(\widetilde{E},\widetilde{F})^{(2)}/4,\\
A_{33}=\big[\widetilde{D},D_2)^{(1)},\widetilde{F}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128, & A_{34}=\big[\widetilde{D},\widetilde{D})^{(2)},D_2\big)^{(1)},\widetilde{K}\big)^{(1)},D_2\big)^{(1)}/64.
\end{array}
$$

Finally we construct the needed affine invariants and T–comitants

$$
\begin{aligned}
\mathcal{F}_1 =&A_2,\\
\mathcal{F}_2 =&-2A_1^2A_3+2A_5(5A_8+3A_9)+A_3(A_8-3A_{10}+3A_{11}+A_{12})-\\
&-A_4(10A_8-3A_9+5A_{10}+5A_{11}+5A_{12}),\\
\mathcal{F}_3 =&-10A_1^2A_3+2A_5(A_8-A_9)-A_4(2A_8+A_9+A_{10}+A_{11}+A_{12})+\\
&+A_3(5A_8+A_{10}-A_{11}+5A_{12}),\\
\mathcal{F}_4 =&\,20A_1^2A_2-A_2(7A_8-4A_9+A_{10}+A_{11}+7A_{12})+A_1(6A_{14}-22A_{15})-4A_{33}+4A_{34},\\
\mathcal{F} =&A_7,\\
\mathcal{B} =&-(3A_8+2A_9+A_{10}+A_{11}+A_{12}),\\
\mathcal{H} =&-(A_4+2A_5),\\
\mathcal{G} =&\widetilde{M}+32\widetilde{H},
\end{aligned}
\tag{2.17}
$$

as well as the needed additional CT–comitants:

$$
\begin{aligned}
\mathcal{B}_1 =&\Big\{\big(T_7,D_2\big)^{(1)}\big[12D_1T_3+2D_1^3+9D_1T_4+36\big(T_1,D_2\big)^{(1)}\big]\\
&-2D_1\big(T_6,D_2\big)^{(1)}\big[D_1^2+12T_3\big]+D_1^2\big[D_1\big(T_8,C_1\big)^{(2)}+6\big(\big(T_6,C_1\big)^{(1)},D_2\big)^{(1)}\big]\Big\}/144,\\
\mathcal{B}_2 =&\Big\{\big(T_7,D_2\big)^{(1)}\big[8T_3\big(T_6,D_2\big)^{(1)}-D_1^2\big(T_8,C_1\big)^{(2)}-4D_1\big(\big(T_6,C_1\big)^{(1)},D_2\big)^{(1)}\big]+\\
&+\Big[\big(T_7,D_2\big)^{(1)}\Big]^2\,(8T_3-3T_4+2D_1^2)\Big\}/384,
\end{aligned}
$$

$$\mathcal{B}_3 = -D_1^2(4D_2^2 + T_8 + 4T_9) + 3D_1D_2(T_6 + 4T_7) - 24T_3(D_2^2 - T_9),$$
$$\mathcal{B}_4 = D_1(T_5 + 2D_2C_1) - 3C_2(D_1^2 + 2T_3).$$

We note that the CT–comitants above actually are GL-invariants (see [30] for detailed definitions) and not affine invariant polynomials. So everywhere in this article when we need to calculate the polynomials $\mathcal{B}_1$–$\mathcal{B}_4$ we shall apply the algorithm described by the next remark.

Remark 2.6. *Assume that we need to evaluate the invariant polynomials constructed above for a quadratic system $S(\boldsymbol{a}, x, y)$ of the form (2.1) corresponding to a point $\boldsymbol{a} \in \mathbb{R}^{12}$. Then all affine invariants and T–comitants (i.e. $\mathcal{T}_i$, $\mathcal{F}_i$ $(i = 1, 2, 3, 4)$, $\mathcal{F}$, $\mathcal{B}$, $\mathcal{H}$ and $\mathcal{G}$) will be calculated for this system directly, except the polynomials $\mathcal{B}_1$–$\mathcal{B}_4$. The last four polynomials will be calculated for any system in the orbit under the translation group action (say, $\tau(x_0, y_0) : x = X + x_0,\ y = Y + y_0$) of the system $S(\boldsymbol{a}, x, y)$, i.e. for the family of systems $S(\boldsymbol{a}(x_0, y_0), X, Y)$.*

This algorithm is needed in order to prove that the conditions given by $\mathcal{B}_1$–$\mathcal{B}_4$ do not depend of the arbitrarily chosen coordinates of the translation vector $\tau(x_0, y_0)$, in other words to prove that these conditions are affine invariant ones.

3 Proof of the Main Theorem

The *Proof* of Main Theorem is organized in two steps:

1) providing that a non-degenerate quadratic system (2.1) has at least one simple real finite singular point, we prove that the conditions given by Main Theorem are necessary and sufficient for the existence of the respective weak singularities of the corresponding types arbitrarily located on the phase plan of the system;

2) in order to complete the proof we show the incompatibility of the conditions given by Main Theorem for a non-degenerate quadratic system (2.1) which does not have any simple finite real singular point.

Observation 3.1. In what follows for simplicity we shall use the notion (cf. [39]) of *fine focus* (respectively *fine saddle*) meaning either a weak focus (respectively a weak saddle) of any order, or a center (respectively an integrable saddle).

3.1 Step 1: The necessity and sufficiency of the conditions for systems with at least one simple real finite singular point

The *statement* (a) follows immediately from Theorem 2.2 as a quadratic system in the case $\mathcal{T}_4 \neq 0$ cannot possess a singular point with zero trace.

Statement (b). Assume $\mathcal{T}_4 = 0$ and $\mathcal{T}_3 \neq 0$. Considering Corollary 2.1 in this case a quadratic system possesses exactly one real simple singular point with zero trace. Due to a translation we replace this point at the origin of coordinates and this leads to the family of systems:

$$\dot{x} = cx + dy + gx^2 + 2hxy + ky^2, \quad \dot{y} = ex - cy + lx^2 + 2mxy + ny^2 \tag{3.1}$$

with $c^2 + de \neq 0$ (as $(0, 0)$ is a simple point). Then via a straightforward computation we obtain:

$$\mathcal{T}_4 = 0 = G_0,\ \mathcal{F}_1 = -G_1/72,\ \mathcal{F}_2 = -2^{-4}3^{-3}G_2\mathcal{F},\ \mathcal{F}_3 = -2^{-4}3^{-3}G_3\mathcal{F},$$
$$\mathcal{F}_4 = 2^{-3}3^{-2}G_4\mathcal{F},\ \mathcal{T}_3 = -8G_5\mathcal{F},\ \operatorname{sign}(\mathcal{T}_2\mathcal{F}) = -\operatorname{sign}(\det \mathcal{A}),$$

where $\mathcal{T}_4$, $\mathcal{T}_3$ and $\mathcal{F}$, $\mathcal{F}_j$, $j = 1, \ldots, 4$ are the affine invariants from (2.3) and (2.17), respectively and G_j $(j = 0, \ldots, 5)$ are the GL-invariants from (2.4).

Since for systems (3.1) the condition $\mathcal{T}_3 \neq 0$ holds, from the expressions above it follows $\mathcal{F} \neq 0$. Therefore considering Lemma 2.1 the statement (b) of the Main Theorem follows obviously.

Next we shall consider step by step each of the families of systems 1), 2), 4), 8), 10), 11), 12), 14a), 14b), 17a) and 17b) (see Table 2) which possess at least one finite real simple singular point.

Remark 3.1. *In what follows for each of the families of systems mentioned above we shall assume that the conditions $\mathcal{T}_4 = \mathcal{T}_3 = 0$ are fulfilled.*

3.1.1 Family of systems with the zero-cycle $\mathcal{D}_S(P,Q) = p+q+r+s$

According to Table 2 we consider the family of systems 1). For the finite singularities of this family (see the third column of Table 2) we obtain, respectively:

$$\rho_1 = c+f, \quad \rho_2 = -c+f+2m, \quad \rho_3 = c-f+2h,$$
$$\rho_4 = c+f+2(m-c)\frac{\beta(\beta-2\gamma)}{\beta^2-4\alpha\gamma} + 2(h-f)\frac{\beta(\beta-2\alpha)}{\beta^2-4\alpha\gamma},$$

where $\alpha = eh-cm$, $\beta = cf-de$, $\gamma = dm-fh$ and $\beta^2-4\alpha\gamma \neq 0$.

On the other hand according to Theorem 2.2 the relations (2.13) are fulfilled, where $\mathcal{G}_0 = \mu_0 = \beta^2-4\alpha\gamma \neq 0$. Clearly the conditions $\mathcal{T}_4 = \mathcal{T}_3 = 0$ (see the remark above) imply the vanishing of two traces. Due to an affine transformation without loss of generality we can assume $\rho_1 = \rho_2 = 0$ that yields $f = -c$ and $m = c$. Then we obtain the systems

$$\dot{x} = cx+dy-cx^2+2hxy-dy^2, \quad \dot{y} = ex-cy-ex^2+2cxy+cy^2, \tag{3.2}$$

and for the singular points $M_1(0,0)$ and $M_2(1,0)$ we have $\Delta_1 = -c^2-de$ and $\Delta_2 = -c^2+de+2eh$, respectively. For systems (3.2) calculations yield:

$$\begin{aligned} &\mathcal{T}_4 = \mathcal{T}_3 = 0, \quad \mathcal{T}_2 = -4(c+h)^2\Delta_1\Delta_2, \quad \mathcal{F}_1 = 2(c+h)(2c^3+c^2e+de^2-2ceh),\\ &\mathcal{B} = -2e^2(c+h)^4, \quad \mathcal{H} = -2(c+h)^2(\Delta_1+\Delta_2), \quad \sigma = 2(c+h)y. \end{aligned} \tag{3.3}$$

3.1.1.1 The case $\mathcal{T}_2 \neq 0$. As for systems (3.2) we have $\rho_1 = \rho_2 = 0$ then considering (2.13) we obtain $\mathcal{T}_2 = \mathcal{G}_0\rho_3\rho_4 \neq 0$. Hence the remaining points could not have zero traces. Thus the point M_i $(i=1,2)$ is either a fine focus (respectively a fine saddle) if and only if $\Delta_i > 0$ (respectively $\Delta_i < 0$). We observe that $\operatorname{sign}\mathcal{T}_2 = -\operatorname{sign}(\Delta_1\Delta_2)$. Then if $\mathcal{T}_2 > 0$ we have $\Delta_1\Delta_2 < 0$ and this means that one singularity is a fine focus, whereas the second one is a fine saddle. If $\mathcal{T}_2 < 0$ we obtain $\Delta_1\Delta_2 > 0$ and since in this case $\operatorname{sign}(\Delta_1+\Delta_2) = -\operatorname{sign}(\mathcal{H})$ we conclude that systems (3.2) possess two fine foci if $\mathcal{H} < 0$ and two fine saddles if $\mathcal{H} < 0$.

It remains to note that according to Lemma 2.2 if $\Delta_i > 0$ (respectively $\Delta_i < 0$) the point M_i $(i=1,2)$ is a weak focus (respectively a weak saddle) of the first order if $\mathcal{F}_1 \neq 0$ and it is a center (respectively an integrable saddle) if $\mathcal{F}_1 = 0$.

We observe that according to (3.3) the relation $\mathcal{B} \leq 0$ holds. Since $\mathcal{T}_2 \neq 0$ the condition $\mathcal{B} = 0$ implies $e = 0$ and then $\Delta_1 = \Delta_2 = -c^2 < 0$, i.e. both weak singularities are fine saddles. However in this case (i.e. when $e = 0$) we obtain $\mathcal{F}_1 = 4c^3(c+h) \neq 0$. Thus we arrive to the conditions provided by the statement (c) of Main Theorem.

3.1.1.2 The case $\mathcal{T}_2 = 0$. Since the singular points $M_1(0,0)$ and $M_2(1,0)$ of systems (3.2) are simple (i.e. $\Delta_1\Delta_2 \neq 0$), according to (3.3) the condition $\mathcal{T}_2 = 0$ yields $c+h = 0$. Therefore $\sigma(a,x,y) = 0$ and systems (3.2) become Hamiltonian:

$$\dot{x} = cx+dy-cx^2-2cxy-dy^2, \quad \dot{y} = ex-cy-ex^2+2cxy+cy^2. \tag{3.4}$$

As it is known for Hamiltonian systems a simple point could be either a center or an integrable saddle. Thus, for the points M_i $(i = 1, \dots, 4)$ of systems (3.4) (see Table 2) we have $\rho_i = 0$ for every $i = 1, \dots, 4$ and calculations yield:

$$\Delta_1 = -(c^2 + de), \quad \Delta_2 = -c^2 - 2ce + de, \quad \Delta_3 = -c^2 + 2cd + de,$$
$$\Delta_4 = \Delta_1\Delta_2\Delta_3/\mu_0, \quad \mu_0 = -(\Delta_1\Delta_2 + \Delta_1\Delta_3 + \Delta_2\Delta_3).$$

Therefore we obtain that $\operatorname{sign}(\mu_0) = \operatorname{sign}(\Delta_1\Delta_2\Delta_3\Delta_4)$. Taking into account that at least one determinant Δ_i $(i = 1, \dots, 4)$ is positive and at least one is negative (as we could have neither four saddles nor four anti-saddles (sf. [11])) we conclude, that for $\mu_0 > 0$ systems (3.4) possess two centers and two integrable saddles.

Assume $\mu_0 < 0$, i.e. there exists an odd number of negative quantities among Δ_i $(i = 1, \dots, 4)$. As a quadratic systems cannot possess three centers we obtain that for $\mu_0 < 0$ systems (3.4) possess one center and three integrable saddles.

Thus as by Table 1 a system has four simple real singularities if and only if $\mathbf{D} < 0$, $\mathbf{R} > 0$ and $\mathbf{S} > 0$, we conclude that the conditions (f_1) and (f_2) of Main Theorem are verified, respectively.

3.1.2 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = p + q + r^c + s^c$

Considering Table 2 for the finite singularities $M_{1,2}(0, \pm i)$, $M_3(1,0)$ and $M_4(x_4,\, y_4)$ of systems 2) we obtain, respectively:

$$\begin{aligned} &\rho_{1,2} = -(a+g) \pm 2i(b+h), \quad \rho_3 = g - a + 2m,\\ &\rho_4 = -(a+g) + 2(g+m)\frac{\beta^2 + 4\gamma^2}{\mu_0} - 2(b+h)\frac{2\beta(\alpha+\gamma)}{\mu_0}, \end{aligned} \tag{3.5}$$

where $\alpha = gm - hl$, $\beta = bg - al$, $\gamma = bh - am$ and $\mu_0 = \beta^2 - 4\alpha\gamma \neq 0$.

According to Theorem 2.2 the relations (2.13) are fulfilled and hence, the condition $\mathcal{T}_4 = \mathcal{T}_3 = 0$ (see Remark 3.1) implies the vanishing of two traces.

3.1.2.1 The case $\mathcal{T}_2 \neq 0$. We claim that in this case the zero traces correspond to the real singularities due to the condition $\mathcal{B} \leq 0$ provided by statement (c) of Main Theorem. Indeed supposing $\rho_1 = \rho_2 = 0$ we obtain $g = -a$, $h = -b$ and then calculation yields

$$\mathcal{T}_4 = \mathcal{T}_3 = 0, \quad \mathcal{T}_2 = 4(a-m)^2\big[4(b^2 + am)^2 + a^2(b + cl)^2\big] > 0, \quad \mathcal{B} = 8a^2(a-m)^4 \geq 0.$$

So $\mathcal{B} \geq 0$ and to satisfy the conditions given by statement (c) we must have $\mathcal{B} = 0$. However according to (c_1) the condition $\mathcal{T}_2 < 0$ is necessary and this contradicts to the expression for $\mathcal{T}_2$ above. Thus our claim is proved and we assume $\rho_3 = \rho_4 = 0$. Considering (3.5) the condition $\rho_3 = 0$ yields $g = a - 2m$. This leads to the systems

$$\begin{aligned} &\dot{x} = a - 2(a-m)x + (a-2m)x^2 + 2hxy + ay^2,\\ &\dot{y} = b - (b+l)x + lx^2 + 2mxy + by^2, \end{aligned} \tag{3.6}$$

for which we calculate

$$\rho_4 = 2\big[al(b+h) - ab^2 + abh - 2a^2m + 2b^2m + 2am^2\big]\Delta_3/\mu_0 \equiv 2W\Delta_3/\mu_0. \tag{3.7}$$

Hence the condition $\rho_4 = 0$ yields $W = 0$ and as W is linear with respect to the parameter l we shall consider three cases: α_1) $a(b+h) \neq 0$; α_2) $a \neq 0$, $b + h = 0$ and α_3) $a = 0$.

$\boldsymbol{\alpha_1}$) For $a(b+h)\neq 0$ the condition $W=0$ yields

$$l=\frac{ab(b-h)+2am(a-m)-2b^2m}{a(b+h)}$$

and then for systems (3.6) we calculate:

$$\begin{aligned}
&\Delta_3=\frac{4(bh-am)(ah+bm)}{a(b+h)},\quad \Delta_4=-\frac{4\Delta_3(bh-am)^2\big[(a-m)^2+(b+h)^2\big]}{(b+h)^2\mu_0},\\
&\mathcal{F}_1=-4(ah+bm)\Delta_3,\quad \mathcal{B}=-8(a^2+b^2+bh-am)^2(ah+bm)^2/a^2,\\
&\mathcal{T}_2=-\Delta_3^{-1}\Delta_4\frac{\mu_0^2(b+h)^2}{(bh-am)^2},\quad \mathcal{H}=-(\Delta_3+\Delta_4)\frac{\mu_0^2(b+h)^2}{2\Delta_3^2(bh-am)^2}\ .
\end{aligned}\tag{3.8}$$

Therefore $\text{sign}\,(\Delta_3\Delta_4)=-\text{sign}\,(\mathcal{T}_2)$ and since $\Delta_3\neq 0$ we have $\mathcal{F}_1\neq 0$. According to Lemma 2.2 each of the points M_3 and M_4 could only be a weak singularity of the first order. Hence if $\mathcal{T}_2>0$ then $\Delta_3\Delta_4<0$ and this means that one point is a weak focus and the second one is a weak saddle, both being of the first order.

In the case $\mathcal{T}_2<0$ we obtain $\Delta_3\Delta_4>0$ and then we have $\text{sign}\,(\Delta_3+\Delta_4)=-\text{sign}\,(\mathcal{H})$. Therefore systems (3.6) possess two weak foci (respectively two weak saddles) of the first order if and only if $\mathcal{H}<0$ (respectively $\mathcal{H}>0$). It remains to note that considering (3.8) we have $\mathcal{B}\leq 0$ and we claim that the condition $\mathcal{B}=0$ holds only if we have two weak saddles of the first order. Indeed assume $\mathcal{B}=0$. As $\Delta_3\neq 0$ (i.e. $ah+bm\neq 0$) and $a\neq 0$ we obtain $m=(a^2+b^2+bh)/a$. Then we get $\Delta_3=\Delta_4=-4(a^2+b^2)^2/a^2<0$, i.e. both points are saddles and our claim is proved. So we get the respective conditions given by the statement (c) of Main Theorem.

$\boldsymbol{\alpha_2}$) Assume now that $a\neq 0$ and $b=-h$. Then we may consider $a=1$ via a time rescaling and considering (3.7) we obtain:

$$W=2(1-m)(h^2+m),\quad \mathcal{T}_2=(1-m)^2\widetilde{W}(h,l,m),$$

where $\widetilde{W}(h,l,m)$ is a polynomial. Since $\mathcal{T}_2\neq 0$ the condition $W=0$ yields $m=-h^2$ and then for systems (3.6) calculations yield:

$$\begin{aligned}
&\Delta_3=-2h(2h^3+h+l)=-\Delta_4\neq 0,\quad \mathcal{T}_2=-\Delta_3\Delta_4(h^2+1)^2/h^2,\\
&\mathcal{F}_1=-4h(h^2+1)\Delta_3,\quad \mathcal{B}=-8h^2(1+h^2)^4.
\end{aligned}$$

We observe that in this case $\Delta_3\Delta_4<0$ and as $\mathcal{F}_1\neq 0$ according to Lemma 2.2 one point is a weak focus and another one is a weak saddle, both being of the first order. On the other hand we have $\mathcal{T}_2>0$ and $\mathcal{B}<0$, i.e. the conditions (c_2) provided by Main Theorem are verified.

$\boldsymbol{\alpha_3}$) If $a=0$ then, considering (3.7), the condition $W=0$ yields $bm=0$. Since $b\neq 0$ (otherwise systems (3.6) become degenerate) we obtain $m=0$. Then for systems (3.6) calculations yield:

$$\begin{aligned}
&\Delta_3=2h(b-l),\quad \Delta_4=-2bh(b-l)/l,\quad \mu_0=4blh^2\neq 0,\quad \mathcal{B}=-2(b-cl)^2(b+h)^4,\\
&\mathcal{T}_2=-\Delta_3\Delta_4\frac{4l^2(b+h)^2}{(b-l)^2},\quad \mathcal{H}=-(\Delta_3+\Delta_4)\frac{2l^2(b+h)^2}{(b-l)^2},\quad \mathcal{F}_1=0.
\end{aligned}$$

So, the condition $\mathcal{F}_1=0$ holds and by Lemma 2.2 the point M_i $(i=1,2)$ will be a center (respectively an integrable saddle) if $\Delta_i>0$ (respectively $\Delta_i<0$). If $\mathcal{T}_2>0$ then $\Delta_3\Delta_4<0$ and therefore only one point is a center and the second one is an integrable saddle. As $\mathcal{B}<0$ this leads to the conditions (c_5) of Main Theorem.

Assume now $\mathcal{T}_2<0$. Then $\text{sign}\,(\Delta_3+\Delta_4)=-\text{sign}\,(\mathcal{H})$ and we conclude that systems (3.6) possess two centers (respectively two integrable saddles) if and only if $\mathcal{H}<0$ (respectively $\mathcal{H}>0$). In this case we get the conditions (c_6) (respectively (c_4)) given by Main Theorem.

3.1.2.2 The case $\mathcal{T}_2 = 0$. According to formulas (2.13) the conditions $\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = 0$ imply the vanishing of three traces. We claim that for a non-degenerate system having two real distinct and two complex finite singularities the traces corresponding to all four points vanish if the conditions above are verified.

Indeed, if a singular point has a respective nonzero trace then necessary it must be real. So considering (3.5) the relations $\rho_1 = \rho_2 = 0$ yield $g = -a$, $h = -b$ and then we calculate

$$\mathcal{T}_4 = \mathcal{T}_3 = 0,\ \mathcal{T}_2 = 4(a-m)^2\big[4(b^2+am)^2+a^2(b+l)^2\big],\ \mathcal{T}_1 = 4(m-a)(b+l)\big[2b(b^2+am)+a^2(b+l)\big].$$

Evidently the condition $\mathcal{T}_2 = 0$ implies $\mathcal{T}_1 = 0$ and this proves our claim.

Thus for systems (3.6) the conditions $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0$ hold and considering (3.5) we obtain $m = a$, $h = -b$ and $g = -a$. This leads to the family of Hamiltonian systems

$$\dot{x} = a - ax^2 - 2bxy + ay^2, \quad \dot{y} = b - (b+l)x + lx^2 + 2axy + by^2 \tag{3.9}$$

for the real singular points of which we calculate

$$\begin{aligned}&\Delta_3 = -2(2a^2 + b^2 - bl), \quad \Delta_4 = -\Delta_3\big[4(a^2+b^2)^2 + a^2(b+l)^2\big]/\mu_0,\\ &\mu_0 = a^2(b+l)^2 - 4(a^2+b^2)(a^2-bl).\end{aligned}$$

Therefore we have $\operatorname{sign}(\Delta_3\Delta_4) = -\operatorname{sign}(\mu_0)$. Moreover calculations yield

$$\mu_0(\Delta_3 + \Delta_4) = 2(a^2+b^2)\Delta_3^2 > 0$$

and this implies $\operatorname{sign}(\Delta_3 + \Delta_4) = \operatorname{sign}(\mu_0)$. Hence, the condition $\mu_0 > 0$ yields $\Delta_3\Delta_4 < 0$ and we obtain one center and one integrable saddle. If $\mu_0 < 0$ then we have $\Delta_3 < 0$, $\Delta_4 < 0$ and hence systems (3.9) have two integrable saddles. So considering Table 1 we conclude that systems (3.9) possess one center and one integrable saddle (respectively two integrable saddles) if the conditions $(f_5), [\alpha]$ (respectively $(f_4), [\alpha]$) provided by Main Theorem are fulfilled.

3.1.3 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 2p + q + r$

Considering Table 2 for the finite singularities $M_{1,2}(0,0)$ (double), $M_3(1,0)$ and $M_4(0,1)$ of systems 4) we obtain, respectively:

$$\rho_{1,2} = c + eu, \quad \rho_3 = -c + 2m + eu, \quad \rho_4 = c + 2h - eu. \tag{3.10}$$

According to Theorem 2.2 the relations (2.13) are fulfilled and hence, the condition $\mathcal{T}_4 = \mathcal{T}_3 = 0$ (see Remark 3.1) implies the vanishing of at least two traces among ρ_i, $i \in \{1,2,3,4\}$.

3.1.3.1 The case $\mathcal{T}_2 \neq 0$. Then only two traces vanish and we have either $\rho_{1,2} = 0$ or $\rho_3 = \rho_4 = 0$. We claim that due to the conditions provided by the statement (c) of Main Theorem the case $\rho_{1,2} = 0$ could not occur.

Indeed, supposing $\rho_{1,2} = 0$ from (3.10) we have $c = -eu$ and then calculation yields:

$$\begin{aligned}&\mu_0 = 4e^2u(h+mu)^2 \neq 0, \quad \mathcal{F} = -\rho_3\rho_4 e(h+mu)(eu^2+eu-h+mu)/2,\\ &\mathcal{T}_4 = \mathcal{T}_3 = 0, \quad \mathcal{T}_2 = \mu_0\rho_3\rho_4, \quad \mathcal{E} = -2e^2(eu^2+eu-h+mu)^4.\end{aligned}$$

Therefore, as $\mu_0\mathcal{T}_2 \neq 0$ the relation $\mathcal{F} = 0$ implies $h = eu + mu + eu^2$ and then $\mathcal{B} = 0$. Hence only the conditions (c_1) could be verified. However in this case we obtain $\mathcal{T}_2 = 16e^2u^4(m+eu)^2(e + 2m + eu)^2 > 0$ than contradicts (c_1). This proves our claim.

Thus the conditions $\rho_3 = \rho_4 = 0$ hold and considering (3.10) we obtain: $c = eu - 2h$, $m = -h$ and this leads to the systems:

$$\begin{aligned}
&\dot{x} = (eu-2h)x + u(eu-2h)y - (eu-2h)x^2 + 2hxy - u(eu-2h)y^2,\\
&\dot{y} = ex + euy - ex^2 - 2hxy - euy^2,
\end{aligned} \tag{3.11}$$

for which calculations yield:

$$\begin{aligned}
&\Delta_3 = 2h(e-2h+eu), \quad \Delta_4 = -u\Delta_3, \quad \mu_0 = u\Delta_3^2 \neq 0,\\
&\mathcal{T}_2 = 4\mu_0(eu-h)^2, \quad \mathcal{B} = -2(1+u)^2(e-2h+eu)^2(eu-h)^4,\\
&\mathcal{F}_1 = 8hu(e-2h+eu)^2(eu-h), \quad \mathcal{H} = -2(\Delta_3+\Delta_4)(eu-h)^2.
\end{aligned}$$

As $\operatorname{sign}(\mu_0) = \operatorname{sign}(u)$ we obtain $\operatorname{sign}(\Delta_3\Delta_4) = -\operatorname{sign}(\mu_0) = -\operatorname{sign}(\mathcal{T}_2)$. Moreover, since $\mathcal{T}_2 \neq 0$ we have $\mathcal{F}_1 \neq 0$ and according to Lemma 2.2 the points M_3 and M_4 are weak singularities of the first order. Therefore, if $\mathcal{T}_2 > 0$ then $\Delta_3\Delta_4 < 0$ and this means that one point is a weak focus and the second one is a weak saddle.

In the case $\mathcal{T}_2 < 0$ we obtain $\Delta_3\Delta_4 > 0$ and since $\operatorname{sign}(\Delta_3+\Delta_4) = -\operatorname{sign}(\mathcal{H})$ systems (3.11) possess two weak foci (respectively weak saddles) of the first order if and only if $\mathcal{H} < 0$ (respectively $\mathcal{H} > 0$).

It remains to note that we have $\mathcal{B} \leq 0$ and $\mathcal{B} = 0$ only if $u = -1$. And in this case we obtain $\Delta_3 = \Delta_4 = -4h^2 < 0$, i.e. both points are weak saddles of the first order. Thus we obtain the respective conditions $(c_1)-(c_3)$ provided by the statement (c) of Main Theorem.

3.1.3.2 The case $\mathcal{T}_2 = 0$. By formulas (2.13) the conditions $\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = 0$ imply the vanishing of three traces. Then clearly we necessarily have $\rho_{1,2} = 0$ (i.e. $c = -eu$) and $\rho_3\rho_4 = 0$. Then we may assume $\rho_4 = 0$ do to the transformation $(x,y) \mapsto (y,x)$ (which replaces the singular points $M_3(1,0) \leftrightarrow M_4(0,1)$) and applying the change $(h,m,e) \mapsto (m,h,-e/u^2)$ and then the additional change $u \to 1/u$. So considering (3.10) we obtain $\rho_4 = 2(h-eu) = 0$, i.e. $h = eu$. Then we get the following family of systems

$$\dot{x} = -eux - eu^2y + eux^2 + 2euxy + eu^2y^2, \quad \dot{y} = ex + euy - ex^2 + 2mxy - euy^2, \tag{3.12}$$

for which we have $\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = 0$ and

$$\begin{aligned}
&\Delta_3 = 2e(e+m)u, \quad \Delta_4 = -u\Delta_3, \quad \rho_3 = 2(m+eu), \quad \rho_4 = 0, \quad \mu_0 = u\Delta_3^2 \neq 0,\\
&\mathcal{T}_1 = \mu_0\rho_3, \quad \mathcal{H} = -\Delta_4\rho_3^2/2, \quad \mathcal{F}_1 = eu\Delta_4\rho_3, \quad \mathcal{B} = -e^2u^4\rho_3^4/8, \quad \mathcal{F} = 0.
\end{aligned}$$

3.1.3.2.1 If $\mathcal{T}_1 \neq 0$ then $\rho_3 \neq 0$ and only the point $M_4(0,1)$ could be a weak singularity for systems (3.12). We observe that $\operatorname{sign}(\Delta_4) = -\operatorname{sign}(\mathcal{H})$ and $\mathcal{F}_1 \neq 0$ due to $\mathcal{T}_1 \neq 0$. According to Lemma 2.2 and Remark 2.3 (the second point with zero trace is the double point $M_{1,2}(0,0)$) we conclude that the point M_4 is a weak saddle (respectively a weak focus) of the first order if and only if $\mathcal{H} > 0$ (respectively $\mathcal{H} < 0$). It remains to note that $\mathcal{F} = 0$ and due to $\Delta_3 \neq 0$ (i.e. $eu \neq 0$) the condition $\mathcal{B} < 0$ holds in this case. This leads respectively to the conditions (d_1) and (d_2) provided by Main Theorem.

3.1.3.2.2 Assume $\mathcal{T}_1 = 0$. Then $\rho_3 = 0$ (i.e. $m = -eu$) and systems (3.12) become Hamiltonian. For the singular points M_3 and M_4 we calculate $\Delta_3 = -2e^2u(u-1)$, $\Delta_4 = -u\Delta_3$, $\mu_0 = u\Delta_3^2$. Therefore we have

$$\operatorname{sign}(\Delta_3\Delta_4) = -\operatorname{sign}(u) = -\operatorname{sign}(\mu_0), \quad \mu_0(\Delta_3+\Delta_4) = 8e^6(1-u)^4u^4.$$

If $\mu_0 > 0$ we have $\Delta_3\Delta_4 < 0$ and we obtain one center and one integrable saddle. If $\mu_0 < 0$ then we have $\Delta_3\Delta_4 > 0$ and $(\Delta_3 + \Delta_4) < 0$. Hence $\Delta_3 < 0$, $\Delta_4 < 0$ and systems (3.12) have two integrable saddles. So considering Table 1 the conditions $(f_5), [\beta]$ and $(f_4), [\beta]$ provided by Main Theorem are verified, respectively.

3.1.4 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 3p + q$

Considering Table 2 for the finite singularities $M_{1,2,3}(0,0)$ (triple) and $M_4(1,0)$ of systems 8) we obtain, respectively:

$$\rho_{1,2,3} = c + eu, \quad \rho_4 = -c + 2m + eu.$$

Clearly the condition $\mathcal{T}_4 = \mathcal{T}_3 = 0$ (see Remark 3.1) implies the vanishing of at least two traces and hence $\rho_{1,2,3} = 0$ and then by formulas (2.13) we get $\mathcal{T}_2 = 0$. So $c = -eu$ and we calculate:

$$\Delta_4 = 2e(h + mu), \quad \rho_4 = 2(m + eu), \quad \mu_0 = (u^2 - v)\Delta_4^2 \neq 0, \quad \mathcal{T}_1 = \mu_0\rho_4, \quad \mathcal{F}_1 = 0.$$

Evidently the simple point $M_4(1,0)$ could not be a weak singularity if $\mathcal{T}_1 \neq 0$. On the other hand, as $\mathcal{F}_1 = 0$, the conditions provided by the statement (d) of Main Theorem could not be verified.

Assuming $\mathcal{T}_1 = 0$ we obtain $m = -eu$ and this leads to the following family of systems:

$$\begin{aligned} \dot{x} &= -eux - eu^2y + eux^2 + 2hxy + u(2h - ev)y^2, \\ \dot{y} &= ex + euy - ex^2 - 2euxy - e(2u^2 - v)y^2, \end{aligned} \tag{3.13}$$

for which we have $\mathcal{T}_i = 0$ $(i = 1, 2, 3, 4)$ and

$$\begin{aligned} &\Delta_4 = 2e(h - eu^2), \quad \rho_4 = 0, \quad \sigma = 2(h - 2eu^2 + ev)y, \quad \mu_0 = (u^2 - v)\Delta_4^2 \neq 0, \\ &\mathcal{H} = -2\Delta_4(h - 2eu^2 + ev)^2, \quad \mathcal{B} = -2e^2(h - 2eu^2 + ev)^4, \quad \mathcal{F}_1 = \mathcal{F} = 0. \end{aligned}$$

3.1.4.1 The case $\sigma \neq 0$. Then systems (3.13) are not Hamiltonian and we have $\mathcal{H} \neq 0$ and $\operatorname{sign}(\Delta_4) = -\operatorname{sign}(\mathcal{H})$. As $\mathcal{F}_1 = 0$ according to Lemma 2.2 and Remark 2.3 (the second point with zero trace is the triple point $M_{1,2,3}(0,0)$) we conclude that the point M_4 is an integrable saddle (respectively a center) if and only if $\mathcal{H} > 0$ (respectively $\mathcal{H} < 0$). It remains to note that due to $\sigma\Delta_3 \neq 0$ the condition $\mathcal{B} < 0$ is fulfilled in this case. This leads respectively to the conditions $(e_3), [\alpha]$ and $(e_4), [\alpha]$ provided by Main Theorem.

3.1.4.2 The case $\sigma = 0$. In this case systems (3.13) become Hamiltonian and considering the relation $h = e(2u^2 - v)$ for the simple singular point $M_4(1,0)$ we calculate:

$$\Delta_4 = 2e^2(u^2 - v), \quad \rho_4 = 0, \quad \Delta_4^3 = 2e^2\mu_0.$$

Therefore $\operatorname{sign}(\Delta_4) = \operatorname{sign}(\mu_0)$ and then the simple point of systems (3.13) will be an integrable saddle (respectively a center) if and only if $\mu_0 < 0$ (respectively $\mu_0 > 0$). So considering Table 1 we get respectively the conditions $(f_6), [\alpha]$ and (f_7) provided by Main Theorem.

3.1.5 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = p + q + r$

Considering Table 2 for the simple finite singularities $M_1(0,0)$, $M_2(1,0)$ and $M_3\Big(\frac{\beta}{2\alpha}, \frac{\beta(2\alpha-\beta)}{4\alpha\gamma}\Big)$ of systems 10) we obtain, respectively:

$$\begin{aligned} &\rho_1 = c + f, \quad \rho_3 = c + f + 2(m - c)\tfrac{\beta}{2\alpha} - 2h\tfrac{\beta(\beta-2\alpha)}{4\alpha\gamma}, \\ &\rho_2 = -c + f + 2m, \quad \mathcal{T}_1 = \mathcal{G}_1 = -8h\alpha\gamma, \quad \mu_0 = 0, \quad \mu_1 = 4\alpha\gamma x, \end{aligned} \tag{3.14}$$

where $\alpha = eh - cm$, $\beta = cf - de$, $\gamma = dm - fh$ and $\alpha\beta\gamma(\beta - 2\alpha) \neq 0$.

As for this family of systems the conditions $\mu_0 = 0$ and $\mu_1 \neq 0$ according to Theorem 2.2 (see formulas (2.14)) we shall consider two subcases: $\mathcal{T}_1 \neq 0$ and $\mathcal{T}_1 = 0$.

3.1.5.1 The case $\mathcal{T}_1 \neq 0$. Then by (3.14) we have $\mathcal{G}_1 \neq 0$ and hence the condition $\mathcal{T}_4 = \mathcal{T}_3 = 0$ (see Remark 3.1) implies the vanishing of at least two traces. Clearly that due to an affine transformation (if necessary) we may assume $\rho_1 = \rho_2 = 0$. So considering (3.14) we obtain $f = -c$ and $m = c$. This leads to the following family of systems:

$$\dot{x} = cx + dy - cx^2 + 2hxy, \quad \dot{y} = ex - cy - ex^2 + 2cxy, \tag{3.15}$$

for which we have $\mathcal{T}_4 = \mathcal{T}_3 = 0$ and

$$\begin{aligned}
&\Delta_1 = -(c^2 + de), \quad \Delta_2 = -c^2 + de + 2eh, \quad \mathcal{T}_1 = -4ch(d + h)(\Delta_1 + \Delta_2), \quad \mathcal{F} = 0,\\
&\mathcal{T}_2 = -4h^2\Delta_1\Delta_2, \quad \mathcal{F}_1 = -2ch(\Delta_1 + \Delta_2), \quad \mathcal{B} = -2e^2h^4, \quad \mathcal{H} = -2h^2(\Delta_1 + \Delta_2).
\end{aligned}$$

We observe that the condition $\Delta_1\Delta_2\mathcal{T}_1 \neq 0$ implies $\mathcal{F}_1\mathcal{H}\mathcal{T}_2 \neq 0$ and then $\operatorname{sign}(\Delta_3\Delta_4) = -\operatorname{sign}(\mathcal{T}_2)$ and $\operatorname{sign}(\Delta_3 + \Delta_4) = -\operatorname{sign}(\mathcal{H})$. Moreover, since $\mathcal{F}_1 \neq 0$ according to Lemma 2.2 the point M_1 as well as the point M_2 could only be a weak singularity of the first order. Therefore, if $\mathcal{T}_2 > 0$ then $\Delta_3\Delta_4 < 0$ and this means that one point is a weak focus and the second one is a weak saddle (both being of the first order). In the case $\mathcal{T}_2 < 0$ we obtain $\Delta_3\Delta_4 > 0$ and systems (3.15) possess two weak saddles (respectively two weak foci) of the first order if and only if $\mathcal{H} > 0$ (respectively $\mathcal{H} < 0$).

It remains to note that we have $\mathcal{B} \leq 0$ and $\mathcal{B} = 0$ if and only if $e = 0$. And in this case we obtain $\Delta_3 = \Delta_4 = -c^2 < 0$, i.e. both points are weak saddles. So the respective conditions $(c_1) - (c_3)$ provided by the statement (c) of Main Theorem are verified.

3.1.5.2 The case $\mathcal{T}_1 = 0$. Since $\mu_1 \neq 0$ considering (3.14) we obtain $h = 0$ and then for systems (3.15) we have $\mathcal{T}_i = 0$ for each $i = 1, \ldots, 4$ and (considering Remark 2.6) we calculate

$$\mathcal{B}_1 = -2cdm\,\rho_1\rho_2\rho_3, \quad \mu_1 = -4cdm^2x \neq 0.$$

Due to the condition $\mu_1 \neq 0$ in order to have at least one zero trace it is necessary and sufficient that $\mathcal{B}_1 = 0$. In this case by Remark 2.4 we may assume $\rho_1 = 0$, i.e. $f = -c$ and then we obtain the systems

$$\dot{x} = cx + dy - cx^2, \quad \dot{y} = ex - cy - ex^2 + 2mxy. \tag{3.16}$$

For these systems calculations yield:

$$\begin{aligned}
&\Delta_1 = -(c^2 + de), \quad \rho_2 = 2(m - c), \quad \rho_3 = (c - m)\Delta_1/(cm), \quad \sigma = 2(m - c)x,\\
&\mathcal{B}_2 = -4d^2(c - m)^4\Delta_1, \quad \mathcal{F}_1 = 2d(c - m)(c^2 + de + 2cm) = -G_1/72, \quad \mathcal{B}_1 = \mathcal{H} = 0,
\end{aligned}$$

where G_1 is one of the GL-invariants associated to the origin of the coordinates (see Lemma 2.1). So if $\mathcal{F}_1 \neq 0$ then $\operatorname{sign}(\Delta_1) = -\operatorname{sign}(\mathcal{B}_2)$ and $G_1 \neq 0$. As for systems (3.16) we have $G_0 = 0$ then by Lemma 2.1 in the case $\mathcal{F}_1 \neq 0$ the singular point $M_1(0, 0)$ is a weak saddle (respectively a weak focus) of the first order if and only if $\mathcal{B}_2 > 0$ (respectively $\mathcal{B}_2 < 0$). As $\mathcal{F}_1 \neq 0$ implies $\sigma \neq 0$ and for these systems $\mathcal{B}_1 = \mathcal{H} = 0$ we get respectively the conditions (e_1) and (e_2) provided by Main Theorem.

Assuming $\mathcal{F}_1 = 0$ we shall consider two subcases: $\sigma \neq 0$ and $\sigma = 0$.

3.1.5.2.1 If $\sigma \neq 0$ then $c - m \neq 0$ and this implies $\rho_2\rho_3 \neq 0$, i.e. none of the points M_2 and M_3 could be a weak singularity. We shall examine the point $M_1(0,0)$. Since the condition $\mathcal{F}_1 = 0$ gives $G_1 = 0$, for systems (3.16) we obtain $G_0 = G_1 = G_2 = G_3 = 0$ and by Lemma 2.1 this point is an integrable saddle (respectively a center) if and only if $\mathcal{B}_2 > 0$ (respectively $\mathcal{B}_2 < 0$). Thus the conditions $(e_3), [\beta]$ and $(e_4), [\beta]$ given by Main Theorem are satisfied, respectively.

3.1.5.2.2 Suppose now $\sigma = 0$. This leads to the Hamiltonian systems

$$\dot{x} = cx + dy - cx^2, \quad \dot{y} = ex - cy - ex^2 + 2cxy, \tag{3.17}$$

for which calculations yield:

$$\Delta_1 = -c^2 - de, \quad \Delta_2 = -c^2 + de, \quad \Delta_3 = \frac{1}{2c^2}\Delta_1\Delta_2.$$

We claim that among the determinants Δ_i $(i = 1,2,3)$ one and only one is positive. Indeed, we could not have $\Delta_1 > 0$ and $\Delta_2 > 0$ (otherwise we obtain the contradictory relation: $-2c^2 > 0$). Therefore we have either $\Delta_1\Delta_2 < 0$ (and then $\Delta_3 < 0$) or $\Delta_1 < 0$ and $\Delta_2 < 0$ (and then $\Delta_3 > 0$). So our claim is proved and this means that systems (3.17) possess one center and two integrable saddles.

On the other hand, considering Table 1 we observe that the conditions (f_3) provided by Main Theorem are verified.

3.1.6 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = p + q^c + r^c$

Considering Table 2 for the simple finite singularities $M_1(0,0)$ and $M_{2,3}(u \pm i, 1)$ of systems 11) we obtain, respectively:

$$\begin{aligned} &\rho_1 = l(u^2+1) + 2(h - gu), \quad \rho_{2,3} = l(u^2+1) - 2mu \pm 2i(g - m), \\ &\mathcal{T}_1 = \mathcal{G}_1 = 8h(u^2+1)(hl - gm)^2, \quad \mu_0 = 0, \quad \mu_1 = 4(u^2+1)(hl - gm)^2 x. \end{aligned} \tag{3.18}$$

As for this family of systems the conditions $\mu_0 = 0$ and $\mu_1 \neq 0$ hold, according to Theorem 2.2 (see formulas (2.14)) we shall consider two subcases: $\mathcal{T}_1 \neq 0$ and $\mathcal{T}_1 = 0$.

3.1.6.1 The case $\mathcal{T}_1 \neq 0$. Then by (3.18) we have $\mathcal{G}_1 \neq 0$ and hence the condition $\mathcal{T}_4 = \mathcal{T}_3 = 0$ (see Remark 3.1) implies the vanishing of at least two traces. Since we have only one real singularity clearly the equalities $\rho_2 = \rho_3 = 0$ have to be fulfilled. Considering (3.18) we obtain $g = m$, $l = 2mu/(u^2+1)$ and then we obtain $\rho_1 = 2h \neq 0$ due to $\mathcal{T}_1 \neq 0$. So if $\mathcal{T}_4 = \mathcal{T}_3 = 0$ and $\mathcal{T}_1 \neq 0$ then the real singular point could not be a weak singularity.

It remains to note that in this case we have

$$\mathcal{T}_3 = 16h^2m^2(2hu - m - mu^2)^2/(u^2+1) = 2h\mathcal{T}_4 \neq 0, \quad \mathcal{B} = 32h^4m^2u^2/(u^2+1)^2.$$

As $\mathcal{T}_3 > 0$ and $\mathcal{B} \geq 0$ we decide that the conditions provided by statement (c) are not verified.

3.1.6.2 The case $\mathcal{T}_1 = 0$. Since $\mu_1 \neq 0$ considering (3.18) we obtain $h = 0$ and then for systems 11) we have $\mathcal{T}_i = 0$ for each $i = 1, \ldots, 4$. On the other hand for these systems with $h = 0$ we calculate:

$$\mathcal{B}_1 = -2g^2m(1+u^2)\,\rho_1\rho_2\rho_3, \quad \mu_1 = 4g^2m^2(1+u^2)x \neq 0, \quad \sigma = l(u^2+1) - 2gu + 2(g-m)x, \quad \mathcal{H} = 0.$$

If $\mathcal{B}_1 \neq 0$ (then $\sigma \neq 0$) we have not weak singularities. On the other hand, as $\mathcal{H} = 0$ we conclude that none of the conditions provided by the statement (e) of Main Theorem is verified.

Since $\mu_1 \neq 0$ the condition $\mathcal{B}_1 = 0$ is equivalent to $\rho_1\rho_2\rho_3 = 0$, where ρ_i are given in (3.18) (setting $h = 0$). So assuming $\mathcal{B}_1 = 0$ we shall consider two subcases: $\sigma \neq 0$ and $\sigma = 0$.

3.1.6.2.1 Suppose first $\sigma \neq 0$. We claim that in this case the condition $\mathcal{B}_1 = 0$ is equivalent to $\rho_1 = 0$ (i.e. only the trace corresponding to the real point vanishes). Indeed, admit the contrary, that $\rho_2 = 0$ (then $\rho_3 = 0$). Considering (3.18) we get $g-m = l(u^2+1)-2mu = 0$ and this evidently implies $\sigma = 0$, i.e. our claim is proved.

Thus we have $\rho_1 = 0$ and this yields $l = 2gu/(u^2+1)$. So we arrive to the following family of systems:

$$\begin{aligned} \dot{x} &= -2gu\,x + g(u^2+1)y + gx^2, \\ \dot{y} &= \frac{2}{u^2+1}(m - 2gu^2 + mu^2)x + 2gu\,y + \frac{2gu}{u^2+1}\,x^2 - 2mxy, \end{aligned} \tag{3.19}$$

for which calculations yield:

$$\begin{aligned} &\Delta_1 = -2gm(1+u^2), \quad \sigma = 2(g-m)x, \quad \mathcal{H} = 0, \quad G_0 = G_2 = G_3 = 0, \\ &\mathcal{B}_2 = -4g^2(1+u^2)^2(g-m)^4\Delta_1, \quad \mathcal{F}_1 = 4gu(g-m)\Delta_1 = -G_1/72. \end{aligned}$$

We observe that the condition $\sigma\Delta_1 \neq 0$ implies $\mathcal{B}_2 \neq 0$ and then $\operatorname{sign}(\Delta_1) = -\operatorname{sign}(\mathcal{B}_2)$.

If $\mathcal{F}_1 \neq 0$ we get $G_1 \neq 0$ and since for systems (3.19) we have $G_0 = 0$, by Lemma 2.1 in this case the singular point $M_1(0,0)$ is a weak saddle (respectively a weak focus) of the first order if and only if $\mathcal{B}_2 > 0$ (respectively $\mathcal{B}_2 < 0$).

If $\mathcal{F}_1 = 0$ then for systems (3.19) we have $G_0 = G_1 = G_2 = G_3 = 0$ and by Lemma 2.1 this point is an integrable saddle (respectively a center) if and only if $\mathcal{B}_2 > 0$ (respectively $\mathcal{B}_2 < 0$). As $\mathcal{H} = \mathcal{B}_1 = 0$ we obtain that the respective conditions (e_1), (e_2), $(e_3), [\beta]$ and $(e_4), [\beta]$ provided by statement (e) of Main Theorem are verified.

3.1.6.2.2 Assume now $\sigma = 0$. Then $m = g \neq 0$ (otherwise systems (3.19) become degenerate) and $l = 2gu/(u^2+1)$. After the time rescaling $t \to t/g$ we get the following family of Hamiltonian systems:

$$\dot{x} = -2u\,x + (u^2+1)y + x^2, \quad \dot{y} = \frac{2(1-u^2)}{1+u^2}\,x + 2u\,y + \frac{2u}{u^2+1}x^2 - 2xy.$$

For the real singular point $M_1(0,0)$ of these systems we have $\Delta_1 = -2(1+u)^2 < 0$, i.e. this point is an integrable saddle. Considering Table 1 we arrive to the conditions $(f_6), [\beta]$ provided by the statement (f) of Main Theorem.

3.1.7 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 2p + q$

Considering Table 2 for the finite singularities $M_{1,2}(0,0)$ (double) and $M_3(1,0)$ of systems 12) we obtain, respectively:

$$\begin{aligned} &\rho_{1,2} = c + eu, \quad \rho_3 = -c + 2m + eu, \quad \mu_0 = 0, \\ &\mathcal{T}_1 = \mathcal{G}_1 = 8h(eh-cm)^2u, \quad \mu_1 = -4(eh-cm)^2ux \neq 0. \end{aligned} \tag{3.20}$$

As for this family of systems the conditions $\mu_0 = 0$ and $\mu_1 \neq 0$ hold (see Remark 2.5) then in accordance with Theorem 2.2 (see formulae (2.14)) we shall consider two subcases: $\mathcal{T}_1 \neq 0$ and $\mathcal{T}_1 = 0$.

3.1.7.1 The case $\mathcal{T}_1 \neq 0$. From (3.20) it follows $\mathcal{G}_1 \neq 0$ and hence the condition $\mathcal{T}_4 = \mathcal{T}_3 = 0$ (see Remark 3.1) implies the vanishing of at least two traces. Therefore the equality $\rho_{1,2} = 0$ has to be fulfilled and considering (3.20) we obtain $c = -eu$. So we get the family of systems

$$\dot{x} = -eux - eu^2y + eux^2 + 2hxy, \quad \dot{y} = ex + euy - ex^2 + 2mxy, \tag{3.21}$$

for which we calculate

$$\Delta_3 = 2e(h+mu),\quad \rho_3 = 2(m+eu),\quad \mathcal{T}_2 = 2hu\rho_3\Delta_3^2,\quad \mathcal{B} = -2e^2(mu-h+eu^2)^4,$$
$$\mathcal{T}_1 = 2hu\Delta_3^2,\quad \mathcal{F}_1 = -2eu(mu-h+eu^2)\Delta_3,\quad \mathcal{F} = -h(mu-h+eu^2)\rho_3\Delta_3/2.$$

We observe that if $\mathcal{T}_2 \neq 0$ the simple point $M_3(0,0)$ could not be a weak singularity.

On the other hand, the conditions provided by statement (c) of Main Theorem could not be satisfied. Indeed, the necessary condition $\mathcal{F} = 0$ due to $\mathcal{T}_2 \neq 0$ implies $h = u(m+eu)$ and then $\mathcal{B} = 0$. However in this case we get $\mathcal{T}_2 > 0$ and this contradicts to the conditions provided by the statement (c).

Assume $\mathcal{T}_2 = 0$. Since $\mathcal{T}_1 \neq 0$ we get $\rho_3 = 0$ (i.e. $m = -eu$) and then we have

$$\Delta_3 = 2e(h-eu^2),\quad \mathcal{T}_1 = 2hu\Delta_3^2,\quad \mathcal{F}_1 = 2ehu\Delta_3,\quad \mathcal{H} = -2h^2\Delta_3,\quad \mathcal{B} = -2e^2h^4,\quad \mathcal{F} = 0.$$

As $\mathcal{T}_1 \neq 0$ we obtain $\operatorname{sign}(\Delta_3) = -\operatorname{sign}(\mathcal{H})$ and $\mathcal{F}_1 \neq 0$. We observe that due to the relation $m = -eu$ systems (3.21) possess two real points with zero traces (one of them being double). So according to Lemma 2.2 and Remark 2.3 we conclude that the point M_3 is a weak saddle (respectively a weak focus) of the first order if and only if $\mathcal{H} > 0$ (respectively $\mathcal{H} < 0$). It remains to note that the condition $\mathcal{T}_1 \neq 0$ implies $\mathcal{B} < 0$ and this leads to the respective conditions provided by the statement (d) of Main Theorem.

3.1.7.2 The case $\mathcal{T}_1 = 0$. Since $\mu_1 \neq 0$ considering (3.20) we obtain $h = 0$ and then for systems 12) we have $\mathcal{T}_i = 0$ for each $i = 1, \dots, 4$. On the other hand for these systems with $h = 0$ we calculate:

$$\begin{gathered}\mathcal{B}_1 = cu\rho_1^2\rho_3\Delta_3,\quad \mu_1 = -4c^2m^2ux \neq 0,\quad \mathcal{H} = 0,\\ \mathcal{B}_2 = c^2u^2(c-m)^2(c-4m-3eu)\rho_1\Delta_3,\quad \sigma = \rho_1 - 2(c-m)x.\end{gathered} \tag{3.22}$$

Clearly if $\mathcal{B}_1 \neq 0$ (this implies $\sigma \neq 0$) then systems (3.21) could not have weak singularities. On the other hand, as $\mathcal{H} = 0$ we decide that none of the conditions provided by the statement (e) of Main Theorem could be verified. Assume $\mathcal{B}_1 = 0$.

3.1.7.2.1 If $\mathcal{B}_2 \neq 0$ then the condition $\mathcal{B}_1 = 0$ yields $\rho_3 = 0$, i.e. according to (3.20) we obtain $c = 2m + eu$. Then after translation of the point M_3 to the origin of coordinates we get the family of systems

$$\dot{x} = (2m+eu)(uy - x - x^2),\quad \dot{y} = -ex + (2m+eu)y - ex^2 + 2mxy, \tag{3.23}$$

for which we calculate

$$\Delta_3 = -2m(2m+eu),\quad \rho_3 = 0,\quad G_0 = G_2 = G_3 = 0,\quad \mathcal{H} = 0,\ \sigma = -2(m+eu)x,$$
$$\mathcal{B}_2 = -4u^2(m+eu)^4(2m+eu)^2\Delta_3,\quad \mathcal{F}_1 = -8mu(m+eu)(2m+eu)^2 = -G_1/72.$$

We observe that the condition $\mathcal{B}_2 \neq 0$ implies $\sigma\mathcal{F}_1 \neq 0$ (then $G_1 \neq 0$) and $\operatorname{sign}(\Delta_3) = -\operatorname{sign}(\mathcal{B}_2)$.

Therefore as for systems (3.23) we have $G_0 = 0$ and $\mathcal{F}_1 \neq 0$ then by Lemma 2.1 the singular point $M_1(0,0)$ is a weak saddle (respectively a weak focus) of the first order if and only if $\mathcal{B}_2 > 0$ (respectively $\mathcal{B}_2 < 0$). This leads to the conditions (e_1) (respectively (e_2)) provided by Main Theorem.

3.1.7.2.2 Assume now $\mathcal{B}_2 = 0$. Since $\mu_1 \neq 0$ considering (3.22) the conditions $\mathcal{B}_1 = \mathcal{B}_2 = 0$ implies $\rho_1 = c + eu = 0$. Indeed assuming $\rho_1 \neq 0$ due to $\Delta_3\mu_1 \neq 0$ we get $(c-m)(c-4m-$

$3eu) = 0 = \rho_3$. Considering (3.20) the relation $\rho_3 = 0$ gives $m = 1/2(c - eu)$ and then we obtain $(c-m)(c-4m-3eu) = -1/2\rho_1^2 = 0$, i.e.we get a contradiction.

Thus $c + eu = 0$ and we claim that in the case $\sigma \neq 0$ the simple singular point could not be a weak singularity. Indeed setting $c = -eu$ we obtain systems (3.21) with $h = 0$ and we calculate: $\sigma = 2(m + eu)x = \rho_3 x \neq 0$, i.e. our claim is proved.

On the other hand, for these systems we have

$$\mathcal{F}_1 = -4e^2mu^3(m + eu), \quad \mu_1 = -4e^2m^2u^3x, \quad \mathcal{B}_2 = 0.$$

So since the condition $\mu_1\sigma \neq 0$ implies $\mathcal{F}_1 \neq 0$, we obtain that the condition $\mathcal{B}_2 = 0$ contradicts to the conditions (e_1) and (e_2) provided by Main Theorem.

Assume now $\sigma = 0$. As for systems 12) (see Table 2) we have $\sigma = c + eu - 2(c-m)x + 2hy$, we get $h = 0$, $m = c = -eu \neq 0$ (otherwise systems 12) become degenerate). Setting $e = 1$ (due to a rescaling) this leads to the family of Hamiltonian systems

$$\dot{x} = -ux - u^2y + ux^2, \quad \dot{y} = x + uy - x^2 - 2uxy.$$

For these systems we obtain $\Delta_3 = -2u^2 < 0$ and hence the simple singular point of the systems above is an integrable saddle. Considering Table 1 we arrive to the conditions $(f_6), [\gamma]$ provided by the statement (f) of Main Theorem.

3.1.8 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = p + q$

According to Table 2 for the systems in this class there exist two canonical forms: $14a$) (when quadratic parts $p_2(x,y)$ and $q_2(x,y)$ are not proportional, i.e. $K = \mathrm{Jacob}\,(p_2,q_2) \neq 0$) and $14b$) (when p_2 and q_2 are proportional, i.e. $K = 0$) and we shall examine each one of these two forms.

3.1.8.1 Canonical systems $14a$). Considering Table 2 for the simple finite singularities $M_1(0,0)$ and $M_2(1,0)$ of these systems we obtain, respectively:

$$\begin{aligned}&\rho_1 = c + f, \quad \rho_2 = -c + f + 2fu, \quad \Delta_1 = cf - de, \quad \Delta_2 = -(2u+1)\Delta_1,\\&\mathcal{T}_2 = \mathcal{G}_2 = -4d^2u^2\Delta_1\Delta_2, \quad \mu_0 = \mu_1 = 0, \quad \mu_2 = -\Delta_1\Delta_2x^2 \neq 0, \quad K = -4u\Delta_1x^2.\end{aligned} \tag{3.24}$$

As for this family of systems the conditions $\mu_0 = \mu_1 = 0$ and $\mu_2 \neq 0$ hold (see Table 1 and Remark 2.5) then in accordance with Theorem 2.2 (see formulas (2.15)) we shall consider two subcases: $\mathcal{T}_2 \neq 0$ and $\mathcal{T}_2 = 0$.

3.1.8.1.1 The case $\mathcal{T}_2 \neq 0$. From (3.24) it follows $\mathcal{G}_2 \neq 0$ and hence considering (2.15) the condition $\mathcal{T}_4 = \mathcal{T}_3 = 0$ (see Remark 3.1) implies the vanishing of both traces, i.e. $\rho_1 = \rho_2 = 0$. Considering (3.24) the relation $\rho_1 = 0$ implies $f = -c$ and then $\rho_2 = -2c(u+1) = 0$. Since for systems $14a$) in this case we have $\mathcal{F}_1 = -2cd(u-1)\Delta_2$, we shall consider two possibilities: $\mathcal{F}_1 \neq 0$ and $\mathcal{F}_1 = 0$.

1) *The subcase* $\mathcal{F}_1 \neq 0$. Then $c \neq 0$ and the condition $\rho_2 = 0$ yields $u = -1$. This leads to the family of systems

$$\dot{x} = cx + dy - cx^2 - 2dxy, \quad \dot{y} = ex - cy - ex^2 + 2cxy, \tag{3.25}$$

for which calculations yield:

$$\Delta_1 = \Delta_2 = -(c^2 + de), \quad \mathcal{F} = 0, \quad \mathcal{H} = -4d^2\Delta_1, \quad \mathcal{T}_3 = -4d^2\Delta_1^2 < 0, \quad \mathcal{B} = -2d^4e^2 \leq 0.$$

We observe that $\operatorname{sign}(\Delta_1) = \operatorname{sign}(\Delta_2) = -\operatorname{sign}(\mathcal{H})$. Since $\mathcal{F}_1 \neq 0$, according to Lemma 2.2 we conclude, that systems (3.25) possess two weak saddles (respectively two weak foci) of the first order if and only if $\mathcal{H} > 0$ (respectively $\mathcal{H} < 0$). It remains to note that the due to $\mathcal{T}_2 < 0$, for systems (3.25) the condition $\mathcal{B} = 0$ implies $e = 0$ and then $\Delta_i < 0$ $(i = 1, 2)$, i.e. both singularities are saddles. Thus we arrive to the conditions (c_1) and (c_3) provided by Main Theorem.

2) *The subcase* $\mathcal{F}_1 = 0$. Due to the condition $\mathcal{T}_2 \neq 0$ (i.e. $d \neq 0$) we get $c(u-1) = 0$ and then the condition $\rho_2 = -2c(u+1) = 0$ yields $c = 0$. Therefore, assuming $d = 1$ due to a time rescaling, we obtain the following family of systems:

$$\dot{x} = y + 2uxy, \quad \dot{y} = ex - ex^2. \tag{3.26}$$

For these systems we have $\rho_1 = \rho_2 = 0$ and

$$\begin{aligned} &\Delta_1 = -e, \quad \Delta_2 = e(1+2u), \quad \mathcal{T}_3 = -4u^2\Delta_1\Delta_2, \quad \mathcal{F} = 0, \\ &\mathcal{H} = -2u^2(\Delta_1 + \Delta_2), \quad \mathcal{E} = -2e^2u^4 < 0. \end{aligned}$$

We observe that $\operatorname{sign}(\Delta_1\Delta_2) = -\operatorname{sign}(\mathcal{T}_3)$. If $\mathcal{T}_3 > 0$ then $\Delta_1\Delta_2 < 0$ and since $\mathcal{F}_1 = 0$ according to Lemma 2.2 systems (3.26) possess one integrable saddle and one center. If $\mathcal{T}_3 < 0$ we obtain $\Delta_1\Delta_2 > 0$ and since $\operatorname{sign}(\Delta_1 + \Delta_2) = -\operatorname{sign}(\mathcal{H})$ we conclude, that systems (3.26) possess two integrable saddles (respectively two centers) if and only if $\mathcal{H} > 0$ (respectively $\mathcal{H} < 0$). As $\mathcal{F} = 0$ and $\mathcal{B} < 0$ this leads to the respective conditions provided by statement (c) of Main Theorem.

3.1.8.1.2 The case $\mathcal{T}_2 = 0$. As $K \neq 0$ considering (3.24) we obtain $d = 0$ and this leads to the family of systems

$$\dot{x} = cx(1-x), \quad \dot{y} = ex + fy - ex^2 + 2fuxy. \tag{3.27}$$

We observe that both finite singularities $M_1(0,0)$ and $M_2(1,0)$ of these systems are located on the invariant lines $x = 0$ and $x = 1$. Therefore the systems above could not have a fine focus.

We shall examine the conditions when systems (3.27) possess weak saddles. For these systems we have $\mathcal{T}_i = 0$, $(i = 1, 2, 3, 4)$, $\mathcal{F}_1 = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B} = \mathcal{H} = \mathcal{F} = 0$ and

$$\begin{gathered} \rho_1 = c + f, \quad \rho_2 = -c + f + 2fu, \quad \Delta_1 = cf, \quad \Delta_2 = -cf(1+2u), \\ \sigma = c + f - 2(c - fu)x, \quad \mathcal{B}_3 = 72c^2\rho_1\rho_2x^2. \end{gathered}$$

Clearly if $\mathcal{B}_3 \neq 0$ systems (3.27) could not have any weak singularity. At the same time we observe that this condition implies $\sigma \neq 0$ and due to the conditions above evidently we get a contradiction with the statement (e) of Main Theorem.

Assume now $\mathcal{B}_3 = 0$. Due to $c \neq 0$ (otherwise we get degenerate systems) this condition gives $\rho_1\rho_2 = 0$ and by Remark 2.4 we may assume $\rho_1 = 0$, i.e. $f = -c$. Then for systems (3.27) in this case we have

$$G_i = 0 \ (i = 0, 1, 2, 3), \quad \rho_2 = -2c(1+u), \quad G_5 = -c^2 = \Delta_1 < 0 \quad \sigma = -2c(1+u)x.$$

1) Assume first $\sigma \neq 0$. Then $\rho_2 \neq 0$ and only one singularity of systems (3.27) could be weak. According to Lemma 2.1 we obtain that the singular point $(0,0)$ is an integrable saddle.

Thus if $\sigma \neq 0$ then systems (3.27) could have only one integrable saddle and this occurs if and only if $\mathcal{B}_3 = 0$. As for these systems the conditions $\mathcal{F}_1 = \mathcal{H} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = \mu_0 = 0$ and $K\mu_2 \neq 0$ are verified, we arrive to the conditions $(e_3), [\gamma]$ provided by Main Theorem.

2) Suppose now $\sigma = 0$. Then $c + f - 2(c - fu)x + 2duy = 0$ and we get $c + f = c - fu = du = 0$. As $K = 4(de - cf)ux^2 \neq 0$ we obtain $f = -c \neq 0$, $d = 0$ and $u = -1$. Setting $c = 1$ (due to a rescaling) we get the family of Hamiltonian systems

$$\dot{x} = x(1-x), \quad \dot{y} = ex - y - ex^2 + 2xy.$$

For the singular points $M_1(0,0)$ and $M_2(1,0)$ we have $\Delta_1 = \Delta_2 = -1 < 0$. So both singularities are integrable saddles. Considering Table 1 we arrive to the conditions $(f_4), [\gamma]$ given by the statement (f) of Main Theorem.

3.1.8.2 Canonical systems $14b$**).** For these systems possessing two real simple singular points $M_1(0,0)$ and $M_2(1,-u)$ (see Table 2) we calculate:

$$\begin{aligned} &\Delta_1 = -g + 2hu - ku^2 = -\Delta_2, \quad \mu_0 = \mu_1 = 0, \quad \mathcal{G} = -8g(gx^2 + 2hxy + ky^2), \\ &\mu_2 = -\Delta_1(gx^2 + 2hxy + ky^2) \neq 0, \quad \mathcal{T}_2 = \mathcal{G}_2 = 4g(h^2 - gk)\Delta_1. \end{aligned} \tag{3.28}$$

So in accordance to Theorem 2.2 (see formulas (2.15)) we shall consider two subcases: $\mathcal{T}_2 \neq 0$ and $\mathcal{T}_2 = 0$.

3.1.8.2.1 The case $\mathcal{T}_2 \neq 0$. Assume $\mathcal{T}_4 = \mathcal{T}_3 = 0$ (see Remark 3.1). We claim that in this case the condition $u \neq 0$ must hold for systems $14b$). Indeed, supposing $u = 0$ for this family we have $\mathcal{T}_2 = 4g^2(gk - h^2) = \mathcal{T}_3/2$ and clearly the conditions $\mathcal{T}_4 = \mathcal{T}_3 = 0$ and $\mathcal{T}_2 \neq 0$ are incompatible.

Thus $u \neq 0$ and via the transformation $(x, y) \mapsto (y - x, -uy)$ we get the following family of systems

$$\dot{x} = cx - ky + gx^2 + 2hxy + ky^2, \quad \dot{y} = x, \tag{3.29}$$

possessing two simple singularities $M_1(0,0)$ and $M_2(0,1)$. Calculations yield:

$$\rho_1 = c, \quad \rho_2 = c + 2h, \quad \Delta_1 = k = -\Delta_2, \quad \mathcal{T}_2 = 4g(h^2 - gk)\Delta_1. \tag{3.30}$$

As $\mathcal{T}_2 \neq 0$ by formulae (2.15) we have $\mathcal{G}_2 \neq 0$ and hence the condition $\mathcal{T}_4 = \mathcal{T}_3 = 0$ (see Remark 3.1) implies $\rho_1 = \rho_2 = 0$. Considering (3.30) we obtain $c = h = 0$ and then for systems (3.29) we have:

$$\mathcal{F} = \mathcal{F}_1 = 0, \quad \mathcal{T}_3 = 4g^2k^2, \quad \mathcal{B} = -2g^4k^2.$$

Since $\Delta_1\Delta_2 < 0$ and $\mathcal{F}_1 = 0$ according to Lemma 2.2 systems (3.29) in this case possess an integrable saddle and a center. As $\mathcal{T}_3 > 0$, $\mathcal{F} = \mathcal{F}_1 = 0$ and $\mathcal{B} =< 0$ we arrive to the conditions (c_5) provided by Main Theorem.

3.1.8.2.2 The case $\mathcal{T}_2 = 0$. Then $g(h^2 - gk) = 0$ and this implies $\mathcal{T}_i = 0$, $(i = 1, 2, 3, 4)$. According to (3.28) the condition $g = 0$ is equivalent to $\mathcal{G} = 0$. We have to consider two subcases: $\mathcal{G} = 0$ and $\mathcal{G} \neq 0$.

1) If $\mathcal{G} = 0$ then $g = 0$ and we get the family of systems

$$\dot{x} = -ku^2x - 2hy + 2hxy + ky^2, \quad \dot{y} = ux + y, \tag{3.31}$$

for which we have

$$\begin{aligned} &\rho_1 = 1 - ku^2, \quad \rho_2 = 1 - 2hu - ku^2, \quad \Delta_1 = u(2h - ku) = -\Delta_2, \quad \mathcal{H} = \mathcal{G} = 0, \\ &\mathcal{B}_1 = 2h\rho_1\rho_2\Delta_1, \quad \mathcal{B}_2 = 2h^3u(\rho_1 + \rho 2)\Delta_1, \quad \mathcal{F}_1 = 2h\Delta_1, \quad \sigma = \rho_1 + 2hy. \end{aligned} \tag{3.32}$$

Clearly if $\mathcal{B}_1 \neq 0$ (then $\sigma \neq 0$) systems (3.31) could not have weak singularities. At the same time as $\mathcal{H} = 0$ none of the conditions provided by the statement (e) of Main Theorem is verified.

Thus in order to have a weak focus the condition $\mathcal{B}_1 = 0$ is necessary.

***a*)** If $\mathcal{F}_1 \neq 0$ then the condition $\mathcal{B}_1 = 0$ yields $\rho_1\rho_2 = 0$ and due to Remark 2.4 we may assume $\rho_1 = 0$. As $u \neq 0$ (due to $\Delta_1 \neq 0$) we obtain $k = 1/u^2$ and for systems (3.31) we have:

$$\begin{aligned} &\Delta_1 = 2hu - 1, \quad \rho_2 = -2hu, \quad \mathcal{B}_2 = -h^2\rho_2^2\Delta_1, \quad G_0 = G_2 = G_3 = 0 = \mathcal{H}, \\ &G_1 = -144h(2hu - 1) = -72\mathcal{F}_1 \neq 0, \ \sigma = 2hy, \end{aligned}$$

i.e. $\operatorname{sign}(\Delta_1) = -\operatorname{sign}(\mathcal{B}_2)$. According to Lemma 2.1 the singular point $M_1(0,0)$ is a weak saddle (respectively a weak focus) of the first order if and only if $\mathcal{B}_2 > 0$ (respectively $\mathcal{B}_2 < 0$). As the conditions $\mathcal{H} = 0$ and $\sigma \neq 0$ hold this leads to the conditions (e_1) (respectively (e_2)) provided by Main Theorem.

b) Suppose $\mathcal{F}_1 = 0$. Considering (3.32) we obtain $h = 0$ and for systems (3.31) we have:

$$\rho_1 = \rho_2 = 1 - ku^2 = \sigma, \quad \Delta_1 = -ku^2 = -\Delta_2, \quad \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = 0, \quad \mu_2 = k^2u^2y^2 \neq 0.$$

If $\sigma \neq 0$ evidently there does not exist any weak singularity. On the other hand as the condition $K = \mathcal{G} = 0$ holds, considering conditions above we decide that no one of the conditions $(e_1)-(e_4)$ provided by Main Theorem are verified.

If $\sigma = 0$ then we arrive to the Hamiltonian systems evidently possessing one integrable saddle and one center. Considering Table 1 the conditions $(f_5), [\gamma]$ provided by Main Theorem hold in this case.

2) Assume now $\mathcal{G} \neq 0$. Then $g \neq 0$ and the condition $\mathcal{T}_2 = 0$ gives $gk - h^2 = 0$. We set a new parameter v as follows: $h = gv$ and therefore we obtain $k = gv^2$. This leads to the systems

$$\dot{x} = -g(1+u^2v^2)x - 2gvy + gx^2 + 2gvxy + gv^2y^2, \quad \dot{y} = ux + y, \tag{3.33}$$

for which considering Remark 2.6 we have:

$$\begin{aligned} &\mathcal{B}_1 = -2gv(g+guv-1)\rho_1\rho_2\Delta_1, \quad \mathcal{F}_1 = -2gv(g+guv-1)\Delta_1, \quad \mathcal{H} = 0, \\ &\mathcal{T}_i = 0, \ (i = 1,2,3,4), \quad \sigma = \rho_1 + 2gx + 2hy, \quad \rho_1 = 1 - g(1+u^2v^2). \end{aligned} \tag{3.34}$$

If $\mathcal{B}_1 \neq 0$ systems (3.33) have not weak singularities. At the same time as $\sigma \neq 0$ (due to $\mathcal{G} \neq 0$, i.e. $g \neq 0$) and $\mathcal{H} = 0$ none of the conditions provided by the statement (e) of Main Theorem is verified.

Assume $\mathcal{B}_1 = 0$.

a) If $\mathcal{F}_1 \neq 0$ then the condition $\mathcal{B}_1 = 0$ yields $\rho_1\rho_2 = 0$ and due to Remark 2.4 we may assume $\rho_1 = 0$. Considering (3.34) we get $g = 1/(1+u^2v^2)$ and then for systems (3.33) calculation yields:

$$\begin{gathered} \Delta_1 = -\frac{(1-uv)^2}{1+u^2v^2}, \quad \rho_2 = \frac{2(1-uv)}{1+u^2v^2}, \quad \mathcal{B}_2 = -\frac{u^2v^4\rho_2^8}{64\Delta_1}, \\ \mathcal{F}_1 = uv^2\rho_2^3/4, \quad G_1 = -72\mathcal{F}_1, \quad G_0 = G_2 = G_3 = 0. \end{gathered}$$

The condition $\mathcal{F}_1 \neq 0$ implies $\rho_2 G_1 \mathcal{B}_2 \neq 0$ and then $\operatorname{sign}(\mathcal{B}_2) = -\operatorname{sign}(\Delta_1)$. As $\rho_2 \neq 0$ the point M_2 could not be a weak singularity. According to Lemma 2.1 the singular point $M_1(0,0)$ is a weak saddle (respectively a weak focus) of the first order if and only if $\mathcal{B}_2 > 0$ (respectively $\mathcal{B}_2 < 0$). So we get respectively the conditions (e_1) and (e_2) provided by Main Theorem.

b) Suppose now $\mathcal{F}_1 = 0$. Considering (3.34) and $g \neq 0$ we obtain $v(g+guv-1) = 0$. Then for systems (3.33) we obtain $\mathcal{H} = \mathcal{F}_1 = \mathcal{B}_1 = \mathcal{B}_2 = 0$ and either

$$\rho_1 = 1-g, \quad \rho_2 = 1+g, \quad \Delta_1 = -g = -\Delta_2, \quad \mu_2 = g^2x^2, \quad \mathcal{B}_3 = 72\rho_1\rho_2\Delta_1^2x^2$$

if $v = 0$, or

$$\begin{gathered} \rho_1 = (1-g)(2g-1)/g, \quad \rho_2 = (1+g)(2g-1)/g, \quad \Delta_1 = -(2g-1)^2/g = -\Delta_2, \\ \mu_2 = (2g-1)^2(x+vy)^2, \quad \mathcal{B}_3 = 72\rho_1\rho_2 g^2(x+vy)^2, \end{gathered}$$

if $v \neq 0$ and $g + guv - 1 = 0$ (then $u = (1-g)/(gv)$).

Clearly if $\mathcal{B}_3 \neq 0$ systems (3.31) in both cases could not have weak singularities. On the other hand, as $\sigma \neq 0$ (due to $\mathcal{G} \neq 0$) and $\mathcal{H} = 0$ none of the conditions provided by the statement (e) of Main Theorem holds.

Assuming $\mathcal{B}_3 = 0$ we obtain $\rho_1\rho_2 = 0$. Due to Remark 2.4 we may consider $\rho_1 = 0$. So in both cases we have $g = 1$ and then we obtain

$$G_0 = G_1 = G_2 = G_3 = 0, \quad G_5 = -1 < 0.$$

So according to Lemma 2.1 in both cases the singular point $M_1(0,0)$ is an integrable saddle.

3.1.9 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = p$

According to Table 2 for the systems in this class having the simple finite singularity $M_1(0,0)$ there exist two canonical forms: $17a)$ (in the case $K = \text{Jacob}\,(p_2, q_2) \neq 0$) and $17b)$ (when $K = 0$).

3.1.9.1 Canonical systems $17a)$.

For these systems (see Table 2) we calculate:

$$\mathcal{T}_4 = -8d^3(c+f)(cf-de)^2, \quad \mathcal{T}_3 = -8d^3(cf-de)^2, \quad \mu_3 = (cf-de)^2x^3 \neq 0.$$

Considering Remark 3.1 and setting $\mathcal{T}_4 = \mathcal{T}_3 = 0$ due to $\mu_3 \neq 0$ we get $d = 0$ and then for systems $17a)$ possessing one singular point $M_1(0,0)$ we obtain

$$\rho_1 = c + f = G_0, \quad \Delta_1 = cf = G_5 \neq 0, \quad G_1 = G_2 = G_3 = 0, \quad \sigma = \rho_1 + 2(2c+f)x,$$
$$\mathcal{T}_i = 0, \ (i = 1,2,3,4), \quad \mathcal{F}_1 = \mathcal{H} = \mathcal{B}_1 = \mathcal{B}_2 = 0, \quad \mathcal{B}_3 = -288c^3\rho_1x^2.$$

If $\mathcal{B}_3 \neq 0$ then evidently the indicated singular point is not a weak one. At the same time as $\sigma \neq 0$ (otherwise we get degenerate systems) and $\mathcal{H} = \mathcal{B}_2 = 0$ clearly we get a contradiction with the statement (e) of Main Theorem.

Assuming $\mathcal{B}_3 = 0$, due to $\Delta_1 \neq 0$ we obtain $\rho_1 = 0$, i.e. $f = -c$. Then $G_5 = -c^2 < 0$ and according to Lemma 2.1 the singular point $M_1(0,0)$ is an integrable saddle. So as for systems $17a)$ the conditions $\mu_0 = 0$ and $K\mu_3 \neq 0$ hold, we arrive to the conditions $(e_3), [\gamma]$ provided by Main Theorem.

3.1.9.2 Canonical systems $17b)$.

Considering Table 2 for these systems we calculate:

$$\begin{aligned} \mathcal{T}_4 &= 8d^2(ld-2m)(ld-m)^3(f+1)(f-de), \quad \mathcal{T}_3 = 8d^2(ld-2m)(ld-m)^3(f-de), \\ \mathcal{B}_2 &= d^2\Phi(d,e,f,l,m,x_0,y_0), \quad \mu_3 = (f-de)(x+dy)^2(clx-cldy+2my), \end{aligned} \tag{3.35}$$

where $\Phi(d,e,f,l,m,x_0,y_0)$ is a polynomial in the parameters of the systems and in x_0 and y_0. Here x_0 and y_0 are the coordinates of the translation vector $\tau(x_0,y_0)$ (see Remark 2.6).

3.1.9.2.1 The case $\mathcal{B}_2 \neq 0$

Then $d \neq 0$ and applying the linear transformation $(x_1,y_1) = (x, x+dy)$ we get the family of systems

$$\dot{x} = y, \quad \dot{y} = ex + fy + 2mxy + ny^2, \tag{3.36}$$

for which we have $\mathcal{T}_4 = 8efm^3n = f\mathcal{T}_3$ and $\mu_3 = -ey^2(2mx+ny) \neq 0$. So the conditions $\mathcal{T}_4 = \mathcal{T}_3 = 0$ (see Remark 3.1) for the family (3.36) gives $mn = 0$. Then for these systems we obtain $\mathcal{T}_i = 0, \ (i = 1,2,3,4)$, $\mathcal{H} = \mathcal{F}_1 = 0$, $\rho_1 = f$, $\Delta_1 = -e$ and either

$$\mathcal{B}_1 = 4n^2\rho_1\Delta_1^2, \quad \mathcal{B}_2 = -4n^4\Delta_1^3, \quad \mu_3 = n\Delta_1y^3 \neq 0, \quad \sigma = f + 2ny \quad \text{if } m = 0, \text{ or}$$

$$\mathcal{B}_1 = 4m^2\rho_1\Delta_1,\ \mathcal{B}_2 = -4m^4\Delta_1,\ \mu_3 = 2m\Delta_1 xy^2 \neq 0,\ \sigma = f + 2mx, \quad \text{if } n = 0.$$

Clearly if $\mathcal{B}_1 \neq 0$ systems (3.36) in both cases could not have weak singularities. On the other hand, as $\sigma \neq 0$ and $\mathcal{H} = 0$ none of the conditions provided by the statement (e) of Main Theorem holds.

Assuming $\mathcal{B}_1 = 0$ due to $\mathcal{B}_2 \neq 0$ we obtain $\rho_1 = 0$, i.e. $f = 0$. We claim that in both cases the point $M_1(0,0)$ is an integrable saddle (respectively a center) if and only if $\mathcal{B}_2 > 0$ (respectively $\mathcal{B}_2 < 0$). Indeed, for systems (3.36) (with $f = 0$ and $mn = 0$) we have $G_0 = G_1 = G_2 = G_3 = 0$ and $G_5 = \Delta_1 = -e$. So according to Lemma 2.1 this point is either an integrable saddle (if $\Delta_1 < 0$), or a center (if $\Delta_1 > 0$) and as $\operatorname{sign}(\Delta_1) = \operatorname{sign}(\mathcal{B}_2)$ our claim is proved.

It remains to note that in this case $\mathcal{H} = \mathcal{F}_1 = 0$ and hence the conditions $(e_3), [\beta]$ and $(e_4), [\beta]$ provided by Main Theorem are verified respectively.

3.1.9.2.2 The case $\mathcal{B}_2 = 0$ We claim that due to the conditions $\mathcal{T}_4 = \mathcal{T}_3 = 0$ we have $d = 0$. Indeed, suppose that $d \neq 0$. As $\mu_3 \neq 0$ considering (3.35) the condition $\mathcal{T}_3 = 0$ implies $(ld - m)(ld - 2m) = 0$. Then for systems $17b)$ we have either

$$\begin{aligned} &\mathcal{B}_2 = 4d^6l^4(de - f)^3 \quad \text{and} \quad \mu_3 = l(f - de)(x + dy)^3 \ \text{ if } \ m = dl, \ \text{ or} \\ &\mathcal{B}_2 = 4d^6l^4(de - f)/4 \quad \text{and} \quad \mu_3 = l(f - de)x(x + dy)^2 \ \text{ if } \ m = dl/2. \end{aligned}$$

Clearly in both cases the condition $d\mu_3 \neq 0$ implies $\mathcal{B}_2 \neq 0$ and this completes the proof of the claim. Thus $d = 0$ and we get the family of systems

$$\dot{x} = x, \quad \dot{y} = ex + fy + lx^2 + 2mxy$$

for which we calculate

$$\begin{aligned} &\rho_1 = 1 + f = G_0, \ \ \Delta_1 = f = G_5, \ \ G_1 = G_2 = G_3 = 0, \ \ \sigma = 1 + f + 2mx, \\ &\mathcal{H} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = 0, \ \ \mathcal{B}_4 = 6\rho_1(lx + 2my)x^2, \ \ \mu_3 = fx^2(lx + 2my) \neq 0. \end{aligned}$$

Evidently to have a weak singularity the condition $\mathcal{B}_4 = 0$ is necessary. On the other hand, if $\mathcal{B}_4 \neq 0$ (then $\sigma \neq 0$) since $K = \mu_2 = 0$ we obtain a contradiction with the conditions provided by the statement (e) of Main Theorem.

So $\mathcal{B}_4 = 0$ (i.e. $f = -1$ that gives $G_0 = 0$) and this implies $G_5 < 0$. So according to Lemma 2.1 we decide that the singular point $(0,0)$ is an integrable saddle. We note that for $f = -1$ we obtain $\sigma = 2mx$. If $\sigma \neq 0$ as the conditions $K = \mu_2 = 0$ and $\mu_3 \neq 0$ are fulfilled, we get the conditions $(e_3), [\varepsilon]$ provided by Main Theorem.

In the case of $\sigma = 0$ the systems above become Hamiltonian having an integrable saddle at the origin of coordinate. Considering Table 1 we arrive to the condition $(f_6), [\delta]$ provided by Main Theorem.

As all families of systems possessing at least one simple finite real singularity are examined, this completes the proof of the Step 1.

3.2 Step 2: incompatibility of the conditions for systems without simple real singularities

According to Table 2 we shall consider step by step the families of systems corresponding to the following forms of the zero–cycle $\mathcal{D}_S(P,Q)$:

$$p^c + q^c + r^c + s^c;\ \ 2p + q^c + r^c;\ \ 2p^c + 2q^c;\ \ 4p;\ \ 3p;\ \ p^c + q^c;\ \ 2p;\ \ 0.$$

Remark 3.2. *According to Corollary 2.1 for the families above we could not have* $\mathcal{T}_4 = 0$ *and* $\mathcal{T}_3 \neq 0$*. Moreover, considering Table 1 we conclude that none of the conditions provided by the statement* (f) *of main Theorem could be verified for any of these families. In other words we need to check only the conditions provided by the statements* (c), (d) *and* (e) *of Main Theorem. So in what follows we shall assume* $\mathcal{T}_4 = \mathcal{T}_3 = 0$ *and* $\sigma \neq 0$*.*

3.2.1 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = p^c + q^c + r^c + s^c$

In Table 2 this class of systems is represented by two canonical forms: $3a)$ and $3b)$.

3.2.1.1 Canonical systems $3a)$**.** Considering Table 2 for the complex singular points $M_{1,2}(0,\pm i)$ and $M_{3,4}(u\pm i, v)$ of these systems we obtain, respectively:

$$\begin{aligned} \rho_{1,2} &= -2(gu+hv) \pm i\Big[2h + \frac{2l(u^2+1)}{v^2+1}\Big] \equiv U_1 \pm iV_1,\\ \rho_{3,4} &= 2mu + \frac{2lv(u^2+1)}{v^2+1} \pm 2i(g+m) \equiv U_2 \pm iV_2. \end{aligned} \tag{3.37}$$

As $\mu_0 \neq 0$ and $\mathcal{T}_4 = \mathcal{T}_3 = 0$ (see the remark above) by formulae (2.13) at least two traces vanish. Due to an affine transformation we may assume $\rho_{1,2} = 0$ and considering (3.37) we get $U_1 = V_1 = 0$. Since $g^2+h^2 \neq 0$ (otherwise we obtain degenerate systems) we may set a new parameter w as follows: $u = hw$, $v = -gw$. This implies $U_1 = 0$ and from $V_1 = 0$ we obtain: $l = -h(1+g^2w^2)/(1+h^2w^2)$. Then for the canonical systems 3a) we obtain:

$$\begin{aligned} \mathcal{B} &= \frac{8g^2(g+m)^2(1+h^2w^2)^2}{(1+g^2w^2)^2}, \quad \mathcal{T}_2 = 4(1+h^2w^2)(g+m)^2\mu_0,\\ \mu_0 &= \frac{4(h^2+gm+g^2h^2w^2+gh^2mw^2)^2}{(1+g^2w^2)(1+h^2w^2)}, \quad \sigma = 2(g+m)x. \end{aligned}$$

As $\mu_0\sigma \neq 0$ (see Table 1 and Remark 3.2) we obtain $\mathcal{T}_2 \neq 0$, i.e. only two traces vanish. So we have to concentrate our attention to the conditions provided by statement (c) of Main Theorem. According to the formula above we have $\mathcal{B} \geq 0$ and then we could be only in the case (c_1) when $\mathcal{B} = 0$. However as $\mu_0 > 0$ we have $\mathcal{T}_2 > 0$ and this contradicts to the condition (c_1).

3.2.1.2 Canonical systems $3b)$**.** Considering Table 2 for the complex singular points $M_{1,2}(0,\pm i)$ and $M_{3,4}(1,\pm ui)$ of these systems we obtain, respectively:

$$\rho_{1,2} = a(u^2-1) - g \pm 2ib \quad \rho_{3,4} = a(u^2-1) + g \pm 2ibu.$$

So we again could not have only one zero trace and the conditions $\mathcal{T}_4 = \mathcal{T}_3 = 0$ imply the vanishing of at least two traces. By the same reasons as above we may consider $\rho_{1,2} = 0$ and we obtain $b = 0$ and $g = a(u^2-1)$. Then calculations yield: $\mathcal{B} = 8a^6(u^2-1)^4$, $\mu_0 = a^2l^2$, $\sigma = 2a(u^2-1)x$.

Due to $\mu_0\sigma \neq 0$ we have $\mathcal{B} > 0$ and obviously we obtain a contradiction with the statement (c) of of Main Theorem.

3.2.2 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 2p + q^c + r^c$

Taking into account Table 2 for the family 5) having one double singular point $M_{1,2}(0,0)$ and two complex points $M_{3,4}(1,\pm i)$ we calculate:

$$\rho_{1,2} = cm + 2en \quad \rho_{3,4} = 2g + cm \pm i(2l + 2em - 2cn).$$

So as $\mu_0 = \mathcal{G}_0 \neq 0$ the conditions $\mathcal{T}_4 = \mathcal{T}_3 = 0$ imply the vanishing of at least two traces (which could coincide).

3.2.2.1 The case $\rho_{1,2}=0$. Then we obtain $cm+2en=0$. Since $m^2+n^2\neq 0$ (see Table 2) we could set a new parameter u as follows: $e=mu$ and $c=-2nu$. Herein for the family of systems 4) we calculate

$$\begin{gathered}\mathcal{T}_2=4u^2(gm+2ln)^2(m^2+4n^2)\widetilde{W},\quad \mathcal{B}=-2u^2(2gn-lm-m^3u-4mn^2u)^4,\\ \mathcal{F}=u(gm+2ln)(2gn-lm-m^3u-4mn^2u)\widetilde{W},\quad \mu_0=u^2(gm+2ln)^2(m^2+4n^2),\\ \mathcal{F}_1=2u^2(gm+2ln)(m^2+4n^2)(2gn-lm-m^3u-4mn^2u),\end{gathered}$$

where $\widetilde{W}=(l+2n^2u)^2+m^2(m^2+5n^2)u^2+2m(lm-gn)u+g^2$.

1) Assume $\mathcal{T}_2\neq 0$. According to Main Theorem we have to consider the conditions provided by the statement (c), in particulary the condition $\mathcal{F}=0$. Due to $\mathcal{T}_2\neq 0$ this condition gives $(2gn-lm-m^3u-4mn^2u)=0$ and then we have $\mathcal{B}=\mathcal{F}_1=0$. The last condition evidently contradicts to the statement (c) of Main Theorem.

2) Suppose now $\mathcal{T}_2=0$. Due to $\mu_0\neq 0$ we obtain $\widetilde{W}=0$ and as $\widetilde{W}$ is of the second degree with respect to the parameter l we calculate: Discriminant$[\widetilde{W},l]=-4(g-mnu)^2$. So in order to force the condition $\widetilde{W}$ we must have $g=mnu$. In this case we obtain $\widetilde{W}=(l+m^2u+2n^2u)^2=0$, i.e. $l=-u(m^2+2n^2)$. However in this case we get $\sigma=0$ and this contradicts to our assumption (see Remark 3.2).

3.2.2.2 The case $\rho_3=\rho_4=0$. Then we get $g=-cm/2$, $l=cn-em$ and calculations yield:

$$\begin{gathered}\mathcal{T}_2=c^2(em-2cn)^2(cm+2en)^2(m^2+4n^2)/4,\quad \mathcal{B}=c^2m^2(cm+2en)^4/8,\\ \mathcal{F}=0,\quad \mu_0=c^2(em-2cn)^2(m^2+4n^2)/4,\quad \sigma=(cm+2en)(1-x),\end{gathered}$$

We observe that the condition $\mu_0\sigma\neq 0$ implies $\mathcal{T}_2>0$ and as $\mathcal{B}\geq 0$ we get a contradiction with the conditions provided by the statement (c) of Main Theorem.

3.2.3 Systems with the zero–cycle $\mathcal{D}_S(P,Q)=2p+2q$

Considering Table 2 for the family 6) having two double singular point $M_{1,2}(0,0)$ and $M_{3,4}(1,0)$ we calculate: $\rho_{1,2}=c+eu\quad \rho_{3,4}=-c+eu+2ev$.

The condition $\mathcal{T}_4=\mathcal{T}_3=0$ (see Remark 3.1) implies either $\rho_1=\rho_2$, or $\rho_3=\rho_4$. Due to an affine transformation we may assume $\rho_1=\rho_2=0$, i.e. $c=-eu$ and then we obtain:

$$\begin{gathered}\mathcal{T}_2=4e^4(k+nu)^2(u+v)^2,\quad \mathcal{B}=-2e^2(eu^2+2euv-n)^4,\quad \mu_0=e^2(k+nu)^2,\\ \mathcal{F}=e^3(k+nu)(u+v)^2(eu^2+2euv-n),\quad \mathcal{T}_1=4e^3(k+nu)^2(u+v).\end{gathered}$$

So if $\mathcal{T}_2\neq 0$ then the condition $\mathcal{F}=0$ (see statement (c) of Main theorem) implies $\mathcal{B}=0$. However in this case we have $\mathcal{T}_2>0$ that contradicts to the conditions provided by the statement (c).

Assume $\mathcal{T}_2=0$. Then evidently $\mathcal{T}_1=0$ and since $\mu_0\neq 0$ we get $v=-u$. This leads to the family of systems

$$\begin{aligned}\dot{x}&=-eux-eu^2y+eux^2+2eu^2xy+ky^2\equiv p(x,y),\\ \dot{y}&=ex+euy-ex^2-2euxy+ny^2\equiv q(x,y),\end{aligned}\tag{3.38}$$

for which we have $\mathcal{T}_i=0,\ (i=1,2,3,4)$ and

$$\mathcal{H}=\mathcal{F}=0,\quad \mathcal{F}_1=-2e^2(k+nu)(n+eu^2),\quad \mathcal{B}=-2e^2(n+eu^2)^4,\quad \sigma=2(n+eu^2)y.$$

As $\sigma\neq 0$ (see Remark 3.2) we have to concentrate our attention to the statement (e) of Main Theorem. Due to $\mu_0\sigma\neq 0$ we have $\mathcal{F}_1\neq 0$ and hence, only the conditions (e_1) or (e_2) provided by

Main Theorem could be satisfied. However we claim that for this family the conditions $\mathcal{B}_1 = 0$ and $\mathcal{B}_2 \neq 0$ are incompatible. Indeed, considering Remark 2.6 for systems (3.38) calculations yield:

$$\mathcal{B}_1(\boldsymbol{a}) = 16e(k+nu)(n+eu^2)^3 y_0^2\big[2q(x_0,y_0) - (eu^2+n)y_0^2\big],$$
$$\mathcal{B}_2(\boldsymbol{a}) = 2e(n+eu^2)(1-2x_0-2uy_0)y_0^{-1}\mathcal{B}_1(\boldsymbol{a}).$$

Evidently for any translation vector $\tau(x_0,y_0)$ the condition $\mathcal{B}_1 = 0$ implies $\mathcal{B}_2 = 0$ and this proves our claim.

3.2.4 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 2p^c + 2q^c$

Considering Table 2 for the family 7) having two double complex singular points $M_{1,2}(0,i)$ and $M_{3,4}(0,-i)$ we calculate: $\mathcal{T}_4 = (al-bg)^2\big[4(b+av)^2 + a^2u^2\big]$, $\mu_0 = (al-bg)^2$.

As $\mu_0 \neq 0$ the condition $\mathcal{T}_4 = 0$ implies $au = 0$ and $b + av = 0$. Since $a^2 + b^2 \neq 0$ we obtain $u = 0$ and $b = -av$. Then we get the family of systems

$$\dot{x} = a + gx^2 + 2avxy + ay^2 \equiv p(x,y), \quad \dot{y} = -av + lx^2 - 2av^2xy - avy^2 \equiv q(x,y),$$

for which calculations yield: $\mathcal{T}_i = 0$ $(i = 1,2,3,4)$ and

$$\mathcal{H} = 0, \;\; \mu_0 = a^2(l+gv)^2 \neq 0, \;\; \mathcal{F}_1 = 8a^2(l+gv)(av^2-g), \;\; \sigma = 2(g-av^2)x \neq 0.$$

Due to $\mu_0\sigma \neq 0$ we have $\mathcal{F}_1 \neq 0$ and the conditions either (e_1) or (e_2) of Main Theorem could be satisfied. However for this family the conditions $\mathcal{B}_1 = 0$ and $\mathcal{B}_2 \neq 0$ are incompatible.

Indeed, considering Remark 2.6 calculations yield:

$$\mathcal{B}_1(\boldsymbol{a}) = -16a(l+gv)(av^2-g)^3 x_0^2\big[2p(x_0,y_0) + (av^2-g)x_0^2\big],$$
$$\mathcal{B}_2(\boldsymbol{a}) = a(av^2-g)(vx_0+y_0)x_0^{-1}\mathcal{B}_1(\boldsymbol{a}).$$

Evidently for any translation vector $\tau(x_0,y_0)$ the condition $\mathcal{B}_1 = 0$ implies $\mathcal{B}_2 = 0$. So we get a contradicts with the conditions provided by the statement (e) of Main Theorem.

3.2.5 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 4p$

This class of systems is represented in Table 2 by three canonical forms possessing the singular point $M_1(0,0)$: 9a) (semi-elementary point), 9b) (non-elementary point) and 9c) (homogeneous systems). We shall examine each one of them assuming $\mathcal{T}_4 = \mathcal{T}_3 = 0$ and $\sigma \neq 0$ (see Remark 3.2).

3.2.5.1 Canonical systems 9a)**.** Considering Table 2 for this family of systems we obtain: $\mathcal{T}_4 = 1 \neq$, i.e. we get a contradiction.

3.2.5.2 Canonical systems 9b)**.** For this family of systems we calculate

$$\mathcal{T}_i = 0 \; (i = 1,2,3,4), \quad \mathcal{H} = 4, \quad \mathcal{B} = -2, \;\; \mathcal{F} = -1, \;\; \sigma = 2(x+y) \neq 0.$$

As $\mathcal{H} > 0$ and $\mathcal{B} < 0$ by statement (e) the condition $\mathcal{F} = 0$ must hold and this contradicts to $\mathcal{F} = -1$.

3.2.5.3 Canonical systems $9c$).

For this family of homogeneous systems we have

$$
\begin{aligned}
&\mathcal{T}_i = 0 \ (i=1,2,3,4), \quad \sigma = 2(g+m)x + 2(h+n)y, \quad \mu_0 = n(4lh^2 - 4ghm + g^2 n)\\
&\mathcal{H} = 4\big[lh(h+n)^2 - ghm(h+3n) - n(hm^2 - g^2 n)\big], \quad \mathcal{B} = \mathcal{F}_1 = 0.
\end{aligned}
$$

As by assumption $\sigma \neq 0$, the statement (e) of Main Theorem has to be considered and since $\mathcal{F}_1 = \mathcal{B} = 0$ the condition $\mathcal{H} = 0$ should be verified.

So we assume $\mathcal{H} = 0$ and we claim that in this case the condition $h(h+n) \neq 0$ has to be fulfilled. Indeed if $h = 0$ then $\mathcal{H} = 4g^2n^2 = 4\mu_0 \neq 0$, whereas in the case $h = -n \neq 0$ we have $\mathcal{H} = 4(g+m)^2n^2 \neq 0$ due to the condition $\sigma = 2(g+m)x \neq 0$.

So our claim is proved and we assume $h(h+n) \neq 0$. Then the condition $\mathcal{H} = 0$ yields $l = \big[ghm(h+3n) + n(hm^2 - g^2n)\big]/\big[h(h+n)^2\big]$. Considering Remark 2.6 we calculate the invariant polynomials $\mathcal{B}_i$ $(i=1,2)$ in the examined case:

$$
\begin{aligned}
\mathcal{B}_1(\boldsymbol{a}) &= \frac{16n}{h(h+n)^2}(gh + 2hm - gn)(gx_0 + mx_0 + hy_0 + ny_0)^2\Phi(g,h,m,n,x_0,y_0),\\
\mathcal{B}_2(\boldsymbol{a}) &= \frac{16n}{h^2(h+n)^2}(gh + 2hm - gn)^2(gx_0 + mx_0 + hy_0 + ny_0)\Phi(g,h,m,n,x_0,y_0).
\end{aligned}
$$

Here $\Phi(g,h,m,n,x_0,y_0)$ is a homogeneous polynomial of degree two in x_0, y_0 and homogeneous of degree four with respect to the parameters of the systems. Evidently for any translation vector $\tau(x_0,y_0)$ the condition $\mathcal{B}_1 = 0$ implies $\mathcal{B}_2 = 0$. So there could neither be verified the conditions $(e_3),[\beta]$ nor $(e_4),[\beta]$. It remains to examine the conditions $(e_3),[\gamma] - (e_3),[\varepsilon]$. However in these three cases we have $\mu_0 = 0$ (we note that $K = 0$ implies $\mu_0 = 0$) whereas for systems $9c$) we have $\mu_0 \neq 0$. So we get a contradiction with the conditions given by the statement (e) of Main Theorem.

3.2.6 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 3p$

Considering Table 2 for the family 13) having the triple singular point $M_{1,2,3}(0,0)$ we calculate:

$$
\mu_0 = 0, \ \mu_1 = 4(lh - gm)^2x, \quad \mathcal{T}_4 = -8l^3h(lh-gm)^2 = l^3\mathcal{T}_1.
$$

3.2.6.1 Assume first $\mathcal{T}_1 \neq 0$. As by Table 1 and Remark 2.5 the condition $\mu_1 \neq 0$ holds, we obtain that the condition $\mathcal{T}_4 = 0$ (see Remark 3.2) implies $l = 0$. Then for systems 13) we have:

$$
\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = 0, \quad \mathcal{T}_1 = -8g^2hm^2, \quad \mathcal{F}_1 = 0.
$$

As $\mathcal{T}_1 \neq 0$ we have to consider the statement (d) of Main Theorem. However the condition $\mathcal{F}_1 = 0$ contradicts to this statement.

3.2.6.2 Suppose now $\mathcal{T}_1 = 0$. Then due to $\mu_1 \neq 0$ we have $h = 0$ and for these systems we calculate

$$
\mathcal{T}_i = 0 \ (i=1,2,3,4), \ \ \mu_1 = 4g^2m^2x, \ \ \mathcal{B}_1 = 2l^3g^2m, \ \ \mathcal{B}_2 = -6l^2g^3m(g+m)^2, \ \ \mathcal{F}_1 = 6lg^2m.
$$

As by assumption $\sigma \neq 0$, we have to consider the statement (e) of Main Theorem. Clearly if $\mathcal{F}_1 \neq 0$ then $\mathcal{B}_1 \neq 0$ and we get a contradiction with this statement.

Assuming $\mathcal{F}_1 = 0$ due to $\mu_1 \neq 0$ we obtain $l = 0$ and then calculation gives

$$
\mathcal{F}_1 = \mathcal{H} = \mathcal{B}_1 = \mathcal{B}_2 = 0, \quad \mathcal{B} = -2g^2(g+m)^4, \quad \sigma = 2(g+m)x.
$$

As $\mu_1\sigma \neq 0$ we have $\mathcal{B} \neq 0$ and we again obtain a contradiction with the statement (e) of Main Theorem.

3.2.7 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = p^c + q^c$

This class of systems is represented in Table 2 by two canonical forms: $15a$) and $15b$), both possessing two complex singular points $M_{1,2}(0,\pm i)$. In accordance with Remark 3.2 we shall examine each of these families assuming $\sigma \neq 0$ and $\mathcal{T}_4 = \mathcal{T}_3 = 0$.

3.2.7.1 Canonical systems $15a$)**.** Considering Table 2 for this family of systems we calculate:

$$\mathcal{T}_4 = \big[h^2u^2 + 4(b+h)^2\big]\mathcal{T}_2, \quad \mathcal{T}_2 = 4m^2(4+u^2)(bh-am)^2, \quad \mu_2 = (bh-am)^2(4+u^2)y^2. \tag{3.39}$$

3.2.7.1.1 Assume first $\mathcal{T}_2 \neq 0$. Then the condition $\mathcal{T}_4 = 0$ implies $hu = b+h = 0$. Setting $b=-h$ and considering the relation $hu=0$ for systems $15a$) we obtain:

$$\mathcal{T}_4 = \mathcal{T}_3 = 0, \quad \mathcal{T}_2 = 4m^2(h^2+am)^2(4+u^2), \quad \mathcal{B} = 8a^2m^4.$$

So the conditions $\mathcal{T}_2 > 0$ and $\mathcal{B} \geq 0$ hold and this contradicts to the statement (c) of Main Theorem.

3.2.7.1.2 Suppose now $\mathcal{T}_2 = 0$. As $\mu_2 \neq 0$ by (3.39) we get $m=0$ and then we calculate:

$$\mathcal{T}_i = 0\ (i=1,2,3,4), \quad \sigma = hu + 2(b+h)y, \quad \mu_2 = b^2h^2(4+u^2)y^2,$$
$$\mathcal{H} = \mathcal{B}_1 = \mathcal{B}_2 = 0, \quad \mathcal{B}_3 = 72b^2\big[4(b+h)^2 + h^2u^2\big]y^2$$

As $\sigma \neq 0$ we should consider the statement (e) of Main Theorem. We observe that the condition $\mu_2\sigma \neq 0$ gives $\mathcal{B}_3 \neq 0$. So as $\mathcal{H} = \mathcal{B}_1 = \mathcal{B}_2 = 0$ evidently we get a contradiction with this statement.

3.2.7.2 Canonical systems $15b$)**.** For this family of systems (see Table 2) calculation gives $\mathcal{T}_4 = 4ag(ag-h^2)(c^2+4h^2) = (c^2+4h^2)\mathcal{T}_2$.

If $\mathcal{T}_2 \neq 0$ then the condition $\mathcal{T}_4 = 0$ yields $c = h = 0$ and therefore we have $\mathcal{T}_4 = \mathcal{T}_3 = 0$, $\mathcal{T}_2 = 4a^2g^2$, $\mathcal{B} = 8a^2g^4$. Hence the condition $\mathcal{T}_2 \neq 0$ implies $\mathcal{B} > 0$ and we get a contradiction with the statement (c) of Main Theorem.

Assume now $\mathcal{T}_2 = 0$. Since $a \neq 0$ (otherwise systems $15b$) become degenerate) we obtain $g(ag-h^2) = 0$ and then we have $\mathcal{T}_i = 0$ $(i=1,2,3,4)$, $\mathcal{H} = 0$ and either

$$\begin{array}{lll} (i) & \mathcal{F}_1 = -2ah, \quad \mathcal{B}_1 = -2ah(c^2+4h^2), & \text{if } g=0, \text{ or} \\ (ii) & \mathcal{F}_1 = -2h(a+ch), \quad \mathcal{B}_1 = -2h(a+ch)(c^2+4h^2), & \text{if } g = h^2/a \neq 0. \end{array}$$

We observe that in both cases the condition $\mathcal{F}_1 \neq 0$ implies $\mathcal{B}_1 \neq 0$ and this contradicts the conditions (e_1) and (e_2) provided by the statement (e) of Main Theorem.

Assume now $\mathcal{F}_1 = 0$. In the case (i) due to $a \neq 0$ we obtain $h=0$ and then we have

$$\sigma = c \neq 0, \quad \mathcal{F}_1 = \mathcal{H} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = K = 0, \quad \mu_2 = a^2y^2, \quad \mathcal{G} = 0.$$

As $K=0$, $\mu_2 \neq 0$ and $\mathcal{G} = 0$ we get a contradiction with the conditions provided by the statement (e) of Main Theorem (see the conditions $(e_3), [\delta]$).

In the case (ii) the condition $\mathcal{F}_1 = 0$ due to $h \neq 0$ implies $a = -ch \neq 0$ and we obtain

$$\mu_2 = h^2(x-cy)^2 \neq 0, \quad \mathcal{F}_1 = \mathcal{H} = \mathcal{B}_1 = \mathcal{B}_2 = 0, \quad \mathcal{B}_3 = 72h^2(c^2+4h^2)(x-cy)^2/c^2 \neq 0$$

Clearly these conditions are in contradiction with the conditions provided by the statement (e) of Main Theorem.

3.2.8 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 2p$

According to Table 2 this class of systems is represented by two canonical forms: 16a) and 16b).

3.2.8.1 Canonical systems 16a)**.** Considering Table 2 for this family of systems we calculate:

$$\mu_0 = \mu_1 = 0,\ \mu_2 = (dl - fg)^2 x^2,\ \mathcal{T}_4 = 4d^2 f^2 (ld - fg)^2,\ \mathcal{T}_2 = 4d^2(ld - fg)^2.$$

If $\mathcal{T}_2 \neq 0$ then the condition $\mathcal{T}_4 = 0$ (see Remark 3.2) gives $f = 0$ and we calculate

$$\mathcal{T}_4 = \mathcal{T}_3 = 0, \quad \mathcal{T}_2 = 4d^4 l^2 \geq 0, \quad \mathcal{F} = -d^3 gl, \quad \mathcal{B} = -2d^2 g^4, \quad \mathcal{F}_1 = 2d^2 gl.$$

As $\mathcal{T}_2 \neq 0$ we have to consider the statement (c) of Main Theorem according to which the condition $\mathcal{F} = 0$ (i.e. $g = 0$) is necessary. However this condition implies $\mathcal{F}_1 = \mathcal{B} = 0$ that contradicts to the statement (c).

Assume $\mathcal{T}_2 = 0$. Since $\mu_2 \neq 0$ we get $d = 0$ and then calculations yield:

$$\mathcal{H} = \mathcal{F}_1 = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = 0, \quad \mathcal{B}_3 = 72 f^2 g^2 x^2, \quad \mu_2 = f^2 g^2 x^2.$$

Thus $\mathcal{B}_3 \neq 0$ due to $\mu_2 \neq 0$ and this leads to a contradiction with the statement (e) of Main Theorem.

3.2.8.2 Canonical systems 16b)**.** Taking into account Table 2 for this family of systems we calculate: $\mu_0 = \mu_1 = 0$ and

$$\mu_2 = (lx^2 + 2mxy + ny^2)U, \quad \mathcal{T}_4 = 4c^2 n(ln - m^2)U, \quad \mathcal{T}_2 = 4n(ln - m^2)U,$$

where $U = d^2 l - 2cdm + c^2 n \neq 0$. If $\mathcal{T}_2 \neq 0$ then the condition $\mathcal{T}_4 = 0$ yields $c = 0$ and calculations yield:

$$\mathcal{T}_4 = \mathcal{T}_3 = 0, \quad \mathcal{T}_2 = 4ld^2 n(ln - m^2), \quad \mathcal{F} = dmn(m^2 - ln), \quad \mathcal{B} = -2d^2 m^4.$$

Therefore due to $\mathcal{T}_2 \neq 0$ the condition $\mathcal{F} = 0$ implies $m = 0$ and then we have $\mathcal{B} = 0$ and $\mathcal{T}_2 = 4d^2 l^2 n^2 > 0$. So we get a contradiction with the conditions provided by the statement (c) of Main Theorem.

Suppose now $\mathcal{T}_2 = 0$. Since $\mu_2 \neq 0$ we get $n(ln - m^2) = 0$ and then $\mathcal{T}_i = 0$ ($i = 1,2,3,4$) and $\mathcal{G} = -8n(lx^2 + 2mxy + ny^2)$.

1) Assume first $\mathcal{G} = 0$. Then $n = 0$ and we have $\mu_2 = d(dl - 2cm)x(lx + 2my) \neq 0$ and

$$\mathcal{F}_1 = 2dm(dl - 2cm), \quad \mathcal{B}_1 = 2c^2 dm(dl - 2cm), \quad \mathcal{B}_2 = 4cd^2 m^3(2cm - dl).$$

If $\mathcal{F}_1 \neq 0$ then according to the statement (e) of Main Theorem the conditions $\mathcal{B}_1 = 0$ and $\mathcal{B}_2 \neq 0$ have to be fulfilled. However obviously the condition $\mathcal{B}_1 = 0$ implies $\mathcal{B}_2 = 0$ and we get a contradiction.

Suppose $\mathcal{F}_1 = 0$. Due to $\mu_2 \neq 0$ this gives $m = 0$ and then we have

$$\mathcal{H} = \mathcal{F}_1 = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = K = \mathcal{G} = 0, \quad \mu_2 = l^2 d^2 x^2 \neq 0, \quad \sigma = c + du \neq 0.$$

We observe that these conditions also contradict to the statement (e) (see (e_3), [δ]) of Main Theorem.

2) Suppose now $\mathcal{G} \neq 0$, i.e $n \neq 0$. We may assume $n = 1$ due to a time rescaling and then the condition $\mathcal{T}_2 = 0$ gives $l = m^2$. Therefore we obtain:

$$\mu_2 = (c - dm)^2 (mx + y)^2, \quad \mathcal{F}_1 = 2m(c - dm)^2, \quad \mathcal{B}_1 = 2c^2 m(c - dm)^2, \quad \mathcal{B}_2 = 4cm^2(c - dm)^3.$$

If $\mathcal{F}_1 \neq 0$ then according to statement (e) of Main Theorem the conditions $\mathcal{B}_1 = 0$ and $\mathcal{B}_2 \neq 0$ should be verified. However clearly the condition $\mathcal{B}_1 = 0$ implies $\mathcal{B}_2 = 0$ and hence we get a contradiction.

Assume now $\mathcal{F}_1 = 0$. Due to $\mu_2 \neq 0$ this gives $m = 0$ and then we have

$$\mathcal{H} = \mathcal{F}_1 = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = 0, \quad \mu_2 = c^2y^2 \neq 0, \quad \mathcal{B}_3 = 72c^2y^2, \quad \mathcal{G} = -8y^2.$$

So due to $\mu_2 \neq 0$ we get $\mathcal{B}_3 \neq 0$ and this leads to the contradiction with the statement (e) of Main Theorem (see the conditions $(e_3), [\delta]$).

3.2.9 Systems with the zero–cycle $\mathcal{D}_S(P,Q) = 0$

We observe that in Table 2 this class of systems is represented by four canonical forms: $18a)$– $18d)$, for which the conditions $\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0$ and $\mu_4 \neq 0$ hold. Moreover in accordance with Remark 3.2 we shall assume $\mathcal{T}_4 = \mathcal{T}_3 = 0$ and $\sigma \neq 0$.

Considering Table 2 for systems $18a)$– $18d)$ calculations yield, respectively:

$18a) \Rightarrow \quad \mathcal{T}_4 = 16h^4(hl-gm)^2, \quad \mu_4 = (hl-gm)^2x^4;$
$18b) \Rightarrow \quad \mathcal{T}_4 = -16m^4n, \quad \mu_4 = y^3(2mx+ny);$
$18c) \Rightarrow \quad \mathcal{T}_4 = 0, \quad \mu_4 = x^3(lx+2my);$
$18d) \Rightarrow \quad \mathcal{T}_4 = 16n^2(m^2-ln)^2, \quad \mu_4 = (lx^2+2mxy+ny^2)^2.$

As $\mu_4 \neq 0$ for all these systems the condition $\mathcal{T}_4 = 0$ leads to the systems for which we have: $\mathcal{T}_i = 0$ $(i = 1,2,3,4)$ and $\mathcal{H} = \mathcal{B}_2 = 0$ (we note that systems $18c)$ are already such systems).

Since for all these families of systems the conditions $\sigma \neq 0$ and $\mu_2 = \mu_3 = 0$ hold, we evidently get a contradiction with the statement (e) of Main Theorem.

Thus all possible cases were examined and hence Main Theorem is proved. ∎

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