GRADIENT FLOWS FOR NON-SMOOTH INTERACTION POTENTIALS

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Abstract. We deal with a nonlocal interaction equation describing the evolution of a particle density under the effect of a general symmetric pairwise interaction potential, not necessarily in convolution form. We describe the case of a convex (or λ-convex) potential, possibly not smooth at several points, generalizing the results of [CDFLS]. We also identify the cases in which the dynamic is still governed by the continuity equation with well-characterized nonlocal velocity field.

1. Introduction

Let us consider a distribution of particles, represented by a Borel probability measure μ on $\mathbb{R}^d$. We introduce the interaction potential $\mathbf{W} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. The value $\mathbf{W}(x, y)$ describes the interaction of two particles of unit mass at the positions x and y. The total energy of a distribution μ under the effect of the potential is given by the interaction energy functional, defined by

$$\mathcal{W}(\mu) := \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{W}(x, y)\, d(\mu \times \mu)(x, y). \tag{1.1}$$

We assume that $\mathbf{W}$ satisfies the following assumptions:

i) $\mathbf{W} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is symmetric, i.e.

$$\mathbf{W}(x, y) = \mathbf{W}(y, x) \quad \text{for every } x, y \in \mathbb{R}^d; \tag{1.2}$$

ii) $\mathbf{W}$ is a λ-convex function for some $\lambda \in \mathbb{R}$, i.e.

$$\text{the function} \quad (x, y) \mapsto \mathbf{W}(x, y) - \frac{\lambda}{2}(|x|^2 + |y|^2) \quad \text{is convex;} \tag{1.3}$$

iii) $\mathbf{W}$ satisfies the quadratic growth condition at infinity, i.e. there exists a constant $C > 0$ such that

$$\mathbf{W}(x, y) \le C(1 + |x|^2 + |y|^2). \tag{1.4}$$

We are interested in the evolution problem given by the continuity equation

$$\partial_t \mu(t) + \operatorname{div}(\boldsymbol{v}(t)\mu(t)) = 0, \qquad \mu(0) = \mu_0, \tag{1.5}$$

describing the dynamics of the particle density, whose total mass is conserved, under the mutual attraction-repulsion force given by (1.1). The velocity vector field $\boldsymbol{v}$ enjoys a nonlocal dependence on μ. In the basic model represented by a C^1 potential $\mathbf{W}$ which depends only on the difference of its variables, so that we may write $\mathbf{W}(x, y) = W(x - y)$, it is given by convolution:

$$\boldsymbol{v}(t) = \nabla W * \mu(t). \tag{1.6}$$

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Under the assumptions (1.2), (1.3), (1.4), in general the function W is not differentiable but only subdifferentiable, therefore it is reasonable to consider a velocity field of the form

$$\boldsymbol{v}(t) = \boldsymbol{\eta}(t) * \mu(t), \tag{1.7}$$

where $\boldsymbol{\eta}$ represents a Borel measurable selection in the subdifferential of W, and we will write $\boldsymbol{\eta} \in \partial W$. In general, such selection is not independent of t. We stress that $x \mapsto \boldsymbol{\eta}(x)$ needs to be a pointwise defined object, since the solutions we consider are probability measures, and since this model typically presents concentration phenomena when starting with absolutely continuous initial data.

In this paper, we are going to analyse equations of the form (1.5)-(1.7) as the gradient flow of the interaction energy (1.1) in the space of Borel probability measures with finite second moment, endowed with the metric-differential structure induced by the so-called Wasserstein distance. This interpretation coming from the optimal transport theory was introduced in [O] for nonlinear diffusion equations and generalized for a large class of functionals including potential, interaction, and internal energy by different authors [CMV, AGS, CMV2], see [V, V2] for related information.

The gradient flow interpretation allows to construct solutions by means of variational schemes based on the euclidean optimal transport distance as originally introduced in [JKO] for the linear Fokker-Planck equation. The convergence of these variational schemes for general functionals was detailed in [AGS]. The results in this monograph, which are quickly summarized in Section 2, apply to the interaction equation (1.5)-(1.6), with a C^1 smooth potential verifying the convexity assumption (1.3) and a growth condition at infinity stricter than (1.4).

On the other hand, these equations have appeared in the literature as simple models of inelastic interactions [MY, BCP, BCCP, T] in which the asymptotic behavior of the equations is given by a total concentration towards a unique Delta Dirac point measure. The typical potential in these models was a power law, $\mathbf{W}(x, y) = |x - y|^\alpha$, $\alpha \geq 0$. Moreover, it was noticed in [LT] that the convergence towards this unique steady state was in finite time for certain range of exponents in the one dimensional case.

Also these equations appear in very simplified swarming or population dynamics models for collective motion of individuals, see [MEBS, BL, BCL, KSUB, BCLR] and the references therein. The interaction potential models the long-range attraction and the short-range repulsion typical in animal groups. In case the potential is fully attractive, equation (1.5) is usually referred as the aggregation equation. For the aggregation equation, finite time blow-up results for weak-L^p solutions, unique up to the blow-up time, have been obtained in the literature [BCL, BLR, CR]. In fact, those results conjectured that solutions tend to concentrate and form Dirac Deltas in finite time under suitable conditions on the interaction potential. On the other hand, the confinement of particles is shown to happen for short-range repulsive long-range attractive potentials under certain conditions [CDFLS2]. Some singular stationary states such as uniform densities on spheres have been identified as stable/unstable for radial perturbations in [BCLR] with sharp conditions on the potential. Finally, in the one dimensional case, stationary states formed by finite number of particles and smooth stationary profiles are found whose stability have been studied in [FR1, FR2] in a suitable sense.

A global-in-time well-posedness theory of measure weak solutions have been developed in [CDFLS] for interaction potentials of the form $\mathbf{W}(x, y) = W(x - y)$ satisfying the assumptions (1.2),(1.3), (1.4), and additionally being C^1-smooth except possibly at the origin. The convexity condition (1.3) restricts the possible singularities of the potential at the origin since it implies that W is Lipschitz, and therefore the possible singularity cannot be worse than $|x|$ locally at the origin. Nevertheless, for a class of potentials in which the local behavior at the origin is like $|x|^\alpha$, $1 \leq \alpha < 2$, the solutions converge towards a Delta Dirac with the full mass at the center of mass of the solution. The condition for blow-up is more general and related to the Osgood criterium for uniqueness of ODEs [BCL, CDFLS, BLR]. Note that the center of mass of the solution is preserved, at least formally, due to the symmetry assumption (1.2).

In this work, we push the ideas started in [CDFLS] further in the direction of giving conditions on the interaction potential to have a global-in-time well-posedness theory of measure solutions. The solutions constructed in Section 2 will be *gradient flow solutions*, as in [AGS], built via the variational schemes based on the optimal transport Wasserstein distance. The crucial point for the analysis in this framework is the identification of the velocity field in the continuity equation satisfied by the limiting curve of measures from the approximating variational scheme. In order to identify it, we need to characterize the sub-differential of the functional defined in (1.1) with respect to the differential structure induced by the Wasserstein metric. The Wasserstein sub-differential of the functional $\mathcal{W}$, which is rigorously introduced in Section 2, is defined through variations along transport maps. It turns out that that the element of minimal norm in this sub-differential, which will be denoted by $\partial^o\mathcal{W}(\cdot)$, is the element that governs the dynamics. Actually, it gives the velocity field via the relation $\boldsymbol{v}(t) = -\partial^o\mathcal{W}(\mu(t))$ for a.e. $t \in (0,\infty)$, which corresponds to the notion of *gradient flow solution*. This notion will be discussed in Section 2, where we will give the precise definition and recall from [AGS, Chapter 11] the main properties, such as the semigroup generation.

In Section 3, we give a characterization of the subdifferential in the general case of the interaction potential $\mathbf{W}(x,y)$ satisfying only the basic assumptions (1.2),(1.3), and (1.4). However, the element of minimal norm in the subdifferential is not fully identified and cannot be universally characterized. Nevertheless, the global well-posedness of the evolution semigroup in measures is obtained.

A distinguished role will be played by the case of a kernel function $\mathbf{W}(x,y)$ which depends only on the difference $x-y$ of its arguments. Hence we will often consider one of the following additional assumptions.

iv) There exists $W : \mathbb{R}^d \to \mathbb{R}$ such that

$$\mathbf{W}(x,y) = W(x-y). \tag{1.8}$$

v) There exists $w : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbf{W}(x,y) = W(x-y) = w(|x-y|). \tag{1.9}$$

The radial hypothesis is frequently made in models, and corresponds to an interaction between particles which depends only on their mutual distance vector. In case $\mathbf{W}(x,y)$ is also radial and convex, we can fully generalize the identification of the element of minimal norm in the subdifferential of the interaction energy done in [CDFLS], regardless of the number of nondifferentiability points of W. We complement our results with explicit examples showing the sharpness of these characterizations.

Before to state the results and in order to fix notations we recall the characterization of subdifferential for λ-convex functions. Given a λ-convex function $V : \mathbb{R}^k \to \mathbb{R}$, a vector $\boldsymbol{\xi}$ belongs to the subdifferential of V at the point $x \in \mathbb{R}^k$ if and only if

$$V(z) - V(x) \geq \langle \boldsymbol{\xi}, z-x\rangle + \frac{\lambda}{2}|z-x|^2 \qquad \forall z \in \mathbb{R}^k, \tag{1.10}$$

and we write $\boldsymbol{\xi} \in \partial V(x)$. In this case, for every $x \in \mathbb{R}^k$, we have that $\partial V(x)$ is a not empty closed convex subset of $\mathbb{R}^k$. We denote by $\partial^o V(x)$ the unique element of minimal euclidean norm in $\partial V(x)$.

The main results. Let us give a brief summary of the results contained in this paper. The main theorem deals with radial-convex potentials and reads as follows.

Let $\mathbf{W}$ *satisfy the three basic assumptions above:* (1.2)*,* (1.3)*, and* (1.4)*. If in addition* $\boldsymbol{W}$ *satisfies* (1.8)*,* (1.9) *and is convex (that is,* $\lambda \geq 0$ *in* (1.3)*), then there exists a unique* gradient flow solution *to the equation*

$$\partial_t \mu(t) - \operatorname{div}\left((\partial^o W * \mu(t))\mu(t)\right) = 0. \tag{1.11}$$

This solution is the gradient flow of the energy $\mathcal{W}$, in the sense that the velocity field in (1.11) satisfies

$$\partial^o W * \mu(t) = \partial^o \mathcal{W}(\mu(t)).$$

On the other hand, when omitting the radial hypothesis (1.9), or when letting the potential be λ-convex but not convex, we show that the evolution of the system under the effect of the potential, that is the gradient flow of $\mathcal{W}$, is characterized by (1.5)-(1.7), where $\boldsymbol{\eta}(t) \in$ is a Borel anti-symmetric selection in ∂W for almost every t. The corresponding rigorous statement is found in Section 2.

About this last result, let us remark that the velocity vector field is still written in terms of a suitable selection $\boldsymbol{\eta}$ in the local subdifferential of W, but such selection $\boldsymbol{\eta}$ is not in general the minimal one in ∂W, and it is not a priori independent of t. By this characterization we also recover the result of [CDFLS], where the only non smoothness point for W is the origin: in such case, for any t we are left with $\boldsymbol{\eta}(t)(x) = \nabla W(x)$ for $x \neq 0$ and $\boldsymbol{\eta}(t)(0) = 0$ for $x = 0$, by anti-symmetry. We stress that, due to the nonlocal structure of the problem, the task of identifying the velocity vector field becomes much more involved when W has several non smoothness points, even if it is λ-convex. Later in Section 4, we will analyse some particular examples, showing that in general it is not possible to write the velocity field in terms of a single selection in ∂W.

Finally, when omitting also the assumption (1.8), we break the convolution structure: in this more general case we show that the velocity is given in terms of an element of $\partial_1 \mathbf{W}$, or equivalently of $\partial_2 \mathbf{W}$ by symmetry, where $\partial_1 \mathbf{W}$ and $\partial_2 \mathbf{W}$ denotes the partial subdifferentials of $\mathbf{W}$ with respect to the first d variables or the last d variables respectively,

$$\boldsymbol{v}(t)(x) = \int_{\mathbb{R}^d} \boldsymbol{\eta}(t)(x,y)\, d\mu(t)(y),$$

where $\boldsymbol{\eta}(t) \in \partial_1 \mathbf{W}$ for almost any t. An additional joint subdifferential condition is also be present in this case, for the rigorous statement we still address to the next section.

Pointwise particle model and asymptotic behavior. In the model case of a system of N point particles, discussed in Section 5, the dynamics are governed by a system of ordinary differential equations. In this case equation (1.5)-(1.7) corresponds to

$$\frac{dx_i}{dt} = \sum_{j=1}^{N} m_j\, \boldsymbol{\eta}(t)(x_j - x_i), \qquad i = 1, \ldots, N,$$

where $x_i(t)$ is the position of the i-th particle and m_i is its mass. It is shown in [CDFLS] that if the attractive strength of the potential is sufficiently high, all the particles collapse to the center of mass in finite time. We will remark how this result is still working under our hypotheses under the same Osgood criterium as in [BCL, CDFLS] for fully attractive potentials. For non-convex non-smooth repulsive-attractive potentials, albeit λ-convex, the analysis leads to non-trivial sets of stationary states with singularities that cannot be treated by the theory in [CDFLS]. Our analysis shows that a very wide range of asymptotic states is indeed possible, we give different explicit examples.

Plan of the paper. In the following Section 2 we introduce the optimal transport framework and the basic properties of our energy functional, in particular the subdifferentiability and the λ-convexity along geodesics. We briefly explain what is a gradient flow in the metric space $\mathscr{P}_2(\mathbb{R}^d)$ and we introduce the notion of gradient flow solution. We present the general well-posedness result of [AGS] and show how it will apply to our interaction models, stating our main results. In Section 3, we make a fine analysis on the Wasserstein subdifferential of $\mathcal{W}$ and find a first characterization of its element of minimal norm. In Section 4 we particularize the characterization to the case of assumption (1.8), which is the convolution case. In particular, we have the strongest result in the case of assumption (1.9). Section 5 relates these arguments to the finite time collapse results in [CDFLS] showing the characterization of the velocity field for particles. Section 6 gives examples of non-smooth non-convex repulsive-attractive potentials, albeit λ-convex, leading to non-trivial sets of stationary states. Finally, the Appendix 7 is devoted to recall technical concepts from the differential calculus in Wasserstein spaces which are needed in Section 3.

2. Wasserstein subdifferential and gradient flow of the interaction energy

2.1. **Optimal transport framework.** We denote by $\mathscr{P}_2(\mathbb{R}^d)$ the space of Borel probability measures over $\mathbb{R}^d$ with finite second moment, i.e., the set of Borel probability measures μ such that

$$\int_{\mathbb{R}^d} |x|^2 \, d\mu(x) < \infty \, .$$

The convergence of probability measures is considered in the narrow sense defined as the weak convergence in the duality with continuous and bounded functions over $\mathbb{R}^d$. Given $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$ and $\gamma \in \Gamma(\mu,\nu)$, where

$$\Gamma(\mu,\nu) := \{\gamma \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d) : \gamma(\Omega \times \mathbb{R}^d) = \mu(\Omega), \gamma(\mathbb{R}^d \times \Omega) = \nu(\Omega), \quad \text{for every Borel set } \Omega \subset \mathbb{R}^d\},$$

the euclidean quadratic transport cost between μ and ν with respect to the transport plan γ is defined by

$$\mathcal{C}(\mu,\nu\,;\gamma) = \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \, d\gamma(x,y)\right)^{1/2} .$$

The "Wasserstein distance" between μ and ν is defined by

$$d_W(\mu,\nu) = \inf_{\gamma\in\Gamma(\mu,\nu)} \mathcal{C}(\mu,\nu\,;\gamma). \tag{2.1}$$

It is well known that the inf in (2.1) is attained by a minimizer. The minimizers in (2.1) are called optimal plans. We denote by $\Gamma_o(\mu,\nu) \subset \Gamma(\mu,\nu)$ the set of optimal plans between μ and ν. It is also well known that $\mu_n \to \mu$ in $\mathscr{P}_2(\mathbb{R}^d)$ (i.e. $d_W(\mu_n,\mu) \to 0$) if and only if μ_n narrowly converges to μ and $\int_{\mathbb{R}^d} |x|^2 \, d\mu_n(x) \to \int_{\mathbb{R}^d} |x|^2 \, d\mu(x)$. The space $\mathscr{P}_2(\mathbb{R}^d)$ endowed with the distance d_W is a complete and separable metric space. For all the details on Wasserstein distance and optimal transportation, we refer to [AGS, V2].

We recall the push forward notation for a map $\mathbf{s} : (\mathbb{R}^d)^m \to (\mathbb{R}^d)^k$, $m, k \geq 1$, and a measure $\mu \in \mathscr{P}((\mathbb{R}^d)^m)$: the measure $\mathbf{s}_{\#}\mu \in \mathscr{P}((\mathbb{R}^d)^k)$ is defined by $\mathbf{s}_{\#}\mu(A) = \mu(\mathbf{s}^{-1}(A))$, where A is a Borel set. A transport plan $\gamma \in \Gamma(\mu, \nu)$ may be induced by a map $\mathbf{s} : \mathbb{R}^d \to \mathbb{R}^d$ such that $\mathbf{s}_{\#}\mu = \nu$. This means that $\gamma = (\mathbf{i}, \mathbf{s})_{\#}\mu$, where $\mathbf{i} : \mathbb{R}^d \to \mathbb{R}^d$ denotes the identity map over $\mathbb{R}^d$ and $(\mathbf{i}, \mathbf{s}) : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is the product map. Finally, π^j will stand for the projection map on the j-th component of a product space. Hence, if γ is a probability measure over a product space (for instance a transport plan), $\pi^j_{\#}\gamma$ is its j-th marginal.

The first properties of the interaction potential functional $\mathcal{W}$ given by (1.1) are contained in the next Proposition.

Proposition 2.1. *Under assumptions* (1.3) *and* (1.4)*, the functional $\mathcal{W}$ is lower semicontinuous with respect to the d_W metric and enjoys the following λ-convexity property: for every $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$ and every $\gamma \in \Gamma(\mu, \nu)$ it holds*

$$\mathcal{W}(\theta^\gamma(t)) \leq (1-t)\mathcal{W}(\mu) + t\mathcal{W}(\nu) - \frac{\lambda}{2} t(1-t)\mathcal{C}^2(\mu, \nu; \gamma), \tag{2.2}$$

where θ^γ denotes the interpolating curve $t \in [0,1] \mapsto \theta^\gamma(t) = ((1-t)\pi^1 + t\pi^2)_{\#}\gamma \in \mathscr{P}_2(\mathbb{R}^d)$.

The lower semicontinuity follows from standard arguments, for the convexity along interpolating curves we refer to [AGS, §9.3]. In particular, since every constant speed Wasserstein geodesic is of the form θ^γ where γ is an optimal plan, then $\mathcal{W}$ is λ-convex along every Wasserstein geodesics. We adapt from [AGS] the definition of the Wasserstein subdifferential for the λ-convex functional $\mathcal{W}$.

Definition 2.2 (The Wasserstein subdifferential of $\mathcal{W}$). *Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. We say that the vector field $\boldsymbol{\xi} \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$ belongs to $\partial\mathcal{W}(\mu)$, the Wasserstein subdifferential of the λ-convex functional $\mathcal{W} : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ at the point μ, if for every $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ there exists $\gamma \in \Gamma_o(\mu, \nu)$ such that*

$$\mathcal{W}(\nu) - \mathcal{W}(\mu) \geq \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}(x), y - x \rangle \, d\gamma(x, y) + \frac{\lambda}{2} \mathcal{C}^2(\mu, \nu\,; \gamma). \tag{2.3}$$

We say that that $\boldsymbol{\xi} \in \partial_S\mathcal{W}(\mu)$, the strong subdifferential of $\mathcal{W}$ at the point μ, if for every $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ and for every admissible plan $\gamma \in \Gamma(\mu, \nu)$, (2.3) *holds.*

The metric slope of the functional $\mathcal{W}$ at the point μ is defined as follows:

$$|\partial\mathcal{W}|(\mu) := \limsup_{\nu \to \mu \text{ in } \mathscr{P}_2(\mathbb{R}^d)} \frac{(\mathcal{W}(\nu) - \mathcal{W}(\mu))^+}{d_W(\nu, \mu)},$$

where $(a)^+$ denotes the positive part of the real number a. Since $\mathcal{W}$ is λ-convex we know that

$$|\partial\mathcal{W}|(\mu) = \min\left\{ \|\boldsymbol{\xi}\|_{L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)} : \boldsymbol{\xi} \in \partial\mathcal{W}(\mu) \right\}. \tag{2.4}$$

Moreover, the element realizing the minimal norm in (2.4) is unique and we denote it by $\partial^o\mathcal{W}(\mu)$ (see [AGS, Chapter 10]). The element of minimal norm in the subdifferential plays a *crucial role*, since it is known to be the velocity vector field of the evolution equation associated to the gradient flow of the functional (1.1) under certain conditions as reviewed next.

2.2. **The gradient flow solution.** As already shown in [CDFLS], we are forced to consider a notion of solution which only assumes that the densities are in fact, measures. Actually, in case of attractive radial potentials verifying assumptions (1.2), (1.3) and (1.4), i.e. $\mathbf{W}(x,y) = w(|x-y|)$, with w increasing, it was shown in [BCL, BLR] that weak-L^p solutions blow-up in finite time. Moreover, these weak-L^p solutions can be uniquely continued as measure solutions, as proved in [CDFLS], leading to a total collapse in a single Delta Dirac at the center of mass in finite time. Furthermore, particle solutions, i.e, solutions corresponding to an initial data composed by a finite number of atoms, remain particle solutions for all times for the evolution of (1.5). Summarizing we can only expect that a regular solution enjoys local in time existence.

Our well-posedness results are based on the following abstract theorem for gradient flow solutions. For all the details we refer to [AGS, Theorem 11.2.1], where some more properties of these solutions are remarked.

Before stating the Theorem, we say that a curve $t \in [0,\infty) \mapsto \mu(t) \in \mathscr{P}_2(\mathbb{R}^d)$ is locally absolutely continuous with locally finite energy, and we denote it by $\mu \in AC^2_{\mathrm{loc}}([0,\infty); \mathscr{P}_2(\mathbb{R}^d))$, if the restriction of μ to the interval $[0,T]$ is absolutely continuous for every $T > 0$ and its metric derivative, which exists for a.e. $t > 0$ defined by

$$|\mu'|(t) := \lim_{s\to t} \frac{d_W(\mu(s),\mu(t))}{|t-s|},$$

belongs to $L^2(0,T)$ for every $T > 0$.

Theorem 2.3. *Let* $\mathbf{W}$ *satisfy the hypotheses* (1.2), (1.3) *and* (1.4). *For any initial datum* $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$, *there exists a unique curve* $\mu \in AC^2_{\mathrm{loc}}([0,\infty); \mathscr{P}_2(\mathbb{R}^d))$ *satisfying*

$$\begin{aligned}
&\partial_t \mu(t) + \operatorname{div}(\boldsymbol{v}(t)\mu(t)) = 0 \ \textit{in}\ \mathcal{D}'((0,\infty)\times\mathbb{R}^d),\\
&\boldsymbol{v}(t) = -\partial^o \mathcal{W}(\mu(t)), \qquad \textit{for a.e.}\quad t>0,\\
&\|\boldsymbol{v}(t)\|_{L^2(\mu(t))} = |\mu'|(t) \qquad \textit{for a.e.}\quad t>0,
\end{aligned}$$

with $\mu(0) = \mu_0$. *The energy identity*

$$\int_a^b \int_{\mathbb{R}^d} |\boldsymbol{v}(t,x)|^2 \, d\mu(t)(x)\, dt + \mathcal{W}(\mu(b)) = \mathcal{W}(\mu(a))$$

holds for all $0 \le a \le b < \infty$. *Moreover, the solution is given by a* λ*-contractive semigroup* $S(t)$ *acting on* $\mathscr{P}_2(\mathbb{R}^d)$, *that is* $\mu(t) = S(t)\mu_0$ *with*

$$d_W(S(t)\mu_0, S(t)\nu_0) \le e^{-\lambda t} d_W(\mu_0,\nu_0), \quad \forall \mu_0,\nu_0 \in \mathscr{P}_2(\mathbb{R}^d)\,.$$

The unique curve given by Theorem 2.3 is called *gradient flow solution* for equation

$$\partial_t \mu(t) = \operatorname{div}(\partial^o \mathcal{W}(\mu(t))\mu(t)). \tag{2.5}$$

Let us finally remark that weak measure solutions as defined in [CDFLS] are equivalent to gradient flow solutions as shown there.

Characterizing the element of minimal norm $\partial^o \mathcal{W}(\mu)$ is then essential to link the constructed solutions to the sought equation (1.5)-(1.6) or (1.5)-(1.7). This characterization was done in [CDFLS] for potentials satisfying (1.2), (1.3), (1.4) and (1.8), being the potential W C^1-smooth except possibly at the origin. Under those assumptions, the authors identified

$\partial^o\mathcal{W}(\mu)$ as $\partial^o W * \mu$, i.e.

$$\partial^o W * \mu(x) = \int_{x\neq y} \nabla W(x-y)\, d\mu(y).$$

The main results in the present work will show that this characterization can be generalized to *convex* potentials satisfying assumptions (1.2), (1.4) and the radial radial hypothesis (1.9), regardless of the number of points of non-differentiability of the potential W. In Theorem 4.5, we show that under those conditions, the formula $\partial^o\mathcal{W}(\mu) = \partial^o W * \mu$ also holds, and the equation takes the form (1.11), which generalizes the standard form of the interaction potential evolution (1.5)-(1.6). In the most general case, i.e. for general potentials satisfying only (1.2), (1.3) and (1.4), we will obtain a characterization in terms of generic Borel measurable selections in $\partial\mathbf{W}$. Therefore, the next two sections are devoted to the study of the velocity field $\partial^o\mathcal{W}$, in order to apply the abstract result above to the aggregation equation. Let us state the results:

- Let $\mathbf{W}$ satisfy (1.2), (1.3) and (1.4). Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. There holds

$$\partial^o\mathcal{W}(\mu) = \int_{\mathbb{R}^d} \boldsymbol{\eta}(\cdot, y)\, d\mu(y) \tag{2.6}$$

  for some selection $\boldsymbol{\eta} \in \partial_1\mathbf{W}$ having the form

$$\boldsymbol{\eta}(x,y) = \frac{1}{2}\left(\boldsymbol{\eta}^1(x,y) + \boldsymbol{\eta}^2(y,x)\right), \tag{2.7}$$

  with the couple $(\boldsymbol{\eta}^1, \boldsymbol{\eta}^2)$ belonging to the joint subdifferential $\partial\mathbf{W}$. This is shown in Theorem 3.4.
- Under the additional assumption (1.8), let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Then we have

$$\partial^o\mathcal{W}(\mu) = \boldsymbol{\eta} * \mu \tag{2.8}$$

  for some $\boldsymbol{\eta} \in \partial W$. This is the characterization following from Corollary 4.2.
- Finally, when the further condition (1.9) holds, and the potential is convex (not only λ-convex for a negative λ) there is

$$\partial^o\mathcal{W}(\mu) = \partial^o W * \mu \tag{2.9}$$

  for all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Here, $\partial^o W$ is the element of minimal norm of the subdifferential of W. This is proven in the subsequent Theorem 4.5.

Remark 2.4. Substituting (2.6), (2.8) or (2.9) in (2.5) and applying Theorem 2.3, one obtains the corresponding well-posedness result. In Section 4 we will show that the selection $\boldsymbol{\eta}$ appearing in (2.8) (and thus also the one in (2.6)) is in general depending on μ. Therefore, in the case of (2.8), the dynamic will be governed by a velocity field of the form (1.7), where the selection depends in general on t. Similarly for the case of (2.6). On the other hand, we stress that a consequence of the last characterization (2.9), when applying Theorem 2.3, is that the selection corresponding to the velocity $\boldsymbol{v}(t)$ does not depend on t.

Remark 2.5. The joint subdifferential constraint (2.7) has a natural interpretation: there is a symmetry in the interaction between particles (action-reaction law).

3. Characterization of the element of minimal norm in the subdifferential

In this section and in the next we analyze the Wasserstein subdifferential of $\mathcal{W}$ and we prove the main core results of this work.

Theorem 3.1. *Consider a Borel measurable selection $(\boldsymbol{\eta}^1, \boldsymbol{\eta}^2) \in \partial \mathbf{W}$, i.e., $(\boldsymbol{\eta}^1, \boldsymbol{\eta}^2) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is a Borel measurable function such that $(\boldsymbol{\eta}^1(x,y), \boldsymbol{\eta}^2(x,y)) \in \partial \mathbf{W}(x,y)$ for every $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. For any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, the map*

$$\boldsymbol{\xi}(x) := \frac{1}{2} \int_{\mathbb{R}^d} (\boldsymbol{\eta}^1(x,y) + \boldsymbol{\eta}^2(y,x))\, d\mu(y) \tag{3.1}$$

belongs to $\partial_S \mathcal{W}(\mu)$. In particular $\partial_S \mathcal{W}(\mu)$ is not empty.

Proof. Since $\mathbf{W}$ is λ-convex, the subdifferential inequality (1.10) in this case can be written as follows

$$\begin{aligned} \mathbf{W}(y_1,y_2) - &\mathbf{W}(x_1,x_2) \\ &\geq \langle (\boldsymbol{\eta}^1(x_1,x_2),\, \boldsymbol{\eta}^2(x_1,x_2)), (y_1 - x_1, y_2 - x_2)\rangle + \frac{\lambda}{2} |(y_1 - x_1, y_2 - x_2)|^2, \end{aligned} \tag{3.2}$$

for every $(x_1, x_2, y_1, y_2) \in (\mathbb{R}^d)^4$. Let $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$ and $\gamma \in \Gamma(\mu,\nu)$. We show that inequality (2.3) holds. Considering the measure $\gamma \times \gamma$, we can write

$$\mathcal{W}(\nu) - \mathcal{W}(\mu) = \frac{1}{2} \int_{(\mathbb{R}^d)^4} (\mathbf{W}(y_1,y_2) - \mathbf{W}(x_1,x_2))\, d(\gamma\times\gamma)(x_1,y_1,x_2,y_2).$$

Hence, by (3.2),

$$\begin{aligned} \mathcal{W}(\nu) - \mathcal{W}(\mu) \geq & \frac{1}{2} \int_{(\mathbb{R}^d)^4} \big[\langle \boldsymbol{\eta}^1(x_1,x_2), (y_1 - x_1)\rangle + \langle \boldsymbol{\eta}^2(x_1,x_2), (y_2 - x_2)\rangle\big]\, d(\gamma\times\gamma)(x_1,y_1,x_2,y_2) \\ & + \frac{\lambda}{4} \int_{(\mathbb{R}^d)^4} (|y_1 - x_1|^2 + |y_2 - x_2|^2)\, d(\gamma\times\gamma)(x_1,y_1,x_2,y_2). \end{aligned}$$

The last term is $\frac{\lambda}{2}\mathcal{C}^2(\mu,\nu;\gamma)$, so that a change of variables gives

$$\begin{aligned} \mathcal{W}(\nu) - \mathcal{W}(\mu) \geq & \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \boldsymbol{\eta}^1(x_1,x_2), (y_1 - x_1)\rangle\, d\gamma(x_1,y_1)\, d\mu(x_2) \\ & + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \boldsymbol{\eta}^2(x_1,x_2), (y_2 - x_2)\rangle\, d\gamma(x_2,y_2)\, d\mu(x_1) + \frac{\lambda}{2}\mathcal{C}^2(\mu,\nu;\gamma) \\ = & \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \boldsymbol{\eta}^1(x,z), (y - x)\rangle\, d\gamma(x,y)\, d\mu(z) \\ & + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \boldsymbol{\eta}^2(z,x), (y - x)\rangle\, d\gamma(x,y)\, d\mu(z) + \frac{\lambda}{2}\mathcal{C}^2(\mu,\nu;\gamma) \\ = & \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle \frac{1}{2}\int_{\mathbb{R}^d} (\boldsymbol{\eta}^1(x,z) + \boldsymbol{\eta}^2(z,x))\, d\mu(z), (y-x) \right\rangle d\gamma(x,y) + \frac{\lambda}{2}\mathcal{C}^2(\mu,\nu;\gamma) \end{aligned}$$

as desired. ☐

In the case of a smooth interaction function $\mathbf{W}$, there is a complete characterization of the strong subdifferential $\partial_S \mathcal{W}(\mu)$ which is single valued.

Proposition 3.2 (**The smooth case**). *Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. If $\mathbf{W} \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies the assumptions* (1.2), (1.3), *and* (1.4), *then the strong Wasserstein subdifferential is a singleton and it is of the form*

$$\begin{aligned}\partial_S \mathcal{W}(\mu) &= \left\{ \int_{\mathbb{R}^d} \nabla_1 \mathbf{W}(\cdot, y)\, d\mu(y) \right\} = \left\{ \int_{\mathbb{R}^d} \nabla_2 \mathbf{W}(y, \cdot)\, d\mu(y) \right\} \\ &= \left\{ \frac{1}{2} \int_{\mathbb{R}^d} \big(\nabla_1 \mathbf{W}(\cdot, y) + \nabla_2 \mathbf{W}(y, \cdot) \big)\, d\mu(y) \right\},\end{aligned} \tag{3.3}$$

where ∇_1 (resp. ∇_2) are the gradients with respect to the first-d (second-d) variables of $\mathbb{R}^d \times \mathbb{R}^d$.

Proof. Since $\partial \mathbf{W}(x, y) = \{\nabla \mathbf{W}(x, y)\}$, by Theorem 3.1 and the symmetry of $\mathbf{W}$ we have that the right hand sides of (3.3) are contained in $\partial_S \mathcal{W}(\mu)$.

In order to prove the opposite inclusion, assume that $\boldsymbol{\xi} \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$ belongs to $\partial_S \mathcal{W}(\mu)$. Let $\mathbf{s} \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$ be an arbitrary vector field, $\nu = \mathbf{s}_\# \mu$ and $\mu_t = (\mathbf{i} + t\mathbf{s})_\# \mu$. Writing (2.3) in correspondence of the plan

$$\gamma_t = (\mathbf{i}, \mathbf{i} + t\mathbf{s})_\# \mu$$

between μ and μ_t, we have

$$\mathcal{W}(\mu_t) - \mathcal{W}(\mu) \geq \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \boldsymbol{\xi}(x), y - x \rangle\, d\gamma_t(x, y) + \frac{\lambda}{2} \mathcal{C}^2(\mu, \mu_t; \gamma_t).$$

Hence, for every $t > 0$

$$\begin{aligned}&\frac{1}{2t} \int_{\mathbb{R}^d \times \mathbb{R}^d} (\mathbf{W}(x + t\mathbf{s}(x), y + t\mathbf{s}(y)) - \mathbf{W}(x, y))\, d(\mu \times \mu)(x, y) \\ &\qquad\qquad \geq \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}(x), \mathbf{s}(x) \rangle\, d\mu(x) + \frac{\lambda}{2} t \|\mathbf{s}\|^2_{L^2(\mu)},\end{aligned}$$

and, by a direct computation

$$\begin{aligned}&\frac{1}{2t} \int_{\mathbb{R}^d \times \mathbb{R}^d} \big(\mathbf{W}(x + t\mathbf{s}(x), y + t\mathbf{s}(y)) - \mathbf{W}(x, y)\big) d(\mu \times \mu)(x, y) \\ &\qquad \geq \frac{\lambda}{4t} \int_{\mathbb{R}^d \times \mathbb{R}^d} \big(|(x + t\mathbf{s}(x), y + t\mathbf{s}(y))|^2 - |(x, y)|^2 \big) d(\mu \times \mu)(x, y) \\ &\qquad\quad + \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}(x), \mathbf{s}(x) \rangle\, d\mu(x) - \lambda \int_{\mathbb{R}^d} \langle x, \mathbf{s}(x) \rangle\, d\mu(x)\,.\end{aligned} \tag{3.4}$$

Since $\mathbf{W}$ is λ-convex, the map

$$t \mapsto \frac{1}{t} (\mathbf{W}(x + t\mathbf{s}(x), y + t\mathbf{s}(y)) - \mathbf{W}(x, y)) - \frac{\lambda}{2t} \big(|(x + t\mathbf{s}(x), y + t\mathbf{s}(y))|^2 - |(x, y)|^2 \big)$$

is nondecreasing in t, for $t > 0$. Taking advantage of the C^1 regularity and the quadratic growth of $\mathbf{W}$, by the monotone convergence theorem, we can pass to the limit in (3.4) as t goes to 0, obtaining

$$\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \nabla \mathbf{W}(x, y), (\mathbf{s}(x), \mathbf{s}(y)) \rangle\, d(\mu \times \mu)(x, y) \geq \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}, \mathbf{s} \rangle\, d\mu. \tag{3.5}$$

Since by the symmetry of $\mathbf{W}$ we have $\nabla_1 \mathbf{W}(x,y) = \nabla_2 \mathbf{W}(y,x)$ for any $x, y \in \mathbb{R}^d$, we can write

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} &\langle \nabla \mathbf{W}(x,y), (\mathbf{s}(x), \mathbf{s}(y))\rangle \, d(\mu\times\mu)(x,y) \\
&= \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \nabla_1 \mathbf{W}(x,y), \mathbf{s}(x)\rangle \, d(\mu\times\mu)(x,y) + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \nabla_1 \mathbf{W}(y,x), \mathbf{s}(y)\rangle \, d(\mu\times\mu)(x,y) \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \nabla_1 \mathbf{W}(x,y), \mathbf{s}(x)\rangle \, d(\mu\times\mu)(x,y).
\end{aligned}$$

This way, (3.5) becomes

$$\int_{\mathbb{R}^d} \left\langle \int_{\mathbb{R}^d} \nabla_1 \mathbf{W}(x,y)\, d\mu(y), \mathbf{s}(x) \right\rangle d\mu(x) \geq \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}, \mathbf{s}\rangle \, d\mu,$$

that is

$$\int_{\mathbb{R}^d} \left\langle \int_{\mathbb{R}^d} \nabla_1 \mathbf{W}(x,y)\, d\mu(y) - \boldsymbol{\xi}(x), \mathbf{s}(x) \right\rangle d\mu(x) \geq 0.$$

Since $\mathbf{s} \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$ is arbitrary we conclude that $\boldsymbol{\xi}(x) = \int_{\mathbb{R}^d} \nabla_1 \mathbf{W}(x,y)\, d\mu(y)$ as elements of $L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$. □

When we drop the assumption of $\mathbf{W} \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ the subdifferential of $\mathcal{W}$ is in general multivalued. In the next Theorem, we characterize the element of minimal norm in the subdifferential of $\mathcal{W}$ at the point μ, which is of the form (3.1).

Remark 3.3. The property stating that every element of the subdifferential of $\mathcal{W}$ at the point μ is of the form (3.1) for a suitable Borel selection of the subdifferential of $\mathbf{W}$ could be in general very difficult and we do not know if it is true.

Theorem 3.4. *Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $\boldsymbol{\xi} = \partial^o \mathcal{W}(\mu)$. Then there exists a measurable selection $(\boldsymbol{\eta}^1, \boldsymbol{\eta}^2) \in \partial \mathbf{W}$ such that*

$$\boldsymbol{\xi}(x) = \frac{1}{2}\int_{\mathbb{R}^d} (\boldsymbol{\eta}^1(x,y) + \boldsymbol{\eta}^2(y,x))\, d\mu(y).$$

The proof of Theorem 3.4 needs several preliminary results. We will make use of a sequence of regularized functionals $\mathcal{W}_n$. We recall that the *Moreau-Yosida* approximation of the function $\mathbf{W}$ is defined as

$$\mathbf{W}_n(x,y) := \inf_{(v,w)\in\mathbb{R}^d\times\mathbb{R}^d} \left\{ \mathbf{W}(v,w) + \frac{n}{2}\, |(x-v, y-w)|^2 \right\}. \tag{3.6}$$

We have $\mathbf{W}_n(x,y) \leq \mathbf{W}(x,y)$ for every $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, $\mathbf{W}_n \in C^{1,1}(\mathbb{R}^d \times \mathbb{R}^d)$ and the sequence $\{\mathbf{W}_n\}_{n\in\mathbb{N}}$ converges pointwise and monotonically to $\mathbf{W}$ as $n \to \infty$.

If the λ-convexity property of $\mathbf{W}$ is satisfied, there exists a constant $K > 0$ such that

$$\mathbf{W}(x,y) \geq -K(1 + |x|^2 + |y|^2), \tag{3.7}$$

and in this case we can show the corresponding bound for $\mathbf{W}_n$, uniformly in n.

Proposition 3.5. *If $\mathbf{W}$ enjoys the λ-convexity assumption* (1.3)*, there exist $\bar{n} \in \mathbb{N}$ and a constant $\bar{K} > 0$ such that for all $n > \bar{n}$*

$$\mathbf{W}_n(x,y) \geq -\bar{K}(1 + |x|^2 + |y|^2)\,.$$

Proof. Indeed, by (3.6) and the estimate (3.7) there holds

$$\mathbf{W}_n(x,y) \geq \inf_{(v,w)\in\mathbb{R}^d\times\mathbb{R}^d} \Big\{ -K(1+|v|^2+|w|^2) + \frac{n}{2}\,|(x-v,y-w)|^2 \Big\}. \tag{3.8}$$

Here we compute the minimum. The gradient of the right hand side is

$$(-2Kv - n(x-v), -2Kw - n(y-w)),$$

and it vanishes when $(v,w) = \frac{n}{n-2K}\,(x,y)$. Substituting in (3.8) we get

$$\mathbf{W}_n(x,y) \geq -K\left(1 + \frac{n^2}{(n-2K)^2}\,(|x|^2+|y|^2) + \frac{4nK^2}{2(n-2K)^2}\,(|x|^2+|y|^2)\right).$$

If we chose for instance $\bar{K} = 2K$, it is then clear that there exists a large enough $\bar{n}$ (depending only on K) such that the desired inequality holds for any $n > \bar{n}$. □

We define the approximating interaction functionals

$$\mathcal{W}_n(\mu) := \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{W}_n(x,y)\,d(\mu\times\mu)(x,y). \tag{3.9}$$

Remark 3.6 (**Semicontinuity properties of $\mathcal{W}$**)**.** Since $\mathbf{W}$ might enjoy a negative quadratic behavior at infinity, it is not true that $\mathbf{W}$ is lower semicontinuous also with respect to the narrow convergence. By the way, it is shown in [CDFLS, §2] that one can choose τ_0 small enough (depending only on $\mathbf{W}$) such that for any $\tau < \tau_0$ and for any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, the functional

$$\nu \mapsto \mathcal{W}(\nu) + \frac{1}{2\tau}\,d_W^2(\nu,\mu), \tag{3.10}$$

is lower semicontinuous with respect to the narrow convegence. Moreover, for $\tau < \tau_0$, minimizers do exist for (3.10). The arguments of [CDFLS] are given for a function $\mathbf{W}$ such that $\mathbf{W}(x,x) = 0$ for any x and assumption *iv)* holds. They can be adapted in a straightforward way if these hypotheses are omitted. Moreover, in the case of the approximating functionals $\mathcal{W}_n$ defined in (3.9), we can choose τ_0 independently of n. Indeed, since Proposition 3.5 gives the bound $\mathbf{W}_n(x,y) \geq -\bar{K}(1+|x|^2+|y|^2)$, for some $\bar{K} > 0$, it is enough to choose τ_0 small enough such that

$$\nu \mapsto \int_{\mathbb{R}^d\times\mathbb{R}^d} -\bar{K}(1+|x|^2+|y|^2)\,d\nu\times\nu(x,y) + \frac{1}{2\tau}\,d_W^2(\mu,\nu)$$

is narrowly lower semicontinuous.

We prove the following more general lower semicontinuity property.

Proposition 3.7. *Let τ_0 be as in* Remark 3.6. *Let $\tau < \tau_0$ and let $\nu \in \mathscr{P}_2(\mathbb{R}^d)$. For any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and for any sequence $\{\mu_n\}_{n\in\mathbb{N}}$ such that μ_n narrowly converges to μ and $\sup_n \int_{\mathbb{R}^d} |x|^2\,d\mu_n(x) < +\infty$, there holds*

$$\mathcal{W}(\mu) + \frac{1}{2\tau}\,d_W^2(\mu,\nu) \leq \liminf_n \left(\mathcal{W}_n(\mu_n) + \frac{1}{2\tau}\,d_W^2(\mu_n,\nu)\right).$$

Moreover, for $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ there holds $\mathcal{W}_n(\mu) \to \mathcal{W}(\mu)$.

Proof. Since $\mathbf{W}_n \geq \mathbf{W}_k$ if $n \geq k$, by Remark 3.6 we have

$$\begin{aligned}\liminf_{n\to\infty}\left(\mathcal{W}_n(\mu_n)+\frac{1}{2\tau}\,d_W^2(\mu_n,\nu)\right) &\geq \liminf_{n\to\infty}\left(\mathcal{W}_k(\mu_n)+\frac{1}{2\tau}\,d_W^2(\mu_n,\nu)\right)\\ &\geq \mathcal{W}_k(\mu)+\frac{1}{2\tau}\,d_W^2(\mu,\nu)\end{aligned}$$

for any fixed $k \in \mathbb{N}$. Now we shall pass to the limit as $k \to \infty$. Notice that $\mathbf{W}_k \nearrow \mathbf{W}$ pointwise and monotonically, and thus by the monotone convergence theorem, $\mathcal{W}_k(\mu)$ converges to $\mathcal{W}(\mu)$. Both statements are proven. □

We recall a suitable notion of convergence of a sequence of vector fields $\boldsymbol{\xi}_n \in L^2(\mathbb{R}^d, \mu_n; \mathbb{R}^d)$.

Definition 3.8. *Let μ_n narrowly convergent to μ and let $\boldsymbol{\xi}_n \in L^2(\mathbb{R}^d, \mu_n; \mathbb{R}^d)$. We say that $\boldsymbol{\xi}_n$ weakly converges to $\boldsymbol{\xi} \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$ if*

$$\int_{\mathbb{R}^d} \langle \boldsymbol{\xi}_n, \zeta\rangle\, d\mu_n \to \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}, \zeta\rangle\, d\mu, \qquad \forall \zeta \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d). \tag{3.11}$$

We say that the convergence is strong if (3.11) *holds and*

$$\int_{\mathbb{R}^d} |\boldsymbol{\xi}_n|^2\, d\mu_n \to \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^2\, d\mu.$$

Remark 3.9. Consider the set $\mathcal{M}_d$ of vector measures of the form $\boldsymbol{\xi}\mu$, with $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $\boldsymbol{\xi} \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$. Reasoning as in [AGS, §5.1], we know that the weak convergence (3.11) is metrizable on every subset $\mathcal{A}$ of $\mathcal{M}_d$ such that

$$\sup_{\mathcal{A}} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|\, d\mu < +\infty.$$

Moreover, by [AGS, Theorem 5.4.4], if $\mathcal{A} \subset \mathcal{M}_d$ is such that

$$\sup_{\mathcal{A}} \int_{\mathbb{R}^d} |\boldsymbol{\xi}|^2\, d\mu < +\infty,$$

then $\mathcal{A}$ is also compact with respect to the weak convergence (3.11).

We also need to define the barycentric projection.

Definition 3.10 (**Disintegration and barycenter**)**.** *Given $\beta \in \Gamma(\mu,\nu)$, we denote by β_x the Borel family of measures over $\mathscr{P}(\mathbb{R}^d)$ such that $\beta = \int_{\mathbb{R}^d} \beta_x\, d\mu(x)$, which disintegrates β with respect to μ. The notation above means that the integral of a Borel function $\varphi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that $\varphi \in L^1(\beta)$, can be sliced as*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(x,y)\, d\beta(x,y) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x,y)\, d\beta_x(y)\, d\mu(x).$$

The barycentric projection of $\beta \in \Gamma(\mu,\nu)$ is defined by

$$\bar{\beta}(x) := \int_{\mathbb{R}^d} y\, d\beta_x(y).$$

For more detail about disintegration see [AFP, Theorem 2.28].

We can prove the following simple

Proposition 3.11. *Let $\{\mu_n\} \subset \mathscr{P}_2(\mathbb{R}^d)$, $\{\nu_n\} \subset \mathscr{P}_2(\mathbb{R}^d)$ be sequences with uniformly bounded second moments and narrowly convergent to μ and ν respectively. For every choice $\gamma_n \in \Gamma_o(\mu_n, \nu_n)$, we have that the sequence $\{\gamma_n\}$ is tight and every limit point with respect to the narrow convergence in $\mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d)$ belongs to $\Gamma_o(\mu, \nu)$. Moreover, if γ is a limit point and γ_{n_k} is a subsequence narrowly convergent to γ, then*

$$\bar{\gamma}_{n_k} \to \bar{\gamma} \quad \textit{weakly in the sense of } \text{Definition 3.8} \textit{ as } k \to +\infty.$$

Proof. The tightness and optimality are contained in [AGS, Proposition 7.1.3]. Let $(\gamma_n)_x$ be the disintegration of γ_n with respect to μ_n and γ_x be the disintegration of γ with respect to μ. Let $\zeta \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ and $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ the function defined by $f(x, y) = \langle \zeta(x), y \rangle$. Since f is continuous and satisfies $|f(x, y)| \leq C|y|$ for every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\sup_n \int_{\mathbb{R}^d \times \mathbb{R}^d} (|x|^2 + |y|^2)\, d\gamma_n(x, y) < +\infty$, by [AGS, Lemma 5.1.7] we have that $\int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y)\, d\gamma_n(x, y) \to \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x, y)\, d\gamma(x, y)$ as $n \to +\infty$. Using this property and the definition of barycenter, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \langle \zeta, \bar{\gamma}_n \rangle\, d\mu_n &= \int_{\mathbb{R}^d} \langle \zeta(x), \int_{\mathbb{R}^d} y\, d(\gamma_n)_x(y) \rangle\, d\mu_n(x) \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \zeta(x), y \rangle\, d\gamma_n(x, y) \to \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \zeta(x), y \rangle\, d\gamma(x, y) \\
&= \int_{\mathbb{R}^d} \langle \zeta(x), \int_{\mathbb{R}^d} y \rangle\, d\gamma_x(y)\, d\mu(x) = \int_{\mathbb{R}^d} \langle \zeta, \bar{\gamma} \rangle\, d\mu.
\end{aligned}$$

□

We recall a definition from [AGS, Chapter 10].

Definition 3.12 (Rescaled plan). *Let $\tau < \tau_0$ (as in* Remark 3.6*). Given $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, let*

$$\mu_\tau = \operatorname{argmin} \left\{ \mathcal{W}(\nu) + \frac{1}{2\tau} d_W(\nu, \mu) : \nu \in \mathscr{P}_2(\mathbb{R}^d) \right\}.$$

Given $\hat{\gamma}_\tau \in \Gamma_o(\mu_\tau, \mu)$, we define the rescaled plan as

$$\gamma_\tau := \left(\pi^1, \frac{\pi^2 - \pi^1}{\tau} \right)_\# \hat{\gamma}_\tau.$$

Next we introduce an abstract result about approximation of the minimal selection in the subdifferential of $\mathcal{W}$. The argument is indeed a direct consequence of the analysis in [AGS, §10.3], but requires the concept of plan subdifferential. Since this is a technical definition, we prefer to postpone a discussion at the end of the paper. Therefore, the proof of the following proposition is given in the appendix.

Proposition 3.13. *Let μ, μ_τ and γ_τ be as in* Definition 3.12*. Then $\mu_\tau \to \mu$ in $\mathscr{P}_2(\mathbb{R}^d)$ as $\tau \to 0$. Moreover, denoting by $\bar{\gamma}_\tau$ the barycenter of γ_τ, we have that $\bar{\gamma}_\tau \in \partial_S \mathcal{W}(\mu_\tau)$ and*

$$\bar{\gamma}_\tau \to \partial^o \mathcal{W}(\mu) \quad \textit{strongly in the sense of } \text{Definition 3.8} \textit{ as } \tau \to 0.$$

Making use of Proposition 3.13 we can prove the following

Lemma 3.14. *Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. There exists a sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathscr{P}_2(\mathbb{R}^d)$, with $\int_{\mathbb{R}^d} |x|^2\, d\mu_n < +\infty$, such that μ_n narrowly converges to μ and*

$$\int_{\mathbb{R}^d} \nabla_1 \mathbf{W}_n(\cdot, y)\, d\mu_n(y) \to \partial^o \mathcal{W}(\mu) \quad \textit{weakly in the sense of } \text{Definition 3.8}.$$

Proof. Let τ_0 be as in Remark 3.6, and consider a measure $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Define, for $\tau \le \tilde{\tau} < \tau_0$,

$$\mu_\tau^h = \operatorname{argmin}\left\{\mathcal{W}_h(\nu) + \frac{1}{2\tau}\, d_W^2(\nu,\mu) : \nu \in \mathscr{P}_2(\mathbb{R}^d)\right\}.$$

By the uniform estimates of [AGS, Lemma 2.2.1],

$$\sup_{h\in\mathbb{N},\, \tau\le\tilde{\tau}} \int_{\mathbb{R}^d} |x|^2\, d\mu_\tau^h < +\infty. \tag{3.12}$$

This shows that the sequence $\{\mu_\tau^h\}_{h\in\mathbb{N}}$ is tight and has bounded second moments. Let μ_τ be a narrow limit point. Proposition 3.7 yields, for any $\nu \in \mathscr{P}_2(\mathbb{R}^d)$,

$$\begin{aligned}
\mathcal{W}(\mu_\tau) + \frac{1}{2\tau}\, d_W^2(\mu_\tau,\mu) &\le \liminf_{h\to\infty}\left(\mathcal{W}_h(\mu_\tau^h) + \frac{1}{2\tau}\, d_W^2(\mu_\tau^h,\mu)\right)\\
&\le \liminf_{h\to\infty}\left(\mathcal{W}_h(\nu) + \frac{1}{2\tau}\, d_W^2(\nu,\mu)\right)\\
&= \mathcal{W}(\nu) + \frac{1}{2\tau}\, d_W^2(\nu,\mu).
\end{aligned}$$

This shows that μ_τ is indeed a minimizer for (3.10). Let $\hat{\gamma}_\tau^h \in \Gamma_o(\mu_\tau^h,\mu)$ and let γ_τ^h be the corresponding rescaled plans (see Definition 3.12). If $\{h(m)\}_{m\in\mathbb{N}} \subset \mathbb{N}$ is a sequence such that $\mu_\tau^{h(m)}$ narrowly converges to μ_τ, since $\{\mu_\tau^{h(m)}\}_{m\in\mathbb{N}}$ has uniformly bounded second moments due to (3.12), by Proposition 3.11 we have (possibly on a further subsequence that we do not relabel) the narrow $\mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d)$ limits

$$\hat{\gamma}_\tau^{h(m)} \rightharpoonup \hat{\gamma}_\tau, \qquad \gamma_\tau^{h(m)} \rightharpoonup \gamma_\tau$$

and also $\bar{\gamma}_\tau^{h(m)} \to \bar{\gamma}_\tau$ weakly in the sense of Definition 3.8. Let $\mathbf{d}$ be a distance which metrizes this weak convergence. Indeed, it is metrizable thanks to Remark 3.9 since

$$\begin{aligned}
\int_{\mathbb{R}^d} |\bar{\gamma}_\tau^{h(m)}(x)|^2\, d\mu_\tau^{h(m)}(x) \;&\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y|^2\, d(\gamma_\tau^{h(m)})_x(y)\, d\mu_\tau^{h(m)}(x) = \int_{\mathbb{R}^d\times\mathbb{R}^d} |y|^2\, d\gamma_\tau^{h(m)}(x,y)\\
&= \int_{\mathbb{R}^d} \frac{|y-x|^2}{\tau^2} d\hat{\gamma}_t^{h(m)}(y) = \frac{1}{\tau^2} d_W^2(\mu_\tau^{h(m)},\mu)
\end{aligned}$$

which is uniformly bounded in m for any fixed $0 < \tau \le \tilde{\tau}$ due to (3.12).

Then, if $\tau(n)$ is a vanishing sequence, we can extract a further subsequence $\{h(n)\}_{n\in\mathbb{N}}$ from $\{h(m)\}_{m\in\mathbb{N}}$ such that,

$$\mathbf{d}(\bar{\gamma}_{\tau(n)}^{h(n)}, \bar{\gamma}_{\tau(n)}) < \frac{1}{n}.$$

Then we have

$$\begin{aligned}
\mathbf{d}(\bar{\gamma}_{\tau(n)}^{h(n)}, \partial^o\mathcal{W}(\mu)) &\le \mathbf{d}(\bar{\gamma}_{\tau(n)}^{h(n)}, \bar{\gamma}_{\tau(n)}) + \mathbf{d}(\bar{\gamma}_{\tau(n)}, \partial^o\mathcal{W}(\mu))\\
&\le \frac{1}{n} + \mathbf{d}(\bar{\gamma}_{\tau(n)}, \partial^o\mathcal{W}(\mu))
\end{aligned}$$

Invoking Proposition 3.13, we know that $\bar{\gamma}_\tau$ converges to $\partial^o\mathcal{W}(\mu)$ weakly in the sense of Definition 3.8 as $\tau \to 0$. Hence, passing to the limit as $n \to \infty$, we see that $\bar{\gamma}_n := \bar{\gamma}_{\tau(n)}^{h(n)}$ weakly converge to $\partial^o\mathcal{W}(\mu)$ in the sense of Definition 3.8.

Finally, by Proposition 3.13 we know that, for any n, $\bar{\gamma}_n \in \partial_S \mathcal{W}_n(\mu_n)$. Since $\mathbf{W}_n$ is $C^{1,1}$, by the characterization of strong subdifferential of Proposition 3.2 we have

$$\bar{\gamma}_n(x) = \int_{\mathbb{R}^d} \nabla_1 \mathbf{W}_n(x,y)\, d\mu(y)$$

and the proof is concluded. $\square$

Proof of **Theorem 3.4**. Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, $\boldsymbol{\xi} = \partial^o \mathcal{W}(\mu)$ and μ_n the sequence given by Lemma 3.14. By Proposition 3.2, the only element of $\partial_S \mathcal{W}_n(\mu_n)$ is given by

$$\boldsymbol{\xi}_n(x) := \int_{\mathbb{R}^d} \nabla_1 \mathbf{W}_n(x,y)\, d\mu_n(y).$$

Let us consider the maps $(\mathbf{i}, \nabla \mathbf{W}_n) : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow (\mathbb{R}^d)^4$ given by

$$(\mathbf{i}, \nabla \mathbf{W}_n)(x,y) = (x, y, \nabla \mathbf{W}_n(x,y)).$$

Introducing the measures

$$\nu_n := (\mathbf{i}, \nabla \mathbf{W}_n)_{\#}(\mu_n \times \mu_n),$$

by Lemma 3.14 we have, for any $\zeta \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned} \int_{(\mathbb{R}^d)^4} \langle v, \zeta(x)\rangle\, d\nu_n(x,y,v,w) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \nabla_1 \mathbf{W}_n(x,y), \zeta(x)\rangle\, d(\mu_n \times \mu_n)(x,y) \\ &= \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}_n, \zeta\rangle\, d\mu_n \longrightarrow \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}, \zeta\rangle\, d\mu. \end{aligned} \tag{3.13}$$

The sequence ν_n is tight. Indeed, from the quadratic growth of $\mathbf{W}$ and $\mathbf{W}_n$ at infinity and the boundedness of μ_n in $\mathscr{P}_2(\mathbb{R}^d)$ we obtain the uniform estimate

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |x|^2\, d(\nabla_1 \mathbf{W}_{n\#}\mu_n)(x) = \sup_n \int_{\mathbb{R}^d} |\nabla_1 \mathbf{W}_n|^2\, d\mu_n(x) < +\infty,$$

which implies the tightness of the marginals of ν_n. Then we can extract a subsequence (that we do not relabel) narrowly converging to some $\nu \in \mathscr{P}((\mathbb{R}^d)^4)$. Moreover, for $\zeta \in C_0^\infty(\mathbb{R}^d)$, due to the linear growth of the integrand we have

$$\lim_{n\to\infty} \int_{(\mathbb{R}^d)^4} \langle v, \zeta(x)\rangle\, d\nu_n(x,y,v,w) = \int_{(\mathbb{R}^d)^4} \langle v, \zeta(x)\rangle\, d\nu(x,y,v,w).$$

The narrow convergence of measures implies that $\mathrm{supp}(\nu)$ is contained in the Kuratowski minimum limit of the supports of ν_n (see for instance [AGS, Proposition 5.1.8]), i.e. for every $(x,y,\boldsymbol{\eta}) \in \mathrm{supp}(\nu)$ there exists a sequence $(x_n, y_n, \boldsymbol{\eta}_n) \in \mathrm{supp}(\nu_n)$ such that $(x_n, y_n, \boldsymbol{\eta}_n)$ converges to $(x,y,\boldsymbol{\eta})$. Since, by definition of ν_n, $\mathrm{supp}(\nu_n) \subset \mathrm{graph}(\partial \mathbf{W}_n)$, then $\mathrm{supp}(\nu) \subset \mathrm{graph}(\partial \mathbf{W})$. Indeed $\boldsymbol{\eta}_n \in \partial \mathbf{W}_n(x_n, y_n)$ and passing to the limit in the subdifferential inequality we obtain that $\boldsymbol{\eta} \in \partial \mathbf{W}(x,y)$.

Disintegrating ν with respect to (x,y), we obtain the measurable family of measures $(x,y) \mapsto \nu_{x,y}$ such that

$$\nu = \int_{\mathbb{R}^d \times \mathbb{R}^d} \nu_{x,y}\, d(\mu \times \mu)(x,y).$$

It follows that $\mathrm{supp}(\nu_{x,y}) \subset \partial\mathbf{W}(x,y)$. In the limit

$$
\begin{aligned}
\lim_{n\to\infty}\int_{\mathbb{R}^d}\langle\boldsymbol{\xi}_n(x),\zeta(x)\rangle\,d\mu_n(x) &= \int_{(\mathbb{R}^d)^4}\langle v,\zeta(x)\rangle\,d\nu(x,y,v,w)\\
&= \frac{1}{2}\int_{(\mathbb{R}^d)^4}\langle v,\zeta(x)\rangle\,d\nu(x,y,v,w) + \frac{1}{2}\int_{(\mathbb{R}^d)^4}\langle w,\zeta(y)\rangle\,d\nu(x,y,v,w)\\
&= \frac{1}{2}\int_{\mathbb{R}^d}\left\langle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\times\mathbb{R}^d} v\,d\nu_{x,y}(v,w)\,d\mu(y),\zeta(x)\right\rangle\,d\mu(x)\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^d}\left\langle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\times\mathbb{R}^d} w\,d\nu_{x,y}(v,w)\,d\mu(x),\zeta(y)\right\rangle\,d\mu(y)\\
&= \int_{\mathbb{R}^d}\left\langle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{2}\,(v\,d\nu_{x,y} + w\,d\nu_{y,x})\,(v,w)\,d\mu(y),\zeta(x)\right\rangle\,d\mu(x).
\end{aligned}
$$

Here we made use of the symmetry of $\mathbf{W}$, which entails

$$
\int_{(\mathbb{R}^d)^4}\phi(v)\varphi(x)\,d\nu(x,y,v,w) = \int_{(\mathbb{R}^d)^4}\phi(w)\varphi(y)\,d\nu(x,y,v,w)
$$

for any functions ϕ,φ, by using this latter equality for ν_n (since $\mathbf{W}_n$ is symmetric) and passing it to the limit. Defining

$$
\boldsymbol{\eta}^1(x,y) := \int_{\mathbb{R}^d\times\mathbb{R}^d} v\,d\nu_{x,y}(v,w), \quad \text{and} \quad \boldsymbol{\eta}^2(x,y) := \int_{\mathbb{R}^d\times\mathbb{R}^d} w\,d\nu_{x,y}(v,w),
$$

we obtain

$$
\lim_{n\to\infty}\int_{\mathbb{R}^d}\langle\boldsymbol{\xi}_n(x),\zeta(x)\rangle\,d\mu_n(x) = \int_{\mathbb{R}^d}\left\langle\int_{\mathbb{R}^d}\frac{1}{2}\,\big(\boldsymbol{\eta}^1(x,y)+\boldsymbol{\eta}^2(y,x)\big)\,d\mu(y),\zeta(x)\right\rangle\,d\mu(x).
$$

By (3.13), we get

$$
\boldsymbol{\xi}(x) = \int_{\mathbb{R}^d}\frac{1}{2}\,\big(\boldsymbol{\eta}^1(x,y)+\boldsymbol{\eta}^2(y,x)\big)\,d\mu(y)
$$

On the other hand, we proved that $\mathrm{supp}(\nu_{x,y}) \subset \partial\mathbf{W}(x,y)$. Since $\boldsymbol{\eta}^1(x,y),\boldsymbol{\eta}^2(x,y)$ are barycenters and $\partial\mathbf{W}(x,y)$ is convex, we have that $\big(\boldsymbol{\eta}^1(x,y),\boldsymbol{\eta}^2(x,y)\big)\in\partial\mathbf{W}(x,y)$. □

4. The convolution case

4.1. **The case of W depending on the difference.** In the case of assumption (1.8) we can particularize the results above.

Lemma 4.1. *Let* (1.8) *hold.* $(\boldsymbol{\eta}^1,\boldsymbol{\eta}^2)\in\partial\mathbf{W}$ *if and only if there exists* $\boldsymbol{\eta}\in\partial W$*, such that* $(\boldsymbol{\eta}^1,\boldsymbol{\eta}^2) = (\boldsymbol{\eta},-\boldsymbol{\eta})$.

Proof. Assume that W is convex, the general case follows considering $x\mapsto W(x)-\frac{\lambda}{2}|x|^2$. If $\boldsymbol{\eta}\in\partial W$ we have for every $(\tilde{x},\tilde{y})\in\mathbb{R}^d\times\mathbb{R}^d$

$$
\begin{aligned}
\mathbf{W}(\tilde{x},\tilde{y})-\mathbf{W}(x,y) = W(\tilde{x}-\tilde{y})-W(x-y) &\geq \langle\boldsymbol{\eta}(x-y),\tilde{x}-\tilde{y}-(x-y)\rangle\\
&= \langle(\boldsymbol{\eta}(x-y),-\boldsymbol{\eta}(x-y)),(\tilde{x},\tilde{y})-(x,y)\rangle\,,
\end{aligned}
$$

which means that $(\boldsymbol{\eta}(x-y),-\boldsymbol{\eta}(x-y))\in\partial\mathbf{W}(x,y)$ by making the abuse of notation $\boldsymbol{\eta}(x,y)\equiv\boldsymbol{\eta}(x-y)$.

On the other hand, if $(\boldsymbol{\eta}^1, \boldsymbol{\eta}^2) \in \partial \mathbf{W}$, then for every $(\tilde{x}, \tilde{y}) \in \mathbb{R}^d \times \mathbb{R}^d$ we have

$$\begin{aligned} W(\tilde{x}-\tilde{y}) - W(x-y) = \mathbf{W}(\tilde{x},\tilde{y}) - \mathbf{W}(x,y) &\geq \left\langle \left(\boldsymbol{\eta}^1(x,y), \boldsymbol{\eta}^2(x,y)\right), ((\tilde{x},\tilde{y}) - (x,y)) \right\rangle \\ &= \langle \boldsymbol{\eta}^1(x,y), (\tilde{x}-x)\rangle + \langle \boldsymbol{\eta}^2(x,y), (\tilde{y}-y)\rangle. \end{aligned} \tag{4.1}$$

Assuming in particular that $x - y = \tilde{x} - \tilde{y}$ the inequality above reduces to

$$0 \geq \langle \boldsymbol{\eta}^1(x,y) + \boldsymbol{\eta}^2(x,y), (\tilde{y}-y)\rangle,$$

and the arbitrariness of $\tilde{y}$ implies that $\boldsymbol{\eta}^1 = -\boldsymbol{\eta}^2$. Using this relation in (4.1) we obtain

$$W(\tilde{x}-\tilde{y}) - W(x-y) \geq \langle \boldsymbol{\eta}^1(x,y), (\tilde{x}-\tilde{y}) - (x-y)\rangle\,,$$

which means that $\boldsymbol{\eta}^1 \in \partial W$. □

Applying the main results of the general case we obtain the following

Corollary 4.2. *Let assumption* (1.8) *hold. If $\boldsymbol{\eta}$ is a Borel measurable anti-symmetric selection of ∂W, then for any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$,*

$$\boldsymbol{\xi}(x) := \int_{\mathbb{R}^d} \boldsymbol{\eta}(x-y)\, d\mu(y) = \boldsymbol{\eta} * \mu \in \partial_S \mathcal{W}(\mu). \tag{4.2}$$

Conversely, if $\boldsymbol{\xi} = \partial^o \mathcal{W}(\mu)$, then there exists a Borel measurable anti-symmetric selection $\boldsymbol{\eta} \in \partial W$ such that

$$\boldsymbol{\xi}(x) = \int_{\mathbb{R}^d} \boldsymbol{\eta}(x-y)\, d\mu(y) = \boldsymbol{\eta} * \mu.$$

Proof. Let $\boldsymbol{\eta}$ be an antisymmetric selection in ∂W. In particular $\boldsymbol{\eta}(x-y) = -\boldsymbol{\eta}(y-x)$ for every $x, y \in \mathbb{R}^d$ and by Lemma 4.1 $(\boldsymbol{\eta}^1(x,y), \boldsymbol{\eta}^2(x,y)) := (\boldsymbol{\eta}(x-y), \boldsymbol{\eta}(y-x)) \in \partial \mathbf{W}(x,y)$. Applying Theorem 3.1 to the Borel measurable selection $(\boldsymbol{\eta}^1, \boldsymbol{\eta}^2)$ just defined, we get (4.2).

Conversely, assuming that $\boldsymbol{\xi} = \partial^o \mathcal{W}(\mu)$, by Theorem 3.4 and Lemma 4.1 we obtain that

$$\boldsymbol{\xi}(x) = \int_{\mathbb{R}^d} \frac{1}{2}(\boldsymbol{\eta}^1(x-y) - \boldsymbol{\eta}^1(y-x))\, d\mu(y)\,.$$

By choosing $\boldsymbol{\eta}(z) = \frac{1}{2}\left(\boldsymbol{\eta}^1(z) - \boldsymbol{\eta}^1(-z)\right)$, we conclude. □

Remark 4.3. We observe that if $\boldsymbol{\eta}$ is a Borel measurable selection of ∂W such that for any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$

$$\boldsymbol{\xi}(x) := \int_{\mathbb{R}^d} \boldsymbol{\eta}(x-y)\, d\mu(y) = \boldsymbol{\eta} * \mu \in \partial_S \mathcal{W}(\mu),$$

then $\boldsymbol{\eta}$ is antisymmetric. Indeed, let $\boldsymbol{\eta} \in \partial W$ be such that

$$\mathcal{W}(\nu) - \mathcal{W}(\mu) \geq \int_{\mathbb{R}^d} \left\langle \int_{\mathbb{R}^d} \boldsymbol{\eta}(x-y)\, d\mu(y), z - x \right\rangle d\gamma(x,z),$$

for $\gamma \in \Gamma(\mu, \nu)$. Choosing $\mu = \delta_{x_1}$ and $\nu = \delta_{x_3}$, $\gamma = \mu \times \nu$, the inequality becomes

$$0 \geq \langle \boldsymbol{\eta}(0), x_2 - x_1\rangle,$$

and since this must hold for any $x_1, x_2 \in \mathbb{R}^d$, we deduce $\boldsymbol{\eta}(0) = 0$. Moreover, taking into account that $\boldsymbol{\eta}(0) = 0$, if $\mu = \frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_2}$, $\nu = \frac{1}{2}\delta_{x_3} + \frac{1}{2}\delta_{x_4}$, with $|x_1 - x_3| \leq |x_1 - x_4|$ and $|x_2 - x_4| \leq |x_2 - x_3|$, the subdifferential inequality reduces to

$$W(x_3 - x_4) - W(x_1 - x_2) \geq \langle \boldsymbol{\eta}(x_1 - x_2), x_3 - x_1\rangle + \langle \boldsymbol{\eta}(x_2 - x_1), x_4 - x_2\rangle.$$

In particular, for $x_3 - x_4 = x_1 - x_2$ we get

$$0 \geq \langle \boldsymbol{\eta}(x_1 - x_2) + \boldsymbol{\eta}(x_2 - x_1), x_3 - x_1 \rangle,$$

which yields $\boldsymbol{\eta}(x_1 - x_2) = -\boldsymbol{\eta}(x_2 - x_1)$ for any $x_1, x_2 \in \mathbb{R}^d$.

Remark 4.4. If $\mu \ll \mathcal{L}^d$ we can conclude that

$$\int_{\mathbb{R}^d} \boldsymbol{\eta}(x-y)\, d\mu(y) \in \partial \mathcal{W}(\mu)$$

for any Borel selection $\boldsymbol{\eta}$ in ∂W. Indeed, in this case the set where ∂W is not a singleton is μ-negligible. That is, in the integral we can restrict to the points where W has a gradient (there is no need to select), and in that case $\nabla W * \mu$ belongs to the Wasserstein subdifferential of $\mathcal{W}$ at μ (it is actually its minimal selection), as shown in [CDFLS].

4.2. **The radial case.** In the radial case, we are able to give a more explicit characterization of the minimal selection of the Wasserstein subdifferential. Before stating our theorem, we recall that

$$\partial^o W(x) = \operatorname{argmin}\{|\mathbf{y}| : \mathbf{y} \in \partial W(x)\}.$$

We have the following

Theorem 4.5. *Let* $\mathbf{W}$ *be convex and such that* (1.9) *holds. Then*

$$\partial^o \mathcal{W}(\mu) = (\partial^o W) * \mu \qquad \forall \mu \in \mathscr{P}_2(\mathbb{R}^d). \tag{4.3}$$

Proof. Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Joining together the results of Theorem 3.4 and corollary 4.2, we know that the $\partial^o \mathcal{W}(\mu)$ has the form of a convolution with an anti-symmetric selection in the subdifferential of W. Hence, in order to find the explicit form of $\partial^o \mathcal{W}(\mu)$, we have to minimize the quantity

$$\|\boldsymbol{\eta} * \mu\|^2_{L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)} = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \boldsymbol{\eta}(x-y)\, d\mu(y) \right|^2 d\mu(x)$$

among all measurable anti-symmetric selections $\boldsymbol{\eta}$ in ∂W. Let us introduce some notation. We define, for any $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$, the singleton sets

$$A_v^+(x) := \{x - v\}, \quad A_v^-(x) := \{x + v\}.$$

Note that by antisymmetry of $\boldsymbol{\eta}$, we get

$$\int_{A_v^\pm(x)} \boldsymbol{\eta}(x-y)\, d\mu(y) = \pm \mu\left(\{x \mp v\}\right) \boldsymbol{\eta}(v)\,. \tag{4.4}$$

Fix a point $v \in \mathbb{R}^d$, different from the origin. We have

$$\left| \int_{\mathbb{R}^d} \boldsymbol{\eta}(x-y)\, d\mu(y) \right|^2 = \left| \int_{A_v^+(x) \cup A_v^-(x)} \boldsymbol{\eta}(x-y)\, d\mu(y) + \int_{\mathbb{R}^d \setminus (A_v^+(x) \cup A_v^-(x))} \boldsymbol{\eta}(x-y)\, d\mu(y) \right|^2$$

We suitably expand the square as $(a+b+c)^2 = a^2 + 2a(b+c) + b^2 + 2b(a+c) - 2ab + c^2$. This way, since $A_v^+(x) \cup (\mathbb{R}^d \setminus (A_v^+(x) \cup A_v^-(x))) = \mathbb{R}^d \setminus A_v^-(x)$ and $A_v^-(x) \cup (\mathbb{R}^d \setminus (A_v^+(x) \cup A_v^-(x))) =$

$\mathbb{R}^d \setminus A_v^+(x)$, the right hand side can be rewritten as

$$
\begin{aligned}
&\left|\int_{A_v^+(x)} \boldsymbol{\eta}(x-y)\,d\mu(y)\right|^2 + 2\left\langle \int_{A_v^-(x)} \boldsymbol{\eta}(x-y)\,d\mu(y), \int_{\mathbb{R}^d\setminus A_v^-(x)} \boldsymbol{\eta}(x-y)\,d\mu(y)\right\rangle \\
&+\left|\int_{A_v^-(x)} \boldsymbol{\eta}(x-y)\,d\mu(y)\right|^2 + 2\left\langle \int_{A_v^+(x)} \boldsymbol{\eta}(x-y)\,d\mu(y), \int_{\mathbb{R}^d\setminus A_v^+(x)} \boldsymbol{\eta}(x-y)\,d\mu(y)\right\rangle \\
&-2\left\langle \int_{A_v^+(x)} \boldsymbol{\eta}(x-y)\,d\mu(y), \int_{A_v^-(x)} \boldsymbol{\eta}(x-y)\,d\mu(y)\right\rangle + R(x),
\end{aligned}
$$

where the remainder term $R(x)$ is given by

$$
R(x) = \left|\int_{\mathbb{R}^d\setminus(A_v^+(x)\cup A_v^-(x))} \boldsymbol{\eta}(x-y)\,d\mu(y)\right|^2,
$$

so that it does not depend on the values of $\boldsymbol{\eta}$ at $\pm v$. Using (4.4), we are left with

$$
\begin{aligned}
\left|\int_{\mathbb{R}^d} \boldsymbol{\eta}(x-y)\,d\mu(y)\right|^2 &= \left(\mu(A_v^+(x)) + \mu(A_v^-(x))\right)^2 |\boldsymbol{\eta}(v)|^2 \\
&\quad + 2\left\langle \int_{A_v^+(x)} \boldsymbol{\eta}(x-y)\,d\mu(y), \int_{\mathbb{R}^d\setminus A_v^+(x)} \boldsymbol{\eta}(x-y)\,d\mu(y)\right\rangle \\
&\quad + 2\left\langle \int_{A_v^-(x)} \boldsymbol{\eta}(x-y)\,d\mu(y), \int_{\mathbb{R}^d\setminus A_v^-(x)} \boldsymbol{\eta}(x-y)\,d\mu(y)\right\rangle + R(x).
\end{aligned}
$$

Making use of the computed identity, we find

$$
\begin{aligned}
\|\boldsymbol{\eta} * \mu\|^2_{L^2(\mathbb{R}^d,\mu;\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} \left(\mu(A_v^+(x)) + \mu(A_v^-(x))\right)^2\,d\mu(x)\right) |\boldsymbol{\eta}(v)|^2 + \mathcal{R} \\
&\quad + 2\int_{\mathbb{R}^d} \left\langle \int_{A_v^+(x)} \boldsymbol{\eta}(x-y)\,d\mu(y), \int_{\mathbb{R}^d\setminus A_v^+(x)} \boldsymbol{\eta}(x-z)\,d\mu(z)\right\rangle d\mu(x) \\
&\quad + 2\int_{\mathbb{R}^d} \left\langle \int_{A_v^-(x)} \boldsymbol{\eta}(x-y)\,d\mu(y), \int_{\mathbb{R}^d\setminus A_v^-(x)} \boldsymbol{\eta}(x-z)\,d\mu(z)\right\rangle d\mu(x) \\
&= \left(\int_{\mathbb{R}^d} \left(\mu(A_v^+(x)) + \mu(A_v^-(x))\right)^2\,d\mu(x)\right) |\boldsymbol{\eta}(v)|^2 + \mathcal{R} \\
&\quad + 2\iint_{\{x-y=v\}} \left\langle \boldsymbol{\eta}(x-y), \int_{\{z\in\mathbb{R}^d: z\neq x-v\}} \boldsymbol{\eta}(x-z)\,d\mu(z)\right\rangle d\mu(x)\,d\mu(y) \\
&\quad + 2\iint_{\{x-y=-v\}} \left\langle \boldsymbol{\eta}(x-y), \int_{\{z\in\mathbb{R}^d: z\neq x+v\}} \boldsymbol{\eta}(x-z)\,d\mu(z)\right\rangle d\mu(x)\,d\mu(y),
\end{aligned}
$$

where $\mathcal{R} = \int_{\mathbb{R}^d} R(x)\,d\mu(x)$. Notice that the integrals with respect to z can be equivalently taken on the set $\{z \in \mathbb{R}^d : z \neq x, y\}$ in the last two lines. Indeed, in the first one the equality $x - y = v$ implies $\{z \in \mathbb{R}^d : z \neq x - v\} = \{z \in \mathbb{R}^d : z \neq y\}$. Similarly for the second one.

Moreover, we can also neglect the set $\{z=x\}$ since $\boldsymbol{\eta}(0)=0$ by anti-symmetry. We exchange x with y in the last integral, so that

$$\begin{aligned}\iint_{\{x-y=-v\}} &\left\langle \boldsymbol{\eta}(x-y), \int_{\{z\neq x,y\}} \boldsymbol{\eta}(x-z)\, d\mu(z)\right\rangle d\mu(x)\, d\mu(y)\\ &= \int\int_{\{y-x=-v\}} \left\langle \boldsymbol{\eta}(y-x), \int_{\{z\neq x,y\}} \boldsymbol{\eta}(y-z)\, d\mu(z)\right\rangle d\mu(y)\, d\mu(x)\\ &= \int\int_{\{x-y=v\}} \left\langle -\boldsymbol{\eta}(x-y), \int_{\{z\neq x,y\}} \boldsymbol{\eta}(y-z)\, d\mu(z)\right\rangle d\mu(y)\, d\mu(x).\end{aligned}$$

Hence from the previous computation we get

$$\begin{aligned}\|\boldsymbol{\eta}*\mu\|^2_{L^2(\mathbb{R}^d,\mu;\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} \left(\mu(A_v^+(x))+\mu(A_v^-(x))\right)^2\, d\mu(x)\right)|\boldsymbol{\eta}(v)|^2+\mathcal{R}\\ &\quad + 2\iint_{\{x-y=v\}}\int_{\{z\neq x,y\}} \langle \boldsymbol{\eta}(v), \boldsymbol{\eta}(x-z)-\boldsymbol{\eta}(y-z)\rangle\, d\mu(z)\, d\mu(x)\, d\mu(y). \end{aligned}\tag{4.5}$$

On the set $\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d : x-y=v\}$, we rewrite the interior integral in the following way

$$\begin{aligned}\int_{\{z\neq x,y\}} \langle \boldsymbol{\eta}(v), \boldsymbol{\eta}(x-z)-\boldsymbol{\eta}(y-z)\rangle\, d\mu(z) &= \int_{\{z\neq x,y,x+v,y-v\}} \langle \boldsymbol{\eta}(v), \boldsymbol{\eta}(x-z)-\boldsymbol{\eta}(y-z)\rangle\, d\mu(z)\\ &\quad + \int_{\{z=x+v\}\cup\{z=y-v\}} \langle \boldsymbol{\eta}(v), \boldsymbol{\eta}(2v)-\boldsymbol{\eta}(v)\rangle\, d\mu(z)\\ &= \int_{\{z\neq x,y,x+v,y-v\}} \langle \boldsymbol{\eta}(v), \boldsymbol{\eta}(x-z)-\boldsymbol{\eta}(y-z)\rangle\, d\mu(z)\\ &\quad + \left(\mu(A_v^-(x))+\mu(A_v^+(y))\right)\left(\langle \boldsymbol{\eta}(v), \boldsymbol{\eta}(2v)\rangle-|\boldsymbol{\eta}(v)|^2\right),\end{aligned}$$

and thus

$$\begin{aligned}\iint_{\{x-y=v\}}&\int_{\{z\neq x,y\}} \langle \boldsymbol{\eta}(v), \boldsymbol{\eta}(x-z)-\boldsymbol{\eta}(y-z)\rangle\, d\mu(z)\, d\mu(x)\, d\mu(y)\\ &= \iint_{\{x-y=v\}}\int_{\{z\neq x,y,x+v,y-v\}} \langle \boldsymbol{\eta}(v), \boldsymbol{\eta}(x-z)-\boldsymbol{\eta}(y-z)\rangle\, d\mu(z)\, d\mu(x)\, d\mu(y)\\ &\quad + 2\left(\int_{\mathbb{R}^d} (\mu(A_v^+(x))\mu(A_v^-(x)))\, d\mu(x)\right)\left(\langle \boldsymbol{\eta}(v), \boldsymbol{\eta}(2v)\rangle-|\boldsymbol{\eta}(v)|^2\right).\end{aligned}$$

Here, in the last term we used

$$\iint_{\{x-y=v\}} \mu(A_v^-(x))\, d\mu(x)\, d\mu(y) = \iint_{\{x-y=v\}} \mu(A_v^+(y))\, d\mu(x)\, d\mu(y) = \int \mu(A_v^+(x))\mu(A_v^-(x))\, d\mu(x)\,.$$

Substituting in (4.5) we get the final expression

$$\begin{aligned}\|\boldsymbol{\eta}*\mu\|^2_{L^2(\mathbb{R}^d,\mu;\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} \left(\mu(A_v^+(x))-\mu(A_v^-(x))\right)^2\, d\mu(x)\right)|\boldsymbol{\eta}(v)|^2+\mathcal{R}\\ &\quad + 2\iint_{\{x-y=v\}}\int_{\{z\neq x,y,x+v,y-v\}} \langle \boldsymbol{\eta}(v), \boldsymbol{\eta}(x-z)-\boldsymbol{\eta}(y-z)\rangle\, d\mu(z)\, d\mu(x)\, d\mu(y)\\ &\quad + 4\left(\int_{\mathbb{R}^d} (\mu(A_v^+(x))\mu(A_v^-(x)))\, d\mu(x)\right)\langle \boldsymbol{\eta}(v), \boldsymbol{\eta}(2v)\rangle.\end{aligned}\tag{4.6}$$

Since W is radial, for all $v \in \mathbb{R}^d$, $\boldsymbol{\eta}(v)$ takes the form

$$\boldsymbol{\eta}(v) = \frac{\eta(|v|)}{|v|}\, v, \tag{4.7}$$

where η denotes the generic selection in the subdifferential of the profile function w in (1.9). Convexity implies that $\eta(|v|) \geq 0$, and thus

$$\langle \boldsymbol{\eta}(v), v\rangle = |\boldsymbol{\eta}(v)|\,|v| \tag{4.8}$$

for all $v \in \mathbb{R}^d$. The convexity hypothesis gives

$$\langle x-y, \boldsymbol{\eta}(x-z) - \boldsymbol{\eta}(y-z)\rangle \geq 0,$$

for any x, y, z. Combining this with (4.7) since $\eta(|v|) \geq 0$, we conclude that

$$\langle \boldsymbol{\eta}(x-y), \boldsymbol{\eta}(x-z) - \boldsymbol{\eta}(y-z)\rangle \geq 0.$$

From this last inequality it is readily seen that the scalar products appearing in (4.6) are (pointwise) nonnegative. Therefore, since the remainder $\mathcal{R}$ is independent on $\boldsymbol{\eta}(v)$ and $\boldsymbol{\eta}(-v)$, it is clear from (4.6) and (4.8) that $\|\boldsymbol{\eta} * \mu\|^2_{L^2(\mathbb{R}^d,\mu;\mathbb{R}^d)}$ at least does not increase if we choose the value of $\boldsymbol{\eta}$ at v to be the one that minimizes $|\boldsymbol{\eta}(v)|$. By the arbitrariness of $v \in \mathbb{R}^d$ we get (4.3). □

Example 4.6. The above result fails if we omit the convexity assumption. Indeed, let us consider a 1-dimensional example. Let

$$\widehat{W}(x) = \frac{1}{2}\,|x^2-1|.$$

Notice that this function is radial and -1-convex, and its subdifferential is

$$\partial W(x) = \begin{cases} x & \text{for } |x| > 1, \\ -x & \text{for } |x| < 1, \\ [-1,1] & \text{for } x = \pm 1. \end{cases}$$

Let us consider the measure $\mu = \frac{1}{3}\delta_{x_1} + \frac{1}{3}\delta_{x_2} + \frac{1}{3}\delta_{x_3}$. We have to minimize the quantity

$$\|\boldsymbol{\eta} * \mu\|_{L^2(\mathbb{R},\mu;\mathbb{R})} = \frac{1}{27}\sum_{j=1}^{3}\left|\sum_{i=1}^{3}\boldsymbol{\eta}(x_j - x_i)\right|^2$$

among all measurable antisymmetric selections $\boldsymbol{\eta}$ in ∂W.

If we let $x_1 = 1$, $x_2 = 0$, $x_3 = 3/4$, the only points where it is needed to select are ± 1, corresponding to $\pm(x_1 - x_2)$, hence expanding the sum above (using the antisymmetry) it is clear that we reduce to find the minimizer of

$$\min\{\boldsymbol{\eta}(x_1-x_2)^2 + \boldsymbol{\eta}(x_1-x_2)(\boldsymbol{\eta}(x_1-x_3) - \boldsymbol{\eta}(x_2-x_3)) : \boldsymbol{\eta}(x_1-x_2) \in [-1,1]\}.$$

Here $x_1 - x_3 = 1/4$ (so that $\boldsymbol{\eta}(x_1-x_3) = -1/4$) and $x_2 - x_3 = -3/4$ (so that $\boldsymbol{\eta}(x_2-x_3) = 3/4$). Then, letting $y = \boldsymbol{\eta}(x_1-x_2)$, we are left with the problem $\min\{y^2 - y : y \in [-1,1]\}$, whose solution is $y = 1/2$. This is different from the element of minimal norm in $\partial W(x_2 - x_1)$, which of course is 0.

We also point out that in this non convex case the choice of the selection is not independent from the measure μ. Indeed, if we change the value of x_3 to be, for instance, $-1/4$, we have $\boldsymbol{\eta}(x_1-x_3) = 5/4$ and $\boldsymbol{\eta}(x_2-x_3) = -1/4$, then we have to solve $\min\{y^2 + 3y/2 : y \in [-1,1]\}$, and the solution is $y = \boldsymbol{\eta}(1) = -\boldsymbol{\eta}(-1) = -3/4$.

Example 4.7. The result of Theorem 4.5 fails if we omit the radial hypothesis on W. As a counterexample we provide a convex function $\mathbf{W}$ satisfying all the assumptions (1.2),(1.3), (1.4), (1.8) and a measure $\mu \in \mathscr{P}_2(\mathbb{R}^2)$ such that

$$\partial^o \mathcal{W}(\mu) \neq (\partial^o W) * \mu.$$

Let the graph of W be a pyramid with vertex in the origin, with varying slopes, given by

$$\nabla W(x,y) = \begin{cases} (1,0) & \text{for} \quad 0 < x < 1,\ -x < y < x, \\ (\theta, 0) & \text{for} \quad x > 1,\ -x < y < x, \end{cases}$$

where $\theta > 2$. W is then defined by symmetry, such that its level sets are squares centered in the origin. Let $\mu = \frac{1}{3}\delta_{x_1} + \frac{1}{3}\delta_{x_2} + \frac{1}{3}\delta_{x_3}$, where

$$x_2 - x_1 = (1,1), \quad x_3 - x_2 = (-1/2 - \varepsilon, 1/2), \quad x_3 - x_1 = (1/2 - \varepsilon, 3/2).$$

Among these points, for small enough $\varepsilon > 0$, ∂W is not a singleton only at $x_2 - x_1$, and in particular it is the convex set K of $\mathbb{R}^2$ defined as

$$K = \{(x,y) \in \mathbb{R}^2 :\ x \geq 0,\ y \geq 0,\ 1 - x \leq y \leq \theta - x\}.$$

We let $\boldsymbol{\eta}_{ij}$, $i,j = 1,2,3$ denote the generic element of ∂W at $x_i - x_j$. In this particular case

$$\partial^o \mathcal{W}(\mu) = \operatorname{argmin} \left\{ \frac{1}{27} \sum_{j=1}^{3} \left| \sum_{i=1}^{3} \boldsymbol{\eta}_{ji} \right|^2 : \boldsymbol{\eta} \in \partial W, \boldsymbol{\eta}_{ji} = -\boldsymbol{\eta}_{ij} \right\}.$$

Since $\boldsymbol{\eta}_{13} = \nabla W(x_1 - x_3), \boldsymbol{\eta}_{23} = \nabla W(x_2 - x_3)$, we are left to minimize with respect to the unique variable $\boldsymbol{\eta}_{21}$, that is, the minimization problem above reduces to

$$\min_{\boldsymbol{\eta}_{21} \in K} | - \boldsymbol{\eta}_{21} + \boldsymbol{\eta}_{13}|^2 + \frac{1}{27} |\boldsymbol{\eta}_{23} + \boldsymbol{\eta}_{21}|^2 + \frac{1}{27} |\boldsymbol{\eta}_{32} + \boldsymbol{\eta}_{31}|^2.$$

We have $\boldsymbol{\eta}_{13} = (0, -\theta)$ and $\boldsymbol{\eta}_{23} = (1,0)$, and hence it is immediate to check the solution is the minimizer of

$$|\boldsymbol{\eta}_{21}|^2 + \langle \boldsymbol{\eta}_{21}, (1,\theta) \rangle. \tag{4.9}$$

If $\boldsymbol{\eta} = \partial^o W$ we have that $\boldsymbol{\eta}_{21}$ is the element of minimal norm in K, that is $(1/2, 1/2)$, and in this case the quantity above takes the value $1 + \theta/2$. But $\theta > 2$, so that the minimum value in K of the quadratic expression (4.9) is 2, attained in a different point, that is $\boldsymbol{\eta}_{21} = (1,0)$.

5. Particle system

As in [CDFLS], the well-posedness result in Theorem 2.3 of measure solutions allows to put in the same framework particle and continuum solutions. Assume that we are given N pointwise particles, each carrying a mass m_i, with $\sum_i m_i = 1$. Let $x_i(t)$ be the position in $\mathbb{R}^d$ of the i-th particle at time t. Let $\boldsymbol{\eta}^1(x,y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\boldsymbol{\eta}^2(x,y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be selections in the subdifferentials $\partial_1 \mathbf{W}$ and $\partial_2 \mathbf{W}$ respectively. We consider the system

$$\frac{dx_i}{dt} = \frac{1}{2} \sum_{j=1}^{N} m_j (\boldsymbol{\eta}^1(x_j - x_i) + \boldsymbol{\eta}^2(x_i - x_j)), \qquad i = 1, \ldots, N.$$

If the x_i are absolutely continuous curves, the empirical measure $\mu(t) = \sum_{i=1}^N m_i \delta_{x_i(t)}$ solves the PDE

$$\frac{d}{dt}\mu(t) - \frac{1}{2}\operatorname{div}\left(\left(\int_{\mathbb{R}^d} \boldsymbol{\eta}^1(\cdot, y) + \boldsymbol{\eta}^2(y, \cdot)\, d\mu(y)\right)\mu(t)\right) = 0.$$

Among these selections, we have the minimal one in $\partial\mathcal{W}(\mu)$, hence we have correspondence between our equation (2.5) and the ODE system above. In particular, if we are in the framework of assumption (1.8), the velocity vector field of the continuity equation is written as a convolution and the corresponding ODEs takes the form

$$\frac{dx_i}{dt} = \sum_{j=1}^N m_j \boldsymbol{\eta}_{ji}, \qquad i = 1, \ldots, N, \tag{5.1}$$

where $\boldsymbol{\eta}_{ij} := \boldsymbol{\eta}(x_i - x_j)$ and $\boldsymbol{\eta}$ is the suitable measurable anti-symmetric selection in ∂W.

Remark 5.1 (**Characterization of the element of minimal norm in the radial convex case for particles**). In the particular case of a system of particles, it is more immediate to see how the proof of Theorem 4.5 works. Actually, it was the origin of the idea in how to get the proof in previous section. We include it since we believe it is an instructive proof. Suppose that W is convex, so that the minimal selection in $\partial\mathcal{W}(\mu)$ is expected, after Theorem 4.5, to be $\partial^o W * \mu$. In fact, we have in this case

$$\mu = \sum_{i=1}^N m_i \delta_{x_i}(x), \qquad m_1 + \ldots + m_N = 1,$$

where $x_i \in \mathbb{R}^d$. After Theorem 3.4, we know that we have to search for the Borel antisymmetric selection $\boldsymbol{\eta}$ in the subdifferential of W that minimizes the $L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$ norm of $\boldsymbol{\eta} * \mu$. We have

$$\|\boldsymbol{\eta} * \mu\|^2_{L^2(\mathbb{R}^d,\mu;\mathbb{R}^d)} = \sum_{j=1}^N m_j \left|\sum_{i=1}^N m_i \boldsymbol{\eta}(x_j, x_i)\right|^2.$$

For simplicity we are letting again $\boldsymbol{\eta}(x_j, x_i) = \boldsymbol{\eta}_{ji}$. Hence we have to solve the problem

$$\min\left\{\sum_{j=1}^N m_j \left|\sum_{i=1}^N m_i \boldsymbol{\eta}_{ji}\right|^2 : \boldsymbol{\eta} \in \partial W, \boldsymbol{\eta}_{ji} = -\boldsymbol{\eta}_{ij}\right\}.$$

Suppose first that W is differentiable in all the points $x_j - x_i$. In this case we have $\boldsymbol{\eta}_{ji} = \nabla W(x_j - x_i)$, and the quantity to minimize simply takes the (single) value

$$\sum_{j=1}^N m_j \left|\sum_{i=1}^N m_i \nabla W(x_j - x_i)\right|^2,$$

which is then the square norm of the minimal selection in $\partial\mathcal{W}$. Otherwise, taking advantage of the anti-symmetry of each selection, let us rearrange the terms highlighting $\boldsymbol{\eta}_{JI}$, where I and J are fixed nonequal numbers between 1 and N, and let $x_J - x_I = x_0$. In order to keep simple, we make the assumption that no other couple (i, j) is such that $x_j - x_i = x_0$. This way, it is not difficult to see that, expanding the squares, we can write

$$\sum_{j=1}^{N} m_j \left|\sum_{i=1}^{N} m_i \boldsymbol{\eta}_{ji}\right|^2 = m_I m_J (m_I + m_J)|\boldsymbol{\eta}_{JI}|^2 + 2 m_I m_J \sum_{\substack{k=1\\ k\neq J,\, I}}^{N} m_k \langle \boldsymbol{\eta}_{JI}, (\boldsymbol{\eta}_{Jk} - \boldsymbol{\eta}_{Ik})\rangle + R, \tag{5.2}$$

where R does not depend on the value of $\boldsymbol{\eta}$ at $\pm x_0$. We claim that each scalar product in the sum above is nonnegative, since by convexity of W, any selection $\boldsymbol{\eta}_{ij}$ in $\partial W(x_i - x_j)$ is monotone in the sense that

$$\langle \boldsymbol{\eta}_{jk} - \boldsymbol{\eta}_{ik}, x_j - x_k - x_i + x_k\rangle = \langle \boldsymbol{\eta}_{jk} - \boldsymbol{\eta}_{ik}, x_j - x_i\rangle \geq 0.$$

By assumption *v)*, W is radial ($W(\cdot) = w(|\cdot|)$), and if $\boldsymbol{\eta}_{ji}$ is a selection in $\partial w(|x_j - x_i|)$, then $\boldsymbol{\eta}_{ji}$ is positive by convexity of W and

$$\langle \boldsymbol{\eta}_{ji}, \boldsymbol{\eta}_{jk} - \boldsymbol{\eta}_{ik}\rangle = \frac{\boldsymbol{\eta}_{ji}}{|x_j - x_i|}\langle x_j - x_i, \boldsymbol{\eta}_{jk} - \boldsymbol{\eta}_{ik}\rangle \geq 0 \quad \forall i, j, k.$$

This shows that the claim is correct. Then if we minimize in (5.2) only with respect to the admissible values of $\boldsymbol{\eta}_{JI}$, we see that the minimum argument is found if we take $\boldsymbol{\eta}_{JI} = \partial^o W(x_J - x_I)$ (and the same for $\boldsymbol{\eta}_{IJ}$, by anti-symmetry). But by construction, such minimum argument does not depend on the value of $\boldsymbol{\eta}$ in the other points. We conclude that if $\boldsymbol{\eta}$ is a solution of the full minimization problem, then $\boldsymbol{\eta}_{JI}$ is forced to be the element of minimal norm in $\partial W(x_J - x_I)$. By the arbitrariness of I, J we conclude: the solution is $\partial^o W * \mu$.

Remark 5.2 (**Collapse**)**.** In the radial-convex case, the finite-time collapse argument in [CDFLS] carries over. Indeed, it is enough to substitute ∇W therein with the new object $\partial^o W$. Consider the equation (5.1) for a system of N particles. We notice that, since $\partial^o W$ is anti-symmetric, the center of mass $\frac{1}{N}\sum_{j=1^N} x_j$ does not move during the evolution. We denote such point by x_0. Then, assuming for simplicity that the weights are uniform, there holds

$$\frac{d(x_i - x_0)}{dt} = -\frac{1}{N}\sum_{j=1}^{N} \partial^o w(|x_i - x_j|)\frac{x_i - x_j}{|x_i - x_j|}.$$

Multiplying by $x_i - x_0$ we find

$$\frac{1}{2}\frac{d}{dt}|x_i - x_0|^2 = -\frac{1}{N}\sum_{j=1}^{N} \partial^o w(|x_i - x_j|)\frac{\langle x_i - x_j, x_i - x_0\rangle}{|x_i - x_j|}. \tag{5.3}$$

Now, let $R(t) := \max_{i\in\{1,\dots,N\}}\{|x_i(t) - x_0|\}$. If $I \in \{1, \dots, N\}$ is such that $|x_I(t) - x_0| = R(t)$, at that time we clearly have $\langle x_I - x_j, x_I - x_0\rangle \geq 0$. For any time and for any I with this property, from (5.3) we see that the derivative of $|x_I - x_0|^2$ is nonpositive, so that its maximum governs the evolution of R:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}R^2(t) &= \max_{\{i: x_i(t) = R(t)\}} -\frac{1}{N}\sum_{j=1}^{N} \partial^o w(|x_i - x_j|)\frac{\langle x_i - x_j, x_i - x_0\rangle}{|x_i - x_j|} \\
&\leq \max_{\{i: x_i(t) = R(t)\}} -\frac{1}{N}\frac{\Omega}{2R(t)}\sum_{j=1}^{N}\langle x_i - x_j, x_i - x_0\rangle \\
&\leq \max_{\{i: x_i(t) = R(t)\}} -\frac{1}{N}\frac{\Omega}{2R(t)}|x_i - x_0|^2 = -\frac{\Omega}{2N}R(t),
\end{aligned}$$

where $\Omega = \inf_{(0,+\infty)} \partial^o w$. Here we used the elementary bound $|x_i(t) - x_j(t)| \le 2R(t)$ and the fact that $\sum_{j=1}^N \langle x_I - x_j, x_I\rangle = |x_I - x_0|^2$ if $I \in \{1,\dots,N\}$ is such that $|x_I(t)| = R(t)$. The solution then reaches the asymptotic state $\mu_\infty = \delta_{x_0}$ in finite time if $\Omega > 0$. We can also generalize the result to the case $\Omega = 0$ under suitable assumptions on the behavior of ω at the origin such that a Gronwall estimate can be achieved. For instance if we ask $\frac{\partial^o w(x)}{x}$ to be decreasing on some interval $(0,\varepsilon)$ we are done. For all the details and a more general discussion on this issue we refer to [CDFLS]. We only remark that if we have finite time collapse in the center of mass for a system of particles, by the general stability properties of gradient flow solutions we can also deduce the finite time collapse for general compactly supported initial data.

6. Examples of asymptotic states in the non convex case

In this section, suppose that assumptions (1.8) and (1.9) hold. We have already seen in the case of particle collapse that, for convex nonnegative potentials, there is the asymptotic state consisting of a single Dirac mass. It is easily seen that any Dirac mass minimizes functional $\mathcal{W}$. So, in this case, by the translation invariance property, the only asymptotic solution is in fact the Dirac mass in the center of mass of the initial datum.

In the repulsive-attractive case, one can expect a much bigger variety of stationary solutions. This is the case as proved in 1D in [FR1, FR2] in which the authors show that the set of stationary states for short-range repulsive long-range attractive potentials can be very large and complicated. Moreover, they give examples in which the stationary states are composed of a finite number of Dirac deltas at points and others in which one has integrable compactly supported stationary solutions. They also show that to have integrable or not stationary states depends in how strong the repulsion is at the origin. Some numerical computations indicates that this is also the case in more dimensions [KSUB]. They show that the set of stable stationary states can be large and with complicated supports arising from instability modes of the uniform distribution on a sphere. Finally, in [BCLR] we have studied the stability for radial perturbations of the uniform distribution on a suitable sphere for general radial repulsive-attractive potentials. We will show a related example below.

In this section we are going to present some examples, based on the available characterization of the velocity vector field of the continuity equation of properties of the set of stationary solutions. In particular, we will consider the (-1)-convex potentials in one dimension given by

$$\widetilde{W}(x) = \frac{1}{2}\,|x^2-1|^2, \qquad \widehat{W}(x) = \frac{1}{2}\,|x^2-1|.$$

Both cases correspond to attractive-repulsive potentials with the same behavior in the repulsive part at the origin. However, the change from repulsive to attractive in one case is smooth and in the other, it is only Lipschitz. In fact, in the two cases there many analogies, but also some different behaviors, as we are going to show with the next propositions. First of all, we search for stationary states made by a finite number of particles.

Proposition 6.1. *There exist two-particles stationary states for $\widetilde{W}$ and $\widehat{W}$.*

Proof. Let $\mu = m_1\delta_{x_1} + m_2\delta_{x_2}$, with $m_1 + m_2 = 1$. For functional $\widetilde{W}$ we have to impose the condition

$$\frac{dx_i}{dt} = m_j \nabla\widetilde{W}(x_j - x_i) = 0, \tag{6.1}$$

for $i, j = 1, 2, i \neq j$. But $\nabla\widetilde{W}(x) = 2x(x^2 - 1)$. Hence it is enough to choose x_1, x_2 such that $|x_1 - x_2| = 1$, independently of the weights. We see that we have infinitely many stationary states made by two Dirac deltas. In the case of $\widehat{W}$, the subdifferential is

$$\partial\widehat{W}(x) = \begin{cases} x & \text{if } |x| > 1, \\ -x & \text{if } |x| < 1, \\ [-1, 1] & \text{if } |x| = 1. \end{cases} \tag{6.2}$$

In order to find a stationary state, we have to solve (6.1) with $\boldsymbol{\eta}$ in place of $\nabla\widetilde{W}$, where $\boldsymbol{\eta}$ is the suitable anti-symmetric selection in $\partial\widehat{W}$ realizing the minimal norm in $\partial\mathcal{W}(\mu)$ (see Corollary 4.2). Such selection is then found minimizing the quantity

$$\|\boldsymbol{\eta} * \mu\|^2_{L^2(\mathbb{R},\mu;\mathbb{R})} = \sum_{j=1}^{2} m_j \left|\sum_{i=1}^{2} m_i \boldsymbol{\eta}(x_j - x_1)\right|^2 = m_1 m_2 (m_1 + m_2)(\boldsymbol{\eta}(x_1 - x_2))^2$$

among the admissible selections $\boldsymbol{\eta}$ (we are using the anti-symmetry). Let again $|x_2 - x_1| = 1$, so that it is clear that the minimum above is zero, attained for $\boldsymbol{\eta}(1) = -\boldsymbol{\eta}(-1) = 0$. And this way, the two equations (6.1) are still satisfied. □

Proposition 6.2. *For both functionals $\widetilde{W}$ and $\widehat{W}$, there are no absolutely continuous stationary states in one dimension.*

Proof. Let us consider functional $\widehat{W}$. The argument is based on the fact that, if μ is absolutely continuous, it does not charge the points of non-differentiability of $\widehat{W}$. For a measure μ to be stationary, we have to verify that the corresponding velocity vector field vanishes. That is $\boldsymbol{\eta} * \mu = 0$, where $\boldsymbol{\eta}$ is the usual optimal selection in $\partial\widehat{W}$, as in Corollary 4.2. Suppose that μ is a stationary state and that $\mu = \rho\mathcal{L}^1$, for some $\rho \in L^1(\mathbb{R})$, then

$$\int_{\{|x-y|>1\}} (x - y)\rho(y)\, dy - \int_{\{|x-y|<1\}} (x - y)\rho(y)\, dy = 0.$$

By the translation invariance property, we can fix without loss of generality the center of mass, so we let $\int_{\mathbb{R}} y\rho(y)\, dy = 0$. We deduce

$$2x \int_{\{|x-y|>1\}} \rho(y)\, dy - x = 2 \int_{\{|x-y|>1\}} y\rho(y)\, dy,$$

hence

$$2x \left(\int_{-\infty}^{x-1} \rho(y)\, dy + \int_{x+1}^{+\infty} \rho(y)\, dy \right) - x = 2 \left(\int_{-\infty}^{x-1} y\rho(y)\, dy + \int_{x+1}^{+\infty} y\rho(y)\, dy \right).$$

Let us denote the term in the parenthesis in the left side by $\Theta(x)$ and let us take the derivative with respect to x. We have

$$2\Theta(x) + 2x(\rho(x-1) - \rho(x+1)) - 1 = 2((x-1)\rho(x-1) - (x+1)\rho(x+1)),$$

which yields $2\Theta(x) - 1 = -2(\rho(x-1) - \rho(x+1))$, that is $\Theta'(x) = -\Theta(x) + \frac{1}{2}$. We find

$$\Theta(x) = ke^{-x} + \frac{1}{2}, \quad k \in \mathbb{R},$$

then $\rho(x-1) - \rho(x+1) = -ke^{-x}$. But the integral of ρ is 1, so $k = 0$ and we are left with $\rho(x-1) = \rho(x+1)$. This is a contradiction, since ρ can not be periodic in this case. The proof for $\widetilde{W}$ is analogous, we omit the details. □

The following are more examples of stationary states

Example 6.3. There are stationary states for functional $\widehat{W}$ of the form

$$\mu = m_1\delta_{x_1} + m_2\delta_{x_2} + m_3\delta x_3,$$

with $m_1 + m_2 + m_3 = 1$. Indeed, we have to verify that

$$\begin{cases} \dfrac{dx_1}{dt} = m_2\boldsymbol{\eta}(x_2 - x_1) + m_3\boldsymbol{\eta}(x_3 - x_1) = 0 \\ \dfrac{dx_2}{dt} = m_1\boldsymbol{\eta}(x_1 - x_2) + m_3\boldsymbol{\eta}(x_3 - x_2) = 0 \\ \dfrac{dx_3}{dt} = m_1\boldsymbol{\eta}(x_1 - x_3) + m_2\boldsymbol{\eta}(x_2 - x_3) = 0, \end{cases} \tag{6.3}$$

where, as usual, $\boldsymbol{\eta}$ represents the anti-symmetric selection in the subdifferential (6.2) given by Corollary 4.2.

For instance, let us search for a solution in the following range

$$x_2 - x_1 = 1, \quad x_3 - x_1 > 1, \quad 0 < x_3 - x_2 < 1. \tag{6.4}$$

We begin searching for the right selection. We use the notation $\boldsymbol{\eta}_{ij} := \boldsymbol{\eta}(x_i - x_j)$. Clearly we need to select only at $\pm(x_2 - x_1)$. Taking the anti-symmetry into account, and recalling that the subdifferential of $\widehat{W}$ is (6.2) and that the relations (6.4) hold, there is

$$\begin{aligned} \|\boldsymbol{\eta} * \mu\|^2_{L^2(\mathbb{R},\mu;\mathbb{R})} &= \sum_{j=1}^{3} m_j \left|\sum_{i=1}^{3} m_i\boldsymbol{\eta}(x_j - x_1)\right|^2 \\ &= m_1(m_2^2\boldsymbol{\eta}_{12}^2 + 2m_2m_3(x_1 - x_3)\boldsymbol{\eta}_{12}) + m_2(m_1^2\boldsymbol{\eta}_{21}^2 + 2m_1m_3(x_3 - x_2)\boldsymbol{\eta}_{21}) + R_1 \\ &= m_1m_2(m_1 + m_2)\boldsymbol{\eta}_{12}^2 + 2m_1m_2m_3(x_1 + x_2 - 2x_3)\boldsymbol{\eta}_{12} + R_2, \end{aligned}$$

where the remainders R_1, R_2 do not depend on the value of $\boldsymbol{\eta}$ at ± 1, hence we only have to minimize with respect to $\boldsymbol{\eta}_{12}$ on the interval $[-1, 1]$. We have a quadratic function, so that if the vertex of the parabola is on the right of the interval $[-1, 1]$, then the minimizer is found for $\boldsymbol{\eta}_{12} = 1$, hence we get $\boldsymbol{\eta}(-1) = 1$ and then, by anti-symmetry, $\boldsymbol{\eta}(1) = -1$. Computing the vertex position, this condition is

$$\frac{m_3(x_3 - x_2) + m_3(x_3 - x_1)}{m_1 + m_2} \geq 1. \tag{6.5}$$

Hence, if such condition holds, making use of (6.2) the first two equations in (6.3) reduce to

$$\begin{cases} -m_2 + m_3(x_3 - x_1) = 0, \\ m_1 - m_3(x_3 - x_2) = 0. \end{cases}$$

We deduce

$$x_3 - x_1 = \frac{m_2}{m_3} \quad \text{and} \quad x_3 - x_2 = \frac{m_1}{m_3},$$

and since $1 = x_2 - x_1 = (x_3 - x_1) - (x_3 - x_2)$, we have

$$m_3 = m_2 - m_1.$$

We may pose, for $0 < \alpha < \frac{1}{4}$,

$$m_1 = \frac{1}{4} - \alpha, \quad m_2 = \frac{1}{2}, \quad m_3 = \frac{1}{4} + \alpha.$$

Then we find

$$x_3 - x_1 = \frac{2}{1+4\alpha}, \quad x_3 - x_2 = \frac{1-4\alpha}{1+4\alpha}. \tag{6.6}$$

This way, the constraints in (6.4) are satisfied. Moreover, it is immediate to check that (6.3) is solved and that

$$\frac{m_3(x_3-x_2)+m_3(x_3-x_1)}{m_1+m_2} = 1,$$

so that (6.5) is verified and the computation is indeed consistent.

In order to find the three points, we can for instance fix the center of mass in the origin:

$$\frac{1-4\alpha}{4}\, x_1 + \frac{1}{2}\, x_2 + \frac{1+4\alpha}{4}\, x_3 = 0.$$

Together with (6.6), this gives

$$x_1 = -1, \quad x_2 = 0, \quad x_3 = \frac{1-4\alpha}{1+4\alpha}.$$

We conclude that, for $0 < \alpha < \frac{1}{4}$,

$$\mu = \left(\frac{1}{4} - \alpha\right)\delta_{-1} + \frac{1}{2}\,\delta_0 + \left(\frac{1}{4}+\alpha\right)\delta_{\frac{1-4\alpha}{1+4\alpha}}$$

is a stationary state.

Remark 6.4. In the case of $\widetilde{W}$, it is very easy to construct an analogous example, solving system (6.3), where this time the actual gradient of $\widetilde{W}$ appears. In both cases, it seems clear that the procedure can be repeated for finding infinitely many stationary states with N Dirac masses for any $N > 3$.

We conclude with an example in two space dimensions. The reference functional is simply the radial of $\widehat{W}$, still denoted by $\widehat{W}$, that is

$$\widehat{W}(x) = \frac{1}{2}\big||x|^2 - 1\big|, \quad x \in \mathbb{R}^2.$$

The functional is still (-1)-convex, and in this case

$$\partial\widehat{W}(x) = \begin{cases} x & \text{if } |x| > 1 \\ -x & \text{if } |x| \le 1 \\ [-1,1]x & \text{if } |x| = 1. \end{cases}$$

Example 6.5. Let σ_R denote the uniform measure on the circumference $\partial B_R(0)$, of radius R, centered in the origin. There exists $R > 0$ such that the measure σ_R is a stationary state for functional $\widehat{W}$.

Indeed, we can show that for any $x \in \partial B_R(0)$ there holds $\boldsymbol{\eta} * \sigma_R = 0$ for a suitable choice of the radius R, $\boldsymbol{\eta}$ being the optimal selection of Corollary 4.2. Explicitly, the convolution is

$$\int_{\{|x-y|>1\}} (x-y)\, d\sigma_R(y) - \int_{\{|x-y|\le 1\}} (x-y)\, d\sigma_R(y),$$

and the set of points $\{y : |x-y| = 1\}$, where one should select, is negligible. Fix x on the circle. We let $(\mathbf{e}_1, \mathbf{e}_2)$ be an orthogonal base in $\mathbb{R}^2$, where $\mathbf{e}_1$ is the direction of x, so that $x = R\mathbf{e}_1$. Hence we have to solve

$$R\mathbf{e}_1\sigma_R(\{|x-y|>1\}) - R\mathbf{e}_1\sigma_R(\{|x-y|\leq 1\}) - \int_{\{|x-y|>1\}} y\,d\sigma_R(y) + \int_{\{|x-y|\leq 1\}} y\,d\sigma_R(y) = 0.$$

We write the integrals in polar coordinates with respect to the origin and the vector $\mathbf{e}_1$. In this system, we let $\alpha = \alpha(R)$ denote the (positive) angle corresponding to the intersection point between $\partial B_R(0)$ and the circle of radius 1 centered in x (see Figure 1 below). In particular

$$\sin\alpha(R) = \frac{\sqrt{R^2 - \frac{1}{4}}}{R^2}. \tag{6.7}$$

Since $y = R\cos\theta\mathbf{e}_1 + R\sin\theta\mathbf{e}_2$, it is immediately seen, by oddness of the sine function, that the equation in the direction of $\mathbf{e}_2$ is identically satisfied. In the direction of $\mathbf{e}_1$ we find

$$2R\int_\alpha^\pi R\,d\theta - 2R\int_0^\alpha R\,d\theta - 2R\int_\alpha^\pi \cos\theta R\,d\theta + 2R\int_0^\alpha \cos\theta R\,d\theta = 0.$$

Hence we have to solve

$$f(R) := \pi - 2\alpha(R) + 2\sin\alpha(R) = 0.$$

If $R < \frac{1}{2}$, the computation does not make sense, but indeed we can not have a stationary state for $R < \frac{1}{2}$, since in this case any point in $\partial B_R(0)$ has distance lower than 1 from x, so that for each of them the effect on x is a repulsion, and x tend to move far from the origin. Taking (6.7) into account, if $R = \frac{1}{2}$, we have $\alpha(R) = \pi$, hence the value of f at $\frac{1}{2}$ is $-\pi$. As R increases from $\frac{1}{2}$ to $+\infty$, the angle $\alpha(R)$ decreases from π to 0. Notice that the function

$$R \mapsto \frac{\sqrt{R^2 - \frac{1}{4}}}{R^2}$$

is increasing from $\frac{1}{2}$ to $\frac{\sqrt{2}}{2}$, where it has its maximum, and is decreasing in $(\frac{\sqrt{2}}{2}, +\infty)$. On the other hand, $\sin(x) - x$ is a decreasing function. Since $f(\sqrt{2}/2) = 2$, we conclude that f has only one zero, found in the interval $(1/2, \sqrt{2}/2)$. If R_0 is the zero, for $R = R_0$ the measure σ_R is stationary.

7. Appendix: vector and plan subdifferential

Here we give a more complete overview about the Wasserstein subdifferential. In [AGS, §10.3], the theory is developed for functionals $\Phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ such that

$$\Phi : \mathscr{P}_2(\mathbb{R}^d) \to (-\infty, +\infty] \text{ is proper and lower semicontinuous in } \mathscr{P}_2(\mathbb{R}^d) \tag{7.1}$$

and

$$\Phi(\cdot) + \frac{1}{2\tau}\,d_W^2(\cdot, \mu) \text{ admits minimizers for any small enough } \tau > 0 \text{ and } \mu \in \mathscr{P}_2(\mathbb{R}^d). \tag{7.2}$$

Indeed, after Remark 3.6 we know that $\mathcal{W}$ satisfies these hypothesis. Hence, we are in the framework of [AGS, §10.3]. We will show how the results therein work for the case of $\mathcal{W}$.

First of all, we remark that Definition 2.2 is equivalent to the following one.

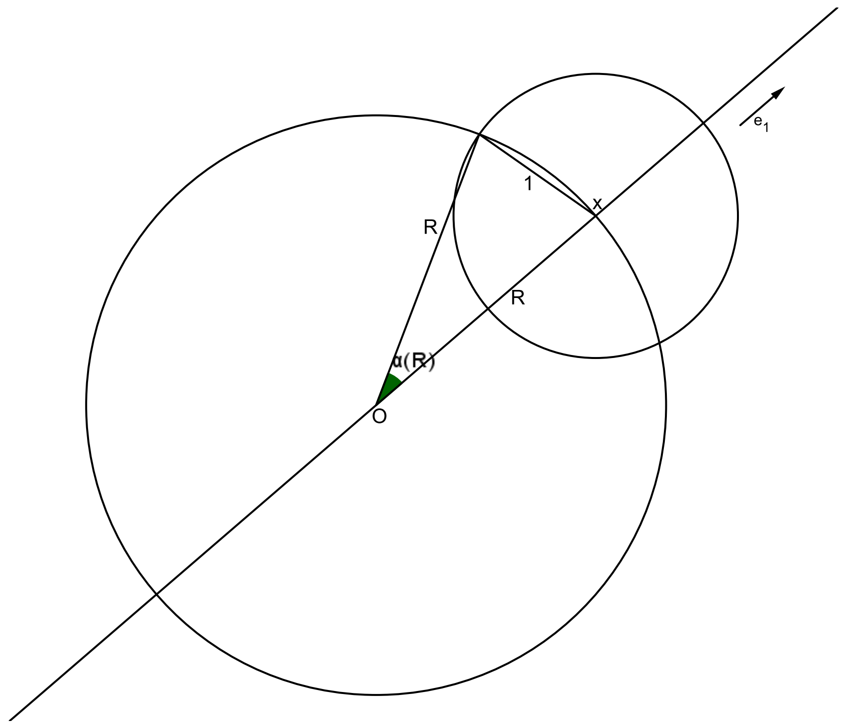

FIGURE 1. The construction of Example 6.5

Proposition 7.1. $\boldsymbol{\xi} \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$ *belongs to the Wasserstein subdifferential of* $\mathcal{W}$ *at* $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ *if and only if*

$$\mathcal{W}(\nu) - \mathcal{W}(\mu) \geq \int_{\mathbb{R}^d} \langle \boldsymbol{\xi}(x), y - x \rangle \, d\gamma(x,y) + o(\mathcal{C}(\mu, \nu; \gamma)) \tag{7.3}$$

as $\nu \to \mu$ *in* $\mathscr{P}_2(\mathbb{R}^d)$, *for some optimal transport plan* $\gamma \in \Gamma_o(\mu, \nu)$. *Moreover,* $\boldsymbol{\xi}$ *is a strong subdifferential if and only if* (7.3) *holds whenever* $\nu \to \mu$ *in* $\mathscr{P}_2(\mathbb{R}^d)$ *and* $\Gamma(\mu, \nu) \ni \gamma \to (\mathbf{i}, \mathbf{i})_{\#}\mu$ *in* $\mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$.

Proof. Let $\gamma \in \Gamma(\mu, \nu)$ and define the interpolating curve $\theta^\gamma(t) = ((1-t)\pi^1 + t\pi^2)_{\#}\gamma$ between μ and ν, so that $\theta(0) = \mu$ and $\theta(1) = \nu$. We take advantage of a property of Wasserstein constant speed geodesics, shown in [AGS, Lemma 7.2.1]: there exists γ^* in $\Gamma_o(\mu, \nu)$ such that $\Gamma_o(\mu, \theta^{\gamma^*}(t))$ contains a unique element for any $t \in [0, 1)$, given by $\gamma_t := (\pi^1, (1-t)\pi^1 + t\pi^2)_{\#}\gamma^*$. Then, (7.3) can be applied in correspondence of γ_t and with $\theta^\gamma(t)$ in place of ν, and together with (2.2), it gives, for $t \to 0$,

$$\begin{aligned}
\mathcal{W}(\nu) - \mathcal{W}(\mu) &\geq \frac{\mathcal{W}(\theta^{\gamma^*}(t)) - \mathcal{W}(\mu)}{t} + \frac{\lambda}{2}(1-t)\mathcal{C}^2(\mu, \nu; \gamma^*) \\
&\geq \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle \boldsymbol{\xi}(x), y - x \rangle \, d\gamma^*(x,y) + \frac{\lambda}{2}(1-t)\mathcal{C}^2(\mu, \nu; \gamma^*) + \frac{1}{t} o(\mathcal{C}(\mu, \theta(t); \gamma_t)).
\end{aligned}$$

Passing to the limit as $t \to 0$, since $\mathcal{C}(\mu, \theta(t); \gamma_t) = t^2 \mathcal{C}(\mu, \nu; \gamma^*)$, one gets (2.3) for the plan $\gamma^* \in \Gamma_o(\mu, \nu)$. One reasons in the same way for the equivalence in the case of strong subdifferentials: indeed, one can define $\theta^\gamma(t)$ for the generic plan $\gamma \in \Gamma(\mu,\nu)$ and use the convexity of $\mathcal{W}$ along any interpolating curve $t \mapsto \theta^\gamma(t)$ (see Proposition 2.1). $\square$

On the other hand, the general definition of subdifferential, given in [AGS, §10.3], is more technical. According to that notion, the subdifferential is in fact a plan $\beta \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$, as in the following

Definition 7.2 (Plan subdifferential). *Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. We say that $\beta \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ belongs to the extended subdifferential of $\mathcal{W}$ at μ if $\pi^1_\#\beta = \mu$ and there holds*

$$\mathcal{W}(\nu) - \mathcal{W}(\mu) \geq \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \langle y, z-x\rangle \, d\boldsymbol{\mu}(x,y,z) + \frac{\lambda}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} |z-x|^2 \, d\boldsymbol{\mu}(x,y,z) \tag{7.4}$$

for some $\boldsymbol{\mu} \in \boldsymbol{\Gamma}_o(\beta,\nu)$ (we obtain a strong subdifferential if the inequality holds for any plan $\boldsymbol{\mu} \in \boldsymbol{\Gamma}(\beta,\nu)$). We write $\beta \in \boldsymbol{\partial}\mathcal{W}(\mu)$ (resp. $\beta \in \boldsymbol{\partial}_S\mathcal{W}(\mu)$). Here the elements of $\boldsymbol{\Gamma}(\beta,\mu)$ are three-plans, that is, measures in $\mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$, such that $\pi^{1,2}_\#\boldsymbol{\mu} = \beta$ and $\pi^3_\#\boldsymbol{\mu} = \nu$. The definition of optimal plan in this case is

$$\boldsymbol{\Gamma}_o(\beta,\nu) := \{\gamma \in \boldsymbol{\Gamma}(\beta,\nu) : \pi^{1,3}\gamma \in \Gamma_o(\mu,\nu)\}.$$

Remark 7.3. Since $\mathcal{W}$ is convex along any linearly interpolating curve, as noticed in Proposition 2.1, from [AGS, Theorem 10.3.6] we learn that we can equivalently define the extended subdifferential of $\mathcal{W}$ by asking inequality (7.4) for any $\boldsymbol{\mu} \in \boldsymbol{\Gamma}_o(\beta,\nu)$.

Remark 7.4. We observe that if β is concentrated on the graph of a vector field $\boldsymbol{\xi}$, we have $\beta = (\mathbf{i},\boldsymbol{\xi})_\#\mu$ and in particular $y = \boldsymbol{\xi}(x)$ for $\boldsymbol{\mu}$-a.e. $(x,y,z) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$. In this case the definition reduces to (2.3).

We recall that, also for extended subdifferentials, there holds

$$|\partial\mathcal{W}|(\mu) = \operatorname{argmin}\left\{\left(\int_{\mathbb{R}^d} |y|^2 \, d\pi^2_\#\beta(y)\right)^{1/2} : \beta \in \boldsymbol{\partial}\mathcal{W}(\mu)\right\}.$$

Moreover, the corresponding minimizer is unique. See [AGS, Theorem 10.3.11]. We denote it by $\boldsymbol{\partial}^o\mathcal{W}(\mu)$.

We have the following

Lemma 7.5. *Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. The following assertions hold:*

$$\beta \in \boldsymbol{\partial}_S\mathcal{W}(\mu) \Rightarrow \bar{\beta} \in \partial_S\mathcal{W}(\mu) \qquad \textit{and} \qquad \beta \in \boldsymbol{\partial}\mathcal{W}(\mu) \Rightarrow \bar{\beta} \in \partial\mathcal{W}(\mu). \tag{7.5}$$

Proof. We begin with the proof for strong subdifferentials. Let $\beta \in \boldsymbol{\partial}_S\mathcal{W}(\mu)$ and we write $\beta = \int_{\mathbb{R}^d} \beta_x \, d\mu(x)$. For any $\gamma \in \Gamma(\mu,\nu)$ there holds

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \bar{\beta}(x), z-x\rangle \, d\gamma(x,z) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \left\langle \int_{\mathbb{R}^d} y \, d\beta_x(y), z-x \right\rangle d\gamma(x,z) \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \langle y, z-x\rangle \, d\beta_x(y)\, d\gamma(x,z).\end{aligned}$$

Moreover, taking into account that $\int_{\mathbb{R}^d} d\beta_x(y) = 1$ for μ-a.e. $x \in \mathbb{R}^d$, we have that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |z-x|^2 \, d\gamma(x,z) = \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} |z-x|^2 \, d\beta_x(y)\, d\gamma(x,z).$$

Let us define the three-plan $\boldsymbol{\mu}$ as

$$\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d}\phi(x,y,z)d\boldsymbol{\mu}(x,y,z):=\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d}\phi(x,y,z)d\beta_x(y)\,d\gamma(x,z)$$

for all continuous functions $\phi:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ with at most quadratic growth at infinity. Then, $\boldsymbol{\mu}$ belongs to $\boldsymbol{\Gamma}(\beta,\nu)$. Making use of (7.4) for this particular choice of $\boldsymbol{\mu}\in\boldsymbol{\Gamma}(\beta,\nu)$, we see that $\bar\beta$ satisfies (2.3). Recalling Remark 7.3, since $\gamma\in\Gamma_o(\mu,\nu)\Rightarrow\boldsymbol{\mu}\in\boldsymbol{\Gamma}_o(\beta,\nu)$, reasoning as done for strong subdifferentials the implication

$$\beta\in\boldsymbol{\partial}\mathcal{W}(\mu)\Rightarrow\bar\beta\in\partial\mathcal{W}(\mu)$$

follows. □

Corollary 7.6. *If β_o is the minimal selection in the extended subdifferential of $\mathcal{W}$ at μ, then it coincides with $(\mathbf{i},\bar\beta_o)_\#\mu$, and $\bar\beta_o$ is the minimal selection in $\partial\mathcal{W}(\mu)$.*

Proof. For any $\beta\in\boldsymbol{\partial}\mathcal{W}(\mu)$, using Jensen's inequality, there holds

$$\int_{\mathbb{R}^d}|y|^2\,d\pi^2_\#\beta(y)=\int_{\mathbb{R}^d\times\mathbb{R}^d}|y|^2\,d\beta(x,y)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|y|^2\,d\beta_x(y)\,d\mu(x)\geq\int_{\mathbb{R}^d}|\bar\beta(x)|^2\,d\mu(x).$$

This shows that the barycentric projection does not increase the norm. Therefore, if β_o is the minimal selection, since $\bar\beta_o\in\partial\mathcal{W}(\mu)$, there has to hold $\beta_o=(\mathbf{i},\bar\beta_o)_\#\mu$. In this case it is also clear that $\bar\beta_o=\partial^o\mathcal{W}(\mu)$. □

Remark 7.7. Because of (7.5), under the same assumptions of Theorem 3.2 one sees that if $\beta\in\boldsymbol{\partial}_S\mathcal{W}(\mu)$, then its barycenter $\bar\beta$ is given by (3.3).

Remark 7.8. In the case of strong subdifferentials, the implication of Lemma 7.5 is true for any functional Φ satisfying the assumptions (7.1) and (7.2). It is shown in Lemma 10.3.4 and Remark 10.3.5 of [AGS] that, given $\mu\in\mathscr{P}_2(\mathbb{R}^d)$, a minimizer μ_τ to $\Phi(\cdot)+\frac{1}{2\tau}d_W^2(\cdot,\mu)$ and a plan $\hat\gamma_\tau\in\Gamma_o(\mu_\tau,\mu)$, there holds

$$\gamma_\tau\in\boldsymbol{\partial}_S\Phi(\mu_\tau),$$

where γ_τ is the rescaled of $\hat\gamma_\tau$ (see Definition 3.12). Moreover, among these rescaled plans, there exists a plan whose barycenter belongs to $\partial_S\Phi(\mu)$. After Lemma 7.5, we may indeed infer that this holds true for the rescaled of any optimal plan in $\Gamma_o(\mu_\tau,\mu)$.

Eventually, we are ready for the proof of Proposition 3.13. We make use of the general convergence properties of rescaled plan subdifferentials shown in [AGS, Theorem 10.3.10], passing to barycenters by means of Lemma 7.5 and Corollary 7.6.

***Proof of* Proposition 3.13**. Let $\tau>0$ be small enough. Once more, let $\mu\in\mathscr{P}_2(\mathbb{R}^d)$, let μ_τ minimize $\mathcal{W}(\cdot)+\frac{1}{2\tau}d_W^2(\cdot,\mu)$ and let $\hat\gamma_\tau\in\Gamma_o(\mu_\tau,\mu)$. Moreover, let γ_τ be the rescaled of $\hat\gamma_\tau$ (as given by Definition 3.12). As a consequence of Theorem 3.1, for any $\mu\in\mathscr{P}_2(\mathbb{R}^d)$ the set $\boldsymbol{\partial}\mathcal{W}(\mu)$ is not empty. Therefore we are in the hypotheses of [AGS, Theorem 10.3.10], which entails, taking into account also [AGS, Remark 10.3.14],

$$\lim_{\tau\to0}\gamma_\tau=\boldsymbol{\partial}^o\mathcal{W}(\mu)\quad\text{in }\mathscr{P}_2(\mathbb{R}^d\times\mathbb{R}^d).$$

But Corollary 7.6 implies that $\boldsymbol{\partial}^o\mathcal{W}(\mu)=(\mathbf{i},\partial^o\mathcal{W}(\mu))_\#\mu$. The convergence above then means that, as $\tau\to0$,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\phi(x,y)\,d(\gamma_\tau)_x(y)\,d\mu_\tau(x)\to\int_{\mathbb{R}^d\times\mathbb{R}^d}\phi(x,y)\,d(\mathbf{i},\partial^o\mathcal{W}(\mu))_\#\mu(x,y)\tag{7.6}$$

for any continuous function $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ with at most quadratic growth at infinity, where $(\gamma_\tau)_x$ denotes the family of measures which disintegrates γ_τ with respect to μ_τ. Letting $\zeta \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$, and choosing $\phi(x,y) = \langle y, \zeta(x)\rangle$ in (7.6), we obtain the convergence in the sense of Definition 3.8 of $\bar{\gamma}_\tau$ to $\partial^o \mathcal{W}(\mu)$. On the other hand, using Jensen inequality (in the same way as in the proof of Corollary 7.6) and (7.6) with $\phi(x,y) = |y|^2$ we obtain

$$\limsup_{\tau\to 0} \int_{\mathbb{R}^d} |\bar{\gamma}_\tau|^2 \, d\mu_\tau \leq \lim_{\tau\to 0} \int_{\mathbb{R}^d\times\mathbb{R}^d} |y|^2 \, d\gamma_\tau = \int_{\mathbb{R}^d\times\mathbb{R}^d} |y|^2 \, d(\mathbf{i}, \partial^o\mathcal{W}(\mu))_{\#}\mu = \int_{\mathbb{R}^d} |\partial^o\mathcal{W}(\mu)|^2 \, d\mu,$$

hence we also have the strong convergence in the sense of Definition 3.8. □

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