A DECOMPOSITION AND WEAK APPROXIMATION OF THE SUB-FRACTIONAL BROWNIAN MOTION.

XAVIER BARDINA* AND DAVID BASCOMPTE

ABSTRACT. We present a decomposition of the sub-fractional Brownian motion into the sum of a fractional Brownian motion plus a stochastic process with absolutely continuous trajectories. The first application we show of this decomposition is the relation between the spaces of integrable functions with respect each one of these three processes. A general result of weak convergence to integrals of $L^2(\mathbb{R}^+)$ functions with respect to standard Brownian motion is proved, and this result permits us to obtain approximations in law of the fractional Brownian motion and the sub-fractional Brownian motion with parameter $H \in (0, 1)$.

1. INTRODUCTION AND PRELIMINARIES

Sub-fractional Brownian motion (sub-fBm for brevity) $S^H = \{S^H(t), t \ge 0\}$ is a centered Gaussian process with covariance function

(1.1)
$$\operatorname{Cov}(S_t^H, S_s^H) = s^H + t^H - \frac{1}{2} \left[(s+t)^H + |s-t|^H \right]$$

where $H \in (0, 2)$.

This process was introduced by Bojdecki *et al.* in 2004 (see [BGT04]) as an intermediate process between standard Brownian motion and fractional Brownian motion. Recall that *fractional Brownian motion* (fBm for short) $B^H = \{B^H(t), t \ge 0\}$ is a centered Gaussian process with covariance function

(1.2)
$$\operatorname{Cov}(B_t^H, B_s^H) = \frac{1}{2} \left(s^H + t^H - |s - t|^H \right)$$

where $H \in (0,2)$. Usually fBm is defined with Hurst parameter belonging to the interval (0,1) with the corresponding covariance, but in order to compare it with sub-fBm we use the stated representation with $H \in (0,2)$. Note that both fBm and sub-fBm are standard Brownian motions for H = 1.

For $H \neq 1$, sub-fBm preserves some of the main properties of fBm, such as longrange dependence, but its increments are not stationary; they are more weakly correlated on non-overlapping intervals than fBm ones, and their covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. For a more detailed discussion of sub-fBm and its properties we refer the reader to [BGT04]. Some properties of this process have also been studied in [Tud08] and [Tud07]. On the other hand there is an extension of sub-fBm in [BGT07].

The main result of this paper (see Theorem 2.2) is the obtainment of a decomposition of the sub-fBm in terms of fBm and another process with absolutely continuous trajectories, $X^H = \{X_t^H, t \ge 0\}$, which is defined by Lei and Nualart in [LN09] by

(1.3)
$$X_t^H = \int_0^\infty (1 - e^{-\theta t}) \theta^{-\frac{1+H}{2}} \, \mathrm{d}W_\theta$$

^{*} Corresponding author

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where W is a standard Brownian motion. Lei and Nualart introduce this process in order to obtain a decomposition of bifractional Brownian motion into the sum of a transformation of X_t^H and a fBm. In [LN09] this process is defined for $H \in (0, 1)$ but one can define it for $H \in (1, 2)$ as we will prove in Proposition 2.1.

The decomposition we state is different for $H \in (0, 1)$ and $H \in (1, 2)$. In the first case, sub-fBm is obtained as a sum of two independent processes, a fBm and the process defined by (1.3), while for $H \in (1, 2)$ is fBm that is decomposed into the sum of the process (1.3) and a sub-fBm, being these independents.

As a first application of the decomposition given in Theorem 2.2, in the third section, it is shown the relation between the spaces of integrable functions with respect each one of the three processes we introduced. In section 4 we will prove a general result of weak convergence, in the space of continuous functions C([0, T]), toward integrals of functions of $L^2(\mathbb{R}^+)$ with respect to standard Brownian motion. This result permits us to obtain approximations in law toward fBm, toward the process defined in (1.3) and, finally, toward a sub-fBm with parameter $H \in (0, 1)$.

Positive constants, denoted by C, with possible subscripts indicating appropriate parameters, may vary from line to line.

2. Decomposition of the sub-fractional Brownian motion.

In this section we prove a decomposition of sub-fBm into the sum of a fBm and the process X^H defined by (1.3). We begin by proving some properties of the process X^H .

Proposition 2.1. The process $X^H = \{X_t^H, t \ge 0\}$ is Gaussian, centered, and its covariance function is:

(2.1)
$$\operatorname{Cov}(X_t^H, X_s^H) = \begin{cases} \frac{\Gamma(1-H)}{H} \left[t^H + s^H - (t+s)^H \right] & si \ H \in (0,1), \\ \frac{\Gamma(2-H)}{H(H-1)} \left[(t+s)^H - t^H - s^H \right] & si \ H \in (1,2). \end{cases}$$

Moreover, X^H has a version with trajectories which are infinitely differentiable on $(0, \infty)$ and absolutely continuous on $[0, \infty)$.

Proof. Clearly X^H is Gaussian and centered, and computing its covariance we obtain:

$$Cov(X_t^H, X_s^H) = \mathbb{E}[X_t^H X_s^H] = \int_0^\infty (1 - e^{-\theta t})(1 - e^{-\theta s})\theta^{-1 - H} d\theta.$$

Integrating by parts we obtain

$$\operatorname{Cov}(X_t^H, X_s^H) = \frac{1}{H} \int_0^\infty \theta^{-H} \left(t e^{-\theta t} + s e^{-\theta s} - (s+t) e^{-\theta(s+t)} \right) \mathrm{d}\theta$$

which, for $H \in (0, 1)$, gives the stated result. For $H \in (1, 2)$ we may integrate by parts a second time, yielding

$$\operatorname{Cov}(X_t^H, X_s^H) = \frac{1}{H(H-1)} \int_0^\infty \theta^{1-H} \left(-t^2 e^{-\theta t} - s^2 e^{-\theta s} + (s+t)^2 e^{-\theta(s+t)} \right) \mathrm{d}\theta,$$

which also gives the stated result.

In order to prove the second part, let us observe that the proof given by Lei and Nualart in [LN09] holds true for $H \in (0, 2)$. In [LN09] is proved that $X_t^H = \int_0^t Y_s ds$ where

$$Y_t = \int_0^\infty \theta^{\frac{1-H}{2}} e^{-\theta t} \mathrm{d}W_\theta, \qquad \mathbb{E}[Y_t^2] = \Gamma(2-H)2^{H-2}t^{H-2}$$

and the *n*-th derivative of the X^H is $(X_t^H)^{(n)} = \int_0^\infty (-1)^{n-1} \theta^{\left(n - \frac{1+H}{2}\right)} e^{-\theta t} dW_\theta$. \Box

From (1.1), (1.2) and (2.1) we can state and prove the following result:

Theorem 2.2. Let B^H be a fBm, S^H a sub-fBm and $W = \{W_t, t \ge 0\}$ a standard Brownian motion. Let X^H be the process given by (1.3). If for $H \in (0,1)$ we suppose that B^H and W are independents, then the processes $\{Y_t^H = C_1 X_t^H + B_t^H, t \ge 0\}$ and $\{S_t^H, t \ge 0\}$ have the same law, where $C_1 = \sqrt{\frac{H}{2\Gamma(1-H)}}$. If for $H \in (1,2)$ we suppose that S^H and W are independents, then the processes $\{Y_t^H = C_2 X_t^H + S_t^H, t \ge 0\}$ and $\{B_t^H, t \ge 0\}$ have the same law, where $C_2 = \sqrt{\frac{H(H-1)}{2\Gamma(2-H)}}$.

Proof. It is clear that the process Y^H is centered and Gaussian in both cases. For $H \in (0, 1)$, from (1.2), (2.1) and using the independence of X^H and B^H we have

$$\begin{aligned} \operatorname{Cov}(Y_t^H, Y_s^H) &= C_1^2 \operatorname{Cov}[X_t^H, X_s^H] + \operatorname{Cov}[B_t^H, B_s^H] \\ &= \frac{1}{2} \left[t^H + s^H - (t+s)^H \right] + \frac{1}{2} \left(s^H + t^H - |s-t|^H \right) \\ &= s^H + t^H - \frac{1}{2} \left[(s+t)^H + |s-t|^H \right], \end{aligned}$$

which completes the proof in this case, and for $H \in (1,2)$, from (1.1), (2.1) and using the independence of X^H and S^H we have

$$\begin{split} \operatorname{Cov}(Y_t^H, Y_s^H) &= C_2^2 \operatorname{Cov}[X_t^H, X_s^H] + \operatorname{Cov}[S_t^H, S_s^H] \\ &= \frac{1}{2} \left[(t+s)^H - t^H - s^H \right] + s^H + t^H - \frac{1}{2} \left[(s+t)^H + |s-t|^H \right] \\ &= \frac{1}{2} \left(s^H + t^H - |s-t|^H \right), \end{split}$$

which completes the proof.

3. Space of integrable functions with respect sub-fractional Brownian motion

Let us consider \mathcal{E} the set of simple functions on [0, T]. Generally, if $U := (U_t, t \in [0, T])$ is a continuous, centered Gaussian process, we denote by \mathcal{H}_U the Hilbert space defined as the closure of \mathcal{E} with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]},\mathbf{1}_{[0,s]}\right\rangle_{\mathcal{H}}=E\left(U_{t}U_{s}\right).$$

In the case of the standard Brownian motion W, the space \mathcal{H}_W is $L^2([0,T])$. On the other hand, for the fractional Brownian motion B^H , the space \mathcal{H}_{B^H} is the set of restrictions to the space of test functions $\mathcal{D}((0,T))$ of the distributions of $W^{\frac{1-H}{2},2}(\mathbb{R})$ with support contained in [0,T] (see [Jol07]). In the case $H \in (0,1)$ all the elements of the domain are functions, and the space \mathcal{H}_{B^H} coincides with the fractional Sobolev space $I_{0^+}^{\frac{1-H}{2}}(L^2([0,T]))$ (see for instance [DÜ99]), but in the case $H \in (1,2)$ this space contains distributions which are not given by any function.

As a direct consequence of Theorem 2.2 we have the following relation between \mathcal{H}_{B^H} , \mathcal{H}_{S^H} and \mathcal{H}_{X^H} , where S^H is the sub-fBm and X^H is the process introduced by Lei and Nualart in [LN09] and defined by (1.3).

Proposition 3.1. For $H \in (0,1)$ the following equality

$$\mathcal{H}_{X^H} \cap \mathcal{H}_{B^H} = \mathcal{H}_{S^H}$$

holds. On the other hand, for $H \in (1,2)$ we have that

$$\mathcal{H}_{X^H} \cap \mathcal{H}_{S^H} = \mathcal{H}_{B^H}.$$

Proof. This proposition is a direct consequence of the two decompositions into the sum of two independent processes proved in Theorem 2.2. \Box

4. Weak convergence results

In this section we prove a result of weak convergence in the space of continuous functions C([0, T]), in the sense of the finite dimensional distributions. We will use this result later in order to prove a convergence result toward sub-fBm using the decomposition we have already shown.

It is well known the result by Stroock (see [Str82]) where it is shown that the family of processes

$$\left\{ x_{\varepsilon}(t) = \frac{1}{\varepsilon} \int_0^t (-1)^{N_{\frac{s}{\varepsilon^2}}} \, \mathrm{d}s, \quad t \in [0,T] \right\},\,$$

defined from the kernels $\theta_{\varepsilon} = \frac{1}{\varepsilon} (-1)^{N_{\frac{s}{\varepsilon^2}}}$ introduced by Kac in [Kac74], converges in law in $\mathcal{C}([0,T])$ to a standard Brownian motion, where $N = \{N_s, s \ge 0\}$ is a standard Poisson process.

A generalization of this result can be found in [Bar01], where it is proved that the family:

(4.1)
$$\left\{ x_{\varepsilon}^{\theta}(t) = \frac{2}{\varepsilon} \int_{0}^{t} e^{i\theta N_{\frac{2s}{\varepsilon^{2}}}} \,\mathrm{d}s, \quad t \in [0,T] \right\}$$

converges in law in $\mathcal{C}([0,T])$ to a complex Brownian motion, for $\theta \in (0,\pi) \cup (\pi, 2\pi)$. Particularly, the real part and the imaginary part of (4.1) tend to independent standard Brownian motions.

Using these tools and based on Theorem 1 of [DJ00], we prove the following result.

Theorem 4.1. Let $f(t, \cdot)$ and $g(t, \cdot)$ be functions of $L^2(\mathbb{R}^+)$ for all $t \in [0, T]$, T > 0, let $\{N_s, s \ge 0\}$ be a standard Poisson process and $\theta \in (0, \pi) \cup (\pi, 2\pi)$. Define the processes Y^f and \tilde{Y}^g , which are given by $Y^f = \{\int_0^\infty f(t, s) dW_s, t \in [0, T]\}$ and $\tilde{Y}^g = \{\int_0^\infty g(t, s) d\tilde{W}_s, t \in [0, T]\}$ and where $W = \{W_s, s \ge 0\}$ and $\tilde{W} =$ $\{\tilde{W}_s, s \ge 0\}$ are independent, standard Brownian motions. We also define the following processes

(4.2)
$$Y_{\varepsilon}^{f} = \left\{ \frac{2}{\varepsilon} \int_{0}^{\infty} f(t,s) \cos\left(\theta N_{\frac{2s}{\varepsilon^{2}}}\right) ds, \quad t \in [0,T] \right\}$$

and

(4.3)
$$\tilde{Y}_{\varepsilon}^{g} = \left\{ \frac{2}{\varepsilon} \int_{0}^{\infty} g(t,s) \sin\left(\theta N_{\frac{2s}{\varepsilon^{2}}}\right) ds, \quad t \in [0,T] \right\}.$$

Then, the finite dimensional distributions of the processes $\{Y_{\varepsilon}^{f}\}$ and $\{\tilde{Y}_{\varepsilon}^{g}\}$ converge in law to the finite dimensional distributions of the processes Y^{f} and \tilde{Y}^{g} .

Proof. Taking into account that the proof is valid for any fixed $t \in [0, T]$, by abuse of notation we will write f(s) instead of f(t, s). Slightly modifying the proof of Theorem 1 in [DJ00], in order to prove the weak convergence, in the sense of the finite dimensional distributions, it suffices to show that

(4.4)
$$\mathbb{E}\left[(Y_{\varepsilon}^{f})^{2}\right] \leq C\left(\int_{0}^{\infty} f^{2}(s) \,\mathrm{d}s\right), \quad \mathbb{E}\left[(\tilde{Y}_{\varepsilon}^{g})^{2}\right] \leq C\left(\int_{0}^{\infty} g^{2}(s) \,\mathrm{d}s\right).$$

Observe that defining

$$Z^f_\varepsilon = Y^f_\varepsilon + i \tilde{Y}^f_\varepsilon = \frac{2}{\varepsilon} \int_0^\infty f(s) e^{i \theta N_{\frac{2s}{\varepsilon^2}}} \mathrm{d}s$$

we have $\mathbb{E}[Z_{\varepsilon}^{f}\bar{Z}_{\varepsilon}^{f}] = \mathbb{E}[(Y_{\varepsilon}^{f})^{2} + (\tilde{Y}_{\varepsilon}^{f})^{2}]$. Therefore if we prove $\mathbb{E}[Z_{\varepsilon}^{f}\bar{Z}_{\varepsilon}^{f}] \leq C \|f\|_{2}^{2}$, where $\|\cdot\|_{2}$ is the $L^{2}(\mathbb{R}^{+})$ norm, the stated convergence follows.

$$\begin{split} \mathbb{E}[Z_{\varepsilon}^{f}\bar{Z}_{\varepsilon}^{f}] &= \mathbb{E}\left[\frac{2}{\varepsilon}\int_{0}^{\infty}f(s)e^{i\theta N\frac{2s}{\varepsilon^{2}}}\mathrm{d}s\frac{2}{\varepsilon}\int_{0}^{\infty}f(r)e^{-i\theta N\frac{2r}{\varepsilon^{2}}}\mathrm{d}r\right] \\ &= \frac{4}{\varepsilon^{2}}\mathbb{E}\left[\int_{0}^{\infty}\int_{0}^{\infty}f(s)f(r)e^{i\theta\left(N\frac{2s}{\varepsilon^{2}}-N\frac{2r}{\varepsilon^{2}}\right)}\mathrm{d}s\,\mathrm{d}r\right] \\ &= \frac{4}{\varepsilon^{2}}\int_{0}^{\infty}\int_{0}^{\infty}\mathbbm{1}_{\{r\leq s\}}f(s)f(r)\mathbb{E}\left[e^{i\theta\left(N\frac{2s}{\varepsilon^{2}}-N\frac{2r}{\varepsilon^{2}}\right)}\right]\mathrm{d}r\,\mathrm{d}s \\ &+ \frac{4}{\varepsilon^{2}}\int_{0}^{\infty}\int_{0}^{\infty}\mathbbm{1}_{\{s\leq r\}}f(s)f(r)\mathbb{E}\left[e^{-i\theta\left(N\frac{2r}{\varepsilon^{2}}-N\frac{2s}{\varepsilon^{2}}\right)}\right]\mathrm{d}s\,\mathrm{d}r. \end{split}$$

Since $\mathbb{E}[e^{i\theta X}] = e^{-2\lambda(1-e^{i\theta})}$ and $\mathbb{E}[e^{-i\theta X}] = e^{-2\lambda(1-e^{-i\theta})}$, being X a Poisson random variable of parameter λ , we obtain

$$\begin{split} \mathbb{E}[Z_{\varepsilon}^{f}\bar{Z}_{\varepsilon}^{f}] &= \frac{4}{\varepsilon^{2}} \int_{0}^{\infty} \int_{0}^{\infty} \mathbb{1}_{\{r \leq s\}} f(s)f(r)e^{-2\frac{s-r}{\varepsilon^{2}}(1-e^{i\theta})} \mathrm{d}r \, \mathrm{d}s \\ &+ \frac{4}{\varepsilon^{2}} \int_{0}^{\infty} \int_{0}^{\infty} \mathbb{1}_{\{s \leq r\}} f(s)f(r)e^{-2\frac{r-s}{\varepsilon^{2}}(1-e^{-i\theta})} \mathrm{d}r \, \mathrm{d}s \\ &\leq \frac{4}{\varepsilon^{2}} \int_{0}^{\infty} \int_{0}^{\infty} \mathbb{1}_{\{r \leq s\}} |f(s)f(r)| \, e^{-2\frac{s-r}{\varepsilon^{2}}(1-\cos\theta)} \mathrm{d}r \, \mathrm{d}s \\ &+ \frac{4}{\varepsilon^{2}} \int_{0}^{\infty} \int_{0}^{\infty} \mathbb{1}_{\{s \leq r\}} |f(s)f(r)| \, e^{-2\frac{r-s}{\varepsilon^{2}}(1-\cos\theta)} \mathrm{d}r \, \mathrm{d}s \end{split}$$

Using the inequality $|f(s)f(r)| \leq \frac{1}{2} (f^2(s) + f^2(r))$ and noting that, by means of a change of variables, the last two integrals are the same leads to

$$\begin{split} \mathbb{E}[Z_{\varepsilon}^{f}\bar{Z}_{\varepsilon}^{f}] &\leq \frac{4}{\varepsilon^{2}} \int_{0}^{\infty} \int_{0}^{\infty} \mathbb{1}_{\{s \leq r\}} \left(f^{2}(s) + f^{2}(r)\right) e^{-2\frac{r-s}{\varepsilon^{2}}(1-\cos\theta)} \mathrm{d}r \, \mathrm{d}s \\ &= \frac{4}{\varepsilon^{2}} \left(\int_{0}^{\infty} f^{2}(s) \int_{s}^{\infty} e^{-2\frac{r-s}{\varepsilon^{2}}(1-\cos\theta)} \mathrm{d}r \, \mathrm{d}s + \int_{0}^{\infty} f^{2}(r) \int_{0}^{r} e^{-2\frac{r-s}{\varepsilon^{2}}(1-\cos\theta)} \mathrm{d}s \, \mathrm{d}r \right) \\ &= 2 \left(\int_{0}^{\infty} f^{2}(s) \left(\frac{1}{1-\cos\theta}\right) \mathrm{d}s + \int_{0}^{\infty} f^{2}(r) \left(\frac{1-e^{-2\frac{r}{\varepsilon^{2}}(1-\cos\theta)}}{1-\cos\theta}\right) \mathrm{d}r \right) \\ &\leq \frac{4}{1-\cos\theta} \int_{0}^{\infty} f^{2}(s) \, \mathrm{d}s. \end{split}$$

Then, the convergence of the finite dimensional distributions has been proved and it remains to prove the independence of the limit processes. We begin by proving that the family $\{Y_{\varepsilon}^{f}\tilde{Y}_{\varepsilon}^{g}\}_{\varepsilon>0}$ is uniformly integrable. Indeed, we will prove that $\sup_{\varepsilon>0} \mathbb{E}\left[(Y_{\varepsilon}^{f}\tilde{Y}_{\varepsilon}^{g})^{2}\right] < \infty$. Using Hölder's inequality we have

$$\sup_{\varepsilon>0} \mathbb{E}\left[(Y_{\varepsilon}^{f} \tilde{Y}_{\varepsilon}^{g})^{2} \right] \leq \sup_{\varepsilon>0} \left(\mathbb{E}[(Y_{\varepsilon}^{f})^{4}] \right)^{\frac{1}{2}} \left(\mathbb{E}[(\tilde{Y}_{\varepsilon}^{g})^{4}] \right)^{\frac{1}{2}}.$$

In order to prove that the last expression is finite, we will show that

(4.5)
$$\mathbb{E}\left[(Y_{\varepsilon}^{f})^{4}\right] \leq C\left(\int_{0}^{\infty} f^{2}(s) \,\mathrm{d}s\right)^{2}, \quad \mathbb{E}\left[(\tilde{Y}_{\varepsilon}^{g})^{4}\right] \leq C\left(\int_{0}^{\infty} g^{2}(s) \,\mathrm{d}s\right)^{2}.$$

Being Z_{ε}^{f} like before, we can prove (4.5) showing that $\mathbb{E}[(Z_{\varepsilon}^{f}\bar{Z}_{\varepsilon}^{f})^{2}] \leq C \|f\|_{2}^{4}$.

$$\mathbb{E}[(Z_{\varepsilon}^{f}\bar{Z}_{\varepsilon}^{f})^{2}] = \frac{16}{\varepsilon^{4}} \mathbb{E}\left[\int_{[0,\infty)^{4}} f(s_{1})\cdots f(s_{4})e^{i\theta\left(N_{\frac{2s_{1}}{\varepsilon^{2}}}+N_{\frac{2s_{2}}{\varepsilon^{2}}}-N_{\frac{2s_{3}}{\varepsilon^{2}}}-N_{\frac{2s_{4}}{\varepsilon^{2}}}\right)} \mathrm{d}s_{1}\cdots \mathrm{d}s_{4}\right]$$
$$= \frac{64}{\varepsilon^{4}}\int_{[0,\infty)^{4}} \mathbb{1}_{\{s_{1}\leq\cdots\leq s_{4}\}}f(s_{1})\cdots f(s_{4})\mathbb{E}\left[E_{1}+\cdots+E_{6}\right]\mathrm{d}s_{1}\cdots \mathrm{d}s_{4}$$

where

$$\begin{split} E_1 &= e^{i\theta \left(N_{\frac{2s_1}{\varepsilon^2}} + N_{\frac{2s_2}{\varepsilon^2}} - N_{\frac{2s_3}{\varepsilon^2}} - N_{\frac{2s_4}{\varepsilon^2}}\right)} = e^{-i\theta \left(N_{\frac{2s_4}{\varepsilon^2}} - N_{\frac{2s_3}{\varepsilon^2}} + 2\left(N_{\frac{2s_3}{\varepsilon^2}} - N_{\frac{2s_2}{\varepsilon^2}}\right) + N_{\frac{2s_2}{\varepsilon^2}} - N_{\frac{2s_1}{\varepsilon^2}}\right)},\\ E_2 &= e^{-i\theta \left(N_{\frac{2s_4}{\varepsilon^2}} - N_{\frac{2s_3}{\varepsilon^2}} + N_{\frac{2s_2}{\varepsilon^2}} - N_{\frac{2s_1}{\varepsilon^2}}\right)}, E_3 &= e^{i\theta \left(N_{\frac{2s_4}{\varepsilon^2}} - N_{\frac{2s_3}{\varepsilon^2}} - \left(N_{\frac{2s_2}{\varepsilon^2}} - N_{\frac{2s_1}{\varepsilon^2}}\right)\right)}, \end{split}$$

 $E_4 = \overline{E_3}, E_5 = \overline{E}_2, E_6 = \overline{E}_1$. To obtain the last expression note that we can arrange s_1, s_2, s_3, s_4 in 24 different ways and due to the symmetry between s_1 and s_2 and between s_3 and s_4 we have 6 possible different situations, each one repeated 4 times. By means of the properties of Poisson process we have

$$\|\mathbb{E}[E_1]\|, \|\mathbb{E}[E_2]\|, \|\mathbb{E}[E_3]\| \le e^{-2\frac{s_4-s_3}{\varepsilon^2}(1-\cos\theta)}e^{-2\frac{s_2-s_1}{\varepsilon^2}(1-\cos\theta)}$$

and we can conclude

$$\mathbb{E}[(Z_{\varepsilon}^{f}\bar{Z}_{\varepsilon}^{f})^{2}] \leq \frac{384}{\varepsilon^{4}} \int_{[0,\infty)^{4}} \mathbb{1}_{\{s_{1} \leq \dots \leq s_{4}\}} |f(s_{1})\cdots f(s_{4})|$$

$$e^{-2\frac{s_{4}-s_{3}}{\varepsilon^{2}}(1-\cos\theta)} e^{-2\frac{s_{2}-s_{1}}{\varepsilon^{2}}(1-\cos\theta)} ds_{1}\cdots ds_{4}$$

$$\leq \frac{384}{2\varepsilon^{2}} \left(\int_{[0,\infty)^{2}} \mathbb{1}_{\{s_{1} \leq s_{2}\}} |f(s_{1})f(s_{2})| e^{-2\frac{s_{2}-s_{1}}{\varepsilon^{2}}(1-\cos\theta)} ds_{1} ds_{2} \right)^{2}$$

$$\leq 3 \left(\frac{4}{1-\cos\theta} \int_{0}^{\infty} f^{2}(s) ds \right)^{2}.$$

Then the family $\{Y^f_{\varepsilon}\tilde{Y}^g_{\varepsilon}\}_{\varepsilon>0}$ is uniformly integrable and consequently

$$\mathbb{E}[Y^f(t)\tilde{Y}^g(s)] = \lim_{\varepsilon \to 0} \mathbb{E}[Y^f_{\varepsilon}(t)\tilde{Y}^g_{\varepsilon}(s)].$$

Since Y^f and \tilde{Y}^g are centered Gaussian processes, in order to prove their independence it suffices to show that the last limit converges to zero as ε tends to zero. To deal with this limit, we observe that

$$\begin{split} \mathbb{E}[(Y_{\varepsilon}^{f}\bar{Y}_{\varepsilon}^{g})^{2}] &= \frac{4}{\varepsilon^{2}}\int_{0}^{\infty}\int_{0}^{\infty}f(s)g(r)\mathbb{E}\left[\cos(\theta N_{\frac{2s}{\varepsilon^{2}}})\sin(\theta N_{\frac{2r}{\varepsilon^{2}}})\right]\mathrm{d}s\mathrm{d}r\\ &= \frac{4}{\varepsilon^{2}}\int_{0}^{\infty}\int_{0}^{\infty}f(s)g(r)\mathbb{1}_{\{s\leq r\}}\mathbb{E}\left[\cos(\theta N_{\frac{2s}{\varepsilon^{2}}})\sin(\theta N_{\frac{2r}{\varepsilon^{2}}})\right]\mathrm{d}s\mathrm{d}r\\ &\quad + \frac{4}{\varepsilon^{2}}\int_{0}^{\infty}\int_{0}^{\infty}f(s)g(r)\mathbb{1}_{\{r\leq s\}}\mathbb{E}\left[\cos(\theta N_{\frac{2s}{\varepsilon^{2}}})\sin(\theta N_{\frac{2r}{\varepsilon^{2}}})\right]\mathrm{d}s\mathrm{d}r\\ &= I_{1}^{\varepsilon} + I_{2}^{\varepsilon}. \end{split}$$

Applying the formula $2\sin a \cos b = \sin(a+b) + \sin(a-b) = \sin(a+b) - \sin(b-a)$ we have

$$\begin{split} I_1^{\varepsilon} &= \frac{2}{\varepsilon^2} \int_0^{\infty} \int_0^{\infty} f(s) g(r) \mathbb{1}_{\{s \le r\}} \mathbb{E} \left[\sin(\theta(N_{\frac{2s}{\varepsilon^2}} + N_{\frac{2r}{\varepsilon^2}})) + \sin(\theta(N_{\frac{2r}{\varepsilon^2}} - N_{\frac{2s}{\varepsilon^2}})) \right] \mathrm{d}s \mathrm{d}r \\ &= I_{1,1}^{\varepsilon} + I_{1,2}^{\varepsilon}, \end{split}$$

$$\begin{split} I_2^{\varepsilon} &= \frac{2}{\varepsilon^2} \int_0^{\infty} \int_0^{\infty} f(s) g(r) \mathbb{1}_{\{r \le s\}} \mathbb{E} \left[\sin(\theta (N_{\frac{2s}{\varepsilon^2}} + N_{\frac{2r}{\varepsilon^2}})) - \sin(\theta (N_{\frac{2s}{\varepsilon^2}} - N_{\frac{2r}{\varepsilon^2}})) \right] \mathrm{d}s \mathrm{d}r \\ &= I_{2,1}^{\varepsilon} - I_{2,2}^{\varepsilon}. \end{split}$$

We proceed to show that $I_{1,1}^{\varepsilon}$ and $I_{2,1}^{\varepsilon}$ converges to zero as ε tends to zero and that $I_{1,2}^{\varepsilon}$ and $I_{2,2}^{\varepsilon}$ have the same (finite) limit, thus obtaining the stated result. We note that

$$\begin{split} I_{1,1}^{\varepsilon} &= \mathrm{Im}\left(\frac{2}{\varepsilon^2} \int_0^{\infty} \int_0^{\infty} f(s)g(r) \mathbbm{1}_{\{s \le r\}} \mathbb{E}\left[e^{i\theta(N_{\frac{2r}{\varepsilon^2}} - N_{\frac{2s}{\varepsilon^2}})} e^{2i\theta N_{\frac{2s}{\varepsilon^2}}}\right] \mathrm{d}s \mathrm{d}r\right) \\ &= \mathrm{Im}(A^{\varepsilon}). \end{split}$$

To find the limit of $I_{1,1}^{\varepsilon}$ we see that $||A^{\varepsilon}||$ converges to zero as ε tends to zero.

$$\begin{split} \|A^{\varepsilon}\| &\leq \frac{2}{\varepsilon^2} \int_0^{\infty} \int_0^{\infty} |f(s)g(r)| \mathbbm{1}_{\{s \leq r\}} e^{-\frac{2}{\varepsilon^2}(r-s)(1-\cos\theta)} e^{-\frac{2}{\varepsilon^2}s(1-\cos2\theta)} \mathrm{d}s \mathrm{d}r \\ &\leq \frac{1}{\varepsilon^2} \int_0^{\infty} \int_0^{\infty} \left(f^2(s) + g^2(r)\right) \mathbbm{1}_{\{s \leq r\}} e^{-\frac{2r}{\varepsilon^2}(1-\cos\theta)} e^{\frac{2s}{\varepsilon^2}(\cos2\theta-\cos\theta)} \mathrm{d}s \mathrm{d}r \\ &= \frac{1}{\varepsilon^2} \int_0^{\infty} f^2(s) e^{\frac{2s}{\varepsilon^2}(\cos2\theta-\cos\theta)} \int_s^{\infty} e^{-\frac{2r}{\varepsilon^2}(1-\cos\theta)} \mathrm{d}r \mathrm{d}s \\ &\quad + \frac{1}{\varepsilon^2} \int_0^{\infty} g^2(r) e^{-\frac{2r}{\varepsilon^2}(1-\cos\theta)} \int_0^r e^{\frac{2s}{\varepsilon^2}(\cos2\theta-\cos\theta)} \mathrm{d}s \mathrm{d}r \\ &= A_1^{\varepsilon} + A_2^{\varepsilon}. \end{split}$$

When $\cos \theta = \cos 2\theta$ it is easy to check the convergence to zero. Otherwise, we integrate obtaining

$$A_1^{\varepsilon} = \frac{1}{2(1-\cos\theta)} \int_0^{\infty} f^2(s) e^{-\frac{2s}{\varepsilon^2}(1-\cos2\theta)} \mathrm{d}s,$$
$$A_2^{\varepsilon} = \frac{1}{2(\cos2\theta - \cos\theta)} \int_0^{\infty} g^2(r) e^{-\frac{2r}{\varepsilon^2}(1-\cos\theta)} \left(e^{\frac{2r}{\varepsilon^2}(\cos2\theta - \cos\theta)} - 1 \right) \mathrm{d}r$$
$$= \frac{1}{2(\cos2\theta - \cos\theta)} \int_0^{\infty} g^2(r) \left(e^{-\frac{2r}{\varepsilon^2}(1-\cos2\theta)} - e^{-\frac{2r}{\varepsilon^2}(1-\cos\theta)} \right) \mathrm{d}r$$

which concludes, as the convergence to zero is easily seen by dominated convergence. In the same manner we can see that $I_{2,1}^{\varepsilon}$ converges to zero.

With respect to the term $I_{1,2}^{\varepsilon}$ we observe that

$$I_{1,2}^{\varepsilon} = \operatorname{Im}\left(\frac{2}{\varepsilon^2} \int_0^{\infty} \int_0^{\infty} f(s)g(r) \mathbb{1}_{\{s \le r\}} e^{-\frac{2}{\varepsilon^2}(r-s)(1-e^{i\theta})} \mathrm{d}s \mathrm{d}r\right)$$

Since $\frac{2}{\varepsilon^2}(1-e^{i\theta})e^{-\frac{2}{\varepsilon^2}(r-s)(1-e^{i\theta})}$ is an approximation of the identity, we have that $I_{1,2}^{\varepsilon}$ converges, as ε tends to zero, to $\operatorname{Im}\left(\frac{1}{1-e^{i\theta}}\int_0^{\infty} f(s)g(s)\mathrm{d}s\right) < \infty$. Clearly the same result is obtained for $I_{2,2}^{\varepsilon}$. This finishes the proof.

4.1. Weak approximation for the fractional Brownian motion from a Poisson process.

We are going to prove a result of weak convergence in $\mathcal{C}([0,T])$ toward fBm, applying Theorem 4.1. In order to do so, we use the following representation of the fBm as the integral of a deterministic kernel with respect to standard Brownian motion

(4.6)
$$B_t^H = \int_0^t \tilde{K}^H(t,s) \,\mathrm{d}W_s,$$

where $H \in (0,2)$, $\tilde{K}^{H}(t,s)$ is defined on the set $\{0 < s < t\}$ and is given by (4.7) $\tilde{K}^{H}(t,s) = d^{H}(t-s)^{\frac{H-1}{2}} + d^{H}\left(\frac{1-H}{2}\right) \int_{s}^{t} (u-s)^{\frac{H-3}{2}} \left(1 - \left(\frac{s}{u}\right)^{\frac{1-H}{2}}\right) du,$

where the normalizing constant d^H is

$$d^{H} = \left(\frac{H\Gamma(\frac{3-H}{2})}{\Gamma(\frac{H+1}{2})\Gamma(2-H)}\right)^{\frac{1}{2}}$$

Since in this section the domain of fBm is restricted to the interval $t \in [0, T]$, we can rewrite the integral representation as

$$B_t^H = \int_0^t \tilde{K}^H(t,s) \, \mathrm{d}W_s = \int_0^T K^H(t,s) \, \mathrm{d}W_s,$$

where $K^{H}(t,s) = \tilde{K}^{H}(t,s) \mathbb{1}_{[0,t]}(s)$.

Applying this representation, since $K^H(t, \cdot) \in L^2(\mathbb{R}^+)$, the following result is a corollary of Theorem 4.1

Corollary 4.2. Let $K^H(t,s) = \tilde{K}^H(t,s)\mathbb{1}_{[0,t]}(s)$, where $\tilde{K}^H(t,s)$ is defined by (4.7), let $\{N_s, s \ge 0\}$ be a standard Poisson process and let $\theta \in (0,\pi) \cup (\pi, 2\pi)$. Then the processes

(4.8)
$$B_{\varepsilon}^{H} = \left\{ \frac{2}{\varepsilon} \int_{0}^{T} K^{H}(t,s) \cos\left(\theta N_{\frac{2s}{\varepsilon^{2}}}\right) ds, \quad t \in [0,T] \right\}$$

and

(4.9)
$$\tilde{B}_{\varepsilon}^{H} = \left\{ \frac{2}{\varepsilon} \int_{0}^{T} K^{H}(t,s) \sin\left(\theta N_{\frac{2s}{\varepsilon^{2}}}\right) ds, \quad t \in [0,T] \right\}$$

converge in law, in the sense of the finite dimensional distributions, toward two independent fractional Brownian motions.

We now proceed to prove the continuity and the tightness of the families of processes defined by (4.8) and (4.9), and consequently, proving the weak convergence in the space C([0, T]).

Theorem 4.3. Under the hypothesis of Corollary 4.2, if moreover one of the following conditions is satisfied:

- (1) $H \in (\frac{1}{2}, 2),$
- (2) $H \in (0, \frac{1}{2}]$ and θ satisfies $\cos((2i+1)\theta) \neq 1$ for all $i \in \mathbb{N}$ such that $i \leq \frac{1}{2} \left[\frac{1}{H}\right]$,

then the processes B_{ε}^{H} and $\tilde{B}_{\varepsilon}^{H}$ converge in law in $\mathcal{C}([0,T])$ toward two independent fractional Brownian motions.

Proof. We first observe that the processes B_{ε}^{H} and $\tilde{B}_{\varepsilon}^{H}$ are continuous. In fact, B_{ε}^{H} and $\tilde{B}_{\varepsilon}^{H}$ are continuous for all $H \in (0, 2)$ and absolutely continuous if $H \in (1, 2)$, since it can be proved that (see Lemma 2.1 in [BF05])

$$|B_{\varepsilon}^{H}(t) - B_{\varepsilon}^{H}(s)| \le C_{H}(t-s)^{\left(\frac{H+1}{2}\right) \wedge 1}$$

and

$$|\tilde{B}_{\varepsilon}^{H}(t) - \tilde{B}_{\varepsilon}^{H}(s)| \le C_{H}(t-s)^{\left(\frac{H+1}{2}\right) \wedge 1}.$$

It only remains to prove the tightness of the families of processes defined by (4.8) and (4.9). Applying Billingsley's criterion (see for instance [Bil68]) and due to

$$\int_0^T \left(K^H(t,r) - K^H(s,r) \right)^2 \mathrm{d}r = \mathbb{E} \left[(B_t^H - B_s^H)^2 \right] = (t-s)^H,$$

it is sufficient to show that

(4.10)
$$\mathbb{E}\left[y_{\varepsilon}^{m}\right] \leq C_{m} \left(\int_{0}^{T} f^{2}(r) \,\mathrm{d}r\right)^{\frac{m}{2}}, \quad \mathbb{E}\left[\tilde{y}_{\varepsilon}^{m}\right] \leq C_{m} \left(\int_{0}^{T} f^{2}(r) \,\mathrm{d}r\right)^{\frac{m}{2}}$$

holds for some *m* satisfying the condition Hm/2 > 1, where $f(r) := K^H(t, r) - K^H(s, r), y_{\varepsilon}^f = \frac{2}{\varepsilon} \int_0^T f(r) \cos(\theta N_{\frac{2r}{\varepsilon^2}}) dr$ and $\tilde{y}_{\varepsilon}^f = \frac{2}{\varepsilon} \int_0^T f(r) \sin(\theta N_{\frac{2r}{\varepsilon^2}}) dr$.

Then, in the case (1), it is sufficient to prove (4.10) for m = 4, which can be seen proving that $\mathbb{E}[(z_{\varepsilon}^{f} \bar{z}_{\varepsilon}^{f})^{2}] \leq C ||f||_{2}^{4}$, where $||\cdot||_{2}$ is the $L^{2}[0,T]$ norm and $z_{\varepsilon}^{f} = y_{\varepsilon}^{f} + i\tilde{y}_{\varepsilon}^{f}$. If we extend f to \mathbb{R}^{+} for zeros, i.e., if we consider $F(r) := f(r)\mathbb{1}_{[0,T]}(r)$, we have proof in Theorem 4.1 that

$$\mathbb{E}[(Z_{\varepsilon}^{F}\bar{Z}_{\varepsilon}^{F})^{2}] \leq 3\left(\frac{4}{1-\cos\theta}\int_{0}^{\infty}F^{2}(s)\mathrm{d}s\right)^{2}.$$

Then,

$$\begin{split} \mathbb{E}[(z_{\varepsilon}^{f}\bar{z}_{\varepsilon}^{f})^{2}] &= \mathbb{E}[(Z_{\varepsilon}^{F}\bar{Z}_{\varepsilon}^{F})^{2}] \\ &\leq 3\left(\frac{4}{1-\cos\theta}\int_{0}^{\infty}F^{2}(s)\mathrm{d}s\right)^{2} = 3\left(\frac{4}{1-\cos\theta}\int_{0}^{T}f^{2}(s)\mathrm{d}s\right)^{2}. \end{split}$$

To prove the result under the hypothesis (2) we must show that (4.10) is satisfied for some even m such that $\frac{Hm}{2} > 1$. If we proceed in the same way as in case (1) we obtain an expression that depends on $1 - \cos((2i+1)\theta)$ for all $i = 0, 1, \ldots, \left[\frac{1}{2H}\right]$ and the constant C_m depends on $\max_{i=0,1,\ldots,\left[\frac{1}{2H}\right]} \frac{1}{1-\cos((2i+1)\theta)}$.

4.2. Approximation in law of the X^H process.

We will give now, for the process X^H defined by (1.3), the same results we have obtained for the fBm.

Corollary 4.4. Let X^H be the process defined by (1.3), let $\{N_s, s \ge 0\}$ be a standard Poisson process and let $\theta \in (0, \pi) \cup (\pi, 2\pi)$. Then the processes

(4.11)
$$X_{\varepsilon}^{H} = \left\{ \frac{2}{\varepsilon} \int_{0}^{\infty} (1 - e^{-st}) s^{-\frac{1+H}{2}} \cos\left(\theta N_{\frac{2s}{\varepsilon^{2}}}\right) ds, \quad t \in [0, T] \right\}$$

and

(4.12)
$$\tilde{X}_{\varepsilon}^{H} = \left\{ \frac{2}{\varepsilon} \int_{0}^{\infty} (1 - e^{-st}) s^{-\frac{1+H}{2}} \sin\left(\theta N_{\frac{2s}{\varepsilon^{2}}}\right) ds, \quad t \in [0, T] \right\}$$

converge in law, in the sense of the finite dimensional distributions, toward two independent processes with the same law that X^H .

Theorem 4.5. Under the hypothesis of Corollary 4.4 the processes defined by (4.11) and (4.12) converge in law in C([0,T]) toward two independent processes with the same law that the process defined by (1.3).

Proof. We first need to show that the processes X_{ε}^{H} and $\tilde{X}_{\varepsilon}^{H}$ are continuous. In fact, they are absolutely continuous. Let us consider for all r > 0 the process

$$Y_r = \frac{2}{\varepsilon} \int_0^\infty s^{\frac{1-H}{2}} e^{-sr} \cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) \mathrm{d}s.$$

This integral exists because, using inequality (4.4), we have

$$\mathbb{E}[Y_r^2] \le C\left(\int_0^\infty s^{1-H} e^{-2sr} \mathrm{d}s\right) = Cr^{H-2}\Gamma(2-H).$$

On the other hand,

$$\mathbb{E}\left[\int_0^t |Y_r| \mathrm{d}r\right] \le \int_0^t (\mathbb{E}[Y_r^2])^{\frac{1}{2}} \mathrm{d}r \le C \int_0^t r^{\frac{H-2}{2}} \mathrm{d}r < \infty$$

since $H \in (0, 2)$.

Let us now observe that $X_{\varepsilon}^{H} = \int_{0}^{t} Y_{r} dr$. Indeed, applying Fubini's theorem,

$$\int_0^t Y_r dr = \frac{2}{\varepsilon} \int_0^\infty s^{\frac{1-H}{2}} \left(\int_0^t e^{-sr} dr \right) \cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) ds$$
$$= \frac{2}{\varepsilon} \int_0^\infty s^{-\frac{1+H}{2}} (1 - e^{-st}) \cos\left(\theta N_{\frac{2s}{\varepsilon^2}}\right) ds$$
$$= X_{\varepsilon}^H.$$

The same proof shows that the process $\tilde{X}_{\varepsilon}^{H}$ is continuous. Next, we prove the convergence only for (4.11). For (4.12) the result is proved similarly.

It suffices to prove the tightness of the family $\{X_{\varepsilon}^{H}\}_{\varepsilon}$. Since $X_{\varepsilon}^{H}(0) = 0$, using Billingsley's criterion we only need to prove that

$$\mathbb{E}\left[|X_{\varepsilon}^{H}(t) - X_{\varepsilon}^{H}(s)|^{4}\right] \le |F(t) - F(s)|^{2}$$

where F is a continuous, non-decreasing function. We observe that

$$\mathbb{E}\left[|X_{\varepsilon}^{H}(t) - X_{\varepsilon}^{H}(s)|^{4}\right] = \mathbb{E}\left[\frac{2}{\varepsilon}\int_{0}^{\infty} \left(\Phi^{H}(t,r) - \Phi^{H}(s,r)\right)\cos(\theta N_{\frac{2r}{\varepsilon^{2}}})\mathrm{d}r\right]^{4}$$

where $\Phi^H(t,r) = (1 - e^{-rt})r^{-\frac{1+H}{2}} \in L^2(\mathbb{R}^+)$. Since $\Phi^H \in L^2(\mathbb{R}^+)$, applying the bound (4.5), which is proved in Theorem 4.1, we obtain

$$\mathbb{E}\left[|X_{\varepsilon}^{H}(t) - X_{\varepsilon}^{H}(s)|^{4}\right] \leq C\left(\int_{0}^{\infty} \left(\Phi^{H}(t, r) - \Phi^{H}(s, r)\right)^{2} \mathrm{d}r\right)^{2}$$
$$= C\left(\int_{0}^{\infty} \left((1 - e^{-rt})^{2}r^{-(1+H)} + (1 - e^{-rs})^{2}r^{-(1+H)}\right) \mathrm{d}r\right)^{2}$$
$$- 2(1 - e^{-rt})(1 - e^{-rs})r^{-(1+H)} \mathrm{d}r\right)^{2}.$$

Using the computations done in Proposition 2.1 and assuming s < t we obtain for $H \in (0,1)$

$$\mathbb{E}\left[|X_{\varepsilon}^{H}(t) - X_{\varepsilon}^{H}(s)|^{4}\right] \leq C \left(2(t+s)^{H} - (2t)^{H} - (2s)^{H}\right)^{2} \\ \leq C \left((2t)^{H} - (2s)^{H}\right)^{2},$$

since s + t < 2t. In the same way, if $H \in (1, 2)$,

$$\mathbb{E}\left[|X_{\varepsilon}^{H}(t) - X_{\varepsilon}^{H}(s)|^{4}\right] \leq C\left((2t)^{H} + (2s)^{H} - 2(t+s)^{H}\right)^{2}$$
$$\leq C\left((2t)^{H} - (2s)^{H}\right)^{2},$$

since s + t > 2s. In both cases we have proved the result with $F(x) = (2x)^{H}$.

4.3. Convergence toward sub-fractional Brownian motion.

To finish this paper, we prove a result of weak convergence to sub-fractional Brownian motion, as a direct conclusion of the previous results.

Theorem 4.6. Let $H \in (0,1)$, let $\{S^H, t \in [0,T]\}$ be a sub-fractional Brownian motion, let $\{X_{\varepsilon}^H(t), t \in [0,T]\}$ be the processes defined by (4.11), let $\{\tilde{B}_{\varepsilon}^H(t), t \in [0,T]\}$ be the processes defined by (4.9) and $C_1 = \sqrt{\frac{H}{2\Gamma(1-H)}}$. Let us assume $\theta \in (0,\pi) \cup (\pi,2\pi)$ and, for $H \in (0,\frac{1}{2}]$, that θ is such that $\cos((2i+1)\theta) \neq 1$ for all $i \in \mathbb{N}$ such that $i \leq \frac{1}{2} \left[\frac{1}{H}\right]$. Then, $\{Y_{\varepsilon}^H(t) = C_1 X_{\varepsilon}^H(t) + B_{\varepsilon}^H(t), t \in [0,T]\}$ weakly converges in $\mathcal{C}([0,T])$ to S^H .

Proof. Applying Theorems **4.3** and **4.5** we know that, respectively, the processes $\tilde{B}_{\varepsilon}^{H}$ and X_{ε}^{H} converge in law in $\mathcal{C}([0,T])$ toward a fBm and the process defined by (1.3). Moreover, applying Theorem **4.1**, we know that the limit laws are independent. Hence, we are under the hypothesis of Theorem **2.2**, which proves the stated result.

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DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA, 08193 BELLATERRA (SPAIN)

E-mail address: bardina@mat.uab.cat, bascompte@mat.uab.cat