# On the Dynamics of a Fluid-Particle Interaction Model: The Bubbling Regime

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#### Abstract

This article deals with the issues of global-in-time existence and asymptotic analysis of a fluid-particle interaction model in the so-called bubbling regime. The mixture occupies the physical space  $\Omega \subset \mathbb{R}^3$  which may be unbounded. The system under investigation describes the evolution of particles dispersed in a viscous compressible fluid and is expressed by the conservation of fluid mass, the balance of momentum and the balance of particle density often referred as the Smoluchowski equation. The coupling between the dispersed and dense phases is obtained through the drag forces that the fluid and the particles exert mutually by action-reaction principle. We show that solutions exist globally in time under reasonable physical assumptions on the initial data, the physical domain, and the external potential. Furthermore, we prove the large-time stabilization of the system towards a unique stationary state fully determined by the masses of the initial density of particles and fluid and the external potential.

### 1 Introduction

Fluid-particle interaction arise in many practical applications in biotechnology, medicine [4], reactive gases formed by fuel droplets in combustion [1, 2, 25, 26], recycling and mineral processes [5, 23], and atmospheric pollution [24]. Aerosols and sprays can be modelled by fluid-particle type interactions in which bubbles of suspended substances are seen as solid particles. Two-phase flow hydrodynamic models have also been proposed [6].

Here, we focus on a particular system derived by formal asymptotics from a mesoscopic description. This is based on a kinetic equation for the particle distribution of Fokker-Planck type coupled to fluid equations. Different macroscopic equations can be obtained as scaling limits, see [7] for a complete description of the modelling

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issues. In these models, the fluid is either incompressible [15, 16] or compressible [7]. The coupling between the kinetic and the fluid equations is obtained through the friction forces that the fluid and the particles exert mutually. The friction force is assumed to follow Stokes law and thus is proportional to the relative velocity vector, i.e., is proportional to the fluctuations of the microscopic velocity  $\xi \in \mathbb{R}^3$  around the fluid velocity field **u**. More precisely, the cloud of particles is described by its distribution function  $f_{\varepsilon}(t, x, \xi)$  on phase space, which is the solution to the dimensionless Vlasov-Fokker-Planck equation

$$\partial_t f_{\varepsilon} + \frac{1}{\sqrt{\varepsilon}} \Big( \xi \cdot \nabla_x f_{\varepsilon} - \nabla_x \Phi \cdot \nabla_{\xi} f_{\varepsilon} \Big) = \frac{1}{\varepsilon} \operatorname{div}_{\xi} \Big( \big( \xi - \sqrt{\varepsilon} u_{\varepsilon} \big) f + \nabla_{\xi} f_{\varepsilon} \Big).$$
(1.1)

Here,  $\epsilon > 0$  is a dimensionless parameter and we have a drag force independent of the fluid density  $\rho_{\varepsilon}$ , but proportional to the relative velocity of the fluid and the particles given by  $\xi - u_{\varepsilon}(t, x)$ . The right hand-side of the moment equation in the Navier-Stokes system takes into account the action of the cloud of particles on the fluid through the forcing term

$$F_{\varepsilon} = \int_{\mathbb{R}^3} \left( \frac{\xi}{\sqrt{\varepsilon}} - u_{\varepsilon}(t, x) \right) f(t, x, \xi) \, d\xi.$$

The density of the particles  $\eta_{\varepsilon}(t, x)$  is related to the probability distribution function  $f_{\varepsilon}(t, x, \xi)$  through the relation

$$\eta_{\varepsilon}(t,x) = \int_{\mathbb{R}^3} f_{\varepsilon}(t,x,\xi) \, d\xi$$

The well-posedness of this kinetic-hyperbolic coupled fluid-particle system has been addressed in [19] in the case of compressible models for the fluid equations. There are two different scaling limits for this model, the so-called: bubbling and flowing regimes. They correspond to the diffusive approximation of the kinetic equation, bubbling regime, written in (1.1), and the strong drag force and strong Brownian motion for the flowing regime. This last regime has been studied in [20] where it is obtained rigorously as the limit from the mesoscopic description for local in time solutions and initial data bounded away from zero for the densities. We also refer to [8] for asymptotic preserving numerical schemes in relation to these scaling limits.

In all the above mentioned studies, the viscosity of the fluid was neglected although it is the source of the drag forces. The viscosity is present in the dimensionless systems although negligible as noted in [7, Remark 3]. In this work, we will deal with the resulting formal macroscopic fluid-particle system obtained through the scaling limit in (1.1) as  $\varepsilon \to 0$  by the standard Hilbert-expansion procedure. In this scaling limit, particles are supposed to have a negligible density with respect to the fluid, and thus, due to buoyancy effects, they will typically move upwards in a system under gravity, from where it bears the name of "bubbling". This situation is typically complemented with no-flux boundary conditions in a bounded domain. More generally, we can ask ourselves under what conditions on the external potential in unbounded domains we can assert convergence towards stationary integrable states. We refer to [7, 8] for a detail account of the physical meaning and validity of the scaling limits.

Summarizing, the state of such flows in this macroscopic description is, in general, characterized by the variables: the total mass density  $\rho(t, x)$ , the velocity field  $\mathbf{u}(t, x)$ , as well as the density of particles in the mixture  $\eta(t, x)$ , depending on the time  $t \in (0, T)$  and the Eulerian spatial coordinate  $x \in \Omega \subset \mathbb{R}^3$ .

In this section, we present the primitive conservation equations governing fluidparticle flows in the bubbling regime. These equations express the conservation of mass, the balance of momentum, and the balance of particle densities often referred as the *Smoluchowski equation*:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{1.2}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(p(\rho) + \eta) - \mu \Delta \mathbf{u} - \lambda \nabla_x \operatorname{div}_x \mathbf{u} = -(\eta + \beta \rho) \nabla_x \Phi, \quad (1.3)$$

$$\partial_t \eta + \operatorname{div}_x(\eta(\mathbf{u} - \nabla_x \Phi)) - \Delta \eta = 0.$$
(1.4)

Here, p denotes the pressure  $p(\rho) = a\rho^{\gamma}$ , a > 0,  $\gamma > 1$ ,  $\beta \neq 0$ , and  $\Phi$  denotes the external potential (typically incorporating gravity and boyancy).

In this paper, we require the potential to satisfy suitable confinement conditions (**HC**) (see Section 2), which does not limit the physical relevance of our results. The viscosity parameters  $\mu > 0$  and  $\lambda + \frac{2}{3}\mu \ge 0$  are nonnegative constants and  $\beta > 0$  if  $\Omega$  is unbounded.

Another macroscopic effect is that the total pressure function in the momentum equation depends on both the particle and the fluid densities  $p(\varrho) + \eta$ . We impose the no-slip boundary condition for the velocity vector leading to no-flux for the fluid density through the boundaries and the no-flux condition for the particle density

$$\mathbf{u}|_{\partial\Omega} = \nabla_x \eta \cdot \nu + \eta \nabla_x \Phi \cdot \nu = 0 \text{ on } (0,T) \times \partial\Omega, \tag{1.5}$$

with  $\nu$  denoting the outer normal vector to the boundary  $\partial\Omega$ . Our problem is supplemented with the initial data  $\{\varrho_0, \mathbf{m}_0, \eta_0\}$  such that

$$\varrho(0,x) = \varrho_0 \in L^{\gamma}(\Omega) \cap L^1_+(\Omega),$$

$$(\varrho \mathbf{u})(0,x) = \mathbf{m}_0 \in L^{\frac{6}{5}}(\Omega) \cap L^1(\Omega),$$

$$\eta(0,x) = \eta_0 \in L^2(\Omega) \cap L^1_+(\Omega).$$
(1.6)

Motivated by the stability arguments in [7], the numerical investigation presented in [8], as well as a number of studies on numerical experiments and scale analysis on the proposed model (see [3]), we investigate the issues of global existence and asymptotic analysis for fluid-particle interaction flows providing a rigorous mathematical theory based on the principle of balance laws. The total energy of the mixture is given by

$$E(\eta, \varrho, \mathbf{u})(t) := \int_{\Omega} \left[ \frac{1}{2} \varrho(t) |\mathbf{u}(t)|^2 + \frac{a}{\gamma - 1} \varrho^{\gamma}(t) + (\eta \log \eta)(t) + (\beta \varrho + \eta)(t) \Phi \right] \mathrm{d}x \quad (1.7)$$

At the formal level, the total energy can be viewed as a Lyapunov function satisfying the *energy inequality* 

$$\frac{dE}{dt} + \int_{\Omega} \left[ \mu |\nabla_x \mathbf{u}|^2 + \lambda |\mathrm{div}_x \mathbf{u}|^2 + |2\nabla_x \sqrt{\eta} + \sqrt{\eta} \nabla_x \Phi|^2 \right] \,\mathrm{d}x \le 0.$$
(1.8)

Therefore, it is reasonable to anticipate that, at least for some sequences  $t_n \to \infty$ ,

$$\eta(t_n) \to \eta_s, \ \varrho(t_n) \to \varrho_s, \ \varrho\mathbf{u}(t_n) \to 0,$$

where  $\eta_s, \varrho_s$  satisfy the stationary problem

$$\nabla_x(p(\varrho_s) + \eta_s) = -(\eta_s + \beta \varrho_s) \nabla_x \Phi \text{ on } \Omega.$$

The energy estimate written in the form

$$E(t) + \int_0^T (\|\nabla_x \mathbf{u}\|_{L^2}^2 + \|\operatorname{div}_x \mathbf{u}\|_{L^2}^2) dt + \int_0^T \int_\Omega |2\nabla_x \sqrt{\eta} + \sqrt{\eta} \nabla_x \Phi|^2 \, \mathrm{d}x dt \le E(0),$$

now implies that

$$|2\nabla_x\sqrt{\eta_s} + \sqrt{\eta_s}\nabla_x\Phi|^2 = 0.$$
(1.9)

The aim of this paper is to show that, in fact, any weak solution converges to a fixed stationary state as time goes to infinity, more precisely,

$$\varrho(t) \to \varrho_s \text{ strongly in } L^{\gamma}(\Omega),$$
  
 $\operatorname{ess\,sup}_{\tau > t} \int_{\Omega} \varrho(\tau) |\mathbf{u}(\tau)|^2 \, \mathrm{d}x \to 0,$ 
  
 $\eta(t) \to \eta_s \text{ strongly in } L^p(\Omega),$ 

as  $t \to \infty$  under the confinement hypothesis on the domain  $\Omega$  and the external potential  $\Phi$  given in **(HC)** (cf. Definition 2.1).

Indeed, it can be shown that the sequences  $(\varrho_n, \eta_n)$  of the time shifts defined as

$$\varrho_n(t,x) := \varrho(t+\tau_n,x), \ \tau_n \to \infty,$$
$$\eta_n(t,x) := \eta(t+\tau_n,x), \ \tau_n \to \infty,$$

contain subsequences, denoted by the same index w.l.o.g., such that

$$\varrho_n \to \varrho_s$$
 strongly in  $L^1_{loc}((0,1) \times \Omega)$ ,

and

$$\eta_n \to \eta_s$$
 strongly in  $L^{p_1}((0,T);L^{p_2})(\Omega)$  for some  $p_1, p_2 > 1$ 

where  $(\varrho_s, \eta_s)$  solve the stationary problem

$$\begin{cases} \nabla_x p(\varrho_s) = -\beta \varrho_s \nabla_x \Phi, \\ \nabla_x \eta_s = -\eta_s \nabla_x \Phi. \end{cases}$$
(1.10)

It is worth noting that the confinement hypothesis is both necessary and sufficient for the stationary problem (1.10) to admit a unique solution  $(\rho_s, \eta_s)$  (Section 4).

The main ingredients of our approach can be formulated as follows:

- A suitable weak formulation of the underlying physical principles is introduced. Physically grounded hypotheses are imposed on the system: The mixture occupies the physical space  $\Omega \subset \mathbb{R}^3$ . The boundary conditions are chosen in such a way so that the dissipation of energy is guaranteed, whereas the pressure of the mixture takes into account both the density of the fluid and the density of particles.
- A priori estimates are established based solely on the boundedness of the initial energy and entropy of the system.

- A suitable approximating scheme is introduced for the construction of the solution based on a two level approximating procedure: The first level involves an *artificial pressure approximation*, whereas the second level approximation employs a *time discretization scheme*. The sequence of approximate solutions is constructed with the aid of a fixed point argument coupling the time-discretized compressible isentropic Navier-Stokes equations to a discretization in time of the equation for  $\eta$ .
- Physically grounded hypotheses are imposed on the domain  $\Omega$  and the external potential  $\Phi$  (confinement hypotheses **(HC)**). The analysis in the present article treats both the case of a *bounded* physical domain  $\Omega$  as well as the case of an *unbounded* domain. We remark that the confinement hypothesis **(HC)** on  $(\Omega, \Phi)$  plays a crucial role in providing control of the negative contribution of the physical entropy  $\eta \log \eta$  in the free-energy bounds for unbounded domains.
- High integrability properties for the density need to be established for the limit passage in the family of approximate solutions and in particular in taking the vanishing artificial pressure limit. We remark that in the present context, the potential  $\Phi$  is not integrable on unbounded domains. To deal with this new difficulty we employ the *Fourier multipliers* in the spirit of [11, 18], while taking into consideration the new features of our problem.
- We remark that both the total fluid mass and the total particle mass are constants of motion. In particular, we are able to conserve the total masses also in the large-time limit allowing us to uniquely determine the long time asymptotics (cf [14]).

The paper is organized as follows. In Section 2 we collect all the necessary hypotheses imposed on the external potential (confinement hypotheses (HC)), we present the notion of *free energy solutions* and the main results of this article. Section 3 is devoted to the proof of the global existence of weak solutions (Theorem 2.1). First, a suitable approximation scheme based on an *artificial pressure approximation* and on a *time discretization scheme* is introduced. The remaining section is devoted to the limit passage in the family of approximate solutions. The most delicate part of the analysis concerned with the vanishing artificial pressure limit is presented in Subsection 3.5. The large time asymptotic analysis is described in Section 4.

### 2 Free-energy solutions and main results

In this work, we analyse the existence and large-time asymptotics of certain type of weak solutions to the two-phase flow problem (1.2)-(1.4) coupled with no-flux boundary conditions (1.5) in two different geometrical constraints of interest in the applications: for bounded domains and for unbounded domains under confinement conditions due to the external potential. We will collect all assumptions concerning the geometry  $\Omega$  and the external potential  $\Phi$  under the generic name of *confinement conditions*. Let us remark that the external potential  $\Phi$  is always defined up to a constant. Therefore, for bounded from below external potentials  $\Phi$ , we can always assume without loss of generality, by adding a suitable constant, that

$$\inf_{x \in \Omega} \Phi(x) = 0. \tag{2.1}$$

**Definition 2.1** Given a domain  $\Omega \in C^{2,\nu}$ ,  $\nu > 0$ ,  $\Omega \subset \mathbb{R}^3$ , and given a bounded from below external potential  $\Phi : \Omega \longrightarrow \mathbb{R}^+_0$  satisfying (2.1), we will say that  $(\Omega, \Phi)$  verifies the confinement hypotheses **(HC)** for the two-phase flow system (1.2)-(1.4) coupled with no-flux boundary conditions (1.5) whenever:

**(HC-Bounded)** If  $\Omega$  is bounded,  $\Phi$  is bounded and Lipschitz continuous in  $\Omega$ and the sub-level sets  $[\Phi < k]$  are connected in  $\Omega$  for any k > 0.

**(HC-Unbounded)** If  $\Omega$  is unbounded, we assume that  $\Phi \in W^{1,\infty}_{loc}(\Omega)$ ,  $\beta > 0$ , the sub-level sets  $[\Phi < k]$  are connected in  $\Omega$  for any k > 0,

$$e^{-\Phi/2} \in L^1(\Omega),$$

and

$$|\Delta\Phi(x)| \le c_1 |\nabla_x \Phi(x)| \le c_2 \Phi(x), \ |x| > R, \tag{2.2}$$

for some large R > 0.

**Remark 2.1** The confinement assumption (HC) has physical relevance in our setting as it is verified for several domains  $\Omega$  with  $\Phi$  being the gravitational potential. For instance,

- 1. when  $\Omega = \{x \in \mathbb{R}^3 \mid (x_1, x_2) \in [a, b]^2, x_3 \in [0, H]\}$  and  $\Phi(x) = gx_3$ , where  $\beta = 1 \frac{\varrho_F}{\varrho_P}$ .
- 2. when  $\Omega = \{x \in \mathbb{R}^3 \mid (x_1, x_2) \in [a, b]^2, x_3 > 0\}$  and  $\Phi(x) = gx_3$ , where  $\beta = 1 \frac{\varrho_F}{\rho_P}$  and  $\varrho_F < \varrho_P$ .
- 3. when  $\Omega = \mathbb{R}^3 \setminus \overline{B(0,R)}$  and  $\Phi(x) = g|x|$ , where B(0,R) is the ball centered at the origin with radius R and  $\beta > 0$ .

Here,  $\varrho_F$  and  $\varrho_P$  are the typical mass density of fluid and particles, respectively. Remark that 1. corresponds to the standard bubbling case (see [7]) in which particles move upwards due to buoyancy.

Let us now specify the type of weak solutions for the two-phase flow system (1.2)-(1.4) we will be dealing with.

**Definition 2.2** Let us assume that  $(\Omega, \Phi)$  satisfy the confinement hypotheses **(HC)**, we say that  $\{\varrho, \mathbf{u}, \eta\}$  is a free-energy solution of problem (1.2)-(1.4) supplemented with boundary data for which (1.5) holds and initial data  $\{\varrho_0, \mathbf{m}_0, \eta_0\}$  satisfying (1.6) provided that the following hold:

•  $\rho \geq 0$  represents a renormalized solution of equation (1.2) on  $(0, \infty) \times \Omega$ : For any test function  $\varphi \in \mathcal{D}([0, T) \times \overline{\Omega})$ , any T > 0, and any b such that

$$b \in L^{\infty} \cap C[0,\infty), \ B(\varrho) = B(1) + \int_{1}^{\varrho} \frac{b(z)}{z^2} \, \mathrm{d}z,$$

the following integral identity holds:

$$\int_{0}^{\infty} \int_{\Omega} \left( B(\varrho) \partial_{t} \varphi + B(\varrho) \mathbf{u} \cdot \nabla_{x} \varphi - b(\varrho) \operatorname{div}_{x} \mathbf{u} \varphi \right) \, \mathrm{d}x \, \mathrm{d}t = -\int_{\Omega} B(\varrho_{0}) \varphi(0, \cdot) \, \mathrm{d}x \quad (2.3)$$

• The balance of momentum holds in distributional sense, namely

$$\int_0^\infty \int_\Omega \left( \rho \mathbf{u} \cdot \partial_t \mathbf{v} + \rho \mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{v} + (p(\rho) + \eta) \operatorname{div}_x \mathbf{v} \right) \, \mathrm{d}x \, \mathrm{d}t = \qquad (2.4)$$

$$\int_0^\infty \int_\Omega (\mu \nabla_x \mathbf{u} \nabla_x \mathbf{v} + \lambda \operatorname{div}_x \mathbf{u} \operatorname{div}_x \mathbf{v} - (\eta + \beta \varrho) \nabla_x \Phi \cdot \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \mathbf{m}_0 \cdot \mathbf{v}(0, \cdot) \, \mathrm{d}x,$$

for any test function  $\mathbf{v} \in \mathcal{D}([0,T); \mathcal{D}(\overline{\Omega}; \mathbb{R}^3))$  and any T > 0 satisfying  $\vec{\varphi}|_{\partial\Omega} = 0$ . All quantities appearing in (2.4) are supposed to be at least integrable. In particular, the velocity field  $\mathbf{u}$  belongs to the space  $L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3))$ , therefore it is legitimate to require  $\mathbf{u}$  to satisfy the boundary conditions (1.5) in the sense of traces.

•  $\eta \ge 0$  is a weak solution of (1.4). That is, the integral identity

$$\int_0^\infty \int_\Omega \eta \partial_t \varphi + \eta \mathbf{u} \cdot \nabla_x \varphi - \eta \nabla_x \Phi \cdot \nabla_x \varphi - \nabla_x \eta \nabla_x \varphi \, \mathrm{d}x dt = -\int_\Omega \eta_0 \varphi(0, \cdot) \, \mathrm{d}x \quad (2.5)$$

is satisfied for test functions  $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega})$  and any T > 0.

All quantities appear in (2.5) must be at least integrable on  $(0,T) \times \Omega$ . In particular,  $\eta$  belongs  $L^2([0,T]; L^3(\Omega)) \cap L^1(0,T; W^{1,\frac{3}{2}}(\Omega))$ .

• Given the total free-energy of the system by

$$E(\varrho, \mathbf{u}, \eta)(t) := \int_{\Omega} \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{a}{\gamma - 1} \varrho^{\gamma} + \eta \log \eta + (\beta \varrho + \eta) \Phi \right) \, \mathrm{d}x,$$

then  $E(\varrho, \mathbf{u}, \eta)(t)$  is finite and bounded by the initial energy of the system, i.e.,  $E(\varrho, \mathbf{u}, \eta)(t) \leq E(\varrho_0, \mathbf{u}_0, \eta_0)$  a.e. t > 0. Moreover, the following free energy-dissipation inequality holds

$$\int_0^\infty \int_\Omega \left( \mu |\nabla_x \mathbf{u}|^2 + \lambda |\mathrm{div}_x \mathbf{u}|^2 + |2\nabla_x \sqrt{\eta} + \sqrt{\eta} \nabla_x \Phi|^2 \right) \, \mathrm{d}x \, dt \le E(\varrho_0, \mathbf{u}_0, \eta_0).$$
(2.6)

Now, we have all ingredients to state the main results of this work.

**Theorem 2.1 (Global Existence)** Let us assume that  $(\Omega, \Phi)$  satisfy the confinement hypotheses **(HC)**. Then, the problem (1.2)-(1.4) supplemented with boundary conditions (1.5) and initial data satisfying (1.6) admits a weak solution  $\{\varrho, \mathbf{u}, \eta\}$  on  $(0, \infty) \times \Omega$  in the sense of Definition 2.2. In addition,

i) the total fluid mass and particle mass given by

$$M_{\varrho}(t) = \int_{\Omega} \varrho(t, \cdot) \, dx \qquad and \qquad M_{\eta}(t) = \int_{\Omega} \eta(t, \cdot) \, dx,$$

respectively, are constants of motion.

*ii)* the density satisfies the higher integrability result

$$\varrho \in L^{\gamma+\Theta}((0,T) \times \Omega), \text{ for any } T > 0,$$

where  $\Theta = \min\{\frac{2}{3}\gamma - 1, \frac{1}{4}\}.$ 

We can now completely characterize the large time behavior of free-energy solutions to (1.2)-(1.7).

**Theorem 2.2 (Large-time Asymptotics)** Let us assume that  $(\Omega, \Phi)$  satisfy the confinement hypotheses **(HC)**. Then, for any free-energy solution  $(\varrho, \mathbf{u}, \eta)$  of the problem (1.2)-(1.4), in the sense of Definition 2.2, there exist universal stationary states  $\varrho_s(x), \eta_s(x)$ , such that

$$\begin{cases} \varrho(t) \to \varrho_s \ strongly \ in \ L^{\gamma}(\Omega), \\ ess \sup_{\tau > t} \int_{\Omega} \varrho(\tau) |\mathbf{u}(\tau)|^2 \ dx \to 0, \\ \eta(t) \to \eta_s \ strongly \ in \ L^{p_2}(\Omega) \ for \ p_2 > 1, \end{cases}$$

as  $t \to \infty$ , where  $(\eta_s, \varrho_s)$  are characterized as the unique free-energy solution of the stationary state problem:

$$\begin{cases} \nabla_x p(\varrho_s) = -\beta \varrho_s \nabla_x \Phi, \\ \nabla_x \eta_s = -\eta_s \nabla_x \Phi, \end{cases} \quad \int_\Omega \varrho_s \ dx = \int_\Omega \varrho_0 \ dx, \quad \int_\Omega \eta_s \ dx = \int_\Omega \eta_0 \ dx. \quad (2.7)$$

given by the formulas

$$\varrho_s = \left(\frac{\gamma - 1}{a\gamma} \left[-\beta \Phi + C_{\varrho}\right]^+\right)^{\frac{1}{\gamma - 1}}, \qquad \eta_s = C_\eta \exp(-\Phi),$$

where  $C_{\eta}$  and  $C_{\varrho}$  are uniquely given by the initial masses due to (2.7).

### 3 Global-in-time existence

This section is devoted to the proof of the existence result (Theorem 2.1) which will be achieved by patching local-in-time solutions with the aid of suitable global estimates.

**Lemma 3.1** The conclusion of Theorem 2.1 holds true on any time-space cylinder  $[0,T) \times \Omega$ , where T > 0 is any given finite time.

Taking Lemma 3.1 for granted, a weak solution verifying Theorem 2.1 can be constructed as follows.

**Proof of Theorem 2.1:** Fix any  $T_0 > 0$ . From Lemma 3.1, we have the existence of a weak solution  $(\rho_1, \mathbf{u}_1, \eta_1)$  on  $[0, 2T_0)$ . To proceed, we will need the function

$$\zeta^{\epsilon}(t) = \begin{cases} 1, & t \in [0, T_0 - \epsilon], \\ \operatorname{dist}(t, T_0), & t \in (T_0 - \epsilon, T_0), \\ 0, & \text{otherwise.} \end{cases}$$

By setting  $\varphi \zeta^{\epsilon}$  as a test-function in the continuity formulation (2.3), we obtain

$$-\int_{T_0-\epsilon}^{T_0}\int_{\Omega}B(\varrho_1)\varphi\epsilon^{-1} dxdt$$
  
+ 
$$\int_0^{T_0}\int_{\Omega}\left(\zeta^{\epsilon}B(\varrho_1)\partial_t\varphi + \zeta^{\epsilon}B(\varrho_1)\mathbf{u}_1\cdot\nabla_x\varphi - \zeta^{\epsilon}b(\varrho_1)\mathrm{div}_x\mathbf{u}_1\varphi\right) dx dt$$
  
= 
$$-\int_{\Omega}B(\varrho_0)\varphi(0,\cdot)\zeta^{\epsilon}(0) dx, \quad \forall \varphi \in C_c^{\infty}([0,T] \times \overline{\Omega}).$$

Sending  $\epsilon \to 0$  in this formulation yields

$$\int_{0}^{T_{0}} \int_{\Omega} \left( B(\varrho_{1})\partial_{t}\varphi + B(\varrho_{1})\mathbf{u}_{1} \cdot \nabla_{x}\varphi - b(\varrho_{1})\operatorname{div}_{x}\mathbf{u}_{1}\varphi \right) \,\mathrm{d}x \,\mathrm{d}t \\ = \int_{\Omega} B(\varrho(T_{0}, \cdot))\varphi(T_{0}, \cdot) \,dxdt - \int_{\Omega} B(\varrho_{0})\varphi(0, \cdot) \,\mathrm{d}x, \quad \forall \varrho \in C_{c}^{\infty}([0, T] \times \overline{\Omega}).$$

Hence,  $(\rho_1, \mathbf{u}_1)$  is a renormalized solution of the continuity equation on the closed interval  $[0, T_0]$  with additional boundary terms at  $t = T_0$ . By applying similar arguments to the momentum equation (2.4) and particle density equation (2.5), we conclude that  $(\rho_1, \mathbf{u}_1, \eta_1)$  is a weak solution on  $[0, T_0]$  (with additional boundary terms at  $t = T_0$ ).

By the various uniform in time bounds (cf. (2.6)),  $(\rho_1, \mathbf{u}_1, \eta_1)$  at time  $T_0$  has sufficient integrability to serve as initial data for a new solution  $(\rho_2, \mathbf{u}_2, \eta_2)$  defined on  $[T_0, 2T_0] \times \Omega$ . The energy of the new solution is less than the initial energy  $E(\rho_0, \mathbf{u}_0, \eta_0)$ and the triple  $(\rho, \mathbf{u}, \eta)$  given by

$$(\varrho, \mathbf{u}, \eta)(t) = (\varrho_i, \mathbf{u}_i, \eta_i), \quad t \in ((i-1)T_0, iT_0],$$

is a weak solution on  $[0, 2T_0] \times \Omega$ . A solution for all times is readily obtained by iterating this process.

In the remaining parts of this section we prove Lemma 3.1. The overall strategy can be outlined as follows: First, we prove that there exists a sequence of approximate solutions  $\{\varrho_{\delta,h}, \mathbf{u}_{\delta,h}, \eta_{\delta,h}\}_{h>0}$  (see the ensuing subsection for details). Then, by first sending  $h \to 0$  and subsequently  $\delta \to 0$  in this sequence, we prove that in the limit we obtain a weak solution to the system (1.2)–(1.4) in the sense of Definition 2.2.

#### 3.1 Approximation scheme

The approximation scheme is realized in two layers. In the first layer we add a regularization term to the pressure.

**Definition 3.1 (Artificial pressure approximation)** For a given  $\delta > 0$ , we say that the triple  $\{\varrho_{\delta}, \mathbf{u}_{\delta}, \eta_{\delta}\}$  is a weak solution to the artificial pressure approximation

scheme provided that  $\{\varrho_{\delta}, \mathbf{u}_{\delta}, \eta_{\delta}\}$  is a weak solution in the sense of Definition 2.2 but with the pressure  $p_{\delta}(\varrho_{\delta})$  given by

$$p_{\delta}(\varrho_{\delta}) = p(\varrho_{\delta}) + \delta \varrho_{\delta}^{6},$$

and initial data  $(\varrho_{\delta,0}, m_{0,\delta}, \eta_{\delta,0}) = (\varrho_0, m_0, \eta_0) \star \kappa_{\delta}$ , with  $\kappa_{\delta}$  denoting the standard smoothing kernel.

The second layer of approximation is a discretization of the equations in time. To define this layer of approximation, we shall make use of the space

$$\mathcal{W}(\Omega) = L^1(\Omega) \cap L^6(\Omega) \times W^{1,2}_0(\Omega) \times W^{1,\frac{3}{2}}(\Omega) \cap L^3(\Omega) \cap L^1(\Omega).$$

**Definition 3.2 (Time discretization scheme)** Let  $\delta > 0$  be fixed. Given a time step h > 0, we discretize the time interval [0,T] in terms of the points  $t^k = kh$ ,  $k = 0, \ldots, M$ , where we assume that Mh = T. Now, we sequentially determine functions

$$\{\varrho_{\delta,h}^k, \mathbf{u}_{\delta,h}^k, \eta_{\delta,h}^k\} \in \mathcal{W}(\Omega), \quad k = 1, \dots, M,$$

such that:

• The time discretized continuity equation,

$$d_t^h[\varrho_{\delta,h}^k] + \operatorname{div}_x(\varrho_{\delta,h}^k \mathbf{u}_{\delta,h}^k) = 0, \qquad (3.1)$$

holds in the sense of distributions on  $\overline{\Omega}$ .

• The time discretized momentum equation with artificial pressure,

$$d_t^h[\varrho_{\delta,h}^k \mathbf{u}_{\delta,h}^k] + \operatorname{div}_x(\varrho_{\delta,h}^k \mathbf{u}_{\delta,h}^k \otimes \mathbf{u}_{\delta,h}^k) - \mu \Delta \mathbf{u}_{\delta,h}^k - \lambda \nabla_x \operatorname{div}_x \mathbf{u}_{\delta,h}^k + \nabla_x \left( p_\delta(\varrho_{\delta,h}^k) + \eta_{\delta,h}^k \right) = -(\beta \varrho_{\delta,h}^k + \eta_{\delta,h}^k) \nabla_x \Phi,$$
(3.2)

holds in the sense of distributions on  $\Omega$ .

• The time discretized particle density equation,

$$d_t^h[\eta_{\delta,h}^k] + \operatorname{div}_x \left( \eta_{\delta,h}^k (\mathbf{u}_{\delta,h}^k - \nabla_x \Phi) \right) - \Delta \eta_{\delta,h}^k = 0,$$
(3.3)

holds in the sense of distributions on  $\overline{\Omega}$ .

In the above equations,  $d_t^h[\phi^k] = \frac{\phi^k - \phi^{k-1}}{h}$  denotes implicit time stepping.

For each fixed h > 0, the time discretized solution  $\{\varrho_{\delta,h}^k, \mathbf{u}_{\delta,h}^k, \eta_{\delta,h}^k\}_{k=1}^M$  is extended to the whole of  $(0, T) \times \Omega$  by setting

$$(\varrho_{\delta,h}, \mathbf{u}_{\delta,h}, \eta_{\delta,h})(t) = (\varrho_{\delta,h}^k, \mathbf{u}_{\delta,h}^k, \eta_{\delta,h}^k), \qquad t \in (t^{k-1}, t^k], \ k = 1, \dots, M.$$
(3.4)

In addition, we set the initial data

$$(\varrho_{\delta,h}(0), (\varrho_{\delta,0}\mathbf{u}_{\delta,h})(0), \eta_{\delta,h}(0)) = (\varrho_{\delta,0}, m_{\delta,0}, \eta_{\delta,0})$$

Next two subsections treat the existence of a well-defined approximation scheme in both relevant cases  $\Omega$  bounded or not under the confinement conditions (HC).

### **3.2** Well-defined Approximations: $\Omega$ bounded

To prove existence of a solution to the time discretized approximate equations (3.1)–(3.3) we will utilize a fixed point argument. For this purpose we define the operator

$$\mathcal{T}_h[\cdot]: \mathcal{W}(\Omega) \to \mathcal{W}(\Omega),$$

as the solution  $(\varrho, \mathbf{u}, \eta) = \mathcal{T}_h[\psi, \mathbf{z}, \zeta]$  to the system of equations:

$$\frac{\varrho - \varrho^{k-1}}{h} + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \qquad (3.5)$$

$$\frac{\varrho \mathbf{u} - \varrho^{k-1} \mathbf{u}^{k-1}}{h} + \operatorname{div}_{x}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - \lambda \nabla_{x} \operatorname{div}_{x} \mathbf{u}$$

$$= -\nabla_{x}(p_{\delta}(\varrho) + \zeta) - (\beta \varrho + \zeta) \nabla_{x} \Phi,$$
(3.6)

and

$$\frac{\eta - \eta^{k-1}}{h} + \operatorname{div}_x \left( \eta(\mathbf{z} - \nabla_x \Phi) \right) - \Delta \eta = 0, \tag{3.7}$$

in the sense of distributions on  $\Omega$ , where  $(\varrho^{k-1}, \mathbf{u}^{k-1}, \eta^{k-1}) \in \mathcal{W}(\Omega)$  is given.

Observe that for a fixed h > 0,  $\delta > 0$ , and  $k = 1, \ldots, M$ , any fixed point of the operator  $\mathcal{T}_h[\cdot]$  will be a distributional solution to the time discretized approximate equations (3.1)–(3.3). i.e the fixed point

$$(\varrho_{\delta,h}^{k}, \mathbf{u}_{\delta,h}^{k}, \eta_{\delta,h}^{k}) = \mathcal{T}_{h} \left[ \varrho_{\delta,h}^{k}, \mathbf{u}_{\delta,h}^{k}, \eta_{\delta,h}^{k} \right]$$

is precisely the desired solution at each time step.

#### **3.2.1** $T_h$ is well-defined

The existence of functions  $(\varrho, \mathbf{u}, \eta) \in \mathcal{W}(\Omega)$  satisfying  $(\varrho, \mathbf{u}, \eta) = \mathcal{T}_h[\psi, \mathbf{z}, \zeta]$  follows from Lemmas 3.2 and 3.4 below.

**Lemma 3.2 ([18], Theorem 6.1)** Let h > 0 be fixed arbitrary and assume that  $\frac{1}{h}\varrho^{k-1} \in L^{\gamma}(\Omega) \cap L^{6}(\Omega)$ ,  $\frac{1}{h}m^{k-1} \in L^{\frac{6}{5}}(\Omega) \cap L^{p}(\Omega)$  for some p > 1,  $\zeta \in W^{1,\frac{3}{2}}(\Omega)$ , and  $\nabla_{x}\Phi \in L^{p}(\Omega)$ , for all p. There exists a weak solution  $(\varrho, \mathbf{u}) \in L^{\gamma}(\Omega) \cap L^{6}(\Omega) \times W_{0}^{1,2}(\Omega)$  of (3.5)-(3.6) satisfying

$$p_{\delta}(\varrho) \in L^2(\Omega), \tag{3.8}$$

and

$$\nabla_x \operatorname{\mathbf{curl}}_x \mathbf{u}, \ \nabla_x \left( \operatorname{div}_x \mathbf{u} - \frac{1}{\lambda + \mu} p_\delta(\varrho) \right), \ \in L^q(K),$$
 (3.9)

where  $q = \frac{15}{11} > \frac{6}{5}$  and  $K \subset \Omega$  is any compact subset.

*Notation.*- Throughout the paper we use overbars to denote weak limits, just to illustrate this notational use, we will often use convexity arguments based on the following classical lemma [11]:

**Lemma 3.3** Let O be a bounded open subset of  $\mathbb{R}^M$  with  $M \ge 1$ . Suppose  $g: \mathbb{R} \to (-\infty, \infty]$  is a lower semicontinuous convex function and  $\{v_n\}_{n\ge 1}$  is a sequence of functions on O for which  $v_n \xrightarrow{} v$  in  $L^1(O)$ ,  $g(v_n) \in L^1(O)$  for each n,  $g(v_n) \xrightarrow{} \overline{g(v)}$  in  $L^1(O)$ . Then  $g(v) \le \overline{g(v)}$  a.e. on O,  $g(v) \in L^1(O)$ , and  $\int_O g(v) dy \le \lim \inf_{n\to\infty} \int_O g(v_n) dy$ . If, in addition, g is strictly convex on an open interval  $(a,b) \subset \mathbb{R}$  and  $g(v) = \overline{g(v)}$  a.e. on O, then, passing to a subsequence if necessary,  $v_n(y) \to v(y)$  for a.e.  $y \in \{y \in O \mid v(y) \in (a,b)\}$ .

**Lemma 3.4** Assume that  $\Omega$  is bounded. Let  $\eta^{k-1} \in L^1(\Omega) \cap W^{1,\frac{3}{2}}(\Omega) \cap \{\eta^{k-1} \ge 0\}$ , and  $\mathbf{z} \in \mathbf{W}_0^{1,2}(\Omega)$  be given functions. Then, for each fixed h > 0, there exists a nonnegative function  $\eta \in W^{1,\frac{3}{2}}(\Omega) \cap L^1(\Omega)$  satisfying (3.7) in the sense of distributions on  $\Omega$ . Moreover,

$$\int_{\Omega} h^{-1} [\eta \log \eta + \eta \Phi] + |2\nabla_x \sqrt{\eta} + \sqrt{\eta} \nabla_x \Phi|^2 \ dx \le C, \tag{3.10}$$

where the constant C > 0 depends on  $\|\mathbf{z}\|_{\mathbf{W}^{1,2}(\Omega)}$ ,  $\|\eta^{k-1}\|_{L^{3}(\Omega)}$ , and  $\|\Phi\|_{W^{1,\infty}(\Omega)}$ .

**Proof:** For each  $\epsilon > 0$ , we let  $\mathbf{z}^{\epsilon} = \mathbf{z} \star \kappa_{\epsilon}$ , where  $\star$  is the convolution product and  $\kappa_{\epsilon}$  is the standard smoothing kernel. From [21, Proposition 4.29], we can assert the existence of a unique weak solution  $\eta^{\epsilon} \in W^{2,2}(\Omega)$ ,  $\eta^{\epsilon} \geq 0$ , to the linear elliptic equation

$$\frac{\eta^{\epsilon} - \eta^{k-1}}{h} + \operatorname{div}_{x} \left( \eta^{\epsilon} (\mathbf{z}^{\epsilon} - \nabla_{x} \Phi) \right) - \Delta \eta^{\epsilon} = 0 \text{ in } \Omega, \qquad (3.11)$$

satisfying the boundary condition

$$\eta^{\epsilon} \nabla_x \Phi \cdot \nu + \nabla_x \eta^{\epsilon} \cdot \nu = 0$$

By integrating over  $\Omega$  (using the boundary condition), we observe that

$$\int_{\Omega} \eta^{\epsilon} \, dx = \int_{\Omega} \eta^{k-1} \, dx \le C,$$

and hence  $\eta^{\epsilon} \in L^1(\Omega)$  independently of  $\epsilon$ .

Let the sequence  $\{B_l\}_{l=1}^{\infty}$  be given by

$$B_{l}(y) = \begin{cases} \log y, & y > l^{-1}, \\ \log l^{-1}, & y \le l^{-1}. \end{cases}$$

Moreover, let  $\Omega_l = \{x \in \Omega : \eta^{\epsilon}(x) > l^{-1}\}$  and  $\Omega_l^c = \Omega \setminus \Omega_l$ . Multiply (3.11) with  $B_l(\eta^{\epsilon})$  and integrate by parts to obtain

$$\int_{\Omega_l} h^{-1} \left[ \eta^{\epsilon} \log \eta^{\epsilon} - \eta^{k-1} \log \eta^{\epsilon} \right] dx - \int_{\Omega_l^c} h^{-1} \left[ \eta^{\epsilon} \log l - \eta^{k-1} \log l \right] dx + \int_{\Omega_l} \nabla_x \Phi \cdot \nabla_x \eta^{\epsilon} + \frac{1}{\eta^{\epsilon}} |\nabla_x \eta^{\epsilon}|^2 - \mathbf{z}^{\epsilon} \nabla_x \eta^{\epsilon} dx = 0.$$
(3.12)

Applying the identity  $\frac{1}{\eta^{\epsilon}} |\nabla_x \eta^{\epsilon}|^2 = 4 |\nabla_x \sqrt{\eta^{\epsilon}}|^2$  and reordering terms in (3.12)

$$\begin{split} &\int_{\Omega_l} h^{-1} \eta^{\epsilon} \log \eta^{\epsilon} + 4 |\nabla_x \sqrt{\eta^{\epsilon}}|^2 \ dx \\ &= \int_{\Omega_l^c} h^{-1} \left[ \eta^{\epsilon} \log l - \eta^{k-1} \log l \right] \ dx + \int_{\Omega_l} h^{-1} \eta^{k-1} \log \eta^{\epsilon} \ dx \\ &\quad - \int_{\Omega_l} 2\sqrt{\eta^{\epsilon}} \nabla_x \Phi \cdot \nabla_x \sqrt{\eta^{\epsilon}} - 2\sqrt{\eta^{\epsilon}} \mathbf{z}^{\epsilon} \cdot \nabla_x \sqrt{\eta^{\epsilon}} \ dx \\ &\leq \int_{\Omega_l^c} h^{-1} \log l^{\frac{1}{l}} \ dx + \int_{\Omega_l} h^{-1} \eta^{k-1} \eta^{\epsilon} \ dx \\ &\quad + 2 \|\nabla_x \sqrt{\eta^{\epsilon}}\|_{L^2(\Omega_l)}^2 \left( \|\sqrt{\eta^{\epsilon}} \nabla_x \Phi\|_{L^2(\Omega_l)} + \|\sqrt{\eta^{\epsilon}} \mathbf{z}^{\epsilon}\|_{L^2(\Omega_l)} \right) \\ &\leq h^{-1} \log l^{\frac{1}{l}} |\Omega_l^c| + \int_{\Omega_l} h^{-1} \eta^{k-1} \eta^{\epsilon} \ dx + 2 \|\sqrt{\eta^{\epsilon}}\|_{L^2(\Omega_l)}^2 \\ &\quad + \|\sqrt{\eta^{\epsilon}} \mathbf{z}^{\epsilon}\|_{L^2(\Omega_l)}^2 + \|\sqrt{\eta^{\epsilon}} \nabla_x \Phi\|_{L^2(\Omega_l)}, \end{split}$$

where the last inequality is the Cauchy inequality. The last term is bounded since  $\eta^{\epsilon}, \eta^{k-1} \in L^1(\Omega)$  and  $\Phi \in W^{1,\infty}(\Omega)$ , independently of  $\epsilon$  and l. Using the Sobolev embedding  $\mathbf{W}^{1,2}(\Omega) \subset \mathbf{L}^6(\Omega)$  together with  $\eta^{k-1} \in L^3(\Omega)$ , we achieve

$$\int_{\Omega_{l}} h^{-1} \eta^{\epsilon} \log \eta^{\epsilon} + 2 |\nabla_{x} \sqrt{\eta^{\epsilon}}|^{2} dx \qquad (3.13)$$

$$\leq C + \int_{\Omega_{l}} \eta^{\epsilon} \left( h^{-1} \eta^{k-1} + |\mathbf{z}^{\epsilon}|^{2} \right) dx \leq C (1 + \|\eta^{\epsilon}\|_{L^{\frac{3}{2}}(\Omega_{l})}).$$

Applying the Hölder inequality, Sobolev embedding, and Young's inequality (with epsilon), respectively, we bound the last term as follows

$$\|\eta^{\epsilon}\|_{L^{\frac{3}{2}}(\Omega_{l})} \leq \|\eta^{\epsilon}\|_{L^{1}(\Omega)}^{\frac{1}{3}} \|\sqrt{\eta^{\epsilon}}\|_{L^{4}(\Omega_{l})}^{\frac{4}{3}} \leq C \|\nabla_{x}\sqrt{\eta^{\epsilon}}\|_{L^{2}(\Omega_{l})}^{\frac{4}{3}} \leq C + \beta \|\nabla_{x}\sqrt{\eta^{\epsilon}}\|_{L^{2}(\Omega_{l})}^{2}.$$

Inserting this expression in (3.13) and fixing  $\beta$  small, we gather

$$\int_{\Omega_l} h^{-1} \eta^{\epsilon} \log \eta^{\epsilon} + |\nabla_x \sqrt{\eta^{\epsilon}}|^2 \, dx \le C, \tag{3.14}$$

where the constant C is independent of both  $\epsilon$  and l. Since  $\Omega$  is bounded, it follows that

$$\eta^{\epsilon} \log \eta^{\epsilon} \in L^1(\Omega), \quad \eta^{\epsilon} \in W^{1, \frac{3}{2}}(\Omega),$$

independently of  $\epsilon$ .

Since  $W^{1,\frac{3}{2}}(\Omega)$  is compactly embedded in  $L^p(\Omega)$  for all p < 3, we can conclude that, passing to a subsequence if necessary,

$$\eta^{\epsilon} \rightharpoonup \eta$$
 in  $W^{1,\frac{3}{2}}(\Omega), \qquad \eta^{\epsilon} \to \eta$  in  $L^{p}(\Omega), \quad p < 3,$ 

as  $\epsilon \to 0$ . Consequently, we can take the limit  $\epsilon \to 0$  in (3.11) to conclude that  $\eta$  satisfies (3.7) in the sense of distributions on  $\Omega$ .

Next, since  $\eta^{\epsilon} \to \eta$  in  $L^{p}(\Omega)$ , p < 3, we can also conclude that

$$\sqrt{\eta^{\epsilon}} \to \sqrt{\eta}$$
 in  $L^q(\Omega), q < 6, \quad \nabla_x \sqrt{\eta^{\epsilon}} \rightharpoonup \nabla_x \sqrt{\eta}$  in  $L^2(\Omega),$ 

as  $\epsilon \to 0$ . Using this, we can send  $\epsilon \to 0$  in (3.14) to obtain

$$\int_{\Omega} h^{-1} \eta \log \eta + \overline{|\nabla_x \sqrt{\eta}|^2} \, dx \le C.$$

By weak lower semi-continuity  $\overline{\|\nabla_x \sqrt{\eta}\|_{L^2(\Omega)}^2} \ge \|\nabla_x \sqrt{\eta}\|_{L^2(\Omega)}^2$ . Thus,

$$\int_{\Omega} h^{-1} \eta \log \eta + |\nabla_x \sqrt{\eta}|^2 \le C.$$

Then, using the identity

$$2\nabla_x \Phi \cdot \nabla_x \eta^\epsilon + \eta^\epsilon |\nabla_x \Phi|^2 + 4 |\nabla_x \sqrt{\eta^\epsilon}|^2 = |2\nabla_x \sqrt{\eta^\epsilon} + \sqrt{\eta^\epsilon} \nabla_x \Phi|^2,$$

we are led to the conclusion

$$\begin{split} \int_{\Omega} h^{-1} \left[ \eta \log \eta + \eta \Phi \right] + \left| 2 \nabla_x \sqrt{\eta^{\epsilon}} + \sqrt{\eta^{\epsilon}} \nabla_x \Phi \right|^2 \, dx \\ &\leq C + \int_{\Omega} h^{-1} \eta \Phi + 4 \nabla_x \sqrt{\eta} \cdot \left( \sqrt{\eta} \nabla_x \Phi \right) + \eta |\nabla_x \Phi|^2 \, dx \\ &\leq C \left( 1 + \|\eta\|_{L^1(\Omega)} \left( \|\Phi\|_{L^{\infty}(\Omega)} + \|\nabla_x \Phi\|_{L^{\infty}(\Omega)}^2 + \|\nabla_x \Phi\|_{L^{\infty}(\Omega)} \|\nabla_x \sqrt{\eta}\|_{L^2(\Omega)} \right) \right) \\ &\leq \tilde{C}, \end{split}$$

which is (3.10) and the proof is complete.

**Remark 3.1** Since  $\nabla_x \sqrt{\eta} \in L^2(\Omega)$ , the set of particle vacuum regions have measure *zero;*  $|\{x \in \Omega; \eta = 0\}| = 0.$ 

#### $\mathcal{T}_h$ admits a fixed point 3.2.2

We now prove that  $\mathcal{T}_h[\cdot]$  admits a fixed point and consequently that the time discretization scheme in Definition 3.2 is well defined. The key observation made is that the  $L^2$  bound on the pressure enables us to obtain an energy equality. This equality in turn yields compactness of the operator  $\mathcal{T}_h[\cdot]$ .

**Lemma 3.5** Assume the case of bounded domain  $\Omega$ . Let  $\{\varrho^{k-1}, m^{k-1}, \eta^{k-1}\} \in L^2(\Omega) \times$  $L^{\frac{6}{5}}(\Omega) \cap L^{p}(\Omega) \times L^{\gamma}(\Omega), p > 1$ , be given functions. Then, for each fixed h > 0 there exists a fixed point  $\{\varrho, \mathbf{u}, \eta\} \in \mathcal{W}(\Omega)$  for the operator  $\mathcal{T}_h[\cdot]$ ; *i.e* 

$$(\varrho, \mathbf{u}, \eta) = \mathcal{T}_h [\varrho, \mathbf{u}, \eta].$$

As a consequence, the time discretization scheme given by Definition 3.2 is welldefined for bounded domains  $\Omega$ .

**Proof:** We will prove the existence of a fixed point by verifying the postulates of the Schauder corollary to the Schaefer fixed point theorem [9]; If the operator  $\mathcal{T}_h[\cdot]$ is continuous, compact, and the set  $\{s \in [0,1], x \in \mathcal{W}(\Omega) : x = \mathcal{T}_h[sx]\}$  is uniformly bounded, then the operator  $\mathcal{T}_h[\cdot]$  admits a fixed point.

First, we observe that the operator  $\mathcal{T}_{h}[\cdot]$  is clearly bounded and continuous.

Next, we prove that the operator  $\mathcal{T}_h[\cdot]$  is compact. For this purpose, let  $\{0, \mathbf{z}_n, \eta_n\}_{n=1}^{\infty}$ be a sequence such that  $(\mathbf{z}_n, \eta_n) \in_b W_0^{1,2}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)$  for all  $n = 1, \ldots, \infty$ , and construct a sequence  $\{\varrho_n, \mathbf{u}_n, \eta_n\}_{n=1}^{\infty}$  by setting

$$(\varrho_n, \mathbf{u}_n, \eta_n) = \mathcal{T}_h[0, \mathbf{z}_n, \zeta_n], \quad n = 1, \dots, \infty.$$

Then, from the previous lemmas we know that  $(\rho_n, \mathbf{u}_n, \eta_n) \in \mathcal{W}(\Omega)$  independently of n. Hence, we have the existence of functions  $(\rho, \mathbf{u}, \eta) \in \mathcal{W}(\Omega)$  such that, by passing along a subsequence if necessary,

$$(\varrho_n, \mathbf{u}_n, \eta_n) \rightharpoonup (\varrho, \mathbf{u}, \eta), \text{ in } \mathcal{W}(\Omega).$$

Moreover, by compact Sobolev embedding, we clearly have the existence of  $(0, \mathbf{z}, \eta) \in$  $\mathcal{W}(\Omega)$  such that  $\zeta_n \to \zeta$  a.e in  $\Omega$  and  $\mathbf{z}_n \to \mathbf{z}$  a.e in  $\Omega$ .

Now, we claim that in fact  $(\varrho_n, \mathbf{u}_n, \eta_n) \to (\varrho, \mathbf{u}, \eta)$  in  $\mathcal{W}(\Omega)$ , where  $(\varrho, \mathbf{u}, \eta) =$  $\mathcal{T}_{h}[0, \mathbf{z}, \zeta]$ , and consequently that the operator  $\mathcal{T}_{h}[\cdot]$  is compact.

To prove this claim, we first note that compactness of  $\eta_n$  in  $W^{1,\frac{3}{2}}(\Omega)$  is immediate from the linearity of (3.3). That is, since  $\eta_n \in W^{1,\frac{3}{2}}(\Omega)$  we have by Sobolev embedding that  $\eta_n \to \eta$  a.e in  $\Omega$  and thus by setting  $\log \eta - \log \eta_n$  (cf. Remark 3.1) as test-function in (3.7) one discovers

$$\lim_{n \to \infty} \int_{\Omega} |\nabla_x \sqrt{\eta_n}|^2 - |\nabla_x \sqrt{\eta}|^2 \, dx = 0,$$

which immediately implies compactness in  $W^{1,\frac{3}{2}}(\Omega)$ . Moreover, in the limit we have that  $\eta$  satisfies

$$\frac{\eta - \eta^{k-1}}{h} + \operatorname{div}_x \left( \eta(\mathbf{z} - \nabla_x \Phi) \right) - \Delta \eta = 0, \qquad (3.15)$$

in the sense of distributions on  $\overline{\Omega}$ .

We continue proving compactness of the operator  $\mathcal{T}_h[\cdot]$  by first proving strong convergence of the density  $\varrho_n \to \varrho$  a.e in  $\Omega$ . Compactness of the velocity  $\mathbf{u}_n$  in  $W_0^{1,2}(\Omega)$  will then follow from an energy equality. In order to prove strong convergence of the density we will need weak sequential continuity of the effective viscous flux. That is,

$$\lim_{n \to \infty} \int_{\Omega} \left[ (\lambda + \mu) \operatorname{div}_{x} \mathbf{u}_{n} - p_{\delta}(\varrho_{n}) \right] \psi \varrho_{n} \, dx = \int_{\Omega} \left[ (\lambda + \mu) \operatorname{div}_{x} \mathbf{u} - \overline{p_{\delta}(\varrho)} \right] \psi \varrho \, dx, \quad \forall \psi \in C_{c}^{\infty}(\Omega).$$
(3.16)

Here, (3.16) is an immediate consequence of (3.9) and compact embedding of Sobolev spaces.

Before we proceed to prove strong convergence of  $\rho_n$ , we first note that there is no problem with taking the limit in (3.1) to obtain in the limit

$$\frac{1}{h}\rho + \operatorname{div}_{x}(\mathbf{u}\rho) = \frac{1}{h}\rho_{k-1}, \qquad (3.17)$$

in the sense of distributions on  $\overline{\Omega}$ . Hence, since in particular  $\rho \in L^6(\Omega)$  and  $\mathbf{u} \in W_0^{1,2}(\Omega)$  we can conclude that, for any  $B \in C[0,\infty) \cap C^1(0,\infty)$ ,

$$\frac{1}{h}\varrho B'(\varrho) + \operatorname{div}_x(B(\varrho)\mathbf{u}) + ((\varrho B'(\varrho) - B(\varrho))\operatorname{div}_x\mathbf{u}) = \frac{1}{h}\varrho_{k-1}B'(\varrho), \quad (3.18)$$

in the sense of distributions on  $\overline{\Omega}$ . Then, by setting  $B(z) = z \log z$  we obtain the identity

$$\int_{\Omega} \varrho_n \log \varrho_n - \varrho \log \varrho + \varrho_n - \varrho \, dx$$
$$= \int_{\Omega} \rho \operatorname{div}_x \mathbf{u} - \varrho_n \operatorname{div}_x \mathbf{u}_n + \varrho_{k-1} (\log \varrho_n - \log \varrho) \, dx.$$

Thus, by taking the limit  $n \to \infty$ , we see that

$$\int_{\Omega} \overline{\rho \log \rho} - \rho \log \rho \, dx$$

$$= \lim_{l \to \infty} \lim_{n \to \infty} \int_{\Omega} \psi_l \rho \operatorname{div}_x \mathbf{u} - \psi_l \rho_n \operatorname{div}_x \mathbf{u}_n \, dx + \int_{\Omega} \rho_{k-1} (\overline{\log \rho} - \log \rho) \, dx,$$
(3.19)

where  $\psi_l \in C_c^{\infty}(\Omega) \cap \{\psi_l \ge 0\}$  is such that  $\psi_l = 1$  on the set  $\{x \in \Omega; \operatorname{dist}(x, \partial\Omega) > \frac{1}{l}\}$ . By an application of (3.16) we see that for each  $l = 1 \dots \infty$ ,

$$\lim_{n \to \infty} \int_{\Omega} \psi_l \varrho \operatorname{div}_x \mathbf{u} - \psi_l \varrho_n \operatorname{div}_x \mathbf{u}_n \, dx = \lim_{n \to \infty} \int_{\Omega} \psi_l \left( p_\delta(\varrho_n) \varrho - p_\delta(\varrho_n) \varrho_n \right) \, dx \le 0,$$

where the last inequality follows from the convexity of  $p_{\delta}(\rho_n)$ . Similarly, the concavity of log  $\rho_n$  yields

$$\int_{\Omega} \varrho_{k-1}(\overline{\log \varrho} - \log \varrho) \, dx \le 0.$$

However, then (3.19) together with the convexity of  $z \log z$  gives

$$0 \le \int_{\Omega} \overline{\rho \log \rho} - \rho \log \rho dx \le 0.$$

which immediately yields  $\rho_n \to \rho$  a.e in  $\Omega$  due to Lemma 3.3.

We are now in a position to prove that  $\mathbf{u}_n \to \mathbf{u}$  in  $W_0^{1,2}(\Omega)$ . For this purpose, we first set  $\mathbf{u}_n$  as test-function in (3.2) to obtain the identity

$$\lim_{n \to \infty} \int_{\Omega} \frac{\varrho_n |\mathbf{u}_n|^2 + \varrho_{k-1} |\mathbf{u}_n|^2}{2h} - \frac{\mathbf{u}_n m_{k-1}}{h} + \mu |\nabla_x \mathbf{u}_n|^2 + \lambda |\operatorname{div}_x \mathbf{u}_n|^2 \, dx$$
$$= \lim_{n \to \infty} \int_{\Omega} \left( p_{\delta}(\varrho_n) + \zeta_n \right) \operatorname{div}_x \mathbf{u}_n - (\zeta_n + \beta \varrho_n) \nabla_x \Phi \cdot \mathbf{u}_n \, dx \qquad (3.20)$$
$$= \int_{\Omega} \left( p_{\delta}(\varrho) + \zeta \right) \operatorname{div}_x \mathbf{u} - (\zeta + \beta \varrho) \nabla_x \Phi \cdot \mathbf{u} \, dx.$$

Note that in addition to the strong convergence of  $\eta_n$  and  $\rho_n$  we really need (3.8) to conclude this.

Now, since  $\rho_n \to \rho$ ,  $\zeta_n \to \zeta$ , and  $\mathbf{u}_n \to \mathbf{u}$  a.e in  $\Omega$  there is no problem with passing to the limit in (3.2) to obtain

$$\frac{\varrho \mathbf{u} - m_{k-1}}{h} + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - \lambda \nabla_x \operatorname{div}_x \mathbf{u}$$
$$= -\nabla_x(p_\delta(\varrho) + \zeta) - (\beta \varrho + \zeta) \nabla_x \Phi,$$

in the sense of distributions on  $\Omega$ . Hence, in view of (3.15) and (3.17), we can conclude that  $(\varrho, \mathbf{u}, \eta) = \mathcal{T}_h[0, \mathbf{z}, \zeta]$ . Moreover, by using  $\mathbf{u}$  as a test-function for this equation we obtain the identity

$$\begin{split} \int_{\Omega} \left[ \frac{\varrho \mathbf{u}^2 + \varrho_{k-1} \mathbf{u}^2}{2h} - \frac{\mathbf{u} m_{k-1}}{h} + \mu |\nabla_x \mathbf{u}|^2 + \lambda |\mathrm{div}_x \mathbf{u}|^2 \right] dx \\ &= \int_{\Omega} (p_{\delta}(\varrho) + \zeta) \mathrm{div}_x \mathbf{u} - (\zeta + \beta \varrho) \nabla_x \Phi \cdot \mathbf{u} \, dx \\ &= \int_{\Omega} \frac{\varrho \overline{\mathbf{u}^2} + \varrho_{k-1} \overline{\mathbf{u}^2}}{2h} - \frac{\mathbf{u} m_{k-1}}{h} + \mu |\overline{\nabla_x \mathbf{u}}|^2 + \lambda |\overline{\mathrm{div}_x \mathbf{u}}|^2 dx, \end{split}$$

where the last equality is (3.20). This can only happen if  $\nabla_x \mathbf{u}_n \to \nabla_x \mathbf{u}$ . Thus, we can conclude that  $\mathcal{T}_h[\cdot]$  is a compact operator.

Finally, let  $s \in [0, 1]$  be arbitrary and assume that there exists a triple  $(\varrho, \mathbf{u}, \eta) \in \mathcal{W}(\Omega)$  such that  $(\varrho, \mathbf{u}, \eta) = \mathcal{T}_h[s\varrho, s\mathbf{u}, s\eta]$ . Then, by setting **u** as test-function in (3.2) we get the identity

$$\int_{\Omega} \frac{\varrho \mathbf{u}^2 + \varrho_{k-1} \mathbf{u}^2}{2h} - \frac{\mathbf{u}m_{k-1}}{h} + \mu |\nabla_x \mathbf{u}|^2 + \lambda |\mathrm{div}_x \mathbf{u}|^2 dx$$

$$= \int_{\Omega} (p_{\delta}(\varrho) + s\eta) \mathrm{div}_x \mathbf{u} - (s\eta + \beta \varrho) \nabla_x \Phi \cdot \mathbf{u} \, dx.$$
(3.21)

Using  $B(z) = \prod_{\delta}(z) := \frac{1}{\gamma - 1}p(\varrho) + \frac{\delta}{5}\varrho^6$  as the renormalization function in (3.18) we also have the identity

$$-\int_{\Omega} p_{\delta}(\varrho) \operatorname{div}_{x} \mathbf{u} \, dx = \frac{1}{h} \int_{\Omega} \Pi_{\delta}'(\varrho) (\varrho - \varrho_{k-1}) \, dx.$$
(3.22)

Similarly, using  $\beta \Phi$  as test-function in (3.5) gives

$$\int_{\Omega} \beta \frac{\varrho - \varrho_{k-1}}{h} \Phi - \beta \varrho \, \mathbf{u} \cdot \nabla_x \Phi dx = 0.$$

Using  $\Phi + \log \eta$  as a test-function in (3.7) (cf. Lemma 3.4) gives

$$0 = \int_{\Omega} \frac{\eta - \eta_{k-1}}{h} (\Phi + \log \eta) - s\eta \mathbf{u} \cdot \nabla_x \Phi + s\eta \operatorname{div}_x \mathbf{u} + |2\nabla_x \sqrt{\eta} + \sqrt{\eta} \nabla_x \Phi|^2 \, dx.$$
(3.23)

By combining (3.22)–(3.23), we obtain the identity

$$\int_{\Omega} p_{\delta}(\varrho) \operatorname{div}_{x} \mathbf{u} + s\eta \operatorname{div}_{x} \mathbf{u} - (s\eta + \beta \varrho) \nabla_{x} \Phi \cdot \mathbf{u} \, dx$$
$$= \int_{\Omega} -\Pi_{\delta}'(\varrho) \frac{(\varrho - \varrho_{k-1})}{h} - \frac{\eta - \eta_{k-1}}{h} (\Phi + \log \eta)$$
$$-\beta \frac{\varrho - \varrho_{k-1}}{h} \Phi - |2\nabla_{x}\sqrt{\eta} + \sqrt{\eta} \nabla_{x} \Phi|^{2} dx.$$

Then, inserting this into (3.21) and reordering terms gives

$$\int_{\Omega} \frac{\varrho \mathbf{u}^{2} + \varrho_{k-1} \mathbf{u}^{2}}{2h} - \frac{\mathbf{u}m_{k-1}}{h} + \frac{\varrho - \varrho_{k-1}}{h} \left(\Pi_{\delta}'(\varrho) + \beta\Phi\right) + \frac{\eta - \eta_{k-1}}{h} (\Phi + \log\eta) dx$$
$$+ \int_{\Omega} \mu |\nabla_{x} \mathbf{u}|^{2} + \lambda |\operatorname{div}_{x} \mathbf{u}|^{2} dx$$
$$= -\int_{\Omega} |2\nabla_{x} \sqrt{\eta} + \sqrt{\eta} \nabla_{x} \Phi|^{2} dx \leq 0.$$
(3.24)

Consequently, we can conclude that

$$\|\eta\|_{W^{1,\frac{3}{2}}(\Omega)} + \|\varrho\|_{L^{\gamma}(\Omega)} + \|\mathbf{u}\|_{W^{1,2}(\Omega)} \le C,$$

where the constant C depends on the data  $\eta_{k-1}, \varrho_{k-1}, m_{k-1}$ , and  $\Phi$ , together with h. However, C does not depend on the parameter s.

We can now conclude the proof since we have proved that the operator  $\mathcal{T}_h[\cdot]$  is bounded, continuous, compact, and that the set  $\{s \in [0,1], x \in \mathcal{W}(\Omega) : x = \mathcal{T}_h[sx]\}$  is uniformly bounded.

#### **3.3** Well-defined Approximations: $\Omega$ unbounded

At this point we have proved that the approximation scheme is well-defined on bounded domains. Given an unbounded domain  $\Omega$  and an external potential  $\Phi$  satisfying the assumptions (**HC**), we can always find an increasing sequence of domains  $\Omega_r$ , with r > 0 such that  $\Omega_r$  are bounded and  $(\Omega_r, \Phi)$  satisfies (**HC**) approximating  $\Omega$  in the sense  $\bigcup_{r>0} \Omega_r = \Omega$ . Using the previous subsection, for any r > 0, there is a solution on  $\Omega_r$ . In this subsection, we prove that we can send  $r \to \infty$  to obtain a solution in  $\Omega$ . The following lemmas will be of use in the sequel.

#### **3.3.1** Consequences of confinement: $\Omega$ unbounded

We show in this part how to control the negative contribution of the physical entropy  $\eta \log \eta$  in the free-energy bounds for unbounded domains  $\Omega$ . Here, the confinement conditions **(HC)** on  $(\Omega, \Phi)$  are crucial. Most of these lemmas can be seen in [10] but we include them here for the sake of completeness. We first start by a classical lemma in kinetic theory.

**Lemma 3.6** Assume that  $(\Omega, \Phi)$  satisfy the hypotheses **(HC)**. For any density  $\eta \in L^1_+(\Omega)$ ,

$$\int_{\Omega} \eta(x) \log_{-} \eta(x) \, dx \leq \frac{1}{2} \int_{\Omega} \Phi(x) \eta(x) \, dx + \frac{1}{e} \int_{\Omega} e^{-\Phi(x)/2} \, dx$$

**Proof:** Let  $\bar{\eta} := \eta \chi_{\{\eta \leq 1\}}$  and  $\bar{M} = \int_{\Omega} \bar{\eta}(x) \, dx \leq \int_{\Omega} \eta(x) \, dx = M$ . Then

$$\int_{\Omega} \bar{\eta}(x) \left( \log \bar{\eta}(x) + \frac{1}{2} \Phi(x) \right) dx = \int_{\Omega} [U(x) \log U(x)] \mu \, dx - \bar{M} \log Z$$

where  $U := \bar{\eta}/\mu$ ,  $\mu(x) = e^{-\Phi(x)/2}/Z$  with  $Z = \int_{\Omega} e^{-\Phi(x)/2} dx$ . The Jensen inequality yields

$$\int_{\Omega} [U(x) \log U(x)] \mu \, dx \ge \left( \int_{\Omega} U(x) \mu \, dx \right) \, \log \left( \int_{\Omega} U(x) \mu \, dx \right) = \bar{M} \, \log \bar{M}$$

and

$$\begin{split} -\int_{\Omega} \eta(x) \, \log_{-} \eta(x) \, dx = & \int_{\Omega} \bar{\eta}(x) \, \log \bar{\eta}(x) \, dx \geq \bar{M} \log \bar{M} - \bar{M} \log Z - \frac{1}{2} \int_{\Omega} \Phi(x) \, \bar{\eta}(x) \, dx \\ \geq & -\frac{Z}{e} - \frac{1}{2} \int_{\Omega} \Phi(x) \, \eta(x) \, dx \; , \end{split}$$

from which the desired claim follows.

We can immediately use this previous lemma to conclude the following consequence.

**Corollary 3.1** Assume that  $(\Omega, \Phi)$  satisfy the hypotheses **(HC)**. For any density  $\eta \in L^1_+(\Omega)$ , if

$$\int_{\Omega} \eta(x) \log \eta(x) \, dx + \int_{\Omega} \Phi(x) \eta(x) \, dx \le C$$

then  $\eta \log \eta \in L^1(\Omega)$  and there exists D > 0 depending on C and  $\Phi$  such that

$$\int_{\Omega} \eta(x) \log_{+} \eta(x) \, dx \le D \qquad and \qquad \int_{\Omega} \Phi(x) \eta(x) \, dx \le D$$

Finally, the above estimates can be used to control the mass of the densities  $\eta$  outside a large ball to avoid loss of mass at infinity.

**Lemma 3.7** Given any domain  $\Omega$  such that  $e^{-\Phi} \in L^1_+(\Omega)$  and any density  $\eta \in L^1_+(\Omega)$ , then

$$\int_{\Omega} \eta(x) \log \eta(x) \, dx + \int_{\Omega} \Phi(x) \eta(x) \, dx \ge \int_{\Omega} \eta(x) \, dx \, \log \left( \frac{\int_{\Omega} \eta(x) \, dx}{\int_{\Omega} e^{-\Phi(x)} \, dx} \right) \, .$$

As a consequence, if  $e^{-\Phi} \in L^1_+(\Omega)$  and

$$\int_{\Omega} \eta(x) \log \eta(x) \, dx + \int_{\Omega} \Phi(x) \eta(x) \, dx \le C \,,$$

then, for any  $\epsilon > 0$  there exists R > 0 depending on C and  $\Phi$  only such that

$$\int_{\Omega \cap (\mathbb{R}^3/B(0,R))} \eta(x) \, dx < \epsilon \, .$$

**Proof:** A direct use of Jensen's inequality shows the first inequality by using the convexity of  $x \mapsto x \log x$ . To show the second claim we start by applying the first inequality to the domain  $\Omega_R^c := \Omega \cap (\mathbb{R}^3/B(0, R))$  from which we obtain

$$\int_{\Omega_R^c} \eta(x) \, dx \, \log\left(\frac{\int_{\Omega_R^c} \eta(x) \, dx}{\int_{\Omega_R^c} e^{-\Phi(x)} \, dx}\right) \le D \tag{3.25}$$

for some D > 0, where Lemma 3.6 and Corollary 3.1 were used. Now, we argue by contradiction, if the second claim were not true, we would have

$$\exists \epsilon > 0 \,\forall R_0 > 0 \,\exists R > R_0 \text{ such that } \int_{\Omega_R^c} \eta(x) \, dx \ge \epsilon_0.$$

Since  $e^{-\Phi} \in L^1_+(\Omega)$ , we can always assume that  $R_0$  is large such that

$$\int_{\Omega_R^c} e^{-\Phi(x)} \, dx \le \int_{\Omega_{R_0}^c} e^{-\Phi(x)} \, dx < \epsilon_0 \le \int_{\Omega_R^c} \eta(x) \, dx$$

and thus due to (3.25),

$$\int_{\Omega_R^c} \eta(x) \, dx \le \int_{\Omega_R^c} e^{-\Phi(x)} \, dx \, e^{D/\epsilon_0} \le \int_{\Omega_{R_0}^c} e^{-\Phi(x)} \, dx \, e^{D/\epsilon_0} \, dx \, e^{D/\epsilon_0} \, dx = 0$$

This leads to a contradiction since the right-hand side can be made arbitrarily small by taking  $R_0$  large enough.

#### 3.3.2 The approximation scheme is well-defined

Notation. - In what follows, we will often obtain a priori estimates for a sequence  $\{v_n\}_{n\geq 1}$  that we write as " $v_n \in_b X$ " for some functional space X. What this really means is that we have a bound on  $||v_n||_X$  that is independent of n.

**Lemma 3.8** Set  $(\Omega, \Phi)$  satisfying **(HC)** with  $\Omega$  unbounded and let  $(\varrho_h^{k-1}, \mathbf{u}_h^{k-1}, \eta_h^{k-1}) \in \mathcal{W}(\Omega)$  be given data. The time discretization scheme (3.1)-(3.3) admits a distributional solution in the sense of Definition 3.2.

**Proof:** Let  $\{\Omega_r\}_{r>0}$  be an increasing sequence of domains such that  $\bigcup_{r>0}\Omega_r = \Omega$ and such that, for each fixed r,  $\Omega_r$  is bounded and  $(\Omega_r, \Phi)$  satisfies **(HC)**. From the results of the previous subsection, we have the existence of a triplet  $(\varrho_r, \mathbf{u}_r, \eta_r)$ satisfying the time discretized equations (3.1)-(3.3) in the sense of distributions on  $\Omega_r$ . Consequently, we can define a family of such solutions:

 $\{\varrho_r, \mathbf{u}_r, \eta_r\}_{R < r < \infty}, \qquad (\varrho_r, \mathbf{u}_r, \eta_r) \in \mathcal{W}(\Omega_r), \text{ for each fixed } r \in (R, \infty),$ 

where R is fixed according to the requirements on the potential (see (2.2)). For this construction, (3.24) yields

$$\int_{\Omega_{r}} \frac{\varrho_{r} \mathbf{u}_{r}^{2} + \varrho^{k-1} \mathbf{u}_{r}^{2}}{2h} + \frac{\varrho_{r}}{h} \left( \Pi_{\delta}'(\varrho_{r}) + \beta \Phi \right) + \frac{\eta_{r}}{h} (\Phi + \log \eta_{r}) dx \\
+ \int_{\Omega_{r}} \mu |\nabla_{x} \mathbf{u}_{r}|^{2} + \lambda |\operatorname{div}_{x} \mathbf{u}_{r}|^{2} + |2\nabla_{x} \sqrt{\eta_{r}} + \sqrt{\eta_{r}} \nabla_{x} \Phi|^{2} dx \\
\leq \int_{\Omega_{r}} \frac{\mathbf{u}_{r} m_{k-1}}{h} + h^{-1} \eta_{k-1} \log \eta_{r} dx \\
\leq \epsilon h^{-1} [\|\eta_{r}\|_{L^{2}(\Omega_{r})}^{2} + \|\mathbf{u}\|_{L^{6}(\Omega_{r})}^{2}] + \frac{1}{4h\epsilon} \left[ \|\eta_{k-1}\|_{L^{2}(\Omega)}^{2} + \|m_{k-1}\|_{L^{\frac{6}{5}}(\Omega)} \right].$$
(3.26)

In addition, integrating (3.3) over  $\Omega_r$ ,

$$\|\eta_r\|_{L^1(\Omega_r)} = \|\eta^{k-1}\|_{L^1(\Omega_r)} \le C,$$

with constant C independent of r. An interpolation inequality and the Sobolev-Poincaré inequality allow us to conclude

$$\|\eta_r\|_{L^2(\Omega_r)} \le \|\eta_r\|_{L^1(\Omega_r)}^{\frac{1}{4}} \|\eta_r\|_{L^3(\Omega_r)}^{\frac{3}{4}} \le C \|\nabla_x \sqrt{\eta_r}\|_{L^2(\Omega_r)}^{\frac{3}{4}},$$
(3.27)

where the constant C is independent of r. By applying (3.27) and the Sobolev-Poincaré inequality to (3.26), we conclude

$$\int_{\Omega_r} \frac{\varrho_r \mathbf{u}_r^2 + \varrho^{k-1} \mathbf{u}_r^2}{2h} + \frac{\varrho_r}{h} \left( \Pi_{\delta}'(\varrho_r) + \beta \Phi \right) + \frac{\eta_r}{h} (\Phi + \log \eta_r) dx$$

$$+ \int_{\Omega_r} (\mu - \epsilon) |\nabla_x \mathbf{u}_r|^2 + \lambda |\operatorname{div}_x \mathbf{u}_r|^2 + |2\nabla_x \sqrt{\eta_r} + \sqrt{\eta_r} \nabla_x \Phi|^2 - \epsilon |\nabla_x \sqrt{\eta_r}|^2 dx \le C,$$
(3.28)

where C is independent of r.

Next, we observe that (3.28) yields

$$\int_{\Omega_r} 4|\nabla_x \sqrt{\eta_r}|^2 + 2\nabla_x \eta_r \nabla_x \Phi + \eta_r |\nabla_x \Phi|^2 \ dx = \int_{\Omega_r} |2\nabla_x \sqrt{\eta_r} + \sqrt{\eta_r} \nabla_x \Phi|^2 \ dx \le C.$$

Reordering terms and integrating by parts

$$\int_{\Omega_r} 4|\nabla_x \sqrt{\eta_r}|^2 + \eta_r |\nabla_x \Phi|^2 \ dxdt \le C + \int_{\Omega_r} \eta_r |\Delta \Phi| \ dx.$$

Then, (2.2) and (3.28) gives

$$\int_{\Omega_R} \eta_r |\Delta \Phi| \, dx + \int_{\Omega_r \setminus \Omega_R} \eta_r |\Delta \Phi| \, dx 
\leq \|\Delta \Phi\|_{L^{\infty}(\Omega_R)} \int_{\Omega_R} \eta_r \, dx + C \int_{\Omega_r \setminus \Omega_R} \eta_r \Phi \, dx \leq C,$$
(3.29)

with C independent of r.

Setting (3.29) into (3.28), fixing  $\epsilon$  small, and applying Corollary 3.1 gives

$$\int_{\Omega_r} \frac{\varrho_r \mathbf{u}_r^2 + \varrho^{k-1} \mathbf{u}_r^2}{2h} + \frac{\varrho_r}{h} \left( \Pi_{\delta}'(\varrho_r) + \beta \Phi \right) + \frac{\eta_r}{h} (\Phi + \log_+ \eta_r) dx$$
$$+ \int_{\Omega_r} \mu |\nabla_x \mathbf{u}_r|^2 + \lambda |\mathrm{div}_x \mathbf{u}_r|^2 + |\nabla_x \sqrt{\eta_r}|^2 + \eta_r |\nabla_x \Phi|^2 dx \le C$$

Since  $\|\sqrt{\eta_r}\|_{L^2(\Omega_r)}^2 = \int_{\Omega_r} \eta_r \ dx \le C$ , the previous estimate allow us to conclude

$$\eta_r \in_b L^3(\Omega_r) \cap W^{1,\frac{3}{2}}(\Omega_r), \tag{3.30}$$

$$\mathbf{u}_r \in_b L^6(\Omega_r) \cap W_0^{1,2}(\Omega_r), \qquad \varrho_r \in_b L^6(\Omega_r) \cap L^{\gamma}(\Omega_r).$$

Next, let  $(\phi, \mathbf{v}, \psi) \in [C_c^{\infty}(\Omega)]^3$  be arbitrary and fix a number  $\overline{r} \in (R, \infty)$  large such that

$$\operatorname{supp}(\phi, \mathbf{v}, \psi) \subset \Omega_r.$$

Then, from the previous bounds, we have the existence of functions  $(\varrho, \mathbf{u}, \eta) \in \mathcal{W}(\Omega)$  such that, along a subsequence,

$$\varrho_r \rightharpoonup \varrho, \quad \text{in } L^6(\Omega_{\bar{r}}), \quad \mathbf{u}_r \rightharpoonup \mathbf{u}, \quad \text{in } W^{1,2}(\Omega_{\bar{r}}), 
\eta_r \rightharpoonup \eta, \quad \text{in } W^{1,\frac{3}{2}}(\Omega_{\bar{r}})).$$
(3.31)

Moreover, by compact Sobolev embedding

$$\eta_r \to \eta, \quad \text{in } L^3_{\text{loc}}(\Omega_{\bar{r}}), \qquad \mathbf{u}_r \to \mathbf{u}, \quad \text{in } L^p_{\text{loc}}(\Omega_{\bar{r}}), \ p < 6.$$
 (3.32)

Using (3.31), (3.32), there is no problem with sending  $r \to 0$  in

$$\int_{\Omega_{\bar{r}}} d_t^h[\varrho_r] \phi - \varrho_r \mathbf{u}_r \nabla_x \phi \, dx = 0,$$
$$\int_{\Omega_{\bar{r}}} d_t^h[\eta_r] \psi + (\mathbf{u}_r - \nabla_x \Phi) \nabla_x \psi - \nabla_x \eta_r \nabla_x \psi \, dx dt = 0,$$

to conclude that  $(\varrho, \mathbf{u}, \eta)$  solves (3.1)-(3.3) in the sense of distributions on  $\Omega$  (Recall that  $(\phi, \psi)$  was chosen arbitrary). Similarly, we can pass to the limit in the momentum equation (3.2) to obtain

$$\int_{\Omega_{\bar{r}}} d_t^h[\rho \mathbf{u}] \mathbf{v} - \rho \mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{v} + \mu \nabla_x \mathbf{u} \nabla_x \mathbf{v} + \lambda \operatorname{div}_x \mathbf{u} \operatorname{div}_x \mathbf{v} \, dx$$
$$+ \int_{\Omega_{\bar{r}}} (\beta \rho + \eta) \nabla_x \Phi \mathbf{v} \, dx = \lim_{r \to \infty} \int_{\Omega_{\bar{r}}} p_\delta(\rho_r) \operatorname{div}_x \mathbf{v} \, dx.$$

Hence, it remains to prove that  $p_{\delta}(\varrho_r) \rightarrow p_{\delta}(\varrho)$  as  $r \rightarrow \infty$ . This problem is also the core problem encountered in the existence analysis of Lions [18]. Due to the regularity properties of  $\eta_r$ , the presence of the  $\eta$  variable does not impose any additional difficulties as compared with [18]. However, some minor modifications are needed to incorporate the unbounded potential  $\Phi$ . The needed modifications for the present stationary case are almost identical to those of the non-stationary case. Thus, we do not give the arguments here but refer the reader to the more general arguments given in Section 4.5.2.

#### **3.4** There exists an artificial pressure solution $(h \rightarrow 0)$

In the previous subsection we proved that, for each h > 0 and  $\delta > 0$ , we can construct functions  $(\varrho_{\delta,h}, \mathbf{u}_{\delta,h}, \eta_{\delta,h})$  according to Definition 3.2 and (3.4). In this section, we prove that the corresponding sequence  $\{(\varrho_{\delta,h}, \mathbf{u}_{\delta,h}, \eta_{\delta,h})\}_{h>0}$ , with  $\delta > 0$  fixed, converges as the time step  $h \to 0$  to an artificial pressure solution in the sense of Definition 3.1.

**Lemma 3.9** [22, Corollary 4, p. 85]. Let  $X \subset B \subset Y$ , be Banach spaces with  $X \subset B$  compactly. Then, for  $1 \leq p < \infty$ ,  $\{v : v \in L^p(0,T;X), v_t \in L^1(0,T;Y)\}$  is compactly embedded in  $L^p(0,T;B)$ .

The following lemma is a variation of a result due to P. L. Lions (see [17]).

**Lemma 3.10** For a given T > 0, divide the time interval (0,T) into M points such that  $(0,T) = \bigcup_{k=1}^{M} (t_{k-1}, t_k]$ , where  $t_k = hk$  and we assume that Mh = T. Let  $\{f_h\}_{h>0}^{\infty}, \{g_h\}_{h>0}^{\infty}$  be two sequences such that:

•  $\{f_h\}_{h>0}, \{g_h\}_{h>0}$  converge weakly to f, g respectively in  $L^{p_1}(0, T; L^{q_1}(\Omega)), L^{p_2}(0, T; L^{q_2}(\Omega))$  where  $1 \le p_1, q_1 \le \infty$ ,

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1.$$

- the mapping  $t \to g_h(t,x)$  is constant on each interval  $(t_{k-1},t_k], k = 1, \ldots, M$ .
- the discrete time derivative satisfies,

$$\frac{g_h(t,x) - g_h(t-h,x)}{h} \in_b L^1(0,T;W^{-1,1}(\Omega)).$$

•  ${f_h}_{h>0}$  satisfies:  $||f_n - f_n(\cdot + \xi, t)||_{L^{p_2}(0,T;L^{q_2}(\Omega))} \to 0$ , as  $|\xi| \to 0$ , uniformly in n.

Then,  $g_h f_h \rightharpoonup gf$  in the sense of distributions on  $(0,T) \times \Omega$ .

#### 3.4.1 Energy estimates

Let  $\delta > 0$  be fixed and let  $\{\varrho_{\delta,h}, \mathbf{u}_{\delta,h}, \eta_{\delta,h}\}_{h>0}$  be a sequence of time discretized solutions constructed according to Definition 3.2 and (3.4). Since (3.24) holds for every k, we can sum this equality over all  $k = 1, \ldots, m$ , for any  $m \in [1, M]$ , to obtain the energy equality

$$E(\varrho_{\delta,h}, \mathbf{u}_{\delta,h}, \eta_{\delta,h})(t^{m}) + \int_{0}^{t^{m}} \int_{\Omega} \mu |\nabla_{x} \mathbf{u}_{\delta,h}|^{2} + \lambda |\operatorname{div}_{x} \mathbf{u}_{\delta,h}|^{2} \, dx dt + \int_{0}^{t^{m}} \int_{\Omega} |2\nabla_{x}\sqrt{\eta_{\delta,h}} + \sqrt{\eta_{\delta,h}}\nabla_{x}\Phi|^{2} \, dx dt + \frac{1}{\gamma - 1} \sum_{k=1}^{m} \int_{\Omega} (\gamma - 1)(\varrho_{\delta,h}^{k})^{\gamma} + (\varrho_{\delta,h}^{k-1})^{\gamma} - \gamma(\varrho_{\delta,h}^{k-1})^{\gamma-1}\varrho_{\delta,h}^{k} \, dx \qquad (3.33) + \frac{\delta}{5} \sum_{k=1}^{m} \int_{\Omega} 5(\varrho_{\delta,h}^{k})^{6} + (\varrho_{\delta,h}^{k-1})^{6} - 6(\varrho_{\delta,h}^{k-1})^{5}\varrho_{\delta,h}^{k} \, dx + \sum_{k=1}^{m} \int_{\Omega} \eta_{k-1} \log\left(\frac{\eta_{k-1}}{\eta_{k}}\right) \, dx = E(\varrho_{\delta,0}, \mathbf{u}_{\delta,0}, \eta_{\delta,0}),$$

where  $t^m = mh \in (0,T)$  and the energy  $E(\cdot, \cdot, \cdot)$  is given by (2.6). By convexity  $(\gamma - 1)a^{\gamma} + b^{\gamma} - \gamma a^{\gamma - 1}b \ge 0$  for all  $a, b \ge 0, \gamma > 1$ . By concavity of  $z \mapsto \log z$ ,

$$\int_{\Omega} \eta_{k-1} \log\left(\frac{\eta_{k-1}}{\eta_k}\right) \ge \int_{\Omega} \eta^{k-1} - \eta^k \ dx = 0.$$

Due to the confinement conditions, this, Corollary 3.1, and (3.33), allow us to conclude the bounds:

$$\varrho_{\delta,h} |\mathbf{u}_{\delta,h}|^{2} \in_{b} L^{\infty}(0,T; L^{1}(\Omega)), 
\mathbf{u}_{\delta,h} \in_{b} L^{2}(0,T; W_{0}^{1,2}(\Omega)), 
\varrho_{\delta,h} \in_{b} L^{\infty}(0,T; L^{\gamma}(\Omega) \cap L^{6}(\Omega)), 
\eta_{\delta,h} \log \eta_{\delta,h} \in_{b} L^{\infty}(0,T; L^{1}(\Omega)).$$
(3.34)

By similar arguments as those leading to (3.30), we conclude

$$\eta_{\delta,h} \in_b L^2(0,T;L^3(\Omega)) \cap L^1(0,T;W^{1,\frac{3}{2}}(\Omega)), \tag{3.35}$$

independent of both h and  $\delta$ .

Utilizing the above h independent bounds, it is a simple exercise to obtain from the system (3.1)-(3.3) the weak time difference bounds

$$d_{t}^{h}[\varrho_{\delta,h}] \in_{b} L^{\infty}(0,T;W^{-1,\frac{3}{2}}(\Omega)),$$
  

$$d_{t}^{h}[\eta_{\delta,h}] \in_{b} L^{1}(0,T;W^{-1,\frac{3}{2}}(\Omega)),$$
  

$$d_{t}^{h}[\varrho_{\delta,h}\mathbf{u}_{\delta,h}] \in_{b} L^{1}(0,T;W^{-1,1}(\Omega)).$$
  
(3.36)

#### **3.4.2** Convergence

In view of (3.34) and (3.35), we have the existence of functions  $(\rho_{\delta}, \mathbf{u}_{\delta}, \eta_{\delta})$  such that, along a subsequence as  $h \to 0$ ,

$$\begin{split} \varrho_{\delta,h} &\stackrel{*}{\rightharpoonup} \varrho_{\delta} & \text{in } L^{\infty}(0,T;L^{\gamma}(\Omega)), \\ \mathbf{u}_{\delta,h} &\rightharpoonup \mathbf{u}_{\delta} & \text{in } L^{2}(0,T;\mathbf{W}_{0}^{1,2}(\Omega)), \\ \eta_{\delta,h} &\rightharpoonup \eta_{\delta} & \text{in } L^{2}(0,T;L^{3}(\Omega)). \end{split}$$

By virtue of (3.36), Lemma 3.10 can be applied to yield

$$\varrho_{\delta,h}\mathbf{u}_{\delta,h} \rightharpoonup \varrho_{\delta}\mathbf{u}_{\delta}, \quad \varrho_{\delta,h}\mathbf{u}_{\delta,h} \otimes \mathbf{u}_{\delta,h} \rightharpoonup \varrho_{\delta}\mathbf{u}_{\delta} \otimes \mathbf{u}_{\delta}, \tag{3.37}$$

in the sense of distributions on  $(0, T) \times \Omega$  as  $h \to 0$ . Here,  $\varrho_{\delta,h} \mathbf{u}_{\delta,h} \otimes \mathbf{u}_{\delta,h} \rightharpoonup \varrho_{\delta} \mathbf{u}_{\delta} \otimes \mathbf{u}_{\delta}$ follows from setting  $g_h = \varrho_{\delta,h} \mathbf{u}_{\delta,h}$  and  $f_h = \mathbf{u}_{h,\delta}$  in Lemma 3.10, where  $g_h \rightharpoonup g = \varrho_{\delta} \mathbf{u}_{\delta}$ from the second last result in (3.37).

Next, since  $\eta_{\delta,h} \in L^1(0,T; L^2(\Omega)) \cap L^1(0,T; W^{\frac{3}{2}}(\Omega)), \partial_t \eta_{\delta,h} \in L^1(0,T; W^{-1,1}(\Omega))$ and  $W^{\frac{3}{2}}$  is compactly embedded in  $L^2$ , we can apply Lemma 3.9 to conclude that

$$\eta_{\delta,h} \to \eta_{\delta} \quad \text{in } L^{2-\epsilon}(0,T; L^{2}_{\text{loc}}(\Omega)), \qquad \sqrt{\eta_{\delta,h}} \to \sqrt{\eta_{\delta}} \quad \text{in } L^{2}(0,T; L^{2}_{\text{loc}}(\Omega)), \quad (3.38)$$

Now, by a straight forward application of the Hölder inequality, we deduce  $\nabla_x \eta_{\delta,h} \in_b L^2(0,T; L^1(\Omega)) \cap L^1(0,T; L^{\frac{3}{2}}(\Omega))$ . The standard interpolation inequality provides an estimate of the form  $\nabla_x \eta_{\delta,h} \in_b L^{\alpha_1}(0,T; L^{\alpha_2}(\Omega))$ . Thus, in particular

$$\nabla_x \eta_{\delta,h} \rightharpoonup \nabla_x \eta_{\delta}, \tag{3.39}$$

in the sense of distributions on  $(0, T) \times \Omega$ .

Now, equipped with (3.37), (3.39), and the bounds (3.34), there is no problem with taking the limit  $h \to 0$  in (3.1) and (3.3) to discover that

$$-\int_{0}^{T}\int_{\Omega}\varrho_{\delta}(\psi_{t}-\mathbf{u}_{\delta}\nabla_{x}\psi)\ dxdt = \int_{\Omega}\varrho_{\delta,0}\psi(0,x)\ dx,\qquad(3.40)$$

for all  $\psi \in C_c^{\infty}([0,T) \times \overline{\Omega}))$  and

$$-\int_{0}^{T}\int_{\Omega}\eta_{\delta}\left(\psi_{t}+(\mathbf{u}_{\delta}-\nabla_{x}\Phi)\nabla_{x}\psi\right)-\nabla_{x}\eta_{\delta}\nabla_{x}\psi\ dxdt=\int_{\Omega}\eta_{\delta,0}\psi(0,x)\ dx\qquad(3.41)$$

for all  $\psi \in C_c^{\infty}([0,T) \times \overline{\Omega}))$ .

Hence, the limiting functions satisfy both the continuity equation and particle equation in the sense of Definition 3.1.

Similarly, we can go to the limit  $h \to 0$  in (3.2) to discover in the limit

$$\begin{split} &\int_0^T \int_\Omega \varrho_\delta \mathbf{u}_\delta \mathbf{v}_t + \varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla_x \mathbf{v}_\delta + \eta_\delta \mathrm{div}_x \mathbf{v} \,\, dx dt \\ &= \int_0^T \int_\Omega \mu \nabla_x \mathbf{u}_\delta \nabla_x \mathbf{v} + \lambda \mathrm{div}_x \mathbf{u}_\delta \mathrm{div}_x \mathbf{v} + (\beta \varrho_\delta + \eta_\delta) \nabla_x \Phi \mathbf{v} \,\, dx dt \\ &\quad - \lim_{h \to 0} \int_0^T \int_\Omega p_\delta(\varrho_{\delta,h}) \mathrm{div}_x \mathbf{v} \,\, dx dt - \int_\Omega \mathbf{m}_0 \mathbf{v}(0,x) \,\, dx, \end{split}$$

for all  $\mathbf{v} \in C_c^{\infty}([0,T) \times \Omega)$ . Thus, in order to conclude existence of an artificial pressure solution in the sense of Definition 3.1, we must prove that in fact

$$\lim_{h \to 0} \int_0^T \int_\Omega p_\delta(\varrho_{\delta,h}) \operatorname{div}_x \mathbf{v} \, dx dt = \int_0^T \int_\Omega p_\delta(\varrho_\delta) \operatorname{div}_x \mathbf{v} \, dx dt.$$
(3.42)

**Lemma 3.11** Fix any  $\delta > 0$  and let  $\{\varrho_{\delta,h}, \mathbf{u}_{\delta,h}, \eta_{\delta,h}\}_{h>0}$  be a sequence of time discretized solutions constructed according to Definition 3.2 and (3.4). Then, there exists a triple  $(\varrho_{\delta}, \mathbf{u}_{\delta}, \eta_{\delta})$  such that as  $h \to 0$ ,  $\varrho_{\delta,h} \rightharpoonup \varrho_{\delta}$  in  $L^{\infty}(0, T; L^{\gamma}(\Omega) \cap L^{6}(\Omega))$ ,  $\varrho_{\delta,h} \rightarrow \varrho_{\delta}$  a.e in  $(0,T) \times \Omega$ ,  $\mathbf{u}_{\delta,h} \rightarrow \mathbf{u}_{\delta}$  in  $L^{2}(0,T; W_{0}^{1,2}(\Omega))$ ,  $\varrho_{\delta,h}\mathbf{u}_{\delta,h} \rightarrow \varrho_{\delta}\mathbf{u}_{\delta}$  in the sense of distributions on  $(0,T) \times \Omega$ ,  $\eta_{\delta,h} \rightarrow \eta_{\delta}$  in  $L^{1}(0,T; W^{1,\frac{3}{2}}(\Omega))$ , and  $\eta_{\delta,h} \rightarrow \eta_{\delta}$  a.e in  $(0,T) \times \Omega$ , where  $(\varrho_{\delta}, \mathbf{u}_{\delta}, \eta_{\delta})$  is a weak solution to the artificial pressure approximation scheme in the sense of Definition 3.1.

**Proof:** In view of the high integrability, and strong convergence properties, of  $\eta_{\delta,h}$ , (3.42) can be proved by the same arguments as those leading to Theorem 7.2 in [18]. Some minor modifications are needed to treat the unbounded potential  $\Phi$ . The arguments needed are identical to those of Section 4.5.2 and will not be given here. From this and the previous results of this section we can conclude existence of an artificial pressure solution. It remains to prove the energy inequality (1.8).

We start with the following calculation

$$\lim_{h \to 0} \int_0^t \int_\Omega 4 |\nabla_x \sqrt{\eta_{\delta,h}}|^2 + 4 \nabla_x \eta_{\delta,h} \nabla_x \Phi + \eta_{\delta,h} |\nabla_x \Phi|^2 \, dx dt$$
$$= \int_0^t \int_\Omega \eta_\delta |\nabla_x \Phi|^2 + 4 \nabla_x \eta_\delta \nabla_x \Phi + 4 \overline{|\nabla_x \sqrt{\eta_\delta}|^2} \, dx dt \qquad (3.43)$$
$$\geq \int_0^t \int_\Omega |2 \nabla_x \sqrt{\eta_\delta} + \sqrt{\eta_\delta} \nabla_x \Phi|^2 \, dx dt,$$

where we have used Lemma 3.3 and (3.39). By taking the limit  $h \to 0$  in (3.33) (using convexity of  $z \mapsto z^{\gamma}$  and  $z \mapsto z \log z$ ), we obtain

$$E(\varrho_{\delta}, \mathbf{u}_{\delta}, \eta_{\delta})(t) + \int_{0}^{t} \int_{\Omega} \mu |\nabla_{x} \mathbf{u}_{\delta}|^{2} + \lambda |\operatorname{div}_{x} \mathbf{u}_{\delta}|^{2} \, dx dt + \int_{0}^{t} \int_{\Omega} |2\nabla_{x} \sqrt{\eta_{\delta}} + \sqrt{\eta_{\delta}} \nabla_{x} \Phi|^{2} \, dx dt \leq E(\varrho_{\delta,0}, \mathbf{u}_{\delta,0}, \eta_{\delta,0}),$$

$$(3.44)$$

for any  $t \in (0, T)$ .

### **3.5** Vanishing artificial pressure limit $(\delta \rightarrow 0)$

In the previous subsection we proved that, for each fixed  $\delta > 0$ , there exists an artificial pressure solution in the sense of Definition 3.1. Throughout this section, we let  $\{\varrho_{\delta}, \mathbf{u}_{\delta}, \eta_{\delta}\}_{\delta>0}$  be a sequence of such solutions. The aim is now to prove that this sequence converges as  $\delta \to 0$  to a weak solution of the fluid–particle interaction model (1.2)-(1.4) in the sense of Definition 2.2. This will then conclude the proof of Lemma 3.1.

#### 3.5.1 Energy bounds

The energy inequality (1.8), together with Sobolev embedding and Corollary 3.1, allow us to conclude the following  $\delta$ -independent bounds

$$\varrho_{\delta} |\mathbf{u}_{\delta}|^{2} \in_{b} L^{\infty}(0,T; L^{1}(\Omega)) \cap L^{2}(0,T; L^{m_{1}}(\Omega)), m_{1} > 1, 
\mathbf{u}_{\delta} \in_{b} L^{2}(0,T; W_{0}^{1,2}(\Omega)), 
\varrho_{\delta} \in_{b} L^{\infty}(0,T; L^{\gamma}(\Omega)), 
\varrho_{\delta} \mathbf{u}_{\delta} \in_{b} L^{\infty}(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega)), 
\eta_{\delta} \log \eta_{\delta} \in_{b} L^{\infty}(0,T; L^{1}(\Omega)).$$
(3.45)

By straight forward applications of the Hölder inequality, using (3.45) and Sobolev embedding, we deduce the bounds

$$\varrho_{\delta} \mathbf{u}_{\delta} \in_{b} L^{2}(0,T; L^{m_{2}}(\Omega)), \quad \varrho_{\delta} \mathbf{u}_{\delta} \otimes \mathbf{u}_{\delta} \in_{b} L^{2}(0,T; L^{c_{2}}(\Omega)),$$

where

$$m_2 = \frac{6\gamma}{6+\gamma}, \qquad c_2 = \frac{3\gamma}{3+\gamma}.$$

From (3.35), we also have that

$$\eta_{\delta} \in_b L^2(0,T;L^3(\Omega)) \cap L^1(0,T;W^{\frac{3}{2}}(\Omega))$$

Using the above  $\delta-\mathrm{independent}$  estimates, we easily deduce the weak time control bounds

$$\partial_t \eta_{\delta} \in_b L^1(0, T; W^{-1, \frac{3}{2}}(\Omega)),$$
  
$$\partial_t \varrho_{\delta} \in_b L^{\infty}(0, T; W^{-1, \frac{2\gamma}{\gamma+1}}(\Omega)),$$
  
$$\partial_t(\varrho_{\delta} \mathbf{u}_{\delta}) \in_b L^1(0, T; W^{-1, 1}(\Omega)).$$

#### **3.5.2** Convergence

The bounds in the previous subsection asserts the existence of functions  $(\rho, \mathbf{u}, \eta)$  such that, passing to a subsequence if necessary,

$$\begin{aligned} \varrho_{\delta} &\stackrel{*}{\rightharpoonup} \varrho_{\delta} & \text{ in } L^{\infty}(0,T;L^{\gamma}(\Omega)), \\ \mathbf{u}_{\delta} &\rightharpoonup \mathbf{u}_{\delta} & \text{ in } L^{2}(0,T;\mathbf{W}_{0}^{1,2}(\Omega)), \\ \eta_{\delta} &\rightharpoonup \eta_{\delta} & \text{ in } L^{2}(0,T;L^{3}(\Omega)). \end{aligned}$$

As in the previous subsection, Lemma 3.10 can be applied to conclude

$$\varrho_{\delta} \mathbf{u}_{\delta} \rightharpoonup \rho \mathbf{u}, \qquad \rho_{\delta} \mathbf{u}_{\delta} \otimes \mathbf{u}_{\delta} \rightharpoonup \rho \mathbf{u} \otimes \mathbf{u},$$
(3.46)

in the sense of distributions on  $(0, T) \times \Omega$ .

By similar arguments as those leading to (3.38), we deduce

$$\eta_{\delta} \to \eta \quad \text{in } L^{2-\epsilon}(0,T; L^2_{\text{loc}}(\Omega)), \qquad \sqrt{\eta_{\delta}} \to \sqrt{\eta} \quad \text{in } L^2(0,T; L^2_{\text{loc}}(\Omega)).$$

and

$$\nabla_x \eta_\delta \rightharpoonup \nabla_x \eta, \tag{3.47}$$

in the sense of distributions on  $(0, T) \times \Omega$ .

Using (3.46), the bounds (3.45), (3.47), and strong convergence of the initial conditions, we can take the limit  $\delta \to 0$  in (3.40) and (3.41) to discover that

$$-\int_0^T \int_\Omega \varrho(\psi_t - \mathbf{u}\nabla_x \psi) \, dx dt = \int_\Omega \varrho_0 \psi(0, x) \, dx,$$

for all  $\psi \in C_c^{\infty}([0,T) \times \overline{\Omega}))$  and

$$-\int_0^T \int_\Omega \eta \left(\psi_t + (\mathbf{u} - \nabla_x \Phi) \nabla_x \psi\right) - \nabla_x \eta \nabla_x \psi \, dx dt = \int_\Omega \eta_0 \psi(0, x) \, dx$$

for all  $\psi \in C_c^{\infty}([0,T) \times \overline{\Omega})$ ). Hence, the limiting functions satisfy both the continuity equation and particle equation in the sense of Definition 2.2.

Similarly, we can go to the limit  $\delta \to 0$  in (3.2) to discover in the limit

$$\begin{split} &\int_0^T \int_\Omega \rho \mathbf{u} \mathbf{v}_t + \rho \mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{v}_\delta + \eta \mathrm{div}_x \mathbf{v} \, dx dt \\ &= \int_0^T \int_\Omega \mu \nabla_x \mathbf{u} \nabla_x \mathbf{v} + \lambda \mathrm{div}_x \mathbf{u} \mathrm{div}_x \mathbf{v} + (\beta \rho + \eta) \nabla_x \Phi \mathbf{v} \, dx dt \\ &- \lim_{\delta \to 0} \int_0^T \int_\Omega p(\rho_\delta) \mathrm{div}_x \mathbf{v} \, dx dt - \int_\Omega \mathbf{m}_0 \mathbf{v}(0, x) \, dx, \end{split}$$

for all  $\mathbf{v} \in C_c^{\infty}([0,T) \times \Omega)$ . Thus, in order to conclude existence of a weak solution in the sense of Definition 2.2, it remains to prove that

$$\lim_{\delta \to 0} \int_0^T \int_\Omega p(\varrho_\delta) \mathrm{div}_x \mathbf{v} \ dx dt = \int_0^T \int_\Omega p(\varrho) \mathrm{div}_x \mathbf{v} \ dx dt,$$

together with the energy inequality (1.8). Consequently, we are faced with a similar situation as in the previous subsection. The main difference is that we now only have  $\gamma > \frac{N}{2}$ . Since  $\eta_{\delta}$  enjoys both high integrability and compactness, the proof follows by a small extension of Feireisl's arguments in [12]. First, we establish higher integrability of the density on the entire domain  $\Omega$ .

**Lemma 3.12** Let  $\{\varrho_{\delta}, \mathbf{u}_{\delta}, \eta_{\delta}\}_{\delta>0}$  be a sequence of artificial pressure solutions in the sense of Definition 3.1. Then, there exists a constant c(T), independent of  $\delta$ , such that

$$\int_0^T \int_\Omega \varrho_\delta^{\gamma+\theta} \, dx dt \le c(T),$$

where  $\Theta = \min\{\frac{2}{3}\gamma - 1, \frac{1}{4}\}.$ 

**Proof:** Since  $\eta_{\delta} \in_b L^2(0,T; L^3(\Omega)) \cap L^1(0,T; W^{1,\frac{3}{2}}(\Omega))$ , the addition of  $\eta_{\delta}$  in the equations does not impose any potential problems. Consequently, if  $\Omega$  is bounded, the proof follows by the classical arguments ([18, 12]).

If  $\Omega$  is unbounded, then  $\nabla_x \Phi$  is no longer integrable and we cannot simply apply existing results. To prove the bound in this case, let  $\Delta^{-1}$  be the inverse Laplacian realized using Fourier multipliers (see [18, 11] for details). For each fixed  $\delta > 0$ , let the test-function  $\mathbf{v}_{\delta}$  be given as

$$\mathbf{v}_{\delta} = \nabla_x \Delta^{-1} \varrho_{\delta}^{\theta}$$

By the requirements on  $\theta$ , we have in particular  $\varrho^{\theta}_{\delta} \in L^{\infty}(0,T; L^{s}(\Omega))$ ,

$$s = \max\left\{\frac{3\gamma}{2\gamma - 3}, 4\gamma\right\}.$$

Thus,  $\mathbf{v}_{\delta} \in L^{\infty}(0,T; W^{1,s}(\Omega)) \cap L^{\infty}(0,T; L^{\infty}(\Omega)).$ 

Next, since  $(\varrho_{\delta}, \mathbf{u}_{\delta})$  is a renormalized solution to the continuity equations, (2.3) with  $B(\varrho_{\delta}) = \varrho_{\delta}^{\theta}$  states

$$\partial_t \varrho^{\theta}_{\delta} = -\mathrm{div}_x (\varrho^{\theta}_{\delta} \mathbf{u}) - (\theta - 1) \varrho^{\theta}_{\delta} \mathrm{div}_x \mathbf{u},$$

in the sense of distributions on  $(0,T) \times \Omega$ . For notational convenience, we observe that

$$\begin{aligned} \|\partial_t \mathbf{v}_{\delta}\|_{L^p(0,T;L^q(\Omega))} &= \|\nabla_x \Delta^{-1} \partial_t \varrho_{\delta}^{\theta}\|_{L^p(0,T;L^q(\Omega))} \\ &\leq \|\varrho_{\delta}^{\theta} \mathbf{u}_{\delta}\|_{L^p(0,T;L^q(\Omega))} + \|\varrho^{\theta} \operatorname{div}_x \mathbf{u}\|_{L^p(0,T;L^r(\Omega))}, \end{aligned}$$

for appropriate  $1 \leq p, q \leq \infty$  and  $r^* = q$ .

Next, we apply  $\mathbf{v}_{\delta}$  as test function for the momentum equation to obtain

$$\begin{split} \int_0^T \int_\Omega a\varrho_{\delta}^{\gamma+\theta} \, dx dt &= -\int_0^T \int_\Omega (\varrho_{\delta} \mathbf{u}_{\delta}) \partial_t \mathbf{v}_{\delta} + \varrho_{\delta} \mathbf{u}_{\delta} \otimes \mathbf{u}_{\delta} : \nabla_x \mathbf{v}_{\delta} \, dx dt \\ &+ \int_0^T \int_\Omega \mu \nabla_x \mathbf{u}_{\delta} \nabla_x \mathbf{v}_{\delta} + \lambda \mathrm{div}_x \mathbf{u}_{\delta} \mathrm{div}_x \mathbf{v}_{\delta} \, dx dt \\ &- \int_0^T \int_\Omega \eta_{\delta} \varrho_{\delta}^{\theta} - (\varrho_{\delta} \beta + \eta_{\delta}) \nabla_x \Phi_{\delta} \mathbf{v}_{\delta} \, dx dt - \int_\Omega \mathbf{m}_0 \mathbf{v}_{\delta}(0, \cdot) \, dx \\ &:= I_1 + I_2 + I_3. \end{split}$$

To conclude it remains to bound  $I_1$ ,  $I_2$ , and  $I_3$ , independently of  $\delta$ . We start with the  $I_1$  term:

$$\begin{aligned} |I_1| &:= \left| \int_0^T \int_{\Omega} \varrho_{\delta} \partial_t \mathbf{v} + \varrho_{\delta} \mathbf{u}_{\delta} \otimes \mathbf{u}_{\delta} : \nabla_x \mathbf{v}_{\delta} \, dx dt \right| \\ &\leq \left\| \varrho_{\delta} \mathbf{u}_{\delta} \right\|_{L^{\infty}(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))} \left\| \varrho_{\delta}^{\theta} \mathbf{u}_{\delta} \right\|_{L^{1}(0,T;L^{\frac{2\gamma}{\gamma-1}}(\Omega))} \\ &+ \left\| \varrho_{\delta} \mathbf{u}_{\delta} \right\|_{L^{2}(0,T;L^{m_{2}}(\Omega))} C(T) \left\| \varrho^{\theta} \operatorname{div}_{x} \mathbf{u} \right\|_{L^{2}(0,T;L^{r}(\Omega))} \\ &+ \left\| \varrho_{\delta} \mathbf{u}_{\delta} \otimes \mathbf{u}_{\delta} \right\|_{L^{2}(0,T;L^{c_{2}}(\Omega))} \left\| \nabla_x \mathbf{v}_{\delta} \right\|_{L^{2}(0,T;L^{c_{2}'}(\Omega))}, \end{aligned}$$

where

$$r = \frac{6\gamma}{7\gamma - 6}, \qquad r^* = (m_2)', \qquad c'_2 = \frac{3\gamma}{2\gamma - 3} \le s.$$

Now, we estimate

$$\begin{aligned} \|\varrho_{\delta}^{\theta}\mathbf{u}_{\delta}\|_{L^{1}(0,T;L^{\frac{2\gamma}{\gamma-1}}(\Omega))} &\leq C(T)\|\varrho_{\delta}^{\theta}\|_{L^{\infty}(0,T;L^{s}(\Omega))}\|\mathbf{u}_{\delta}\|_{L^{2}(0,T;L^{6}(\Omega))},\\ \|\varrho_{\delta}^{\theta}\operatorname{div}_{x}\mathbf{u}_{\delta}\|_{L^{2}(0,T;L^{r}(\Omega))} &\leq C\|\operatorname{div}_{x}\mathbf{u}_{\delta}\|_{L^{2}(0,T;L^{2}(\Omega))}\|\varrho_{\delta}^{\theta}\|_{L^{\infty}(0,T;L^{s}(\Omega))}\end{aligned}$$

and hence conclude that  $|I_1| \leq C(T)$ . Next, we easily deduce the bound  $|I_2| \leq C(T)$ , and it only remains to bound  $I_3$ .

$$|I_{3}| \leq \|\eta_{\delta}\|_{L^{2}(0,T;L^{3}(\Omega))} \|\varrho^{\theta}\|_{L^{2}(0,T;L^{\frac{3}{2}}(\Omega))} + \|(\beta\varrho_{\delta} + \eta_{\delta})\nabla_{x}\Phi\|_{L^{1}(0,T;L^{1}(\Omega))} \|\mathbf{v}_{\delta}\|_{L^{\infty}(0,T;L^{\infty}(\Omega))} + \|\mathbf{m}_{0}\|_{L^{1}(\Omega)} \|\mathbf{v}_{\delta}\|_{L^{\infty}(\Omega)} \leq C(T) \left(1 + \|(\beta\varrho_{\delta} + \eta_{\delta})\nabla_{x}\Phi\|_{L^{\infty}(0,T;L^{1}(\Omega))}\right).$$

Using the energy estimate (3.44), Corollary 3.1, and the requirements (2.2) on the potential, we readily deduce

$$\begin{split} \sup_{t \in (0,T)} &\int_{\Omega} |\beta \varrho_{\delta} + \eta_{\delta}| |\nabla_x \Phi| \ dx \\ &\leq \|\nabla_x \Phi\|_{L^{\infty}(B(0,R))} \sup_{t \in (0,T)} \int_{B(0,R)} \beta \varrho_{\delta} + \eta_{\delta} \ dx \\ &+ C \sup_{t \in (0,T)} \int_{\Omega \setminus B(0,R)} (\beta \varrho_{\delta} + \eta_{\delta}) \Phi \ dx dt \leq C(T), \end{split}$$

which brings the proof to an end.

The following lemma concludes the proof of Lemma 3.1.

**Lemma 3.13** Let  $\{\varrho_{\delta}, \mathbf{u}_{\delta}, \eta_{\delta}\}_{\delta>0}$  be a sequence of artificial pressure solutions in the sense of Definition 3.1. Then, there exists a triple  $(\varrho, \mathbf{u}, \eta)$  such that as  $\delta \to 0$ ,  $\varrho_{\delta} \rightharpoonup \varrho$  in  $L^{\infty}(0, T; L^{\gamma}(\Omega))$ ,  $\varrho_{\delta} \rightarrow \varrho$  a.e in  $(0, T) \times \Omega$ ,  $\mathbf{u}_{\delta} \rightharpoonup \mathbf{u}$  in  $L^{2}(0, T; W_{0}^{1,2}(\Omega))$ ,  $\varrho_{\delta} \mathbf{u}_{\delta} \rightarrow \varrho \mathbf{u}$  in the sense of distributions on  $(0, T) \times \Omega$ ,  $\eta_{\delta} \rightharpoonup \eta$  in  $L^{1}(0, T; W^{1,\frac{3}{2}}(\Omega))$ , and  $\eta_{\delta} \rightarrow \eta$  a.e in  $(0, T) \times \Omega$ , where  $(\varrho, \mathbf{u}, \eta)$  is a weak solution to the particle interaction model in the sense of Definition 2.2.

Moreover, the total fluid mass and particle mass given by

$$M_{\varrho}(t) = \int_{\Omega} \varrho(t, \cdot) \, dx \quad and \quad M_{\eta}(t) = \int_{\Omega} \eta(t, \cdot) \, dx,$$

respectively, are constants of motion.

**Proof:** 1. Since  $\eta_{\delta} \to \eta$  a.e in  $(0,T) \times \Omega$ , the presence of the  $\eta_{\delta}$  variable does not impose any extra difficulties as compared with the corresponding situation for the barotropic compressible Navier–Stokes equations. Consequently, strong convergence of the density and existence of a solution follows as in [12, Section 4.2]. In the case  $\Omega$  bounded, this concludes the proof. For an unbounded domain, the potential is non-integrable and it remains to prove the energy estimate (1.8). Since the mappings  $z \mapsto z^{\gamma}$  and  $z \mapsto z \log z$  are convex, we can apply Lemma 3.3 to conclude that, for a.e  $t \in (0, T)$ ,

$$E(\varrho, \mathbf{u}, \eta) \le \lim_{\delta \to 0} E(\varrho_{\delta}, \mathbf{u}_{\delta}, \eta_{\delta}).$$
(3.48)

Next, similar to (3.43), we deduce,

$$\int_{0}^{T} \int_{K} |2\nabla_{x}\sqrt{\eta} + \sqrt{\eta}\nabla_{x}\Phi|^{2} dxdt$$

$$\leq \lim_{\delta \to 0} \left[ \int_{0}^{T} \int_{K} 4|\nabla_{x}\sqrt{\eta_{\delta}}|^{2} + 4\nabla_{x}\eta_{\delta}\nabla_{x}\Phi + \eta_{\delta}|\nabla_{x}\Phi|^{2} dxdt \right]$$

$$= \lim_{\delta \to 0} \left[ \int_{0}^{T} \int_{\Omega} |2\nabla_{x}\sqrt{\eta_{\delta}} + \sqrt{\eta_{\delta}}\nabla_{x}\Phi|^{2} dxdt \right]$$

$$\leq E(\varrho_{0}, \mathbf{u}_{0}, \eta_{0}).$$
(3.49)

for any compact K.

T

Finally, using (3.48) and (3.49), we can send  $\delta \to 0$  in (3.44) to obtain

$$E(\varrho, \mathbf{u}, \eta)(t) + \int_0^t \int_\Omega \mu |\nabla_x \mathbf{u}|^2 + \lambda |\mathrm{div}_x \mathbf{u}|^2 \, dx dt + \int_0^t \int_\Omega \eta \, |\nabla_x \Phi + \nabla_x \log \eta|^2 \, dx dt \leq E(\varrho_0, \mathbf{u}_0, \eta_0).$$

2. Let us now prove that  $M_{\varrho}$  and  $M_{\eta}$  are constants of motion. To prove this, we make use of Lemma 3.7 and conclude that for any  $\epsilon > 0$  there exists R > 0 such that

$$\int_{\Omega} \eta_{\delta,h} \, dx = \int_{\Omega \cap B(0,R)} \eta_{\delta,h} \, dx + \mathcal{O}(\epsilon).$$

By sending  $\delta, h \to 0$ , we gather

$$M_{\varrho}(0) = \int_{\Omega} \eta_0 \ dx = \lim_{\delta, h \to 0} \int_{\Omega} \eta_{\delta, h} \ dx = \int_{\Omega \cap B(0, R)} \eta \ dx + \mathcal{O}(\epsilon),$$

which implies that  $M_{\eta}$  is a constant of motion.

From the energy estimate (3.33) and Corollary 3.1 it follows that (for a.e  $t \in (0, T)$ )

$$\int_{\Omega} \varrho_{h,\delta} \Phi \ dx \le C.$$

By the requirements of the potential (2.2), we can for any M > 0 determine a radius R such that  $\Phi(x) \ge M$ , for all |x| > R. Hence,

$$\int_{\Omega} \varrho_{\delta,h} \, dx = \int_{B(0,R)} \varrho_{\delta,h} \, dx + \int_{\Omega \setminus B(0,R)} \varrho_{\delta,h} \, dx,$$

where

$$\int_{\Omega \setminus B(0,R)} \varrho_{\delta,h} \, dx \le M^{-1} \int_{\Omega \setminus B(0,R)} \Phi \varrho_{\delta,h} \, dx \le M^{-1} C$$

Since this holds for any M > 0, we conclude that  $M_{\varrho}(t)$  is a constant of motion.

## 4 Large-time Asymptotics

The analysis in the previous section yields

Lemma 4.1 Under the hypotheses of Theorem 2.2, we have

$$\lim_{\tau \to \infty} \int_{\tau-1}^{\tau+2} \|\nabla \mathbf{u}\|_{L^{p_1}(\Omega)}^2 + \|\varrho |\mathbf{u}|^2 \|_{L^{p_2}(\Omega)} + \|\varrho |\mathbf{u}|\|_{L^{p_3}(\Omega)}^2 + \||\eta| \mathbf{u}|\|_{L^{p_4}(\Omega)} dt = 0,$$

$$\lim_{\tau \to \infty} \int_{\tau-1}^{\tau+2} \int_{\Omega} |2\nabla_x \sqrt{\eta} + \sqrt{\eta} \nabla_x \Phi|^2 \, dx dt = 0,$$
(4.1)

and

$$\int_{\tau-1}^{\tau+2} \int_{\Omega} \varrho^{\gamma+\theta} \, dx dt \le C, \quad \text{for all } \tau > 1.$$

Here,

$$p_1 = 2, \quad p_2 = \frac{3\gamma}{\gamma+3}, \quad p_3 = \frac{6\gamma}{\gamma+6}, \quad p_4 = 4,$$
  
 $\theta = \min\left\{\frac{2}{3}\gamma - 1, \frac{1}{4}\right\}.$ 

Let the sequence  $\{t_n\}_{n=1}^{\infty}$  be such that  $t_n \to \infty$  and define the sequences:

$$\mathbf{u}_{n}(t,x) = \mathbf{u}(t+t_{n},x), \quad \varrho_{n}(t,x) = \varrho(t+t_{n},x), \quad \eta_{n}(t,x) = \eta(t+t_{n},x), \quad t \in (-1,2), \quad x \in \Omega.$$

Observe that, for each fixed n, the triple  $(\rho_n, \mathbf{u}_n, \eta_n)$  is a weak solution to particle interaction model in the sense of Definition 2.2.

By virtue of the previous lemma, and bounded energy, we can conclude the existence of functions  $\rho_s$ ,  $\overline{p}$ , and  $\eta_s$ , such that

$$\varrho_n = \varrho(t + \tau_n) \to \varrho_s \text{ weakly in } L^{\gamma}((-1, 2) \times \Omega) 
p_n = p(t + \tau_n) \to \overline{p} \text{ weakly in } L^1((-1, 2) \times \Omega) 
\eta_n = \eta(t + \tau_n) \to \eta_s \text{ weakly in } L^2((-1, 2), L^3(\Omega)).$$
(4.2)

**Lemma 4.2** Given (4.2) and the hypothesis of Theorem 2.2,  $\eta_n \to \eta_s$  a.e in  $(-1,2) \times \Omega$ , where  $\eta_s$  is given by

$$\eta_s = C_\eta \exp(-\Phi), \quad C_\eta = \frac{\int_\Omega \exp(-\Phi) \, dx}{\int_\Omega \eta_0 \, dx},$$

which is the unique solution to the problem

$$\nabla_x \eta_s = -\eta_s \nabla_x \Phi, \quad \int_\Omega \eta_s \ dx = \int_\Omega \eta_0 \ dx. \tag{4.3}$$

**Proof:** From the particle equation (1.4), we observe that

$$\partial_t \eta_n = -\operatorname{div}_x(\eta_n \mathbf{u}_n) + \operatorname{div}_x \left[ \sqrt{\eta_n} \left( \sqrt{\eta_n} \nabla_x \Phi + 2 \nabla_x \sqrt{\eta} \right) \right]$$

in the sense of distributions on  $(-1, 2) \times \Omega$ . Hence,

$$\int_{-1}^{2} \|\partial_t \eta_n\|_{W^{-1,1}(\Omega)} dt \le \|\eta_n \mathbf{u}_n\|_{L^1(-1,2;L^4(\Omega))} + \|\eta_n\|_{L^2(-1,2;L^2(\Omega))} \|\sqrt{\eta_n} \nabla_x \Phi + 2\nabla_x \sqrt{\eta}\|_{L^2(-1,2;L^2(\Omega))}.$$

Lemma 4.1 tells us that the right-hand side converges to zero as  $n \to \infty$ . Since in addition  $\eta_n \in_b L^1(-1, 2; W^{\frac{3}{2}}(\Omega))$ , Lemma 3.9 can be applied to conclude that  $\eta_n \to \eta_s$  a.e in  $(-1, 2) \times \Omega$ . In addition, as in (3.39), we find that  $\nabla_x \eta_n \to \nabla_x \eta$  in the sense of distributions on  $(-1, 2) \times \Omega$ .

Then, as in (3.49), we calculate

$$0 \leq \int_{-1}^{2} \int_{K} |\nabla_{x}\sqrt{\eta_{s}} + \sqrt{\eta_{s}}\nabla_{x}\Phi|^{2} dxdt$$
$$\leq \lim_{n \to \infty} \left[ \int_{-1}^{2} \int_{K} |2\nabla_{x}\sqrt{\eta_{n}} + \sqrt{\eta_{n}}\nabla_{x}\Phi|^{2} dxdt \right] \leq 0,$$

for any compact K. The last equality follows directly from (4.1). Hence, we can conclude

$$\nabla_x \sqrt{\eta_s} = -\sqrt{\eta_s} \nabla_x \Phi$$
 a.e in  $(0, T) \times \Omega$ .

Clearly, this means that also

$$\nabla_x \eta_s = -\eta_s \nabla_x \Phi \quad \text{a.e in } (0,T) \times \Omega.$$
(4.4)

Next, we let  $n \to \infty$  in (1.4) and apply Lemma 4.1 and (4.4) to obtain

$$\int_{-1}^{2} \int_{\Omega} \eta_{s} \phi_{t} \, dx dt = 0, \quad \forall \phi \in C_{c}^{\infty}((-1,2) \times \Omega).$$

Hence,  $\eta_s$  is *independent of time*. Using this, we deduce

$$\int_{\Omega} \eta_0 \, dx = \lim_{n \to \infty} \frac{1}{3} \int_{-1}^2 \int_{\Omega} \eta_n \, dx dt$$
$$= \lim_{n \to \infty} \left[ \frac{1}{3} \int_{-1}^2 \int_{B(0,R)} \eta_n \, dx dt + \frac{1}{3} \int_{-1}^2 \int_{\Omega \setminus B(0,R)} \eta_n \, dx dt \right]$$
$$= \int_{B(0,R)} \eta_s \, dx + \lim_{n \to \infty} \left[ \frac{1}{3} \int_{-1}^2 \int_{\Omega \setminus B(0,R)} \eta_n \, dx dt \right]$$

for any ball B(0, R) of finite radius R. Now, in view of (2.2), for any M > 0 there exists a radius R such that  $\Phi \ge M$ . Consequently,

$$\frac{1}{3} \int_{-1}^{2} \int_{\Omega \setminus B(0,R)} \eta_n \, dx dt \le \frac{1}{M} \int_{\Omega} \eta_n \Phi \, dx dt \le M^{-1}D,$$

where D is the constant appearing in Corollary 3.1. Letting  $M \to \infty$ , we conclude

$$\int_{\Omega} \eta_s \ dx = \int_{\Omega} \eta_0 \ dx. \tag{4.5}$$

Since (4.3) is linear in  $\eta_s$  and  $\nabla_x \eta_s$ , this concludes the proof.

By taking the limit  $n \to \infty$  in the continuity equation (1.2), keeping in mind Lemma 4.1, we obtain

$$\int_{-1}^{2} \int_{\Omega} \varrho_{s} \phi_{t} \, dx dt = 0, \quad \forall \phi \in C_{c}^{\infty}((-1,2) \times \Omega).$$

Thus,  $\rho_s$  is independent of time. By virtue of Corollary 3.1 and the energy estimate (2.6), we have that  $\beta \rho_s \Phi \in L^{\infty}(0,T; L^1(\Omega))$ . Then, we can redo the arguments leading to (4.5), with  $\beta \rho$  replacing  $\eta$ , to obtain

$$\int_{\Omega} \varrho_s \, dx = \int_{\Omega} \varrho_0 \, dx. \tag{4.6}$$

Note that we do not lose mass in the large-time limit (cf [14]).

**Lemma 4.3** Given the convergences (4.2),  $\rho_n \to \rho_s$  in  $L^{\gamma}((-1,2) \times \Omega)$ , where

$$\varrho_s = \left(\frac{\gamma - 1}{a\gamma} \left[-\beta \Phi + C_{\varrho}\right]^+\right)^{\frac{1}{\gamma - 1}},\tag{4.7}$$

 $C_{\varrho}$  is uniquely determined by  $\int_{\Omega} \varrho_0 dx$ , and  $\varrho_s$  is the unique solution of

$$\nabla_x p(\varrho_s) = -\beta \varrho_s \nabla_x \Phi \quad in \ \Omega, \quad \int_\Omega \varrho_s \ dx = \int_\Omega \varrho_0 \ dx$$

**Proof:** With the already obtained bounds, there is no problem with passing to the limit in the momentum equation (1.3) to obtain

$$\nabla_x \overline{p} + \nabla_x \eta = -\beta \varrho_s \nabla_x \Phi - \eta_s \nabla_x \Phi,$$

in the sense of distributions on  $\Omega$ . By virtue of the previous lemma,

$$\nabla_x \overline{p} = -\beta \varrho_s \nabla_x \Phi.$$

However, then we are in the same situation as in [14] and the arguments of [14, Proposition 4.1] can be applied to obtain strong convergence  $\rho_n \to \rho_s$  in  $L^{\gamma}((-1,2) \times \Omega)$ . Consequently,  $\bar{p} = p(\rho_s)$ , and, in view of (4.6),  $\rho_s$  solves

$$\nabla_x p(\varrho_s) = -\beta \varrho_s \nabla_x \Phi, \qquad \int_\Omega \varrho_s \ dx = \int_\Omega \varrho_0 \ dx$$

in the sense of distributions. According to [13, Theorem 2.1],  $\rho_s$  is then of the form (4.7), where  $C_{\rho}$  is uniquely determined by  $\int_{\Omega} \rho_s dx$ .

Lemmas 5.2 and 5.3 concludes the proof of Theorem 2.2.

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