VARIATION FOR THE RIESZ TRANSFORM AND UNIFORM RECTIFIABILITY

ALBERT MAS AND XAVIER TOLSA

Abstract. For $1 \leq n < d$ integers and $\rho > 2$, we prove that an n-dimensional Ahlfors-David regular measure μ in $\mathbb{R}^d$ is uniformly n-rectifiable if and only if the ρ-variation for the Riesz transform with respect to μ is a bounded operator in $L^2(\mu)$. This result can be considered as a partial solution to a well known open problem posed by G. David and S. Semmes which relates the $L^2(\mu)$ boundedness of the Riesz transform to the uniform rectifiability of μ.

1. Introduction

In this paper we characterize the notion of uniform rectifiability in the sense of David and Semmes [DS2] in terms of the L^2 boundedness of the ρ-variation for the Riesz transform, with $\rho > 2$.

Given $1 \leq n < d$ integers and a Radon measure μ in $\mathbb{R}^d$, one defines the n-dimensional Riesz transform of a function $f \in L^1(\mu)$ by $R^\mu f(x) = \lim_{\epsilon \searrow 0} R^\mu_\epsilon f(x)$ (whenever the limit exists), where

$$R^\mu_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{x-y}{|x-y|^{n+1}}\, f(y)\, d\mu(y), \qquad x \in \mathbb{R}^d.$$

We will use the notation $\mathcal{R}^\mu f(x) := \{R^\mu_\epsilon f(x)\}_{\epsilon>0}$. When $d = 2$ (i.e., μ is a Radon measure in $\mathbb{C}$), one defines the Cauchy transform of $f \in L^1(\mu)$ by $C^\mu f(x) = \lim_{\epsilon \searrow 0} C^\mu_\epsilon f(x)$ (whenever the limit exists), where

$$C^\mu_\epsilon f(x) = \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, d\mu(y), \qquad x \in \mathbb{C}.$$

To avoid the problem of existence of the preceding limits, it is useful to consider the maximal operators $R^\mu_* f(x) = \sup_{\epsilon>0} |R^\mu_\epsilon f(x)|$ and $C^\mu_* f(x) = \sup_{\epsilon>0} |C^\mu_\epsilon f(x)|$. Notice that the Cauchy transform coincides with the 1-dimensional Riesz transform in $\mathbb{R}^2$ modulo conjugation, since $1/x = \overline{x}/|x|^2$ for all $x \in \mathbb{C} \setminus \{0\}$.

The Cauchy and Riesz transforms are two very important examples of singular integral operators with a Calderón-Zygmund kernel. Given $d \geq 2$, the kernels $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ that we consider in this paper satisfy

$$|K(x)| \leq \frac{C}{|x|^n}, \quad |\partial_{x^i} K(x)| \leq \frac{C}{|x|^{n+1}} \quad \text{and} \quad |\partial_{x^i}\partial_{x^j} K(x)| \leq \frac{C}{|x|^{n+2}}, \tag{1}$$

for all $1 \leq i, j \leq d$ and $x = (x^1, \ldots, x^d) \in \mathbb{R}^d \setminus \{0\}$, where $1 \leq n < d$ is some integer and $C > 0$ is some constant; and moreover $K(-x) = -K(x)$ for all $x \neq 0$ (i.e. K is odd). Notice that

Date: September, 2011.

2010 *Mathematics Subject Classification.* Primary 42B20, 42B25.

Key words and phrases. ρ-variation and oscillation, Calderón-Zygmund singular integrals, Riesz transform, uniform rectifiability.

Both authors are partially supported by grants 2009SGR-000420 (Generalitat de Catalunya) and MTM2010-16232 (Spain). Albert Mas is also supported by grant AP2006-02416 (FPU program, Spain).

the n-dimensional Riesz transform corresponds to the vector kernel $(x^1,\dots,x^d)/|x|^{n+1}$, and the Cauchy transform to $(x^1,-x^2)/|x|^2$ (so, we may consider K to be any scalar component of these vector kernels). For $f\in L^1(\mu)$ and $x\in\mathbb{R}^d$, we set

$$T^\mu_\epsilon f(x)\equiv T_\epsilon(f\mu)(x):=\int_{|x-y|>\epsilon} K(x-y)f(y)\,d\mu(y),$$

and we denote $\mathcal{T}^\mu f(x)=\{T^\mu_\epsilon f(x)\}_{\epsilon>0}$.

Definition 1.1 (ρ-variation and oscillation)**.** *Let $\mathcal{F}:=\{F_\epsilon\}_{\epsilon>0}$ be a family of functions defined on $\mathbb{R}^d$. Given $\rho>0$, the* ρ-variation *of $\mathcal{F}$ at $x\in\mathbb{R}^d$ is defined by*

$$\mathcal{V}_\rho(\mathcal{F})(x):=\sup_{\{\epsilon_m\}}\biggl(\sum_{m\in\mathbb{Z}}|F_{\epsilon_{m+1}}(x)-F_{\epsilon_m}(x)|^\rho\biggr)^{1/\rho},$$

where the pointwise supremum is taken over all decreasing sequences $\{\epsilon_m\}_{m\in\mathbb{Z}}\subset(0,\infty)$. Fix a decreasing sequence $\{r_m\}_{m\in\mathbb{Z}}\subset(0,\infty)$. The oscillation *of $\mathcal{F}$ at $x\in\mathbb{R}^d$ is defined by*

$$\mathcal{O}(\mathcal{F})(x):=\sup_{\{\epsilon_m\},\{\delta_m\}}\biggl(\sum_{m\in\mathbb{Z}}|F_{\epsilon_m}(x)-F_{\delta_m}(x)|^2\biggr)^{1/2},$$

where the pointwise supremum is taken over all sequences $\{\epsilon_m\}_{m\in\mathbb{Z}}$ and $\{\delta_m\}_{m\in\mathbb{Z}}$ such that $r_{m+1}\leq\epsilon_m\leq\delta_m\leq r_m$ for all $m\in\mathbb{Z}$.

The ρ-variation and oscillation for martingales and some families of operators have been studied in many recent papers on probability, ergodic theory, and harmonic analysis (see [Lp], [Bo], [JKRW], [CJRW1], [JSW], [LT], and [OSTTW], for example). In this paper we are interested in the ρ-variation and oscillation of the family $\mathcal{T}^\mu f$. That is, given a Radon measure μ in $\mathbb{R}^d$ and $f\in L^1(\mu)$ we will deal with

$$(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f(x):=\mathcal{V}_\rho(\mathcal{T}^\mu f)(x),\quad(\mathcal{O}\circ\mathcal{T}^\mu)f(x):=\mathcal{O}(\mathcal{T}^\mu f)(x).$$

We are specially interested in the case $\mathcal{T}^\mu=\mathcal{R}^\mu$. Notice, by the way, that $T^\mu_*f(x)\leq(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f(x)$ for any compactly supported function $f\in L^1(\mu)$ and all $x\in\mathbb{R}^d$.

When μ coincides with the Lebesgue measure in the real line and $K(x)=1/x$ is the kernel of the Hilbert transform, Campbell, Jones, Reinhold and Wierdl [CJRW1] showed that $\mathcal{V}_\rho\circ\mathcal{T}^\mu$ and $\mathcal{O}\circ\mathcal{T}^\mu$ are bounded in $L^p(\mu)$, for $1<p<\infty$, and of weak type $(1,1)$. This result was extended to other singular integral operators in higher dimensions in [CJRW2]. The case of the Cauchy transform and other odd Calderón-Zygmund operators on Lipschitz graphs was studied recently in [MT].

Let us turn our attention to uniform rectifiability now. Recall that a Radon measure μ in $\mathbb{R}^d$ is called n-rectifiable if there exists a countable family of n-dimensional C^1 submanifolds $\{M_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}^d$ such that $\mu(E\setminus\bigcup_{i\in\mathbb{N}}M_i)=0$. Moreover, μ is said to be n-dimensional Ahlfors-David regular, or simply AD regular, if there exists some constant $C>0$ such that $C^{-1}r^n\leq\mu(B(x,r))\leq Cr^n$ for all $x\in\operatorname{supp}\mu$ and $0<r\leq\operatorname{diam}(\operatorname{supp}\mu)$. One also says that μ is uniformly n-rectifiable if there exist $\theta,M>0$ so that, for each $x\in\operatorname{supp}\mu$ and $r>0$, there is a Lipschitz mapping g from the n-dimensional ball $B^n(0,r)\subset\mathbb{R}^n$ into $\mathbb{R}^d$ such that $\operatorname{Lip}(g)\leq M$ and $\mu\bigl(B(x,r)\cap g(B^n(0,r))\bigr)\geq\theta r^n$, where $\operatorname{Lip}(g)$ stands for the Lipschitz constant of g. In particular, uniform rectifiability implies rectifiability. Given a set $E\subset\mathbb{R}^d$, we denote by $\mathcal{H}^n_E$ the n-dimensional Hausdorff measure restricted to E. Then E is called, respectively, n-rectifiable, AD regular, or uniformly n-rectifiable if $\mathcal{H}^n_E$ is so. By the Lebesgue differentiation theorem, any n-dimensional AD regular measure μ is of the form $\mu=f\mathcal{H}^n_{\operatorname{supp}\mu}$ with $C^{-1}\leq f(x)\leq C$ for some constant $C>0$ and all $x\in\operatorname{supp}\mu$.

G. David and S. Semmes asked more than twenty years ago the following question, which is still open (see, for example, [Pa, Chapter 7]):

Question 1.2. *Is it true that an n-dimensional AD regular measure μ is uniformly n-rectifiable if and only if R_*^μ is bounded in $L^2(\mu)$?*

Some comments are in order. By the results in [DS1], the "only if" implication of the question above is already known to hold. Also in [DS1], G. David and S. Semmes gave a positive answer to Question 1.2 if one replaces the L^2 boundedness of R_*^μ by the L^2 boundedness of T_*^μ for a wide class of odd kernels K. In the case $n = 1$ (in particular, for the Cauchy transform), the "if" implication was proved by P. Mattila, M. Melnikov and J. Verdera in [MMV] using the notion of curvature of measures. Later on, G. David and J. C. Léger [Lé] proved that the L^2 boundedness C_*^μ implies that μ is rectifiable, even without the AD regularity assumption (with $n = 1$).

When μ is the n-dimensional Hausdorff measure on a set $E \subset \mathbb{R}^d$ such that $\mu(E) < \infty$, the rectifiability of μ is also related with the existence μ-a.e. of the principal value of the Riesz transform of μ, that is, the existence of $R^\mu 1(x) = \lim_{\epsilon \searrow 0} R_\epsilon^\mu 1(x)$ for μ-a.e. $x \in E$. In [MPr], P. Mattila and D. Preiss proved that, under the additional assumption that

$$\liminf_{r \to 0} r^{-n} \mu(B(x, r)) > 0 \qquad \text{for } \mu\text{-a.e. } x \in E, \tag{2}$$

the rectifiability of E is equivalent to the existence of $R^\mu 1(x)$ μ-a.e. $x \in E$. Later on, in [To3] X. Tolsa removed the assumption (2) and proved the result in full generality. Let us mention that, for the case $n = 1$ and $d = 2$ (that is, for the Cauchy transform), the analogous results had been obtained previously by [Ma2] under the assumption (2), and in [To1], in full generality, by using the notion of curvature of measures.

In this paper we prove the following:

Theorem 1.3. *Let $1 \leq n < d$ and $\rho > 2$. An n-dimensional AD regular Radon measure μ in $\mathbb{R}^d$ is uniformly n-rectifiable if and only if $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is a bounded operator in $L^2(\mu)$. Moreover, if μ is n-uniformly rectifiable, then for any kernel K satisfying* (1)*, the operator $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ is bounded in $L^2(\mu)$.*

Let us compare this result with the David-Semmes Question 1.2. Notice that the preceding theorem asserts that if we replace the $L^2(\mu)$ boundedness of R_*^μ by the stronger assumption that $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is bounded in $L^2(\mu)$, then μ must be uniformly rectifiable. On the other hand, the theorem claims that the variation for odd singular integral operators with any kernel satisfying (1), in particular for the n-dimensional Riesz transforms, is bounded in $L^2(\mu)$.

A natural question then arises. Given an arbitrary measure μ on $\mathbb{R}^d$, without atoms say, does the $L^2(\mu)$ boundedness of R_*^μ implies the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$, for $\rho > 2$? By the results of [MMV] and Theorem 1.3, this is true in the case $n = 1$ if μ is AD regular 1-dimensional. Clearly, a positive answer in the general case $n \geq 1$ would solve the David-Semmes problem in the affirmative. Nevertheless, such an approach to try to solve this problem looks quite difficult. In fact, we recall that is not even known if the $L^2(\mu)$ boundedness of R_*^μ ensures the μ-a.e. existence of the principal values of $R^\mu 1$, which is a necessary condition for the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$.

Concerning the proof of Theorem 1.3, in our previous paper [MT] we showed that, if μ stands for the n-dimensional Hausdorff-measure on an n-dimensional Lipschitz graph, then the ρ-variation for Riesz transforms and odd Calderón-Zygmund operators with smooth truncations are bounded in $L^2(\mu)$. This is a fundamental step to prove that $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ and, more generally, $\mathcal{V}_\rho \circ \mathcal{T}^\mu$, are bounded in $L^2(\mu)$ if μ is uniformly n-rectifiable. Another basic tool in our arguments is the geometric corona decomposition of uniformly rectifiable measures

introduced by David and Semmes in [DS1], which, roughly speaking, describes how $\operatorname{supp}(\mu)$ can be approximated at different scales by n-dimensional Lipschitz graphs.

The proof of the fact that the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ implies the uniform rectifiability of μ is not so laborious as the one of the converse implication. As remarked above, if $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is bounded in $L^2(\mu)$, then the principal values of $R^\mu 1$ exist μ-a.e., which implies the n-rectifiability of μ, by the results of [MPr] or [To3]. However, this is not enough to ensure the *uniform* n-rectifiability of μ. We will prove the uniform n-rectifiability by arguments partially inspired by some of the techniques in [To4].

Finally, let us remark that Theorem 1.3 follows from a more general result, namely Theorem 2.3 below, which also deals with the variation for Riesz transforms and odd Calderón-Zygmund operators with smooth truncations.

As usual, in the paper the letter 'C' stands for some constant which may change its value at different occurrences, and which quite often only depends on n and d. The notation $A \lesssim B$ ($A \gtrsim B$) means that there is some fixed constant C such that $A \leq CB$ ($A \geq CB$), with C as above. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

2. Preliminaries

2.1. The main theorem.

Definition 2.1 (families of truncations)**.** *Let $\chi_\mathbb{R} := \chi_{[1,\infty)}$ and let $\varphi_\mathbb{R} : [0,+\infty) \to [0,+\infty)$ be a non decreasing $\mathcal{C}^2$ function with $\chi_{[4,\infty)} \leq \varphi_\mathbb{R} \leq \chi_{[1/4,\infty)}$. Suppose moreover that $|\varphi_\mathbb{R}'|$ is bounded below away from zero in $[1/3,3]$, i.e., $\chi_{[1/3,3]} \leq C|\varphi_\mathbb{R}'|$ for some $C > 0$.*

Given $x \in \mathbb{R}^d$, and $0 < \epsilon \leq \delta$, we set

$$\chi_\epsilon(x) := \chi_\mathbb{R}(|x|/\epsilon) \quad \text{and} \quad \chi_\epsilon^\delta(x) := \chi_\epsilon(x) - \chi_\delta(x),$$
$$\varphi_\epsilon(x) := \varphi_\mathbb{R}(|x|^2/\epsilon^2) \quad \text{and} \quad \varphi_\epsilon^\delta(x) := \varphi_\epsilon(x) - \varphi_\delta(x).$$

Notice that, for any finite Radon measure μ, $T_\epsilon \mu(x) = (K\chi_\epsilon * \mu)(x)$. Given $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$, we denote $\widetilde{x} = (x^1, \ldots, x^n, 0, \ldots, 0) \in \mathbb{R}^d$, and we set $\widetilde{\varphi}_\epsilon(x) := \varphi_\epsilon(\widetilde{x})$ and $\widetilde{\varphi}_\epsilon^\delta(x) := \varphi_\epsilon^\delta(\widetilde{x})$. Finally, for $f \in L^1(\mu)$ we set $\mathcal{T}^\mu f \equiv \mathcal{T}(f\mu) := \{T_\epsilon^\mu f\}_{\epsilon>0}$,

$$T_{\varphi_\epsilon}^\mu f(x) \equiv T_{\varphi_\epsilon}(f\mu)(x) := (K\varphi_\epsilon * \mu)(x) \quad \text{and} \quad \mathcal{T}_\varphi^\mu f \equiv \mathcal{T}_\varphi(f\mu) := \{T_{\varphi_\epsilon}^\mu f\}_{\epsilon>0},$$
$$T_{\widetilde{\varphi}_\epsilon}^\mu f(x) \equiv T_{\widetilde{\varphi}_\epsilon}(f\mu)(x) := (K\widetilde{\varphi}_\epsilon * \mu)(x) \quad \text{and} \quad \mathcal{T}_{\widetilde{\varphi}}^\mu f \equiv \mathcal{T}_{\widetilde{\varphi}}(f\mu) := \{T_{\widetilde{\varphi}_\epsilon}^\mu f\}_{\epsilon>0}.$$

***Remark* 2.2.** In the definition, the choice of $[4,\infty)$, $[1/4,\infty)$, and $[1/3,3]$ is not specially relevant, it is just for definiteness. One can replace the preceding intervals by other suitable intervals, and all the proofs in the paper remain almost the same.

We will prove the following.

Theorem 2.3 (Main Theorem)**.** *Let $1 \leq n < d$ be integers. Let μ be an n-dimensional AD regular Radon measure on $\mathbb{R}^d$. The following are equivalent:*

- *(a) μ is uniformly n-rectifiable.*
- *(b) For any K satisfying (1) and any $\rho > 2$, the operator $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ is bounded in $L^p(\mu)$ for all $1 < p < \infty$, and from $L^1(\mu)$ into $L^{1,\infty}(\mu)$.*
- *(c) For any K satisfying (1) and any $\rho > 2$, the operator $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ is bounded in $L^2(\mu)$.*
- *(d) For some $\rho > 0$, the operator $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is bounded in $L^2(\mu)$.*
- *(e) For $K(x) = x/|x|^{n+1}$ and some $\rho > 0$, the operator $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ is bounded in $L^2(\mu)$.*

Clearly, Theorem 1.3 is a direct consequence of the preceding result.

***Remark* 2.4.** Let $\{r_m\}_{m\in\mathbb{Z}} \subset (0,\infty)$ be a fixed decreasing sequence defining $\mathcal{O}$. Then, the implications $(a) \Rightarrow (b), \ldots, (e)$ in the theorem above still hold if one replaces $\mathcal{V}_\rho$ by $\mathcal{O}$. If there exists $C > 0$ such that $C^{-1} r_m \leq r_m - r_{m+1} \leq C r_m$ for all $m \in \mathbb{Z}$, then the implications $(b), \ldots, (e) \Rightarrow (a)$ also hold (so Theorem 2.3 remains true replacing $\mathcal{V}_\rho$ by $\mathcal{O}$), but we do not know if they are still true without this additional assumption (see Remark 6.9).

Notice that, by Theorem 2.3, besides $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ and $\mathcal{O} \circ \mathcal{R}^\mu$, the operators $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi$ and $\mathcal{O} \circ \mathcal{T}^\mu_\varphi$ for $K(x) = x/|x|^{n+1}$ characterize completely the n-AD regular measures μ which are uniformly n-rectifiable.

One of the main ingredients for the proof of Theorem 2.3 is the following result, which strengthens one of the endpoint estimates obtained in [MT]. Let $M(\mathbb{R}^d)$ be the space of finite real Radon measures on $\mathbb{R}^d$, with the norm induced by the variation of measures.

Theorem 2.5. *Let $\rho > 2$ and let μ be the n-dimensional Hausdorff measure restricted to an n-dimensional Lipschitz graph. Then, $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is a bounded operator from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$. In particular, $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi$ is of weak type $(1,1)$. The bound of the norm of this operator only depends on n, d, K, ρ, $\varphi_{\mathbb{R}}$, and the maximal slope of Γ.*

By an n-dimensional Lipschitz graph $\Gamma \subset \mathbb{R}^d$ we mean any translation and rotation of a set of the type $\{x \in \mathbb{R}^d : x = (y, A(y)),\, y \in \mathbb{R}^n\}$, where $A : \mathbb{R}^n \to \mathbb{R}^{d-n}$ is some Lipschitz function with Lipschitz constant $\mathrm{Lip}(A)$, which coincides with the maximal slope of Γ.

***Remark* 2.6.** The theorem above remains valid if one replaces $\mathcal{V}_\rho$ by $\mathcal{O}$. Moreover, the norm of $\mathcal{O} \circ \mathcal{T}^\mu_\varphi$ is bounded independently of the sequence that defines $\mathcal{O}$.

The plan to prove Theorem 2.3 is the following: in Section 3 we deal with Theorem 2.5, which is used in the subsequent Section 4 to obtain the implication $(a) \Longrightarrow (b)$ of Theorem 2.3. In Section 5 we prove $(a) \Longrightarrow (c)$ in Theorem 5.1, and in Section 6 we prove Theorem 6.8, which gives $(d) \Longrightarrow (a)$ and $(e) \Longrightarrow (a)$, and finishes the proof of Theorem 2.3, taking into account that the implications $(b) \Longrightarrow (e)$ and $(c) \Longrightarrow (d)$ are trivial.

Theorems 2.3 and 2.5 are stated in terms of $\mathcal{V}_\rho$, but they also hold for $\mathcal{O}$, as remarked above. However, we will only give the proof of these results for $\mathcal{V}_\rho$, because the case of $\mathcal{O}$ follows by very similar arguments and computations.

2.2. **Calderón-Zygmund decomposition for measures.** Given a cube $Q \subset \mathbb{R}^d$ and $a > 0$, we denote by $\ell(Q)$ the side length of Q and by aQ the cube concentric with Q with side length $a\ell(Q)$. The cubes that we consider in this paper have sides parallel to the coordinate axes in $\mathbb{R}^d$.

A proof of the following result can be found in [To5, Chapter 2] or [M, Lemma 5.1.2].

Lemma 2.7 (Calderón-Zygmund decomposition)**.** *Assume that $\mu := \mathcal{H}^n_{\Gamma\cap B}$, where Γ is an n-dimensional Lipschitz graph and $B \subset \mathbb{R}^d$ is some fixed ball. For any $\nu \in M(\mathbb{R}^d)$ with compact support and any $\lambda > 2^{d+1}\|\nu\|/\|\mu\|$, the following holds:*

(a) There exists a finite or countable collection of almost disjoint cubes $\{Q_j\}_j \subset \mathbb{R}^d$ (that is, $\sum_j \chi_{Q_j} \leq C$) and a function $f \in L^1(\mu)$ such that

$$|\nu|(Q_j) > 2^{-d-1}\lambda\mu(2Q_j), \tag{3}$$

$$|\nu|(\eta Q_j) \leq 2^{-d-1}\lambda\mu(2\eta Q_j) \quad \text{for } \eta > 2, \tag{4}$$

$$\nu = f\mu \text{ in } \mathbb{R}^d \setminus \textstyle\bigcup_j Q_j \text{ with } |f| \leq \lambda \ \mu\text{-a.e.} \tag{5}$$

(b) *For each j, let $R_j := 6Q_j$ and denote $w_j := \chi_{Q_j}\big(\sum_k \chi_{Q_k}\big)^{-1}$. Then, there exists a family of functions $\{b_j\}_j$ with* $\operatorname{supp} b_j \subset R_j$ *and with constant sign satisfying*

$$\int b_j \, d\mu = \int w_j \, d\nu, \tag{6}$$

$$\|b_j\|_{L^\infty(\mu)} \mu(R_j) \leq C|\nu|(Q_j), \text{ and} \tag{7}$$

$$\textstyle\sum_j |b_j| \leq C_0 \lambda \quad \text{(where } C_0 \text{ is some absolute constant).} \tag{8}$$

2.3. **Dyadic lattices.** For the study of the uniformly rectifiable measures we will use the "dyadic cubes" built by G. David in [Da, Appendix 1] (see also [DS2, Chapter 3 of Part I]). These dyadic cubes are not true cubes, but they play this role with respect to a given n-dimenasional AD regular Radon measure μ, in a sense. To distinguish them from the usual cubes, we will call them *μ-cubes*.

Let us explain which are the precise results and properties about the lattice of dyadic μ-cubes. Given an n-dimensional AD regular Radon measure μ in $\mathbb{R}^d$ (for simplicity, we may assume $\operatorname{diam}(\operatorname{supp}\mu) = \infty$), for each $j \in \mathbb{Z}$ there exists a family $\mathcal{D}_j$ of Borel subsets of $\operatorname{supp}\mu$ (the dyadic μ-cubes of the j-th generation) such that:

(a) each $\mathcal{D}_j$ is a partition of $\operatorname{supp}\mu$, i.e. $\operatorname{supp}\mu = \bigcup_{Q\in\mathcal{D}_j} Q$ and $Q \cap Q' = \emptyset$ whenever $Q, Q' \in \mathcal{D}_j$ and $Q \neq Q'$;
(b) if $Q \in \mathcal{D}_j$ and $Q' \in \mathcal{D}_k$ with $k \leq j$, then either $Q \subset Q'$ or $Q \cap Q' = \emptyset$;
(c) for all $j \in \mathbb{Z}$ and $Q \in \mathcal{D}_j$, we have $2^{-j} \lesssim \operatorname{diam}(Q) \leq 2^{-j}$ and $\mu(Q) \approx 2^{-jn}$;
(d) there exists $C > 0$ such that, for all $j \in \mathbb{Z}$, $Q \in \mathcal{D}_j$, and $0 < \tau < 1$,

$$\begin{aligned}\mu\big(\{x \in Q : \operatorname{dist}(x, \operatorname{supp}\mu \setminus Q) \leq \tau 2^{-j}\}\big) \\ + \mu\big(\{x \in \operatorname{supp}\mu \setminus Q : \operatorname{dist}(x, Q) \leq \tau 2^{-j}\}\big) \leq C\tau^{1/C} 2^{-jn}.\end{aligned} \tag{9}$$

This property is usually called the *small boundaries condition*. From (9), it follows that there is a point $z_Q \in Q$ (the center of Q) such that $\operatorname{dist}(z_Q, \operatorname{supp}\mu \setminus Q) \gtrsim 2^{-j}$ (see [DS2, Lemma 3.5 of Part I]).

We denote $\mathcal{D} := \bigcup_{j\in\mathbb{Z}} \mathcal{D}_j$. For $Q \in \mathcal{D}_j$, we define the side length of Q as $\ell(Q) = 2^{-j}$. Notice that $\ell(Q) \lesssim \operatorname{diam}(Q) \leq \ell(Q)$. Actually it may happen that a μ-cube Q belongs to $\mathcal{D}_j \cap \mathcal{D}_k$ with $j \neq k$. In this case, $\ell(Q)$ is not well defined. However, this problem can be solved in many ways. For example, the reader may think that a μ-cube is not only a subset of $\operatorname{supp}\mu$, but a couple (Q, j), where Q is a subset of $\operatorname{supp}\mu$ and $j \in \mathbb{Z}$ is such that $Q \in \mathcal{D}_j$.

Given $a > 1$ and $Q \in \mathcal{D}$, we set $aQ := \big\{x \in \operatorname{supp}\mu : \operatorname{dist}(x, Q) \leq (a-1)\ell(Q)\big\}$. Observe that $\operatorname{diam}(aQ) \leq \operatorname{diam}(Q) + 2(a-1)\ell(Q) \leq (2a-1)\ell(Q)$.

2.4. **Corona decomposition.** Given an n-dimensional AD regular Radon measure μ on $\mathbb{R}^d$, let $\mathcal{D} := \{Q \in \mathcal{D}_j : j \in \mathbb{Z}\}$ be the dyadic lattice associated to μ introduced in Subsection 2.3. Following [DS2, Definitions 3.13 and 3.19 of Part I], one says that μ admits a corona decomposition if, for each $\eta > 0$ and $\theta > 0$, one can find a triple $(\mathcal{B}, \mathcal{G}, \mathrm{Trs})$, where $\mathcal{B}$ and $\mathcal{G}$ are two subsets of $\mathcal{D}$ (the "bad μ-cubes" and the "good μ-cubes") and Trs is a family of subsets $S \subset \mathcal{G}$ (that we will call *trees*), which satisfy the following conditions::

(a) $\mathcal{D} = \mathcal{B} \cup \mathcal{G}$ and $\mathcal{B} \cap \mathcal{G} = \emptyset$.
(b) $\mathcal{B}$ satisfies a Carleson packing condition, i.e., $\sum_{Q\in\mathcal{B}:\, Q\subset R} \mu(Q) \lesssim \mu(R)$ for all $R \in \mathcal{D}$.
(c) $\mathcal{G} = \biguplus_{S\in\mathrm{Trs}} S$, i.e., any $Q \in \mathcal{G}$ belongs to only one $S \in \mathrm{Trs}$.
(d) Each $S \in \mathrm{Trs}$ is *coherent*. This means that each $S \in \mathrm{Trs}$ has a unique maximal element Q_S which contains all other elements of S as subsets, that $Q' \in S$ as soon as $Q' \in \mathcal{D}$ satisfies $Q \subset Q' \subset Q_S$ for some $Q \in S$, and that if $Q \in S$ then either all of

the children of Q lie in S or none of them do (if $Q \in \mathcal{D}_j$, the *children* of Q is defined as the collection of μ-cubes $Q' \in \mathcal{D}_{j+1}$ such that $Q' \subset Q$).

(e) The maximal μ-cubes Q_S, for $S \in \mathrm{Trs}$, satisfy a Carleson packing condition. That is, $\sum_{S \in \mathrm{Trs}:\, Q_S \subset R} \mu(Q_S) \lesssim \mu(R)$ for all $R \in \mathcal{D}$.

(f) For each $S \in \mathrm{Trs}$, there exists an n-dimensional Lipschitz graph Γ_S with constant smaller than η such that $\mathrm{dist}(x, \Gamma_S) \leq \theta\, \mathrm{diam}(Q)$ whenever $x \in 2Q$ and $Q \in S$ (one can replace "$x \in 2Q$" by "$x \in C_{cor} Q$" for any constant $C_{cor} \geq 2$ given in advance, by [DS2, Lemma 3.31 of Part I]).

It is shown in [DS1] (see also [DS2]) that if μ is uniformly rectifiable then it admits a corona decomposition for all parameters $k > 2$ and $\eta, \theta > 0$. Conversely, the existence of a corona decomposition for a single set of parameters $k > 2$ and $\eta, \theta > 0$ implies that μ is uniformly rectifiable.

2.5. **The α and β coefficients.** Let μ be an n-dimensional AD regular Radon measure in $\mathbb{R}^d$ and $\mathcal{D}$ as in Subsection 2.3. Given $1 \leq p < \infty$ and a μ-cube $Q \in \mathcal{D}$, one sets (see [DS2])

$$\beta_{p,\mu}(Q) = \inf_L \left\{ \frac{1}{\ell(Q)^n} \int_{2Q} \left(\frac{\mathrm{dist}(y, L)}{\ell(Q)} \right)^p d\mu(y) \right\}^{1/p},$$

where the infimum is taken over all n-planes L in $\mathbb{R}^d$. For $p = \infty$ one replaces the L^p norm by the supremum norm. The $\beta_{\infty,\mu}$ coefficients were first introduced by P. Jones in his celebrated work on rectifiability [Jn], while the $\beta_{p,\mu}$'s for $1 \leq p < \infty$ were introduced by G. David and S. Semmes in their pioneering work on uniform rectifiability (see [DS1] for example).

Other coefficients that have been proved useful in the study of uniform rectifiability and boundedness of Calderón-Zygmund operators are the α coefficients introduced in [To4]. Let $F \subset \mathbb{R}^d$ be the closure of an open set. Given two finite Radon measures σ, ν on $\mathbb{R}^d$, one sets $\mathrm{dist}_F(\sigma, \nu) := \sup\bigl\{ \bigl| \int f\, d\sigma - \int f\, d\nu \bigr| : \mathrm{Lip}(f) \leq 1, \mathrm{supp} f \subset F \bigr\}$. Finally, given a μ-cube $Q \in \mathcal{D}$, consider the closed ball $B_Q := B(z_Q, 6\sqrt{d}\ell(Q))$, where z_Q denotes the center of Q. Then one defines

$$\alpha_\mu(Q) := \frac{1}{\ell(Q)^{n+1}} \inf_{c \geq 0, L} \mathrm{dist}_{B_Q}(\mu,\, c\mathcal{H}^n_L),$$

where the infimum is taken over all constants $c \geq 0$ and all n-planes L in $\mathbb{R}^d$.

The following result characterizes the uniform rectifiability of μ in terms of the α and β coefficients (see [DS1] for $(a) \Longleftrightarrow (b)$ and [To4] for $(a) \Longleftrightarrow (c)$).

Theorem 2.8. *Let $p \in [1, 2]$ and let μ be an n-dimensional AD regular Radon measure in $\mathbb{R}^d$. The following are equivalent:*

(a) *μ is uniformly n-rectifiable.*

(b) *$\sum_{Q \in \mathcal{D}:\, Q \subset R} \beta_{p,\mu}(Q)^2 \ell(Q)^n \lesssim \ell(R)^n$ for all μ-cubes $R \in \mathcal{D}$.*

(c) *$\sum_{Q \in \mathcal{D}:\, Q \subset R} \alpha_\mu(Q)^2 \ell(Q)^n \lesssim \ell(R)^n$ for all μ-cubes $R \in \mathcal{D}$.*

For the case $\mu = \mathcal{H}^n_\Gamma$ for some Lipschitz graph $\Gamma = \{x \in \mathbb{R}^d : x = (y, A(y)),\, y \in \mathbb{R}^n\}$, one can take $\mathcal{D} = \{\widetilde{Q} \times \mathbb{R}^{d-n} \cap \Gamma : \widetilde{Q} \in \mathcal{D}(\mathbb{R}^n)\}$, where $\mathcal{D}(\mathbb{R}^n)$ denotes the standard dyadic lattice of $\mathbb{R}^n$. For $Q = (\widetilde{Q} \times \mathbb{R}^{d-n}) \cap \Gamma \in \mathcal{D}$, we set

$$\widetilde{\alpha}_\mu(Q) := \frac{1}{\ell(\widetilde{Q})^{n+1}} \inf_{c \geq 0, L} \mathrm{dist}_{6\widetilde{Q} \times \mathbb{R}^{d-n}}(\mu,\, c\mathcal{H}^n_L),$$

where the infimum is taken over all constants $c \geq 0$ and all n-planes L in $\mathbb{R}^d$. Then, it is easy to show that $\widetilde{\alpha}_\mu(Q) \approx \alpha_\mu(Q)$ for all $Q \in \mathcal{D}$.

One can also define $\widetilde{\beta}_{p,\mu}(Q)$ in an analogous manner. By Theorem 2.8,

$$\sum_{Q\in\mathcal{D}:\,Q\subset R} (\widetilde{\beta}_{p,\mu}(Q)^2 + \widetilde{\alpha}_\mu(Q)^2)\ell(Q)^n \leq C\ell(R)^n \tag{10}$$

for all $R \in \mathcal{D}$, with C independent of R. Moreover, one can also show that this last inequality also holds replacing Q and R by k_1Q and k_2R for any $k_1, k_2 \geq 1$ given in advance, where $kQ := (k\widetilde{Q} \times \mathbb{R}^{d-n}) \cap \Gamma$ for $k > 0$.

3. If Γ is an n-dimensional Lipschitz graph, then $\mathcal{V}_\rho \circ \mathcal{T}_\varphi : M(\mathbb{R}^d) \to L^{1,\infty}(\mathcal{H}^n_\Gamma)$ is a bounded operator

The following result is contained in [MT, Theorem 1.1] (see also [M, Main Theorem 3.0.1]).

Theorem 3.1. *Let $\rho > 2$ and let μ be the n-dimensional Hausdorff measure restricted to an n-dimensional Lipschitz graph. Then, the operator $\mathcal{V}_\rho \circ \mathcal{T}^\mu_{\widetilde{\varphi}}$ is bounded in $L^2(\mu)$. The bound of the norm only depends on n, d, K, ρ, $\varphi_{\mathbb{R}}$, and the slope of the graph.*

By very similar techniques to the ones used in the proof of the theorem above, one can prove the following.

Theorem 3.2. *Let $\rho > 2$ and let μ be the n-dimensional Hausdorff measure restricted to an n-dimensional Lipschitz graph. Then, the operator $\mathcal{V}_\rho \circ \mathcal{T}^\mu_{\varphi}$ is bounded in $L^2(\mu)$. The bound of the norm only depends on n, d, K, ρ, $\varphi_{\mathbb{R}}$, and the slope of the graph.*

Sketch of the proof. The first step consists in obtaining the following basic estimate: Fix a cube $\widetilde{P} \subset \mathbb{R}^n$. Set $\Gamma := \{x \in \mathbb{R}^d : x = (y, A(y)),\, y \in \mathbb{R}^n\}$, where $A : \mathbb{R}^n \to \mathbb{R}^{d-n}$ is a Lipschitz function supported in $\widetilde{P}$, and set $P := (\widetilde{P} \times \mathbb{R}^{d-n}) \cap \Gamma$. Set $\mu := f\mathcal{H}^n_\Gamma$, where $f(x) = 1$ for all $x \in \Gamma \setminus P$ and $C_0^{-1} \leq f(x) \leq C_0$ for all $x \in P$, for some constant $C_0 > 0$.

For each $x \in \Gamma$, define

$$W\mu(x)^2 := \sum_{m\in\mathbb{Z}} |(K\varphi_{2^{-m}} * \mu)(x) - (K\widetilde{\varphi}_{2^{-m}} * \mu)(x)|^2. \tag{11}$$

and

$$S\mu(x)^2 := \sup_{\{\epsilon_m\}} \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}:\,\epsilon_m,\epsilon_{m+1}\in I_j} |(K\varphi^{\epsilon_m}_{\epsilon_{m+1}} * \mu)(x)|^2, \tag{12}$$

where $I_j = [2^{-j-1}, 2^{-j})$ and the supremum is taken over all decreasing sequences of positive numbers $\{\epsilon_m\}_{m\in\mathbb{Z}}$. Then, we claim that

$$\|W\mu\|^2_{L^2(\mu)} + \|S\mu\|^2_{L^2(\mu)} \lesssim \sum_{Q\in\mathcal{D}} \big(\,\widetilde{\alpha}_\mu(C_1Q)^2 + \widetilde{\beta}_{2,\mu}(Q)^2\,\big)\ell(Q)^n, \tag{13}$$

where $C_1 > 0$ only depends on C_0, n, d, K, $\varphi_{\mathbb{R}}$, and $\mathrm{Lip}(A)$, and where $\mathcal{D}$ denotes the dyadic lattice associated to $\mathcal{H}^n_\Gamma$ defined below Theorem 2.8.

Let us prove the claim. If we define $\widetilde{S}\mu$ like $S\mu$ but replacing $\varphi^{\epsilon_m}_{\epsilon_{m+1}}$ by $\widetilde{\varphi}^{\epsilon_m}_{\epsilon_{m+1}}$, in the proof of Theorem 3.1 in [MT] it is shown that $\|\widetilde{S}\mu\|^2_{L^2(\mu)}$ is bounded above by the right hand side of (13). The proof for $\|S\mu\|^2_{L^2(\mu)}$ is almost the same.

Let us deal now with $W\mu$. Fix $D := (\widetilde{D} \times \mathbb{R}^{d-n}) \cap \Gamma \in \mathcal{D}$ with $\ell(D) = 2^{-m}$ and $x \in D$. Let L_D be an n-plane that minimizes $\widetilde{\alpha}_\mu(C_1D)$, where $C_1 > 0$ is some constant big enough which will be fixed later, and let $\sigma_D := c_D\mathcal{H}^n_{L_D}$ be a minimizing measure for $\widetilde{\alpha}_\mu(C_1D)$. Let L^x_D be the n-plane parallel to L_D which contains x, and set $\sigma^x_D := c_D\mathcal{H}^n_{L^x_D}$.

Since $x \in D$ and $\ell(D) = 2^{-m}$, $(\varphi_{2^{-m}}(x-\cdot) - \widetilde{\varphi}_{2^{-m}}(x-\cdot))K(x-\cdot)$ is a function supported in $C_1\widetilde{D} \times \mathbb{R}^{d-n}$ (for some constant C_1 big enough) and with Lipschitz constant smaller than $C2^{m(n+1)}$. Moreover, by the antisymmetry of the function $(\varphi_{2^{-m}}(x-\cdot)-\widetilde{\varphi}_{2^{-m}}(x-\cdot))K(x-\cdot)$, and since σ_D^x is a multiple of the n-dimensional Hausdorff measure on an n-plane which contains x, we have $(K\varphi_{2^{-m}} * \sigma_D^x)(x) - (K\widetilde{\varphi}_{2^{-m}} * \sigma_D^x)(x) = 0$. Therefore,

$$\begin{aligned}(K\varphi_{2^{-m}} * \mu)(x) - (K\widetilde{\varphi}_{2^{-m}} * \mu)(x) &= (K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * \mu)(x)\\ &= (K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * (\mu - \sigma_D))(x) + (K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * (\sigma_D - \sigma_D^x))(x).\end{aligned} \tag{14}$$

Using the definition of $\widetilde{\alpha}_\mu$, we get

$$|(K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * (\mu - \sigma_D))(x)| \lesssim 2^{m(n+1)}\mathrm{dist}_{C_1\widetilde{D}\times\mathbb{R}^{d-n}}(\mu, \sigma_D) \lesssim \widetilde{\alpha}_\mu(C_1 D). \tag{15}$$

Since L_D^x is a translation of L_D, by standard estimates it is not hard to show that

$$|(K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * (\sigma_D - \sigma_D^x))(x)| \lesssim 2^m \mathrm{dist}(x, L_D) = \mathrm{dist}(x, L_D)/\ell(D). \tag{16}$$

Let $\mathrm{dist}_{\mathcal{H}}(E, F)$ denote the Hausdorff distance of two given sets $E, F \subset \mathbb{R}^d$, and set $\widetilde{B}_D := 6\widetilde{D}\times\mathbb{R}^{d-n}$. If L_D^1 and L_D^2 denote a minimizing n-plane for $\widetilde{\beta}_{1,\mu}(D)$ and $\widetilde{\beta}_{2,\mu}(D)$, respectively, one can show that $\mathrm{dist}_{\mathcal{H}}(L_D \cap \widetilde{B}_D, L_D^1 \cap \widetilde{B}_D) \lesssim \widetilde{\alpha}_\mu(D)\ell(D)$ and that $\mathrm{dist}_{\mathcal{H}}(L_D^1 \cap \widetilde{B}_D, L_D^2 \cap \widetilde{B}_D) \lesssim \widetilde{\beta}_{2,\mu}(D)\ell(D)$. This easily implies that $\mathrm{dist}(x, L_D) \lesssim \mathrm{dist}(x, L_D^2) + \widetilde{\beta}_{2,\mu}(D)\ell(D) + \widetilde{\alpha}_\mu(D)\ell(D)$ for all $x \in D$. Applying this to (16), and using also (15) and (14), we obtain

$$\begin{aligned}\|W\mu\|_{L^2(\mu)}^2 &= \int \sum_{m\in\mathbb{Z}} |(K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * \mu)(x)|^2 \, d\mu(x)\\ &= \sum_{m\in\mathbb{Z}} \sum_{D\in\mathcal{D}:\,\ell(D)=2^{-m}} \int_D |(K(\varphi_{2^{-m}} - \widetilde{\varphi}_{2^{-m}}) * \mu)(x)|^2 \, d\mu(x)\\ &\lesssim \sum_{m\in\mathbb{Z}} \sum_{D\in\mathcal{D}:\,\ell(D)=2^{-m}} \int_D \big(\mathrm{dist}(x, L_D^2)/\ell(D) + \widetilde{\beta}_{2,\mu}(D) + \widetilde{\alpha}_\mu(C_1 D)\big)^2 \, d\mu(x)\\ &\lesssim \sum_{D\in\mathcal{D}} \big(\widetilde{\alpha}_\mu(C_1 D)^2 + \widetilde{\beta}_{2,\mu}(D)^2\big)\ell(D)^n,\end{aligned}$$

which proves (13).

Let now μ be as in Theorem 3.2. Using (13) and Theorem 3.1, one can show that there exists $C > 0$ such that, for any cube $\widetilde{D} \subset \mathbb{R}^n$ and any $g \in L^\infty(\mu)$ supported in D (where $D := \widetilde{D} \times \mathbb{R}^{d-n}$),

$$\int_D \big((\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu)g\big)^2 \, d\mu \le C\|g\|_{L^\infty(\mu)}^2 \mu(D).$$

This yields the endpoint estimates $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu : H^1(\mu) \to L^1(\mu)$ and $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu : L^\infty(\mu) \to BMO(\mu)$, where $H^1(\mu)$ denotes the atomic Hardy space related to μ. Then, by interpolation, one finally deduces that $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ is bounded in $L^2(\mu)$. Since this part of the proof is analogous to the one in the proof of Theorem 3.1 (see [MT, Theorem 1.1]), we omit it. □

3.1. **Proof of Theorem 2.5.** The proof of Theorem 2.5 uses the Calderón-Zygmund decomposition of Lemma 2.7 and rather standard arguments. Set $\mu := \mathcal{H}^n_{\Gamma\cap B}$, where is some fixed ball $B \subset \mathbb{R}^d$. Let $\nu \in M(\mathbb{R}^d)$ be a finite Radon measure with compact support and $\lambda > 2^{d+1}\|\nu\|/\|\mu\|$. We will show that

$$\mu\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > \lambda\}\big) \le \frac{C}{\lambda}\|\nu\|, \tag{17}$$

where $C > 0$ depends on n, d, K, ρ and Γ, but not on B. Let us check that this implies that $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mathcal{H}^n_\Gamma)$. First, we show that (17) also holds for ν without compact support. Set $\nu_N = \chi_{B(0,N)}\,\nu$ and let N_0 be such that $\operatorname{supp}\mu \subset B(0,N_0)$. Then it is not hard to show that, for $x \in \operatorname{supp}\mu$,

$$|(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_N(x)| \leq C\,\frac{|\nu|(\mathbb{R}^d \setminus B(0,N))}{N - N_0},$$

thus $(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_N(x) \to (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x)$ for all $x \in \operatorname{supp}\mu$, and since the estimate (17) holds by assumption for ν_N, letting $N \to \infty$, we deduce that it also holds for ν. Now, by increasing the size of the ball B and by monotone convergence, we deduce that $\mathcal{H}^n_\Gamma\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > \lambda\}\big) \leq C\lambda^{-1}\|\nu\|$, as desired.

To prove (17) for $\nu \in M(\mathbb{R}^d)$ with compact support, let $\{Q_j\}_j$ be the almost disjoint family of cubes of Lemma 2.7, and set $\Omega := \bigcup_j Q_j$ and $R_j := 6Q_j$. Then we can write $\nu = g\mu + \nu_b$, with

$$g\mu = \chi_{\mathbb{R}^d\setminus\Omega}\nu + \sum_j b_j\mu \quad \text{and} \quad \nu_b = \sum_j \nu_b^j := \sum_j (w_j\nu - b_j\mu),$$

where the functions b_j satisfy (6), (7), (8) and $w_j = \chi_{Q_j}\big(\sum_k \chi_{Q_k}\big)^{-1}$.

By the subadditivity of $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$, we have

$$\begin{aligned}\mu\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > \lambda\}\big)\\ \leq \mu\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi)g(x) > \lambda/2\}\big) + \mu\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b(x) > \lambda/2\}\big).\end{aligned} \tag{18}$$

Since $\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}^n_\Gamma}_\varphi$ is bounded in $L^2(\mathcal{H}^n_\Gamma)$ by Theorem 3.2, it is easy to show that $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi$ is bounded in $L^2(\mu)$, with a bound independent of B. Notice that $|g| \leq C\lambda$ by (5) and (8). Then, using (7),

$$\begin{aligned}\mu\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi)g(x) > \lambda/2\}\big) &\lesssim \frac{1}{\lambda^2}\int |(\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi)g|^2\,d\mu \lesssim \frac{1}{\lambda^2}\int |g|^2\,d\mu\\ &\lesssim \frac{1}{\lambda}\int |g|\,d\mu \lesssim \frac{1}{\lambda}\left(|\nu|(\mathbb{R}^d\setminus\Omega) + \sum_j \int_{R_j} |b_j|\,d\mu\right)\\ &\lesssim \frac{1}{\lambda}\left(|\nu|(\mathbb{R}^d\setminus\Omega) + \sum_j |\nu|(Q_j)\right) \lesssim \frac{\|\nu\|}{\lambda}.\end{aligned} \tag{19}$$

Let $\widehat{\Omega} := \bigcup_j 2Q_j$. By (3), we have $\mu(\widehat{\Omega}) \leq \sum_j \mu(2Q_j) \lesssim \lambda^{-1}\sum_j |\nu|(Q_j) \lesssim \lambda^{-1}\|\nu\|$. We are going to show now that

$$\mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b(x) > \lambda/2\}\big) \leq \frac{C}{\lambda}\,\|\nu\|, \tag{20}$$

and then (17) is a direct consequence of (18), (19), (20) and the estimate $\mu(\widehat{\Omega}) \lesssim \lambda^{-1}\|\nu\|$. Since $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is sublinear,

$$\begin{aligned}\mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} : (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b(x) > \lambda/2\}\big) &\lesssim \frac{1}{\lambda}\sum_j \int_{\mathbb{R}^d\setminus\widehat{\Omega}} (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b^j\,d\mu\\ &\leq \frac{1}{\lambda}\sum_j \int_{\mathbb{R}^d\setminus 2R_j} (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b^j\,d\mu + \frac{1}{\lambda}\sum_j \int_{2R_j\setminus 2Q_j} (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b^j\,d\mu.\end{aligned} \tag{21}$$

We are going to estimate the two terms on the right of (21) separately. Let us start with the first one. Given j and $x \in \operatorname{supp}\mu \setminus 2R_j$, let $\{\epsilon_m\}_{m\in\mathbb{Z}}$ be a decreasing sequence of positive

numbers (which depends on j and x, i.e. $\epsilon_m \equiv \epsilon_m(j,x)$) such that

$$(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu_b^j(x) \leq 2\bigg(\sum_{m\in\mathbb{Z}} |(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x)|^\rho\bigg)^{1/\rho}. \tag{22}$$

If we set $I_k := [2^{-k-1}, 2^{-k})$, we can decompose $\mathbb{Z} = \mathcal{S} \cup \mathcal{L}$, where

$$\mathcal{L} := \{m \in \mathbb{Z} : \epsilon_m \in I_k,\ \epsilon_{m+1} \in I_i,\ \text{for } i > k\},$$
$$\mathcal{S} := \bigcup_{k\in\mathbb{Z}} \mathcal{S}_k, \quad \mathcal{S}_k := \{m \in \mathbb{Z} : \epsilon_m, \epsilon_{m+1} \in I_k\}.$$

Let z_j denote the center of Q_j (and of R_j). Then, since $\nu_b^j(R_j) = 0$ and $\mathrm{supp}\nu_b^j \subset R_j$,

$$\begin{aligned} |(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x)| &= \left| \int \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)K(x-y)\,d\nu_b^j(y)\right| \\ &\leq \int \left|\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)K(x-y) - \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-z_j)K(x-z_j)\right| d|\nu_b^j|(y). \end{aligned} \tag{23}$$

If $m \in \mathcal{L}$, it is easy to see that $|\nabla(\varphi_{\epsilon_{m+1}}^{\epsilon_m}K)(t)| \leq |\nabla(\varphi_{\epsilon_{m+1}}K)(t)| + |\nabla(\varphi_{\epsilon_m}K)(t)| \lesssim |t|^{-n-1}$ for all $t \in \mathbb{R}^d \setminus \{0\}$. Moreover, since $x \in \mathbb{R}^d \setminus 2R_j$ and $\mathrm{supp}\nu_b^j \subset R_j$, there are finitely many $m \in \mathcal{L}$ (which depends only on n and d) such that $(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x) \neq 0$, and this number only depends on n and d. On the other hand, if $m \in \mathcal{S}_k$, it is not hard to show that $|\nabla(\varphi_{\epsilon_{m+1}}^{\epsilon_m}K)(t)| \lesssim 2^k|\epsilon_m - \epsilon_{m+1}||t|^{-n-1}$. Actually, this follows from the fact that $(\varphi_{\epsilon_{m+1}}^{\epsilon_m}K)(t) \neq 0$ only if $|t| \approx 2^{-k}$ and the estimates

$$\begin{aligned} |\varphi_{\epsilon_{m+1}}^{\epsilon_m}(t)| &= \left|\varphi_{\mathbb{R}}\left(\frac{|t|}{\epsilon_{m+1}}\right) - \varphi_{\mathbb{R}}\left(\frac{|t|}{\epsilon_m}\right)\right| \leq \|\varphi_{\mathbb{R}}'\|_{L^\infty(\mathbb{R})}\left|\frac{|t|}{\epsilon_{m+1}} - \frac{|t|}{\epsilon_m}\right| \\ &= \|\varphi_{\mathbb{R}}'\|_\infty |t|\,\frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_m\epsilon_{m+1}} \lesssim 2^k|\epsilon_m - \epsilon_{m+1}| \end{aligned} \tag{24}$$

and

$$\begin{aligned} \left|\partial_{t^i}\big(\varphi_{\epsilon_{m+1}}^{\epsilon_m}(t)\big)\right| &\leq \left|\varphi_{\mathbb{R}}'\left(\frac{|t|}{\epsilon_m}\right)\frac{1}{\epsilon_m} - \varphi_{\mathbb{R}}'\left(\frac{|t|}{\epsilon_{m+1}}\right)\frac{1}{\epsilon_{m+1}}\right| \\ &\leq \left|\varphi_{\mathbb{R}}'\left(\frac{|t|}{\epsilon_m}\right)\right|\left|\frac{1}{\epsilon_m} - \frac{1}{\epsilon_{m+1}}\right| + \left|\varphi_{\mathbb{R}}'\left(\frac{|t|}{\epsilon_m}\right) - \varphi_{\mathbb{R}}'\left(\frac{|t|}{\epsilon_{m+1}}\right)\right|\frac{1}{\epsilon_{m+1}} \\ &\leq \left(\|\varphi_{\mathbb{R}}'\|_\infty + \|\varphi_{\mathbb{R}}''\|_\infty\frac{|t|}{\epsilon_{m+1}}\right)\frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_m\epsilon_{m+1}} \lesssim 2^k(\epsilon_m - \epsilon_{m+1})|t|^{-1}, \end{aligned} \tag{25}$$

where $1 \leq i \leq d$ and t^i denotes the i'th coordinate of $t \in \mathbb{R}^d$ (recall that $\epsilon_m \approx \epsilon_{m+1} \approx 2^{-k}$ for $m \in \mathcal{S}_k$ and we assumed $|t| \approx 2^{-k}$). Similarly to the case $m \in \mathcal{L}$, there are finitely many $k \in \mathbb{Z}$ such that $\mathrm{supp}\varphi_{2^{-k-1}}^{2^{-k}}(x-\cdot) \cap R_j \neq \emptyset$, and the number only depends on n and d (notice that $\mathrm{supp}\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-\cdot) \subset \mathrm{supp}\varphi_{2^{-k-1}}^{2^{-k}}(x-\cdot)$ for all $m \in \mathcal{S}_k$).

From these estimates and remarks, and (22), (23), we obtain

$$\begin{aligned}(\mathcal{V}_\rho\circ\mathcal{T}_\varphi)\nu_b^j(x) &\lesssim \sum_{k\in\mathbb{Z}}\sum_{m\in\mathcal{S}_k}|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m}*\nu_b^j)(x)| + \sum_{m\in\mathcal{L}}|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m}*\nu_b^j)(x)|\\ &\lesssim \sum_{k\in\mathbb{Z}:\,\operatorname{supp}\varphi_{2^{-k-1}}^{2^{-k}}(x-\cdot)\cap R_j\neq\emptyset}\ \sum_{m\in\mathcal{S}_k} 2^k|\epsilon_m-\epsilon_{m+1}||x-z_j|^{-n-1}\ell(R_j)\|\nu_b^j\|\\ &\quad+\sum_{m\in\mathcal{L}:\,\operatorname{supp}\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-\cdot)\cap R_j\neq\emptyset}|x-z_j|^{-n-1}\ell(R_j)\|\nu_b^j\|\lesssim |x-z_j|^{-n-1}\ell(R_j)\|\nu_b^j\|\end{aligned}$$

for all j and $x\in\operatorname{supp}\mu\setminus 2R_j$. Therefore, using that μ has n-dimensional growth, that $\|\nu_b^j\|\lesssim|\nu|(Q_j)$, and that the Q_j's are semidisjoint,

$$\sum_j\int_{\mathbb{R}^d\setminus 2R_j}(\mathcal{V}_\rho\circ\mathcal{T}_\varphi)\nu_b^j\,d\mu\lesssim\sum_j\ell(R_j)\|\nu_b^j\|\int_{\mathbb{R}^d\setminus 2R_j}|x-z_j|^{-n-1}\,d\mu\lesssim\sum_j\|\nu_b^j\|\lesssim\|\nu\|. \tag{26}$$

Let us now estimate the second term on the right hand side of (21). As above, given j and $x\in 2R_j\setminus 2Q_j$, let $\{\epsilon_m\}_{m\in\mathbb{Z}}$ be a decreasing sequence of positive numbers such that

$$(\mathcal{V}_\rho\circ\mathcal{T}_\varphi)(w_j\nu)(x)\leq 2\Big(\sum_{m\in\mathbb{Z}}|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m}*(w_j\nu))(x)|^\rho\Big)^{1/\rho},$$

where $w_j=\chi_{Q_j}\big(\sum_k\chi_{Q_k}\big)^{-1}$. Since $\rho>2$, $\mathcal{V}_\rho\circ\mathcal{T}_\varphi$ is sublinear, and since $\nu_b^j=w_j\nu-b_j\mu$, for $x\in 2R_j\setminus 2Q_j$ we have

$$\begin{aligned}(\mathcal{V}_\rho\circ\mathcal{T}_\varphi)\nu_b^j(x)&\leq(\mathcal{V}_\rho\circ\mathcal{T}_\varphi)(w_j\nu)(x)+(\mathcal{V}_\rho\circ\mathcal{T})(b_j\mu)(x)\\ &\leq 2\sum_{m\in\mathbb{Z}}\big|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m}*(w_j\nu))(x)\big|+(\mathcal{V}_\rho\circ\mathcal{T}_\varphi^\mu)b_j(x)\\ &\lesssim|\nu|(Q_j)|x-z_j|^{-n}+(\mathcal{V}_\rho\circ\mathcal{T}_\varphi^\mu)b_j(x).\end{aligned}$$

Since $\mathcal{V}_\rho\circ\mathcal{T}_\varphi^\mu$ is bounded in $L^2(\mu)$, using the estimate above and Cauchy-Schwarz we get

$$\begin{aligned}\sum_j\int_{2R_j\setminus 2Q_j}(\mathcal{V}_\rho\circ\mathcal{T}_\varphi)\nu_b^j\,d\mu&\lesssim\sum_j\int_{2R_j\setminus 2Q_j}\frac{|\nu|(Q_j)}{|x-z_j|^n}\,d\mu(x)+\sum_j\int_{2R_j\setminus 2Q_j}(\mathcal{V}_\rho\circ\mathcal{T}_\varphi^\mu)b_j\,d\mu\\ &\lesssim\sum_j|\nu|(Q_j)\frac{\mu(2R_j)}{\ell(Q_j)^n}+\sum_j\|(\mathcal{V}_\rho\circ\mathcal{T}_\varphi^\mu)b_j\|_{L^2(\mu)}\mu(2R_j)^{1/2}\\ &\lesssim\sum_j|\nu|(Q_j)+\sum_j\|b_j\|_{L^\infty(\mu)}\mu(R_j)\lesssim\sum_j|\nu|(Q_j)\lesssim\|\nu\|.\end{aligned}$$

Together with (26) and (21), this proves (20), and Theorem 2.5 follows.

4. If μ is a uniformly n-rectifiable measure, then $\mathcal{V}_\rho\circ\mathcal{T}_\varphi^\mu: L^p(\mu)\to L^p(\mu)$ is a bounded operator for $1<p<\infty$

The purpose of this section consists in proving the following theorem and the subsequent corollary.

Theorem 4.1. *Let μ be an n-dimensional AD regular Radon measure in $\mathbb{R}^d$ and let $\rho>2$. Assume that there exist constants C_0 and C_1 such that, for each ball B centered on $\operatorname{supp}\mu$, there is a set $F=F_B$ such that:*

(a) $\mu(F\cap B)\geq C_0\mu(B)$,

(b) $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ *is bounded from* $M(\mathbb{R}^d)$ *to* $L^{1,\infty}(\mathcal{H}^n_F)$ *with constant bounded by* C_1.

Then $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ *is bounded from* $M(\mathbb{R}^d)$ *to* $L^{1,\infty}(\mu)$, *and* $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi$ *is a bounded operator in* $L^p(\mu)$ *for all* $1 < p < \infty$.

Corollary 4.2. *If* μ *is an* n*-dimensional AD regular uniformly* n*-rectifiable measure, then* $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi$ *is a bounded operator in* $L^p(\mu)$ *for all* $1 < p < \infty$ *and* $\rho > 2$. *Moreover, the operator* $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ *is bounded from* $M(\mathbb{R}^d)$ *to* $L^{1,\infty}(\mu)$, *so* $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi$ *is also of weak type* $(1,1)$.

Proof. Recall from [DS2, Definition 1.26] that a Radon measure ν in $\mathbb{R}^d$ has $BPLG$ (*big pieces of Lipschitz graphs*) if ν is n-dimensional AD regular and if there exist constants $C_1 > 0$ and $\theta > 0$ such that, for any $x \in \text{supp}\nu$ and $0 < r < \text{diam}(\text{supp}\nu)$, there is (a rotation and translation of) an n-dimensional Lipschitz graph Γ with constant less than C_1 such that $\nu(\Gamma \cap B(x,r)) \geq \theta r^n$. Thus, if ν has $BPLG$, the assumption (a) of Theorem 4.1 is satisfied for ν by taking $F = \Gamma$, while Theorem 2.5 implies that the assumption (b) holds with a uniform constant. Therefore, from Theorem 4.1 we deduce that, if ν has $BPLG$ and $\rho > 2$, then $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\nu)$.

Similarly, a measure ν has $(BP)^2LG$ (*big pieces of big pieces of Lipschitz graphs*) if there exist constants C_g, θ, and $0 < \alpha \leq 1$ so that, if B is any ball centered on $\text{supp}\nu$, then there is an n-dimensional AD regular set $F \subset \mathbb{R}^d$ (with constant bounded by C_g) such that $\nu(F \cap B) \geq \alpha\nu(B)$ and such that $\mathcal{H}^n_F$ has $BPLG$ with uniform constants. So $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is a bounded operator from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mathcal{H}^n_F)$, by the comments above. Hence, we can apply once again Theorem 4.1 to ν (now (b) is satisfied for the big pieces F of ν), and we deduce that, for any measure ν which has $(BP)^2LG$, $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\nu)$. Similar arguments yield that $\mathcal{V}_\rho \circ \mathcal{T}^\nu_\varphi$ is a bounded operator in $L^p(\nu)$ for all $1 < p < \infty$.

Finally, from [DS2, page 22] and the remark given in [DS2, page 16], we know that if μ is n-dimensional AD regular, then being uniformly n-rectifiable is equivalent to having $(BP)^2LG$. Therefore, the corollary is proved by applying the comments above to $\nu = \mu$. □

Since the arguments for proving Theorem 4.1 are more or less standard in Calderón-Zygmund theory, for the sake of shortness we will only sketch its proof (see [To5, Chapter 2] or [DS2, Proposition 1.28 of Part I] for a similar argument).

***Sketch of the proof of* Theorem 4.1.** The proof follows by the so-called *good* λ *inequality* method. Fix $\rho > 2$ and let M^μ denote the Hardy-Littlewood maximal operator

$$M^\mu\nu(x) := \sup_{r>0} \frac{|\nu|(B(x,r))}{\mu(B(x,r))}, \qquad \text{for } \nu \in M(\mathbb{R}^d) \text{ and } x \in \text{supp}\mu.$$

The good λ *inequality*: one shows that there exists some absolute constant $\eta > 0$ such that for all $\epsilon > 0$ there exists $\delta := \delta(\epsilon) > 0$ such that

$$\begin{aligned} \mu\big(\{x \in \mathbb{R}^d \,:\, (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > (1+\epsilon)\lambda,\ M^\mu\nu(x) \leq \delta\lambda\}\big) \\ \leq (1-\eta)\mu\big(\{x \in \mathbb{R}^d \,:\, (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)\nu(x) > \lambda\}\big) \end{aligned} \tag{27}$$

for all $\lambda > 0$ and $\nu \in M(\mathbb{R}^d)$. It is easy to check that this implies that $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$, and that $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi$ is bounded in $L^p(\mu)$ for all $1 < p < \infty$, by standard arguments (recall that M^μ is bounded in these spaces).

The proof of (27) is quite standard. The interested reader may look at [M, Theorem 5.2.1] for the detailed proof, or to [To5, Chapter 2] for similar arguments. The only point that we should mention is that, in order to pursue the good λ inequality method, one needs the following estimate: let $\nu \in M(\mathbb{R}^d)$, consider a ball $B \subset \mathbb{R}^d$ and take $x, z \in B$. Then,

$$\big|(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2B}\nu)(x) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d \setminus 2B}\nu)(z)\big| \lesssim M^\mu\nu(x). \tag{28}$$

We finish the sketch of the proof of Theorem 4.1 by showing (28). Since $x, z \in B$ and $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$ is sublinear and positive, by the mean value theorem,

$$\begin{aligned}
&\big|(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d\setminus 2B}\nu)(x) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d\setminus 2B}\nu)(z)\big| \\
&\qquad \leq \sup_{\epsilon_m} \left(\sum_{m\in\mathbb{Z}} |(K\varphi^{\epsilon_m}_{\epsilon_{m+1}} * (\chi_{\mathbb{R}^d\setminus 2B}\nu))(x) - (K\varphi^{\epsilon_m}_{\epsilon_{m+1}} * (\chi_{\mathbb{R}^d\setminus 2B}\nu))(z)|^\rho \right)^{1/\rho} \\
&\qquad \leq \sup_{\epsilon_m} \left(\sum_{m\in\mathbb{Z}} \left(\int_{B_m(x,z)} |\nabla(\varphi^{\epsilon_m}_{\epsilon_{m+1}} K)(u_{x,z}(y)-y)||x-z|\,d|\nu|(y)\right)^\rho\right)^{1/\rho},
\end{aligned} \tag{29}$$

where $B_m(x,z) := (\mathbb{R}^d \setminus 2B) \cap (\mathrm{supp}\varphi^{\epsilon_m}_{\epsilon_{m+1}}(x-\cdot) \cup \mathrm{supp}\varphi^{\epsilon_m}_{\epsilon_{m+1}}(z-\cdot))$ and $u_{x,z}(y)$ is some point lying on the segment joining x and z. For each x and z, let $\epsilon_m \equiv \epsilon_m(x,z)$ be a sequence that realizes the supremum in the right hand side of (29). Given $\epsilon_m > 0$, let $j(\epsilon_m)$ denote the integer such that $\epsilon_m \in [2^{-j(\epsilon_m)-1}, 2^{-j(\epsilon_m)})$. For $j \in \mathbb{Z}$ set $I_j := [2^{-j-1}, 2^{-j})$. As usual, we decompose $\mathbb{Z} = \mathcal{S} \cup \mathcal{L}$, where

$$\begin{aligned}
&\mathcal{S} := \bigcup_{j\in\mathbb{Z}} \mathcal{S}_j, \quad \mathcal{S}_j := \{m \in \mathbb{Z} : \epsilon_m, \epsilon_{m+1} \in I_j\}, \\
&\mathcal{L} := \{m \in \mathbb{Z} : \epsilon_m \in I_i,\, \epsilon_{m+1} \in I_j \text{ for } i < j\}.
\end{aligned}$$

Notice that if $2^{-j+2} < r(B)$, where $r(B)$ denotes the radius of B, then $B_m(x.z) = \emptyset$ for all $m \in \mathcal{S}_j$. Therefore, we can assume that $j \leq \log_2(4/r(B))$. If $m \in \mathcal{S}_j$, then $B_m(x,z) \subset B(x, 2^{-j+3})$, and for $t \in \mathrm{supp}(\varphi^{\epsilon_m}_{\epsilon_{m+1}} K)$ we have that $|\nabla(\varphi^{\epsilon_m}_{\epsilon_{m+1}} K)(t)| \lesssim 2^{j(n+2)}|\epsilon_m - \epsilon_{m+1}|$ (see (24) and (25)). If $m \in \mathcal{L}$, we easily have $|\nabla(\varphi^{\epsilon_m}_{\epsilon_{m+1}} K)(t)| \lesssim |t|^{-n-1}$. Therefore, using (29), that $\rho > 2$, that the sets $B_m(x,z)$ have bounded overlap for $m \in \mathcal{L}$, and that $|x-z| \lesssim r(B)$, we get

$$\begin{aligned}
&\big|(\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d\setminus 2B}\nu)(x) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi)(\chi_{\mathbb{R}^d\setminus 2B}\nu)(z)\big| \\
&\quad \lesssim \sum_{j\leq \log_2(4/r(B))} \sum_{m\in\mathcal{S}_j} |x-z| 2^{j(n+2)} |\epsilon_m - \epsilon_{m+1}| \int_{B(x,2^{-j+3})} d|\nu|(y) \\
&\qquad + |x-z| \sum_{m\in\mathcal{L}} \int_{B_m(x,z)} |x-y|^{-n-1}\, d|\nu|(y) \\
&\quad \lesssim \sum_{j\leq \log_2(4/r(B))} r(B) 2^{j(n+1)} \int_{B(x,2^{-j+3})} d|\nu|(y) + r(B) \int_{\mathbb{R}^d\setminus 2B} \frac{d|\nu|(y)}{|x-y|^{n+1}} \\
&\quad \lesssim \sum_{j\leq \log_2(4/r(B))} \frac{r(B)2^j}{\mu(B(x,2^{-j+3}))} \int_{B(x,2^{-j+3})} d|\nu|(y) \\
&\qquad + r(B) \sum_{k\geq 1} \int_{2^{k+2}r(B)\geq |x-y|\geq 2^{k-1}r(B)} \frac{d|\nu|(y)}{|x-y|^{n+1}} \\
&\quad \lesssim M^\mu\nu(x) + \sum_{k\geq 1} \frac{2^{-k}}{\mu(B(x,2^{k+2}r(B_i)))} \int_{B(x,2^{k+2}r(B))} d|\nu|(y) \lesssim M^\mu\nu(x).
\end{aligned}$$

□

***Remark* 4.3.** Notice that, to prove (28), it is a key fact that we are considering smooth truncations (given by $\varphi_\mathbb{R}$) in the definition of $\mathcal{T}_\varphi$. These computations are no longer valid if one replaces $\mathcal{T}_\varphi$ by $\mathcal{T}$.

5. If μ is a uniformly n-rectifiable measure, then $\mathcal{V}_\rho \circ \mathcal{T}^\mu : L^2(\mu) \to L^2(\mu)$ is a bounded operator

This section is devoted to the proof of the following result.

Theorem 5.1. *Let $\rho > 2$ and let μ be an n-dimensional AD regular Radon measure on $\mathbb{R}^d$. If μ is uniformly n-rectifiable, then $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ is a bounded operator in $L^2(\mu)$.*

5.1. **Short and long variation.** Given $j \in \mathbb{Z}$, set $I_j := [2^{-j-1}, 2^{-j})$. Then, using the triangle inequality, we can split the variation operator into the so-called short variation and long variation operators, i.e., $(\mathcal{V}_\rho \circ \mathcal{T}^\mu)f(x) \leq (\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f(x) + (\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu)f(x)$, where

$$
\begin{aligned}
(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f(x) &:= \sup_{\{\epsilon_m\}} \left(\sum_{j\in\mathbb{Z}} \sum_{\epsilon_m, \epsilon_{m+1} \in I_j} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^\rho \right)^{1/\rho}, \\
(\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu)f(x) &:= \sup_{\{\epsilon_m\}} \left(\sum_{\substack{m\in\mathbb{Z}:\, \epsilon_m \in I_j,\, \epsilon_{m+1}\in I_k \\ \text{for some } j<k}} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^\rho \right)^{1/\rho},
\end{aligned} \tag{30}
$$

and, in both cases, the pointwise supremum is taken over all the sequences of positive numbers $\{\epsilon_m\}_{m\in\mathbb{Z}}$ decreasing to zero. To prove Theorem 5.1 we will show that both the short and long variation operators are bounded in $L^2(\mu)$.

5.2. $L^2(\mu)$ **boundedness of $\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu$.** The $L^2(\mu)$-norm of the long variation operator $\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu$ can be handled by comparing it with its smoothed version $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$, using Corollary 4.2, and estimating the error terms by the short variation operator.

Lemma 5.2. *We have $\|(\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu)f\|_{L^2(\mu)} \lesssim \|(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f\|_{L^2(\mu)} + \|f\|_{L^2(\mu)}$.*

Proof. We decompose

$$
\begin{aligned}
\big((\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu)f(x)\big)^\rho &= \sup_{\{\epsilon_m\}} \sum_{\substack{m\in\mathbb{Z}:\, \epsilon_m \in I_j,\, \epsilon_{m+1}\in I_k \\ \text{for some } j<k}} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^\rho \\
&\lesssim \sup_{\{\epsilon_m\}} \sum_{\substack{m\in\mathbb{Z}: \\ \epsilon_m \in I_j,\, \epsilon_{m+1}\in I_k \\ \text{for some } j<k}} \Big(|(K(\chi_{\epsilon_{m+1}}^{\epsilon_m} - \varphi_{\epsilon_{m+1}}^{\epsilon_m}) * (f\mu))(x)|^\rho + |(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^\rho \Big) \\
&\lesssim \sup_{\{\epsilon_m\}} \sum_{\substack{m\in\mathbb{Z}:\, \epsilon_m \in I_j,\, \epsilon_{m+1}\in I_k \\ \text{for some } j<k}} |(K(\chi_{\epsilon_{m+1}}^{\epsilon_m} - \varphi_{\epsilon_{m+1}}^{\epsilon_m}) * (f\mu))(x)|^\rho + \big((\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu)f(x)\big)^\rho.
\end{aligned} \tag{31}
$$

For simplicity, we denote by $\big((\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}_{\chi-\varphi}^\mu)f(x)\big)^\rho$ the first term on the right hand side of (31). Notice that, given $\epsilon, \delta > 0$, we have $\chi_\epsilon^\delta - \varphi_\epsilon^\delta = (\chi_\epsilon - \varphi_\epsilon) - (\chi_\delta - \varphi_\delta)$. Recall that, in the definition of $\varphi_{\mathbb{R}}$ in Definition 2.1, we have taken $\chi_{[4,\infty)} \leq \varphi_{\mathbb{R}} \leq \chi_{[1/4,\infty)}$. Hence, given $t \geq 0$,

$$
\chi_{\mathbb{R}}(t) - \varphi_{\mathbb{R}}(t) = \chi_{[1,\infty)}(t) - \int_{1/4}^4 \varphi_{\mathbb{R}}'(s)\chi_{[s,\infty)}(t)\,ds = \int_{1/4}^4 \varphi_{\mathbb{R}}'(s)\big(\chi_{[1,\infty)}(t) - \chi_{[s,\infty)}(t)\big)\,ds
$$

(that is, $\chi_{\mathbb{R}} - \varphi_{\mathbb{R}}$ is a convex combination of $\chi_{[1,\infty)} - \chi_{[s,\infty)}$ for $1/4 \leq s \leq 4$), and thus, by Fubini's theorem,

$$\begin{aligned}\big(K(\chi_\epsilon - \varphi_\epsilon) * (f\mu)\big)(x) &= \int \big(\chi_{\mathbb{R}}(|x-y|^2/\epsilon^2) - \varphi_{\mathbb{R}}(|x-y|^2/\epsilon^2)\big)K(x-y)f(y)\,d\mu(y)\\ &= \int_{1/4}^4 \varphi'_{\mathbb{R}}(s)\int \Big(\chi_{[1,\infty)}(|x-y|^2/\epsilon^2) - \chi_{[s,\infty)}(|x-y|^2/\epsilon^2)\Big)K(x-y)f(y)\,d\mu(y)\,ds\\ &= \int_{1/4}^4 \varphi'_{\mathbb{R}}(s)\int \chi_\epsilon^{\epsilon\sqrt{s}}(x-y)K(x-y)f(y)\,d\mu(y)\,ds = \int_{1/4}^4 \varphi'_{\mathbb{R}}(s)\big((K\chi_\epsilon^{\epsilon\sqrt{s}} * (f\mu))(x)\big)\,ds.\end{aligned}$$

Therefore, by the triangle inequality and Minkowski's integral inequality, we get

$$\begin{aligned}\|(\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu_{\chi-\varphi})f\|_{L^2(\mu)} &\leq 2\Big\| \sup_{\{\epsilon_m \in I_m:\, m\in\mathbb{Z}\}} \Big(\sum_{m\in\mathbb{Z}} |(K(\chi_{\epsilon_m} - \varphi_{\epsilon_m}) * (f\mu))(x)|^\rho\Big)^{1/\rho}\Big\|_{L^2(\mu)}\\ &\leq 2\int_{1/4}^4 \varphi'_{\mathbb{R}}(s)\Big\| \sup_{\{\epsilon_m \in I_m:\, m\in\mathbb{Z}\}} \Big(\sum_{m\in\mathbb{Z}} |(K\chi_{\epsilon_m}^{\epsilon_m\sqrt{s}} * (f\mu))(x)|^\rho\Big)^{1/\rho}\Big\|_{L^2(\mu)} ds.\end{aligned}$$

One can easily verify that $\sup_{\{\epsilon_m \in I_m:\, m\in\mathbb{Z}\}} \big(\sum_{m\in\mathbb{Z}} |(K\chi_{\epsilon_m}^{\epsilon_m\sqrt{s}} * (f\mu))(x)|^\rho\big)^{1/\rho} \lesssim (\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f(x)$ for all $s \in [1/4, 4]$ with uniform bounds. Hence

$$\|(\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu_{\chi-\varphi})f\|_{L^2(\mu)} \lesssim \int_{1/4}^4 \varphi'_{\mathbb{R}}(s)\|(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f\|_{L^2(\mu)}\,ds \lesssim \|(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f\|_{L^2(\mu)}. \tag{32}$$

Finally, using (31), (32), and Corollary 4.2,

$$\begin{aligned}\|(\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu)f\|_{L^2(\mu)} &\lesssim \|(\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu_{\chi-\varphi})f\|_{L^2(\mu)} + \|(\mathcal{V}_\rho \circ \mathcal{T}^\mu_{\varphi})f\|_{L^2(\mu)}\\ &\lesssim \|(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f\|_{L^2(\mu)} + \|f\|_{L^2(\mu)}.\end{aligned}$$

□

Thus, to prove Theorem 5.1, it only remains to show the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu$.

5.3. **$L^2(\mu)$ boundedness of $\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu$.** Given $f \in L^2(\mu)$ and $x \in \operatorname{supp}\mu$, let $\{\epsilon_m\}_{m\in\mathbb{Z}}$ be a decreasing sequence of positive numbers (depending on x) such that

$$\big((\mathcal{V}_2^{\mathcal{S}} \circ \mathcal{T}^\mu)f(x)\big)^2 \leq 2\sum_{j\in\mathbb{Z}} \sum_{\epsilon_m, \epsilon_{m+1} \in I_j} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2.$$

Given $D \in \mathcal{D}_j$ and $x \in D$, we set $\mathcal{S}_D(x) := \{m \in \mathbb{Z} :\, \epsilon_m, \epsilon_{m+1} \in I_j\}$. Since $\rho \geq 2$, we have

$$\begin{aligned}\|(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f\|^2_{L^2(\mu)} \leq \|(\mathcal{V}_2^{\mathcal{S}} \circ \mathcal{T}^\mu)f\|^2_{L^2(\mu)} &\lesssim \int \sum_{j\in\mathbb{Z}} \sum_{\epsilon_m, \epsilon_{m+1} \in I_j} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2\,d\mu(x)\\ &= \sum_{D\in\mathcal{D}} \int_D \sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2\,d\mu(x).\end{aligned}$$

Let η and θ be two positive numbers that will be fixed below (see the proofs of Claims 5.4 and 5.5). Consider a corona decomposition of μ with parameters η and θ as in Subsection

2.4. Then, we can decompose $\mathcal{D} = \mathcal{B} \cup (\bigcup_{S\in \mathrm{Trs}} S)$, so that

$$\begin{aligned}\|(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu) f\|_{L^2(\mu)}^2 &\lesssim \sum_{D\in\mathcal{B}} \int_D \sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2 \, d\mu(x) \\ &\quad + \sum_{S\in\mathrm{Trs}} \sum_{D\in S} \int_D \sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2 \, d\mu(x).\end{aligned} \tag{33}$$

Since the μ-cubes in $\mathcal{B}$ satisfy a Carleson packing condition, we can use Carleson's embedding theorem to estimate the sum on the right hand side of (33) over the μ-cubes in $\mathcal{B}$. More precisely, if we set $m_D^\mu f := \mu(D)^{-1} \int_D f \, d\mu$ for $D \in \mathcal{D}$, we have

$$\begin{aligned}&\sum_{D\in\mathcal{B}} \int_D \sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2 \, d\mu(x) \\ &\qquad \leq \sum_{D\in\mathcal{B}} \int_D \sum_{m\in\mathcal{S}_D(x)} \left(\int_{\epsilon_{m+1}\leq |x-y|\leq \epsilon_m} |K(x-y)||f(y)| \, d\mu(y) \right)^2 d\mu(x) \\ &\qquad \lesssim \sum_{D\in\mathcal{B}} \int_D \left(\frac{1}{\ell(D)^n} \int_{5D} |f| \, d\mu \right)^2 d\mu \approx \sum_{D\in\mathcal{B}} \left(m_{5D}^\mu |f|\right)^2 \mu(D) \lesssim \|f\|_{L^2(\mu)}^2.\end{aligned} \tag{34}$$

Now we are going to estimate now the second term on the right hand side of (33), that is the sum over the μ-cubes in S, for all $S \in \mathrm{Trs}$. To this end, we need to introduce some notation. Given $R \in \mathcal{D}_j$ for some $j \in \mathbb{Z}$, let $P(R)$ denote the μ-cube in $\mathcal{D}_{j-1}$ which contains R (the *parent* of R), and set

$$\begin{aligned}&\mathrm{Ch}(R) := \{Q \in \mathcal{D}_{j+1} : Q \subset R\}, \\ &V(R) := \{Q \in \mathcal{D}_j : Q \cap B(y, \ell(R)) \neq \emptyset \text{ for some } y \in R\}\end{aligned} \tag{35}$$

($\mathrm{Ch}(R)$ are the *children* of R, and $V(R)$ stands for the *vicinity* of R). Notice that $P(R)$ is a μ-cube but $\mathrm{Ch}(R)$ and $V(R)$ are collections of μ-cubes. It is not hard to show that the number of μ-cubes in $\mathrm{Ch}(R)$ and $V(R)$ is bounded by some constant depending only on n and the AD regularity constant of μ. If $R \in S$ for some $S \in \mathrm{Trs}$, we denote by $\mathrm{Tr}(R)$ the set of μ-cubes $Q \in S$ such that $Q \subset R$ (the *tree* of R). Otherwise, i.e., if $R \in \mathcal{B}$, we set $\mathrm{Tr}(R) := \emptyset$. Finally, if $\mathrm{Tr}(R) \neq \emptyset$, let $\mathrm{Stp}(R)$ denote the set of μ-cubes $Q \in \mathcal{B} \cup (\mathcal{G} \setminus \mathrm{Tr}(R))$ such that $Q \subset R$ and $P(Q) \in \mathrm{Tr}(R)$ (the *stopping* μ-cubes relative to R), so actually $Q \subsetneq R$. On the other hand, if $R \in \mathcal{B}$, we set $\mathrm{Stp}(R) := \{R\}$.

Fix $S \in \mathrm{Trs}$, $D \in S$, and $x \in D$. To deal with the second term on the right hand side of (33), we have to estimate the sum $\sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2$. By the definition of $\mathcal{S}_D(x)$, we have

$$\sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)|^2 = \sum_{m\in\mathcal{S}_D(x)} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (\chi_{\widetilde{D}} f\mu))(x)|^2, \tag{36}$$

where $\widetilde{D} := \bigcup_{R\in V(D)} R$. Since this union of μ-cubes is disjoint, we can decompose the function $\chi_{\widetilde{D}} f$ using a Haar basis adapted to $\mathcal{D}$ in the following manner:

$$\chi_{\widetilde{D}} f = \sum_{R\in V(D)} \left((m_R^\mu f)\chi_R + \sum_{Q\in\mathrm{Tr}(R)} \Delta_Q f + \sum_{Q\in\mathrm{Stp}(R)} \widetilde{\Delta}_Q f \right), \tag{37}$$

where we have set

$$\Delta_Q f := \sum_{U\in\mathrm{Ch}(Q)} \chi_U (m_U^\mu f - m_Q^\mu f), \quad \text{and} \quad \widetilde{\Delta}_Q f := \sum_{U\in\mathrm{Ch}(Q)} \chi_U (f - m_Q^\mu f) = \chi_Q (f - m_Q^\mu f).$$

Using (37), we split the left hand side of (36) as follows:

$$\begin{aligned}\sum_{m\in\mathcal{S}_D(x)} |(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (f\mu))(x)|^2 \lesssim{}& \sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)} (K\chi^{\epsilon_m}_{\epsilon_{m+1}} * ((m^\mu_R f)\chi_R\mu))(x)\Big|^2 \\ &+ \sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)}\sum_{Q\in \mathrm{Tr}(R)} (K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (\Delta_Q f\mu))(x)\Big|^2 \\ &+ \sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)}\sum_{Q\in \mathrm{Stp}(R)} (K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (\widetilde{\Delta}_Q f\mu))(x)\Big|^2.\end{aligned} \tag{38}$$

In the following subsections, we will estimate each part separately.

5.3.1. **Estimate of $\sum_{m\in\mathcal{S}_D(x)} \big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (\Delta_Q f\mu))(x)\big|^2$ from (38).**

Lemma 5.3. *Under the notation above, we have*

$$\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)} (K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (\Delta_Q f\mu))(x)\Big|^2 d\mu(x) \lesssim \|f\|^2_{L^2(\mu)}.$$

Proof. Let $C_0 > 0$ be a small constant to be fixed below. Given $m \in \mathcal{S}_D(x)$ set $A_m(x) := A(x, \epsilon_{m+1}, \epsilon_m)$, and given $R \in V(D)$ let

$$\begin{aligned} J^{1,R}_m &:= \{Q \in \mathrm{Tr}(R) : Q \cap A_m(x) \neq \emptyset,\, \ell(Q) > C_0(\epsilon_m - \epsilon_{m+1})\},\\ J^{2,R}_m &:= \{Q \in \mathrm{Tr}(R) : Q \cap A_m(x) \neq \emptyset,\, \ell(Q) \leq C_0(\epsilon_m - \epsilon_{m+1})\}.\end{aligned}$$

For $Q \in J^{1,R}_m$, we write $|(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (\Delta_Q f\mu))(x)| \lesssim \ell(D)^{-n}\|\chi_{A_m(x)}\Delta_Q f\|_{L^1(\mu)}$. The following claim will be proved in Subsection 5.3.2 below.

Claim 5.4. *The following estimate holds: $\sum_{Q\in J^{1,R}_m} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1/2}$.*

Using that $V(D)$ has finitely many elements (depending only on n and the AD regularity constant of μ), Cauchy-Schwarz inequality, Claim 5.4, and the previous estimate, we obtain

$$\begin{aligned}&\sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)}\sum_{Q\in J^{1,R}_m} (K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (\Delta_Q f\mu))(x)\Big|^2 \\ &\qquad\lesssim \sum_{R\in V(D)}\sum_{m\in\mathcal{S}_D(x)} \bigg(\sum_{Q\in J^{1,R}_m} \ell(D)^{-n}\|\chi_{A_m(x)}\Delta_Q f\|_{L^1(\mu)}\bigg)^2 \\ &\qquad\lesssim \sum_{R\in V(D)}\sum_{m\in\mathcal{S}_D(x)} \bigg(\sum_{Q\in J^{1,R}_m} \ell(Q)^{n-1/2}\bigg)\bigg(\sum_{Q\in J^{1,R}_m} \frac{\|\chi_{A_m(x)}\Delta_Q f\|^2_{L^1(\mu)}}{\ell(D)^{2n}\ell(Q)^{n-1/2}}\bigg) \\ &\qquad\lesssim \sum_{R\in V(D)}\sum_{m\in\mathcal{S}_D(x)}\sum_{Q\in\mathrm{Tr}(R)} \frac{\|\chi_{A_m(x)}\Delta_Q f\|^2_{L^1(\mu)}}{\ell(D)^{n+1/2}\ell(Q)^{n-1/2}} \\ &\qquad\lesssim \sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)} \left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2} \frac{\|\Delta_Q f\|^2_{L^1(\mu)}}{\ell(D)^n\ell(Q)^n}.\end{aligned} \tag{39}$$

We deal now with the μ-cubes $Q \in J_m^{2,R}$. Let z_Q denote the center of Q. Since $\int \Delta_Q f\, d\mu = 0$, we can decompose

$$\begin{aligned}(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (\Delta_Q f\mu))(x) &= \int \big(\chi_{A_m(x)}(y)K(x-y) - \chi_{A_m(x)}(z_Q)K(x-z_Q)\big)\Delta_Q f(y)\, d\mu(y)\\ &= \int \chi_{A_m(x)}(y)\Big(K(x-y) - K(x-z_Q)\Big)\Delta_Q f(y)\, d\mu(y)\\ &\quad + \int \Big(\chi_{A_m(x)}(y) - \chi_{A_m(x)}(z_Q)\Big)K(x-z_Q)\Delta_Q f(y)\, d\mu(y)\\ &=: T_m^{1,\mu}(\Delta_Q f)(x) + T_m^{2,\mu}(\Delta_Q f)(x).\end{aligned} \tag{40}$$

For the first term on the right hand side of the last equality, we have the standard estimate (by assuming C_0 small enough, so any $Q \in J_m^{2,R}$ is far from x)

$$|T_m^{1,\mu}(\Delta_Q f)(x)| \lesssim \int_{A_m(x)} \frac{|y-z_Q|}{|x-y|^{n+1}}\, |\Delta_Q f(y)|\, d\mu(y) \lesssim \frac{\ell(Q)}{\ell(D)^{n+1}} \|\chi_{A_m(x)}\Delta_Q f\|_{L^1(\mu)}.$$

From this estimate and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)} \sum_{Q\in J_m^{2,R}} T_m^{1,\mu}(\Delta_Q f)(x)\Big|^2\\ &\qquad \lesssim \sum_{R\in V(D)} \sum_{m\in\mathcal{S}_D(x)} \Big(\sum_{Q\in J_m^{2,R}} \frac{\ell(Q)}{\ell(D)^{n+1}} \|\chi_{A_m(x)}\Delta_Q f\|_{L^1(\mu)}\Big)^2\\ &\qquad \lesssim \sum_{R\in V(D)} \Big(\sum_{Q\in \mathrm{Tr}(R)} \frac{\ell(Q)}{\ell(D)^{n+1}} \sum_{m\in\mathcal{S}_D(x)} \|\chi_{A_m(x)}\Delta_Q f\|_{L^1(\mu)}\Big)^2\\ &\qquad \lesssim \sum_{R\in V(D)} \Big(\sum_{Q\in \mathrm{Tr}(R)} \frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}}\Big)\Big(\sum_{Q\in \mathrm{Tr}(R)} \frac{\|\Delta_Q f\|_{L^1(\mu)}^2}{\ell(Q)^{n-1}\ell(D)^{n+1}}\Big).\end{aligned}$$

Since $\ell(R) = \ell(D)$ for all $R \in V(D)$, we have $\sum_{Q\in\mathrm{Tr}(R)} \big(\frac{\ell(Q)}{\ell(D)}\big)^{n+1} \le \sum_{Q\in\mathcal{D}:\, Q\subset R} \big(\frac{\ell(Q)}{\ell(R)}\big)^{n+1} \lesssim 1$. Thus, using that $t \lesssim \sqrt{t}$ for all $t \lesssim 1$, we conclude

$$\sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)} \sum_{Q\in J_m^{2,R}} T_m^{1,\mu}(\Delta_Q f)(x)\Big|^2 \lesssim \sum_{R\in V(D)} \sum_{Q\in\mathrm{Tr}(R)} \left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2} \frac{\|\Delta_Q f\|_{L^1(\mu)}^2}{\ell(Q)^n \ell(D)^n}. \tag{41}$$

We deal now with the second term on the right hand side of (40). Given $Q \in J_m^{2,R}$, since $\mathrm{supp}(\Delta_Q f) \subset Q$, if $Q \subset A_m(x)$ or $Q \subset (A_m(x))^c$ then we obviously have $\chi_{A_m(x)}(y) - \chi_{A_m(x)}(z_Q) = 0$ for all $y \in \mathrm{supp}(\Delta_Q f)$. Therefore, to estimate the sum of $T_m^{2,\mu}(\Delta_Q f)(x)$ over all $Q \in J_m^{2,R}$, we can replace $J_m^{2,R}$ by

$$J_m^{3,R} := \{Q \in \mathrm{Tr}(R) :\, Q \cap A_m(x) \neq \emptyset,\, Q \cap (A_m(x))^c \neq \emptyset,\, \ell(Q) \le C_0(\epsilon_m - \epsilon_{m+1})\}.$$

For $m \in \mathcal{S}_D(x)$ and $Q \in J_m^{3,R}$, we will use the estimate $|T_m^{2,\mu}(\Delta_Q f)(x)| \lesssim \ell(D)^{-n}\|\Delta_Q f\|_{L^1(\mu)}$.

Claim 5.5. *The following holds:* $\sum_{Q\in J_m^{3,R}} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1}(\epsilon_m - \epsilon_{m+1})^{1/2}$.

Hence, using that $V(D)$ has finitely many terms, Cauchy-Schwarz inequality, assuming Claim 5.5 (see Subsection 5.3.2), and by the previous estimate, we deduce

$$\begin{aligned}
\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in J_m^{2,R}} T_m^{2,\mu}(\Delta_Q f)(x)\Big|^2 &\lesssim \sum_{R\in V(D)}\sum_{m\in\mathcal{S}_D(x)}\Big(\sum_{Q\in J_m^{3,R}}\frac{\|\Delta_Q f\|_{L^1(\mu)}}{\ell(D)^n}\Big)^2\\
&\leq \sum_{R\in V(D)}\sum_{m\in\mathcal{S}_D(x)}\Big(\sum_{Q\in J_m^{3,R}}\frac{\ell(Q)^{n-1/2}}{\ell(D)^{n-1/2}}\Big)\Big(\sum_{Q\in J_m^{3,R}}\frac{\ell(Q)^{1/2-n}}{\ell(D)^{n+1/2}}\,\|\Delta_Q f\|_{L^1(\mu)}^2\Big)\\
&\lesssim \sum_{R\in V(D)}\sum_{m\in\mathcal{S}_D(x)}\left(\frac{\epsilon_m-\epsilon_{m+1}}{\ell(D)}\right)^{1/2}\sum_{Q\in J_m^{3,R}}\frac{\ell(Q)^{1/2-n}}{\ell(D)^{n+1/2}}\,\|\Delta_Q f\|_{L^1(\mu)}^2\\
&\leq \sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}\frac{\ell(Q)^{1/2-n}}{\ell(D)^{n+1/2}}\,\|\Delta_Q f\|_{L^1(\mu)}^2\sum_{\substack{m\in\mathcal{S}_D(x):\,A_m(x)\cap Q\neq\emptyset,\\ \ell(Q)\leq C_0(\epsilon_m-\epsilon_{m+1})}}\left(\frac{\epsilon_m-\epsilon_{m+1}}{\ell(D)}\right)^{1/2}.
\end{aligned}$$

The sum over m on the right hand side of the last inequality can be easily bounded by some constant depending on C_0, thus we finally obtain

$$\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in J_m^{2,R}} T_m^{2,\mu}(\Delta_Q f)(x)\Big|^2 \lesssim \sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}\left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2}\frac{\|\Delta_Q f\|_{L^1(\mu)}^2}{\ell(Q)^n\ell(D)^n}. \tag{42}$$

Finally, combining (39), (40), (41), and (42), we conclude

$$\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(\Delta_Q f\mu))(x)\Big|^2 \lesssim \sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}\left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2}\frac{\|\Delta_Q f\|_{L^1(\mu)}^2}{\ell(Q)^n\ell(D)^n}, \tag{43}$$

Since $\|\Delta_Q f\|_{L^1(\mu)}\lesssim \|\Delta_Q f\|_{L^2(\mu)}\ell(Q)^{n/2}$ by Hölder's inequality, since $V(D)$ has finitely many terms, and since $\ell(R)=\ell(D)$ for all $R\in V(D)$, we get

$$\begin{aligned}
\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D\sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*(\Delta_Q f\mu))(x)\Big|^2 d\mu(x)&\\
\lesssim \sum_{S\in\mathrm{Trs}}\sum_{D\in S}\sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}\left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2}\|\Delta_Q f\|_{L^2(\mu)}^2&\\
\leq \sum_{S\in\mathrm{Trs}}\sum_{Q\in S}\sum_{R\in\mathcal{D}:\,R\supset Q}\sum_{D\in V(R)}\left(\frac{\ell(Q)}{\ell(R)}\right)^{1/2}\|\Delta_Q f\|_{L^2(\mu)}^2&\\
\lesssim \sum_{S\in\mathrm{Trs}}\sum_{Q\in S}\|\Delta_Q f\|_{L^2(\mu)}^2 \leq \sum_{Q\in\mathcal{D}}\|\Delta_Q f\|_{L^2(\mu)}^2 \leq \|f\|_{L^2(\mu)}^2.&
\end{aligned}$$

To complete the proof of Lemma 5.3, it only remains to show Claims 5.4 and 5.5. □

5.3.2. **Proof of Claims 5.4 and 5.5**. First of all, we need an auxiliary result whose easy proof is left for the reader.

Lemma 5.6. *Let $\Gamma:=\{x\in\mathbb{R}^d : x=(y,A(y)),\, y\in\mathbb{R}^n\}$ be the graph of a Lipschitz function $A:\mathbb{R}^n\to\mathbb{R}^{d-n}$ such that $\mathrm{Lip}(A)$ is small enough. Then, $\mathcal{H}^n_\Gamma(A^d(z,a,b))\lesssim (b-a)b^{n-1}$ for all $0<a\leq b$ and $z\in\Gamma$.*

***Remark* 5.7.** Actually, to obtain the conclusion of the lemma, one only needs $\mathrm{Lip}(A) < 1$ (see [M, Lemma 4.1.9]). Let us mention that this assumption is sharp in the sense that if $\mathrm{Lip}(A) \geq 1$ then the lemma fails. However, we do not need this stronger version for our purposes.

Claims 5.4 and 5.5 follow from the next lemma, which will be proved using Lemma 5.6.

Lemma 5.8. *Let $C_0 > 0$ be some constant depending only on n, d, and the AD regularity constant of μ, and consider $x \in D \in \mathcal{D}_j$ for some $j \in \mathbb{Z}$. Let $\epsilon \in [2^{-j-1}, 2^{-j})$. Given $k \geq j$ and $R \in V(D)$, set*

$$\Lambda_k := \{Q \in \mathrm{Tr}(R) \cap \mathcal{D}_k : \, Q \subset A(x, \epsilon - C_0 2^{-k}, \epsilon + C_0 2^{-k})\}.$$

Then, $\mu\big(\bigcup_{Q\in\Lambda_k} Q\big) \lesssim 2^{-k}\ell(D)^{n-1} \approx 2^{-k-j(n-1)}$.

Proof. First of all, we can assume $k \gg j$ (otherwise, the claim follows easily using the AD regularity of μ), thus we may assume that $\mathrm{dist}(x, Q) \geq \frac{3}{4}\,\epsilon$. For simplicity, set $S \equiv \mathrm{Tr}(R)$. By the property (f) of the corona decomposition of μ, there exists a (rotation and translation of an) n-dimensional Lipschitz graph Γ_S with $\mathrm{Lip}(\Gamma_S) \leq \eta$ such that $\mathrm{dist}(y, \Gamma_S) \leq \theta\,\mathrm{diam}(Q)$ whenever $y \in C_{cor}Q$ and $Q \in S$, for some given constant $C_{cor} \geq 2$. Since $x \in D$ and $R \in V(D)$, we have $x \in C_{cor}Q$ assuming C_{cor} big enough, and so $\mathrm{dist}(x, \Gamma_S) \leq \theta\,\mathrm{diam}(Q)$. Hence, if η and θ are small enough, one can easily modify Γ_S inside $B(x, \frac{1}{4}\,\epsilon)$ to obtain a Lipschitz graph Γ_S^x such that $x \in \Gamma_S^x$, and moreover

$$\mathrm{Lip}(\Gamma_S^x) \leq \eta' \text{ for some } \eta' \text{ small enough}, \quad \text{and} \quad \Gamma_S^x \setminus B(x, \epsilon/4) = \Gamma_S \setminus B(x, \epsilon/4). \tag{44}$$

Using that $\mathrm{dist}(x, Q) \geq \frac{3}{4}\,\epsilon$ for all $Q \in \Lambda_k$, that $\mathrm{dist}(z_Q, \Gamma_S) \leq \theta\,\mathrm{diam}(Q)$ for the centre z_Q of Q, and the last part of (44), we deduce that $\mathrm{dist}(z_Q, \Gamma_S^x) \leq \theta\,\mathrm{diam}(Q)$ for all $Q \in \Lambda_k$. So $B(z_Q, \theta\,\mathrm{diam}(Q)) \cap \Gamma_S^x \neq \emptyset$, which in turn yields $\mathcal{H}^n\big(\Gamma_S^x \cap B(z_Q, 2\theta\,\mathrm{diam}(Q))\big) \gtrsim (\theta\,\mathrm{diam}(Q))^n$. Therefore, since $\{B(z_Q, 2\theta\,\mathrm{diam}(Q))\}_{Q\in\Lambda_k}$ is a family with finite overlap bounded by some constant depending only on n, θ, and the AD regularity constant of μ, we have

$$\begin{aligned}
\mu\Big(\bigcup_{Q\in\Lambda_k} Q\Big) &\approx \sum_{Q\in\Lambda_k} \ell(Q)^n \lesssim \theta^{-n} \sum_{Q\in\Lambda_k} \mathcal{H}^n\big(\Gamma_S^x \cap B(z_Q, 2\theta\,\mathrm{diam}(Q))\big) \\
&\lesssim \theta^{-n}\mathcal{H}^n_{\Gamma_S^x}\Big(\bigcup_{Q\in\Lambda_k} B(z_Q, 2\theta\,\mathrm{diam}(Q))\Big) \\
&\lesssim \theta^{-n}\mathcal{H}^n_{\Gamma_S^x}\big(A(x, \epsilon - C_0 2^{-k}, \epsilon + C_0 2^{-k})\big) \lesssim \theta^{-n} 2^{-k-j(n-1)},
\end{aligned}$$

where we used Lemma 5.6 and that $\epsilon \approx 2^{-j}$ in the last inequality. The lemma is proved. □

***Proof of* Claim 5.4.** Recall that $J_m^{1,R} := \{Q \in \mathrm{Tr}(R) : \, Q \cap A_m(x) \neq \emptyset,\, \ell(Q) \geq C_0(\epsilon_m - \epsilon_{m+1})\}$, where $R \in V(D)$ and $D \in \mathcal{D}_j$. We have to check that $\sum_{Q\in J_m^{1,R}} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1/2}$. We will split the sum into different scales and we will apply Lemma 5.8 at each scale.

Given $i \in \mathbb{Z}$ such that $2^{-i} \geq C_0(\epsilon_m - \epsilon_{m+1})$, the number of μ-cubes $Q \in \mathcal{D}_i$ such that $Q \subset R$ and $Q \cap A_m(x) \neq \emptyset$ is bounded by $C\ell(R)^{n-1}2^{i(n-1)} \approx 2^{-j(n-1)+i(n-1)}$, since for all these μ-cubes, $Q \subset A(x, \epsilon_{m+1} - C2^{-i}, \epsilon_m + C2^{-i}) \subset A(x, \epsilon_m - C2^{-i+1}, \epsilon_m + C2^{-i+1})$ for some constant $C > 0$ big enough, and then by Lemma 5.8, $\mu\big(\bigcup_{Q\in J_m^{1,R}\cap\mathcal{D}_i} Q\big) \lesssim 2^{-i}\ell(D)^{n-1}$.

Therefore,

$$\sum_{Q\in J_m^{1,R}} \ell(Q)^{n-1/2} = \sum_{i\in\mathbb{Z}:\, i\geq j} 2^{i/2} \sum_{Q\in J_m^{1,R}\cap\mathcal{D}_i} \ell(Q)^n \lesssim \sum_{i\in\mathbb{Z}:\, i\geq j} 2^{i/2}2^{-i}\ell(D)^{n-1}$$
$$\approx 2^{-j/2}\ell(D)^{n-1} = \ell(D)^{n-1/2}.$$

□

***Proof of* Claim 5.5.** Recall that $J_m^{3,R} := \{Q \in \mathrm{Tr}(R) : Q \cap A_m(x) \neq \emptyset,\, Q \cap (A_m(x))^c \neq \emptyset,\, \ell(Q) \leq C_0(\epsilon_m - \epsilon_{m+1})\}$, where $R \in V(D)$ and $D \in \mathcal{D}_j$. We have to check that

$$\sum_{Q\in J_m^{3,R}} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1}(\epsilon_m - \epsilon_{m+1})^{1/2}.$$

As before, we will split the sum into the different scales and we will apply Lemma 5.8 at each scale. Given $i \in \mathbb{Z}$ such that $2^{-i} \leq C_0(\epsilon_m - \epsilon_{m+1})$, since for any $Q \in J_m^{3,R}\cap\mathcal{D}_i$ we have $Q \subset A(x, \epsilon_{m+1} - C2^{-i}, \epsilon_{m+1} + C2^{-i}) \cup A(x, \epsilon_m - C2^{-i}, \epsilon_m + C2^{-i})$ for some constant $C > 0$ big enough, by Lemma 5.8 applied to both annuli we have $\mu\big(\bigcup_{Q\in J_m^{3,R}\cap\mathcal{D}_i} Q\big) \lesssim 2^{-i}\ell(D)^{n-1}$. Therefore,

$$\begin{aligned}\sum_{Q\in J_m^{3,R}} \ell(Q)^{n-1/2} &= \sum_{i\in\mathbb{Z}:\, i\geq -\log_2(C_0(\epsilon_m-\epsilon_{m+1}))} 2^{i/2} \sum_{Q\in J_m^{3,R}\cap\mathcal{D}_i} \ell(Q)^n \\ &\lesssim \sum_{i\in\mathbb{Z}:\, i\geq -\log_2(C_0(\epsilon_m-\epsilon_{m+1}))} 2^{-i/2}\ell(D)^{n-1} \approx (\epsilon_m - \epsilon_{m+1})^{1/2}\ell(D)^{n-1}.\end{aligned}$$

□

5.3.3. **Estimate of $\sum_{m\in\mathcal{S}_D(x)} \big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Stp}(R)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (\widetilde{\Delta}_Q f\mu))(x)\big|^2$ from (38).**

Lemma 5.9. *Under the notation above, we have*

$$\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)} \left|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Stp}(R)} (K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (\widetilde{\Delta}_Q f\mu))(x)\right|^2 d\mu(x) \lesssim \|f\|_{L^2(\mu)}^2.$$

Proof. Given $R \in V(D)$, consider a μ-cube $Q \in \mathrm{Stp}(R)$. If $\mathrm{Tr}(R) \neq \emptyset$, then $Q \in \mathcal{B} \cup (\mathcal{G} \setminus \mathrm{Tr}(R))$, $Q \subset R$ and $P(Q) \in \mathrm{Tr}(R)$ (in particular, $Q \subsetneq R$). Take $S \in \mathrm{Trs}$ such that $R \in S$. By property (f) of the corona decomposition (see Subsection 2.4), we have $\mathrm{dist}(y, \Gamma_S) \leq \theta\mathrm{diam}(P(Q))$ for all $y \in C_{cor}P(Q)$. Hence, $\mathrm{dist}(y, \Gamma_S) \leq C\theta\mathrm{diam}(Q)$ for all $y \in C_{cor}Q$. On the other hand, if $\mathrm{Tr}(R) = \emptyset$ we have set $\mathrm{Stp}(R) = \{R\}$. In this case, we have $R \in \mathcal{B}$. Take S such that $D \in S$. Since $R \in V(D)$, we have $R \subset C_{cor}D$ if C_{cor} is chosen big enough, and thus $\mathrm{dist}(y, \Gamma_S) \leq C\theta\mathrm{diam}(R)$ for all $y \in C'R$, where C is as above and C' depends on C_{cor}.

Taking into account the comments above, one can prove the following claims using similar arguments to the ones in the proof of Claims 5.4 and 5.5.

Claim 5.10. *Let $x \in D \in \mathcal{D}$, $R \in V(D)$, and $m \in \mathcal{S}_D(x)$. If we set $J_m^{1,R} := \{Q \in \mathrm{Stp}(R) : Q \cap A_m(x) \neq \emptyset,\, \ell(Q) \geq C_0(\epsilon_m - \epsilon_{m+1})\}$, then $\sum_{Q\in J_m^{1,R}} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1/2}$.*

Claim 5.11. *Let $x \in D \in \mathcal{D}$, $R \in V(D)$, and $m \in \mathcal{S}_D(x)$. If we set $J_m^{3,R} := \{Q \in \mathrm{Stp}(R) : Q \cap A_m(x) \neq \emptyset,\, Q \cap (A_m(x))^c \neq \emptyset,\, \ell(Q) \leq C_0(\epsilon_m - \epsilon_{m+1})\}$, then $\sum_{Q\in J_m^{3,R}} \ell(Q)^{n-1/2} \lesssim \ell(D)^{n-1}(\epsilon_m - \epsilon_{m+1})^{1/2}$.*

The only properties of $\Delta_Q f$ that we used to obtain (43) were that $\Delta_Q f$ is supported in Q and that $\int \Delta_Q f \, d\mu = 0$. The function $\widetilde{\Delta}_Q f$ is also supported in Q and has vanishing integral. Thus, if we replace $\mathrm{Tr}(R)$ by $\mathrm{Stp}(R)$, Claims 5.4 and 5.5 by Claims 5.10 and 5.11, and $\Delta_Q f$ by $\widetilde{\Delta}_Q f$, the same arguments that gave us (43) yield the following estimate:

$$\sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)} \sum_{Q\in \mathrm{Stp}(R)} (K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (\widetilde{\Delta}_Q f\mu))(x)\Big|^2 \lesssim \sum_{R\in V(D)} \sum_{Q\in \mathrm{Stp}(R)} \frac{\ell(Q)^{1/2-n}}{\ell(D)^{1/2+n}} \, \|\widetilde{\Delta}_Q f\|^2_{L^1(\mu)}. \tag{45}$$

Below we will use that $\|\widetilde{\Delta}_Q f\|^2_{L^1(\mu)} \ell(Q)^{-n} = \big(\int_Q |f - m^\mu_Q f| \, d\mu\big)^2 \ell(Q)^{-n} \lesssim \big(m^\mu_Q |f|\big)^2 \mu(Q)$. Notice that, by the definition of $\mathrm{Stp}(R)$ and since the corona decomposition is coherent (property (d)), any $Q \in \mathrm{Stp}(R)$ is actually a maximal μ-cube Q_S of some $S \in \mathrm{Trs}$ or $Q \in \mathcal{B}$ (and in this case $\mathrm{Tr}(R)$ is empty). Hence, if we integrate (45) in D, we sum over all $D \in S \in \mathrm{Trs}$, and we change the order of summation, we get

$$\begin{aligned}
&\sum_{S\in\mathrm{Trs}} \sum_{D\in S} \int_D \sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)} \sum_{Q\in \mathrm{Stp}(R)} (K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (\widetilde{\Delta}_Q f\mu))(x)\Big|^2 d\mu(x) \\
&\qquad \lesssim \sum_{S\in\mathrm{Trs}} \sum_{D\in S} \sum_{R\in V(D)} \sum_{Q\in \mathrm{Stp}(R)} \left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2} \frac{\|\widetilde{\Delta}_Q f\|^2_{L^1(\mu)}}{\ell(Q)^n} \\
&\qquad \lesssim \sum_{D\in\mathcal{D}} \sum_{R\in V(D)} \sum_{S\in\mathrm{Trs}:\, Q_S\subset R} \left(\frac{\ell(Q_S)}{\ell(D)}\right)^{1/2} \big(m^\mu_{Q_S}|f|\big)^2 \mu(Q_S) \\
&\qquad\quad + \sum_{D\in\mathcal{D}} \sum_{R\in V(D)} \sum_{Q\in\mathcal{B}:\, Q\subset R} \left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2} \big(m^\mu_{Q}|f|\big)^2 \mu(Q) \\
&\qquad = \sum_{S\in\mathrm{Trs}} \sum_{R\in\mathcal{D}:\, R\supset Q_S} \sum_{D\in V(R)} \left(\frac{\ell(Q_S)}{\ell(R)}\right)^{1/2} \big(m^\mu_{Q_S}|f|\big)^2 \mu(Q_S) \\
&\qquad\quad + \sum_{Q\in\mathcal{B}} \sum_{R\in\mathcal{D}:\, R\supset Q} \sum_{D\in V(R)} \left(\frac{\ell(Q)}{\ell(R)}\right)^{1/2} \big(m^\mu_{Q}|f|\big)^2 \mu(Q).
\end{aligned}$$

Finally, using that $V(R)$ has finitely many elements, and that the μ-cubes Q_S with $S \in \mathrm{Trs}$ and the μ-cubes $Q \in \mathcal{B}$ satisfy a Carleson packing condition (so we can apply Carleson's embedding theorem), we deduce

$$\begin{aligned}
&\sum_{S\in\mathrm{Trs}} \sum_{D\in S} \int_D \sum_{m\in\mathcal{S}_D(x)} \Big| \sum_{R\in V(D)} \sum_{Q\in \mathrm{Stp}(R)} (K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (\widetilde{\Delta}_Q f\mu))(x)\Big|^2 d\mu(x) \\
&\quad \lesssim \sum_{S\in\mathrm{Trs}} \big(m^\mu_{Q_S}|f|\big)^2 \mu(Q_S) \sum_{R\in\mathcal{D}:\, R\supset Q_S} \frac{\ell(Q_S)^{1/2}}{\ell(R)^{1/2}} + \sum_{Q\in\mathcal{B}} \big(m^\mu_{Q}|f|\big)^2 \mu(Q) \sum_{R\in\mathcal{D}:\, R\supset Q} \frac{\ell(Q)^{1/2}}{\ell(R)^{1/2}} \\
&\quad \lesssim \sum_{S\in\mathrm{Trs}} \big(m^\mu_{Q_S}|f|\big)^2 \mu(Q_S) + \sum_{Q\in\mathcal{B}} \big(m^\mu_{Q}|f|\big)^2 \mu(Q) \lesssim \|f\|^2_{L^2(\mu)}.
\end{aligned}$$

□

5.3.4. **Estimate of $\sum_{m\in\mathcal{S}_D(x)} \big|\sum_{R\in V(D)} (K\chi^{\epsilon_m}_{\epsilon_{m+1}} * ((m^\mu_R f)\chi_R\mu))(x)\big|^2$ from (38)**. We will need the following auxiliary lemma, which we prove for completeness, despite we think it is already known.

Lemma 5.12. *Given $D \in \mathcal{D}$ and $f \in L^2(\mu)$, set $a_D(f) := \sum_{R\in V(D)} |m_R^\mu f - m_D^\mu f|$. Then, there exists $C > 0$ depending only n and the AD regularity constant of μ such that*

$$\sum_{D\in\mathcal{D}} (a_D(f))^2 \mu(D) \leq C\|f\|^2_{L^2(\mu)}.$$

Proof. By subtracting a constant if necessary, we can assume that f has mean zero. Consider the representation of f with respect to the Haar basis associated to $\mathcal{D}$, that is $f = \sum_{Q\in\mathcal{D}} \Delta_Q f$. For $m \in \mathbb{Z}$, we define the function $u_m = \sum_{Q\in\mathcal{D}_m} \Delta_Q f$, so $f = \sum_{m\in\mathbb{Z}} u_m$ and the equality holds in $L^2(\mu)$. Given $j \in \mathbb{Z}$, define the operator

$$S_j(f) := \Big(\sum_{D\in\mathcal{D}_j} (a_D(f))^2 \chi_D \Big)^{1/2}.$$

We will prove that there exists a sequence $\{\sigma(k)\}_{k\in\mathbb{Z}}$ such that

$$\sum_{k\in\mathbb{Z}} \sigma(k) \leq C < \infty \quad \text{and} \quad \|S_j(u_m)\|_{L^2(\mu)} \lesssim \sigma(|m-j|)\|u_m\|_{L^2(\mu)}. \tag{46}$$

Assume for the moment that (46) holds. Then, since each S_j is sublinear, by Cauchy-Schwarz inequality and the orthogonality of the u_m's,

$$\begin{aligned}
\sum_{D\in\mathcal{D}} (a_D(f))^2\mu(D) &= \sum_{j\in\mathbb{Z}} \int \sum_{D\in\mathcal{D}_j} (a_D(f))^2 \chi_D \, d\mu = \sum_{j\in\mathbb{Z}} \|S_j(f)\|^2_{L^2(\mu)} \\
&= \sum_{j\in\mathbb{Z}} \Big\| S_j\Big(\sum_{m\in\mathbb{Z}} u_m \Big) \Big\|^2_{L^2(\mu)} \leq \sum_{j\in\mathbb{Z}} \Big(\sum_{m\in\mathbb{Z}} \|S_j(u_m)\|_{L^2(\mu)} \Big)^2 \\
&\leq \sum_{j\in\mathbb{Z}} \Big(\sum_{m\in\mathbb{Z}} \sigma(|m-j|) \Big) \Big(\sum_{m\in\mathbb{Z}} \sigma(|m-j|)^{-1} \|S_j(u_m)\|^2_{L^2(\mu)} \Big) \\
&\lesssim \sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \sigma(|m-j|)\|u_m\|^2_{L^2(\mu)} = \sum_{m\in\mathbb{Z}} \|u_m\|^2_{L^2(\mu)} \sum_{j\in\mathbb{Z}} \sigma(|m-j|) \\
&\lesssim \sum_{m\in\mathbb{Z}} \|u_m\|^2_{L^2(\mu)} = \|f\|^2_{L^2(\mu)},
\end{aligned}$$

and the lemma follows. Let us verify (46) now. By definition,

$$\|S_j(u_m)\|^2_{L^2(\mu)} = \sum_{D\in\mathcal{D}_j} \Big(\sum_{R\in V(D)} \Big| \sum_{Q\in\mathcal{D}_m} \int \Delta_Q f \Big(\frac{\chi_R}{\mu(R)} - \frac{\chi_D}{\mu(D)} \Big) d\mu \Big| \Big)^2 \mu(D). \tag{47}$$

Assume first that $m \geq j$. If $D \in \mathcal{D}_j$, $R \in V(D)$, and $Q \in \mathcal{D}_m$, then either $Q \cap R = \emptyset$ or $Q \subset R$. In both cases, since $\Delta_Q f$ has mean zero and is supported in Q, we have $\int \Delta_Q f \, \chi_R \, d\mu = 0$. Thus, the right hand side of (47) vanishes (obviously $D \in V(D)$), and (46) follows.

Assume now that $m < j$. Set $\widetilde{D} := \bigcup_{R\in V(D)} R$. Recall that $\Delta_Q f := \sum_{U\in \mathrm{Ch}(Q)} \chi_U (m_U^\mu f - m_Q^\mu f)$, so $\Delta_Q f$ is constant in each $U \in \mathrm{Ch}(Q)$. Hence, if for some $U \in \mathrm{Ch}(Q)$ we have $\widetilde{D} \subset U$ or $\widetilde{D} \subset \mathrm{supp}\mu \setminus U$, then $(R \cup D) \subset U$ or $(R \cup D) \cap U = \emptyset$ for all $R \in V(D)$, and so

$$\int \chi_U (m_U^\mu f - m_Q^\mu f) \Big(\frac{\chi_R}{\mu(R)} - \frac{\chi_D}{\mu(D)} \Big) d\mu = (m_U^\mu f - m_Q^\mu f) \int_U \Big(\frac{\chi_R}{\mu(R)} - \frac{\chi_D}{\mu(D)} \Big) d\mu = 0$$

for all $R \in V(D)$. Therefore, if we set $m_{U,Q}^\mu f := (m_U^\mu f - m_Q^\mu f)$, using that $V(D)$ has finitely many elements and that $\int |\mu(R)^{-1}\chi_R - \mu(D)^{-1}\chi_D| \, d\mu \leq 2$ for all $R \in V(D)$, we deduce from

(47) that

$$\|S_j(u_m)\|^2_{L^2(\mu)} = \sum_{D\in\mathcal{D}_j}\Big(\sum_{R\in V(D)}\Big|\sum_{Q\in\mathcal{D}_m}\int \sum_{\substack{U\in\mathrm{Ch}(Q):\\ \widetilde{D}\cap U\neq\emptyset,\\ \widetilde{D}\cap U^c\neq\emptyset}} \chi_U\, m^\mu_{U,Q} f\Big(\frac{\chi_R}{\mu(R)} - \frac{\chi_D}{\mu(D)}\Big)\,d\mu\Big|\Big)^2\mu(D)$$

$$\lesssim \sum_{D\in\mathcal{D}_j}\Big(\sum_{Q\in\mathcal{D}_m}\sum_{\substack{U\in\mathrm{Ch}(Q):\,\widetilde{D}\cap U\neq\emptyset,\\ \widetilde{D}\cap U^c\neq\emptyset}} |m^\mu_{U,Q} f|\Big)^2\mu(D) \tag{48}$$

$$= \sum_{D\in\mathcal{D}_j}\Big(\sum_{\substack{U\in\mathcal{D}_{m+1}:\,\widetilde{D}\cap U\neq\emptyset,\\ \widetilde{D}\cap U^c\neq\emptyset}} |m^\mu_{U,P(U)} f|\Big)^2\mu(D).$$

It is not hard to show that, since $m<j$ and $D\in\mathcal{D}_j$, the number of μ-cubes $U\in\mathcal{D}_{m+1}$ such that $\widetilde{D}\cap U\neq\emptyset$ and $\widetilde{D}\cap U^c\neq\emptyset$ is bounded by some constant depending only on n and the AD regularity constant of μ (but not on the precise value of m). Hence,

$$\sum_{D\in\mathcal{D}_j}\Big(\sum_{\substack{U\in\mathcal{D}_{m+1}:\,\widetilde{D}\cap U\neq\emptyset,\\ \widetilde{D}\cap U^c\neq\emptyset}} |m^\mu_{U,P(U)} f|\Big)^2\mu(D) \lesssim \sum_{D\in\mathcal{D}_j}\sum_{\substack{U\in\mathcal{D}_{m+1}:\,\widetilde{D}\cap U\neq\emptyset,\\ \widetilde{D}\cap U^c\neq\emptyset}} |m^\mu_{U,P(U)} f|^2\mu(D)$$

$$= \sum_{U\in\mathcal{D}_{m+1}} |m^\mu_{U,P(U)} f|^2\,\mu\Big(\bigcup_{\substack{D\in\mathcal{D}_j:\,\widetilde{D}\cap U\neq\emptyset,\\ \widetilde{D}\cap U^c\neq\emptyset}} D\Big). \tag{49}$$

Fix $U\in\mathcal{D}_{m+1}$. Recall that $\widetilde{D}:=\bigcup_{R\in V(D)} R$, so $\mathrm{diam}(\widetilde{D})\approx\mathrm{diam}(D)$. Thus, there exists a constant $\tau_0>0$ such that

$$\begin{aligned}\bigcup_{D\in\mathcal{D}_j:\,\widetilde{D}\cap U\neq\emptyset,\,\widetilde{D}\cap U^c\neq\emptyset} D &\subset \{x\in U:\,\mathrm{dist}(x,\mathrm{supp}\mu\setminus U)\leq \tau_0\ell(D)\}\\ &\quad\cup\{x\in \mathrm{supp}\mu\setminus U:\,\mathrm{dist}(x,U)\leq\tau_0\ell(D)\}\\ &= \{x\in U:\,\mathrm{dist}(x,\mathrm{supp}\mu\setminus U)\leq \tau_0 2^{m-j+1}\ell(U)\}\\ &\quad\cup\{x\in \mathrm{supp}\mu\setminus U:\,\mathrm{dist}(x,U)\leq\tau_0 2^{m-j+1}\ell(U)\}.\end{aligned}$$

If $m\ll j$, then $\tau:=\tau_0 2^{m-j+1}<1$, so we can apply the *small boundaries condition* (9) of Subsection 2.3 to obtain $\mu\big(\bigcup_{D\in\mathcal{D}_j:\,\widetilde{D}\cap U\neq\emptyset,\,\widetilde{D}\cap U^c\neq\emptyset} D\big)\leq C\tau^{1/C}2^{-mn}$. On the contrary, if $|m-j|\lesssim 1$, then $\tau^{1/C}\approx 1$, so $\mu\big(\bigcup_{D\in\mathcal{D}_j:\,\widetilde{D}\cap U\neq\emptyset,\,\widetilde{D}\cap U^c\neq\emptyset} D\big)\leq\mu(C_1U)\lesssim 2^{-mn}\approx\tau^{1/C}2^{-mn}$, for some big constant $C_1>0$. Thus, in any case, $\mu\big(\bigcup_{D\in\mathcal{D}_j:\,\widetilde{D}\cap U\neq\emptyset,\,\widetilde{D}\cap U^c\neq\emptyset} D\big)\lesssim 2^{(m-j)/C}\ell(U)^n$, and combining this with (49) and (48) we conclude that, for $m<j$,

$$\begin{aligned}\|S_j(u_m)\|^2_{L^2(\mu)} &\lesssim 2^{(m-j)/C}\sum_{U\in\mathcal{D}_{m+1}} |m^\mu_U f - m^\mu_{P(U)} f|^2\ell(U)^n\\ &\approx 2^{(m-j)/C}\int\sum_{U\in\mathcal{D}_{m+1}}\chi_U|m^\mu_U f - m^\mu_{P(U)} f|^2\,d\mu = 2^{-|m-j|/C}\|u_m\|^2_{L^2(\mu)},\end{aligned}$$

which gives (46) with $\sigma(k)=2^{-\frac{|k|}{2C}}$ and finishes the proof of the lemma. □

Lemma 5.13. *Under the notation above, we have*

$$\sum_{S\in \mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)}\left|\sum_{R\in V(D)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^\mu f)\chi_R\mu))(x)\right|^2 d\mu(x)\lesssim \|f\|_{L^2(\mu)}^2.$$

Proof. Recall that, given $D\in\mathcal{D}$, we have set $\widetilde{D}:=\bigcup_{R\in V(D)}R$. For $x\in D$, we have

$$\begin{aligned}\sum_{m\in\mathcal{S}_D(x)}\left|\sum_{R\in V(D)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^\mu f)\chi_R\mu))(x)\right|^2 &\lesssim \sum_{m\in\mathcal{S}_D(x)}\left|(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_D^\mu f)\chi_{\widetilde{D}}\mu))(x)\right|^2\\ &\quad+\sum_{m\in\mathcal{S}_D(x)}\left|\sum_{R\in V(D)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^\mu f-m_D^\mu f)\chi_R\mu))(x)\right|^2.\end{aligned}\tag{50}$$

We are going to estimate the two terms on the right hand side of (50) separately. For the second one, recall also that, given $m\in\mathcal{S}_D(x)$, we have set $A_m(x):=A(x,\epsilon_{m+1},\epsilon_m)$. We write

$$\begin{aligned}|(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^\mu f-m_D^\mu f)\chi_R\mu))(x)| &\le |m_R^\mu f-m_D^\mu f|\int_{A_m(x)}|K(x-y)|\chi_R(y)\,d\mu(y)\\ &\lesssim |m_R^\mu f-m_D^\mu f|\,\mu(A_m(x)\cap R)\ell(D)^{-n}.\end{aligned}$$

Therefore, interchanging the order of summation,

$$\begin{aligned}&\sum_{m\in\mathcal{S}_D(x)}\left|\sum_{R\in V(D)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^\mu f-m_D^\mu f)\chi_R\mu))(x)\right|^2\\ &\quad\lesssim\left(\sum_{m\in\mathcal{S}_D(x)}\sum_{R\in V(D)}|m_R^\mu f-m_D^\mu f|\,\mu(A_m(x)\cap R)\ell(D)^{-n}\right)^2\\ &\quad\le\left(\sum_{R\in V(D)}|m_R^\mu f-m_D^\mu f|\,\frac{\mu(R)}{\ell(D)^n}\right)^2\approx\left(\sum_{R\in V(D)}|m_R^\mu f-m_D^\mu f|\right)^2=(a_D(f))^2,\end{aligned}$$

where $a_D(f)$ are the coefficients introduced in Lemma 5.12. If we integrate on D and sum over all $D\in S$ and $S\in\mathrm{Trs}$, we can apply Lemma 5.12, and we finally obtain

$$\begin{aligned}&\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D\sum_{m\in\mathcal{S}_D(x)}\left|\sum_{R\in V(D)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^\mu f-m_D^\mu f)\chi_R\mu))(x)\right|^2 d\mu(x)\\ &\qquad\lesssim\sum_{D\in\mathcal{D}}(a_D(f))^2\mu(D)\lesssim\|f\|_{L^2(\mu)}^2.\end{aligned}\tag{51}$$

Let us estimate now the first term on the right hand side of (50). Let L_D be a minimizing n-plane for $\alpha_\mu(D)$ and let L_D^x be the n-plane parallel to L_D which contains x. Given $z\in\mathbb{R}^d$, let p_0^x denote the orthogonal projection onto L_D^x. Let $g_1,g_2:\mathbb{R}\to[0,1]$ be such that $\mathrm{supp}g_1\subset(-2\varepsilon\ell(D),2\varepsilon\ell(D))$, $\mathrm{supp}g_2\subset(-\ell(D)\varepsilon,\ell(D)\varepsilon)^c$, and $g_1+g_2=1$, where $\varepsilon>0$ is some fixed constant small enough. For $z\in\mathbb{R}^d$, consider the projection onto L_D^x given by

$$p^x(z):=\left(x+(p_0^x(z)-x)\frac{|z-x|}{|p_0^x(z)-x|}\right)g_2(|p_0^x(z)-x|)+p_0^x(z)g_1(|p_0^x(z)-x|).\tag{52}$$

Since $\mathrm{supp}g_2$ does not contain the origin, p^x is well defined. Moreover, if $z\in\mathbb{R}^d$ is such that $g_2(|p_0^x(z)-x|)=1$, then $|z-x|=|p^x(z)-x|$.

Let $C_* > 0$ be a small constant which will be fixed below. Assume that $\alpha_\mu(10D) \geq C_*$. Then, we can easily estimate

$$\begin{aligned}\sum_{m\in\mathcal{S}_D(x)} \left|(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * ((m^\mu_D f)\chi_{\widetilde{D}}\mu))(x)\right|^2 &= |m^\mu_D f|^2 \sum_{m\in\mathcal{S}_\mathcal{D}(x)} \left|\int_{A_m(x)\cap\widetilde{D}} K(x-y)\,d\mu(y)\right|^2 \\ &\lesssim |m^\mu_D f|^2 \Big(\sum_{m\in\mathcal{S}_\mathcal{D}(x)} \int_{A_m(x)\cap\widetilde{D}} |K(x-y)|\,d\mu(y)\Big)^2 \\ &\lesssim |m^\mu_D f|^2 \Big(\int_{\widetilde{D}} \ell(D)^{-n}\,d\mu(y)\Big)^2 \lesssim |m^\mu_D f|^2 \lesssim |m^\mu_D f|^2 \alpha_\mu(10D)^2.\end{aligned} \tag{53}$$

From now on, we assume that $\alpha_\mu(10D) < C_*$. By assuming C_* small enough, it is not difficult to show that then the distance between $\widetilde{D}$ and L^x_D is smaller than $\ell(D)/1000$. Moreover, p^x restricted to $\{y \in A_m(x) : \operatorname{dist}(y, L^x_D) \leq \ell(D)/1000\}$ is a Lipschitz function with Lipschitz constant depending only n, d, and the AD regularity constant of μ. Furthermore, by taking ε small enough, we have

$$p^x(z) = x + (p^x_0(z) - x)\frac{|z-x|}{|p^x_0(z)-x|} \tag{54}$$

for all $z \in \{y \in \widetilde{D} \cap A_m(x) : \operatorname{dist}(y, L^x_D) \leq \ell(D)/1000\} \subset \operatorname{supp}\mu$.

Recall that $D \in S$ for some $S \in \mathrm{Trs}$. Let Q_S be the maximal μ-cube of S, and set $\nu_x := p^x_\sharp(\chi_{40Q_S}\mu)$. Then, since $\operatorname{supp}\mu \cap A_m(x) \subset \widetilde{D}$ by the construction of $\widetilde{D}$,

$$\begin{aligned}(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * ((m^\mu_D f)\chi_{\widetilde{D}}\mu))(x) &= (m^\mu_D f)\int_{A_m(x)} K(x-y)\,d\mu(y) \\ &= (m^\mu_D f)\int_{A_m(x)} K(x-y)\,d(\mu-\nu_x)(y) + (m^\mu_D f)\int_{A_m(x)} K(x-y)\,d\nu_x(y) \\ &=: U1_m(x) + U2_m(x).\end{aligned} \tag{55}$$

Claim 5.14. *Under the notation above, we have*

$$\sum_{m\in\mathcal{S}_D(x)} |U1_m(x)|^2 \lesssim |m^\mu_D f|^2 \left(\beta_{1,\mu}(D)^2 + \alpha_\mu(D)^2 + \left(\frac{\operatorname{dist}(x, L_D)}{\ell(D)}\right)^2\right).$$

***Proof of* Claim 5.14.** By (54), $y \in A_m(x)$ if and only if $p^x(y) \in A_m(x)$ in the integral defining $U1_m(x)$. Since $|y - p^x(y)| \lesssim \operatorname{dist}(y, L^x_D) \leq \operatorname{dist}(y, L_D) + \operatorname{dist}(x, L_D)$ for all $y \in \operatorname{sup}\mu \cap A_m(x)$,

$$\begin{aligned}|U1_m(x)| &\leq |m^\mu_D f| \int_{A_m(x)} |K(x-y) - K(x-p^x(y))|\,d\mu(y) \\ &\lesssim \frac{|m^\mu_D f|}{\ell(D)^{n+1}} \int_{A_m(x)} |y - p^x(y)|\,d\mu(y) \\ &\lesssim \frac{|m^\mu_D f|}{\ell(D)^{n+1}} \int_{A_m(x)} (\operatorname{dist}(y, L_D) + \operatorname{dist}(x, L_D))\,d\mu(y).\end{aligned}$$

If L_D^1 denotes a minimizing n-plane for $\beta_1(D)$, then $\mathrm{dist}_{\mathcal{H}}(L_D \cap B_D, L_D^1 \cap B_D) \lesssim \alpha_\mu(D)\ell(D)$, so $\mathrm{dist}(y, L_D) \lesssim \mathrm{dist}(y, L_D^1) + \alpha_\mu(D)\ell(D)$ for $y \in CD$ (see [To4]). Therefore,

$$\begin{aligned}\sum_{m\in\mathcal{S}_D(x)} |U1_m(x)|^2 &\lesssim \left(\frac{|m_D^\mu f|}{\ell(D)^{n+1}} \sum_{m\in\mathcal{S}_D(x)} \int_{A_m(x)} (\mathrm{dist}(y, L_D) + \mathrm{dist}(x, L_D))\, d\mu(y)\right)^2 \\ &\lesssim |m_D^\mu f|^2 \left(\ell(D)^{-n-1} \int_{CD} (\mathrm{dist}(y, L_D) + \mathrm{dist}(x, L_D))\, d\mu(y)\right)^2 \\ &\lesssim |m_D^\mu f|^2 \left(\beta_{1,\mu}(D)^2 + \alpha_\mu(D)^2 + \left(\frac{\mathrm{dist}(x, L_D)}{\ell(D)}\right)^2\right).\end{aligned}$$

□

Let us consider $U2_m(x)$ now. We can assume that ν_x is absolutely continuous with respect to $\mathcal{H}^n_{L_D^x}$ (for example, by convolving it with an approximation of the identity and making a limiting argument). Let h_x be the corresponding density, so $\nu_x = h_x \mathcal{H}^n_{L_D^x}$. We may also assume that $h_x \in L^2(\mathcal{H}^n_{L_D^x})$. So,

$$U2_m(x) = (m_D^\mu f) \int_{A_m(x)} K(x-y)\, d\nu_x(y) = (m_D^\mu f) \int_{A_m(x)} K(x-y) h_x(y)\, d\mathcal{H}^n_{L_D^x}(y).$$

At this point, we need to introduce a wavelet basis.

Definition 5.15. *Let $\mathcal{D}^n$ denote the standard dyadic lattice of $\mathbb{R}^n$. Let $\{\psi_Q^k\}_{Q\in\mathcal{D}^n,\, k=1,\dots,2^n-1}$ be an orthonormal basis of $\mathcal{C}^1$ wavelets on $\mathbb{R}^n$ in the following manner (see* [Da, Part I]*):*

- *(a) $\psi_Q^k : \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^1$ function for all $Q \in \mathcal{D}^n$ and $k = 1, \dots, 2^n - 1$.*
- *(b) There exists $C > 1$ and $\psi_0 : [0, C]^n \to \mathbb{R}$ with $\|\psi_0\|_2 = 1$, $\|\psi_0\|_\infty \lesssim 1$, and such that, for any $Q \in \mathcal{D}^n$ and $k = 1, \dots, 2^n - 1$, there exists $l \in \mathbb{Z}^n$ such that $\psi_Q^k(y) = \psi_0(y/\ell(Q) - l)\ell(Q)^{-n/2}$ for all $y \in \mathbb{R}^n$.*
- *(c) $\|\psi_Q^k\|_2 = 1$, $\int \psi_Q^k\, d\mathcal{L}^n = 0$ and $\int \psi_Q^k \psi_R^l\, d\mathcal{L}^n = 0$, for all $Q, R \in \mathcal{D}^n$ and $k, l = 1, \dots, 2^n - 1$ such that $(Q, k) \neq (R, l)$, where $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$.*
- *(d) $\mathrm{supp}\psi_Q^k \subset C_w Q$ for all $Q \in \mathcal{D}^n$ and $k = 1, \dots, 2^n - 1$, where $C_w > 1$ is some fixed constant (which depends on n). In particular, for any $j \in \mathbb{Z}$ the supports of the functions in $\bigcup_{Q\in\mathcal{D}^n:\, \ell(Q)=2^{-j}} \{\psi_Q^k\}_{k=1,\dots,2^n-1}$ have finite overlap.*
- *(e) $\|\psi_Q^k\|_\infty \lesssim \ell(Q)^{-n/2}$ and $\|\nabla\psi_Q^k\|_\infty \lesssim \ell(Q)^{-n/2-1}$ for all $Q \in \mathcal{D}^n$, $k = 1, \dots, 2^n - 1$.*
- *(f) If $h \in L^2(\mathcal{L}^n)$, then $h = \sum_{Q\in\mathcal{D}^n,\, k=1,\dots,2^n-1} \Delta_Q^k h$, where $\Delta_Q^k h := \big(\int h\psi_Q^k\, d\mathcal{L}^n\big)\psi_Q^k$.*

In order to reduce the notation, we may think that a cube of $\mathcal{D}^n$ is not only a subset of $\mathbb{R}^n$, but a couple (Q, k), where Q is a subset of $\mathbb{R}^n$ and $k = 1, \dots, 2^n - 1$. In particular, there exist $2^n - 1$ cubes in $\mathcal{D}^n$ such that the subsets that they represent in $\mathbb{R}^n$ coincide. We make this abuse of notation to avoid using the superscript k in the previous definition. Then, we can rewrite the wavelet basis as $\{\psi_Q\}_{Q\in\mathcal{D}^n}$, with the evident adjustments of the properties $(a), \dots, (f)$ in Definition 5.15.

Let $\mathcal{D}_x^{n,0}$ be a fixed dyadic lattice of the n-plane L_D^x, and let $\{\psi_Q\}_{Q\in\mathcal{D}_x^{n,0}}$ be a wavelet basis as the one introduced in Definition 5.15 but defined on L_D^x. Denote by E_D^x the n-dimensional vector space which defines L_D^x, and let $\{Q_k^0\}_{k\in\mathbb{Z}}$ be a fixed sequence of nested dyadic cubes in E_D^x having the origin as a common vertex and such that $\ell(Q_k^0) = 2^{-k}$ for all $k \in \mathbb{Z}$. Given $s \in E_D^x$, set $\mathcal{D}_x^{n,s} := \{s + Q : Q \in \mathcal{D}_x^{n,0}\}$ (notice that, for any $k \in \mathbb{Z}$, the family $\{Q \in \mathcal{D}_x^{n,s} : \ell(Q) = 2^{-k}\}$ is periodic in the parameter s), For any $Q \in \mathcal{D}_x^{n,0}$ and $y \in L_D^x$, if

$Q' = s + Q \in \mathcal{D}_x^{n,s}$, we define $\psi_{Q'}(y) \equiv \psi_{s+Q}(y) := \psi_Q(y-s)$. Then $\{\psi_{Q'}\}_{Q' \in \mathcal{D}_x^{n,s}}$ is also a wavelet basis defined on L_D^x. Consider the decomposition of h_x with respect to this basis,

$$h_x = \sum_{Q \in \mathcal{D}_x^{n,s}} \Delta_Q^\psi h_x = \sum_{Q \in \mathcal{D}_x^{n,0}} \Delta_{Q,s}^\psi h_x, \tag{56}$$

where $\Delta_{Q,s}^\psi h_x(z) := \big(\int h_x(y)\psi_Q(y-s)\,d\mu(y)\big)\,\psi_Q(z-s)$ (recall that, for any $Q \in \mathcal{D}_x^{n,s}$, $\int \psi_Q\,d\mathcal{H}^n_{L_D^x} = 0$). We set $J(Q_S) := -\log_2(\ell(Q_S))$, and given $Q \in \mathcal{D}_x^{n,s}$, we set $J(Q) := -\log_2(\ell(Q))$ and $J'(Q) := \max\{J(Q_S), J(Q)\}$. Given $\Omega \subset E_D^x$, denote by $m_{s\in\Omega} g$ the average of a function $g : E_D^x \to \mathbb{R}$ over all $s \in \Omega$ and with respect to $\mathcal{H}^n_{E_D^x}$. Then, by the periodicity of $\{\psi_Q\}_{Q\in\mathcal{D}_x^{n,s}}$ in the parameter s (recall Definition 5.15(b)) and (56), we can write

$$h_x = m_{s\in Q^0_{J(Q_S)}}(h_x) = \sum_{Q\in\mathcal{D}_x^{n,0}} m_{s\in Q^0_{J(Q_S)}}(\Delta_{Q,s}^\psi h_x) = \sum_{Q\in\mathcal{D}_x^{n,0}} m_{s\in Q^0_{J'(Q)}}(\Delta_{Q,s}^\psi h_x).$$

Set $J := \{Q \in \mathcal{D}_x^{n,0} : \operatorname{supp}\psi_Q(\cdot - s) \cap \operatorname{supp}\chi_{2^{-j-1}}^{2^{-j}}(x-\cdot) \neq \emptyset \text{ for some } s \in Q^0_{J'(Q)}\}$. Then,

$$U2_m(x) = (m_D^\mu f)\int_{A_m(x)} K(x-y)\sum_{Q\in J} m_{s\in Q^0_{J'(Q)}}(\Delta_{Q,s}^\psi h_x(y))\,d\mathcal{H}^n_{L_D^x}(y). \tag{57}$$

Recall that $D \in \mathcal{D}_j$ and $m \in \mathcal{S}_D(x)$. Since $x \in D$ and $\ell(D) = 2^{-j}$, if $Q \in J$, then $D \subset B(x, C_a\ell(Q))$ or $Q \subset B(x, C_a\ell(D))$ for some constant $C_a > 0$ big enough. In particular, if $\ell(Q) \gtrsim \ell(D)$ then $D \subset B(z_Q, C_a\ell(Q))$, and if $\ell(Q) \leq C\ell(D)$ with $C > 0$ small enough then $Q \subset B(z_D, C_a\ell(D))$, where z_Q denotes the center of $Q \subset L_D^x$ and z_D denotes the center of $D \in \mathcal{D}$. We define

$$\begin{aligned} J_1 &:= \{Q \in J : \ell(Q) \leq C\ell(D)\} \subset \{Q \in \mathcal{D}_x^{n,0} : Q \subset B(z_D, C_a\ell(D))\}, \text{ and} \\ J_2 &:= J \setminus J_1 \subset \{Q \in \mathcal{D}_x^{n,0} : D \subset B(z_Q, C_a\ell(Q))\}. \end{aligned}$$

Then, using (57), that $\operatorname{supp}\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-\cdot) \subset \operatorname{supp}\chi_{2^{-j-1}}^{2^{-j}}(x-\cdot)$ for all $m \in \mathcal{S}_D(x)$, that $\int_{A_m(x)} K(x-y)\,d\mathcal{H}^n_{L_D^x}(y) = 0$ by antisymmetry, and that $J'(Q) = J(Q)$ for all $Q \in J_1$ (because $D \subset Q_S$), if x' denotes some fixed point in $A(x, 2^{-j-1}, 2^{-j}) \cap L_D^x$, we have

$$\begin{aligned} U2_m(x) &= (m_D^\mu f)\int_{A_m(x)} K(x-y)\sum_{Q\in J_1} m_{s\in Q^0_{J(Q)}}(\Delta_{Q,s}^\psi h_x(y))\,d\mathcal{H}^n_{L_D^x}(y) \\ &\quad + (m_D^\mu f)\int_{A_m(x)} K(x-y)\sum_{Q\in J_2} m_{s\in Q^0_{J'(Q)}}\big(\Delta_{Q,s}^\psi h_x(y) - \Delta_{Q,s}^\psi h_x(x')\big)\,d\mathcal{H}^n_{L_D^x}(y) \\ &=: U3_m(x) + U4_m(x). \end{aligned} \tag{58}$$

Claim 5.16. *Under the notation above, we have*

$$\sum_{m\in\mathcal{S}_D(x)} |U4_m(x)|^2 \lesssim |m_D^\mu f|^2 \sum_{Q\in J_2}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2}\ell(Q)^{-n}\big(m_{s\in Q^0_{J'(Q)}}\|\Delta_{Q,s}^\psi h_x\|_2\big)^2.$$

***Proof of* Claim 5.16.** By property (e) of the wavelet basis in Definition 5.15, we have $|\Delta_{Q,s}^\psi h_x(y) - \Delta_{Q,s}^\psi h_x(x')| \leq \|\nabla(\Delta_{Q,s}^\psi h_x)\|_\infty |x'-y| \lesssim \|\Delta_{Q,s}^\psi h_x\|_2 |x'-y|\ell(Q)^{-n/2-1}$. Moreover,

if $y \in A_m(x)$, then $|x' - y| \lesssim \ell(D)$. Therefore,

$$\begin{aligned}|U4_m(x)| &\leq \sum_{Q\in J_2} |m_D^\mu f| \int_{A_m(x)} |K(x-y)| m_{s\in Q^0_{J'(Q)}}\big(\big|\Delta^\psi_{Q,s}h_x(y) - \Delta^\psi_{Q,s}h_x(x')\big|\big)\,d\mathcal{H}^n_{L^x_D}(y)\\ &\lesssim \sum_{Q\in J_2} |m_D^\mu f| m_{s\in Q^0_{J'(Q)}}\big(\|\Delta^\psi_{Q,s}h_x\|_2\big)\ell(D)^{1-n}\ell(Q)^{-n/2-1}\mathcal{H}^n_{L^x_D}(A_m(x)),\end{aligned}$$

and then, by Cauchy-Schwarz inequality and since $J_2 \subset \{Q \in \mathcal{D}^{n,0}_x : D \subset B(z_Q, C_a\ell(Q))\}$ (in particular, $\ell(D)/\ell(Q) \lesssim (\ell(D)/\ell(Q))^{1/2}$),

$$\begin{aligned}\sum_{m\in\mathcal{S}_D(x)} |U4_m(x)|^2 &\lesssim \bigg(\sum_{m\in\mathcal{S}_D(x)}\sum_{Q\in J_2} |m_D^\mu f|\, m_{s\in Q^0_{J'(Q)}}\big(\|\Delta^\psi_{Q,s}h_x\|_2\big)\frac{\ell(Q)^{n/2+1}}{\ell(D)^{n-1}}\,\mathcal{H}^n_{L^x_D}(A_m(x))\bigg)^2\\ &\leq \bigg(\sum_{Q\in J_2} |m_D^\mu f|\, m_{s\in Q^0_{J'(Q)}}\big(\|\Delta^\psi_{Q,s}h_x\|_2\big)\ell(D)\ell(Q)^{n/2+1}\bigg)^2\\ &\leq \bigg(\sum_{Q\in J_2}\frac{\ell(D)}{\ell(Q)}\bigg)\bigg(\sum_{Q\in J_2} |m_D^\mu f|^2\big(m_{s\in Q^0_{J'(Q)}}\|\Delta^\psi_{Q,s}h_x\|_2\big)^2\frac{\ell(D)}{\ell(Q)^{n+1}}\bigg)\\ &\lesssim |m_D^\mu f|^2\sum_{Q\in J_2}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2}\ell(Q)^{-n}\big(m_{s\in Q^0_{J'(Q)}}\|\Delta^\psi_{Q,s}h_x\|_2\big)^2.\end{aligned}$$

□

We are going to estimate $U3_m(x)$ with techniques very similar to the ones used in Subsections 5.3.1 and 5.3.3. First of all, let $b_* > 0$ be a small constant which will be fixed later on, and consider the family $\mathcal{P} := \{Q \in \mathcal{D}^{n,0}_x : \ell(Q) \leq \ell(D)\}$. Let Stp denote the set of cubes $Q \in \mathcal{P}$ such that there exists $R_Q \in \mathcal{D}$ with $\ell(R_Q) = \ell(Q)$, $10R_Q \cap (p^x)^{-1}(\mathrm{supp}\psi_Q) \neq \emptyset$, and

$$\sum_{R\in\mathcal{D}:\, R_Q\subset R,\, \ell(R)\leq\ell(D)} \alpha_\mu(10R) \geq b_* \quad \text{but} \quad \sum_{R\in\mathcal{D}:\, P(R_Q)\subset R,\, \ell(R)\leq\ell(D)} \alpha_\mu(10R) < b_*. \tag{59}$$

Observe that if Q and Q' are different and belong to Stp, then $Q \cap Q' = \emptyset$. Notice also that $D \notin$ Stp because we assumed $\alpha_\mu(10D) < C_*$. Finally, denote by Tr the set of cubes $Q \in \mathcal{P} \setminus$ Stp such that $R \notin$ Stp for all $R \in \mathcal{P}$ with $R \supset Q$. Then $\mathcal{P} = \mathrm{Tr} \cup \bigcup_{Q\in\mathrm{Stp}}\{R \in \mathcal{P} : R \subset Q\}$. By taking C_* small enough we can assume that, if $R \in J_1 \cap \mathcal{P}$ and $R \subset Q$ for some $Q \in$ Stp, then $Q \in J_1$. So we write

$$\begin{aligned}&\sum_{Q\in J_1} m_{s\in Q^0_{J(Q)}}(\Delta^\psi_{Q,s}h_x)\\ &\qquad= \sum_{Q\in J_1\cap\mathrm{Tr}} m_{s\in Q^0_{J(Q)}}(\Delta^\psi_{Q,s}h_x) + \sum_{Q\in J_1\cap\mathrm{Stp}}\sum_{R\in J_1\cap\mathcal{P}:\, R\subset Q} m_{s\in Q^0_{J(Q)}}(\Delta^\psi_{R,s}h_x)\end{aligned}$$

Set $\widetilde{\Delta}^\psi_{Q,s}h_x := \sum_{R\in\mathcal{P}:\, R\subset Q}\Delta^\psi_{R,s}h_x$. Then, using the definition of J_1 and J, we can split

$$\begin{aligned}U3_m(x) &= (m_D^\mu f)\int_{A_m(x)} K(x-y)\sum_{Q\in J_1\cap\mathrm{Tr}} m_{s\in Q^0_{J(Q)}}(\Delta^\psi_{Q,s}h_x(y))\,d\mathcal{H}^n_{L^x_D}(y)\\ &\quad + (m_D^\mu f)\int_{A_m(x)} K(x-y)\sum_{Q\in J_1\cap\mathrm{Stp}} m_{s\in Q^0_{J(Q)}}(\widetilde{\Delta}^\psi_{Q,s}h_x(y))\,d\mathcal{H}^n_{L^x_D}(y)\\ &=: U3^a_m(x) + U3^b_m(x).\end{aligned} \tag{60}$$

Claim 5.17. *Under the notation above, we have*

$$\sum_{m\in\mathcal{S}_D(x)} |U3^a_m(x)|^2 \lesssim |m^\mu_D f|^2 \sum_{Q\in J_1\cap \mathrm{Tr}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2} \|m_{s\in Q^0_{J(Q)}}(\Delta^\psi_{Q,s}h_x)\|_2^2\, \ell(D)^{-n}.$$

For simplicity of notation, we have set $\|\cdot\|_p := \|\cdot\|_{L^p(\mathcal{H}^n_{L^x_D})}$.

***Proof of* Claim 5.17.** Notice that $\mathcal{H}^n_{L^x_D}(A_m(x)) \lesssim (\epsilon_m - \epsilon_{m+1})\ell(D)^{n-1}$. Moreover, the function $m_{s\in Q^0_{J(Q)}}(\Delta^\psi_{Q,s}h_x)$ is supported in CQ and has vanishing integral, because the same holds for each $\Delta^\psi_{Q,s}h_x$ with $s\in Q^0_{J(Q)}$. Hence, the sum $\sum_{m\in\mathcal{S}_D(x)}|U3^a_m(x)|^2$ can be estimated using arguments very similar to the ones in Subsection 5.3.1 (see (43)), and the analogues of Lemma 5.6 and Claims 5.4 and 5.5 for $\mathcal{H}^n_{L^x_D}$ follow easily. One obtains the expected estimate. □

Claim 5.18. *Under the notation above, we have*

$$\sum_{m\in\mathcal{S}_D(x)} |U3^b_m(x)|^2 \lesssim |m^\mu_D f|^2 \sum_{Q\in J_1\cap \mathrm{Stp}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2} \frac{\|m_{s\in Q^0_{J(Q)}}(\widetilde{\Delta}^\psi_{Q,s}h_x)\|_1^2}{\ell(D)^n\ell(Q)^n}.$$

***Proof of* Claim 5.18.** Since $m_{s\in Q^0_{J(Q)}}(\widetilde{\Delta}^\psi_{Q,s}h_x)$ has vanishing integral and it is supported in a neighbourhood of Q, the term $U3^b_m(x)$ can be estimated in the same manner (but now we do not use the estimate $\|m_{s\in Q^0_{J(Q)}}(\widetilde{\Delta}^\psi_{Q,s}h_x)\|_1^2 \lesssim \ell(Q)^n\|m_{s\in Q^0_{J(Q)}}(\widetilde{\Delta}^\psi_{Q,s}h_x)\|_2^2$), and one obtains the expected estimate (compare with (45)). □

Recall that we have fixed $x\in D\in S\in \mathrm{Trs}$, and we denote by Q_S the maximal μ-cube in S from the corona decomposition, so $D\subset Q_S$. The following lemma, whose proof is given in Subsection 5.3.5, yields the suitable estimates for $m_{s\in Q^0_{J'(Q)}}(\Delta^\psi_{Q,s}h_x)$ and $m_{s\in Q^0_{J'(Q)}}(\widetilde{\Delta}^\psi_{Q,s}h_x)$.

Lemma 5.19. *Assume that $\alpha_\mu(D) < C_*$, for some constant $C_* > 0$ small enough. Given $Q\in\mathcal{D}^{n,0}_x$, there exists constants $C_1, C_2 > 1$ depending on C_* and b_* (see (59))such that,*

(a) *if $Q\in J_2$ and $\ell(Q) > \ell(Q_S)$, then $m_{s\in Q^0_{J'(Q)}}(\|\Delta^\psi_{Q,s}h_x\|_2) \lesssim \ell(Q_S)^n\ell(Q)^{-n/2}$,*

(b) *if $Q\in J_2$ and $\ell(Q)\le \ell(Q_S)$, then*

$$m_{s\in Q^0_{J'(Q)}}\big(\|\Delta^\psi_{Q,s}h_x\|_2\big) \lesssim \bigg(\sum_{R\in\mathcal{D}:\, D\subset R\subset B(z_Q,C_1\ell(Q))} \alpha_\mu(C_1R) + \frac{\mathrm{dist}(x,L_D)}{\ell(D)}\bigg)\ell(Q)^{n/2},$$

(c) *if $Q\in J_1\cap \mathrm{Tr}$, then there exists $Q_0\equiv Q_0(x,Q)\in\mathcal{D}$ depending on x and $Q\in\mathcal{D}^{n,0}_x$ such that $Q_0\subset C_2D$, $\ell(Q_0)\approx\ell(Q)$, $Q_0\cap(p^x)^{-1}(\mathrm{supp}\psi_Q)\neq\emptyset$ and*

$$\|m_{s\in Q^0_{J(Q)}}(\Delta^\psi_{Q,s}h_x)\|_2 \lesssim \bigg(\sum_{R\in\mathcal{D}:\, Q_0\subset R\subset C_2D} \alpha_\mu(C_2R) + \frac{\mathrm{dist}(x,L_D)}{\ell(D)}\bigg)\ell(Q)^{n/2}, \quad \textit{and}$$

(d) *if $Q\in J_1\cap\mathrm{Stp}$, then $\|m_{s\in Q^0_{J(Q)}}(\widetilde{\Delta}^\psi_{Q,s}h_x)\|_1\lesssim\ell(Q)^n$.*

We are ready to put all the estimates together to bound the first term on the right hand side of (50). From (53), (55), (58), and (60) we have

$$\begin{aligned}\sum_{m\in\mathcal{S}_D(x)} \left|(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * ((m_D^\mu f)\chi_{\widetilde{D}}\mu))(x)\right|^2 &\lesssim |m_D^\mu f|^2\alpha_\mu(10D)^2 \\ &+ \sum_{m\in\mathcal{S}_D(x)} (|U1_m(x)|^2 + |U3_m^a(x)|^2 + |U3_m^b(x)|^2 + |U4_m(x)|^2).\end{aligned} \tag{61}$$

Let us deal with $U1_m(x)$ (the term $|m_D^\mu f|^2\alpha_\mu(10D)^2$ above is handled in the same manner). If L_D^1 and L_D^2 denote a minimizing n-plane for $\beta_{1,\mu}(D)$ and $\beta_{2,\mu}(D)$, respectively, one can show that $\operatorname{dist}_{\mathcal{H}}(L_D\cap B_D, L_D^1\cap B_D)\lesssim \alpha_\mu(D)\ell(D)$ and $\operatorname{dist}_{\mathcal{H}}(L_D^1\cap B_D, L_D^2\cap B_D)\lesssim \beta_{2,\mu}(D)\ell(D)$, so we have $\operatorname{dist}(x,L_D)\lesssim \operatorname{dist}(x,L_D^2)+\beta_{2,\mu}(D)\ell(D)+\alpha_\mu(D)\ell(D)$ for $x\in D$. Then, by Claim 5.14 and Carleson's embedding theorem,

$$\begin{aligned}&\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)} |U1_m|^2\,d\mu \\ &\qquad\lesssim \sum_{D\in\mathcal{D}}\int_D |m_D^\mu f|^2\left(\beta_{1,\mu}(D)^2+\alpha_\mu(D)^2+\left(\frac{\operatorname{dist}(x,L_D)}{\ell(D)}\right)^2\right)d\mu(x) \\ &\qquad\lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2\ell(D)^n\big(\beta_{1,\mu}(D)^2+\alpha_\mu(D)^2+\beta_{2,\mu}(D)^2\big)\lesssim \|f\|_{L^2(\mu)}^2.\end{aligned} \tag{62}$$

For the case of $U3_m^a(x)$, by Claim 5.17 and Lemma 5.19(c) applied to the μ-cubes in $J_1\cap\mathrm{Tr}$, we have

$$\begin{aligned}&\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)} |U3_m^a|^2\,d\mu \\ &\qquad\lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2\int_D \sum_{Q\in J_1\cap\mathrm{Tr}}\left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2}\frac{\|m_{s\in Q^0_{J(Q)}}(\Delta_{Q,s}^\psi h_x)\|_2^2}{\ell(D)^n}\,d\mu(x) \\ &\qquad\lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2\int_D \sum_{Q\in J_1\cap\mathrm{Tr}}\left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2}\Bigg(\sum_{\substack{R\in\mathcal{D}:\\ Q_0(x,Q)\subset R\subset C_2D}}\alpha_\mu(C_2R)\Bigg)^2 d\mu(x) \\ &\qquad\quad+\sum_{D\in\mathcal{D}} |m_D^\mu f|^2\int_D \sum_{Q\in J_1\cap\mathrm{Tr}}\left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2}\left(\frac{\operatorname{dist}(x,L_D)}{\ell(D)}\right)^2 d\mu(x) =: S_1+S_2.\end{aligned}$$

Recall that $J_1\subset\{Q\in\mathcal{D}_x^{n,0}: Q\subset B(z_D,C_a\ell(D))\}$. Then $\sum_{Q\in J_1}(\ell(Q)/\ell(D))^{n+1/2}\lesssim 1$, and since $\operatorname{dist}(x,L_D)\lesssim \operatorname{dist}(x,L_D^2)+\beta_{2,\mu}(D)\ell(D)+\alpha_\mu(D)\ell(D)$ for $x\in D$, then $S_2\lesssim \sum_{D\in\mathcal{D}}|m_D^\mu f|^2(\beta_{2,\mu}(D)^2+\alpha_\mu(D)^2)\ell(D)^n$, and hence $S_2\le C\|f\|_{L^2(\mu)}^2$, by Carleson's embedding theorem. For S_1, since $\ell(Q)\approx\ell(Q_0(x,Q))$ (recall the definition of $Q_0\equiv Q_0(x,Q)$ in Lemma 5.19(c)), $Q_0(x,Q)\subset C_2D$, and every $Q_0\in\mathcal{D}$ intersects $(p^x)^{-1}(\operatorname{supp}\psi_Q)$ for finitely many cubes $Q\in\mathcal{D}_x^{n,0}$ (with a bound for the number of such cubes Q independent of x and

Q_0), we have

$$\begin{aligned}\sum_{Q\in J_1\cap \mathrm{Tr}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2} &\Bigg(\sum_{R\in\mathcal{D}:\, Q_0(x,Q)\subset R\subset C_2D} \alpha_\mu(C_2R)\Bigg)^2 \\ &= \sum_{P\in\mathcal{D}:\, P\subset C_2D}\ \sum_{\substack{Q\in\mathcal{D}_x^{n,0}:\, Q\subset B(z_D,C_a\ell(D)),\\ Q_0(x,Q)=P}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2}\Bigg(\sum_{R\in\mathcal{D}:\, P\subset R\subset C_2D} \alpha_\mu(C_2R)\Bigg)^2 \\ &\lesssim \sum_{P\in\mathcal{D}:\, P\subset C_2D} \left(\frac{\ell(P)}{\ell(D)}\right)^{n+1/2}\Bigg(\sum_{R\in\mathcal{D}:\, P\subset R\subset C_2D} \alpha_\mu(C_2R)\Bigg)^2.\end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}&\sum_{P\in\mathcal{D}:\, P\subset C_2D} \left(\frac{\ell(P)}{\ell(D)}\right)^{n+1/2}\Bigg(\sum_{R\in\mathcal{D}:\, P\subset R\subset C_2D} \alpha_\mu(C_2R)\Bigg)^2 \\ &\qquad\lesssim \sum_{P\in\mathcal{D}:\, P\subset C_2D} \left(\frac{\ell(P)}{\ell(D)}\right)^{n+1/2} \log_2\left(\frac{\ell(D)}{\ell(P)}\right) \sum_{R\in\mathcal{D}:\, P\subset R\subset C_2D} \alpha_\mu(C_2R)^2 \\ &\qquad\lesssim \sum_{R\in\mathcal{D}:\, R\subset C_2D} \alpha_\mu(C_2R)^2 \sum_{P\in\mathcal{D}:\, P\subset R} \left(\frac{\ell(P)}{\ell(D)}\right)^{n+1/4} \\ &\qquad\lesssim \sum_{R\in\mathcal{D}:\, R\subset C_2D} \alpha_\mu(C_2R)^2 \left(\frac{\ell(R)}{\ell(D)}\right)^{n+1/4} =: \lambda_1(D)^2.\end{aligned} \tag{63}$$

By standard arguments one can easily show that these λ_1 coefficients satisfy a Carleson packing condition, so by (63) and Carleson's embedding theorem we obtain $S_1 \lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \ell(D)^n \lambda_1(D)^2 \lesssim \|f\|^2_{L^2(\mu)}$, which combined with $S_2 \lesssim \|f\|^2_{L^2(\mu)}$ yields

$$\sum_{S\in\mathrm{Trs}} \sum_{D\in S} \int_D \sum_{m\in\mathcal{S}_D(x)} |U3_m^a|^2\,d\mu \lesssim \|f\|^2_{L^2(\mu)}. \tag{64}$$

Let us deal now with $U3_m^b$. By Claim 5.18 and Lemma 5.19(d) applied to the μ-cubes in $J_1\cap \mathrm{Stp}$, we have

$$\begin{aligned}&\sum_{S\in\mathrm{Trs}} \sum_{D\in S} \int_D \sum_{m\in\mathcal{S}_D(x)} |U3_m^b|^2\,d\mu \\ &\qquad\lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \int_D \sum_{Q\in J_1\cap\mathrm{Stp}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{1/2} \frac{\|m_{s\in Q^0_{J(Q)}}(\widetilde{\Delta}^\psi_{Q,s} h_x)\|_1^2}{\ell(D)^n \ell(Q)^n}\,d\mu(x) \\ &\qquad\lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \int_D \sum_{Q\in J_1\cap\mathrm{Stp}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2} d\mu.\end{aligned}$$

Given $D\in\mathcal{D}$, consider the family $\Lambda_D := \{R\in\mathcal{D}:\, R = R_Q \text{ for some } x\in D \text{ and some } Q\in J_1\cap\mathrm{Stp}\}$ (see the definition of R_Q in (59)). Observe that every $R\in\mathcal{D}$ intersects $(p^x)^{-1}(Q\cap L_D^x)$ for finitely many μ-cubes $Q\in\mathcal{D}_x^{n,0}$ such that $\ell(Q)=\ell(R)$. Thus, simlilarly to what we

did for $Q \in J_1 \cap \mathrm{Tr}$ in the case of $U3_m^a$, we have

$$\sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \int_D \sum_{Q\in J_1\cap \mathrm{Stp}} \left(\frac{\ell(Q)}{\ell(D)}\right)^{n+1/2} d\mu \lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \int_D \sum_{R\in\Lambda_D} \left(\frac{\ell(R)}{\ell(D)}\right)^{n+1/2} d\mu$$
$$\lesssim \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \sum_{R\in\Lambda_D} \left(\frac{\ell(R)}{\ell(D)}\right)^{n+1/2} \mu(D) = \sum_{D\in\mathcal{D}} |m_D^\mu f|^2 \lambda_2(D)^2 \mu(D),$$

where we have set $\lambda_2(D)^2 := \sum_{R\in\Lambda_D} (\ell(R)/\ell(D))^{n+1/2}$. Since the α_μ's satisfy a Carleson packing condition, it is not hard to show that the same holds for the λ_2's. Indeed, since for any $R \in \Lambda_D$ we have $\sum_{R'\in\mathcal{D}:\, R\subset R',\, \ell(R)\leq \ell(D)} \alpha_\mu(10R') \geq b_*$ by (59), then

$$\lambda_2(D)^2 \leq b_*^{-2} \sum_{R\in\Lambda_D} \left(\frac{\ell(R)}{\ell(D)}\right)^{n+1/2} \Bigg(\sum_{R'\in\mathcal{D}:\, R\subset R',\, \ell(R)\leq\ell(D)} \alpha_\mu(10R') \Bigg)^2,$$

and we can proceed as in (63). Hence, putting these estimates together and using Carleson's embedding theorem for the λ_2's, we obtain

$$\sum_{S\in\mathrm{Trs}} \sum_{D\in S} \int_D \sum_{m\in\mathcal{S}_D(x)} |U3_m^b|^2 \, d\mu \lesssim \|f\|^2_{L^2(\mu)}. \tag{65}$$

We deal now with $U4_m(x)$. By Claim 5.16 and Lemma 5.19(a) and (b) applied to the cubes in J_2,

$$\sum_{S\in\mathrm{Trs}} \sum_{D\in S} \int_D \sum_{m\in\mathcal{S}_D(x)} |U4_m|^2 \, d\mu$$
$$\lesssim \sum_{S\in\mathrm{Trs}} \sum_{D\in S} |m_D^\mu f|^2 \int_D \sum_{Q\in J_2} \left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2} \frac{m_{s\in Q^0_{J'(Q)}} \big(\|\Delta^\psi_{Q,s} h_x\|_2\big)^2}{\ell(Q)^n} \, d\mu$$
$$\lesssim \sum_{S\in\mathrm{Trs}} \sum_{D\in S} |m_D^\mu f|^2 \int_D \sum_{Q\in J_2:\, \ell(Q)\leq\ell(Q_S)} \left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2} \tag{66}$$
$$\Bigg[\Bigg(\sum_{R\in\mathcal{D}:\, D\subset R\subset B(z_Q, C_1\ell(Q))} \alpha_\mu(C_1 R) \Bigg)^2 + \left(\frac{\mathrm{dist}(x, L_D)}{\ell(D)}\right)^2 \Bigg] d\mu$$
$$+ \sum_{S\in\mathrm{Trs}} \sum_{D\in S} |m_D^\mu f|^2 \int_D \sum_{Q\in J_2:\, \ell(Q)>\ell(Q_S)} \left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2} \frac{\ell(Q_S)^{2n}}{\ell(Q)^{2n}} \, d\mu =: S_3 + S_4.$$

Regarding S_3, since $\mathrm{dist}(x, L_D) \lesssim \mathrm{dist}(x, L_D^2) + \beta_{2,\mu}(D)\ell(D) + \alpha_\mu(D)\ell(D)$ for $x \in D$ and $\sum_{Q\in J_2} (\ell(D)/\ell(Q))^{1/2} \lesssim 1$, the second term in the definition of S_3 is bounded by $\sum_{D\in\mathcal{D}} |m_D^\mu f|^2 (\beta_{2,\mu}(D)^2 + \alpha_\mu(D)^2)\ell(D)^n$, and hence by $C\|f\|^2_{L^2(\mu)}$, by Carleson's embedding

theorem. For the first term in S_3, by Cauchy-Schwarz inequality,

$$\sum_{S\in \mathrm{Trs}}\sum_{D\in S}|m_D^\mu f|^2\int_D \sum_{Q\in J_2:\,\ell(Q)\leq \ell(Q_S)}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2}\Bigg(\sum_{\substack{R\in\mathcal{D}:\\ D\subset R\subset B(z_Q,C_1\ell(Q))}}\alpha_\mu(C_1R)\Bigg)^2 d\mu$$

$$\lesssim \sum_{S\in \mathrm{Trs}}\sum_{D\in S}|m_D^\mu f|^2\int_D \sum_{\substack{Q\in J_2:\\ \ell(Q)\leq \ell(Q_S)}}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2}\log_2\left(\frac{\ell(Q)}{\ell(D)}\right)\sum_{\substack{R\in\mathcal{D}:\\ D\subset R\subset B(z_Q,C_1\ell(Q))}}\alpha_\mu(C_1R)^2\,d\mu$$

$$\lesssim \sum_{D\in\mathcal{D}}|m_D^\mu f|^2\int_D \sum_{\substack{R\in\mathcal{D}:\\ D\subset R}}\alpha_\mu(C_1R)^2\sum_{\substack{Q\in\mathcal{D}_x^{n,0}:\\ R\subset B(z_Q,C_1\ell(Q))}}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/4} d\mu.$$

Notice that $\sum_{Q\in\mathcal{D}_x^{n,0}:\,R\subset B(z_Q,C_1\ell(Q))}\big(\ell(D)/\ell(Q)\big)^{1/4}\lesssim \big(\ell(D)/\ell(R)\big)^{1/4}$, thus the right side of the preceeding inequality is bounded above by

$$\sum_{D\in\mathcal{D}}|m_D^\mu f|^2\ell(D)^n\sum_{R\in\mathcal{D}:\,D\subset R}\alpha_\mu(C_1R)^2\left(\frac{\ell(D)}{\ell(R)}\right)^{1/4}=:\sum_{D\in\mathcal{D}}|m_D^\mu f|^2\ell(D)^n\lambda_3(D)^2. \tag{67}$$

By standard arguments one can show that the λ_3's satisfy a Carleson packing condition, so by Carleson's embedding theorem again, the last term in (67) is bounded by $C\|f\|^2_{L^2(\mu)}$. Thus we obtain $S_3\lesssim \|f\|^2_{L^2(\mu)}$.

The estimate of S_4 from (66) is easier:

$$S_4\lesssim \sum_{S\in \mathrm{Trs}}\sum_{D\in S}|m_D^\mu f|^2\int_D \sum_{\substack{Q\in\mathcal{D}_x^{n,0}:\,\ell(Q)>\ell(Q_S),\\ D\subset B(z_Q,C_1\ell(Q))}}\frac{\ell(D)^{1/2}\ell(Q_S)^{2n}}{\ell(Q)^{2n+1/2}}\,d\mu(x).$$

As before, $\sum_{Q\in\mathcal{D}_x^{n,0}:\,\ell(Q)>\ell(Q_S),\,D\subset B(z_Q,C_1\ell(Q))}\ell(Q)^{-2n-1/2}\lesssim \ell(Q_S)^{-2n-1/2}$, thus

$$\begin{aligned}S_4&\lesssim \sum_{S\in \mathrm{Trs}}\sum_{D\in S}|m_D^\mu f|^2\ell(D)^n\left(\frac{\ell(D)}{\ell(Q_S)}\right)^{1/2}\lesssim \sum_{D\in\mathcal{D}}|m_D^\mu f|^2\ell(D)^n\sum_{S\in\mathrm{Trs}:\,S\ni D}\left(\frac{\ell(D)}{\ell(Q_S)}\right)^{1/2}\\ &=:\sum_{D\in\mathcal{D}}|m_D^\mu f|^2\ell(D)^n\lambda_4(D)^2.\end{aligned}$$

Similarly to the case of the λ_3 coefficients, one can show that the λ_4's also satisfy a Carleson packing condition, thus $S_4\lesssim \|f\|^2_{L^2(\mu)}$ by Carleson's embedding theorem. Actually, if one defines $\widehat{\alpha}_\mu(Q)=1$ if $Q=Q_S$ for some $S\in\mathrm{Trs}$ and $\widehat{\alpha}_\mu(Q)=0$ otherwise, using the packing condition for the μ-cubes Q_S with $S\in\mathrm{Trs}$, one can easily verify that the $\widehat{\alpha}_\mu$'s satisfy a Carleson packing condition. Then,

$$\lambda_4(D)^2=\sum_{S\in\mathrm{Trs}:\,D\subset Q_S}\left(\frac{\ell(D)}{\ell(Q_S)}\right)^{1/2}\widehat{\alpha}_\mu(Q_S)^2=\sum_{Q\in\mathcal{D}:\,D\subset Q}\left(\frac{\ell(D)}{\ell(Q)}\right)^{1/2}\widehat{\alpha}_\mu(Q)^2,$$

and we can argue as in the case of the λ_3's in (67).

By the estimates of S_3 and S_4, we obtain

$$\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D\sum_{m\in\mathcal{S}_D(x)}|U4_m|^2\,d\mu\lesssim\|f\|^2_{L^2(\mu)}. \tag{68}$$

Finally, plugging (62), (64), (65), and (68) in (61), and combining the result with (50) and (51), we conclude that

$$\sum_{S\in \mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)}\left|\sum_{R\in V(D)}(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*((m_R^\mu f)\chi_R\mu))(x)\right|^2 d\mu(x)\lesssim \|f\|^2_{L^2(\mu)},$$

and Lemma 5.13 is finally proved, except for Lemma 5.19. □

5.3.5. **Proof of Lemma 5.19**.

***Proof of* Lemma 5.19**(a)**.** By Definition 5.15(e), for any $s\in Q^0_{J'(Q)}$ we have

$$\begin{aligned}\|\Delta^\psi_{Q,s}h_x\|_\infty &\lesssim |\langle h_x,\psi_{s+Q}\rangle|\ell(Q)^{-n/2}\lesssim \ell(Q)^{-n}\int h_x\,d\mathcal{H}^n_{L^x_D}=\ell(Q)^{-n}\int d\nu_x\\ &=\ell(Q)^{-n}\int d(p^x_\sharp(\chi_{40Q_S}\mu))=\ell(Q)^{-n}\int_{40Q_S}d\mu\lesssim\frac{\ell(Q_S)^n}{\ell(Q)^n}.\end{aligned}$$

Hence, $\|\Delta^\psi_{Q,s}h_x\|_2\le\|\Delta^\psi_{Q,s}h_x\|_\infty\mathcal{L}^n(\mathrm{supp}\psi_{s+Q})^{1/2}\lesssim\ell(Q_S)^n\ell(Q)^{-n/2}$ for all $s\in Q^0_{J'(Q)}$, and Lemma 5.19(a) follows by taking the average over $s\in Q^0_{J'(Q)}$. □

***Proof of* Lemma 5.19**(b)**.** Since $D\subset B(z_Q,C_a\ell(Q))$, $D\in S$, and $\ell(D)\lesssim\ell(Q)\le\ell(Q_S)$, by taking C_{cor} big enough (see property (f) in Subsection 2.4), we can assume that μ is well approximated by Γ_S in a neighborhood of Q. We are going to show that, for each $s\in Q^0_{J'(Q)}$,

$$\|\Delta^\psi_{Q,s}h_x\|_2\lesssim\Bigg(\sum_{R\in\mathcal{D}:\,D\subset R\subset B(z_Q,C_1\ell(Q))}\alpha_\mu(C_1R)+\frac{\mathrm{dist}(x,L_D)}{\ell(D)}\Bigg)\ell(Q)^{n/2},\tag{69}$$

and Lemma 5.19(b) will follow by taking the average over $s\in Q^0_{J'(Q)}$.

Fix $Q\in J_2$, so $D\subset B(z_Q,C_a\ell(Q))$ with $\ell(Q)\le\ell(Q_S)$, and $s\in Q^0_{J'(Q)}$. Take $Q'\in\mathcal{D}$ such that $\ell(Q)=\ell(Q')$ and $Q\subset B(z_{Q'},3\ell(Q))$. Recall that $\mathrm{supp}\psi_{s+Q}\subset CQ$ and $|\nabla\psi_{s+Q}|\lesssim\ell(Q)^{-n/2-1}$. Let $\phi_{s+Q'}$ be an extension of ψ_{s+Q}, i.e., let $\phi_{s+Q'}:\mathbb{R}^d\to\mathbb{R}$ be such that $\mathrm{supp}\phi_{s+Q'}\subset B_{Q'}\subset\mathbb{R}^d$, $|\nabla\phi_{s+Q'}|\lesssim\ell(Q')^{-n/2-1}$ and $\phi_{s+Q'}=\psi_{s+Q}$ in L^x_D.

Let $L_{Q'}$ be a minimizing n-plane for $\alpha_\mu(C_1Q')$, where $C_1>1$ is some big constant to be fixed below, and let $L^x_{Q'}$ be the n-plane parallel to $L_{Q'}$ which contains x. Let $\sigma_{Q'}:=c_{Q'}\mathcal{H}^n_{L_{Q'}}$ be a minimizing measure for $\alpha_\mu(C_1Q')$ and define $\sigma^x_{Q'}:=c_{Q'}\mathcal{H}^n_{L^x_{Q'}}$. Finally, set $\sigma:=c_{Q'}\mathcal{H}^n_{L^x_D}$. Since ψ_{s+Q} has vanishing integral in L^x_D, we also have $\int\phi_{s+Q'}\,d\mathcal{H}^n_{L^x_D}=0$. Hence,

$$\begin{aligned}\|\Delta^\psi_{Q,s}h_x\|_2=\|\langle h_x,\psi_{s+Q}\rangle\psi_{s+Q}\|_2=|\langle h_x,\psi_{s+Q}\rangle|&=\Big|\int_{L^x_D}\phi_{s+Q}(y)\,d\nu_x(y)\Big|\\ =\Big|\int\phi_{s+Q'}(y)\,d(\nu_x-\sigma)(y)\Big|&\lesssim\ell(Q)^{-n/2-1}\mathrm{dist}_{B_{Q'}}(\nu_x,\sigma).\end{aligned}\tag{70}$$

We can assume that

$$\sum_{R\in\mathcal{D}:\,D\subset R\subset B(z_Q,C_1\ell(Q))}\alpha_\mu(C_1R)\le b_*,\tag{71}$$

otherwise Lemma 5.19(b) follows easily. By assuming (71) one can show that the angle between L^x_D and $L^x_{Q'}$ is small. By the triangle inequality, we have

$$\mathrm{dist}_{B_{Q'}}(\nu_x,\sigma)\le\mathrm{dist}_{B_{Q'}}(\nu_x,p^x_\sharp\sigma^x_{Q'})+\mathrm{dist}_{B_{Q'}}(p^x_\sharp\sigma^x_{Q'},\sigma).\tag{72}$$

To deal with the first term on the right hand side of (72), let h be a Lipschitz function such that $\operatorname{supp} h \subset B_{Q'}$ and $\operatorname{Lip}(h) \leq 1$. Then, using that $\operatorname{supp}\mu$ is well approximated in CQ' by a Lipschitz graph Γ_S with small slope, the function $h \circ p^x$ restricted to $\operatorname{supp}\mu \cup L^x_Q$ can be extended to a Lipschitz function supported in $B_{C_1Q'}$ (if C_1 is big enough) with $\operatorname{Lip}(h \circ p^x)$ bounded by a constant which only depends on n, d, and $\operatorname{Lip}(\Gamma_S)$. Therefore,

$$\begin{aligned}(73)\quad &\Big|\int_{B_{Q'}} h\, d(\nu_x - p^x_\sharp \sigma^x_{Q'})\Big| = \Big|\int_{B_{C_1Q'}} h\circ p^x\, d(\mu - \sigma^x_{Q'})\Big| \lesssim \operatorname{dist}_{B_{C_1Q'}}(\mu, \sigma^x_{Q'})\\ &\leq \operatorname{dist}_{B_{C_1Q'}}(\mu, \sigma_{Q'}) + \operatorname{dist}_{B_{C_1Q'}}(\sigma_{Q'}, \sigma^x_{Q'}) \lesssim \alpha_\mu(C_1Q')\ell(Q)^{n+1} + \operatorname{dist}(x, L_{Q'})\ell(Q)^n.\end{aligned}$$

Since $x \in D$ and $D \subset C_1Q'$ (if $C_1 > C_b$), by [To4, Remark 5.3] we have

$$\operatorname{dist}(x, L_{Q'}) \lesssim \sum_{R\in\mathcal{D}:\, D\subset R\subset C_1Q'} \alpha_\mu(R)\ell(R) + \operatorname{dist}(x, L_D). \tag{74}$$

Taking the supremum over all possible Lipschitz functions h in (73) and using that $\ell(D) \leq \ell(R) \lesssim \ell(Q)$ in the sum above, we get

$$\operatorname{dist}_{B_{Q'}}(\nu_x, p^x_\sharp \sigma^x_{Q'}) \lesssim \sum_{R\in\mathcal{D}:\, D\subset R\subset C_1Q'} \alpha_\mu(C_1R)\ell(Q)^{n+1} + \frac{\operatorname{dist}(x, L_D)}{\ell(D)}\,\ell(Q)^{n+1}. \tag{75}$$

To estimate the second term on the right hand side of (72), notice that $p^x_\sharp \sigma = \sigma$ because $p^x|_{L^x_D} = \mathrm{Id}$. Hence, as in (73),

$$\begin{aligned}\operatorname{dist}_{B_{Q'}}(p^x_\sharp \sigma^x_{Q'}, \sigma) &= \operatorname{dist}_{B_{Q'}}(p^x_\sharp \sigma^x_{Q'}, p^x_\sharp \sigma) \lesssim \operatorname{dist}_{B_{C_1Q'}}(\sigma^x_{Q'}, \sigma)\\ &\leq \operatorname{dist}_{B_{C_1Q'}}(\sigma^x_{Q'}, \sigma_{Q'}) + \operatorname{dist}_{B_{C_1Q'}}(\sigma_{Q'}, \sigma)\\ &\lesssim \operatorname{dist}_{B_{C_1Q'}}(\mathcal{H}^n_{L^x_{Q'}}, \mathcal{H}^n_{L_{Q'}}) + \operatorname{dist}_{B_{C_1Q'}}(\mathcal{H}^n_{L_{Q'}}, \mathcal{H}^n_{L_D}) + \operatorname{dist}_{B_{C_1Q'}}(\mathcal{H}^n_{L_D}, \mathcal{H}^n_{L^x_D})\\ &\lesssim \operatorname{dist}(x, L_{Q'})\ell(Q)^n + \operatorname{dist}_{B_{C_1Q'}}(\mathcal{H}^n_{L_{Q'}}, \mathcal{H}^n_{L_D}) + \operatorname{dist}(x, L_D)\ell(Q)^n.\end{aligned}$$

The term $\operatorname{dist}_{B_{C_1Q'}}(\mathcal{H}^n_{L_{Q'}}, \mathcal{H}^n_{L_D})$ can be estimated using the intermediate μ-cubes between D and C_1Q' (similarly to (75)), and we obtain

$$\operatorname{dist}_{B_{C_1Q}}(\mathcal{H}^n_{L_Q}, \mathcal{H}^n_{L_D}) \lesssim \sum_{R\in\mathcal{D}:\, D\subset R\subset C_1Q} \alpha_\mu(C_1R)\ell(Q)^{n+1}.$$

Thus, by (74) and since $\ell(D) \lesssim \ell(Q)$,

$$\operatorname{dist}_{B_{Q'}}(p^x_\sharp \sigma^x_{Q'}, \sigma) \lesssim \sum_{R\in\mathcal{D}:\, D\subset R\subset C_1Q'} \alpha_\mu(C_1R)\ell(Q)^{n+1} + \frac{\operatorname{dist}(x, L_D)}{\ell(D)}\,\ell(Q)^{n+1}.$$

Then, (69) follows by plugging this last inequality and (75) in (72) combined with (70), and recalling that $\ell(Q) \approx \ell(Q')$. Thus we are done with Lemma 5.19(b).

□

***Proof of* Lemma 5.19**(c)**.** Given $Q \in J_1 \cap \mathrm{Tr}$, using (59) we have

$$\sum_{R'\in\mathcal{D}:\, R\subset R',\, \ell(R')\leq \ell(D)} \alpha_\mu(10R') < b_*$$

for all $R \in \mathcal{D}$ with $\ell(R) = \ell(Q)$ and such that $R \cap (p^x)^{-1}(\operatorname{supp}\psi_{s+Q}) \neq \emptyset$ for all $s \in Q^0_{J(Q)}$. By assuming b_* small enough, we are going to show that for some $Q_0(x, Q) \in \mathcal{D}$ as in the

statement (c) and all $s \in Q^0_{J(Q)}$ we have

$$\|\Delta^\psi_{Q,s} h_x\|_2 \lesssim \Big(\sum_{R\in\mathcal{D}:\, Q_0\subset R\subset C_2 D} \alpha_\mu(C_2R) + \frac{\operatorname{dist}(x, L_D)}{\ell(D)} \Big) \ell(Q)^{n/2} \tag{76}$$

As before, Lemma 5.19(c) will follow by averaging over $s \in Q^0_{J(Q)}$, and noting that $\|m_{s\in Q^0_{J(Q)}}(\Delta^\psi_{Q,s} h_x)\|_2 \leq m_{s\in Q^0_{J(Q)}} \|\Delta^\psi_{Q,s} h_x\|_2$ by Minkowski's integral inequality.

Take $Q \in J_1 \cap \mathrm{Tr}$. Let C_2 be some big constant which will be fixed later on, and let $Q_0 \in \mathcal{D}$ be a minimal μ-cube such that $C_2 Q_0$ contains $\operatorname{supp}\mu \cap (p^x)^{-1}(\operatorname{supp}\psi_{s+Q} \cap L^x_D)$ for all $s \in J(Q)$. We can assume that $Q_0 \subset C_2 D$ if C_2 is big enough and, by (59), we may also suppose that $\sum_{R\in\mathcal{D}:\, Q_0\subset R\subset C_2 D} \alpha_\mu(C_2R)$ is small enough. Hence, if L_{Q_0} is a minimizing n-plane for $\beta_{\infty,\mu}(C_2Q_0)$, the angle between L_{Q_0} and L^x_D is also small enough, since it is bounded by $\sum_{R\in\mathcal{D}:\, Q_0\subset R\subset C_2 D} \alpha_\mu(C_2R)$ (see [To4, Lemma 5.2] for a related argument). It is not hard to show that then

$$\operatorname{diam}(\Gamma \cap (p^x)^{-1}(Q \cap L^x_D)) \lesssim \ell(Q). \tag{77}$$

Let L_{Q_0} and $\sigma_{Q_0} := c_{Q_0} \mathcal{H}^n_{L_{Q_0}}$ be a minimizing n-plane and measure for $\alpha_\mu(C_2Q_0)$, respectively. Fix $z_{Q_0} \in L_{Q_0} \cap B_{C_2Q_0}$ and let L_r be an n-plane parallel to L^x_D which contains z_{Q_0}. Finally, define the measures $\sigma_r := c_{Q_0} \mathcal{H}^n_{L_r}$ and $\sigma' := c_{Q_0} \mathcal{H}^n_{L^x_D}$.

Since σ' is a multiple of $\mathcal{H}^n_{L^x_D}$, similarly to (70) and using the triangle inequality,

$$\begin{aligned} \|\Delta^\psi_{Q,s} h_x\|_2 \ell(Q)^{n/2+1} &\lesssim \operatorname{dist}_{B_Q}(\nu_x, \sigma') \\ &\leq \operatorname{dist}_{B_Q}(\nu_x, p^x_\sharp \sigma_{Q_0}) + \operatorname{dist}_{B_Q}(p^x_\sharp \sigma_{Q_0}, p^x_\sharp \sigma_r) + \operatorname{dist}_{B_Q}(p^x_\sharp \sigma_r, \sigma'), \end{aligned} \tag{78}$$

where we have set $B_Q := B(z_Q, 3\ell(Q)) \subset \mathbb{R}^d$ (for these computations, we may also assume that $\ell(Q)$ is small enough in comparison with $\ell(D)$).

Arguing as in (73), if C_2 is big enough, we have

$$\operatorname{dist}_{B_Q}(\nu_x, p^x_\sharp \sigma_{Q_0}) = \operatorname{dist}_{B_Q}(p^x_\sharp \mu, p^x_\sharp \sigma_{Q_0}) \lesssim \alpha_\mu(C_2Q_0) \ell(Q)^{n+1}, \tag{79}$$

and

$$\operatorname{dist}_{B_Q}(p^x_\sharp \sigma_{Q_0}, p^x_\sharp \sigma_r) \lesssim \operatorname{dist}_{B_{C_2Q_0}}(\sigma_{Q_0}, \sigma_r) \lesssim \operatorname{dist}_{\mathcal{H}}(L_{Q_0} \cap B_{C_2Q_0}, L_r \cap B_{C_2Q_0}) \ell(Q)^n.$$

Let γ be the angle between L_r and L_{Q_0} (which is the same as the one between L_D and L_{Q_0}). Since $z_{Q_0} \in L_{Q_0} \cap L_r \cap B_{C_2Q_0}$, we have $\operatorname{dist}_{\mathcal{H}}(L_{Q_0} \cap B_{C_2Q_0}, L_r \cap B_{C_2Q_0}) \lesssim \sin(\gamma)\ell(Q)$, and it is not difficult to show that $\sin(\gamma) \lesssim \sum_{R\in\mathcal{D}:\, Q_0\subset R\subset C_2 D} \alpha_\mu(C_2R)$. Thus,

$$\operatorname{dist}_{B_Q}(p^x_\sharp \sigma_{Q_0}, p^x_\sharp \sigma_r) \lesssim \sum_{R\in\mathcal{D}:\, Q_0\subset R\subset C_2 D} \alpha_\mu(C_2R) \ell(Q)^{n+1}. \tag{80}$$

Let us estimate the last term on the right hand side of (78). Since $c_{Q_0} \lesssim 1$, we have $\operatorname{dist}_{B_Q}(p^x_\sharp \sigma_r, \sigma') \lesssim \operatorname{dist}_{B_Q}(p^x_\sharp \mathcal{H}^n_{L_r}, \mathcal{H}^n_{L^x_D})$. Let h be a 1-Lipschitz function supported in B_Q and such that Set $d := \operatorname{dist}(z_{Q_0}, L^x_D)$. Since $Q \in J_1 \subset J$ and $\ell(Q) \leq C\ell(D)$, if C is small enough then $\operatorname{dist}(x, B_Q) \gtrsim \ell(D)$. Without loss of generality, we may assume that $x = 0$ and that $L^x_D = \mathbb{R}^n \times \{0\}^{d-n}$, so $L_r = z_{Q_0} + \mathbb{R}^n \times \{0\}^{d-n}$. Thus, if we set $z'_{Q_0} := (z^{n+1}_{Q_0}, \ldots, z^d_{Q_0})$, we have that $d = |z'_{Q_0}|$ and p^x restricted to $L_r \cap B_Q$ can be written in the following manner: $p^x : y = (y^1, \ldots, y^n, z'_{Q_0}) \mapsto (F(y^1, \ldots, y^n), 0)$, where $F : \mathbb{R}^n \setminus \{0\}^n \to \mathbb{R}^n$ is defined by

$$F(y) = y \frac{\sqrt{|y|^2 + d^2}}{|y|} = y \sqrt{1 + \frac{d^2}{|y|^2}}.$$

Therefore, $\int h\,d(p^x_\sharp \mathcal{H}^n_{L_r}) = \int h\circ p^x\,d\mathcal{H}^n_{L_r} = \int_{\mathbb{R}^n} (h\circ p^x)(y, z'_{Q_0})\,dy = \int_{\mathbb{R}^n} h(F(y),0)\,dy$, and we also have $\int h\,d\mathcal{H}^n_{L^x_D} = \int_{\mathbb{R}^n} h((y,0))\,dy = \int_{\mathbb{R}^n} h(F(y),0)J(F)(y)\,dy$ by a change of variables, where $J(F)$ denotes the Jacobian of F. Hence

$$\left|\int h\,d(p^x_\sharp \mathcal{H}^n_{L_r} - \mathcal{H}^n_{L^x_D})\right| \lesssim \int_{\mathbb{R}^n} |h(F(y),0)||1-J(F)(y)|\,dy. \tag{81}$$

Notice that, because of the assumptions on $\mathrm{supp} h(F(\cdot),0)$ and since $z_{Q_0}\in B_{C_2Q_0}$ and $Q_0\subset C_2D$, we have $d\lesssim |y|$ for all $y\in \mathrm{supp} h(F(\cdot),0)$. If F_i denotes the i'th coordinate of F, it is straightforward to check that $\partial_{y^j}F_i(y) = -d^2y^iy^j|y|^{-3}(|y|^2+d^2)^{-1/2}$ if $i\neq j$ and $\partial_{y^i}F_i(y) = (1+d^2/|y|^2)^{1/2} - d^2(y^i)^2|y|^{-3}(|y|^2+d^2)^{-1/2}$. Thus, we easily obtain

$$|1-J(F)(y)| \lesssim d/|y| \lesssim d/\ell(D) \tag{82}$$

for all $y\in \mathrm{supp} h(F(\cdot),0)$. Since $\mathrm{diam}(\mathrm{supp} h(F(\cdot),0)) \lesssim \ell(Q)$ and $h((F(\cdot),0))$ is Lipschitz, using (82) and taking the supremum in (81) over all such functions h, we have $\mathrm{dist}_{B_Q}(p^x_\sharp \mathcal{H}^n_{L_r}, \mathcal{H}^n_{L^x_D}) \lesssim \ell(Q)^{n+1}d/\ell(D)$. Finally, by [To4, Remark 5.3] and since $z_{Q_0}\in L_{Q_0}$,

$$d\lesssim \mathrm{dist}(z_{Q_0},L_D) + \mathrm{dist}(L_D,L^x_D) \lesssim \sum_{R\in\mathcal{D}:\,Q_0\subset R\subset C_2D} \alpha_\mu(C_2R)\ell(R) + \mathrm{dist}(x,L_D),$$

and thus

$$\mathrm{dist}_{B_Q}(p^x_\sharp \mathcal{H}^n_{L_r}, \mathcal{H}^n_{L^x_D}) \lesssim \sum_{R\in\mathcal{D}:\,Q_0\subset R\subset C_2D} \alpha_\mu(C_2R)\ell(Q)^{n+1} + \frac{\mathrm{dist}(x,L_D)}{\ell(D)}\,\ell(Q)^{n+1}. \tag{83}$$

Finally, (76) follows by applying (79), (80), and (83) to (78), which yields Lemma 5.19(c). □

***Proof of* Lemma 5.19**(d)**.** This is the key point where taking averages of dyadic lattices with respect to the parameter s is necessary. Given $Q\in J_1\cap \mathrm{Stp}$, we have to show that $\|m_{s\in Q^0_{J(Q)}}(\widetilde{\Delta}^\psi_{Q,s}h_x)\|_1 \lesssim \ell(Q)^n$. Unlike in $(a),\ldots,(c)$, the estimate in (d) does not hold for a particular choice of s in general but, as we will see, it holds in average. Recall that, for a fixed $s\in Q^0_{J(Q)}$,

$$\begin{aligned}
\widetilde{\Delta}^\psi_{Q,s}h_x &= \sum_{R\in\mathcal{P}:\,R\subset Q} \Delta^\psi_{R,s}h_x \\
&= \sum_{\substack{R\in\mathcal{P}:\,\mathrm{supp}\psi_R\cap Q\neq\emptyset\\ \ell(R)\leq\ell(Q)}} \chi_{s+Q}\,\Delta^\psi_{R,s}h_x - \sum_{\substack{R\in\mathcal{P}:\,\mathrm{supp}\psi_R\cap Q\neq\emptyset\\ \ell(R)\leq\ell(Q),\,R\not\subset Q}} \chi_{s+Q}\,\Delta^\psi_{R,s}h_x \\
&\quad + \sum_{\substack{R\in\mathcal{P}:\\ R\subset Q}} \chi_{(s+Q)^c}\,\Delta^\psi_{R,s}h_x =: I_s + II_s + III_s.
\end{aligned}$$

We are going to estimate I_s, II_s, and III_s separately. For the case of I_s, we have

$$\chi_{s+Q}\,h_x = \chi_{s+Q} \sum_{R\in\mathcal{D}^{n,0}_x:\,\ell(R)>\ell(Q)} \Delta^\psi_{R,s}h_x + \chi_{s+Q} \sum_{R\in\mathcal{D}^{n,0}_x:\,\ell(R)\leq\ell(Q)} \Delta^\psi_{R,s}h_x = \chi_{s+Q}\,I'_s + I_s,$$

where we have set $I'_s := \sum_{R\in\mathcal{D}^{n,0}_x:\,\ell(R)>\ell(Q)} \Delta^\psi_{R,s}h_x$. On one hand, since $Q\in J_1\cap\mathrm{Stp}$, (59) holds. Thus, using that $\sum_{R\in\mathcal{D}:\,P(R_Q)\subset R,\,\ell(R)\leq\ell(D)} \alpha_\mu(10R) < b_*$, one can show that

$$\|\chi_{s+Q}\,h_x\|_1 \lesssim \ell(Q)^n \tag{84}$$

(see above (77) for a related argument). On the other hand, since $\|\chi_{s+Q}\, h_x\|_1 \lesssim \ell(Q)^n$, it is known that then $\|\chi_{s+Q} I'_s\|_1 \lesssim \ell(Q)^n$ (see [Da, Part I], in particular pay attention to the last sum in equation (46) of Part I). Combining these estimates, we conclude that $\|I_s\|_1 \lesssim \ell(Q)^n$.

Let us now deal with II_s. First of all, split II_s into different scales, that is

$$\sum_{\substack{R\in\mathcal{P}:\,\mathrm{supp}\psi_R\cap Q\neq\emptyset\\ \ell(R)\leq\ell(Q),\,R\not\subset Q}} \chi_{s+Q}\,\Delta^{\psi}_{R,s} h_x = \sum_{k\geq J(Q)}\ \sum_{\substack{R\in\mathcal{P}:\,\mathrm{supp}\psi_R\cap Q\neq\emptyset\\ \ell(R)=2^{-k},\,R\not\subset Q}} \chi_{s+Q}\,\Delta^{\psi}_{R,s} h_x.$$

Observe that if $k \geq J(Q)$, $\mathrm{supp}\psi_R \cap Q \neq \emptyset$, $\ell(R) = 2^{-k}$, and $R \not\subset Q$, then $s + R \subset U_{C2^{-k}}(s + \partial Q)$, where $C > 1$ is some fixed constant and $U_{C2^{-k}}(s + \partial Q) := \{z \in L^x_D : \mathrm{dist}(z, s + \partial Q) < C2^{-k}\}$. Hence, using Definition 5.15(e) and the definition of h_x, we get

$$\|II_s\|_1 \leq \sum_{k\geq J(Q)}\ \sum_{\substack{R\in\mathcal{P}:\,\mathrm{supp}\psi_R\cap Q\neq\emptyset\\ \ell(R)=2^{-k},\,R\not\subset Q}} \|\Delta^{\psi}_{R,s} h_x\|_1 \lesssim \sum_{k\geq J(Q)} \nu_x(U_{C2^{-k}}(s + \partial Q)).$$

The case of III_s can be dealt with very similar techniques, and then one obtains the same estimate. Therefore,

$$\begin{aligned}\|m_{s\in Q^0_{J(Q)}}(\widetilde{\Delta}^{\psi}_{Q,s} h_x)\|_1 &= \|m_{s\in Q^0_{J(Q)}}(I_s + II_s + III_s)\|_1 \leq m_{s\in Q^0_{J(Q)}}\|I_s + II_s + III_s\|_1\\ &\lesssim \ell(Q)^n + m_{s\in Q^0_{J(Q)}}\Big(\sum_{k\geq J(Q)} \nu_x(U_{C2^{-k}}(s + \partial Q))\Big).\end{aligned} \tag{85}$$

Using Fubini's theorem, it is not difficult to show that

$$m_{s\in Q^0_{J(Q)}}\nu_x(U_{C2^{-k}}(s + \partial Q))\big) \lesssim 2^{-k}\ell(Q)^{-1}\nu_x(CQ)$$

for all for $k \geq J(Q)$ (see [To2, Lemma 7.5] for example, for a related argument). Since $Q \in \mathrm{Stp}$, then (59) holds and then, as in (84), we have $\nu_x(CQ) \lesssim \ell(Q)^n$, thus

$$m_{s\in Q^0_{J(Q)}}\Big(\sum_{k\geq J(Q)} \nu_x(U_{C2^{-k}}(s + \partial Q))\Big) \lesssim \ell(Q)^n.$$

If we combine this last estimate with (85), we are done. □

5.3.6. **Final estimates**. From Lemmas 5.3, 5.9, and 5.13, we obtain the following:

$$\begin{aligned}&\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Tr}(R)}(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (\Delta_Q f)\mu)(x)\Big|^2 d\mu(x)\\ &\quad+\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}\sum_{Q\in\mathrm{Stp}(R)}(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (\widetilde{\Delta}_Q f)\mu)(x)\Big|^2 d\mu(x)\\ &\qquad+\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)}\Big|\sum_{R\in V(D)}(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (m^{\mu}_R f)\chi_R\mu)(x)\Big|^2 d\mu(x) \lesssim \|f\|^2_{L^2(\mu)}.\end{aligned}$$

Combining this estimate with (38), we deduce

$$\sum_{S\in\mathrm{Trs}}\sum_{D\in S}\int_D \sum_{m\in\mathcal{S}_D(x)}\big|(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (f\mu))(x)\big|^2 d\mu(x) \lesssim \|f\|^2_{L^2(\mu)}.$$

Finally, using (33) and (34), we conclude that

$$\|(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu) f\|_{L^2(\mu)}^2 \lesssim \sum_{D\in\mathcal{D}} \int_D \sum_{m\in\mathcal{S}_D(x)} \left|(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mu))(x)\right|^2 d\mu(x) \lesssim \|f\|_{L^2(\mu)}^2.$$

This finishes the proof of Theorem 5.1.

6. If $\mathcal{V}_\rho \circ \mathcal{R}^\mu : L^2(\mu) \to L^2(\mu)$ is a bounded operator, then μ is a uniformly n-rectifiable measure

Let $C_\mu > 0$ be the AD regularity constant of an AD regular measure μ, that is $C_\mu^{-1} r^n \leq \mu(B(x,r)) \leq C_\mu r^n$ for all $x \in \mathrm{supp}\mu$ and $0 \leq r < \mathrm{diam}(\mathrm{supp}\mu)$. For simplicity of notation, we may assume that $\mathrm{diam}(\mathrm{supp}\mu) = \infty$ (the general case follows with minor modifications in our arguments). As before, we denote by $\mathcal{D}$ the dyadic lattice of μ-cubes introduced in Subsection 2.3.

In this section, we set $K(x) = x|x|^{-n-1}$ for $x \neq 0$. Recall that, given $\epsilon > 0$, a Radon measure μ, and $f \in L^1(\mu)$, we have set $\mathcal{R}^\mu f := \{R_\epsilon^\mu f\}_{\epsilon>0}$, where

$$R_\epsilon^\mu f(x) = \int_{|x-y|>\epsilon} K(x-y) f(y)\, d\mu(y).$$

In order to prove the main theorem of this section, namely Theorem 6.8, we need first to introduce some notation and state some preliminary results.

Definition 6.1 (Special truncation of the Riesz transform)**.** *For $\epsilon > 0$, let φ_ϵ be as in Definition 2.1. Given $m \in \mathbb{Z}$ and a Radon measure μ in $\mathbb{R}^d$, we set*

$$S_m\mu(x) := \int \big(\varphi_{2^{-m-1}}(x-y) - \varphi_{2^{-m}}(x-y)\big) K(x-y)\, d\mu(y).$$

Lemma 6.2 (Lemma 5.8 of [DS1])**.** *Given $Q \in \mathcal{D}$, there exist $n+1$ points $x_0, \ldots, x_n$ in Q (and thus in $\mathrm{supp}\mu$) such that $\mathrm{dist}(x_j, L_{j-1}) \geq C\ell(Q)$, where L_k denotes the k-plane passing through $x_0, \ldots, x_k$, and where C depends only on n and C_μ.*

Lemma 6.3 (Lemma 7.4 and Remark 7.5 of [To4])**.** *Let $Q \in \mathcal{D}$ and $x_0, \ldots, x_n \in Q$ be like in Lemma 6.2. Denote $r = \mathrm{diam}(Q)$, and let $m, p \in \mathbb{Z}$ be such that $t \geq s > 4r$ for $t = 2^{-p}$ and $s = 2^{-m}$. Suppose that $A(x_0, 2^{-m-1/2}, 2^{-m+1/2}) \cap \mathrm{supp}\mu \neq \emptyset$. Then any point $x_{n+1} \in 3Q$ satisfies*

$$\mathrm{dist}(x_{n+1}, L_0) \lesssim s \sum_{j=1}^{n+1} \sum_{k=p}^{m} |S_k\mu(x_j) - S_k\mu(x_0)| + \frac{r^2}{s} + \frac{rs}{t}, \tag{86}$$

where L_0 is the n-plane passing through $x_0, \ldots, x_n$.

The following proposition is a direct consequence of the techniques used in the last section of [To4]. We give the proof for completeness.

Proposition 6.4. *Given $\epsilon_0 > 0$, there exist $\delta_0 > 0$ and $m_0, k_0 \in \mathbb{N}$ depending on ϵ_0, n, and C_μ such that, for all $i \in \mathbb{Z}$ and all $Q \in \mathcal{D}_i$ with $\beta_{1,\mu}(Q) > \epsilon_0$, there exist $k \in \mathbb{Z}$ with $|k| \leq k_0$ and $P \in \mathcal{D}_{i+k+m_0}$ such that $P \subset 4Q$ and $|S_{i+k}\mu(x)| \geq \delta_0$ for all $x \in P$.*

Proof. Fix $\epsilon_0 > 0$. Let $Q \in \mathcal{D}_i$ such that $\beta_{1,\mu}(Q) > \epsilon_0$. Take points $x_0, \ldots, x_n$ in Q as in Lemma 6.2, denote $r = \mathrm{diam}Q$, and let $m \in \mathbb{Z}$ to be fixed below such that $4r < 2^{-m} =: s$

and $A(x_0, 2^{-m-1/2}, 2^{-m+1/2}) \cap \operatorname{supp}\mu \neq \emptyset$ (we assume $\operatorname{diam}(\operatorname{supp}\mu) = \infty$). By Lemma 6.3, for $t := 2^{-p} \geq s$ to be fixed below and all $x_{n+1} \in 3Q$,

$$\begin{aligned}\operatorname{dist}(x_{n+1}, L_0) &\lesssim s \sum_{j=1}^{n+1} \sum_{k=p}^{m} |S_k\mu(x_j) - S_k\mu(x_0)| + \frac{r^2}{s} + \frac{rs}{t} \\ &\lesssim s \sum_{k=p}^{m} \sum_{j=0}^{n+1} |S_k\mu(x_j)| + \frac{r^2}{s} + \frac{rs}{t}.\end{aligned}$$

Then, by integrating on $x_{n+1} \in 3Q$, for some constant $C_1 > 0$ depending only on n and C_μ

$$\begin{aligned}\epsilon_0 < \beta_{1,\mu}(Q) &\leq \frac{1}{\ell(Q)^n} \int_{3Q} \frac{\operatorname{dist}(x_{n+1}, L_0)}{\ell(Q)}\, d\mu(x_{n+1}) \\ &\leq C_1 \biggl(\frac{s}{r} \sum_{k=p}^{m} \biggl(\frac{1}{\ell(Q)^n} \int_{3Q} |S_k\mu(x_{n+1})|\, d\mu(x_{n+1}) + \sum_{j=0}^{n} |S_k\mu(x_j)| \biggr) + \frac{r}{s} + \frac{s}{t} \biggr).\end{aligned}$$

Thus,

$$\frac{r}{s} \biggl(\frac{\epsilon_0}{C_1} - \frac{r}{s} - \frac{s}{t} \biggr) \leq \sum_{k=p}^{m} \biggl(\int_{3Q} \frac{|S_k\mu(x_{n+1})|}{\ell(Q)^n}\, d\mu(x_{n+1}) + \sum_{j=0}^{n} |S_k\mu(x_j)| \biggr).$$

We can easily choose s and t big enough (depending on r, ϵ_0, and C_1) such that, for some constant $\epsilon_1 > 0$ depending only on ϵ_0, n and C_μ,

$$0 < \epsilon_1 \leq \sum_{k=p}^{m} \biggl(\int_{3Q} \frac{|S_k\mu(x_{n+1})|}{\ell(Q)^n}\, d\mu(x_{n+1}) + \sum_{j=0}^{n} |S_k\mu(x_j)| \biggr). \tag{87}$$

Notice that, since $t = 2^{-p}$ and $s = 2^{-m}$ where chosen depending on $r \approx 2^{-i}$, the sum on the right hand side of (87) has a finite number of terms which only depends on ϵ_0, n and C_μ. Therefore, there exists $k_0 \in \mathbb{N}$ and $C_2 > 0$ depending only on ϵ_0, n and C_μ such that, for some negative integer k with $|k| \leq k_0$ and some $j = 0, \ldots, n$,

$$\epsilon_1 \leq C_2 \biggl(\frac{1}{\ell(Q)^n} \int_{3Q} |S_{i+k}\mu|\, d\mu + |S_{i+k}\mu(x_j)| \biggr),$$

which implies that there exists C_3 (depending on C_2) and $z \in 3Q$ such that $\epsilon_1 \leq C_3 |S_{i+k}\mu(z)|$.

Given $x \in \operatorname{supp}\mu$, if $|x - z| \leq 2^{-i-k}$, then

$$\begin{aligned}|S_{i+k}\mu(x) - S_{i+k}\mu(z)| &\leq \int_{|y-z| \lesssim 2^{-i-k}} \|\nabla(\varphi_{i+k} K)\|_\infty |x - z|\, d\mu(y) \\ &\lesssim 2^{(i+k)(n+1)} |x - z| \int_{|y-z| \lesssim 2^{-i-k}} d\mu(y) \lesssim 2^{i+k} |x - z|.\end{aligned}$$

Hence if $|x - z| \leq C_4 2^{-i-k}$ with $C_4 > 0$ small enough, we have $C_3 |S_{i+k}\mu(x) - S_{i+k}\mu(z)| \leq \epsilon_1/2$, so $\epsilon_1/2 \leq C_3 |S_{i+k}\mu(x)|$. Therefore, there exist $m_0 \in \mathbb{N}$ depending on C_4 (and thus on ϵ_0, n, and C_μ) and $P \in \mathcal{D}_{i+k+m_0}$ such that $\epsilon_1/2 \leq C_3 |S_{i+k}\mu(x)|$ for all $x \in P$. We can also assume that $P \subset 4Q$ by taking C_4 small enough, and since $|k| \leq k_0$ we have $\ell(P) \approx \ell(Q)$. The proposition follows by setting $\delta_0 := \epsilon_1/(2C_3) > 0$. □

Definition 6.5. *Given $\epsilon_0 > 0$, let $\delta_0, m_0 > 0$ be as in Proposition 6.4. Set*

$$\mathcal{B} := \{Q \in \mathcal{D} : \beta_{1,\mu}(Q) > \epsilon_0\}, \quad \widetilde{\mathcal{B}} := \bigcup_{k \in \mathbb{Z}} \{Q \in \mathcal{D}_{k+m_0} : |S_k\mu(x)| \geq \delta_0 \text{ for all } x \in Q\}.$$

Given $P, R \in \mathcal{D}$ with $P \subset R$, we set $F_P^R = \sum_{Q \in \widetilde{\mathcal{B}}:\, P \subset Q \subset R} \chi_Q$ and $F^R = \sum_{Q \in \widetilde{\mathcal{B}}:\, Q \subset R} \chi_Q$.

Lemma 6.6. *Let $\rho > 0$. Assume that there exists $C_0 > 0$ such that, for all $R \in \mathcal{D}$,*

$$\int_R \big(F^R\big)^{2/\rho}\, d\mu \leq C_0 \mu(R). \tag{88}$$

Then, there exists $C > 0$ such that $\sum_{Q \in \widetilde{\mathcal{B}}:\, Q \subset R} \mu(Q) \leq C\mu(R)$ for all $R \in \mathcal{D}$.

Proof. Let $M > 1$ big enough (it will be fixed below). For $R \in \mathcal{D}$, set

$$\begin{aligned}
\mathrm{Tree}(R) &:= \big\{Q \in \widetilde{\mathcal{B}} : Q \subset R,\, \chi_Q F_Q^R \leq M\chi_Q\big\},\\
\mathrm{Top}_0(R) &:= \big\{P \in \widetilde{\mathcal{B}} : P \subset R,\, \chi_P F_P^R > M\chi_P,\ \text{and } \chi_Q F_Q^R \leq M\chi_Q\\
&\qquad\qquad \text{for all } Q \in \widetilde{\mathcal{B}} \text{ such that } P \subsetneq Q \subset R\big\}.
\end{aligned}$$

For $m \geq 1$, set $\mathrm{Top}_m(R) := \bigcup_{P \in \mathrm{Top}_{m-1}(R)} \mathrm{Top}_0(P)$, and $\mathrm{Top}(R) := \bigcup_{m \geq 0} \mathrm{Top}_m(P)$. Notice that if $R \in \widetilde{\mathcal{B}}$ then $R \in \mathrm{Tree}(R)$, because $M > 1$. Notice also that

$$\{Q \in \widetilde{\mathcal{B}} : Q \subset R\} = \mathrm{Tree}(R) \cup \Big(\bigcup_{P \in \mathrm{Top}(R)} \mathrm{Tree}(P)\Big), \tag{89}$$

and the union is disjoint.

Fix $R \in \mathcal{D}$. Then, by (89),

$$\begin{aligned}
\sum_{Q \in \widetilde{\mathcal{B}}:\, Q \subset R} \mu(Q) &= \sum_{Q \in \mathrm{Tree}(R)} \mu(Q) + \sum_{P \in \mathrm{Top}(R)} \sum_{Q \in \mathrm{Tree}(P)} \mu(Q)\\
&= \int_R \sum_{Q \in \mathrm{Tree}(R)} \chi_Q\, d\mu + \int_R \sum_{P \in \mathrm{Top}(R)} \sum_{Q \in \mathrm{Tree}(P)} \chi_Q\, d\mu.
\end{aligned} \tag{90}$$

Given $x \in R$ and $P \in \mathcal{D}$ such that $P \subset R$, by the definition of $\mathrm{Tree}(P)$, we have

$$\sum_{Q \in \mathrm{Tree}(P)} \chi_Q(x) \leq M\chi_P(x).$$

Therefore, by (90),

$$\sum_{Q \in \widetilde{\mathcal{B}}:\, Q \subset R} \mu(Q) \leq M\mu(R) + \int_R \sum_{P \in \mathrm{Top}(R)} M\chi_P\, d\mu = M\Big(\mu(R) + \sum_{m \geq 0} \sum_{P \in \mathrm{Top}_m(R)} \mu(P)\Big). \tag{91}$$

We are going to prove that, if M is big enough,

$$\sum_{P \in \mathrm{Top}_m(R)} \mu(P) \leq 2^{-m}\mu(R) \tag{92}$$

for all $m \geq 0$, and then, by (91), we will finally obtain

$$\sum_{Q \in \widetilde{\mathcal{B}}:\, Q \subset R} \mu(Q) \leq M\mu(R) + M\sum_{m \geq 0} 2^{-m}\mu(R) \leq 3M\mu(R),$$

and the lemma will be proven.

Notice that, if $P, P' \in \mathrm{Top}_0(R)$ are different, then $P \cap P' = \emptyset$ because of the last condition in the definition of $\mathrm{Top}_0(R)$. So, to verify (92), it is enough to show that, for all $m \geq 0$,

$$\sum_{P \in \mathrm{Top}_{m+1}(R)} \mu(P) < \frac{1}{2} \sum_{P \in \mathrm{Top}_m(R)} \mu(P). \tag{93}$$

We have

$$\sum_{P\in \mathrm{Top}_{m+1}(R)} \mu(P) = \sum_{P\in \mathrm{Top}_m(R)} \sum_{Q\in \mathrm{Top}_0(P)} \mu(Q) \tag{94}$$

and $\sum_{Q\in \mathrm{Top}_0(P)} \chi_Q = \chi_U$, where $U := \bigcup_{Q\in \mathrm{Top}_0(P)} Q \subset P$. If $x \in U$, there exists $Q \in \mathrm{Top}_0(P)$ such that $x \in Q$, so $1 = \chi_Q(x) < M^{-2/\rho}\big(F^P_Q(x)\big)^{2/\rho} \leq M^{-2/\rho}\big(F^P(x)\big)^{2/\rho}$, and then using (88) we have

$$\sum_{Q\in \mathrm{Top}_0(P)} \mu(Q) = \int_P \sum_{Q\in \mathrm{Top}_0(P)} \chi_Q \,d\mu = \int_U 1\,d\mu < M^{-2/\rho} \int_P \big(F^P\big)^{2/\rho}\,d\mu \leq \frac{C_0}{M^{2/\rho}}\,\mu(P),$$

which, in combination with (94), yields (93) by taking $M > (2C_0)^{\rho/2}$. □

Lemma 6.7. *Assume that, for some $C_1 > 0$, $\sum_{Q\in\widetilde{\mathcal{B}}:\, Q\subset R} \mu(Q) \leq C_1\mu(R)$ for all $R \in \mathcal{D}$. Then there exists $C_2 > 0$ such that $\sum_{Q\in\mathcal{B}:\, Q\subset R} \mu(Q) \leq C_2\mu(R)$ for all $R \in \mathcal{D}$.*

Proof. Given $Q \in \mathcal{B}$, by Proposition 6.4, there exists $P_Q \in \mathcal{D}_{k+m_0}$ for some $k \in \mathbb{Z}$ such that $P_Q \subset 4Q$, $\mu(P_Q) \geq C_0\mu(Q)$, and $|S_k\mu(x)| \geq \delta_0$ for all $x \in P_Q$, where $C_0 > 0$ is some small constant. Thus, in particular, $P_Q \in \widetilde{\mathcal{B}}$ for all $Q \in \mathcal{B}$. Since $P_Q \subset 4Q$ and $\mu(P_Q) \geq C_0\mu(Q)$ for all $Q \in \mathcal{B}$, given $P \in \widetilde{\mathcal{B}}$ there are finitely many μ-cubes $Q \in \mathcal{B}$ such that $P_Q = P$, and the number of such μ-cubes is bounded above by a constant depending only on n, C_0, and C_μ. Hence, since $4R$ is contained in the union of a bounded number of μ-cubes with side length $\ell(R)$,

$$\sum_{Q\in\mathcal{B}:\, Q\subset R} \mu(Q) \leq C_0^{-1} \sum_{Q\in\mathcal{B}:\, Q\subset R} \mu(P_Q) \lesssim \sum_{P\in\widetilde{\mathcal{B}}:\, P\subset 4R} \mu(P) \leq C_1\mu(R)$$

for all $R \in \mathcal{D}$, as wished. □

Theorem 6.8. *Let $\rho > 0$. Given an n-dimensional AD regular measure μ, if $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is a bounded operator in $L^2(\mu)$, then μ is uniformly n-rectifiable.*

Proof. It is easy to see that, if $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is a bounded operator in $L^2(\mu)$, then R^μ_* is also bounded in $L^2(\mu)$. By Theorem 1.2 in [DS2, Part III, Chapter 1], in order to show that μ is uniformly n-rectifiable, it is enough to show that μ satisfies the Weak Geometric Lemma, i.e., that for any $\epsilon_0 > 0$, the set $\mathcal{B}$ is a Carleson set. In other words, it suffices to show that there exists a constant $C > 0$ depending on ϵ_0 such that $\sum_{Q\in\mathcal{B}:\, Q\subset R} \mu(Q) \leq C\mu(R)$ for all $R \in \mathcal{D}$. By Lemma 6.7 and Lemma 6.6, this holds if, for some $\rho > 0$, there exists $C > 0$ depending on ϵ_0 such that, for all $R \in \mathcal{D}$,

$$\int_R \big(F^R\big)^{2/\rho}\,d\mu \leq C\mu(R). \tag{95}$$

Notice that, for $m \in \mathbb{Z}$ and $f \in L^1(\mu)$, $S_m(f\mu) = T^\mu_{\varphi_{2^{-m-1}}} f - T^\mu_{\varphi_{2^{-m}}} f$, where S_m is introduced in Definition 6.1 and $T^\mu_{\varphi_\epsilon}$ is as in Definition 2.1 (remember that now K denotes the Riesz kernel), thus

$$\sum_{k\in\mathbb{Z}} |S_k(f\mu)(x)|^\rho \leq \big((\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi) f(x)\big)^\rho. \tag{96}$$

We may assume that $\rho \geq 1$, since $(\mathcal{V}_{\widetilde{\rho}} \circ \mathcal{R}^\mu) f(x) \leq (\mathcal{V}_\rho \circ \mathcal{R}^\mu) f(x)$ for $\widetilde{\rho} \geq \rho$, and then the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ for some $\rho > 0$ implies the $L^2(\mu)$ boundedness of $\mathcal{V}_{\widetilde{\rho}} \circ \mathcal{R}^\mu$ for all $\widetilde{\rho} \geq \rho$. Since $\varphi_{\mathbb{R}}\big(2^{2m}t^2\big)$ is a convex combination of the functions $\chi_{\{s\in\mathbb{R}\,:\, s>\epsilon\}}(t)$ for $\epsilon > 0$, using that $\rho \geq 1$ and Minkowski's integral inequality, it is not hard to show that the $L^2(\mu)$

boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ implies the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi$ (see Subsection 5.2, or [CJRW1, Lemma 2.4], for a similar argument). Therefore, for any $M > 0$, we have

$$\|(\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi)\chi_{MR}\|^2_{L^2(\mu)} \leq C\mu(MR) \leq C\mu(R) \text{ for all } R \in \mathcal{D}. \tag{97}$$

Fix $\epsilon_0 > 0$, let $\delta_0, m_0 > 0$ be as in Proposition 6.4, and let $R \in \mathcal{D}$. Given $x \in R$ and $k \in \mathbb{Z}$, for any $Q \in \mathcal{D}_{k+m_0} \cap \widetilde{\mathcal{B}}$ such that $x \in Q \subset R$ we have $|S_k\mu(x)| \geq \delta_0$. Notice that, since $Q \in \mathcal{D}_{k+m_0}$ and $Q \subset R$, there exists $M > 1$ depending only on n and m_0 such that $\delta_0 \leq |S_k\mu(x)| = |S_k(\chi_{MR}\mu)(x)|$. Therefore, using (96) and that for each $k \in \mathbb{Z}$ there is at most one μ-cube $Q \in \mathcal{D}_{k+m_0}$ such that $x \in Q \subset R$,

$$\begin{aligned} F^R(x) &= \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_{k+m_0}\cap\widetilde{\mathcal{B}}\,:\,x\in Q\subset R} \chi_Q(x) \leq \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathcal{D}_{k+m_0}\cap\widetilde{\mathcal{B}}\,:\,x\in Q\subset R} \delta_0^{-\rho}|S_k(\chi_{MR}\mu)(x)|^\rho \\ &\leq \delta_0^{-\rho}\sum_{k\in\mathbb{Z}} |S_k(\chi_{MR}\mu)(x)|^\rho \leq \delta_0^{-\rho}\big((\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi)\chi_{MR}(x)\big)^\rho \end{aligned} \tag{98}$$

and then, by (97),

$$\int_R \big(F^R\big)^{2/\rho}\,d\mu \leq \delta_0^{-2}\int_R \big((\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi)\chi_{MR}\big)^2\,d\mu \leq \delta_0^{-2}\|(\mathcal{V}_\rho \circ \mathcal{T}^\mu_\varphi)\chi_{MR}\|^2_{L^2(\mu)} \leq C\mu(R)$$

for all $R \in \mathcal{D}$. This yields (95), and the theorem follows. □

***Remark* 6.9.** Let $\{r_m\}_{m\in\mathbb{Z}} \subset (0,\infty)$ be a fixed decreasing sequence defining $\mathcal{O}$. If there exists $C > 0$ such that $C^{-1}r_m \leq r_m - r_{m+1} \leq Cr_m$ for all $m \in \mathbb{Z}$, then the last inequality in (98) still holds if we replace $\mathcal{V}_\rho$ by $\mathcal{O}$ (by taking from the beginning $\rho = 2$). Hence, Theorem 6.8 still holds replacing $\mathcal{V}_\rho$ by $\mathcal{O}$ for this particular sequence $\{r_m\}_{m\in\mathbb{Z}}$. However, we do not know if it holds for any $\{r_m\}_{m\in\mathbb{Z}} \subset (0,\infty)$.

References

[Bo] J. Bourgain, *Pointwise ergodic theorems for arithmetic sets,* Inst. Hautes Etudes Sci. Publ. Math. 69 (1989), 5–45.

[CJRW1] J. Campbell, R. L. Jones, K. Reinhold, and M. Wierdl, *Oscillation and variation for the Hilbert transform,* Duke Math. J. 105 (2000), 59–83.

[CJRW2] J. Campbell, R. L. Jones, K. Reinhold, and M. Wierdl, *Oscillation and variation for singular integrals in higher dimensions,* Trans. Amer. Math. Soc. 35 (2003), 2115–2137.

[Da] G. David, *Wavelets and singular integrals on curves and surfaces,* Lecture Notes in Math. 1465, Springer-Verlag, Berlin, 1991.

[DS1] G. David and S. Semmes, *Singular integrals and rectifiable sets in R^n: au-delà des graphes lipschitziens,* Astérisque No. 193 (1991).

[DS2] G. David and S. Semmes, *Analysis of and on uniformly rectifiable sets,* Mathematical Surveys and Monographs, 38, American Mathematical Society, Providence, RI, 1993.

[Jn] P. W. Jones, *Rectifiable sets and the travelling salesman problem,* Invent. Math. 102 (1990), 1–15.

[JKRW] R. L. Jones, R. Kaufman, J. Rosenblatt, and M. Wierdl, *Oscillation in ergodic theory,* Ergodic Theory and Dynam. Sys. 18 (1998), 889–936.

[JSW] R. L. Jones, A. Seeger, and J. Wright, *Strong variational and jump inequalities in harmonic analysis,* Trans. Amer. Math. Soc. 360 (2008), 6711–6742.

[LT] M. Lacey and E. Terwilleger, *A Wiener-Wintner theorem for the Hilbert transform,* Ark. Mat. 46 (2008), 2, 315–336.

[Lé] J. C. Léger, *Menger curvature and rectifiability*, Ann. of Math., 149 (1999), pp. 831–869.

[Lp] D. Lépingle, *La variation d'ordre p des semi-martingales,* Z. Wahrscheinlichkeitstheorie Verw. Gebiete 36 (1976), 295–316.

[M] A. Mas, *Variation for Riesz transforms and analytic and Lipschitz harmonic capacities,* Doctoral dissertation. Departament de Matemàtiques, Universitat Autònoma de Barcelona (2011).

[MT] A. Mas and X. Tolsa, *Variation and oscillation for singular integrals with odd kernel on Lipschitz graphs,* provisionally accepted in Proc. London Math. Soc. (2011).

[Ma1] P. Mattila. *Geometry of sets and measures in Euclidean spaces,* Cambridge Stud. Adv. Math. 44, Cambridge Univ. Press, Cambridge, 1995.

[Ma2] P. Mattila, *Cauchy singular integrals and rectifiability of measures in the plane*, Adv. Math. 115 (1995), pp. 1–34.

[MMV] P. Mattila, M. S. Melnikov and J. Verdera, *The Cauchy integral, analytic capacity, and uniform rectifiability,* Ann. of Math. (2) 144 (1996), 127–136.

[MPr] P. Mattila and D. Preiss, *Rectifiable measures in $\mathbb{R}^n$ and existence of principal values for singular integrals,* J. London Math. Soc. (2) 52 (1998), no. 3, pp. 482–496.

[OSTTW] R. Oberlin, A. Seeger, T. Tao, C. Thiele, and J. Wright *A variation norm Carleson theorem*, accepted in J. Eur. Math. Soc.

[Pa] H. Pajot, *Analytic capacity, rectifiability, Menger curvature and the Cauchy integral,* Lecture Notes in Math. 1799, Springer, 2002.

[To1] X. Tolsa. *Principal values for the Cauchy integral and rectifiability,* Proc. Amer. Math. Soc. 128(7) (2000), pp. 2111–2119.

[To2] X. Tolsa, *The semiadditivity of continuous analytic capacity and the inner boundary conjecture,* Amer. J. Math. 126 (2004), pp. 523–567.

[To3] X. Tolsa, *Principal values for Riesz transforms and rectifiability,* J. Funct. Anal., vol. 254(7) (2008), pp. 1811–1863.

[To4] X. Tolsa, *Uniform rectifiability, Calderón-Zygmund operators with odd kernel, and quasiorthogonality,* Proc. London Math. Soc. 98(2) (2009), 393–426.

[To5] X. Tolsa, *Analytic capacity and Calderón-Zygmund theory with non doubling measures,* in preparation (2011).

Albert Mas. Departament de Matemàtiques, Universitat Autònoma de Barcelona. 08193 Barcelona, Catalonia. Spain

E-mail address: amblesa@mat.uab.cat

Xavier Tolsa. Institució Catalana de Recerca i Estudis Avançats (ICREA) and Departament de Matemàtiques, Universitat Autònoma de Barcelona. 08193 Barcelona, Catalonia

E-mail address: xtolsa@mat.uab.cat