EXISTENCE AND UNIQUENESS OF LIMIT CYCLES FOR GENERALIZED φ-LAPLACIAN LIÉNARD EQUATIONS

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ABSTRACT. Liénard equation $x'' + f(x)x' + g(x) = 0$, appears as a simplified model in many problems of science and engineering. Since the first half of 20th century, many papers have appeared giving existence a uniqueness conditions for the limit cycles that a Liénard equation exhibits. In this paper we extend some of these results for the case of the generalized φ-laplacian Liénard equation $(\varphi(x'))' + f(x)\psi(x') + g(x) = 0$. This generalization appears when other derivations, different from the classic one, are considered, such as the relativistic one. Our results apply, for example, to the relativistic van der Pol equation $\left(x'/\sqrt{1-(x'/c)^2}\right)' + \mu(x^2-1)x' + x = 0$.

1. Introduction

Liénard equation,

$$x'' + f(x)x' + g(x) = 0, \tag{1}$$

appears as simplified model in many domains in science and engineering. It was intensively studied during the first half of 20th century as it can be used to model oscillating circuits or simple pendulums. In the simple pendulum case, f and g represents the friction and acceleration terms. One of the first models where this equation appears was introduced by Balthasar van der Pol. See [12]. Considering the equation

$$x'' + \mu(x^2-1)x' + x = 0,$$

for modeling the oscillations of a triode vacuum tube. See [7] for other references about more applications.

The first results on the existence and uniqueness of periodic solutions on the Liénard equation appear in [8, 13]. For some results on the existence and uniqueness of limit cycles, some papers like [16] or [17] and the books [18] and [19] could also be referred. Additionally, more current references on related problems are [3] and [9].

In this work, some criteria are presented for existence and uniqueness results on limit cycles for the generalized φ-laplacian Liénard equation

$$(\varphi(x'))' + f(x)\psi(x') + g(x) = 0. \tag{2}$$

Besides the obvious mathematical interest of this generalization, our main motivation for considering such equation comes from some relativistic models studied before. Special Relativity imposes a universal bound for the propagation speed of any gravitational or electromagnetic wave. If c is the speed of light in the vacuum, in the framework

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of Special Relativity the momentum of a particle with unitary rest mass is given by $\varphi(x') = \frac{x'}{\sqrt{1-\frac{x^2}{c^2}}}$ (see for instance [6]). The harmonic relativistic oscillator

$$\left(\frac{x'}{\sqrt{1-(x'/c)^2}}\right)' + x = 0$$

is a classical topicstudied by several authors (see for instance [5, 11]). Other authors have included damping terms and nonlinear forces. An example is the forced pendulum with relativistic effects, which model can be expressed as

$$\left(\frac{x'}{\sqrt{1-\frac{x^2}{c^2}}}\right)' + kx' + a\sin x = p(t) \tag{3}$$

where $p(t)$ a periodic function, is presented in [14] (see also [15]). In this work, conditions on the function p are given, for which equation (3) presents periodic orbits.

In the case of Liénard equation (1), it is usual to apply some change of variables to express the equation as the planar system

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x) \end{cases}$$

or

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -g(x) - f(x)y. \end{cases}$$

In this work a variation of this approach is considered. Our results apply, after a time rescaling, to system

$$\begin{cases} \dot{x} = y\varphi'(y), \\ \dot{y} = -g(x) - f(x)\psi(y). \end{cases} \tag{4}$$

Before stating the results some necessary hypotheses are introduced. All the functions in (2) should be at least locally Lipschitz continuous, $\mathcal{C}^{0,1}$, except $\varphi(y)$ that should be in $\mathcal{C}^{1,1}$. These properties assure the existence and uniqueness of a solution for any initial value problem associated to system (4). More regularity of each function is required in some concrete results.

Let $\mathcal{D}$ be the greater connected domain, neighborhood of the origin, for which the four functions in (2), $\varphi(y)$, $\psi(y)$, $f(x)$ and $g(x)$, are well defined. In this way $\mathcal{D}$ can be consider as $\mathcal{D} = (x_1, x_2) \times (y_1, y_2)$ where $x_1, y_1 \in \mathbb{R}^- \cup \{-\infty\}$ and $x_2, y_2 \in \mathbb{R}^+ \cup \{+\infty\}$. Along this paper all the limits considered for x or y tending to x_i or y_i with $i = 1, 2$ are limits from the interior of the intervals of definition. It means that, for example, we denote $y \to y_2$ instead of $y \to {y_2}^-$.

Based on the results of the classical Liénard equation (1) given in [16, 17, 19] the following conditions, denoted by (H), are established.

(H_0) $f(x)$, $g(x)$ and $\psi(y)$ are of class $\mathcal{C}^{0,1}(\mathbb{R})$ and $\varphi(y)$ is of class $\mathcal{C}^{1,1}(\mathbb{R})$.
(H_1) $xg(x) > 0$ for all $x \in (x_1, x_2) \setminus \{0\}$ and $g(0) = 0$.
(H_2) $f(0) \neq 0$.
(H_3) $\mathrm{Dom}(\varphi) \subseteq \mathrm{Dom}(\psi)$.
(H_4) $\psi(0) = 0$.
(H_5) $\varphi'(y) \in \mathbb{R}^+ \setminus \{0\}$ for all $y \in (y_1, y_2)$ and $\varphi(0) = 0$.

(H_0) represents the regularity condition. (H_1) and (H_2) are inherited from the referred classic results. $(H_3 - H_5)$ are the most basic hypotheses that we impose to $\varphi(y)$ and $\psi(y)$. We have included the extra condition $\varphi(0) = 0$ for simplicity and symmetry reasons. Without lost of generality, the case $\varphi(0) \neq 0$ can also be considered after doing a translation of this function.

Briefly the aim of this work is to provide conditions on functions f, g, φ and ψ such that system (4) has at least a periodic orbit and, moreover, if it exists, when it is unique. Next two results summarizes these properties.

Theorem 1. *[Existence Theorem] Consider system* (4) *under the hypotheses* (H). *Additionally, the next properties hold.*

(i) $y\psi(y)f(x) \leq 0$ *in a neighborhood of the origin,* $I_x \times I_y = [x_-, x_+] \times [y_-, y_+] \subset \mathcal{D}$, *except for a finite number of points where it vanishes.*
(ii) There exist δ *and* η *in* $\mathbb{R}$, *with* $x_1 < \eta < 0 < \delta < x_2$, *such that* $f(x) > 0$ *for all* $x \in (x_1, x_2) \setminus [\eta, \delta]$.
(iii) For each $i = 1, 2$ *there exists* λ_i *in* $\mathbb{R}^+ \cup \{+\infty\}$ *such that, if* $|x_i| = +\infty$, *then* $\liminf_{x\to x_i} x(|g(x)| + f(x)) = \lambda_i$, *and if* $x_i \in \mathbb{R}$, *then* $\liminf_{x\to x_i} |x - x_i|(|g(x)| + f(x)) = \lambda_i$.
(iv) $y\psi(y) > 0$ *for all* $y \neq 0$.
(v) For $i = 1, 2$, $\lim_{y\to y_i} \psi(y)/(y\varphi'(y)) \in \mathbb{R}$.
(vi) The integral $\int_\eta^\delta f(x)dx$ *is positive or, alternatively, there exists* $y_0 \in (y_1, y_2)$ *such that* $-\psi(y_0) \in \left[\liminf_{x\to x_i} g(x)/f(x), \limsup_{x\to x_i} g(x)/f(x)\right]$ *for at least one of the* x_i *and there exists* U, *neighborhood of* y_0, *such that* $\operatorname{sign}(\psi'(y))$ *is constant almost for every* $y \in U$.

Then system (4) *has at least a periodic orbit contained in* $\mathcal{D}$.

Theorem 2. *[Unicity Theorem] Consider system* (4) *under the hypotheses* (H). *Additionally, the next properties hold.*

(i) $f, g \in \mathcal{C}^{0,1}((x_1, x_2))$ *and* $\varphi, \psi \in \mathcal{C}^{1,1}((y_1, y_2))$ *with* $x_1, y_1 \in \mathbb{R}^- \cup \{-\infty\}$ *and* $x_2, y_2 \in \mathbb{R}^+ \cup \{+\infty\}$.
(ii) There exist $a < 0 < b$ *such that* $f(x) < 0$ *when* $x \in (a, b)$ *and* $f(x) > 0$ *when* $x \in (x_1, x_2) \setminus [a, b]$,
(iii) $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) > 0$, *for all* $x \in (x_1, x_2) \setminus I_0$ *where* $I_0 \subset [a, b]$ *such that* I_0 *contains the origin and* $I_0 = (a, x_0)$ *or* $I_0 = (x_0, b)$ *with* x_0 *satisfying that* $\int_0^{x_0} g(s)ds = \min\left\{\int_0^a g(s)ds, \int_0^b g(s)ds\right\}$.
(iv) $\psi'(y) > 0$ *and* $\dfrac{d}{dy}\left(\dfrac{\psi'(y)}{y\varphi'(y)}\right) < 0$, *for all* y *in* $(y_1, y_2) \setminus \{0\}$.

Then system (4) *has at most one limit cycle. Moreover, when it exists, it is stable.*

In Section 2, we present the different kinds of functions that we can find in our study. Section 3 is devoted to introduce a new way to compactify the domain $\mathcal{D}$ to $\widetilde{\mathcal{D}} = (-1, 1) \times (-1, 1)$. This compactification allows us to consider the *boundary of* $\mathcal{D}$ as the inverse of the boundary of $\widetilde{\mathcal{D}}$. Hence, we can study the behavior of the differential equation (4) close to the boundary of $\mathcal{D}$. Moreover, this compactification allows us to consider all the cases in a unified way. So we restrict our study to the case $\mathcal{D} = \widetilde{\mathcal{D}}$. In Section 4, we show some first integrals of the particular cases of (4) when the friction term vanishes, $f(x) = 0$,

$$\begin{cases} \dot{x} = y\varphi'(y), \\ \dot{y} = -g(x), \end{cases} \tag{5}$$

or when the acceleration term vanishes, $g(x) = 0$,

$$\begin{cases} \dot{x} = y\varphi'(y), \\ \dot{y} = -f(x)\psi(y). \end{cases} \tag{6}$$

We use both first integrals as state functions of system (4).

These first sections include all the technical results needed to prove the main results. Hence, in Section 5, we prove the Existence Theorem, Theorem 1. The proof follows from the Poincaré-Bendixson theorem, see [2], because the statement ensures that the origin and the boundary of $\mathcal{D}$ have the same stability, in fact both are repellors. Proposition 6 studies the stability of the origin and Propositions 20 and 21 deal with the stability of the boundary of $\mathcal{D}$. Finally, Section 6 is devoted to prove the uniqueness of limit cycle in the whole space, Theorem 2. The proof, as it is done in [19], is obtained by contradiction, computing the integral of the divergence of the vector field between any two consecutive limit cycles. We remark that our proof does not need any restriction on the location of the limit cycles.

2. Function families

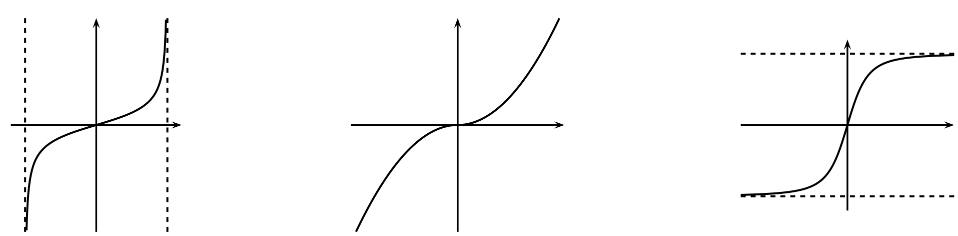
In this paper we only consider three basic different behaviors of $\varphi(y)$ over y_i for $i = 1, 2$. We say that

(a) $\varphi(y)$ is *singular* over y_i if $y_i \in \mathbb{R}$ and $\lim_{y \to y_i} \varphi(y) = \pm\infty$,

(b) $\varphi(y)$ is *non-bounded regular* over y_i if $y_i = \pm\infty$ and $\lim_{y \to y_i} \varphi(y) = \pm\infty$, and,

(c) $\varphi(y)$ is *bounded regular* over y_i if $y_i = \pm\infty$ and $\lim_{y \to y_i} \varphi(y) \in \mathbb{R}^+$.

For shortness, we denote above properties by S_i, NB_i and B_i, respectively. Some graphical representations of this basic functions are showed in Figure 1.



(a) Singular (b) Non-bounded Regular (c) Bounded Regular

Figure 1. Basic φ functions

The most representative function of the singular case could be the relativistic operator, $\varphi(s) = s/\sqrt{1 - s^2}$. An example of the non-bounded regular case is the p-laplacian operator, $\varphi(s) = |s|^{p-1}s$. And for the bounded regular case, we can use, for example, the mean curvature operator, $\varphi(s) = s/\sqrt{1 + s^2}$. These three examples are well known in the literature about φ-laplacian problems. See for example [1], [10] or [4], respectively.

Although the previous examples are all symmetric we do not ask for any symmetry to the function $\varphi(y)$, nor a symmetric behavior at the boundary of the domain. Therefore some mixed cases can also be considered. Hence the results of this paper apply also for functions like $\varphi(s) = s/(1 - s)$ or $\varphi(s) = e^s - 1$. See Figure 2.

We also consider different kinds of function $f(x)$ in terms of the type of its domain of definition. So we say that we are in a *finite (infinite) case on x_i*, denoted by F_i (I_i),

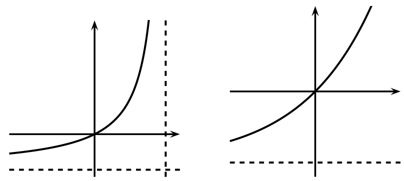


FIGURE 2. Some examples of mixed behavior at the boundary of the domain

if $x_i \in \mathbb{R}$ ($x_i \notin \mathbb{R}$). The results of this paper also apply when we have not symmetry in the behavior of $f(x)$ at both x_i, $i = 1, 2$, at the same time, as in the case of the function φ. An example of this situation is the function $f(s) = \frac{s^2}{1-s} - 1$, shown in Figure 3.

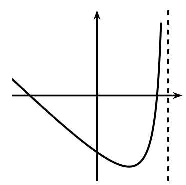


FIGURE 3. A mixed function $f(x)$

The functions g and ψ are actually determined by the hypotheses (H) and, as it can be seen in the next section, they do not play an special role in the compactification. Hence it is not necessary to study their different behaviors at the boundary of the domain.

3. A Polygonal Compactification

The main tool of this work is a transformation of the domain of definition, $\mathcal{D} = (x_1, x_2) \times (y_1, y_2)$, of the generalized Liénard differential equation (4). Next proposition allows us to unify all the different behaviors detailed in the previous section via a transformation to the square $(-1, 1) \times (-1, 1)$. We consider it as a polygonal compactification because the closure of the new domain is a compact set which boundary is a polygon.

Proposition 3. *Given system* (4) *satisfying* (H) *and defined in* $\mathcal{D} = (x_1, x_2) \times (y_1, y_2)$ *where* $x_1, y_1 \in \mathbb{R}^- \cup \{-\infty\}$ *and* $x_2, y_2 \in \mathbb{R}^+ \cup \{+\infty\}$*, there exists a change of variables of class* $\mathcal{C}^1$ *such that* (4) *writes as*

$$\left\{ \begin{array}{rcl} \dot{w} & = & \chi(z), \\ \dot{z} & = & -\widetilde{g}(w) - \widetilde{f}(w)\widetilde{\psi}(z), \end{array} \right. \tag{7}$$

the new domain of definition is $\widetilde{\mathcal{D}} = (-1, 1) \times (-1, 1)$ *and the functions* $\widetilde{f}$, $\widetilde{g}$, $\widetilde{\psi}$ *and* χ *satisfy the following properties.*

($\widetilde{H}_0$*)* $\widetilde{f}(w)$, $\widetilde{g}(w)$ *and* $\widetilde{\psi}(z)$, $\chi(z)$ *are of class* $\mathcal{C}^{0,1}(\mathbb{R})$.

($\widetilde{H}_1$*)* $w\widetilde{g}(w) > 0$ *for all* $w \in (-1, 1) \setminus \{0\}$ *and* $\widetilde{g}(0) = 0$.

($\widetilde{H}_2$*)* $\widetilde{f}(0) \neq 0$.

($\widetilde{H}_3$*)* $\mathrm{Dom}(\chi) \subseteq \mathrm{Dom}(\widetilde{\psi})$.

($\widetilde{H}_4$*)* $\widetilde{\psi}(0) = 0$.

($\widetilde{H}_5$*)* $z\chi(z) > 0$ *for all* $z \in (-1, 1)$ *and* $\chi(0) = 0$.

From now on we use either system (4) defined in $\mathcal{D}$ or system (7) defined in $\widetilde{\mathcal{D}}$. A graphical interpretation of the last result can be seen in Figure 4.

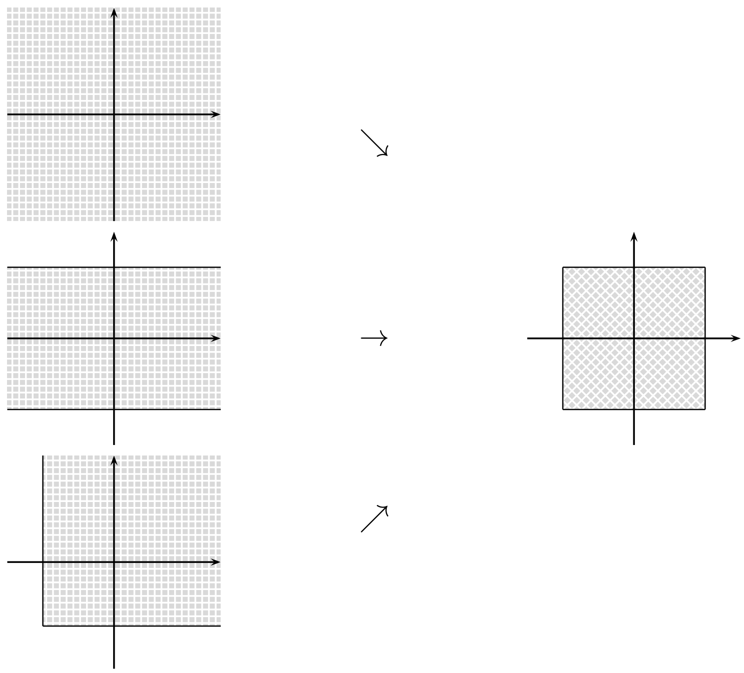


FIGURE 4. Some examples of compactified boundaries

Proof of Proposition 3. All the functions, f, g, φ and ϕ are functions of one variable. Taking into account the symmetry of the hypotheses (H) with respect to the origin we can consider different changes of variables for the positive and the negative axes. These changes define a global piecewise change of class $\mathcal{C}^1$ for the variable x and another one for the variable y. Hence system (4) is equivalent to system (7) and all the conditions of hypotheses (H) are transformed to the equivalent conditions of hypotheses $(\widetilde{H})$. From the above considerations we only show the changes corresponding to first quadrant, that is $x > 0$ and $y > 0$. The other follow analogously.

Following the classification of Section 2 we consider all possible cases F_2, I_2, S_2, NB_2 and B_2 because the changes of variables are different for each type.

For the type F_2, let us consider the change of variable

$$w = \frac{x_2 x}{(x_2 - 1)x + x_2} \text{ which inverse is } x = \frac{x_2 w}{(1 - x_2)w + x_2},$$

that transforms (4) to

$$\begin{cases} \dot{w} &= y\varphi'(y), \\ \dot{y} &= -\widetilde{g}(w) - \widetilde{f}(w)\psi(y) \end{cases}$$

where

$$\widetilde{g}(w) = \frac{x_2^2\, g\left(\dfrac{x_2 w}{(1 + x_2)w + x_2}\right)}{((1 - x_2)w + x_2)^2} \quad \text{and} \quad \widetilde{f}(w) = \frac{x_2^2\, f\left(\dfrac{x_2 w}{(1 + x_2)w + x_2}\right)}{((1 - x_2)w + x_2)^2}$$

and the corresponding side of the boundary of the transformed $\mathcal{D}$ is the line $w = 1$.

For the type I_2, we consider the change of variable

$$w = \frac{x}{1+|x|} \text{ which inverse is } x = \frac{w}{1-|w|}, \tag{8}$$

that transforms (4) to

$$\begin{cases} \dot{w} &= y\varphi'(y), \\ \dot{y} &= -\widetilde{g}(w) - \widetilde{f}(w)\psi(y) \end{cases}$$

where

$$\widetilde{g}(w) = \frac{g\left(\dfrac{w}{1-|w|}\right)}{(1-|w|)^2} \quad \text{and} \quad \widetilde{f}(w) = \frac{f\left(\dfrac{w}{1-|w|}\right)}{(1-|w|)^2}.$$

Moreover, the line $w = 1$ contains the corresponding side of the boundary of the transformed $\mathcal{D}$.

For the type S_2, consider the change of variable

$$z = \frac{y_2^2 y}{y^2 - y_2(2-y_2)y + y_2^2}$$

which inverse is

$$y = \frac{y_2}{2z}\left((2-y_2)z + y_2 - \sqrt{y_2(z-1)((y_2-4)z - y_2)}\right),$$

and (4) writes as

$$\begin{cases} \dot{x} &= \dfrac{y_2}{2z}\left((2-y_2)z + y_2 - \sqrt{y_2(z-1)((y_2-4)z - y_2)}\right)\widetilde{\varphi}'(z), \\ \dot{z} &= -g(x) - f(x)\widetilde{\psi}(z), \end{cases}$$

where

$$\widetilde{\varphi}(z) = \varphi\left(\frac{y_2}{2z}\left((2-y_2)z + y_2 - \sqrt{y_2(z-1)((y_2-4)z - y_2)}\right)\right),$$
$$\widetilde{\psi}(z) = \psi\left(\frac{y_2}{2z}\left((2-y_2)z + y_2 - \sqrt{y_2(z-1)((y_2-4)z - y_2)}\right)\right) \text{ and}$$

$z = 1$ contains the boundary of the transformed $\mathcal{D}$.

For the type NB_2, let us consider the change of variable

$$z = \frac{y}{1+|y|} \text{ which inverse is } y = \frac{z}{1-|z|}, \tag{9}$$

and (4) writes as

$$\begin{cases} \dot{x} &= z\dfrac{\widetilde{\varphi}'(z)}{1-|z|}, \\ \dot{z} &= -g(x) - f(x)\widetilde{\psi}(z) \end{cases}$$

where

$$\widetilde{\varphi}(z) = \varphi\left(\frac{z}{1-|z|}\right), \quad \widetilde{\psi}(z) = \psi\left(\frac{z}{1-|z|}\right)$$

and $z = 1$ contains a piece of the boundary of the transformed $\mathcal{D}$.

Finally, for the type B_2 if $\lim_{y\to y_2} \varphi(y) = \nu_2$, the change of variable needed is

$$z = \frac{\varphi\left(\dfrac{\nu_2}{\varphi'(0)}y\right)}{\nu_2} \text{ which inverse is } y = \frac{\varphi'(0)}{\nu_2}\varphi^{-1}(\nu_2 z),$$

that allow us to express (4) as follows

$$\begin{cases} \dot{x} &= \gamma(z), \\ \dot{z} &= -g(x) - f(x)\widetilde{\psi}(z), \end{cases}$$

where

$$\gamma(z) = \frac{(\varphi'(0))^2}{\nu_2}\varphi^{-1}(\nu_2 z)\frac{\varphi'\left(\dfrac{\varphi'(0)}{\nu_2}\varphi^{-1}(\nu_2 z)\right)}{\varphi'\left(\varphi^{-1}(\nu_2 z)\right)}, \quad \widetilde{\psi}(z) = \psi\left(\frac{\varphi'(0)}{\nu_2}\varphi^{-1}(\nu_2 z)\right)$$

and again $z = 1$ contains the boundary of the transformed $\mathcal{D}$.

The proof ends checking that the global piecewise changes are all $\mathcal{C}^1$. This is done because any of the previous changes are of class $\mathcal{C}^1$ for $x \neq 0$ or $y \neq 0$. Moreover, the left and right derivatives at the origin coincides, in fact, $w'(0^+) = w'(0^-) = 1$ and $z'(0^+) = z'(0^-) = 1$. Therefore we combine the changes depending on the behavior at $\mathbb{R}^+$ and $\mathbb{R}^-$. □

Under hypothesis (H_0), system (4) satisfies the sufficient conditions to assure existence and unicity of any initial value problem in $\mathcal{D}$. Then, from the previous proposition, this is also done for the equivalent system (7) in the corresponding $\widetilde{\mathcal{D}}$.

4. State functions

In a general framework, a differential equation can be thought as a dynamical system. In that case, a function of state, also called state function, E is a property of the system that depends only on the current state of it. That is, the value of the function E at some point is independent of the processes undergone by the system to arrive to this value. State functions usually appear in physical and chemical systems, for example the mass, the energy, the entropy and the temperature, among others.

The state functions, for system (4), described in this section are constructed as first integrals in the null friction and the null acceleration cases, (5) and (6) respectively. They are very helpful in the study of system (4), particularly in the proof of the Uniqueness Theorem, Theorem 2.

Lemma 4. *[Primary Energy Function] The function $E(x,y) = G(x) + \Phi(y)$, where $G(x) = \int_0^x g(u)du$ and $\Phi(y) = \int_0^y v\varphi'(v)dv$, is a first integral of system (5) in $\mathcal{D}$. Moreover $E(0,0) = 0$ and the origin is a local center.*

Proof. Straightforward computations show that E is well defined and $\dot{E} \equiv 0$ over the solutions of system (5). The existence of the first integral and the monodromic structure of the origin gives the center property □

Likewise, next result holds.

Lemma 5. *[Secondary Energy Function] The function $J(x,y) = F(x) + \Psi(y)$, where $F(x) = \int_0^x f(u)du$ and $\Psi(y) = \int_0^y \frac{v\varphi'(v)}{\psi(y)}dv$, is a first integral of (6) in $\mathcal{D}$. Moreover $J(0,0) = 0$.*

5. Existence results

Theorem 1, the Existence Theorem, is a consequence of the results of this section. In order to simplify the reading of all the results we refer to system (4), defined in $\mathcal{D}$, for the functions f, g, φ, ψ and to system (7), defined in $\widetilde{\mathcal{D}}$, for the corresponding transformed functions $\widetilde{f}, \widetilde{g}, \chi, \widetilde{\psi}$, via the change of variables of Proposition 3. Additionally we also assume the hypothesis (H) on (4) and $(\widetilde{H})$ on (7).

5.1. **Local stability of the origin.** Let us show some conditions so that system (4) has a singular point at the origin which stability can be determined.

Proposition 6. *Assume that f and ψ vanish only on a finite number of points in neighborhoods of the origin $I_x = [x_-, x_+] \subset (x_1, x_2)$ and $I_y = [y_-, y_+] \subset (y_1, y_2)$, respectively. If* $\operatorname{sign} f(x)$ *and* $\operatorname{sign} y\psi(y)$ *are constant in I_x and I_y, then the origin is a repellor (attractor) when $y\psi(y)f(x) \leq 0$ (≥ 0). Moreover the basin of repulsion (attraction) contains the biggest level curve of the Primary Energy Function E, defined in Lemma 4, completely contained in $I_x \times I_y$.*

Proof. By equation (4) we have

$$\dot{E} = g(x)\dot{x} + y\varphi'(y)\dot{y} = g(x)y\varphi'(y) + y\varphi'(y)(-g(x) - f(x)\psi(y)) = -y\psi(y)\varphi'(y)f(x).$$

As $\varphi'(y) > 0$ for every $y \in (y_1, y_2) \setminus \{0\}$ the sign of $\dot{E}$ is constant in a neighborhood of the origin. Thus we can assure that the stability of the origin, applying Hartman's Theorem, is done by the sign of $y\psi(y)f(x)$. □

5.2. **Stability of the infinity.** This section is devoted to prove that the boundary of $\mathcal{D}$ is a repellor under the hypotheses of Theorem 1. First, we study how the compatification transforms the hypotheses. Second, Proposition 14 explains the behavior of the orbits close to the boundaries of the compactified domain $\widetilde{\mathcal{D}}$ and Definition 15 introduces the notion of regular and singular points in the boundary. Propositions 17 and 18 establish the dynamics of the finite points and Corollary 19 shows that their ω-limit remains in $\widetilde{\mathcal{D}}$ or is the full boundary. Finally, we prove that the boundary is a repellor considering two cases. Proposition 20 deals with the case with singular points on the boundary different from the vertex and Proposition 21 without them.

Lemma 7. *If $y\psi(y) > 0$ then $z\widetilde{\psi}(z) > 0$.*

Lemma 8. *If there exist $\delta, \eta \in \mathbb{R}$, with $x_1 < \eta < 0 < \delta < x_2$, where $f(x) > 0$ for all $x \in (x_1, x_2) \setminus [\eta, \delta]$, we have that there exist $\widetilde{\delta}, \widetilde{\eta} \in \mathbb{R}$, with $-1 < \widetilde{\eta} < 0 < \widetilde{\delta} < 1$, satisfying $\widetilde{f}(w) > 0$ for all $w \in (-1, 1) \setminus [\widetilde{\eta}, \widetilde{\delta}]$.*

Lemma 9. *Assume that there exists $y_0 \in (y_1, y_2)$ such that $-\psi(y_0) \in \left[\liminf_{x \to x_i} \frac{g(x)}{f(x)}, \limsup_{x \to x_i} \frac{g(x)}{f(x)}\right]$ for at least one of the x_i and a neighborhood U of y_0 where* $\operatorname{sign}(\psi'(y))$ *is constant almost for every y in U. Then this properties are also satisfied in the compactified domain.*

The proofs of the above lemmas involve straightforward computations, using the appropriate change of variables from the proof of Proposition 3.

Lemma 10. *Assume that there exist $\delta, \eta \in \mathbb{R}$, with $x_1 < \eta < 0 < \delta < x_2$, such that $f(x) > 0$ for all $x \in (x_1, x_2) \setminus [\eta, \delta]$, and $\lambda_i \in \mathbb{R}^+ \cup \{+\infty\}$ such that, for $i = 1, 2$,*

$$\begin{aligned} &\text{if } |x_i| = +\infty, \quad \liminf_{x \to x_i} x(|g(x)| + f(x)) = \lambda_i, \text{ or} \\ &\text{if } x_i \in \mathbb{R}, \qquad \liminf_{x \to x_i} |x - x_i|(|g(x)| + f(x)) = \lambda_i. \end{aligned}$$

Then there exist $\widetilde{\lambda}_i \in \mathbb{R}^+ \cup \{+\infty\}$ such that $\liminf_{w \to w_i} |w - w_i|(|\widetilde{g}(x)| + \widetilde{f}(x)) = \widetilde{\lambda}_i$, for $i = 1, 2$.

Proof. The statement follows immediately for $x_i \in \mathbb{R}$, so we only prove it for $|x_2| = +\infty$. The case $|x_1| = +\infty$ is analogous. From the change of variables (8) we have

$$\liminf_{w\to 1} |1-w|(|\widetilde{g}(w)| + \widetilde{f}(w)) = \liminf_{x\to x_2} \left(1 - \frac{x}{1+x}\right) \frac{g(x)+f(x)}{\left(1-\dfrac{x}{1+x}\right)^2} =$$

$$= \liminf_{x\to x_2} (1+|x|)(|g(x)| + f(x)) = \beta + \lambda.$$

And the above expression is positive because $\beta = \liminf_{x\to x_2}(|g(x)| + f(x)) \geq 0$, since $|g(x)|$ and $f(x)$ are positive functions for $|x|$ large enough. □

Lemma 11. *The function χ satisfies $\lim_{|z|\to 1} |\chi(z)| = +\infty$.*

Proof. We restrict the proof to the case $z > 0$. The other cases are analogous. Following the structure of proof of Proposition 3 we consider the types S_2, NB_2 and B_2.

For S_2 we have $\chi(z) = \dfrac{y_2}{2z}\left((2-y_2)z + y_2 - \zeta\right)\widetilde{\varphi}'(z)$ with

$$\widetilde{\varphi}'(z) = \frac{\left((2-y_2)z + y_2 - AAA\right) y_2^2}{2z^2\zeta} \varphi'\left(\frac{y_2}{2z}\left((2-y_2)z + y_2 - \zeta\right)\right)$$

and $\zeta = \sqrt{y_2(z-1)((y_2-4)z - y_2)}$.

As we are in the singular type, $\lim_{y\to y_2} \varphi'(y) = +\infty$. If not, φ can be regularly extended from $y = y_2$, which contradicts the maximality of the domain $\mathcal{D}$. Thus,

$$\lim_{z\to 1} \chi(z) = \lim_{y\to y_2} y\varphi'(y) \frac{\left(y^2 - y_2(2-y_2)y + y_2^2\right)^2}{y_2^2(y_2-y)(y_2+y)} = +\infty.$$

For NB_2 we have $\chi(z) = z\widetilde{\varphi}'(z)/(1-z) = z\varphi'\left(z/(1-z)\right)/(1-z)^3$ with $z \in (0,1)$ and

$$\lim_{z\to z_2} \chi(z) = \lim_{y\to y_2} y(1+y)^2\varphi'(y) = +\infty$$

because $\lim_{y\to y_2} y^2\varphi'(y) = +\infty$. We show this assertion by contradiction. Assume that $\lim_{y\to y_2} y^2\varphi'(y) = \alpha$ where α is a real number. Then there exists $y_0 > 0$ such that $\varphi'(y) < (\alpha+1)/y^2$ for all $y > y_0$. So, it follows that

$$\varphi(y) = \int_0^y \varphi'(v)dv = \int_0^{y_0} \varphi'(v)dv + \int_{y_0}^y \varphi'(v)dv = \varphi(y_0) + \int_{y_0}^y \varphi'(v)dv <$$

$$< \varphi(y_0) + \int_{y_0}^{+\infty} \frac{\alpha+1}{v^2} dv = \varphi(y_0) + \left.\frac{-(\alpha+1)}{v}\right|_{y_0}^{+\infty} = \varphi(y_0) + \frac{\alpha+1}{y_0}$$

for all $y \in (0, +\infty)$. It means that φ is a bounded function, which contradicts the condition of being of type NB_2.

We conclude considering the type B_2. We have

$$\lim_{z\to 1} \chi(z) = \lim_{z\to z_2} \frac{\varphi'(0)^2}{\nu_2} \varphi^{-1}(\nu_2 z) \frac{\varphi'\left(\dfrac{\varphi'(0)}{\nu_2}\varphi^{-1}(\nu_2 z)\right)}{\varphi'(\varphi^{-1}(\nu_2 z))} = \lim_{y\to+\infty} \varphi'(0) y \frac{\varphi'(y)}{\varphi'\left(\dfrac{\nu_2}{\varphi'(0)} y\right)} = +\infty,$$

because

$$\lim_{y\to+\infty}\frac{\varphi'(y)}{\varphi'\left(\dfrac{\nu_2}{\varphi'(0)}y\right)}=1.$$

This equality follows from $\lim\limits_{y\to+\infty}\varphi'(y)=0$, hence $\varphi'(y)$ is a Cauchy function. Let $\varepsilon\in(0,1)$ and $\delta=\frac{\varepsilon}{3}$. Thus there exists y_0 such that for all $y>y_0$, $|\varphi'(y)-\varphi'(\nu_2 y/\varphi'(0))|<\delta$ holds. Therefore,

$$\min\left\{1-\delta,\frac{1}{1+\delta}\right\}<\frac{\varphi'(y)}{\varphi'\left(\dfrac{\nu_2}{\varphi'(0)}y\right)}<\max\left\{1+\delta,\frac{1}{1-\delta}\right\}$$

and, consequently,

$$1-\frac{\varepsilon}{2}<1-\frac{\varepsilon}{3}\leq\frac{\varphi'(y)}{\varphi'\left(\dfrac{\nu_2}{\varphi'(0)}y\right)}\leq\frac{3}{3-\varepsilon}<1+\frac{\varepsilon}{2}.$$

□

Lemma 12. *If the limit* $\lim\limits_{y\to y_i}\dfrac{\psi(y)}{y\varphi'(y)}$ *is real then* $\lim\limits_{z\to z_i}\dfrac{\widetilde{\psi}(z)}{\chi(z)}=0$*, for* $i=1,2$.

Proof. Let us just consider the case $i=2$, that is $z_2=1$. The case $i=1$ follows analogously. The different types that should be considered are S_2, NB_2 and B_2.

We start with type S_2,

$$\lim_{z\to1}\frac{\widetilde{\psi}(z)}{\chi(z)}=\lim_{y\to y_2}\frac{y_2^2(y_2-y)(y_2+y)}{(y^2-y_2(2-y_2)y+y_2^2)^2}\frac{\psi(y)}{y\varphi'(y)}=\frac{2}{y_2^3}\lim_{y\to y_2}(y_2-y)\frac{\psi(y)}{y\varphi'(y)}=0.$$

For type NB_2, let us consider the change (9) and we obtain

$$\lim_{z\to z_2}\frac{\widetilde{\psi}(z)}{\chi(z)}=\lim_{z\to z_2}(1-z)\frac{\psi\left(\dfrac{z}{1-z}\right)}{\dfrac{z}{1-z}\varphi'\left(\dfrac{z}{1-z}\right)}=\lim_{y\to+\infty}\frac{1}{1+y}\frac{\psi(y)}{y\varphi'(y)}=0.$$

Finally, let us consider type B_2. Since $\lim\limits_{y\to+\infty}\varphi'(y)=0$, we have

$$\lim_{z\to z_2}\frac{\widetilde{\psi}(z)}{\chi(z)}=\lim_{y\to+\infty}\frac{\psi(y)\varphi'\left(\dfrac{\nu_2}{\varphi'(0)}y\right)}{\varphi'(0)y\varphi'(y)}=\frac{1}{\varphi'(0)}\lim_{y\to+\infty}\frac{\psi(y)}{y\varphi'(y)}\varphi'\left(\frac{\nu_2}{\varphi'(0)}y\right)=0.$$

□

Lemma 13. *Let h be a function of class $\mathcal{C}^{0,1}([0,1))$, if* $\liminf\limits_{s\to1}(1-s)h(s)>0$ *then* $\int_0^1 h(s)ds=+\infty$.

Proof. From the statement, $\liminf\limits_{s\to1}(1-s)h(s)=\lambda$ for a positive real number λ. Thus, there exists $\varepsilon\in(0,1)$ such that $h(s)\geq\lambda/(2(1-s))$, for all $s\in(1-\varepsilon,1)$. Hence,

$$\int_0^1 h(s)ds=\int_0^{1-\varepsilon}h(s)ds+\int_{1-\varepsilon}^1 h(s)ds.$$

The proof ends because $h(s)$ is bounded in $[0,1-\varepsilon]$ and

$$\int_{1-\varepsilon}^{1} h(s)ds \geq \int_{1-\varepsilon}^{1} \frac{\lambda}{2}\frac{1}{1-s}ds = +\infty.$$

□

Next result extends the dynamics to the boundary of $\widetilde{D}$. Then we study the behavior of the orbits in each quadrant.

Proposition 14. *Consider system* (7) *under the hypotheses* $(\widetilde{H})$*. Additionally we assume that*

(i) there exist real numbers $\widetilde{\delta},\widetilde{\eta}$*, such that* $-1<\widetilde{\eta}<0<\widetilde{\delta}<1$*, and* $\widetilde{f}(w)>0$ *for all* $w\in(-1,1)\setminus[\widetilde{\eta},\widetilde{\delta}]$*,*

(ii) there exists $\widetilde{\lambda}_i\in\mathbb{R}^+\cup\{+\infty\}$ *with* $\liminf_{w\to w_i}|w-w_i|(|\widetilde{g}(x)|+\widetilde{f}(x))=\widetilde{\lambda}_i$*, for* $i=1,2$*,*

(iii) $z\widetilde{\psi}(z)>0$ *for all* $z\neq 0$*,*

(iv) $\lim_{|z|\to 1}|\chi(z)|=+\infty$*, and*

(v) $\lim_{z\to z_i}\widetilde{\psi}(z)/\chi(z)=0$*, for* $i=1,2$*.*

Then for $i=1,2$*, given* $w_0,z_0\in(-1,1)$ *such that* $-\widetilde{\psi}(z_0)\notin\left[\liminf_{w\to w_i}\frac{\widetilde{g}(x)}{\widetilde{f}(x)},\limsup_{w\to w_i}\frac{\widetilde{g}(x)}{\widetilde{f}(x)}\right]$*, the vector field defined by* (7) *is topologically equivalent to*

$$\begin{cases}\dot{w}=\operatorname{sign}(z_i),\\ \dot{z}=0,\end{cases}\qquad or\qquad \begin{cases}\dot{w}=0,\\ \dot{z}=\lim_{w\to w_i}\operatorname{sign}(-\widetilde{g}(w)-\widetilde{f}(w)\widetilde{\psi}(z_0))\end{cases}$$

in a neighborhood of (w_0,z_i) *or* (w_i,z_0)*, respectively.*

Proof. First we prove the equivalence for neighborhoods of points $(w_0,\pm1)$. For any $z\neq 0$, we can rewrite the system (7) with a positive time rescaling as

$$\begin{cases}\dot{w}=\operatorname{sign}(\chi(z)),\\ \dot{z}=\frac{-\widetilde{g}(w)-\widetilde{f}(w)\widetilde{\psi}(z)}{|\chi(z)|}.\end{cases}\tag{10}$$

For any fixed $w_0\in(-1,1)$, applying (iv) and (v), we have $\dot{z}\to 0$ when $z\to\pm1$. It means that the segments of the boundary of $\widetilde{\mathcal{D}}$ contained in $\{z=\pm1\}$ are invariant for (10). Using hypotheses $(\widetilde{H})$, $\chi(z)$ and z have the same sign, so the proof, for this case, ends.

Finally, we only prove the equivalence for neighborhoods of points $(1,z_0)$. For the points $(-1,z_0)$ the proof is analogous. For any z_0 satisfying $-\widetilde{\psi}(z_0)\notin\left[\liminf_{w\to1}\widetilde{g}(w)/\widetilde{f}(w),\right.$ $\left.\limsup_{w\to1}\widetilde{g}(w)/\widetilde{f}(w)\right]$ there exists a neighborhood of $(1,z_0)$ in $\widetilde{\mathcal{D}}$ such that $\widetilde{g}(w)+\widetilde{f}(w)\widetilde{\psi}(z)$ does not vanishes. In this neighborhood, system (7) is equivalent to

$$\begin{cases}\dot{w}=\dfrac{\chi(z)}{|-\widetilde{g}(w)-\widetilde{f}(w)\widetilde{\psi}(z)|},\\ \dot{z}=\operatorname{sign}(-\widetilde{g}(w)-\widetilde{f}(w)\widetilde{\psi}(z)).\end{cases}$$

Hence, the equivalence follows similarly to the previous case. □

Definition 15. *We say that the points* $(w_0,\pm1)$ *or* $(\pm1,z_0)$*, are regular points in the boundary when they satisfy the properties of the Proposition 14. The other points, including the vertex, are called singular.*

Remark 16. *Proposition 14 extends the dynamical behavior of system* (7) *in* $\widetilde{\mathcal{D}}$ *to the regular points of its closure. See a possible phase portrait in Figure 5.*

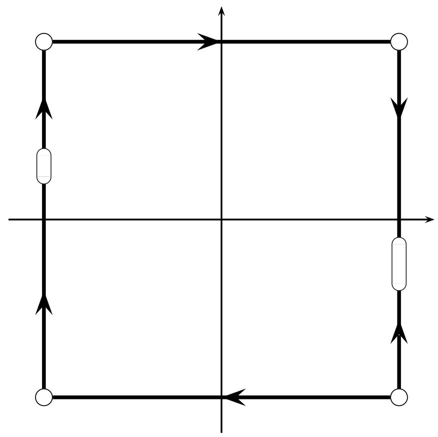


FIGURE 5. An example of a phase portrait on the boundary. The rounded regions represent the set of singular points

Proposition 17. *Under the assumptions of Proposition 14, the positive orbit of every point in* $[0,1)\times(0,1)$ *(resp. in* $(-1,0]\times(-1,0)$*) cuts transversally, in finite time, the segment* $(0,1)\times\{0\}$ *(resp.* $(-1,0)\times\{0\}$*). See Figure 6.*

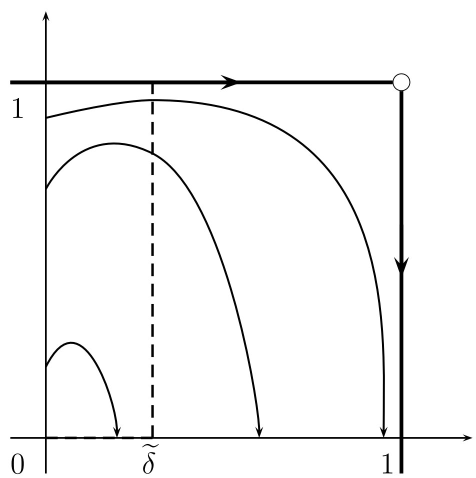


FIGURE 6. Phase portrait of system (7) on the first quadrant

Proof. We only prove the result on the first quadrant. The other follows by symmetry.

The first component of the vector field on the points over the positive z-axis is greater than zero. Proposition 14 provides a Flow Box argument for the vector field (7) in the neighborhood of the points $(0,1)\times\{1\}$. As there are no critical points with $w>0$,

$z > 0$ in $\widetilde{\mathcal{D}}$, the proof ends, from the Poincaré-Bendixson Theorem, showing that there are no orbits tending to $(1, z_0)$ with $z_0 > 0$. We prove it by contradiction.

In a neighborhood of the segment $\{1\} \times (0, 1)$ in $\widetilde{\mathcal{D}}$, condition (ii) implies that $\lim_{w\to 1}(1-w)\widetilde{g}(w)$ or $\lim_{w\to 1}(1 - w)\widetilde{f}(w)$ is strictly positive. Then, one of the state functions of Section 4, after the changes of variables of Proposition 3,

$$\widetilde{E}(w, z) = \widetilde{G}(w) + \widetilde{\Phi}(z) = \int_0^w \widetilde{g}(u)du + \int_0^z \chi(v)dv$$
$$\widetilde{J}(w, z) = \widetilde{F}(w) + \widetilde{\Psi}(z) = \int_0^w \widetilde{f}(u)du + \int_0^z \frac{\chi(v)}{\widetilde{\psi}(v)}dv,$$

goes to infinity when (w, z) tends to the boundary of $\widetilde{\mathcal{D}}$ by Lemma 13.

Therefore, if an orbit goes to the boundary then one of the state functions goes to infinity. This contradicts with the fact that both state functions decrease over the solutions of the vector field on the region $w > \widetilde{\delta}$ and $z > 0$. Because, by (i) and $(\widetilde{H})$,

$$\dot{\widetilde{E}}(w, z) = \widetilde{g}(w)\dot{w} + \chi(z)\dot{z} = -\widetilde{f}(w)\widetilde{\psi}(z)\chi(z) < 0$$

and

$$\dot{\widetilde{J}}(w, z) = \widetilde{f}(w)\dot{w} + \frac{\chi(z)}{\widetilde{\psi}(z)}\dot{z} = -\widetilde{g}(w)\frac{\chi(z)}{\widetilde{\psi}(z)} < 0.$$

□

Proposition 18. *Under the assumptions of Proposition 14, the positive orbit of every point in* $(0, 1) \times (-1, 0]$ *(resp. in* $(-1, 0) \times [0, 1)$*) cuts transversally, in finite time, the segment* $\{0\} \times (-1, 0)$ *(resp.* $\{0\} \times (0, 1)$*). See Figure 7.*

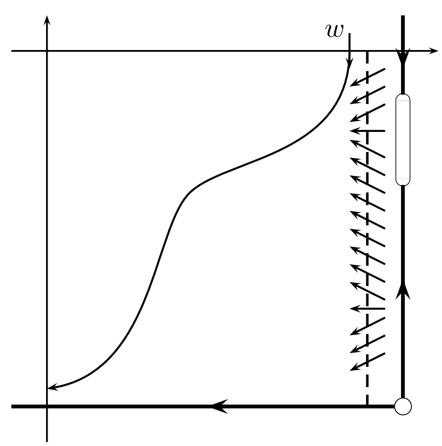


FIGURE 7. Phase portrait of system (7) on the fourth quadrant

Proof. We only prove the result on the fourth quadrant, the other follows by symmetry.

The second component of the vector field on the points over the positive w-axis is negative. Proposition 14 provides a Flow Box argument for the vector field (7) in the neighborhood of the points $(0, 1) \times \{-1\}$. As there are no critical points with $w > 0$, $z < 0$ in $\widetilde{\mathcal{D}}$, the proof ends, from the Poincaré-Bendixson Theorem, using the condition $\dot{w} = \chi(z) < 0$ on the fourth quadrant. □

The last two propositions imply the next corollary.

Corollary 19. *The ω-limit of a finite point of $\widetilde{\mathcal{D}}$ can not be partially contained in the corresponding boundary.*

Finally, the proof of Theorem 1 follows from the next two propositions. In the first result, the boundary always presents singular points besides the vertex, while in the second does not.

Proposition 20. *Consider system* (4) *under the hypotheses* (H) *and satisfying that*

(i) there exist $\delta, \eta \in \mathbb{R}$, with $x_1 < \eta < 0 < \delta < x_2$, such that $f(x) > 0$ for all $x \in (x_1, x_2) \setminus [\eta, \delta]$,

(ii) for each $i = 1, 2$ there exists $\lambda_i \in \mathbb{R}^+ \cup \{+\infty\}$ such that, if $|x_i| = +\infty$, then $\liminf_{x \to x_i} x(|g(x)| + f(x)) = \lambda_i$, and if $x_i \in \mathbb{R}$, then $\liminf_{x \to x_i} |x - x_i|(|g(x)| + f(x)) = \lambda_i$,

(iii) $y\psi(y) > 0$ for all $y \neq 0$,

(iv) for $i = 1, 2$, $\lim_{y \to y_i} \psi(y)/(y\varphi'(y)) \in \mathbb{R}$,

(v) there exists $y_0 \in (y_1, y_2)$ such that $-\psi(y_0) \in \left[\liminf_{x \to x_i} g(x)/f(x), \limsup_{x \to x_i} g(x)/f(x)\right]$ for at least one of the x_i, $i = 1, 2$, and there exists U, neighborhood of y_0, such that $\operatorname{sign}(\psi'(y))$ is constant almost for every $y \in U$.

Then, the boundary of $\mathcal{D}$ is a repellor.

Proof. Applying Lemmas 7 to 12 to system (7), we can compactify and we are on the hypotheses of Propositions 14. Then, from statement (v), see Lemma 9, there exists $z_0 \in (-1, 1)$ such that $-\widetilde{\psi}(z_0)$ is in $I_i = \left[\liminf_{w \to w_i} \widetilde{g}(w)/\widetilde{f}(w), \limsup_{w \to w_i} \widetilde{g}(w)/\widetilde{f}(w)\right]$ for at least one of the w_i and there exists a neighborhood, $\widetilde{U}$, of z_0 such that $\operatorname{sign}(\widetilde{\psi}'(z))$ is constant almost for every $z \in \widetilde{U}$.

We only prove the case $w_2 = 1$ and $\widetilde{\psi}'(z) > 0$. The other cases follow similarly.

The proof is done in two steps. In the first one, we study the behavior of the vector field close to $(1, z_0)$ and, in second place, we construct a negatively invariant region that proves that the infinity is a repellor. For the first step we distinguish two different cases, when I_2 is a proper interval or it reduces to a point.

If we are in the first case, we can assume that $\widetilde{\phi}(z_0)$ is in the interior of I_2. From the definition of lim inf and lim sup, there exists a sequence $\{w_n\}_{n=1}^{+\infty} \subset (0, 1)$ such that $\lim_{n \to +\infty} w_n = 1$ and $-\widetilde{g}(w_n) - \widetilde{f}(w_n)\widetilde{\psi}(z_0) = 0$. Moreover, there exists $\varepsilon > 0$ such that $(z_0 - \varepsilon, z_0 + \varepsilon) \subset \widetilde{U}$ and $\widetilde{\psi}(z_0 - \delta) < \widetilde{\psi}(z_0) < \widetilde{\psi}(z_0 + \delta)$, for all δ in $(0, \varepsilon)$. Hence, it follows that for all $n \in \mathbb{N}$ and $\delta \in (0, \varepsilon)$ we have $\dot{z}(w_n, z_0 - \delta) > 0$ and $\dot{z}(w_n, z_0 + \delta) < 0$, and consequently there exists an orbit, $\Gamma(z_0)$, which α-limit set is the point $(1, z_0)$. See Figure 8.

In the second case, $-\widetilde{\psi}(z_0) = \lim_{w \to 1} \widetilde{g}(w)/\widetilde{f}(w)$ and $\widetilde{\psi}'(z) > 0$ in $\widetilde{U}$. Then the branch of $\widetilde{\psi}^{-1}\left(-\widetilde{g}(w)/\widetilde{f}(w)\right)$ defined in $\widetilde{U}$ is a well defined function for all $w \in (1 - \gamma, 1)$ for a positive small enough γ. Similar arguments as the ones in the previous case imply that $\Gamma(z_0)$ also exists. See Figure 9.

For the second step, Propositions 17 and 18 imply that the orbit $\Gamma(z_0)$ touches the positive w-axis, in finite time, passing through all quadrants in counterclockwise direction. We call $(\widehat{w}, 0)$ the first time that this happens and $(\widehat{w}, \widehat{z})$ the first time that the orbit $\Gamma(z_0)$ cuts the straight line $w = \widehat{w}$ in the fourth quadrant. Then, by $(\widetilde{H}_5)$, the

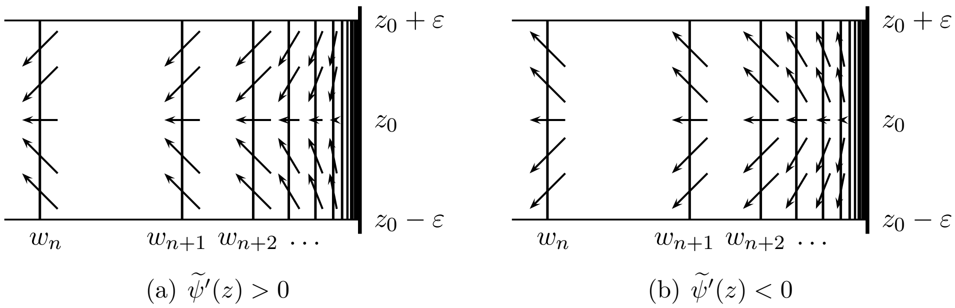


FIGURE 8. Behavior of the flux near to a continuum of singular points in the boundary

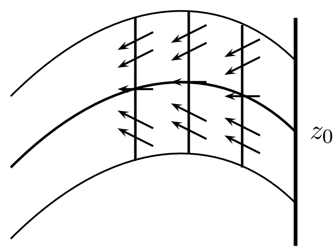


FIGURE 9. Behaviour of the flux near to an isolated singular point in the boundary

region defined by $\Gamma(z_0)$ between $(\widehat{w}, \widehat{z})$ and $(\widehat{w}, 0)$ and the segment with those endpoints, $\widehat{S}$, is positively invariant. The proof ends because the positive orbits of all points in the complement of this region in $\widetilde{D}$ cross the segment $\widehat{S}$. See Figure 10.

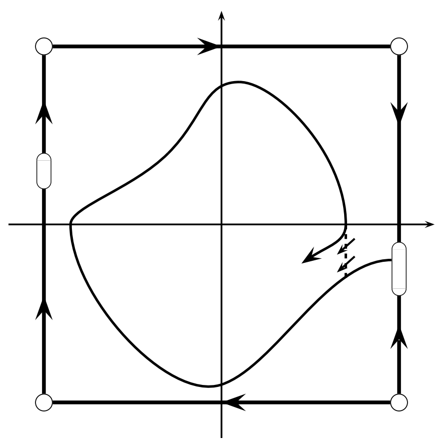


FIGURE 10. Positively invariant region when the boundary has singular points different from the vertex

□

Proposition 21. *Consider system* (4) *under the hypotheses* (H) *and satisfying that*

(i) *there exist* δ *and* η *in* $\mathbb{R}$, *with* $x_1 < \eta < 0 < \delta < x_2$, *such that* $f(x) > 0$ *for all* $x \in (x_1, x_2) \setminus [\eta, \delta]$ *and the integral* $\int_\eta^\delta f(x)dx$ *is positive,* *(ii) for each* $i = 1, 2$ *there exists* λ_i *in* $\mathbb{R}^+ \cup \{+\infty\}$ *such that, if* $|x_i| = +\infty$*, then* $\liminf_{x\to x_i} x(|g(x)| + f(x)) = \lambda_i$*, and if* $x_i \in \mathbb{R}$*, then* $\liminf_{x\to x_i} |x - x_i|(|g(x)| + f(x)) = \lambda_i$*,*

(iii) $y\psi(y) > 0$ *for all* $y \neq 0$*,*

(iv) for $i = 1, 2$*,* $\lim_{y\to y_i} \psi(y)/(y\varphi'(y)) \in \mathbb{R}$*.*

Then the boundary of $\mathcal{D}$ *is a repellor.*

Proof. System (4) from Proposition 3 and Lemmas 7 to 12, writes as the compactified equivalent system (7) in $\widetilde{\mathcal{D}}$. In order to obtain a contradiction, we suppose that the boundary of $\mathcal{D}$ is not a repellor. Hence, Propositions 14, 17 and 18 ensure the existence of a return map close to the boundary. More concretely, for $\varepsilon > 0$ small enough, there exists an orbit, Γ_ε, that starts at $(\widetilde{\eta}, z_0)$ with $z_0 \in (1-\varepsilon, 1)$, cuts after a time T the segment $\{\widetilde{\eta}\} \times (0, 1)$ at $(\widetilde{\eta}, z_T)$ with $z_T \in [z_0, 1)$ and remains, for positive time, in $(-1, 1) \times (-1, 1) \setminus (-1+\varepsilon, 1-\varepsilon) \times (-1+\varepsilon, 1-\varepsilon)$. See Figure 11. Let us denote by $(\widetilde{\eta}, z_0)$, $(\widetilde{\delta}, z_1)$, $(\widetilde{\delta}, z_2)$, $(\widetilde{\eta}, z_3)$ and $(\widetilde{\eta}, z_T)$ the consecutive cutting points of Γ_ε with the segments $\{\widetilde{\eta}\} \times (-1, 1)$ and $\{\widetilde{\delta}\} \times (-1, 1)$. Consequently, $z_0, z_1, z_T \in (1-\varepsilon, 1)$ and $z_2, z_3 \in (-1, -1+\varepsilon)$.

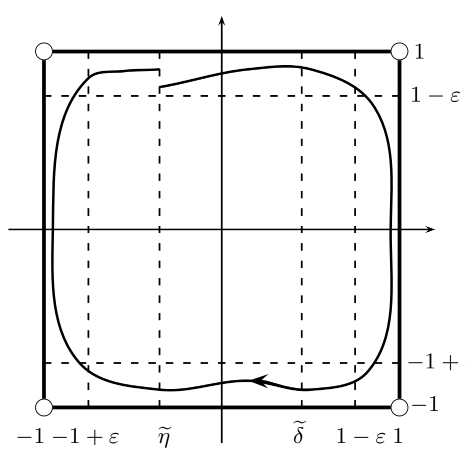


FIGURE 11. Phase portrait, close boundary, of Γ_ε when the boundary of $\widetilde{\mathcal{D}}$ is not a repellor

The contradiction is obtained checking that the primary energy function $\widetilde{E}$, see Lemma 4, in $(\widetilde{\eta}, z_T)$ is lower than in $(\widetilde{\eta}, z_0)$. Because, as $z\dfrac{\partial \widetilde{E}}{\partial z} = z\chi(z) > 0$ for all $z \neq 0$, $\widetilde{E}$ grows when $|z|$ grows but $z_T \geq z_0$. So, we conclude proving that $\widetilde{E}$ decreases when the orbit passes through the consecutive cutting points defined before.

Straightforward computations show that, from (i), there exist $\widetilde{\eta}, \widetilde{\delta}$ satisfying $-1 < \widetilde{\eta} < 0 < \widetilde{\delta} < 1$ and $\int_{\widetilde{\eta}}^{\widetilde{\delta}} \widetilde{f}(s)ds > 0$. Then there exists a positive real number A such that $-\int_{\widetilde{\eta}}^{\widetilde{\delta}} \widetilde{f}(s)ds + 2M(\widetilde{\delta} - \widetilde{\eta})/A < 0$, where $M = \max_{w\in(\widetilde{\eta},\widetilde{\delta})} |g(w)|$. By hypotheses (H), (iv) and Lemma 12, we obtain that $\lim_{z\to z_i} \chi(z) = +\infty$ and $\lim_{z\to z_i} \widetilde{\psi}(z)/\chi(z) = 0$ or, equivalently, $\lim_{z\to z_i} \chi(z)/\widetilde{\psi}(z) = +\infty$, for $i = 1, 2$. Therefore we fix $\varepsilon > 0$, small enough, such that $\chi(z) > A^2$ and $\frac{\chi(z)}{\widetilde{\psi}(z)} > A$ for all $z \in (1-\varepsilon, 1) \cup (-1, -1+\varepsilon)$.

Hence, writing equation (7) as $dz/dw = -\widetilde{g}(w)/\widetilde{\psi}(z) - \widetilde{f}(w)\widetilde{\psi}(z)/\chi(z)$, we can estimate the differences

$$z_1 - z_0 = \int_{\widetilde{\eta}}^{\widetilde{\delta}} \left(-\frac{\widetilde{g}(w)}{\widetilde{\psi}(z)} - \frac{\widetilde{f}(w)\widetilde{\psi}(z)}{\chi(z)} \right) dw \leq \frac{1}{A} \left(\int_{\widetilde{\eta}}^{\widetilde{\delta}} -\widetilde{f}(w)dw + \frac{M}{A}(\widetilde{\delta} - \widetilde{\eta}) \right) < 0,$$

$$\widetilde{G}(\widetilde{\delta}) - \widetilde{G}(\widetilde{\eta}) = \int_{\widetilde{\eta}}^{\widetilde{\delta}} \widetilde{g}(w)dw \leq M(\widetilde{\delta} - \widetilde{\eta}) \text{ and } \widetilde{\Phi}(z_1) - \widetilde{\Phi}(z_0) = -\int_{z_1}^{z_0} \chi(z)dz \leq -A^2(z_0 - z_1),$$

where the $\widetilde{G}$ and $\widetilde{\Phi}$ are the compactified functions defined in Lemma 4. Thus, the primary energy function satisfies

$$\widetilde{E}(\widetilde{\delta}, z_1) - \widetilde{E}(\widetilde{\eta}, z_0) \leq M(\widetilde{\delta} - \widetilde{\eta}) + A^2(z_1 - z_0) \leq 2M(\widetilde{\delta} - \widetilde{\eta}) - A \int_{\widetilde{\eta}}^{\widetilde{\delta}} \widetilde{f}(w)dw < 0.$$

So, the energy decreases from $(\widetilde{\eta}, z_0)$ to $(\widetilde{\delta}, z_1)$ and from $(\widetilde{\delta}, z_2)$ to $(\widetilde{\eta}, z_3)$, applying the same argument replacing z_0 and z_1 by z_2 and z_3, respectively.

Finally, the energy also decreases from $(\widetilde{\delta}, z_1)$ to $(\widetilde{\delta}, z_2)$ and from $(\widetilde{\eta}, z_3)$ to $(\widetilde{\eta}, z_T)$ because $\widetilde{f}(w) > 0$ and $\dot{\widetilde{E}} = -\widetilde{f}(w)\chi(z)\widetilde{\psi}(z) \leq 0$, for all $w \in (-1,1) \setminus [\widetilde{\eta}, \widetilde{\delta}]$ and $z \in (-1,1)$. □

6. Uniqueness of limit cycle

This section is devoted to prove the Unicity result, Theorem 2.

Proposition 6 shows that the origin is the unique singular point, which is a repellor. Additionally, there is no periodic orbits entirely contained in $(a,b) \times (y_1, y_2)$ because $\dot{E} = -f(x)y\varphi'(y)\psi(y) > 0$, for all $(x,y) \in (a,b) \times (y_1,y_2) \setminus \{(0,0)\}$. Moreover, all the periodic orbits contain the region $\{(x,y) \in \mathcal{D} : 0 \leq E(x,y) \leq \min(G(a), G(b))\}$, because it is negatively invariant. In the proof we assume that $G(a) \leq G(b)$, in other case we can change (x,y) by $(-x,-y)$.

The proof is done by the method of comparison. Let us suppose that we have two different limit cycles, Γ_1 and Γ_2. Then we prove that the integral of the divergence of the vector field (4), between them, is different from zero, in fact it is negative. This contradicts the existence of two limit cycles because it implies that both orbits have the same stability.

Taking into account the above considerations, there are three possible configurations of Γ_1 and Γ_2 in terms of the position of a, b and x_0. See Figure 12. We only present the proof when Γ_1 contains the segment $(a, x_0) \times \{0\}$ and Γ_2 contains the segment $(a,b) \times \{0\}$. See Figure 12(b). The proof follows similarly for the other two cases.

The integral of the divergence of equation (4),

$$\operatorname{div} X = \frac{d}{dx}\left(y\varphi'(y)\right) + \frac{d}{dy}\left(-g(x) - f(x)\psi(y)\right) = -f(x)\psi'(y),$$

between both periodic orbits is computed decomposing the region in five different regions G_i, $i = 1, \ldots, 5$. See them in Figure 13.

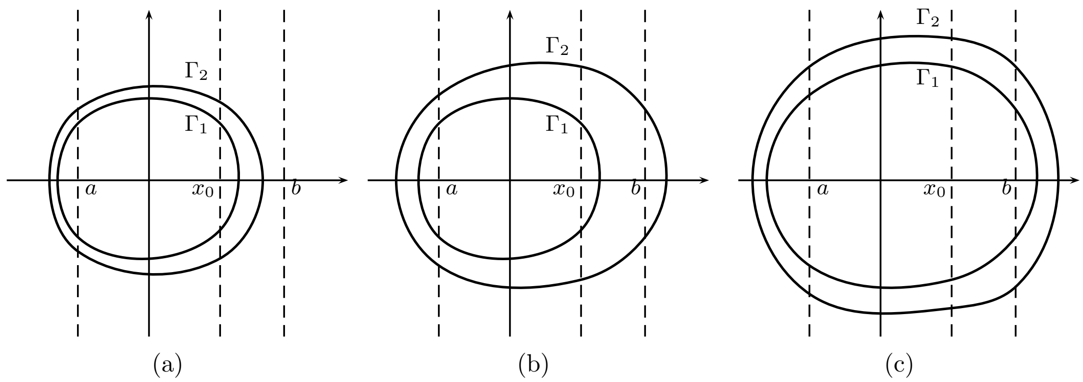


FIGURE 12. Relative positions between Γ_1 and Γ_2 with respect to the intervals (a, b) and (a, x_0)

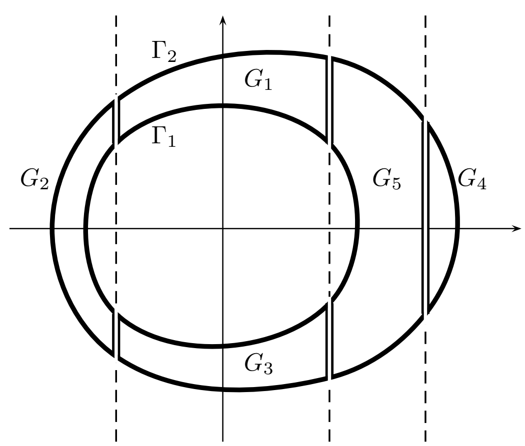


FIGURE 13. Decomposition of the enclosed region between Γ_1 and Γ_2

The different time reparametrizations that we use along the periodic orbits, in each region, are

$$dt = \begin{cases} \dfrac{1}{y\varphi'(y)}\,dx & \text{if } (x,y) \in \Gamma_{|G_1 \cup G_3}, \\ -\dfrac{1}{g(x)+f(x)\psi(y)}\,dy & \text{if } (x,y) \in \Gamma_{|G_2 \cup G_4}, \\ -\dfrac{\psi(y)}{g(x)y\varphi'(y)}\,dJ & \text{if } (x,y) \in \Gamma_{|G_5}, \end{cases}$$

where J is the secondary energy function, defined in Lemma 5. Then, by the Green's Lemma, the integral of the divergence along the curve $\Gamma_1 - \Gamma_2$ can be written as

$$\begin{aligned}\int_{\Gamma_1-\Gamma_2} \operatorname{div} X dt = & \iint_{G_1\cup G_3} -\frac{d}{dy}\left(\frac{f(x)\psi'(y)}{y\varphi'(y)}\right) dx\, dy+ \\ & + \iint_{G_2\cup G_4} \frac{d}{dx}\left(-\frac{f(x)\psi'(y)}{g(x)+f(x)\psi(y)}\right) dy\, dx+ \\ & + \iint_{G_5} \frac{d}{dx}\left(-\frac{f(x)\psi'(y)\psi(y)}{g(x)y\varphi'(y)}\right) dJ\, dx = \\ = & \iint_{G_1\cup G_3} \Delta_1\, dx\, dy + \iint_{G_2\cup G_4} \Delta_2\, dy\, dx + \iint_{G_5} \Delta_5\, dJ\, dx,\end{aligned}$$

where

$$\begin{aligned}\Delta_1 &= -f(x)\frac{d}{dy}\left(\frac{\psi'(y)}{y\varphi'(y)}\right), && \text{for } x\in(a,b),\ y\in(y_1,y_2)\setminus\{0\},\\ \Delta_2 &= -\psi'(y)\frac{g(x)^2}{(g(x)+f(x)\psi(y))^2}\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right), && \text{for } x\in(x_1,x_2)\setminus[a,b],\ y\in(y_1,y_2)\setminus\{0\},\\ \Delta_5 &= -\frac{\psi'(y)\psi(y)}{y\varphi'(y)}\left(-\frac{f(x)}{g(x)}\right), && \text{for } x\in(x_0,b),\ y\in(y_1,y_2)\setminus\{0\}.\end{aligned}$$

The proof ends because, from the statements, it can be checked that all the integrands, Δ_1, Δ_2 and Δ_5, are negative in each region where they are considered.

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