p-COMPACT GROUPS AS SUBGROUPS OF MAXIMAL RANK OF KAC-MOODY GROUPS

JAUME AGUADÉ

1. Introduction

In [28], Kitchloo constructed a map $f : BX \to BK^{\wedge}_p$ where K is a certain Kac-Moody group of rank two, X is a rank two mod p finite loop space and f is such that it induces an isomorphism between even dimensional mod p cohomology groups. Here B denotes the classifying space functor and $(-)^{\wedge}_p$ denotes the Bousfield-Kan $\mathbb{F}_p$-completion functor ([8]).

This space X —or rather the triple $(X^{\wedge}_p, BX^{\wedge}_p, e)$ where $e : X \simeq \Omega BX$— is a particular example of what is known as a p-compact group. These objects were introduced by Dwyer and Wilkerson in [15] as the homotopy theoretical framework to study finite loop spaces and compact Lie groups from a homotopy point of view. The foundational paper [15] together with its many sequels by Dwyer-Wilkerson and other authors represent now an active, well established research area which contains some of the most important recent advances in homotopy theory.

While p-compact groups are nowadays reasonably well understood objects, our understanding of Kac-Moody groups and their classifying spaces from a homotopy point of view is far from satisfactory. The work of Kitchloo in [28] started a project which has also involved Broto, Saumell, Ruiz and the present author and has produced a series of results ([2], [3], [10]) which show interesting similarities between this theory and the theory of p-compact groups, as well as non trivial challenging differences.

The goal of this paper is to extend the construction of Kitchloo that we have recalled above to produce rank-preserving maps $BX \to BK^{\wedge}_p$ for a wide family of p-compact groups X. These maps can be understood as the homotopy analogues to monomorphisms, in a sense that will be made precise in section 13. We prove:

Theorem 1.1. *Let p be a prime and let X be a simply connected p-compact group with Weyl group W_X. Assume that the order of W_X is prime to p and W_X is generated by pseudoreflections of order two. Then, there exists a Kac-Moody group K of the same rank as X and a monomorphism from X to $K^{\wedge}_p$.*

Actually, this paper contains more information than what is contained in this theorem. The proof of 1.1 is constructive and contains a concrete description of the groups K involved, and the properties of these groups are investigated beyond what is strictly necessary to prove 1.1. Also, the main technical ingredient in the proof (theorem 8.3) —as well as our review of the classic work of Coxeter and Shephard ([13], [35]) on reflection groups— may have some independent interest. It should be

The author is partially supported by grants MTM204-06686, PR2007-0097 and SGR2005-00606.

pointed out that we provide counterexamples (see section 15) which show that the hypothesis on W_X in theorem 1.1 cannot be relaxed.

Our motivation for the research done in this paper has several aspects. First of all, this paper can be viewed as a further step in understanding Kac-Moody groups and their classifying spaces from a homotopy point of view. Secondly, we think that it is interesting to relate p-compact groups —which are purely "homotopical" objects which "live at a single prime"— to other "analytical", "integral" objects as are Kac-Moody groups. On the other hand, the maps $f : BX \to BK_p^\wedge$ that we construct in theorem 1.1 give rise to new families of "homogeneous spaces" which may be interesting on their own. For instance, the original map of Kitchloo in [28] produced a fibration which looks like a particular case of the Anick fibration ([6]) —and which it has been conjectured to be equal to the Anick fibration. In the same way, the maps in 1.1 yield fibrations which may be related to Anick's work and could be worth of a closer investigation.

The author would like to thank PIMS at the University of British Columbia and the Department of Mathematics of Kyoto University where parts of this work were completed.

2. Kac-Moody groups

Kac-Moody groups are certain connected topological groups whose construction and basic properties can be read in [25], [26] and [32]. Simply connected compact Lie groups are particular examples of Kac-Moody groups, but most Kac-Moody groups are infinite dimensional. We review here very quickly the basic facts about Kac-Moody groups that we will need in our study and we refer to the works of Kac and Peterson for any further study of these objects.

A *generalized Cartan matrix* is a square matrix $A = (a_{ij})$ with $a_{ii} = 2$, such that a_{ij} are non-positive integers for $i \neq j$ and $a_{ij} = 0$ implies $a_{ji} = 0$. Such a matrix provides enough data to define a Lie algebra $\mathfrak{g}(A)$ which is, in general, infinite-dimensional. Actually, $\mathfrak{g}(A)$ is finite dimensional if and only if A is positive definite. These Lie algebras are called Kac-Moody Lie algebras and Kac proved that they can be integrated in some way that we do not need to discuss here to give rise to simply connected topological groups $G(A)$ which are just the complex semisimple Lie group associated to A if the Cartan matrix A is positive definite. $G(A)$ has a canonical involution and the *unitary form* $K(A)$ is defined as the fixed point set of this involution. Along this paper, the words "Kac-Moody group" refer to the topological group $K(A)$ associated to any generalized Cartan matrix A. The rank of $K(A)$ is the size of the matrix A. We point out that Kac-Moody groups are connected and simply connected by construction. The center of a Kac-Moody group is well understood and we can talk about "adjoint forms" of Kac-Moody groups. Other properties of these groups will be introduced when needed.

Let us introduce the three families of Kac-Moody groups that we will study in this paper. They depend on two positive integral parameters a, b. For any $n \geq 2$ we denote by $K_n^1(a, b)$ the Kac-Moody group of rang n associated to the generalized

Cartan matrix

$$\begin{pmatrix} 2 & -a & & & & & \\ -b & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & \ddots & & & \\ & & & & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix}$$

The entries of this matrix are equal to 2 at the diagonal and equal to -1 at the lines immediately above and below the diagonal, except for the first 2×2 block. All other entries are zero. We assume $ab \geq 4$ so that this matrix is never positive definite and $K_n^1(a,b)$ is not a Lie group. The groups $K_2^1(a,b)$ were studied thoroughly in [28], [3] and [2] where they were denoted $K(a,b)$.

For $n \geq 3$ we denote by $K_n^2(a,b)$ the Kac-Moody group of rang n associated to the generalized Cartan matrix

$$\begin{pmatrix} 2 & -a & -1 & & & & & \\ -b & 2 & -1 & & & & & \\ -1 & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & & \ddots & & & \\ & & & & & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}$$

The entries of this matrix are equal to 2 at the diagonal and equal to -1 at the lines immediately above and below the diagonal, except for the first 3×3 block. All other entries are zero. This matrix is never positive definite and so $K_n^2(a,b)$ is not a Lie group.

For $n \geq 5$, $K_n^3(a,b)$ is the Kac-Moody group of rank n associated to the generalized Cartan matrix

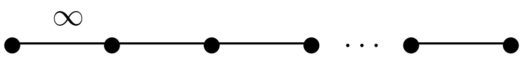
$$\begin{pmatrix} 2 & -a & -1 & & & & -1 \\ -b & 2 & -1 & & & & 0 \\ -1 & -1 & 2 & -1 & & & 0 \\ & & -1 & 2 & -1 & & 0 \\ & & & & \ddots & & \\ -1 & 0 & 0 & 0 & \cdots & 0 & 2 \end{pmatrix}$$

consisting of the Cartan matrix for $K_{n-1}^2(a,b)$ with an extra row and column. Again, $K_n^3(a,b)$ is never a Lie group.

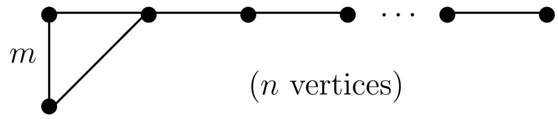
3. Coxeter diagrams and Weyl groups

Each Kac-Moody group K has a maximal torus of finite rank —which is a maximal connected abelian subgroup of K and all such subgroups are conjugate— and a corresponding Weyl group W_K which is a Coxeter group. W_K is finite if and only if K is a Lie group. The entries in the Cartan matrix (a_{ij}) give immediately a presentation for W_K in the following way. If the rank is n then W_K has generators $w_1, \ldots, w_n$ and relations $w_i^2 = 1$ for all i and $(w_i w_j)^{m_{ij}} = 1$ where $m_{ij} = 2,3,4,6,0$ if $a_{ij}a_{ji} = 0,1,2,3$ or > 3, respectively. Then, the Coxeter diagram for K is defined as the graph with vertices $1, \ldots, n$, the vertex i is joined by an edge to the vertex j if $m_{ij} > 2$ and this edge has a label m_{ij} if $m_{ij} > 3$. If $m_{ij} = 0$ then we use ∞ as a label instead of 0.

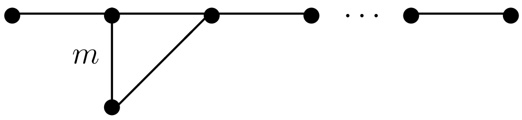
The Coxeter diagrams for the groups K_n^1, K_n^2, K_n^3 are as follows. The diagram for $K_n^1(a,b)$ is



The diagram for $K^2_n(a,b)$ is



m is the order of w_1w_2. In this diagram, we order the vertices in the way that the vertices in the top row are $1,3,4,\dots,n$ and the bottom vertex is 2. The diagram for $K^3_n(a,b)$ is



The bottom vertex is 2 and the top left vertex is the last one.

The action of the Weyl group W_K on the Lie algebra of the maximal torus gives a well defined representation of W_K in $GL_n(\mathbb{Z})$. Each generator w_j acts on a certain basis $h_1,\dots,h_n$ as

$$w_jh_i = h_i - a_{ij}h_j. \tag{1}$$

As an example, the three generators of the Weyl group of $K^2_3(a,b)$ as elements in $GL_3(\mathbb{Z})$ are given by

$$w_1 = \begin{pmatrix} -1 & b & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 0 & 0 \\ a & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

4. Parabolic subgroups and a homotopy decomposition of BK

If K is any Kac-Moody group of rank n and I is a proper subset of $\{1,\dots,n\}$ we can consider the *parabolic* subgroup P_I of K (see [25]). If I is empty, then P_I is just the standard maximal torus of K. We write $P_I(K)$ if we want to indicate which Kac-Moody group we are considering.

A fundamental result in the homotopy theory of the classifying spaces of Kac-Moody groups is the following result which follows from Mitchell's theory of topological Tits buildings ([29], see also [28] and [10]): If the Weyl group W is *infinite* then there is a homotopy equivalence

$$BK \simeq \operatorname*{hocolim}_I BP_I.$$

In this way, we have a kind of inductive way to study the spaces BK. Notice that the groups P_I are not, properly speaking, Kac-Moody groups but many of the concepts of Kac-Moody groups can be applied as if P_I were a Kac-Moody group of rank n with Weyl group W_I generated by the reflections w_i, $i \in I$. Also, if W_I is infinite, then the homotopy colimit decomposition above holds true for BP_I and BP_I is equivalent to the homotopy colimit of the spaces BP_J for $J \subsetneq I$. Hence, in the homotopy decomposition of BK given above, we only need to consider the parabolic subgroups with *finite* Weyl group, i.e. the parabolic subgroups of K which are Lie groups:

$$BK \simeq \operatorname*{hocolim}_{W_I \text{ finite}} BP_I. \tag{2}$$

Moreover ([10]), if W_K is infinite, then we have homotopy equivalences

$$\operatorname*{hocolim}_{I} BW_I \simeq BW, \quad \operatorname*{hocolim}_{I} W_I \backslash W \simeq *$$

and

$$\operatorname*{hocolim}_{W_I \text{ finite}} BW_I \simeq BW, \quad \operatorname*{hocolim}_{W_I \text{ finite}} W_I \backslash W \simeq *. \tag{3}$$

For a Kac-Moody group K we define its *finiteness width* $fw(K)$ by $fw(K) = r$ if there are exactly r maximal parabolic subgroups of K which are compact Lie groups.

Notice that if $K = K_n^i(a,b)$ is a Kac-Moody group belonging to the families that we introduced in section 2 and $ab > 4$ then the Weyl group W_I of a parabolic subgroup P_I of K is finite if and only if $\{1,2\}$ is not a subset of I. Hence, these Kac-Moody groups have always finiteness width equal to 2.

If $fw(K) = 2$, then in the homotopy decomposition of BK we only need to consider two maximal parabolic subgroups P_I and P_J and one sees easily that BK is then a push out

$$BK \simeq \operatorname{hocolim}\{BP_I \leftarrow BP_{I\cap J} \rightarrow BP_J\}.$$

5. The structure of the maximal parabolic subgroups

To study the structure of the parabolic subgroups of the Kac-Moody groups $K_n^i(a,b)$ we can use the presentation of Kac-Moody groups that is described in [26]. If K has rank n, then there are group homomorphisms $\varphi_i\colon SU(2) \to K$, $i = 1,\dots,n$, such that their images generate K. If D is the unit disc in $\mathbb{C}$ and we write

$$z_i(u) = \varphi_i \begin{pmatrix} u & (1-||u||^2)^{1/2} \\ -(1-||u||^2)^{1/2} & \bar{u} \end{pmatrix}$$

for $u \in D$, then K has a presentation with generators $\{z_i(u) \mid u \in D,\, i = 1,\dots,n\}$ and certain relations (see [25] for the full details). In particular, we want to mention that the standard maximal torus in K is

$$T_K = \{z_1(\lambda_1)\cdots z_n(\lambda_n) \mid \lambda_1,\dots,\lambda_n \in S^1\}$$

and the maximal torus centralizes each $SU(2)_i = \varphi_i(SU(2)) = \langle z_i(u) \mid u \in D\rangle \subset K$. Moreover, the action of the maximal torus on each $SU(2)_i$ can be explicitly read out from the Cartan matrix (a_{ij}) of K. We have:

$$z_i(\lambda)\, z_j(u)\, z_i(\lambda)^{-1} = z_j(\lambda^{a_{ij}} u)\, z_j(\lambda^{-a_{ij}}), \quad \lambda \in S^1,\, u \in D.$$

We are interested in the structure of the maximal parabolic subgroups of the Kac-Moody groups $K_n^i(a,b)$ which are Lie groups. In each case, there are only two such parabolic subgroups and if we denote by $\underline{r}$ the complement of $\{r\}$ in $\{1,\dots,n\}$, they are $P_{\underline{1}}$ and $P_{\underline{2}}$. It turns out that some of these groups are isomorphic to the groups in a family of compact Lie groups that was studied by Notbohm and Smith in their work on fake Lie groups ([31]) where they described and classified all compact connected Lie groups locally isomorphic to $S^1 \times SU(n)$. Let us briefly review the results of [31] which we will use here.

For each $n, s, k \geq 1$ with s dividing n, Notbohm-Smith define a Lie group $FU_{s,k}(n)$ as a quotient

$$FU_{s,k}(n) = (S^1 \times SU(n))/C_s$$

where C_s is the central cyclic subgroup generated by

$$(\omega^{-k}, \omega \cdot \mathrm{id}), \quad \omega = \exp(2\pi i/s).$$

These Lie groups have the following properties ([31]):

i) $FU_{s,k}(n)$ is a compact connected Lie group locally isomorphic to $S^1 \times SU(n)$ and any compact connected Lie group locally isomorphic to $S^1 \times SU(n)$ is isomorphic to some $FU_{s,k}(n)$, for some integers s, k.
ii) $\pi_1(FU_{s,k}(n)) \cong \mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z}$ with $r = (s, k)$.
iii) $FU_{s,k}(n) \cong FU_{s',k'}(n)$ if and only if $s = s'$, $(s, k) = (s', k')$ and $k \equiv \pm k'$ (mod s).
iv) $FU_{1,k}(n) \cong S^1 \times SU(n)$, $FU_{n,1}(n) \cong U(n)$.
v) These groups $FU_{s,k}(n)$ are determined by the representation

$$\sigma_{s,k} : \Sigma_n \longrightarrow GL_n(\mathbb{Z})$$

given by the action of the Weyl group on the maximal torus. These representations are described in [31].

The structure of the parabolic subgroups of $K_2^1(a, b)$ can be found in [2]. For the other groups introduced in section 2, we have:

Proposition 5.1. *$P_{\underline{2}}(K_n^i(a, b)) \cong FU_{s,k}(n)$ for $i = 2, 3$ and $P_{\underline{1}}(K_n^1(a, b)) \cong FU_{s,k}(n)$, where the integers s and k are determined in the following way: $l = (n, r)$, $t = r/l$, $s = n/l$, $kt \equiv 1$ (mod s) and*

$$r = \begin{cases} b + 2 - n & \text{if } i = 2, 3 \\ a & \text{if } i = 1. \end{cases}$$

Proof. To simplify the notation, let us write $P_{\underline{2}} = P_{\underline{2}}(K_n^i(a, b))$ for $i = 2, 3$ and $P_{\underline{1}} = P_{\underline{1}}(K_n^1(a, b))$.

Using the presentation of K discussed at the beginning of this section, we have that $P_{\underline{2}}$ is generated by $z_2(\lambda)$ for $\lambda \in S^1$ and $z_i(u)$ for $u \in D$ and $i \neq 2$. Hence, $P_{\underline{2}}$ is a semidirect product $S^1 \ltimes SU(n)$ and, as said, the action of S^1 on $SU(n)$ in this semidirect product is given by the entries in the Cartan matrix. Explicitly, an elementary computation shows that $P_{\underline{2}} = S^1 \ltimes SU(n)$ with action

$$A^\lambda = M_\lambda A M_\lambda^{-1}, \quad A \in SU(n), \lambda \in S^1$$

where M_λ is a diagonal matrix with diagonal entries $\lambda^{-a}, 1, \lambda, \ldots, \lambda$. A similar analysis applies to $P_{\underline{1}}$ and we obtain that $P_{\underline{1}}$ can be described in exactly the same way as $P_{\underline{2}}$, with M_λ now equal to the diagonal matrix with diagonal entries $\lambda^{-a}, 1, \ldots, 1$.

We define a homomorphism

$$\phi : S^1 \times SU(n) \longrightarrow S^1 \ltimes SU(n)$$

$$\phi(\lambda, A) = (\lambda^s, \lambda^t M_{\lambda^s} A).$$

Then, the choices of s and t ensure that the matrix $\lambda^t M_{\lambda^s}$ has determinant one. Also, it is straightforward to check that ϕ is a homomorphism and is surjective. The kernel of ϕ is $\{(\lambda, \lambda^{-t} \cdot \mathrm{id}) \mid \lambda^s = 1\}$ and this coincides with the subgroup C_s in the definition of $FU_{s,k}(n)$ above. ☐

In particular, in the rank three case there are only two possibilities for the groups $FU_{s,k}(3)$ and we have

$$P_1(K_3^2(a,b)) \cong \begin{cases} S^1 \times SU(3) \text{ if } a \equiv 1\,(3) \\ U(3) \text{ if } a \equiv 0, 2\,(3) \end{cases}$$

Since a and b can be interchanged in $K_n^2(a,b)$ we have also a description of the parabolic $P_{\underline{1}}(K_n^2(a,b))$. However, $P_{\underline{2}}(K_n^1(a,b))$ and $P_{\underline{1}}(K_n^3(a,b))$ do not admit a description in this way. These groups have the form $S^1 \ltimes (SU(2) \times SU(n-1))$ and the action of S^1 is such that S^1 and $SU(2)$ generate a group $U(2)$ while S^1 and $SU(n-1)$ generate a fake $F_{s,k}(n-1)$ with s, k depending on n and a (resp. b) like before. On the other side, the (non-maximal) parabolic subgroup $P_{\underline{1}}(K_n^2(a,b)) \cap P_{\underline{2}}(K_n^2(a,b))$ has an easier description:

Proposition 5.2. $P_{\underline{1}}(K_n^2(a,b)) \cap P_{\underline{2}}(K_n^2(a,b)) \cong S^1 \times U(n-1)$.

Proof. From the presentation of K described above, we see that the parabolic in the statement is a semidirect product $(S^1 \times S^1) \ltimes SU(n-1)$ with action given by

$$A^\lambda = N_\lambda A N_\lambda^{-1}, \quad A \in SU(n-1),\ \lambda \in S^1 \times S^1$$

where N_λ is the diagonal matrix with diagonal entries $\lambda^{-1}, 1, \ldots, 1$. Then one easily checks that

$$\phi : S^1 \times U(n-1) \longrightarrow (S^1 \times S^1) \ltimes SU(n-1)$$
$$\phi(\lambda, A) = (\lambda, \lambda^{-1} \det(A), N_{\det A} A)$$

is an isomorphism. ☐

6. Reflections and pseudoreflections

If F is a commutative domain, an element $\phi \in GL_n(F)$ is a *pseudoreflection* if ϕ has finite order and $\phi - I$ has rank ≤ 1.

Pseudoreflections are important in invariant theory because if F is a field and W is a finite subgroup of $GL_n(F)$, then a necessary condition for the ring of invariants $F[t_1, \ldots, t_n]^W$ to be a polynomial algebra is that W is generated by pseudoreflections. Moreover, if F has characteristic zero or prime to the order of W, then this condition is also sufficient.

Pseudoreflections are important in the theory of Kac-Moody groups because equation (1) shows that the standard generators of the Weyl group of a Kac-Moody group are pseudoreflections (of order two) in $GL_n(\mathbb{Z})$.

Pseudoreflections are important in homotopy theory because the Weyl group of a p-compact group of rank n is a pseudoreflection group in $GL_n(\mathbb{Z}_p)$. ($\mathbb{Z}_p$ is the ring of p-adic integers.) We will explain this in more detail later.

The concept of a pseudoreflection is an obvious generalization of the classic concept of an orthogonal or unitary reflection. Let V be an Euclidean (unitary) space, i.e. a finite dimensional vector space over $\mathbb{R}$ ($\mathbb{C}$) with a positive definite quadratic (Hermitic) form. Then, a *reflection* on V is a linear map $\phi : V \to V$ such that there exists a vector $v \in V$ and a root of unity θ such that ϕ is given by

$$\phi(x) = x - (1-\theta)\,\frac{x \cdot v}{v \cdot v}\,v. \tag{4}$$

In the Euclidean case the order of a (non trivial) reflection is always two, while in the unitary case a reflection can have order equal to any positive integer. Is is clear that a reflection is an orthogonal (unitary) transformation and it is also clear that a reflection is also a pseudoreflection. Conversely, if ϕ is a pseudoreflection on V, then we can always find a positive definite form on V such that ϕ leaves this form invariant and so ϕ is a reflection with respect to this form. More in general, if W is a *finite* group generated by pseudoreflections in V then, by averaging over W, we can find a positive definite form invariant by W and so W is a group generated by true reflections. In this sense, over a Euclidean or unitary space V, the concepts "finite group generated by pseudoreflections" and "finite group generated by reflections" are equivalent. However, this is no longer true if we consider *infinite* groups, like the Weyl groups of Kac-Moody groups which are not Lie groups.

A square matrix is called *symmetrizable* if there is an invertible diagonal matrix D and a symmetric matrix B such that $A = DB$. By comparing (1) and (4) we observe that a necessary condition for the standard generators of the Weyl group of a Kac-Moody group K to be reflections is that the Cartan matrix of K is symmetrizable. In particular, if $a \neq b$ then the Weyl groups of $K_n^i(a,b)$, $i = 2,3$, are *not* reflection groups, even if they certainly are pseudoreflection groups, because one can easily check that the Cartan matrices of these Kac-Moody groups are symmetrizable if and only if $a = b$.

7. Finite unitary reflection groups and p-compact groups

The irreducible finite unitary reflection groups were classified by Shephard and Todd in 1954 ([36], see also [12], [11], [35], [13]). Their list contains three infinite families of groups and 34 more groups: 19 groups in dimension 2, 5 groups in dimension 3, 5 groups in dimension 4, one group in each dimension 5, 7 and 8, and two groups in dimension 6. If necessary, we will denote by $\mathcal{R}_i$ the finite unitary reflection group which appears as entry i in the list of Shephard and Todd. Besides the classification theorem, the paper [36] contains a large amount of information about all these groups.

In the list of Shephard and Todd of finite unitary reflection groups we can also find the classic *real* reflection groups. Since a real reflection must have order two, the real reflection groups are generated by reflections of order two. But this property does not characterize the real reflection groups among the unitary reflection groups, for there are some finite unitary reflection groups which are generated by reflections of order two but are not real. They are relevant for the purposes of the present paper and were studied by Coxeter in [13].

To simplify the notation, let us introduce the following terminology. A *group of type NCOT* will be a finite group in $GL_n(\mathbb{C})$ for any n which is generated by reflections of order two and which does not satisfy the crystallographic condition. Two such groups which are conjugate in $GL_n(\mathbb{C})$ will be considered as equivalent. The irreducible groups of type NCOT are (see [13]):

(1) The two real non-crystallographic groups usually denoted I_4, I_5. $I_4 = \mathcal{R}_{23}$ of order 120 is the symmetry group of the icosahedron, while $I_5 = \mathcal{R}_{30}$ of order 14,400 is the symmetry group of the the 120-cell in dimension 4.
(2) The groups $G(m, m, n)$ in dimension $n > 2$ with $m > 2$, of order $m^{n-1}n!$, and the groups $G(m, m, 2)$ in dimension 2 for $m \neq 2, 3, 4, 6$, of order $2m$, which are a particular case of the family $\mathcal{R}_2$ in the list of Shephard-Todd.
(3) $\mathcal{R}_{24}$ of order 336 and $\mathcal{R}_{27}$ of order 2,160, both in dimension 3.
(4) $\mathcal{R}_{29}$ of order 7680 in dimension 4, $\mathcal{R}_{33}$ of order 51,840 in dimension 5 and $\mathcal{R}_{34}$ of order 39,191,040 in dimension 6.

We will say more about these groups in a following section of this paper.

Shephard ([35]) extended to unitary reflection groups the Coxeter graph of a real reflection group. The *Shephard graph* of a unitary reflection group contains:

(1) A *node* for each generating reflection, with a label k if the reflection has order $k > 2$.
(2) An *edge* between any two non commuting reflections, with a label m if the two reflections generate a group of order $2m > 6$.
(3) If the graph contains a *circuit*, this circuit has a certain label l (see [35] for details).

Coxeter ([13]) changed the way to label the circuits in the Shephard graph. Anyway, the only labeled circuit that we will use is a triangle with a label l and in this case the meaning of l is as follows: If the nodes of the triangle correspond to reflections R_1, R_2, R_3 then l is the order of $R_1R_2R_3R_2$.

Then, Shephard shows ([35]) that the graph contains enough information to uniquely determine the reflecting hyperplanes and so the graph determines the reflection group up to equivalence (it is not, however, always possible to find a set of reflections corresponding to a given graph).

Clark and Ewing studied ([11], see also [7]) the character fields and Schur indices of each group in the Shephard-Todd list and in this way we know, for each group $\mathcal{R}_i$ in the list, the smallest number field where we can realize the representation of $\mathcal{R}_i$. This is significant because it allows a classification of non-modular pseudoreflection groups in $GL_n(\mathbb{Z}_p)$ and $GL_n(\mathbb{F}_p)$:

Proposition 7.1. *(*[17]*) Let p be an odd prime. Then for any positive integers n and m with m prime to p the following sets are in bijective correspondence:*

(1) *Conjugacy classes of subgroups $W \subset GL_n(\mathbb{F}_p)$ such that W is generated by pseudoreflections and $|W| = m$.*
(2) *Conjugacy classes of subgroups $W \subset GL_n(\mathbb{Z}_p)$ such that W is generated by pseudoreflections and $|W| = m$.*

(3) *Conjugacy classes of subgroups $W \subset GL_n(\mathbb{C})$ such that W is generated by pseudoreflections, $|W| = m$ and the character field of W can be embedded in $\mathbb{Q}_p$.* □

The investigation of the homotopy properties of compact connected Lie groups and their classifying spaces eventually led to the concept of p-compact group (Dwyer and Wilkerson, [15]). Recall that a *loop space* X is a triple (X, BX, e) in which X is a space, BX is a connected pointed space, and $e : X \to \Omega BX$ is a homotopy equivalence. Given a prime p, a p-compact group is an $\mathbb{F}_p$-complete loop space X with finite mod p cohomology and such that $\pi_0 X$ is a finite p-group. In particular, the p-completion of a compact connected Lie group is an example of a p-compact group, but there are examples of *exotic* p-compact groups, i.e. p-compact groups which do not come from a compact Lie group. Is is remarkable that the complete classification of all p-compact groups has been obtained recently ([4] for p odd and [5] for $p = 2$) as a culmination of a long series of very relevant investigations in homotopy theory.

For a p-compact group X it is possible to define the concepts of maximal torus T_X and Weyl group W_X. The action of W_X on $H^2(BT_X; \mathbb{Z}_p)$ gives a representation $W_X \subset GL_n(\mathbb{Z}_p)$ and it turns out that W_X is a pseudoreflection group. In this paper, we do not need to get involved with the deepest theory of p-compact groups, because we will only deal with the simplest case, namely the non-modular p-compact groups, also called Clark-Ewing spaces ([11]). We briefly discuss here the construction of these spaces.

Let p be an odd prime and let W be a unitary reflection group (i.e., $W = \mathcal{R}_i$ for some i) of order prime to p, which can be realized in $GL_n(\mathbb{Z}_p)$. Let T be the $\mathbb{F}_p$-completion of a torus of rank n. Then, W acts on BT and the Clark-Ewing space associated to p and W is defined as

$$BX(W, p) = (BT \times_W EW)^\wedge_p.$$

This space (or, rather, the triple (X, BX, e)) is a p-compact group with maximal torus T and Weyl group W. The mod p cohomology of $BX(W, p)$ coincides with the ring of invariants $H^*(BT; \mathbb{F}_p)^W$ which is a polynomial algebra $\mathbb{F}_p[x_1, \dots, x_n]$ because W is generated by pseudoreflections and W has order prime to p. We refer to [11] for the full details of this construction and to [17] for a deeper study of these spaces.

8. The p-completed Weyl group

Let K be any Kac-Moody group with infinite Weyl group $W \subset GL_n(\mathbb{Z})$ and let p be a prime. For any $n > 0$ we can consider the *mod p^i Weyl group of K*, $W_{p^n} \subset GL_n(\mathbb{Z}/p^n\mathbb{Z})$ defined as the reduction mod p^i of W. Using the tower of finite groups $\{W_{p^i}\}_{i>0}$ we can define the *p-completed Weyl group of K* as the inverse limit

$$W^\wedge_p = \lim_{\leftarrow} \{W_{p^i}\}_{i>0} \subset GL_n(\mathbb{Z}_p).$$

Example 8.1. *If $K = K_2^1(a, b)$ then both W_p and $W^\wedge_p$ can be explicitly computed. See [28] and [2] for full details. Since we are assuming $ab > 4$, W is an infinite dihedral group and W_p is a finite dihedral group of order $2k$. If we assume, for simplicity, that p is odd, then k is given as follows:*

(1) $k = p$ *if* $p|ab-4$;
(2) $k = 2p$ *if* p *divides* a *or* b *but not both.*
(3) k *is the multiplicative order of the roots of* $X^2-(ab-2)X+1$ *in* $\mathbb{F}_{p^2}$ *in all other cases.*

From this we can also deduce the structure of $W_p^\wedge$ *(see* [3]*):* $W_p^\wedge$ *is an extension of* $\mathbb{Z}/2\mathbb{Z}$ *by* $(W_p^\wedge)^+ = W_p^\wedge \cap SL_n\mathbb{Z}_p$ *and*

$$(W_p^\wedge)^+ \cong \mathbb{Z}_p \times \mathbb{Z}/l\mathbb{Z}$$

where l *is given as follows:*

(1) $l = 1$ *if* $p|ab-4$;
(2) $l = 2$ *if* p *divides* a *or* b *but not both.*
(3) l *is the multiplicative order of the roots of* $X^2-(ab-2)X+1$ *in* $\mathbb{F}_{p^2}$ *in all other cases.* □

We observe that $W \subset W_p^\wedge$ and we see that in general $W_p^\wedge$ may be larger than W. The homotopy theoretical relevance of the group $W_p^\wedge$ is due to theorem 8.2 below. Let us introduce first some notation. Let T be the standard maximal torus of K and let us denote by T_{p^∞} and T_{p^i} the p-torsion subgroup of T and the subgroup of T of p^i-th roots of unity, respectively. We denote by

$$\mathrm{Map}_1(BT_{p^\infty}, BK_p^\wedge)$$

the space of all maps $f : BT_{p^\infty} \to BK_p^\wedge$ such that, for all i, the restriction of f to BT_{p^i} is homotopic to some map $B\phi_i$ where ϕ_i is an *injective* homomorphism $\phi_i : T_{p^i} \to K$. We denote by

$$[BT_{p^\infty}, BK_p^\wedge]_1$$

the set of components of $\mathrm{Map}_1(BT_{p^\infty}, BK_p^\wedge)$. We extend these notations to BT_{p^i} and we write $\mathrm{Map}_1(BT_{p^i}, BK_p^\wedge)$ and $\mathrm{Map}_1(BT_{p^i}, BK_p^\wedge)$ whose meaning should now be obvious.

The linear group $GL_n(\mathbb{Z}/p^i\mathbb{Z})$ acts on BT_{p^i} and the linear group $GL_n(\mathbb{Z}_p)$ acts on BT_{p^∞}. There is a map

$$\Phi : GL_n(\mathbb{Z}_p) \to [BT_{p^\infty}, BK_p^\wedge]_1$$

which sends a matrix M to the map $B(i \circ M)$ where $i : T_{p^\infty} \to K$ is the standard inclusion.

Now we can state the two main results of this section, theorems 8.2 and 8.3, which I learned from N. Kitchloo.

Theorem 8.2. Φ *factors through a bijection:*

$$W_p^\wedge \backslash GL_n(\mathbb{Z}_p) \xrightarrow{\cong} [BT_{p^\infty}, BK_p^\wedge]_1.$$

This theorem is a corollary of a more general result which gives information about the homotopy type of the spaces $\mathrm{Map}_1(BT_{p^\infty}, BK_p^\wedge)$. Let us consider the finite discrete orbit spaces

$$D_i = W_{p^i} \backslash GL_n(\mathbb{Z}/p^i\mathbb{Z}),$$

$i > 0$, with the obvious maps $D_1 \leftarrow D_2 \leftarrow D_3 \leftarrow \cdots$

Theorem 8.3. *There is an inverse tower of pointed connected finite complexes*

$$Y_1 \leftarrow Y_2 \leftarrow Y_3 \leftarrow \cdots$$

such that there is a homotopy equivalence

$$\mathrm{Map}_1(BT_{p^\infty}, BK_p^\wedge) \simeq BT_p^\wedge \times \operatorname*{holim}_i \{(Y_i)_p^\wedge \times D_i\}.$$

This equivalence is compatible with the quotient map $GL_n(\mathbb{Z}_p) \to W_p^\wedge \backslash GL_n(\mathbb{Z}_p)$.

Proof of 8.2. The homotopy equivalence in theorem 8.3 reduces the proof of 8.2 to a computation of π_0 of a homotopy limit. We have a surjection

$$[BT_{p^\infty}, BK_p^\wedge]_1 = \pi_0 \mathrm{Map}_1(BT_{p^\infty}, BK_p^\wedge) \to \varprojlim_i \{D_i\}$$

whose fibres are in one-to-one correspondence to the set $\varprojlim^1_i \{\pi_1((Y_i)_p^\wedge)\}$. The spaces Y_i may well be $\mathbb{F}_p$-bad in the sense of Bousfield-Kan ([8]) but nevertheless these obstruction sets are trivial because of the following argument. Goerss proved ([19]) that for spaces of finite type the (Bousfield-Kan) p-completion has the same weak homotopy type as the Sullivan p-profinite completion ([40]). Now, the homotopy groups of the p-profinite completion of a space are pro-p-groups ([40]) and it is well known that $\varprojlim^1$ vanishes on towers of pro-p-groups. Hence, the obstruction sets are trivial and the map above is a bijection. Moreover, an elementary argument using that $\{W_{p^i}\}$ is a tower of epimorphisms, shows that $\varprojlim\{D_i\} = W_p^\wedge \backslash GL_n(\mathbb{Z}_p)$. The information contained in the last sentence of 8.3 completes the proof. □

The proof of 8.3 relies on an important result of Broto and Kitchloo which extends to Kac-Moody groups some properties that were known for compact Lie groups. In [10] Broto-Kitchloo define a certain class $\mathcal{G}$ of topological groups. This class contains all (a) Kac-Moody groups; (b) parabolic subgroups of Kac-Moody groups; (c) normalizers of maximal tori in (a) and (b); (c) the adjoint forms of the groups in (a), (b), (c); (d) Weyl groups of Kac-Moody groups. Then,

Theorem 8.4 (Broto-Kitchloo)**.** *If π is a finite p-group then*

a) *For any $K \in \mathcal{G}$, there is a homotopy equivalence*

$$\coprod_{\rho \in \mathrm{Rep}(\pi, K)} (BC_K(\rho))_p^\wedge \xrightarrow{\simeq} \mathrm{Map}(B\pi, BK_p^\wedge)$$

where $C_K(\rho)$ denotes the centralizer in K of the image of ρ. In particular, $[B\pi, BK_p^\wedge] \cong \mathrm{Rep}(\pi, K)$.

b) *If $\{P_I\}$ denotes the poset of parabolic subgroups of a Kac-Moody group K, then there is a homotopy equivalence*

$$\big(\operatorname*{hocolim}_{W_I \text{ finite}} \mathrm{Map}(B\pi, BP_{I_p^\wedge})\big)_p^\wedge \xrightarrow{\simeq} \mathrm{Map}(B\pi, BK_p^\wedge).$$

Part (a) of this result is theorem C in [10]. Part (b) is a consequence of 4.2 in [9], using the results of appendix 7 in [10]. □

Proof of 8.3. Let I be any index such that W_I is finite. By 8.4 applied to the Lie group P_I we know that any map $BT_{p^i} \to BP_I{}_p^\wedge$ is homotopic to $B\rho$ for some homomorphism $\rho : T_{p^i} \to P_I$. It is a well known property of compact connected Lie groups that if ρ injective and i is large enough, then ρ factorizes through the maximal torus T and has centralizer equal to T. Hence, each component of $\mathrm{Map}_1(BT_{p^i}, BP_I{}_p^\wedge)$ for large i is $BT_p^\wedge$. Also, the set of components of $\mathrm{Map}_1(BT_{p^i}, BP_I{}_p^\wedge)$ is indexed by the automorphisms of T_{p^i} modulo the action of the Weyl group of P_I which is W_I. We have

$$\mathrm{Map}_1(BT_{p^i}, BP_I{}_p^\wedge) \simeq BT_p^\wedge \times \left(W_I \backslash GL_n(\mathbb{Z}/p^i\mathbb{Z})\right).$$

Since we only consider indices I such thatW_I is finite, we can assume that i is large enough such that the composition $W_I \hookrightarrow W \twoheadrightarrow W_{p^i}$ is injective for all I. Then, a choice of a set theoretical section of $GL_n(\mathbb{Z}/p^i\mathbb{Z}) \to W_{p^i} \backslash GL_n(\mathbb{Z}/p^i\mathbb{Z})$ gives a bijection

$$W_I \backslash GL_n(\mathbb{Z}/p^i\mathbb{Z}) \cong (W_I \backslash W_{p^i}) \times (W_{p^i} \backslash GL_n(\mathbb{Z}/p^i\mathbb{Z})) = (W_I \backslash W_{p^i}) \times D_i$$

compatible with the maps corresponding to inclusions $I \subset J$. We obtain

$$\operatorname*{hocolim}_I \{W_I \backslash GL_n(\mathbb{Z}/p^i\mathbb{Z})\} = \operatorname*{hocolim}_I \{W_I \backslash W_{p^i}\} \times D_i.$$

Then we define

$$Y_i = \operatorname*{hocolim}_I \{W_I \backslash W_{p^i}\},$$

which is a finite complex. It is possible to give a description of these spaces Y_i as follows.

Lemma 8.5. *Let N_i be the kernel of the projection $W \to W_{p^i}$. Assume that i is large enough so that $W_I \hookrightarrow W_{p^i}$ for all finite W_I. Then $Y_i \simeq BN_i$. In particular, the spaces Y_i are connected.*

Proof. Let us first introduce some notation. Let $\mathcal{I}$ be the category of indices I with W_I finite, with a single map $I \to J$ if and only if $I \subset J$. For any discrete group G, let us denote by $\mathcal{C}(G)$ the category with a single object with endomorphism monoid equal to G. Let F be the functor from $\mathcal{I}$ to CAT which sends the object $I \in \mathcal{I}$ to the category $\mathcal{C}(W_I)$, and the map $I \hookrightarrow J$ to the functor induced by the inclusion $W_I \subset W_J$. Let $\mathcal{D} = \mathcal{I} \int F$ be the Grothendieck construction as in [41]. In this particular instance, the category $\mathcal{I} \int F$ is quite simple: its objects are the same as the objects of $\mathcal{I}$ and the morphisms are given by

$$\mathrm{Map}(I, J) = \begin{cases} \varnothing & \text{if } I \not\subset J \\ W_J & \text{if } W_I \subset W_J. \end{cases}$$

If Y is any functor from $\mathcal{D}$ to spaces, we can use proposition 0.2 in [37] to compute its homotopy colimit:

$$\operatorname*{hocolim}_{\mathcal{D}} Y \simeq \operatorname*{hocolim}_{\mathcal{I}} \operatorname*{hocolim}_{\mathcal{C}(W_I)} Y(I).$$

Let Z be the functor from $\mathcal{C}(N_i) \times \mathcal{D}$ to spaces which sends each object to the discrete space W and the morphism (h, w), $h \in N_i$, $w \in W_J$, to the map $g \mapsto wgh^{-1}$, $g \in W$. Let us compute now the homotopy colimit of Z in two different ways:

$$\operatorname*{hocolim}_{\mathcal{C}(N_i)\times\mathcal{D}} Z = \operatorname*{hocolim}_{\mathcal{C}(N_i)} \operatorname*{hocolim}_{\mathcal{D}} Z \simeq \operatorname*{hocolim}_{\mathcal{C}(N_i)} \operatorname*{hocolim}_{\mathcal{I}} (W_I \backslash W) \simeq \operatorname*{hocolim}_{\mathcal{C}(N_i)} * \simeq BN_i$$

Here we have used the colimit decomposition of BW for W infinite as seen in formula 3 of section 4. On the other side,

$$\operatorname*{hocolim}_{\mathcal{C}(N_i)\times\mathcal{D}} Z = \operatorname*{hocolim}_{\mathcal{D}} \operatorname*{hocolim}_{\mathcal{C}(N_i)} Z \simeq \operatorname*{hocolim}_{\mathcal{D}}(W/N_i) \simeq \operatorname*{hocolim}_{\mathcal{I}}(W_I\backslash W_{p^i}) \simeq Y_i.$$

□

Let us continue now with the proof of 8.3. By 8.4 we have

$$\begin{aligned}\mathrm{Map}_1(BT_{p^i}, BK_p^\wedge) \simeq& (\operatorname*{hocolim}_{I}\{\mathrm{Map}_1(BT_{p^i}, BP_{I_p}^\wedge)\})_p^\wedge\\ \simeq& BT_p^\wedge \times (\operatorname*{hocolim}_{I}\{W_I\backslash GL_n(\mathbb{Z}/p^i\mathbb{Z})\})_p^\wedge\\ \simeq& BT_p^\wedge \times ((Y_i)_p^\wedge \times D_i).\end{aligned}$$

Finally, we notice that $BT_{p^\infty} = \mathrm{hocolim}_i\{BT_{p^i}\}$ and we can complete the proof as follows:

$$\begin{aligned}\mathrm{Map}_1(BT_{p^\infty}, BK_p^\wedge) =& \operatorname*{holim}_{i}\{\mathrm{Map}_1(BT_{p^i}, BK_p^\wedge)\}\\ \simeq& BT_p^\wedge \times (\operatorname*{holim}_{i}\{(Y_i)_p^\wedge \times D_i\}).\end{aligned}$$

To check that the last sentence in the theorem is true is not difficult if we follow the behavior of $GL_n(\mathbb{Z}_p)$ along each step of the proof. □

9. The groups $G(m, m, n)$

For $n, m \geq 2$, $G(m, m, n)$ is defined as the group of linear transformations of $\mathbb{C}^n$ given by

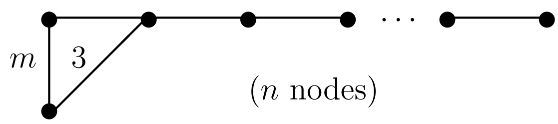
$$x_i \mapsto \theta^{r_i} x_{\sigma(i)} \quad i = 1, \dots, n,$$

where $x_1, \dots, x_n$ is the standard basis of $\mathbb{C}^n$, $\sigma \in \Sigma_n$ is any permutation of $\{1, \dots, n\}$, $\theta = \exp(2\pi i/m)$ and the integers r_i satisfy $r_1 + \ldots + r_n \equiv 0 \bmod m$. As an abstract group, it is isomorphic to a semidirect product $\Sigma_n \ltimes (\mathbb{Z}/m\mathbb{Z})^{n-1}$. If $n = 2$ then $G(m, m, 2)$ is a dihedral group. Clearly, $G(m, m, n)$ does not depend on the choice of θ, which can be any primitive m-th root of unity.

$G(m, m, n)$ is generated by unitary reflections of order two with respect to the hyperplanes

$$\begin{aligned} x_i - x_{i-1} &= 0, \quad i = 2, \dots, n\\ x_1 - \theta x_2 &= 0,\end{aligned}$$

and so it is a finite unitary reflection group. Its Shephard graph with respect to these generating reflections is as follows:



Assume now $n > 2$. The group $G(m, m, n)$ admits another description which will be more useful to us. Let $c = -1 - \theta^{-1}$ and consider the pseudoreflections $w_1, \dots, w_n$ on $\mathbb{C}^n$ given by

$$w_j(x_i) = x_i - a_{ij} x_j, \quad i, j = 1, \dots, n$$

where (a_{ij}) is the matrix

$$\begin{pmatrix} 2 & -c & -1 & & & & & \\ -\bar{c} & 2 & -1 & & & & & \\ -1 & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & & \ddots & & & \\ & & & & & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}$$

To prove that $w_1, \ldots, w_n$ generate a group which is equivalent to $G(m, m, n)$ we could prove that $w_1, \ldots, w_n$ leave invariant a positive definite Hermitic form, show that the graph associated to $w_1, \ldots, w_n$ is the same as the Shephard graph of $G(m, m, n)$ discussed above and then use the result of Shephard that the graph determines the group. Or we can provide an explicit equivalence between the two representations, as follows. Let us consider the basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$ given by

$$\begin{aligned} v_1 &= (1, -1, 0, \ldots, 0), \\ v_2 &= (\theta, -1, 0, \ldots, 0), \\ v_i &= (\theta, -1, \theta - 1, \ldots, \theta - 1, 0, \ldots, 0), \quad i = 3, \ldots, n, \quad (n - i \text{ zeroes}). \end{aligned}$$

It is easy to check that these vectors have the following properties:

(1) $v_2 = w_1(v_1)$, $v_i = w_i(v_{i-1})$ for $i = 3 \ldots, n$.
(2) $w_i(v_1) = v_1$ for $i \neq 1, 2$.
(3) $w_2(v_1) = \theta^{-1} v_2$.
(4) $w_1(v_j) = v_j = w_2(v_j)$ for $j \neq 1, 2$.
(5) $w_i(v_j) = v_j$ for $i \neq 1, 2$, $j \neq i, i - 1$.

It follows that, with respect to the basis $v_1, \ldots, v_n$, the pseudoreflections $w_1, \ldots, w_n$ behave as the generating reflections of $G(m, m, n)$.

If $n > 2$ then the character field for $G(m, m, n)$ is $\mathbb{Q}(\theta)$ and this group can be realized over the p-adic integers if and only if $\mathbb{Z}_p$ contains an m-th root of unity. We have thus a Clark-Ewing p-compact group $BX(G(m, m, n), p)$ for any $p > n > 2$ and any $m | p - 1$. The restriction $p > n$ is necessary because the Clark-Ewing space $BX(G, p)$ is only defined when p does not divide the order of W which in this case is equal to $m^{n-1} n!$. Actually, if $p < n$ (and $m | p - 1$), there is also a p-compact group with Weyl group equal to $G(m, m, n)$, but it has to be constructed by different methods (see [30]).

The case $n = 2$ is special because the character field is $\mathbb{Q}(\theta + \theta^{-1})$ ([11]) and the Clark-Ewing p-compact group $BX(G(m, m, 2), p)$ exists for any m and any odd $p \equiv \pm 1 \bmod m$ ([11]). When $n = 2$ we can choose the basis of $\mathbb{C}^2$ given by $v_1 = (1, -1)$, $v_2 = (1 + \theta, -1 - \bar{\theta})$ and the generating reflections become

$$w_1 = \begin{pmatrix} -1 & c \\ 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

with $c = -2 - \theta - \bar{\theta}$.

10. The remaining irreducible groups of type NCOT

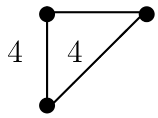
We use the classic notation $\tau = (1+\sqrt{5})/2$, $\omega = (-1+i\sqrt{3})/2$. Let us discuss the five non-real groups first. In dimension 3 we have the groups $\mathcal{R}_{24}$ and $\mathcal{R}_{27}$ of orders 336 and 2.160, respectively. According to Coxeter ([13]), these groups are generated by three pseudoreflections of order two in $\mathbb{C}^3$ given by the matrices

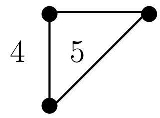
$$w_1 = \begin{pmatrix} -1 & \bar{c} & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & 0 & 0 \\ c & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

where c is the complex number

$$\begin{aligned} \mathcal{R}_{24} &: \quad c = (-1+i\sqrt{7})/2 \\ \mathcal{R}_{27} &: \quad c = -1-\tau\omega^2 \end{aligned}$$

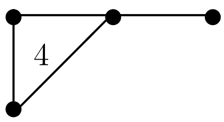
Is is easy to check that the Shephard graphs for these groups are as follows:

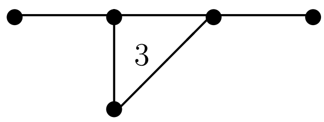
$\mathcal{R}_{24}$: 

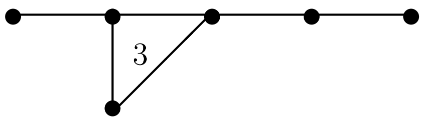
$\mathcal{R}_{27}$: 

In the case of $\mathcal{R}_{24}$ the above representation is defined over $\mathbb{Z}_p$ if and only if -7 is a quadratic residue mod p and this happens if and only if $p \equiv 1,2,4 \bmod 7$. In the case of $\mathcal{R}_{27}$ we notice that $c+\bar{c} = -2+\tau$ and so the above representation is defined over $\mathbb{Z}_p$ if and only if both 5 and -3 are quadratic residues mod p. This happens if and only if $p \equiv 1,4 \bmod 15$. This coincides with the results of [11] about existence of Clark-Ewing p-compact groups associated to the finite reflection groups $\mathcal{R}_{24}$ and $\mathcal{R}_{27}$: $BX(\mathcal{R}_{24},p)$ exists for $p \equiv 1,2,4 \bmod 7$ and $BX(\mathcal{R}_{27},p)$ exists for $p \equiv 1,4 \bmod 15$.

The Shephard graphs for the groups $\mathcal{R}_{29}$, $\mathcal{R}_{33}$ and $\mathcal{R}_{34}$ have the form of a triangle with "tails":

$\mathcal{R}_{29}$: 

$\mathcal{R}_{33}$: 

$\mathcal{R}_{34}$: 

and the representations can be given as in the case of dimension three, with some appropriate values for the complex parameter c (see [13]):

$$\begin{aligned} \mathcal{R}_{29} &: \quad c = i \\ \mathcal{R}_{33}, \mathcal{R}_{34} &: \quad c = \omega. \end{aligned}$$

i is a p-adic integer if and only if $p \equiv 1 \bmod 4$ and $\omega \in \mathbb{Z}_p$ if and only if $p \equiv 1 \bmod 3$. Hence, we know for which primes p the given representation of these reflection groups is defined over $\mathbb{Z}_p$ (this is in accordance to the results of [11]). For these primes, we can construct the corresponding Clark-Ewing p-compact groups, except that in the case of $\mathcal{R}_{29}$ we have to exclude the prime 5 because it divides the order of $\mathcal{R}_{29}$ and in the case of $\mathcal{R}_{34}$ we have to exclude the prime 7 for the same reason. Thus, the Clark-Ewing space $BX(\mathcal{R}_{29}, p)$ exists for $p \equiv 1 \bmod 4$, $p \neq 5$, the Clark-Ewing space $BX(\mathcal{R}_{33}, p)$ exists for $p \equiv 1 \bmod 3$ and the Clark-Ewing space $BX(\mathcal{R}_{34}, p)$ exists for $p \equiv 1 \bmod 3$, $p \neq 7$. Actually, there is a 5-compact group with Weyl group $\mathcal{R}_{29}$ and a 7-compact group with Weyl group $\mathcal{R}_{34}$ (see [1]), but they cannot be constructed by the Clark-Ewing method as explained above and will not be used in this paper, except for the proof of 15.1.

It only remains to discuss the two real reflection groups $\mathcal{R}_{23}$ and $\mathcal{R}_{30}$. The reader can find an explicit description of the generating reflections of these groups in page 71 of the book [20]. $\mathcal{R}_{30}$ is the symmetry group of the regular polytope in dimension four known as the 120-cell and it can be generated by orthogonal reflections with respect to the hyperplanes

$$\begin{aligned}(\tau+1)x_1 + x_2 - \tau x_3 &= 0\\ -(\tau+1)x_1 + x_2 + \tau x_3 &= 0\\ \tau x_1 - (\tau+1)x_2 + x_3 &= 0\\ -\tau x_1 - (\tau+1)x_3 + x_4 &= 0\end{aligned}$$

$\mathcal{R}_{23}$ is the symmetry group of the icosahedron and can be generated by orthogonal reflections with respect to the first three hyperplanes above. In both cases, if we take a basis consisting of unitary normal vectors to the reflecting hyperplanes, we obtain the following generators for $\mathcal{R}_{23}$ and $\mathcal{R}_{30}$:

$$\mathcal{R}_{23} = \left\langle \begin{pmatrix} -1 & \tau & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 0 & 0\\ \tau & -1 & 1\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & -1\end{pmatrix} \right\rangle,$$

$$\mathcal{R}_{30} = \left\langle \begin{pmatrix} -1 & \tau & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0\\ \tau & -1 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & -1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1 & -1\end{pmatrix} \right\rangle.$$

We see that these representations can be realized over $\mathbb{Z}_p$ if and only if $\tau \in \mathbb{Z}_p$ and this happens if and only if $p \equiv \pm 1 \bmod 5$. Hence, for these primes we have the Clark-Ewing spaces $BX(\mathcal{R}_{23}, p)$ and $BX(\mathcal{R}_{30}, p)$.

For each group $\mathcal{R}_i$ in this section, we see that the inclusion $\mathcal{R}_i \hookrightarrow GL_n(\mathbb{Z}_p)$ depends on some choice of elements in $\mathbb{Z}_p$ like $\sqrt{-7}$, $\sqrt{5}$ or $\sqrt{-1}$. These choices are not unique but different choices produce conjugate groups inside $GL_n(\mathbb{Z}_p)$, but not necessarily conjugate representations. This can be checked directly or using proposition 7.1.

11. Cohomology of BK

Let K be a Kac-Moody group with standard maximal torus T and Weyl group W. Recall from section 4 that $fw(K)$ is the number of maximal parabolic subgroups of K which are compact Lie groups.

Let us fix a prime p. In this section we want to compute $H^*(BK; \mathbb{F}_p)$ under some suitable hypothesis. For simplicity, we write $H^*(-) = H^*(-; \mathbb{F}_p)$ throughout the remainder of this paper.

We say that K is *non-modular* if the order of the mod p Weyl group W_p of K is prime to p. Notice that the action of W on T gives an action of the finite group $W_p \subset GL_n(\mathbb{F}_p)$ on $H^*(BT)$. As usual, we denote by $H^*(BT)^{W_p}$ the subalgebra of invariant elements. Then,

Theorem 11.1. *Let p be a prime and let K be a non-modular Kac-Moody group which is a product of Kac-Moody groups with $fw(K) = 2$. Then,*

1. *$H^{even}(K)$ is a polynomial algebra.*
2. *$BT \to BK$ induces an isomorphism $H^{even}(BK) \cong H^*(BT)^{W_p}$.*
3. *$H^*(BK)$ is a finitely generated free $H^*(BT)^{W_p}$-module.*

Proof. Clearly, it is enough to prove the theorem for a single Kac-Moody group K with $fw(K) = 2$. First of all, W_p is a subgroup of $GL_n(\mathbb{F}_p)$ generated by pseudoreflections of order two and the non-modularity hypothesis implies that p is odd and $H^*(BT)^{W_p}$ is a polynomial algebra. Hence, (1) follows from (2). To simplify the notation, let us write $S = H^*(BT)$, $\mathcal{I} = H^*(BT)^{W_p}$.

Since $fw(K) = 2$ the colimit decomposition of section 4 reduces to a homotopy push out:

$$BK \simeq \operatorname{hocolim}\{BP_I \leftarrow BP_{I\cap J} \to BP_J\} \tag{5}$$

which yields a Mayer-Vietoris exact sequence in cohomology. In this push out the groups P_I, P_J are compact connected Lie groups with Weyl groups W_I, W_J respectively.

Let L denote any of the indices $I, J, I\cap J$. The finite group W_L satisfies the following properties:

(a) W_L is a finite group of order prime to p.
(b) S^{W_L} is a polynomial algebra $S^{W_L} \simeq \mathbb{F}_p[x_1^L, \dots, x_n^L]$.
(c) $\mathbb{F}_p \otimes_{\mathcal{I}} S^{W_L} = (\mathbb{F}_p \otimes_{\mathcal{I}} S)^{W_L}$.
(d) S^{W_L} is a finitely generated free $\mathcal{I}$-module.
(e) $H^*(BP_L) \xrightarrow{\cong} S^{W_L}$.

(a) follows from the well known fact that the kernel of the mod p reduction homomorphism $GL_n(\mathbb{Z}_p) \to GL_n(\mathbb{F}_p)$ is torsion free for all odd primes p. Hence, the homomorphism $W_L \hookrightarrow W \twoheadrightarrow W_p$ is a monomorphism.

(b) follows from the theory of pseudoreflection groups: W_L is a finite pseudoreflection group of order prime to p and so its ring of (modular) invariants is a polynomial algebra.

To obtain (c) we just need to use the averaging homomorphism $A : S \to S^{W_L}$ defined by $A(x) = (1/|W_L|)\Sigma_{g\in W_L} gx$ which is $\mathcal{I}$-linear.

(d) is also well known (see, for instance, [38] 6.7.11).

(e) is a well known consequence of the existence of the transfer (see [18]).

The Mayer-Vietoris cohomology exact sequence induced by the push out (5) reads:

$$0 \to H^{\text{even}}(BK) \to S^{W_I} \oplus S^{W_J} \to S^{W_{I\cap J}} \to H^{\text{odd}}(BK) \to 0.$$

Notice that this is an exact sequence of $\mathcal{I}$-modules. Part (2) of the theorem follows immediately because W_I and W_J generate W.

Since $S^{W_{I\cap J}}$ is a finitely generated $\mathcal{I}$-module by (d), so is $H^{\text{odd}}(BK)$. It remains to prove that $H^{\text{odd}}(BK)$ is a free $\mathcal{I}$-module. By [38] 6.1.1, it is enough to prove that $\text{Tor}_1^{\mathcal{I}}(\mathbb{F}_p, H^{\text{odd}}(BK)) = 0$. By [38] 6.7.11, the above exact sequence is a free resolution of $H^{\text{odd}}(BK)$ as an $\mathcal{I}$-module, and thus it can be used to compute $\text{Tor}^{\mathcal{I}}(-, H^{\text{odd}}(BK))$. We obtain

$$\text{Tor}_1^{\mathcal{I}}(\mathbb{F}_p, H^{\text{odd}}(BK)) = \frac{\text{Ker}\left(\mathbb{F}_p \otimes_{\mathcal{I}} (S^{W_I} \oplus S^{W_J}) \to \mathbb{F}_p \otimes_{\mathcal{I}} S^{W_{I\cap J}}\right)}{\text{Im}\left(\mathbb{F}_p \otimes_{\mathcal{I}} \mathcal{I} \to \mathbb{F}_p \otimes_{\mathcal{I}} (S^{W_I} \oplus S^{W_J})\right)}.$$

Then, $\mathbb{F}_p \otimes_{\mathcal{I}} \mathcal{I} \to \mathbb{F}_p \otimes_{\mathcal{I}} (S^{W_I} \oplus S^{W_J})$ is a monomorphism and property (c) above shows that

$$\text{Ker}\left(\mathbb{F}_p \otimes_{\mathcal{I}} (S^{W_I} \oplus S^{W_J}) \to \mathbb{F}_p \otimes_{\mathcal{I}} S^{W_{I\cap J}}\right) = (\mathbb{F}_p \otimes_{\mathcal{I}} S)^{W_p}.$$

The fact that the order of W_p is prime to p implies ([38], 7.5.2) that the W_p representation $\mathbb{F}_p \otimes_{\mathcal{I}} S$ is the regular representation of W_p. Hence $(\mathbb{F}_p \otimes_{\mathcal{I}} S)^W = \mathbb{F}_p$ and $\text{Tor}_1^{\mathcal{I}}(\mathbb{F}_p, H^{\text{odd}}) = 0$. Part (3) of the theorem follows. $\square$

Therefore, the mod p cohomology of BK as a $H^*(BT)^{W_p}$-module is well understood, under the hypothesis of non-modularity and finiteness width equal to two. To obtain the degrees of a set of free $H^*(BT)^{W_p}$-generators of $H^*(BK)$ we just need to perform a Poincaré series computation. For a graded $\mathbb{F}_p$-vector space V, let us denote its Poincaré series as

$$\mathcal{P}(V) = \sum_{i=0}^{\infty} (\dim(V^i))\, z^i.$$

The Mayer-Vietoris exact sequence that we have used in the proof of the theorem above yields the identity

$$\mathcal{P}(\Sigma^{-1} H^{\text{odd}}(BK)) = \mathcal{P}(\mathcal{I}) + \mathcal{P}(S^{W_{I\cap J}}) - \mathcal{P}(S^{W_I}) - \mathcal{P}(S^{W_J})$$

and we get that if $\{e_i\}$ is a set of free generators of $H^{\text{odd}}(BK)$ over $\mathcal{I}$, and each e_i has degree $|e_i|$, then

$$\sum z^{|e_i|} = z\left[1 + \frac{\mathcal{P}(S^{W_{I\cap J}}) - \mathcal{P}(S^{W_I}) - \mathcal{P}(S^{W_J})}{\mathcal{P}(\mathcal{I})}\right].$$

Example 11.2. *Let us consider the Kac-Moody groups $K_n^2(a,b)$ with parameters a, b such that the mod p Weyl group is equal to $G(m,m,n)$. Then, the degrees of the generators of the rings of invariants $\mathcal{I}$, S^{W_I}, S^{W_J}, $S^{W_{I\cap J}}$ are well known:*

$\mathcal{I}$ has generators in degrees $2m, 4m, \ldots, 2(n-1)m, 2n$

S^{W_I} and S^{W_J} have generators in degrees $2, 4, \ldots, 2n$

$S^{W_{I\cap J}}$ has generators in degrees $2, 2, 4, \ldots, 2n-2$.

Then,

$$\mathcal{P}(H^{odd}(BK))/\mathcal{P}(\mathcal{I}) = z\left[q_{m-1}(z^2)\, q_{m-1}(z^4) \cdots q_{m-1}(z^{2(n-1)})\, \left(q_{n-1}(z^2) - 2\right) + 1\right],$$

where $q_s(z) = 1 + z + z^2 + \cdots + z^s$. We see that, in general, the rank of $H^*(BK)$ over $H^*(BT)^{W_p}$ is very large. The problem of completely determining the algebraic structure of $H^*(BK_n^2(a,b))$ seems to be approachable only when and $n = 2$ and indeed this structure was elucidated in [2].

12. Mod p Weyl groups

In this section we denote $W_p(K)$ the mod p Weyl group of a Kac-Moody group K. The following theorem computes the mod p Weyl groups of the Kac-Moody groups $K^i_n(a,b)$ that we have introduced in section 2, under some suitable condition on the integral parameters a and b.

Theorem 12.1. *Let p be an odd prime and let a, b be positive integers.*

(1) *If $(a+1)(b+1)\equiv 1\,(p)$, then $W_p(K^2_n(a,b)) = G(m,m,n)$ and $W_p(K^1_2(a,b)) = G(m,m,2)$ for m equal to the order of $-a-1$ in $\mathbb{F}_p^*$.*

(2) *If $a+1\equiv 0\,(p)$ and $P(X) = X^2+(b+2)X+1$ is irreducible over $\mathbb{F}_p$, then $W_p(K^1_2(a,b)) = G(m,m,2)$ where m is the order in $\mathbb{F}_p^*$ of the roots of $P(X)$. The same conclusion holds if we interchange a and b.*

(3) *If $ab\equiv 2\,(p)$ and $a+b\equiv -1\,(p)$, then $p\equiv 1,2,4\,(7)$ and $W_p(K^2_3(a,b)) = \mathcal{R}_{24}$.*

(4) *If $d = a+b+2$ satisfies $d^2-d-1\equiv 0\,(p)$ and $e = (-1-b)/d$ satisfies $e^2+e+1\equiv 0\,(p)$, then $p\equiv 1,4\,(15)$ and $W_p(K^2_3(a,b)) = \mathcal{R}_{27}$.*

(5) *If $ab\equiv 1\,(p)$ and $a^2\equiv -1\,(p)$, then $p\equiv 1\,(4)$ and $W_p(K^2_4(a,b)) = \mathcal{R}_{29}$.*

(6) *If $ab\equiv 1\,(p)$ and $a^2+a+1\equiv 0\,(p)$, then $p\equiv 1\,(3)$, $W_p(K^3_5(a,b)) = \mathcal{R}_{33}$ and $W_p(K^3_6(a,b)) = \mathcal{R}_{34}$.*

(7) *If $a\equiv b\,(p)$ and $a^2-a-1\equiv 0\,(p)$, then $p\equiv \pm 1\,(5)$ and $W_p(K^1_3(a,b)) = \mathcal{R}_{23}$ and $W_p(K^1_4(a,b)) = \mathcal{R}_{30}$.*

Since the p-adic reflection groups $G(m,m,n)$ and $\mathcal{R}_i$ are only defined up to conjugation, the notation $W_p = \mathcal{R}_i$ should be understood as saying that the groups W_p and $\mathcal{R}_i$ are conjugate.

Proof. The theorem follows from inspection of the generators of the groups of type NCOT given in sections 9 and 10 and comparison to the generators of $W(K)$ as described in section 3, and using 7.1. All further details are straightforward and are left to the reader. □

What we will use in the proof of the main theorem of this paper is the following corollary:

Corollary 12.2. *Let p be a prime and let $\mathcal{R}\subset GL_n(\mathbb{Z}_p)$ be a finite pseudoreflection group of order prime to p generated by pseudoreflections of order two. Then, there exist a Kac-Moody group K of rank n such that $W_p(K) = \mathcal{R}$. K can be chosen to be a direct product of Kac-Moody groups with finiteness width equal to two.*

Proof. By 7.1 and the Shephard-Todd classification of finite unitary reflection groups, $\mathcal{R}$ decomposes as a direct product of the Weyl group of a compact connected Lie group and irreducible groups of type NCOT. The irreducible groups of type NCOT are $G(m,m,r)$, $\mathcal{R}_{23}$, $\mathcal{R}_{24}$, $\mathcal{R}_{27}$, $\mathcal{R}_{29}$, $\mathcal{R}_{30}$, $\mathcal{R}_{33}$ and $\mathcal{R}_{34}$ and each of these irreducible groups can only appear if the prime p satisfies some congruences that were made explicit in sections 9 and 10. Now we can use theorem 12.1 to see that in each case we can find suitable parameters a, b such that we can define K as the direct product of a Lie group and Kac-Moody groups of type $K^i_r(a,b)$. We only comment the case of $G(m,m,2)$ for $m|p+1$. Let $\theta\in\mathbb{F}_{p^2}$ be a primitive m-th root of unity. Since $m|p+1$ we see that the Frobenius automorphism of $\mathbb{F}_{p^2}$ fixes $\eta = \theta+\theta^{-1}$. Hence, $\eta\in\mathbb{F}_p$.

Take any positive integers a, b such that $a \equiv -1\,(p)$, $b \equiv -2-\eta\,(p)$, $ab > 4$. Then $W_p(K_2^1(a,b)) = G(m,m,2)$ by 12.1(2). $\square$

13. Monomorphisms from a homotopy point of view

A fundamental step in the study of finite loop spaces by Dwyer-Wilkerson ([15]) and others is a homotopical interpretation of some group-theoretical concepts. For instance, in the theory of p-compact groups (see [15]), a *homomorphism* from X to Y is a pointed map $BX \to BY$, an *element of order* p in X is a pointed map $B\mathbb{Z}/pZ \to BX$ and a *monomorphism* is a homomorphism $BX \to BY$ whose homotopy fibre is $\mathbb{F}_p$-finite. We use the same ideas to define a notion of monomorphism which is appropriate when we deal with both Kac-Moody groups and p-compact groups. We need a concept from the Dror Farjoun theory of localization (see [14]):

Definition 13.1. *(*[14]*) A space X is $B\mathbb{Z}/p\mathbb{Z}$-null if the evaluation map*

$$e : \mathrm{Map}(B\mathbb{Z}/p\mathbb{Z}, X) \to X$$

is a weak equivalence. This is the same as saying that the space of pointed maps $\mathrm{Map}_*(B\mathbb{Z}/p\mathbb{Z}, X)$ *is weakly contractible.*

As a consequence of the theorem of Miller solving the Sullivan conjecture, the fact that a space is $B\mathbb{Z}/p\mathbb{Z}$-null can be decided, under some mild hypothesis, by just checking the mod p cohomology of this space:

Proposition 13.2. *(*[34]*, 8.6.2) If X is a connected nilpotent space such that $H^*(X)$ is of finite type and $\pi_1 X$ is finite, then the following two conditions are equivalent:*

(1) *$H^*(X)$ is locally finite as a module over the Steenrod algebra.*
(2) *X is $B\mathbb{Z}/p\mathbb{Z}$-null.* $\square$

A module M over the Steenrod algebra is *locally finite* if for any $x \in M$ only a finite number of Steenrod operations act non-trivially on x.

If $\mathcal{X} = (X, BX, e)$ is a loop space, we say that it is *connected* if X is connected. We say that $\mathcal{X}$ is *p-complete* if both X and BX are $\mathbb{F}_p$-complete spaces in the sense of Bousfield-Kan. If $\mathcal{X}$ is connected, then X is $\mathbb{F}_p$-complete if and only if BX is $\mathbb{F}_p$-complete (see the argument in [15] 11.9). p-compact groups and $\mathbb{F}_p$-completions of Kac-Moody groups are our main examples of p-complete loop spaces.

Definition 13.3. *Let $\mathcal{X} = (X, BX, e_X)$, $\mathcal{Y} = (Y, BY, e_Y)$ be p-complete loop spaces. A* homomorphism *$f : \mathcal{X} \to \mathcal{Y}$ is a pointed map $f : BX \to BY$. A* monomorphism *is a pointed map $f : BX \to BY$ such that its homotopy fibre is $B\mathbb{Z}/p\mathbb{Z}$-null.*

As evidence that this definition is plausible, we point out the following facts.

Proposition 13.4. *(a) If $\mathcal{X}$ and $\mathcal{Y}$ are p-compact groups, then a monomorphism $f : \mathcal{X} \to \mathcal{Y}$ as defined above is the same as a monomorphism of p-compact groups in the sense of* [15].

(b) Let $\phi : L \to K$ be an (algebraic, continuous) homomorphism between Kac-Moody groups. Then, $\mathrm{Ker}\,\phi$ *contains no element of order p if and only if the homotopy fibre of the induced map $B\phi : BL_p^\wedge \to BK_p^\wedge$ is $B\mathbb{Z}/p\mathbb{Z}$-null.*

Proof. Part (a) is a particular case of proposition 3.2 in [16].

Part (b) is a consequence of theorem 8.4. Let F be the fibre of $B\phi : BL^\wedge_p \to BK^\wedge_p$ and consider the induced fibration of mapping spaces:

$$\mathrm{Map}(B\mathbb{Z}/p\mathbb{Z}, F) \to \mathrm{Map}(B\mathbb{Z}/p\mathbb{Z}, BL^\wedge_p) \to \mathrm{Map}(B\mathbb{Z}/p\mathbb{Z}, BK^\wedge_p).$$

By 8.4, $[B\mathbb{Z}/p\mathbb{Z}, BL^\wedge_p] = \mathrm{Rep}(\mathbb{Z}/p\mathbb{Z}, L)$, $[B\mathbb{Z}/p\mathbb{Z}, BK^\wedge_p] = \mathrm{Rep}(\mathbb{Z}/p\mathbb{Z}, K)$. Hence, If $\mathrm{Ker}\,\phi$ contains no element of order p we have a fibration

$$\mathrm{Map}(B\mathbb{Z}/p\mathbb{Z}, F) \to \mathrm{Map}(B\mathbb{Z}/p\mathbb{Z}, BL^\wedge_p)_0 \to \mathrm{Map}(B\mathbb{Z}/p\mathbb{Z}, BK^\wedge_p)_0$$

where $\mathrm{Map}(-,-)_0$ denotes the space of null homotopic maps.

By 8.4 again, $\mathrm{Map}(B\mathbb{Z}/p\mathbb{Z}, BL^\wedge_p)_0 \simeq BL^\wedge_p$, $\mathrm{Map}(B\mathbb{Z}/p\mathbb{Z}, BK^\wedge_p)_0 \simeq BK^\wedge_p$ and the fibration above shows that F is $B\mathbb{Z}/p\mathbb{Z}$-null. Conversely, if F is $B\mathbb{Z}/p\mathbb{Z}$-null, then

$$[B\mathbb{Z}/p\mathbb{Z}, BL^\wedge_p] \to [B\mathbb{Z}/p\mathbb{Z}, BK^\wedge_p]$$

has trivial kernel and $\mathrm{Ker}\,\phi$ contains no element of order p. □

14. Proof of theorem 1.1

Now we have all ingredients of the proof of the main result of this paper.

Proof of 1.1. We have a simply connected p-compact group X with Weyl group W_X of order prime to p generated by reflections of order two. In particular, p must be odd. The classification of p-compact groups for p odd ([4], theorem 1.2) tells us that $X \cong G^\wedge_p \times Y$ where G is a compact simply connected Lie group and Y is a p-compact group of Clark-Ewing type (see section 7), associated to a pseudoreflection group $\mathcal{R} \subset GL_r(\mathbb{Z}_p)$ of order prime to p and generated by pseudoreflections of order two. Since G is itself a Kac-Moody group, this reduces the proof of 1.1 to the case in which X is an irreducible p-compact group of Clark-Ewing type.

By corollary 12.2, there is a Kac-Moody group K such that $W_p(K) = W_X \subset GL_n(\mathbb{Z}/p\mathbb{Z})$ and $fw(K) = 2$. Let T_{p^∞} be the standard p-discrete maximal torus of K. T_{p^∞} is also the standard p-discrete maximal torus of X and we will construct a map $BX \to BK^\wedge_p$ extending the inclusion $T_{p^\infty} \hookrightarrow BK^\wedge_p$. We know that $BX = (BT_{p^\infty} \times_{W_X} EW_X)^\wedge_p$, therefore, to construct this map, we need to check two facts:

1. The inclusion $T_{p^\infty} \hookrightarrow BK^\wedge_p$ is W_X-equivariant up to homotopy.
2. The obstructions to extend a map $T_{p^\infty} \hookrightarrow BK^\wedge_p$ which is W_X-equivariant up to homotopy to a map $(BT_{p^\infty} \times_{W_X} EW_X) \to BK^\wedge_p$ vanish.

To solve (1) we use the following argument. Since W_X has order prime to p, we can inductively lift W_X to $W_{p^i}(K)$ for all i and eventually we obtain an inclusion $W_X \hookrightarrow W^\wedge_p(K) \subset GL_n(\mathbb{Z}_p)$. Proposition 7.1 shows that this inclusion must be conjugate to the standard inclusion $W_X \subset GL_n(\mathbb{Z}_p)$. Hence, we can assume that $W_X \subset W^\wedge_p(K)$. Then, proposition 8.2 implies that $i : T_{p^\infty} \hookrightarrow BK^\wedge_p$ is W_X-equivariant up to homotopy.

To solve (2) we notice that the obstructions to an extension $(BT_{p^\infty} \times_{W_X} EW_X) \to BK^\wedge_p$ belong to

$$H^{r+1}(W_X; \pi_r(\mathrm{Map}(BT_{p^\infty}, BK^\wedge_p)_i), \quad r \geq 1.$$

(see [42]). By proposition 8.3 we know that the groups $\pi_r(\mathrm{Map}(BT_{p^\infty}, BK_p^\wedge)_i)$ are pro-p-groups. Then, the following lemma solves the problem.

Lemma 14.1. *Let $r \geq 2$ and let A be a pro-p-group, abelian if $r > 2$. Let G be a finite group of order prime to p acting on A. Then, $H^r(G; A)$ is trivial.*

Proof. If A is abelian, the lemma is well known: $H^r(G; A)$ is the standard cohomology of the group G with coefficients in the G-module A, the order of G annihilates $H^r(G; A)$ and at the same time, multiplication by the order of G is an automorphism of A. Hence, $H^r(G; A) = 0$.

If A is not abelian, then $H^2(G; A)$ is the set of extension of G by A (see [39]). Since the classic Schur-Zassenhaus theorem extends to profinite groups (see theorem 2.3.15 in [33]), we know that there cannot be non trivial extensions of G by the pro-p-group A. □

Hence, the obstructions must vanish and we have a map $BX \to BK_p^\wedge$ extending (up to homotopy) the identity between the p-completions of the maximal tori. We need to prove that the fibre F of this map is $B\mathbb{Z}/p\mathbb{Z}$-null. Since F is simply connected, by proposition 13.2, we just need to check that $H^*(F)$ is locally finite.

Consider the principal fibration $X \to K_p^\wedge \to F$. We know that $H^*(X)$ is an exterior algebra on n transgressive generators. Also, theorem 11.1 tells us that $BX \to BK_p^\wedge$ is surjective in cohomology. Hence, the Serre spectral sequence for $X \to K_p^\wedge \to F$ must collapse at E_2 and we obtain that $H^*(F)$ injects in $H^*(K)$. If we apply 13.4.b to $L = \{1\}$ we see that $H^*(K)$ is locally finite. The proof is complete. □

15. Final remarks

In this final section we want to show that theorem 1.1 is, in some sense, the best possible result that we can expect. To do this, we will provide counterexamples to show that the hypothesis on W_X cannot be removed and that we really need to use Kac-Moody groups since compact Lie groups would not do the job.

Proposition 15.1. *Theorem 1.1 becomes false if it is modified in any of these ways:*

(1) *If we omit the hypothesis that W_X is generated by reflections of order two.*
(2) *If we omit the hypothesis that W_X has order prime to p.*
(3) *If we add that K is a compact Lie group.*

Proof. To prove (1), let X be the p-compact group of rank 1 with Weyl group $W_X = \mathbb{Z}/m\mathbb{Z}$ for some large prime p and some large $m|p-1$. By [11] such a p-compact group exists. In rank one, the only available Kac-Moody group is $SU(2)$. But a map $BX \to BSU(2)_p^\wedge$ must be trivial in cohomology and its fibre cannot be $B\mathbb{Z}/p\mathbb{Z}$-null.

It is an easy consequence from the theory of p-compact groups (see [16]) that a monomorphism $f : BX \to BY$ between connected p-compact groups of the same rank induces a monomorphism $W_X \hookrightarrow W_Y$. Let X be a p-compact group of rank two with Weyl group $W_X = D_{2m}$ a dihedral group of order $m|p-1$ for some large prime p and some large m. By [11] such a p-compact group exists. If there is a monomorphism $BX \to BG_p^\wedge$ for some compact connected Lie group G, then W_G

must contain a large dihedral group. But the order of the Weyl group of a Lie group of rank two is bounded. This proves part (3).

The most interesting counterexample is the one that we need to prove part (2). Along this paper, we have used the fact that the order of W_X is prime to p in several crucial points but the reader might have got the impression that this could be just a way to simplify the proofs and that some more careful argument could be used to handle the "modular" case. Part (2) claims that this is not so.

Let $p = 5$ and let BX be the classifying space of the 5-compact group of rank four with Weyl group $W_X = \mathcal{R}_{29}$. The existence of this 5-compact group was established in [1]. We will show that there is no monomorphism from BX to $BK_5^\wedge$ for any Kac-Moody group K of rank 4. Let $f : BX \to BK_5^\wedge$ be any map. The Weyl group W_X contains a cyclic group π of order 5 which acts on the maximal torus T_X through its irreducible representation of degree 4. Let $N_{X,5}(T_X)$ be the 5-normalizer of X, which is the subgroup of the normalizer of T_X in X which is an extension of π by T_X. Inside $N_{X,5}(T_X)$ let us consider the group N of order 5^5 which is an split extension of $\pi \subset W_X$ by an elementary abelian 5-group $V = (\mathbb{Z}/5\mathbb{Z})^4 \subset T_X$. N is non abelian and there is a map $BN \to BX$. Composing with f we obtain a map $g : BN \to BK_5^\wedge$.

By theorem 8.4 we know that g is homotopic to the map induced by some representation $\rho : N \to K$. By [10], theorem 5.2, ρ is equivalent to some homomorphism $\rho : N \to P_I$ where P_I is a parabolic of K with finite Weyl group. Hence, P_I is a compact connected Lie group of rank 4.

Assume first that P_I is a proper parabolic in K. Then P_I is a split extension of a torus of rank at least one by a compact connected Lie group of rank at most 3. Since the minimal non-trivial $\mathbb{Q}$-representation of a cyclic group of prime order p is in degree $p-1$, we deduce that the Weyl group of P_I has order prime to 5. This implies (see lemma A.1 in [22]) that $\rho(N)$ must be contained in some maximal torus of P_I. Since N is non abelian, we deduce that ρ is not a monomorphism. If we assume now that the fibre of $f : BX \to BK_5^\wedge$ is $\mathbb{Z}/5\mathbb{Z}$-null, we deduce that there is a nullhomotopic map $B\mathbb{Z}/5\mathbb{Z} \to BN \to BX$. But ([15] 9.9) $BN_{X,5}(T_X) \to BX$ is a monomorphism. Hence, we have an algebraic monomorphism $\mathbb{Z}/5\mathbb{Z} \hookrightarrow N_{X,5}(T_X)$ inducing a nullhomotopic map $B\mathbb{Z}/5\mathbb{Z} \to BN_{X,5}(T_X)$, a contradiction.

Hence, $P_I = K$ and K is itself a compact Lie group of rank 4. As above, a monomorphism $BX \to BK_5^\wedge$ implies the existence of a monomorphism $W_X \hookrightarrow W_K$. But $W_X = \mathcal{R}_{29}$ has order 7680 and inspecting the orders of the Weyl groups of all compact Lie groups of rank 4 we cannot find any with a Weyl group of order divisible by 7680. □

References

[1] J. Aguadé, *Constructing modular clasifying spaces*, Israel J. Math., 66 (1989), 23–40.

[2] J. Aguadé, C. Broto, N. Kitchloo, L. Saumell, *Cohomology of classifying spaces of central quotients of rank two Kac-Moody groups.* J. Math. Kyoto Univ. 45 (2005), no. 3, 449–488.

[3] J. Aguadé, A. Ruiz, *Maps between classifying spaces of Kac-Moody groups*, Adv. in Math., 178 (2003), 66–98.

[4] K.K.S. Andersen, J. Grodal, J.M. Møller, A. Viruel, *The classification of p-compact groups for p odd*, Ann. Math. 167 (2008), 95–210.

[5] K.K.S. Andersen, J. Grodal, *The classification of 2-compact groups*, (preprint).

[6] D. Anick, *Differential algebras in topology.* Research Notes in Mathematics, 3. A K Peters, Ltd., Wellesley, MA, 1993

[7] M. Benard, *Shur indices and splitting fields of the unitary reflection groups*, J. Algebra 38 (1976), 318–342.

[8] A.K. Bousfield, D.M. Kan, *Homotopy limits, completions and localizations*, Lect. Notes Math. 304, Springer Verlag, Berlin, Heidelberg, New York 1972.

[9] C. Broto, R. Levi, B. Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc., 16 (2003), 779-856.

[10] C. Broto, N. Kitchloo, *Classifying spaces of Kac-Moody groups*, Math. Z., 240 (2002), 621–649.

[11] A. Clark, J. Ewing, *The realization of polynomial algebras as cohomology rings*, Pacific J. Math. 50 (1974), 425–434.

[12] A. Cohen, *Finite complex reflection groups*, Ann. Sci. École Norm. Sup. 9 (1976), 379–436.

[13] H.S.M. Coxeter, *Groups generated by unitary reflections of period two*, Canadian J. Math. 9 (1957), 243–272.

[14] E. Dror Farjoun, *Cellular spaces, null spaces and homotopy localization.* Lect. Notes in Math., 1622. Springer-Verlag, Berlin, 1996.

[15] W. G. Dwyer, C. W. Wilkerson, *Homotopy fixed points methods for Lie groups and finite loop spaces*, Ann. Math. 139 (1994), 395–442.

[16] W. G. Dwyer, C. W. Wilkerson, *Product splittings for p-compact groups*, Fund. Math. 147 (1995), 279–300.

[17] W. G. Dwyer, H. Miller, C. W. Wilkerson, *Homotopical uniqueness of classifying spaces*, Topology, **31** (1992), 29–45.

[18] M. Feshbach, *The transfer and compact Lie groups*, Trans. Amer. Math. Soc., 251 (1979), 139–169.

[19] P. Goerss, *Comparing completions of a space at a prime*, Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997), 65–102, Contemp. Math., 220, Amer. Math. Soc., Providence, RI, 1998.

[20] L.C. Grove, C.T. Benson, *Finite reflection groups.* Second edition. Graduate Texts in Mathematics, 99. Springer-Verlag, New York, 1985.

[21] S. Jackowski, J. McClure, B. Oliver, *Homotopy theory of classifying spaces of compact Lie groups.* Algebraic topology and its applications, 81–123, Math. Sci. Res. Inst. Publ., 27, Springer, New York-Berlin, 1994.

[22] S. Jackowski, J. McClure, B. Oliver, *Homotopy classification of self-maps of BG via G-actions. II.* Ann. of Math. (2) 135 (1992), no. 2, 227–270.

[23] V.G. Kac, *Infinite-dimensional Lie algebras.* Cambridge University Press, Cambridge, 1990.

[24] V.G. Kac (ed.) *Infinite-dimensional groups with applications.* Papers from the conference held at the Mathematical Sciences Research Institute, Berkeley, Calif., May 10–15, 1984. Math. Sci. Res. Inst. Publ., 4. Springer-Verlag, New York-Berlin, 1985.

[25] V.G. Kac, *Constructing groups associated to infinite dimensional Lie algebras*, Infinite Dimensional groups with applications (V. Kac, ed.), MSRI Publications, vol. 4, Springer-Verlag, New York, 1985, pp. 167–216.

[26] V.G. Kac, D.H. Peterson, *Defining relations of certain infinite-dimensional groups.* The mathematical heritage of Élie Cartan (Lyon, 1984). Astérisque 1985, Numero Hors Serie, 165–208.

[27] V.G. Kac, D.H. Peterson, *On geometric invariant theory for infinite-dimensional groups*, in *Algebraic Groups, Utrecht 1986*, Lecture Notes in Math. 1271, 109–142. Springer-Verlag, Berlin-New York, 1987.

[28] N. Kitchloo, *Topology of Kac-Moody groups.* Thesis, MIT, 1998.

[29] S.A. Mitchell, *Quillen's theorem on buildings and the loops on a symmetric space*, Enseign. Math. (2) 34 (1988), no. 1-2, 123–166.

[30] D. Notbohm, *Topological realization of a family of pseudoreflection groups*, Fund. Math., 155 (1998), 1–31.

[31] D. Notbohm, L. Smith, *Fake Lie groups and maximal tori III*, Math. Ann. 290 (1991), 629–642.

[32] D.H. Peterson, V.G. Kac, *Infinite flag varieties and conjugacy theorems*, Proc. Nat. Acad. Sci. U.S.A. 80 (1983), no. 6 i., 1778–1782.

[33] L. Ribes, P. Zalesskii, *Profinite groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 40. Springer-Verlag, Berlin, 2000.

[34] L. Schwartz, *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*, Chicago Lect. Math., The University of Chicago Press, 1994.

[35] G.C. Shephard, *Unitary groups generated by reflections*, Canadian J. Math. 5 (1953), 364–383.

[36] G.C. Shephard, J.A. Todd, *Finite unitary reflection groups*, Canadian J. Math. 6 (1954), 274–304.

[37] J. Słomińska, *Homotopy colimits on E-I-categories.* Algebraic topology Poznań 1989, 273–294, Lecture Notes in Math., 1474, Springer, Berlin, 1991.

[38] L. Smith, *Polynomial Invariants of Finite Groups.* A K Peters, Wellesley, Mass., 1995

[39] T.A. Springer, *Nonabelian H^2 in Galois cohomology.* 1966 Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965) pp. 164–182 Amer. Math. Soc., Providence, R.I.

[40] D. Sullivan, *Genetics of homotopy theory and the Adams conjecture*, Ann. Math. 100 (1974), 1–79.

[41] R.W. Thomason, *Homotopy colimits in the category of small categories*, Math. Proc. Cambridge Philos. Soc. 85 (1979), no. 1, 91–109.

[42] Z. Wojtkowiak, *On maps from* $\mathrm{ho}\varinjlim F$ *to* Z. Algebraic topology, Barcelona, 1986, 227–236, Lect. Notes in Math., 1298, Springer, Berlin, 1987.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Cerdanyola del Vallès, Spain.

E-mail address: aguade@mat.uab.cat