# Refined long-time asymptotics for some polymeric fluid flow models

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**Abstract**

We consider a polymeric fluid model, consisting of the incompressible Navier-Stokes equations coupled to a non-symmetric Fokker-Planck equation. First, steady states and exponential convergence to it in relative entropy are proved for the linear Fokker-Planck equation in the Hookean case. The FENE model is also addressed proving the existence of stationary states and the convergence towards them in suitable weighted norms. Then, using the "entropy method" exponential convergence to the steady state is established for the coupled model in the Hookean case under some smallness assumption. The results continue and expand the analysis of [JLLO] in both the Hookean and the FENE models.

## 1 Introduction

We consider a coupled microscopic-macroscopic model for a dilute solution of polymers in a homogeneous fluid. The incompressible Navier-Stokes equations for the macroscopic flow shall be coupled via the stress tensor to a microscopic model for the polymer chains distributed within the fluid (cf. [BAH, BCAH, DE, OP] for the physical background of such models). Let us briefly review the coupled model for the polymer distribution within a macroscopic flow. After putting the system in non-dimensional form and setting all remaining dimensionless parameters equal to one for notational simplicity, it reads as follows

$$\frac{\partial u}{\partial t}(t,x) + (u(t,x)\cdot\nabla_x)u(t,x) = \Delta_x u(t,x) - \nabla_x p(t,x) + \mathrm{div}_x \tau(t,x)\,, \tag{1.1}$$

$$\mathrm{div}_x u = 0\,, \tag{1.2}$$

$$\tau(t,x) = \int_{\mathbb{R}^d} (X\otimes\nabla_X \Pi(X))\,\psi(t,x,X)\,dX\,, \tag{1.3}$$

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where $u(t,x)$, $x \in \Omega \subset \mathbb{R}^d$, $d \geq 2$ is the velocity field of the fluid, $p(t,x)$ the pressure, and $\tau(t,x)$ the stress tensor. This system is coupled through Eq. (1.3) to the following microscopic model for the polymer evolution. Here, the polymers are modeled as dumbbells of length and orientation given by the vector $X \in \mathbb{R}^d$. The Fokker-Planck–type evolution equation for $\psi(t,x,X)$, the probability density of dumbbells w.r.t. the (microscopic) extension $X$ at time $t$ and (macroscopic) position $x$ reads

$$\begin{aligned} \frac{\partial \psi}{\partial t}(t,x,X) &+ (u(t,x)\cdot \nabla_x)\,\psi(t,x,X) \\ &= - \operatorname{div}_X \left( \left[ \nabla_x \otimes u(t,x)\cdot X - \frac{1}{2}\nabla_X \Pi(X) \right] \psi(t,x,X) \right) \\ &+ \frac{1}{2}\Delta_X\,\psi(t,x,X)\,. \end{aligned} \tag{1.4}$$

Here, $\nabla_X \Pi(X)$ denotes the recovering force field between the two beads of the dumbbells modelled as a spring with potential $\Pi(X)$. The term $\nabla_x \otimes u(t,x)\cdot X$ comes from the deformation of the dumbbell extensions due to the stress forces produced by the inhomogeneous flow field $u$. Actually, the derivatives in the Fokker-Planck equation involving the velocity $u$ can be considered as the Eulerian terms coming from a microscopic Lagrangian description, see [BAH, OP, DLY, HCDL] for a full discussion of the model. The model of *Hookean dumbbells* is obtained by setting the elastic spring potential $\Pi$ as $\Pi(X) = |X|^2$, while the *finite extensible nonlinear elasticity* (FENE) model of polymeric fluids comes from choosing

$$\Pi(X) = -\frac{b}{2}\ln\left(1 - \frac{|X|^2}{b}\right),$$

with $b \geq 2$.

Under suitable boundary conditions, the equations (1.1)-(1.4) admit special solutions in the form of homogeneous flows (i.e. $\nabla_x \otimes u = \kappa$) with a stationary distribution of the polymer extensions. The stability of such solutions will be one topic of this paper. But first we shall focus on the $x$–homogeneous equation associated to (1.4) and the coupled system (1.1)-(1.4). Using the entropy–entropy dissipation method we shall analyze its large-time behavior. In particular we prove its exponential convergence to equilibrium solutions in the form of homogeneous stationary flows.

The goal of this paper is to answer several open questions posed in [JLLO], more precisely:

A. Hookean case: existence and uniqueness of stationary states for a general deformation matrix $\kappa$ and convergence in relative entropy to them, both for the $x$–homogenous case and in the coupled case under the assumption of a small deformation matrix.

B. FENE case: existence, uniqueness, and asymptotic stability for the $x$–homogenous case in weighted $L^2$–spaces in a more general setting than in [JLLO].

But we cannot yet conclude large time convergence in relative entropy in the FENE case, as we are still lacking certain bounds on the stationary states, see §2.4. This relative entropy convergence result would immediately imply asymptotic stability results of these homogeneous stationary flows for the coupled system for the FENE case by repeating the arguments of [JLLO, Theorem 1] or [JLLO, Proposition 9].

The paper is organized as follows: In §2 we study the linear Fokker-Planck equation (1.4) for the dumbbell distribution in a given, homogeneous flow field $\nabla_x \otimes u = \kappa$. Stationary states $\psi_\infty$ and exponential convergence of $\psi(t, X)$ towards them are established for Hookean dumbbells (for all matrices $\kappa$ such that $2\kappa - I$ is stable) and the FENE model (under some smallness conditions on $\kappa$). In §3 we analyze the coupled system in the Hookean case with non-homogeneous Dirichlet boundary conditions on $u$. Under a smallness assumption on $\kappa$ and $X$-moments on $\psi$, we prove exponential convergence of $(u(t), \psi(t))$ to the homogeneous stationary flow $(u_\infty, \psi_\infty)$.

# 2 Homogeneous flow with a given velocity field

In this section we consider the case that the velocity field $u$ is given and that there exists an arbitrary (but constant in $t$ and $x$) matrix $\kappa \in \mathbb{R}^{d\times d}$ such that

$$u(t, x) = \kappa x\,. \tag{2.1}$$

Then we rewrite Eq. (1.4) for the unknown $\psi = \psi(t, X)$, which is now independent of the space variable $x$:

$$\begin{aligned} \frac{\partial \psi}{\partial t}(t, X) &= L\psi(t, X)\,, \quad t > 0\,, \quad X \in \mathbb{R}^d, \\ L\psi(X) &:= \frac{1}{2}\mathrm{div}_X\left([\nabla_X \Pi(X) - 2\kappa X]\,\psi(X)\right) + \frac{1}{2}\Delta_X \psi(X), \\ \psi(0, X) &= \psi_0(X)\,. \end{aligned} \tag{2.2}$$

Assuming $\psi_0 \geq 0$ implies $\psi(t, X) \geq 0$ for $t > 0$ by a parabolic maximum principle. Moreover, the divergence form of (2.2) implies that $\psi$ stays normalized under time evolution: $\int \psi(t, X)\, dX = \int \psi_0(X)\, dX = 1$.

Now we shall analyze the large-time behavior of (2.2) for three types of (given) potentials $\Pi(X)$: *Hookean dumbbells*, § 2.2, the *finite extensible nonlinear elasticity* (FENE) model of polymeric fluids, § 2.4, and general (radially symmetric) potentials in the special case that $\kappa$ is a normal matrix, §2.3. To this end we shall apply the entropy–entropy dissipation method (cf. [AMTU, BE84, MV, ACJ], e.g.).

## 2.1 Entropy–entropy dissipation method

Let us briefly summarize for later reference the main definitions and the steps of the entropy–entropy dissipation method. It aims at deriving estimates for the relative entropy of the solution $\psi(t)$ w.r.t. the steady state $\psi_\infty$. For $\psi$, $\varphi$ two probability densities on $\mathbb{R}^d$ the (logarithmic) relative entropy is defined as

$$e(\psi|\varphi) := \int_{\mathbb{R}^d} \psi(X) \ln \frac{\psi(X)}{\varphi(X)}\, dX \geq 0\,.$$

Since it satisfies the Csiszár-Kullback inequality [Cs, KL]

$$\|\psi - \varphi\|^2_{L^1(\mathbb{R}^d)} \leq 2e(\psi|\varphi)\,, \tag{2.3}$$

it is a measure for the "distance" of $\psi$ to $\varphi$.

To apply the entropy–entropy dissipation method for non-symmetric Fokker-Planck equations (cf. §2.4 of [AMTU]) we shall proceed in three steps: Firstly, we shall prove the existence of a unique normalized steady state of (2.2). In several cases it is possible to derive an explicit formula for $\psi_\infty$ (cf. §§2.2, 2.3).

In the second step we use the unique normalized steady state $\psi_\infty$ to split the drift vector field in (2.2) as

$$\nabla\Pi(X) - 2\kappa X = \nabla A(X) + \vec{F}(X), \tag{2.4}$$

with

$$A(X) := -\ln(\psi_\infty(X))\,. \tag{2.5}$$

Since $\psi_\infty$ is a steady state of (2.2), we obtain from

$$\mathrm{div}_X\left(\nabla_X A(X)\psi_\infty + \nabla_X\psi_\infty\right) = 0 \tag{2.6}$$

that the (non-gradient) vector field $\vec{F}$ satisfies

$$\mathrm{div}_X\left(\vec{F}\psi_\infty\right) = 0\,. \tag{2.7}$$

Accordingly, finding the steady state $\psi_\infty$ of (2.2) is equivalent to decomposing the given vector field (2.4) into a gradient field ($\nabla A$) and a divergence-free field ($\vec{F}$, in the sense of (2.7)). This resembles the Helmholtz-Hodge decomposition in incompressible fluid mechanics, and we shall illustrate this analogy in subsequent examples. Another consequence of the above splitting of the drift field is the decomposition of the generator $L$ into its symmetric and anti-symmetric parts in $L^2(\psi_\infty^{-1}dX)$:

$$L^{\mathrm{s}}\psi := \frac{1}{2}\mathrm{div}_X\left(\nabla A(X)\psi + \nabla_X\psi\right) = \frac{1}{2}\mathrm{div}_X\left(\psi_\infty\nabla_X\frac{\psi}{\psi_\infty}\right), \tag{2.8}$$

$$L^{\mathrm{as}}\psi := \frac{1}{2}\mathrm{div}_X\left(\vec{F}(X)\psi\right). \tag{2.9}$$

Note that $L^s \le 0$. Moreover, $\psi_\infty$ is not only the steady state of the non-symmetric Fokker-Planck equation (2.2) but also of its "symmetric part" $\psi_t = L^s\psi$.

The third step consists in applying Theorem 2.19 of [AMTU]: The entropy decay of solutions to the non-symmetric Fokker-Planck equation (2.2) is at least as fast as the decay rate for the corresponding symmetric Fokker-Planck equation $\psi_t = L^s\psi$.

## 2.2 Hookean dumbbells

Here we assume that

$$\Pi(X) = \frac{|X|^2}{2}\,. \tag{2.10}$$

The resulting model (2.2), (2.10) was already analyzed in §2.1 of [JLLO]. There the authors established a unique (normalized) steady state $\psi_\infty$ and the exponential decay of the relative entropy for the two cases: $\kappa$ either symmetric or antisymmetric. But the generic case was left as an open problem. The following results close this gap in a unified approach. The next proposition constitutes the first step of the entropy-entropy method:

**Proposition 2.1** *Let the potential* $\Pi$ *be defined by* (2.10) *and let the eigenvalues of the matrix* $\kappa$ *satisfy* $\Re\lambda_j(\kappa) < \frac{1}{2},\ j = 1, ..., d$. *Then there exists a unique normalized steady state* $\psi_\infty$ *for Eq.* (2.2). *It has the form*

$$\psi_\infty(X) = (2\pi)^{-d/2}(\det\Sigma)^{-1/2}\exp\left(-\frac{1}{2}X^T\Sigma^{-1}X\right), \tag{2.11}$$

*with the symmetric, positive definite matrix* $\Sigma$ *given by*

$$\Sigma = 2\int_0^\infty e^{-(I-2\kappa)\theta}e^{-(I-2\kappa^T)\theta}\,d\theta\,. \tag{2.12}$$

*Here,* $\kappa^T$ *denotes the transpose of* $\kappa$ *and* $I$ *the identity matrix. Moreover, the spectral condition that the eigenvalues of the matrix* $\kappa$ *satisfy* $\Re\lambda_j(\kappa) < \frac{1}{2},\ j = 1, ..., d$ *is necessary for the existence of a stationary normalized solution of the form* (2.11).

**Remark 2.2** *For* $\kappa$ *normal,* (2.12) *simplifies to*

$$\Sigma^{-1} = I - 2\kappa^s\,, \tag{2.13}$$

*with* $\kappa^s := (\kappa + \kappa^T)/2$.

Proposition 2.1 makes use of the following lemma (cf. [Br, SZ], §2.2 of [HJ]):

**Lemma 2.3** *Consider the* continuous Lyapunov equation

$$B\Sigma + \Sigma B^H + Q = 0 \tag{2.14}$$

*for the* $d \times d$ *matrix* $\Sigma$ *with a given hermitian and positive definite* $d \times d$ *matrix* $Q$. *A necessary and sufficient condition for the existence of a positive definite, hermitian solution is that the* $d \times d$ *matrix* $B$ *is stable (i.e.* $\Re\lambda_j(B) < 0,\ j = 1, ..., d$). *Then, the unique solution is given by*

$$\Sigma = \int_0^\infty e^{B\theta}Q\,e^{B^H\theta}\,d\theta\,. \tag{2.15}$$

The solution of (2.14) can be computed by a standard numerical algorithm [BS], which is also implemented in *MATLAB*, e.g.

**Proof of Prop. 2.1** The stationary version of Eq. (2.2) with Hookean potential (2.10) reads

$$-\mathrm{div}_X\big((2\kappa - I)\,X\psi(X)\big) + \Delta_X\psi(X) = 0\,. \tag{2.16}$$

From this it is natural to assume that $2\kappa - I$ is a stable matrix for a confinement on $\psi$ to exist. Next we Fourier transform (2.16), denoting $\widehat{\psi}(\xi) := \mathcal{F}_{X\to\xi}\psi(X)$:

$$\xi^T\,(I - 2\kappa)\,\nabla_\xi\widehat{\psi}(\xi) = -|\xi|^2\widehat{\psi}(\xi)\,. \tag{2.17}$$

Using the ansatz:

$$\widehat{\psi}(\xi) = \frac{1}{Z}\exp\left(-\frac{1}{2}\xi^T\Sigma\,\xi\right),$$

with a positive definite, symmetric matrix $\Sigma$ and normalization constant $Z$, Eq. (2.17) reduces to

$$\xi^T \left( (I - 2\kappa)\, \Sigma - I \right) \xi = 0\,, \ \forall \xi \in \mathbb{R}^d\,, \tag{2.18}$$

which is equivalent to $0 = -2\left((I-2\kappa)\,\Sigma - I\right)^s = -\,(I-2\kappa)\,\Sigma - \Sigma\,(I-2\kappa)^T + 2I$. This is a continuous Lyapunov equation for $\Sigma$. Then, Lemma 2.3 guarantees the existence of a unique positive definite and symmetric matrix $\Sigma$, since $2\kappa - I$ is stable and $2I$ is positive definite and symmetric. Inverse Fourier transformation and normalization yields (2.11). (2.13) is readily obtained by diagonalizing the normal matrix $\kappa$.

Uniqueness of the steady state in the weighted space $L^2(\mathbb{R}^d; \psi_\infty^{-1} dX)$ directly follows from the convergence result of Th. 2.5. The fact that the spectral condition is necessary is included in Lemma 2.3. ■

An expression closely related to (2.12) is given in [JLLO], Remark 10 for the stationary stress tensor $\tau_\infty(x)$ in a homogeneous stationary flow.

So far we have established the existence of a unique normalized steady state of Gaussian shape $\psi_\infty(X) = \exp\left(-A(X)\right)$ where $A(X)$ is a quadratic polynomial (cf. (2.11)). In order to prove exponential convergence of $\psi(t)$ to the steady state $\psi_\infty$, we apply the entropy-entropy dissipation method for "non-symmetric diffusion equations" as outlined in §2.

First we rewrite (2.2), (2.10) in the following "split form":

$$\frac{\partial \psi}{\partial t}(t,X) = \frac{1}{2}\mathrm{div}_X \left( \left( \nabla_X A(X) + \vec{F}(X) \right) \psi(t,X) \right) + \frac{1}{2}\Delta_X \psi(t,X)\,, \tag{2.19}$$

with $A(X) := -\ln(\psi_\infty(X)) = \frac{1}{2} X^T \Sigma^{-1} X + const$ and

$$\vec{F}(X) = (I - 2\kappa)X - \nabla_X A(X) = (I - 2\kappa - \Sigma^{-1})X\,. \tag{2.20}$$

**Corollary 2.4** *Under the assumptions of Prop.* 2.1*, Eqs.* (2.2)*,* (2.10) *can be written in the "split-form"* (2.19)*. In addition to* (2.7)*, the (non-gradient) vector field* $\vec{F}$ *defined by* (2.20) *also satisfies*

$$\mathrm{div}_X(\vec{F}) = 0\,. \tag{2.21}$$

*As a consequence the splitting in* (2.19) *provides the "pointwise" Helmholtz-Hodge decomposition of the vector field*

$$\nabla \Pi - 2\kappa X = \nabla A + \vec{F}.$$

**Proof of Cor. 2.4** Eq. 2.21 follows from

$$0 = \mathrm{div}_X \left[ \vec{F} \mathrm{e}^{-A(X)} \right] = \mathrm{e}^{-A(X)} \left( \mathrm{div}_X \vec{F} - \nabla_X A \cdot \vec{F} \right)\,,$$

where

$$\nabla_X A \cdot \vec{F} = X^T \Sigma^{-1} (I - 2\kappa - \Sigma^{-1}) X = \left( X^T \Sigma^{-1} \right) \left[ (I - 2\kappa)\Sigma - I \right] \left( \Sigma^{-1} X \right) = 0$$

by (2.18).
Using (2.20), Eq. (2.21) implies $\mathrm{Tr}(I - 2\kappa - \Sigma^{-1}) = 0$, and hence $\mathrm{Tr}\Sigma^{-1} \in \mathbb{R}$ is always explicitly computable:

$$\mathrm{Tr}\Sigma^{-1} = \mathrm{Tr}(I - 2\kappa^s) = d - 2\sum_{j=1}^{d} \Re\lambda_j(\kappa) > 0.$$

Compare this expression to (2.13), which holds in the special case $\kappa$ normal (Remark 2.2). ■

By using the third step of the entropy-entropy method in §2.1, we can prove the following:

**Theorem 2.5** *Let $\psi_0 \in L^1(\mathbb{R}^d)$ be a probability density with $e(\psi_0|\psi_\infty) < \infty$. Under the assumptions of Prop.* 2.1*, it holds exponential convergence of $\psi(t)$ towards $\psi_\infty$ in relative entropy with rate $\lambda_{\min}(\Sigma^{-1}) > 0$:*

$$e(\psi(t)|\psi_\infty) \leq e^{-\lambda_{\min}(\Sigma^{-1})t} e(\psi_0|\psi_\infty)\,, \quad t \geq 0\,. \tag{2.22}$$

**Proof of Thm. 2.5** The decomposition of the generator $L$ (cf. (2.8)) simplifies in the Hookean case to

$$L^s\psi = \frac{1}{2}\mathrm{div}(\Sigma^{-1}X\psi + \nabla\psi)\,, \quad L^{as}\psi = \frac{1}{2}\vec{F}\cdot\nabla\psi\,.$$

§2.4 of [AMTU] now applies directly to the non-symmetric Fokker-Planck equation (2.19): The entropy decay of its solution is at least as fast as that of the symmetric counterpart $\psi_t = L^s\psi$. Using the *Bakry-Emery convexity condition* (cf. [BE84, BE85, AMTU]) for $A(X)$, the decay rate is given by $\lambda_{\min}(\mathrm{Hess}(A)) = \lambda_{\min}(\Sigma^{-1}) > 0$, the minimal eigenvalue of the matrix $\Sigma^{-1}$. Hence we have proved the thesis. ■

In the two special cases of $\kappa$ either symmetric or antisymmetric we recover the results of [JLLO]: For $\kappa$ symmetric, we have $\Sigma = (I - 2\kappa)^{-1}$ and the decay rate of $e(t)$ is $1 - 2\lambda_{\max}(\kappa) > 0$, just as in Prop. $1(iv)$ of [JLLO]. For $\kappa$ antisymmetric, we obtain $\Sigma = I$, $\psi_\infty(X) = (2\pi)^{-d/2}e^{-\Pi(X)}$ and the decay rate is 1 (like in Prop. $1(i)$ of [JLLO]).

**Remark 2.6** *The entropy decay rate of Theorem* 2.5 *is actually sharp which can be seen as follows: "Optimal functions" for the entropy decay of the symmetric Fokker-Planck equation $\psi_t = L^s\psi$ (with the quadratic potential $\frac{1}{2}X^T\Sigma^{-1}X$) are the shifted Maxwellians $\mu(X) := \rho_\infty(X - \xi e_1)$, $\xi \in \mathbb{R}\setminus\{0\}$, where $e_1$ is an eigenvector of $\Sigma^{-1}$ for $\lambda_{\min}(\Sigma^{-1})$ (cf.* §3.5 *of* [AMTU]*). This means that the entropy decay for the symmetric Fokker-Planck equation with $\psi_0 = \mu$ is exactly exponential with the rate $\lambda_{\min}(\Sigma^{-1})$. Note that $\mu$ is also an "optimal function" of the corresponding Logarithmic Sobolev inequality which makes it an equality.*

*Now we recall that, for a non-symmetric Fokker-Planck equation, the relative entropy $e(\psi_0|\psi_\infty)$ and the entropy dissipation*

$$\frac{d}{dt}e(\psi(t)|\psi_\infty)\Big|_{t=0} = -\frac{1}{2}\int_{\mathbb{R}^d}\left|\nabla\frac{\psi_0}{\psi_\infty}\right|^2\frac{\psi_\infty^2}{\psi_0}\,dX$$

*both coincide with the terms in its symmetric counterpart – the entropy by definition and the entropy dissipation because of*

$$\int(L^{as}\psi_0)\ln\frac{\psi_0}{\psi_\infty}\,dX = -\frac{1}{2}\int\psi_\infty\vec{F}\cdot\nabla\frac{\psi_0}{\psi_\infty}\,dX = \frac{1}{2}\int\mathrm{div}(\vec{F}\psi_\infty)\frac{\psi_0}{\psi_\infty}\,dX = 0$$

*(cf.* §2.4 *of* [AMTU]*). Hence, for $\psi_0 = \mu$ and $t = 0$ the time-derivative of both sides in* (2.22) *coincide. And this rules out any better decay rate in Theorem* 2.5*.*

**Remark 2.7** *In* [ACJ, AC] *an alternative entropy method for non-symmetric diffusion equations was developed. There, the exponential decay rate of* $e(t)$ *is estimated by the uniform convexity of*

$$\frac{\partial^2 A}{\partial X^2} - \frac{1}{2}\left(\frac{\partial \vec{F}}{\partial X} + \left(\frac{\partial \vec{F}}{\partial X}\right)^T\right).$$

*In the Hookean case this lower convexity bound is* $\tilde{\lambda} := \lambda_{\min}(2\Sigma^{-1} - I + 2\kappa^s)$ *with* $\Sigma = \Sigma(\kappa)$ *given by* (2.12). *As discussed in Remark* 2.6 *it has to satisfy* $\tilde{\lambda} \leq \lambda_{\min}(\Sigma^{-1})$ *(which is easily verified numerically). Hence, this approach does not yield a "better" decay rate for the Hookean dumbbell model having a homogeneous vector field* $\vec{F}(X) = (I - 2\kappa - \Sigma^{-1})X$. *This was to be expected from the examples given in* [ACJ], *where improved decay rates were obtained only from highly non-homogeneous vector fields* $\vec{F}(X)$.

### 2.3 General potential with a normal matrix $\kappa$

Here we consider (2.2) with a radially symmetric potential $\Pi(X) = \pi(|X|)$. While we present here the whole space problem $X \in \mathbb{R}^d$, the same argument applies to bounded domain models as is §2.4. For $\kappa$ normal we have the following generalization of Prop. 2.1 (cf. (2.13)):

**Proposition 2.8** *Let* $\kappa$ *be a normal matrix, and let the potential* $\Pi$ *and* $\kappa$ *satisfy* $\exp[-\Pi(X) + X^T\kappa^s X] \in L^1(\mathbb{R}^d)$. *Then,*

$$\psi_\infty(X) = C\, e^{-\Pi(X) + X^T\kappa^s X}, \tag{2.23}$$

*with some appropriate constant* $C$, *is a normalized steady state of* (2.2).

**Proof.-** Using the matrix decomposition $\kappa = \kappa^{\rm s} + \kappa^{\rm as}$, a straightforward computation yields for all $X \in \mathbb{R}^d$:

$$L\psi_\infty = -\mathrm{div}(\kappa^{\rm as} X \psi_\infty) = X^T\kappa^{\rm as}\left[2\kappa^{\rm s}X - \frac{X}{|X|}\pi'(|X|)\right]\psi_\infty = \frac{1}{2}X^T[\kappa, \kappa^T]X\psi_\infty = 0.$$

■

We remark that for the FENE model, this form of the steady state could also have been deduced from the estimate (102) in [JLLO].

Now we proceed as in §2.2 and define

$$A(X) := -\ln(\psi_\infty(X)) = \Pi(X) - X^T\kappa^s X + const, \qquad \vec{F}(X) = -2\kappa^{\rm as}X$$

as the coefficients of the Fokker-Planck equation in "split form" (2.19). It satisfies $\mathrm{div}\vec{F} = 0$, $\vec{F}\cdot\nabla A = 0$. The entropy-entropy dissipation method then yields again:

**Theorem 2.9** *Let* $\psi_0 \in L^1(\mathbb{R}^d)$ *be a probability density with* $e(\psi_0|\psi_\infty) < \infty$ *and assume that* $\lambda := \inf_{X\in\mathbb{R}^d}[\lambda_{\min}(\mathrm{Hess}(\Pi(X)) - 2\kappa^s)] > 0$. *Under the assumptions of Proposition* 2.8, *it holds exponential convergence of* $\psi(t)$ *towards* $\psi_\infty$ *in relative entropy with rate* $\lambda$*:*

$$e(\psi(t)|\psi_\infty) \leq e^{-\lambda t} e(\psi_0|\psi_\infty), \quad t \geq 0.$$

We remark that an alternative decay rate can be obtained by considering $A$ as an $L^\infty$-perturbation of the uniformly convex potential $\Pi$ and applying a Holley-Stroock perturbation argument for logarithmic Sobolev inequalities (cf. [HS, AMTU]). Particularly for bounded domain models (like the FENE model of §2.4) this may yield a better decay rate.

**Corollary 2.10** *Under the assumptions of Theorem* 2.9, $\psi_\infty$ *from* (2.23) *is the* unique *normalized steady state of* (2.2).

Let us finally mention a weaker condition for uniqueness of the steady state: Let the coefficient $A(X)$ be such that the operator $L^s$ has a positive spectral gap when considered on $L^2(\psi^{-1}dX)$. This would then imply exponential convergence of $\psi(t)$ towards $\psi_\infty$ in the $L^2(\psi^{-1}dX)$–norm (cf. [AMTU] for details).

## 2.4 FENE potential

Here, we will improve on the hypotheses for the existence, uniqueness, and stability of stationary states compared to §2.1 of [JLLO]. More precisely, the results in [JLLO] show that being $\kappa$ a general traceless matrix with $\kappa^s$ "small" (i.e. $|\kappa^s| < 1/2$), then stationary states exist and asymptotic stability is obtained by a Holley-Stroock perturbation argument. We will show that the existence, uniqueness of stationary states and their asymptotic stability can be established from a pure linear operator theory point of view in weighted Sobolev spaces. This leads to the answer to these questions under less restrictive hypotheses than in [JLLO]. However, we do not know in general how to prove convergence in relative entropy due to the lack of pointwise control of the behavior close to the boundary of the stationary states. Now, we consider Eq. (2.2)

$$\frac{\partial \psi}{\partial t} = \frac{1}{2}\mathrm{div}_X\left(\left[\nabla_X \Pi(X) - 2\kappa X\right]\psi\right) + \frac{1}{2}\Delta_X \psi\,, \tag{2.24}$$

with the potential $\Pi$ given by

$$\Pi(X) = -\frac{b}{2}\ln\left(1 - \frac{|X|^2}{b}\right)\,, \tag{2.25}$$

for some $b \geq 2$ (cf. §1.1 of [JLLO] for a discussion of this parameter bound). In this model the polymer chains are assumed to have finite extensibility. This is reflected by $\Pi(X) \to +\infty$ for $|X|^2 \to b$. Hence, it is natural to study the problem in the ball of radius $\sqrt{b}$, $\mathcal{B} = \mathcal{B}(0, \sqrt{b})$ with a no-flux boundary condition on $\partial\mathcal{B}$. Then $\psi$ also satisfies a homogeneous Dirichlet boundary condition (cf. §1.1 of [JLLO] for details).

Our first goal (cf. §2.1) is to prove that

$$\begin{aligned} L\psi(X) := \frac{1}{2}\mathrm{div}_X\left(\left[\nabla_X \Pi(X) - 2\kappa X\right]\psi(X)\right) + \frac{1}{2}\Delta_X\psi(X) &= 0\,, \quad \mathcal{B}\,, \\ \psi &= 0\,, \quad \partial\mathcal{B} \end{aligned} \tag{2.26}$$

has a unique normalized solution. In contrast to §2.2, §2.3 we do *not* know here the explicit form of the steady state (at least for $\kappa$ not normal). This prevents us from using for (2.24), (2.25) –at least at the very beginning– the canonical decomposition (2.4), (2.5) of its drift field. As an alternative, we shall rather use a perturbation argument to establish the existence of a steady state. In particular we shall discuss the following four cases:

(i) $\kappa = \kappa^{\rm s}$, $\kappa = \kappa^{\rm as}$, or the commutator $[\kappa, \kappa^T]$ small,

(ii) $\kappa$ has a small distance to the set of normal matrices.

We first use an auxiliary decomposition of $\kappa$ as $\kappa = \kappa_1 + \kappa_2$, with $\kappa_1$ a normal matrix to be chosen later. Inspired by the steady state function in the case $\kappa$ normal (cf. (2.23)) we set

$$\mu(X) := C\, e^{-\Pi(X) + X^T \kappa_1^{\rm s} X} = C\, \left(1 - \frac{|X|^2}{b}\right)^{\frac{b}{2}} e^{X^T \kappa_1^{\rm s} X}\,, \tag{2.27}$$

with $\kappa_1^{\rm s} := (\kappa_1 + \kappa_1^T)/2$ and the normalization $\int_{\mathcal{B}} \mu\, dX = 1$. Clearly, $\mu = 0$ on $\partial\mathcal{B}$, and for $b > 2$ it also takes homogeneous Neumann boundary values.

In analogy to [AU] we define the following weighted $H_0^1$–space:

$$\mathcal{V} := \left\{ \psi \,\Big|\, \frac{\psi}{\mu},\, \nabla \frac{\psi}{\mu} \in L^2(\mathcal{B}; \mu dX),\, \psi\big|_{\partial\mathcal{B}} = 0 \right\}$$

with its obvious norm

$$\|\psi\|_{\mathcal{V}}^2 = \left\| \frac{\psi}{\mu} \right\|^2_{L^2(\mathcal{B};\mu dX)} + \left\| \nabla \frac{\psi}{\mu} \right\|^2_{L^2(\mathcal{B};\mu dX)}\,.$$

The Hilbert space $\mathcal{V}$ (cf. [Tr], §3.2.2) is a dense subset of $\mathcal{H} := L^2(\mathcal{B}; \mu^{-1} dX)$. We shall denote the $\mathcal{H}$–inner product by $\langle \cdot, \cdot \rangle$. Note that the space $\mathcal{V}$ is independent of the decomposition $\kappa = \kappa_1 + \kappa_2$ (with equivalent norms) which is seen as follows: Since $\frac{1}{\alpha} \le e^{X^T \kappa_1^{\rm s} X} \le \alpha$ on $\mathcal{B}$ for some $\alpha > 0$, this independence is trivial concerning $\frac{\psi}{\mu} \in L^2(\mathcal{B}; \mu dX)$. For the second term in the definition of $\mathcal{V}$ we use

$$\nabla \frac{\psi}{\mu} = \frac{\nabla\psi + \psi \nabla \Pi}{\mu} - \frac{\psi}{\mu} 2\kappa_1^{\rm s} X\,. \tag{2.28}$$

Hence, the second term in (2.28) is always in $L^2(\mathcal{B}; \mu dX)$, and for the first term we conclude as before.

Following the decomposition of $\kappa$, we decompose $L$ as $L = L_1 + L_2$:

$$\begin{aligned} L_1\psi &:= \frac{1}{2} \mathrm{div}\left( [\nabla\Pi - 2\kappa_1 X]\psi + \nabla_X \psi \right) = \frac{1}{2} \mathrm{div}\left( \mu \nabla \frac{\psi}{\mu} \right)\,, \\ L_2\psi &:= -\mathrm{div}\left( \kappa_2 X \psi \right)\,. \end{aligned}$$

Next we define the associated quadratic forms:

$$\begin{aligned} q_1(\psi, \varphi) &:= -\langle L_1\psi, \varphi \rangle = \frac{1}{2} \int_{\mathcal{B}} \nabla \frac{\psi}{\mu} \cdot \nabla \frac{\varphi}{\mu} \mu\, dX\,, \\ q_2(\psi, \varphi) &:= -\langle L_2\psi, \varphi \rangle = -\int_{\mathcal{B}} \frac{\psi}{\mu} \left( \nabla \frac{\varphi}{\mu} \right) \cdot \kappa_2 X \mu\, dX\,, \\ q(\psi, \varphi) &:= q_1(\psi, \varphi) + q_2(\psi, \varphi)\,, \end{aligned} \tag{2.29}$$

which are all bounded on $\mathcal{V}^2$. Note that $L_1$ (with form domain $\mathcal{V}$) is symmetric in $\mathcal{H}$, but $L_2$ is in general *not* anti-symmetric. From (2.29) it follows that the kernel of $L_1$ is spanned by $\mu$.

Using these quadratic forms we shall now give a weak reformulation of the steady-state problem (2.26):

**Proposition 2.11** *The weak formulation of* (2.26) *reads: Find* $\phi \in \mu^\perp$ *such that*

$$q(\phi, \varphi) = -q_2(\mu, \varphi)\,, \qquad \forall\, \varphi \in \mu^\perp\,, \tag{2.30}$$

*with* $\mu^\perp$ *the closed subset of* $\mathcal{V}$ *defined by*

$$\mu^\perp := \left\{ \psi \in \mathcal{V} \,\Big|\, \int_{\mathcal{B}} \psi \, dX = 0 \right\}.$$

*The weak solution of* (2.26) *is then* $\psi := \phi + \mu \in \mathcal{V}$.

Note that $\langle \psi, \mu \rangle = \int_{\mathcal{B}} \psi \, dX = 0$ characterizes the orthogonal complement of $\mu$ in $\mathcal{H}$.
**Proof.-** The problem to solve reads

$$L\psi = 0\,, \qquad \text{with} \quad \int_{\mathcal{B}} \psi \, dX = 1,\ \psi \in \mathcal{V}\,. \tag{2.31}$$

In order to cope with this normalization we proceed as in [AGGS] and introduce $\phi := \psi - \mu \in \mu^\perp$. It satisfies

$$L\phi = -L\mu = -L_2\mu\,, \qquad \phi \in \mu^\perp\,. \tag{2.32}$$

Taking the $\mathcal{H}$–inner product with $\varphi \in \mu^\perp$ yields the weak formulation (2.30). ■

**Lemma 2.12**

*(a)* $L_1$ *has a spectral gap* $\lambda_1 > 0$.

*(b)* $L_1$ *gives rise to the following Poincaré inequality:*

$$\left\| \frac{\psi}{\mu} \right\|^2_{L^2(\mathcal{B};\mu dX)} \leq \frac{1}{\lambda_1} \left\| \nabla \frac{\psi}{\mu} \right\|^2_{L^2(\mathcal{B};\mu dX)} \qquad \forall\, \psi \in \mu^\perp\,. \tag{2.33}$$

**Proof.-** (a) Since $\Pi$ is an "infinitely deep potential well" (i.e. $\Pi(X) \to \infty$ as $|X| \to \sqrt{b}$), $L_1$ has a positive spectral gap (for any choice of $\kappa_1$ !): This spectral gap $\lambda_1$ can be estimated with either of the following two arguments. First one could use the Bakry-Emery-condition [BE84, AMTU] for the potential $\Pi(X) - X^T \kappa_1^s X$ yielding

$$\lambda_1 > \lambda_{BE} := \min_{X \in \mathcal{B}} [\lambda_{\min}(\mathrm{Hess}(\Pi(X)) - 2\kappa_1^s)] = \lambda_{\min}(I - 2\kappa_1^s)\,.$$

This yields a spectral gap if $\kappa_1^s < \frac{1}{2}$. An alternative estimate for the log-Sobolev constant of $L_1$ and hence for its spectral gap is obtained by considering $-X^T \kappa_1^s X$ as an $L^\infty(\mathcal{B})$–perturbation of the potential $\Pi$. $\Pi$ is uniformly convex with

$$\min_{X \in \mathcal{B}} [\lambda_{\min}(\mathrm{Hess}\, \Pi(X))] = 1.$$

Using

$$b\lambda_{\min}(\kappa_1^s) \leq X^T \kappa_1^s X \leq b\lambda_{\max}(\kappa_1^s)\,, \quad X \in \mathcal{B}\,,$$

the Holley-Stroock perturbation argument [HS, AMTU] yields for any $\kappa_1^s$

$$\lambda_1 > e^{-b\left[\lambda_{\max}(\kappa_1^s) - \lambda_{\min}(\kappa_1^s)\right]} > 0\,.$$

We remark that neither of these estimates is sharp for the considered $\Pi$.

(b) The spectral gap of $L_1$ gives rise to the Poincaré inequality (2.33) (e.g. put $g = \frac{\psi}{\mu}$ in §3.3 of [AMTU]). ■

**Proposition 2.13** *Let the spectral gap of $L_1$ and the matrix decomposition of $\kappa$ satisfy*

$$\sqrt{b}\,\|\kappa_2\|_2 < \frac{\sqrt{\lambda_1}}{2}\,. \tag{2.34}$$

*Then, the stationary Fokker-Planck equation* (2.26), (2.25) *admits a unique normalized weak solution $\psi_\infty = \phi + \mu \in \mathcal{V}$.*

**Proof.-** We estimate with the Poincaré inequality for $\psi \in \mu^\perp$:

$$|q_2(\psi,\psi)| \le \sqrt{b}\,\|\kappa_2\|_2 \left\|\frac{\psi}{\mu}\right\|_{L^2(\mu dX)} \left\|\nabla\frac{\psi}{\mu}\right\|_{L^2(\mu dX)} \le \sqrt{\frac{b}{\lambda_1}}\,\|\kappa_2\|_2 \left\|\nabla\frac{\psi}{\mu}\right\|^2_{L^2(\mu dX)}\,. \tag{2.35}$$

Hence, $q$ is coercive on $\mu^\perp$:

$$q(\psi,\psi) \ge \left(\frac{1}{2} - \sqrt{\frac{b}{\lambda_1}}\|\kappa_2\|_2\right) \left\|\nabla\frac{\psi}{\mu}\right\|^2_{L^2(\mu dX)} \ge \left(\frac{1}{2} - \sqrt{\frac{b}{\lambda_1}}\|\kappa_2\|_2\right) \frac{\lambda_1}{1+\lambda_1}\,\|\psi\|^2_{\mathcal{V}}\,, \tag{2.36}$$

and the assertion follows from the Lax-Milgram lemma applied to (2.30). ∎

We remark that the weak solution $\psi_\infty$ is independent of the decomposition $\kappa = \kappa_1 + \kappa_2$: Otherwise this would contradict the unique solvability of the weak formulation (2.30).

We shall now illustrate condition (2.34) for several typical decompositions of the shear matrix $\kappa$:

**Example 2.14** *Choose $\kappa_1 = \kappa^{\rm s}$, $\kappa_2 = \kappa^{\rm as}$, and hence $\mu = C\,e^{-\Pi(X)+X^T\kappa^{\rm s}X}$. Then, condition* (2.34) *reads*

$$\sqrt{b}\,\|\kappa^{\rm as}\|_2 < \frac{\sqrt{\lambda_1}}{2}\,. \tag{2.37}$$

To derive an alternative condition, $q_2$ can be rewritten here as

$$\begin{aligned} q_2(\psi,\psi) &= -\frac{1}{2}\int_{\mathcal{B}} \nabla^T \left(\frac{\psi}{\mu}\right)^2 \kappa^{\rm as} X\mu\, dX = \frac{1}{2}\int_{\mathcal{B}} \left(\frac{\psi}{\mu}\right)^2 {\rm div}(\kappa^{\rm as}X\mu)\, dX \\ &= \int_{\mathcal{B}} \left(\frac{\psi}{\mu}\right)^2 X^T\kappa^{\rm s}\kappa^{\rm as}X\mu\, dX = -\frac{1}{4}\int_{\mathcal{B}} \left(\frac{\psi}{\mu}\right)^2 X^T[\kappa,\kappa^T]X\mu\, dX\,. \end{aligned}$$

Estimating as in (2.35) yields the following alternative condition for Prop. 2.13 to hold:

$$b\,\|[\kappa,\kappa^T]\|_2 < 2\lambda_1\,. \tag{2.38}$$

**Example 2.15** *Choose $\kappa_1 = \kappa^{\rm as}$, $\kappa_2 = \kappa^{\rm s}$, and hence $\mu = C\,e^{-\Pi(X)}$, $\lambda_1 > \lambda_{BE} = 1$. Then, condition* (2.34) *reads*

$$\sqrt{b}\,\|\kappa^{\rm s}\|_2 < \frac{\sqrt{\lambda_1}}{2}\,. \tag{2.39}$$

**Example 2.16** *With* §2.3 *in mind, another obvious option is to choose $\kappa_1$ as the* closest normal matrix *to $\kappa$* [Ru]*, and $\kappa_2$ as the non-normal remainder. We refer to* [La] *for estimates between this non-normal remainder and the commutator $[\kappa,\kappa^T]$.*

Next we turn to the large-time convergence of the Fokker-Planck solution $\psi(t)$ towards the steady state $\psi_\infty$:

**Theorem 2.17** *Let $\psi_0 \in \mathcal{H}$. Then, the Fokker-Planck equation* (2.24), (2.25) *has a unique weak solution $\psi \in L^2((0,T),\mathcal{V}) \cap H^1((0,T),\mathcal{V}') \cap C([0,T],\mathcal{H})$ for any $T > 0$. Moreover, $\int \psi_0\, dX = \int \psi(t)\, dX$, $\forall\, t \geq 0$. For $\psi_0$ normalized and under the assumptions of Proposition* 2.13 *it satisfies*

$$\|\psi(t) - \psi_\infty\|_{\mathcal{H}} \leq e^{-\lambda_1\left(\frac{1}{2} - \sqrt{\frac{b}{\lambda_1}}\|\kappa_2\|_2\right)t}\|\psi_0 - \psi_\infty\|_{\mathcal{H}}\,, \quad t > 0\,,$$

*and analogously under assumption* (2.38). *Moreover, $\psi_\infty(X) \geq 0$.*

**Proof.-** Using $q(\psi,\varphi) = -\langle L\psi, \varphi\rangle$ we see that $L \in \mathcal{L}(\mathcal{V},\mathcal{V}')$. Since $q$ satisfies on $\mathcal{V}$ the following modified coercivity estimate (use (2.29), (2.35) and Young's inequality):

$$q(\psi,\psi) \geq \left(\frac{1}{2} - \varepsilon\right)\|\psi\|^2_{\mathcal{V}} - C(\varepsilon)\|\psi\|^2_{\mathcal{H}}\,,$$

standard parabolic theory (cf. §11.1 of [RR], e.g.) yields the first assertion. Moreover, this solution satisfies a.e. in $(0,T)$:

$$\frac{d}{dt}\|\psi(t)\|^2_{\mathcal{H}} = 2\,{}_{\mathcal{V}'}\langle\psi'(t),\psi(t)\rangle_{\mathcal{V}} = -2q(\psi(t),\psi(t)).$$

For normalized $\psi_0$ we have $\psi(t) - \psi_\infty \in \mu^\perp$, a.e. in $(0,\infty)$ by using $\mu$ as test function in the weak formulation of the equation. Hence, (2.36) and the Poincaré inequality yield

$$\frac{d}{dt}\|\psi(t) - \psi_\infty\|^2_{\mathcal{H}} \leq -2\lambda_1\left(\frac{1}{2} - \sqrt{\frac{b}{\lambda_1}}\|\kappa_2\|_2\right)\|\psi(t) - \psi_\infty\|^2_{\mathcal{H}}\,, \quad \text{a.e. in } (0,T)\,,$$

and the exponential convergence follows.

The fact that $\psi(t) - \psi_\infty \in \mu^\perp$ and $\psi \in C([0,T],\mathcal{H})$ imply the conservation of mass. To prove the non-negativity of $\psi_\infty$ we choose an arbitrary non-negative, normalized $\psi_0 \in \mathcal{H}$. $\psi(t,X) \geq 0$ then implies $\psi_\infty(X) \geq 0$. ∎

We remark that the existence part of the above theorem (in an equivalent norm) was already sketched in Appendix B of [JLLO]. We only included it for the sake of completeness.

Following the procedure of Remark 13 in [JLLO], one can deduce that the weak solutions from Prop. 2.13 and Th. 2.17 then satisfy the *no-flux boundary condition* in the following sense:

$$\int_{\partial\mathcal{B}}\left(\frac{1}{2}\nabla\frac{\psi(t)}{\mu} - \kappa_2 X\frac{\psi(t)}{\mu}\right)\cdot n\chi\mu\, dS = 0 \quad \forall\,\chi \in H^1(\mathcal{B};\mu\, dX)\,,$$

with $n$ being the unit outward normal vector on $\partial\mathcal{B}$.

One strategy to extend the above large time convergence to the logarithmic relative entropy (w.r.t. the stationary state $\psi_\infty$) would be to apply a Holley-Stroock perturbation argument. To this end we would have to show that there exist constants $C_1, C_2$ such that

$$0 < C_1 \leq \psi_\infty/\mu \leq C_2\,, \tag{2.40}$$

as done in [JLLO, Proposition 10, Lemma 6] in their case. However, we do not know how to show these bounds in the present case.

## 3 Coupled model: large time behavior

In this section, we derive exponential convergence results towards homogeneous stationary flow solutions of the coupled problem

$$\begin{aligned} \frac{\partial u}{\partial t} + (u\cdot\nabla_x)u &= \Delta_x u - \nabla_x p + \mathrm{div}_x \int_{\mathbb{R}^d} (X \otimes \nabla_X \Pi(X))\psi \, dX \,, \\ \mathrm{div}_x u &= 0 \,, \\ \frac{\partial \psi}{\partial t} + u\cdot\nabla_x \psi &= -\mathrm{div}_X \left( [\nabla_x u \cdot X - \frac{1}{2}\nabla_X \Pi(X)]\,\psi \right) + \frac{1}{2}\Delta_X \psi \,. \end{aligned} \tag{3.1}$$

More precisely, this problem is posed in a bounded spatial domain $x \in \Omega$ and in the configuration space $B \subset \mathbb{R}^d$ with $B = \mathbb{R}^d$ in the Hookean case or $B = \mathcal{B}(0, \sqrt{b})$, the Euclidean ball centered at 0 with radius $\sqrt{b}$, in the FENE model. These equations have to be complemented by boundary conditions in such a way that $u_\infty = \kappa x$, with $\kappa$ any traceless real matrix, and $\psi_\infty = \psi_\infty(X)$ form a stationary solution. Here, $\psi_\infty$ is given by the stationary solution obtained in Section 2 either in the Hookean or in the FENE case. These solutions were called *homogeneous stationary flows* in [JLLO, §3.3]. These boundary conditions amount to $u = u_\infty$ on $\partial\Omega$ and $\psi = 0$ on $\partial\mathcal{B}(0, \sqrt{b})$ in the FENE case or decay at infinity of the solution $\psi$ in the Hookean case. The latter condition is usually imposed by the class of solutions we work with. We will refer to these boundary conditions as *non-homogeneous stationary Dirichlet boundary conditions* as in [JLLO].

In this section we will concentrate on the Hookean case, in particular on the long time asymptotics of smooth solutions to (3.1), where $u$ satisfies the above boundary conditions and $\psi$ has a fast decay at infinity. Such solutions in the Hookean case are known to exist for small initial data when $\kappa = 0$ [LLZ] with a smallness condition in some suitably chosen high-order Sobolev space. Similar results are quite likely to hold for the above non-homogeneous boundary conditions in a suitable neighbourhood of the steady state $(u_\infty, \psi_\infty)$. Let us remark that in these "close to equilibrium" results, one obtains (see [LLZ]) that the deformation matrix $\nabla_x \otimes u^s := \frac{1}{2}\big(\nabla_x \otimes u + (\nabla_x \otimes u)^T\big)$ is globally bounded, i.e., that

$$D := \sup_{0<t<\infty} \|\nabla_x \otimes u^s(t,\cdot)\|_{L^\infty(\Omega)} < \infty. \tag{3.2}$$

We will make this bounded deformation matrix assumption in the rest of this section. In far-from-equilibrium situations, however, the existence of such global solutions remains an open problem.

The objective of this section is to improve over the results obtained in [JLLO] in the coupled case. More precisely, we will show an exponential rate of convergence in the Hookean case under some smallness assumption, a result stated as an open problem in [JLLO, §3.3].

One of the technical ingredients is the following generalization of the Csiszár-Kullback inequality obtained by a similar proof to the standard Csiszár-Kullback inequality [Cs, KL]. It can also be derived from the Csiszár-Kullback inequality and standard moment interpolation.

**Lemma 3.1** *Let $\psi, \varphi \in L^1_+(\mathbb{R}^d)$ with unit mass such that $\varphi > 0$ with bounded fourth moments, i.e., $|X|^4(\psi + \varphi) \in L^1(\mathbb{R}^d)$. Then, the following inequality holds:*

$$\||X|^2(\psi - \varphi)\|^2_{L^1(\mathbb{R}^d)} \le 2\, e(\psi|\varphi)\, \max\left( \int_{\mathbb{R}^d} |X|^4 \psi \, dX, \int_{\mathbb{R}^d} |X|^4 \varphi \, dX \right). \tag{3.3}$$

**Proof.-** By a Taylor expansion at order two of $s \log s$ at $s = 1$ and the normalization of mass, we can write the relative entropy for $\psi$ and $\varphi$ as

$$e(\psi|\varphi) = \int_{\mathbb{R}^d} \frac{\psi}{\varphi} \ln \frac{\psi}{\varphi} \, \varphi \, dX = \frac{1}{2} \int_{\mathcal{A}} \frac{1}{\xi} (\psi - \varphi)^2 \, dX \tag{3.4}$$

where $\mathcal{A} := \{X \in \mathbb{R}^d : \psi(X) \neq \varphi(X)\}$ and $\xi(X)$ lies between $\psi(X)$ and $\varphi(X)$, i.e., $0 \leq \min(\psi, \varphi) < \xi < \max(\psi, \varphi)$ in $\mathcal{A}$, and thus $\xi > 0$ in $\mathcal{A}$. By Hölder's inequality and since $\mathcal{A}$ is measurable, we get

$$\int_{\mathcal{A}} |X|^2 |\psi - \varphi| \, dX \leq \left( \int_{\mathcal{A}} \frac{1}{\xi} (\psi - \varphi)^2 \, dX \right)^{1/2} \left( \int_{\mathcal{A}} |X|^4 \xi \, dX \right)^{1/2},$$

from which the stated inequality (3.3) is obtained. ∎

Now, let us show that the assumption of a bounded deformation matrix (3.2), implies the boundedness in space and time of all moments of the distribution function $\psi$ and in particular of the stress tensor in the Hookean case, if they are initially bounded. In order to verify this, we multiply the equation for $\psi$ by $|X|^{2n}$, with $n \in \mathbb{N}$ and integrate to obtain

$$\begin{aligned} \frac{\partial m_{2n}(\psi)}{\partial t} + u \cdot \nabla_x m_{2n}(\psi) = & 2n \int_{\mathbb{R}^d} X^T (\nabla_x \otimes u^s) X \, |X|^{2n-2} \psi \, dX \\ & - n m_{2n}(\psi) + n(2n - 2 + d) m_{2n-2}(\psi) \, , \end{aligned}$$

where

$$m_{2n}(\psi)(t, x) := \int_{\mathbb{R}^d} |X|^{2n} \psi(t, x, X) \, dX.$$

Using the assumption (3.2), we get

$$\frac{\partial m_{2n}(\psi)}{\partial t} + u \cdot \nabla_x m_{2n}(\psi) \leq \, n(2D - 1) m_{2n}(\psi) + n(2n - 2 + d) m_{2n-2}(\psi)$$

for all $n \in \mathbb{N}$. Assuming that the inhomogeneity is small enough, i.e., $D < \frac{1}{2}$, then we get

$$\frac{d}{dt} M_{2n}(\psi) \leq -A_n M_{2n}(\psi) + B_n M_{2n-2}(\psi) \, ,$$

where $M_{2n}(\psi)(t, x) = m_{2n}(\psi)(t, \Phi_t(x))$ with $\Phi_t$ the flow map associated to the velocity field $u$, i.e.,

$$\begin{cases} \dfrac{d\Phi_t(x)}{dt} & = u(t, \Phi_t(x)) \qquad\qquad t \geq 0 \, , \\ \Phi_0(x) & = x \qquad\qquad\qquad\quad x \in \mathbb{R}^d \, . \end{cases}$$

Now, a simple induction argument starting at $n = 1$ for which $M_0(\psi) = 1$, implies that

$$M_{2n} := \max_{0 \leq t < \infty} \left\| \int_{\mathbb{R}^d} |X|^{2n} \psi(t, x, X) \, dX \right\|_{L^\infty(\Omega)} < \infty \tag{3.5}$$

if initially $m_{2n}(\psi_0) \in L^\infty(\Omega)$, i.e.,

$$\left\| \int_{\mathbb{R}^d} |X|^{2n} \psi_0(x, X) \, dX \right\|_{L^\infty(\Omega)} < \infty. \tag{3.6}$$

Actually, the assumption $D < \frac{1}{2}$ involving (3.2) could be slightly weakened to

$$\sup_{0<t<\infty} \sup_{x\in\Omega} \lambda_{\max}(\nabla_x \otimes u^s(t,x)) < \frac{1}{2},$$

which is a closely related analogue of $2\kappa - I$ having to be a stable matrix (cf. the condition in Prop. 2.1).

Now, with these estimates together with the entropy-entropy dissipation procedure applied in [JLLO, §3.3.1], we can deduce an exponential convergence result towards homogeneous stationary flows in the Hookean case with small enough initial data. Now we define the total relative entropy of a solution of the coupled model $(u, \psi)$ to the homogeneous stationary flow $(u_\infty, \psi_\infty)$ as:

$$E(t) = \frac{1}{2}\int_\Omega |\bar{u}(t)|^2\, dx + \int_\Omega \int_{\mathbb{R}^d} \psi(t) \ln\left(\frac{\psi(t)}{\psi_\infty}\right) dX\, dx,$$

with $\bar{u} := u - u_\infty$. Then one has the following formula for its evolution, see [JLLO, Appendix A]:

$$\frac{dE}{dt} + \int_\Omega |\nabla_x \otimes \bar{u}|^2\, dx + \frac{1}{2}\int_\Omega \int_{\mathbb{R}^d} \psi \left|\nabla_X \ln\left(\frac{\psi}{\psi_\infty}\right)\right|^2 dX\, dx = \tag{3.7}$$
$$- \int_\Omega \bar{u}^T \kappa \bar{u}\, dx - \int_\Omega \int_{\mathbb{R}^d} \nabla_X \ln\left(\frac{\psi_\infty}{e^{-\Pi(X)}}\right)^T (\nabla_x \otimes \bar{u}) X\, \bar{\psi}\, dX dx$$

with $\bar{\psi} := \psi - \psi_\infty$, $\Pi(X) = \frac{1}{2}|X|^2$.

**Theorem 3.2** *Let us consider the homogeneous stationary flow $(u_\infty = \kappa x, \psi_\infty)$ with the traceless matrix $\kappa$ satisfying $\Re\lambda_j(\kappa) < \frac{1}{2}$, $j = 1, ..., d$ and $\psi_\infty$ given by (2.11). Let $(u, \psi)$ be a given, smooth, fast-decaying at infinity solution to the system (3.1) with non-homogeneous Dirichlet boundary conditions such that the deformation matrix is uniformly bounded (in the sense of (3.2)). Then, the solution converges exponentially fast towards $(u_\infty, \psi_\infty)$, provided a "smallness" condition holds for the solution and $\kappa^s$ as specified in (3.8). More precisely, u converges exponentially fast to $u_\infty$ in $L^2(\Omega)$ and the total relative entropy of $\psi$ w.r.t. $\psi_\infty$ converges exponentially fast to 0.*

**Proof.-** We proceed as in [JLLO, Theorem 1] for the FENE case. The potential $\frac{1}{2}X^T\Sigma^{-1}X = C - \ln\psi_\infty$ (cf. (2.11)) satisfies the Bakry-Emery condition with constant $\lambda = \lambda_{min}(\Sigma^{-1})$. Using the resulting Logarithmic Sobolev inequality for the measure $\psi_\infty dX$ in (3.7) we obtain

$$\frac{dE}{dt} + \int_\Omega |\nabla_x \otimes \bar{u}|^2\, dx + \lambda \int_\Omega \int_{\mathbb{R}^d} \psi \ln\left(\frac{\psi}{\psi_\infty}\right) dX\, dx$$
$$\leq |\kappa^s| \int_\Omega |\bar{u}|^2\, dx + \int_\Omega |\nabla_x \otimes \bar{u}| \int_{\mathbb{R}^d} \left|\nabla_X \ln\left(\frac{\psi_\infty}{e^{-\Pi(X)}}\right)\right| |X|\, |\bar{\psi}|\, dX dx.$$

Applying Young's inequality with $\epsilon < 1$ to be chosen, using the Poincaré inequality on $\Omega$, and the explicit formula of $\psi_\infty$, we get

$$\frac{dE}{dt} + \frac{1-\epsilon}{C_P^2}\int_\Omega |\bar{u}|^2\, dx + \lambda \int_\Omega \int_{\mathbb{R}^d} \psi \ln\left(\frac{\psi}{\psi_\infty}\right) dX\, dx \leq$$
$$|\kappa^s| \int_\Omega |\bar{u}|^2\, dx + \frac{|I - \Sigma^{-1}|^2}{4\epsilon} \int_\Omega \left(\int_{\mathbb{R}^d} |X|^2\, |\bar{\psi}|\, dX\right)^2 dx.$$

Now, we apply the inequality (3.3) proved in Lemma 3.1 to obtain

$$\frac{dE}{dt}+\frac{1-\epsilon}{C_P^2}\int_\Omega |\bar{u}|^2\,dx+\lambda\int_\Omega\int_{\mathbb{R}^d}\psi\ln\left(\frac{\psi}{\psi_\infty}\right)dX\,dx\le$$
$$|\kappa^s|\int_\Omega|\bar{u}|^2\,dx+\frac{|I-\Sigma^{-1}|^2}{2\epsilon}\max(M_4,M_4^\infty)\int_\Omega\int_{\mathbb{R}^d}\psi\ln\left(\frac{\psi}{\psi_\infty}\right)dX\,dx,$$

where $M_4$ is given by (3.5) and $M_4^\infty$ is the fourth moment of $\psi_\infty$. From here it is clear that, if $\epsilon$ can be chosen such that

$$\epsilon<1-C_P^2|\kappa^s|\ \text{ and }\ \epsilon>\frac{|I-\Sigma^{-1}|^2}{2\lambda}\max(M_4,M_4^\infty),$$

or equivalently if

$$C_P^2|\kappa^s|+\frac{|I-\Sigma^{-1}|^2}{2\lambda}\max(M_4,M_4^\infty)<1\,, \tag{3.8}$$

then exponential convergence holds. ■

The previous result is the main new addition to the results in [JLLO] concerning the long time asymptotics of the coupled problem. Concerning the FENE case, we have shown in §2.4 the existence of the stationary state $\psi_\infty$ in several new situation not covered in [JLLO]. Those stationary states $\psi_\infty$ that eventually verify the bounds (2.40), give rise to a Logarithmic Sobolev inequality via a Holley-Stroock perturbation argument. In these cases we can derive the corresponding long time asymptotics result of the coupled problem just by repeating the proof of [JLLO, Theorem 1] or [JLLO, Proposition 9].

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