Generalized Hessians of $C^{1,1}$ - functions and second-order viscosity subjets

Luc BARBET, Aris DANIILIDIS, Pierpaolo SORAVIA

Abstract. Given a $C^{1,1}$–function $f : U \to \mathbb{R}$ (where $U \subset \mathbb{R}^n$ open) we deal with the question of whether or not at a given $x_0 \in U$ there exists a local minorant φ of f of class C^2 that satisfies $\varphi(x_0) = f(x_0)$, $D\varphi(x_0) = Df(x_0)$ and $D^2\varphi(x_0) \in \mathcal{H}f(x_0)$ (the generalized Hessian of f at x_0). This question is motivated by the second-order viscosity theory of the PDE, since for nonsmooth functions, an analogous result between subgradients and first-order viscosity subjets is known to hold in every separable Asplund space. In this work we show that the aforementioned second–order result holds true whenever $\mathcal{H}f(x_0)$ has a minimum with respect to the semidefinite cone (thus in particular, in one dimension), but it fails in two dimensions even for piecewise polynomial functions. We extend this result by introducing a new notion of directional minimum of $\mathcal{H}f(x_0)$.

Key words. Nonsmooth analysis, generalized Hessian, second-order viscosity subjet.

AMS Subject Classification. *Primary* 49J52 ; *Secondary* 49L25, 49J53.

1 Introduction

Rademacher's theorem asserts that every Lipschitz continuous function $f : U \to \mathbb{R}^k$, where U is a nonempty open subset of $\mathbb{R}^n$, is almost everywhere differentiable with respect to the Lebesgue measure, thus in particular, the set $\operatorname{dom} Df$ of points where the derivative of f exists is dense in U. Given any $x_0 \in U$, one of the primary notions of standard variational analysis is the notion of *generalized Jacobian* $Jf(x_0)$ (respectively, *subdifferential* $\partial f(x_0)$, if $k = 1$, *i.e.* $f : U \to \mathbb{R}$) defined as follows:

$$Jf(x_0) := \left\{ q \in \mathbb{R}^{n\times k} : \, q = \lim_{x_n \to x_0} Df(x_n), \, x_n \in \operatorname{dom} Df \right\}, \tag{1}$$

that is, $Jf(x_0)$ is the set of all possible limits of derivatives of f (identified to their Jacobian matrix) at points in $\operatorname{dom} Df$ converging to x_0 (see for example [2], [8], [10], [7]). Note that the fact that the derivatives exist in a dense set together with the boundedness of their norms (by the Lipschitz constant of f) guarantee that the generalized Jacobian $Jf(x_0)$ is always nonempty and compact. Another standard notion in the literature is the notion of the *Clarke Jacobian* [2, Section 2.6], denoted by $J^o f(x_0)$, (respectively, *Clarke subdifferential* $\partial^o f(x_0)$) which is defined as the convex hull of $Jf(x_0)$.

In case of real–valued functions, the Clarke subdifferential is known to contain the so–called *Fréchet* (or *regular* [16]) subdifferential

$$\hat{\partial} f(x_0) := \left\{ p \in \mathbb{R}^n : \, \liminf_{x \to x_0,\, x \neq x_0} \frac{f(x) - f(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \geq 0 \right\}. \tag{2}$$

The above subdifferential $\hat{\partial} f(x_0)$ is sometimes called *viscosity* subdifferential, because in every separable Asplund space (more generally, in every Banach space admitting a Fréchet differentiable renorming) the elements of $\hat{\partial} f(x_0)$ can be identified to the *first–order viscosity subjets* in the following sense (see [1], [6], [9] and references therein):

$p \in \hat{\partial} f(x_0)$ if, and only if, there exists a C^1 local minorant φ of f satisfying

$$\begin{cases} \varphi(x_0) = f(x_0) \\ D\varphi(x_0) = p. \end{cases} \tag{3}$$

In the above formula, since x_0 is a local minimizer of the (nonsmooth) function $f - \varphi$, the second line of (3) actually interprets the first-order necessary optimality condition $0 \in \hat{\partial}(f - \varphi)(x_0)$. At the same time, formula (3) relates subdifferential theory to the notion of first–order viscosity subsolutions of **P**artial **D**ifferential **E**quations (see [3] e.g.). The last two decades the second–order viscosity theory has been developed to cover the fully nonlinear partial differential equations (see [4] for details). A natural question is thus arising, on whether or not the previous correspondence can be extended to the second–order setting. This has been one of the motivations of this work.

In the sequel, $f : U \subset \mathbb{R}^n \to \mathbb{R}$ denotes a $C^{1,1}$ function (*i.e.*, f everywhere differentiable and $Df : U \to \mathbb{R}^n$ locally Lipschitz continuous). Then formula (1) applied to the derivative function Df gives the so–called *generalized Hessian* of f at x_0

$$\mathcal{H}f(x_0) := \left\{ A \in \mathbb{R}^{n\times n} : A = \lim_{x_n \to x_0} D^2 f(x_n),\, x_n \in \operatorname{dom} D^2 f \right\}, \tag{4}$$

and its convex envelope defines the *Clarke Hessian* $\mathcal{H}^o f(x_0)$. In the literature of nonsmooth analysis there are many works on this topic, see for example [7], [15], [10] and references therein. See also [14] for applications to positive semidefinite optimization, as well as [11], [12] for some generalizations. In this setting, a second-order result —analogous to (3)— reads as follows:

($\mathcal{Q}$) Does there exist a local minorant φ of f around x_0 of class C^2 satisfying

$$\text{(i)} \quad \varphi(x_0) = f(x_0) \qquad \text{(ii)} \quad D\varphi(x_0) = Df(x_0) \qquad \text{(iii)} \quad D^2\varphi(x_0) \in \mathcal{H}f(x_0)\ ?$$

Since x_0 is a local minimizer of the function $f - \varphi$, a relaxation of (iii) to the following condition

$$\text{(iv)} \quad \text{for some } A \in \mathcal{H}f(x_0) \text{ we have} \quad A \succeq D^2\varphi(x_0) \qquad \text{(cone relaxation)}$$

would correspond to a second-order necessary optimality condition for $f - \varphi$. (As usual, the notation $A \succeq D^2\varphi(x_0)$ means that the (symmetric) matrix $A - D^2\varphi(x_0)$ has non-negative eigenvalues.) This weaker version of ($\mathcal{Q}$) will be always true (Remark 10). However, ($\mathcal{Q}$) is much more demanding, since it asserts equality: $A = D^2\varphi(x_0)$.

The above question has a positive answer whenever the generalized Hessian has a minimum element (Corollary 4), that is, whenever there exists $A_* \in \mathcal{H}f(x_0)$ such that for all $A \in \mathcal{H}f(x_0)$ we have $A \succeq A_*$. This is always the case in dimension one (Corollary 5), but notice that the element of $\mathcal{H}f(x_0)$ that satisfy ($\mathcal{Q}$) might hold be unique (Example 6). In $\mathbb{R}^n$ the existence of a minimum for the generalized Hessian is a very restrictive condition and can only guaranteed in very particular cases (Theorem 17).

Our sufficient condition is then extended to the case where $\mathcal{H}f(x_0)$ admits a *directional minimum* (Theorem 3), a notion that we make precise in Section 2. In what follows we show that directional minima may not exist (Example 7) or may not be unique (Example 8), while under an additional regularity assumption, they characterize the elements of $\mathcal{H}f(x_0)$ for which $(\mathcal{Q})$ has a positive answer (Theorem 23).

However $(\mathcal{Q})$ has a negative answer in general: we provide an example of a $C^{1,1}$ function in $\mathbb{R}^2$ for which $(\mathcal{Q})$ fails (Example 9), even if condition (iii) is relaxed as follows:

$$\text{(iii)}' \quad D^2\varphi(x_0) \in \mathcal{H}^o f(x_0) \qquad \text{(Clarke relaxation).}$$

At this stage, let us observe that a stronger statement asserting that all generalized Hessians can be represented as second derivatives of such minorants would not have any chance to hold. To see this, let us consider the following simple one-dimensional example:

$$f(x) = \begin{cases} \frac{1}{2}x^2, & \text{if } x \geq 0 \\ x^2, & \text{if } x \leq 0. \end{cases} \tag{5}$$

Indeed, one easily sees that $\mathcal{H}^o f(0) = [1, 2]$ while any C^2 minorant φ of f with $\varphi(0) = f(0) = 0$ and $\varphi'(0) = f'(0) = 0$ should necessarily satisfy $\varphi''(0) \leq 1$. This example seems to indicate that in the one-dimensional case —where $(\mathcal{Q})$ has a positive answer— the only element of the Clarke Hessian with this property is its minimum. Even if this comes about to hold for most functions we meet in practice, an easy example reveals that this assertion is not true in general (Example 6). Example 7 reflects the same situation of non-uniqueness in n-dimensions but the generalized Hessian does not have directional minima. In Example 16, all matrices in the generalized Hessian are diagonal, there is no minimum element and $(\mathcal{Q})$ is satisfied by a unique element.

Notation. In the sequel, for any set $K \subset \mathbb{R}^m$, we denote by $\overline{K}$ (respectively, $\text{int}K$) the topological closure (respectively, interior) of K. Further, given $t \in \mathbb{R}$ and $a \in \mathbb{R}^m$ we shall use the abbreviate notation $a + tK$ for the set $\{a + tx : x \in K\}$ and we denote by $B(x, r)$ the closed ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$. We denote by $\|x\|$ the norm of $x \in \mathbb{R}^n$ and by $\langle x, y \rangle$ the scalar product of $x, y \in \mathbb{R}^n$. Let $W^\top$ stand for the transpose of any $n \times k$ matrix W. Under this notation, identifying the vector $x \in \mathbb{R}^n$ with an $n \times 1$ matrix, we have $x^\top \cdot x = \langle x, x \rangle = \|x\|^2 \in \mathbb{R}$, while $x \cdot x^\top$ gives rise to a symmetric $n \times n$ matrix.

Further, given a real–valued function f defined on some subset of $\mathbb{R}^m$, we shall denote by $\text{dom}\, f$ its domain and by $Df(x)$ (respectively, $D^2f(x)$) the first (respectively, second) derivative of the function f at a point $x \in \mathbb{R}^m$, whenever they exist. We also denote by $\text{dom}\, Df$ and $\text{dom}\, D^2f$ the domains of Df and D^2f respectively. In case $m = 1$, the simplified notation $f'(x)$ (respectively, $f''(x)$) will be in use. If the domain of f is a closed interval $[a, b]$ of $\mathbb{R}$ we say that f is (twice) differentiable at $x = a$ if the (second) right derivative exists, and we keep the same notation $f'(a)$ (resp. $f''(a)$) to denote the right derivatives.

Let further S^n denote the set of symmetric matrices. This set is partially ordered by the cone S^n_+ of positive semidefinite matrices: for $A, B \in S^n$ we write $A \succeq B$ whenever $A - B \in S^n_+$. Given $K \subset S^n$, if there exists $A_* \in K$ with the property $K \subset A_* + S^+_n$, then A_* is the minimum of K with respect to the aforementioned order and will be denoted by $\min K$ (note that $\min K$ might not exist).

2 Main results

We consider the set of local C^2 minorants of f at x_0 coinciding with f at zero (and first) order at x_0,

$$\mathcal{M}_f(x_0) \,=\, \left\{ \varphi \in C^2(V) : \varphi \le f,\, \varphi(x_0) = f(x_0) \right\}, \tag{6}$$

where V is some neighborhood of x_0. Note that $\varphi \in \mathcal{M}_f(x_0)$ implies $D\varphi(x_0) = Df(x_0)$. According to [4], the set of *second-order subjets* of f at x_0 is defined by

$$\mathcal{J}_f(x_0) = \{ (D\varphi(x_0), D^2\varphi(x_0)) : \varphi \in \mathcal{M}_f(x_0) \} \subset \mathbb{R}^n \times S^n.$$

In our analysis we shall rather use the projection of the above set onto S^n, that is,

$$\mathcal{J}_f^2(x_0) \,=\, \left\{ D^2\varphi(x_0) : \varphi \in \mathcal{M}_f(x_0) \right\}. \tag{7}$$

With this notation, $(\mathcal{Q})$ can be reformulated as follows:

$(\mathcal{Q})$ Is it true that for any $C^{1,1}$ function f and any $x_0 \in \operatorname{dom} f$

$$\mathcal{H}f(x_0) \cap \mathcal{J}_f^2(x_0) \neq \emptyset\, ?$$

Note that $(\mathcal{Q})$ obviously yields $\mathcal{H}^o f(x_0) \cap \mathcal{J}_f^2(x_0) \neq \emptyset$ and that, thanks to the convexity of $\mathcal{H}^o f(x_0)$ and $\mathcal{J}_f^2(x_0)$, the latter is always a convex set.

We first deal with a positive result to the problem $(\mathcal{Q})$ in the case that the generalized Hessian of f at x_0 has *directional minimum.* Let us define formally this notion, for any $C^{1,1}$ function $f : \mathbb{R}^n \to \mathbb{R}$. Given $X \in \mathcal{H}f(x_0)$ we denote

$$\mathcal{T}_{f,x_0}(X) = \{v \in \mathbb{R}^n : \exists \{x_n\}_n \subset \operatorname{dom} D^2 f,\;\; x_n \to x_0,\; D^2 f(x_n) \to X,\; \frac{x_n - x_0}{\|x_n - x_0\|} \to v\}.$$

Definition 1 (Directional minimum element of a generalized Hessian)**.** We say that $M \in \mathcal{H}f(x_0)$ is a directional minimum of $\mathcal{H}f(x_0)$ if for all $X \in \mathcal{H}f(x_0)$ and all $v \in \mathcal{T}_{f,x_0}(X)$ we have

$$\langle Mv, v\rangle \,\le\, \langle Xv, v\rangle\,.$$

Obviously, a minimum element of $\mathcal{H}f(x_0)$ (with respect to the cone S^n_+ of the positive semidefinite matrices) is necessarily a directional minimum, since the above inequality should then hold for all $v \in \mathbb{R}^n$, and not only for those in $\mathcal{T}_{f,x_0}(X)$. Thus the notion of directional minimum is weaker. Notice that since $f \in C^{1,1}$, for all unit vectors v we have

$$K_{f,x_0}(v) := \mathcal{T}_{f,x_0}^{-1}(v) \,=\, \{X \in \mathcal{H}f(x_0) : v \in \mathcal{T}_{f,x_0}(X)\} \neq \emptyset\,.$$

In particular, M is a directional minimum of $\mathcal{H}f(x_0)$ if and only if for all $v \in \mathbb{R}^n$ and $X \in K_{f,x_0}(v)$ we have

$$\langle Mv, v\rangle \,\le\, \langle Xv, v\rangle\,.$$

We specify better how to construct elements in $K_{f,x_0}(v)$, using the following notation: for a unit vector $v \in \mathbb{R}^n$ define

$$C_v = \{y \in \mathbb{R}^n : \langle y, v\rangle > 0, \langle y, v\rangle^2 > \|y - \langle y, v\rangle\, v\,\|\}\,.$$

Notice that C_v is a nonempty open set satisfying $0 \in \overline{C_v} \setminus C_v$ and $tv \in C_v$ for all $t > 0$.

Lemma 2 (Convergence within a prescribed tangent)**.** *If $\{x_n\}_n \subset C_v$ is such that $x_n \to 0$, then $x_n/\|x_n\| \to v$.*

Proof. Let $\{x_n\}_n$ be a sequence as in the statement, then by definition of C_v if we set

$$r_n = \langle \frac{x_n}{\|x_n\|}, v\rangle \in [0,1]$$

we obtain

$$1 \geq r_n^2 > \frac{1}{\|x_n\|}\left\| \frac{x_n}{\|x_n\|} - \langle \frac{x_n}{\|x_n\|}\,,\, v\rangle\, v \right\| \geq \frac{1}{\|x_n\|}(1-r_n) \geq 0.$$

As $n \to +\infty$ we get the conclusion. □

In the sequel we define for all $x_0 \in \mathbb{R}^n$, $r > 0$ and $v \in \mathbb{R}^n$ with $\|v\| = 1$ the set

$$C^r_{x_0,v} = (x_0 + C_v) \cap B(x_0, r)\,.$$

Our main sufficient condition for solving problem $(\mathcal{Q})$ is contained in the following result.

Theorem 3 (Hessian with directional minimum element)**.**
Let U be an open subset of $\mathbb{R}^n$, $x_0 \in U$ and $f : U \to \mathbb{R}$ be a $C^{1,1}$ function.
(i) If a matrix $M \in S^n$ satisfies $\langle Mv, v\rangle \leq \langle Xv, v\rangle$ for all $X \in \mathcal{H}f(x_0)$ and $v \in \mathcal{T}_{f,x_0}(X)$, then there exists $\varphi \in \mathcal{M}_f(x_0)$ such that $M = D^2\varphi(x_0)$.
(ii) If in addition $M \in \mathcal{H}f(x_0)$ (ie. M is a directional minimum of $\mathcal{H}f(x_0)$), then $(\mathcal{Q})$ holds.

Proof. There is no loss of generality to assume that U is convex and that $f(x_0) = 0$ and $Df(x_0) = 0$. Consider now some $r_0 > 0$ such that Df is Lipschitz continuous on $B(x_0, r_0) \subset U$. Let $M \in S^n$ be as in the statement and fix $d \in \mathbb{R}^n$ with $\|d\| = 1$ and $r \in [0, r_0]$. We define

$$\varphi_d(r) := \begin{cases} \inf\limits_{x \in C^r_{x_0,d}} \; \inf\limits_{T \in \mathcal{H}f(x)\cup\{M\}} \langle Td, d\rangle, & \text{if } r > 0 \\[2ex] \langle Md, d\rangle, & \text{if } r = 0\,. \end{cases} \tag{8}$$

Note that φ_d is a real-valued nonincreasing function, so that

$$\varepsilon_d(r) := \varphi_d(0) - \varphi_d(r) \geq 0$$

defines a nondecreasing function which is in fact upper bounded by a constant independent of d. Using the upper semicontinuity of the multifunction $x \mapsto \mathcal{H}f(x)$ at x_0 we deduce that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathcal{H}f(B(x_0, \delta)) \subset \mathcal{H}f(x_0) + B(0, \varepsilon).$$

We now claim, shrinking $\delta > 0$ if necessary, that the following stronger conclusion holds:

$$x \in C^\delta_{x_0,d} \implies \mathcal{H}f(x) \subset K_{f,x_0}(d) + B(0,\varepsilon)\,.$$

Indeed if this is not the case, we can find $\bar{\varepsilon} > 0$ and a sequence $\{x_n\}_n \subset C^{r_0}_{x_0,v}$ satisfying $x_n \neq x_0$, $x_n \to x_0$ and $(x_n - x_0)/\|x_n - x_0\| \to d$ such that there exist $A_n \in \mathcal{H}f(x_n)$ with $A_n \notin K_{f,x_0}(d) + B(0, \bar{\varepsilon})$. By definition of $\mathcal{H}f(x_n)$ we can slightly modify the sequence $\{x_n\}_n$ to a sequence $\{y_n\}_n \subset C^{r_0}_{x_0,d} \cap \operatorname{dom} D^2 f$ so that $y_n \to x_0$ and $A_n = D^2 f(y_n)$. Taking a subsequence if necessary, we may assume $A_n \to X \in S^n$. Using the upper semicontinuity of $x \mapsto \mathcal{H}f(x)$ we deduce $X \in \mathcal{H}f(x_0)$, which in view of the fact that $(y_n - x_0)/\|y_n - x_0\| \to d$ yields $X \in K_{f,x_0}(d)$. This provides a contradiction.

For some $x \in C^{\delta}_{x_0,d}$ and $T \in \mathcal{H}f(x) \cup \{M\}$ (both depending on ε and d) we have $\varphi_d(\delta) \geq \langle Td, d\rangle - \varepsilon$. Let us assume that $T \in \mathcal{H}f(x)$ (the case $T = M$ is simpler). Since T can be written as $T_0 + \varepsilon T_1$ where $T_0 \in K_{f,x_0}(d)$ and $T_1 \in S^n \cap B(0,1)$, we get $\varphi_d(\delta) \geq \langle T_0 d, d\rangle + \varepsilon\langle T_1 d, d\rangle - \varepsilon \geq \langle Md, d\rangle - 2\varepsilon$, *i.e.*, $\varepsilon_d(\delta) \leq 2\varepsilon$. We conclude that the function

$$\varepsilon(r) := \sup_{\|d\|=1} \varepsilon_d(r) \tag{9}$$

(which is real-valued and nondecreasing) satisfies:

$$\lim_{r \to 0} \varepsilon(r) = 0. \tag{10}$$

Consider the function

$$f_1(x) := f(x) - \frac{1}{2}\langle M(x - x_0), x - x_0\rangle \qquad \text{for } x \in U.$$

Note that f_1 is $C^{1,1}$, $f_1(x_0) = 0$ and $Df_1(x_0) = 0$. Further, fix any $x \in U \setminus \{x_0\}$ (with $[x_0, x] \subset U$) and apply the second-order expansion of f_1 (*cf.* [7, Theorem 2.3]). Then for some $z \in (x_0, x)$ and $S \in \operatorname{co} \mathcal{H}f_1(z)$ we have

$$f_1(x) - f_1(x_0) \;=\; \frac{1}{2}\,\langle S(x - x_0), x - x_0\rangle \;=\; \frac{1}{2}\,\langle Sd, d\rangle\,\|x - x_0\|^2, \tag{11}$$

where $d := (x - x_0)/\|x - x_0\|$. Then there exist $\{\lambda_i\}_{i=1}^k \subset [0,1]$ with $\sum_{i=1}^k \lambda_i = 1$ and $\{T_i\}_{i=1}^k \subset \mathcal{H}f(z)$ such that

$$S = \sum_{i=1}^k \lambda_i (T_i - M).$$

Set $r := \|x - x_0\| \geq \|z - x_0\|$. From (8), (9) and (11) we deduce

$$f_1(x) - f_1(x_0) \;=\; \frac{1}{2}\,\sum_{i=1}^k \lambda_i \left(\langle T_i d, d\,\rangle - \langle Md, d\,\rangle\right) \|x - x_0\|^2 \;\geq\; \frac{1}{2}\left(\varphi_d(r) - \varphi_d(0)\right) r^2 \geq -\varepsilon(r) r^2,$$

which in view of Lemma 25 (Appendix) yields

$$f_1(x) - f_1(x_0) \;\geq\; -\psi(\|x - x_0\|),$$

where $\psi(0) = \psi'(0) = \psi''(0) = 0$ for some C^2 function ψ, and whenever $\|x - x_0\|$ sufficiently small. Defining

$$x \mapsto \varphi(x) := \frac{1}{2}\langle M(x - x_0), x - x_0\rangle - \psi(\|x - x_0\|)$$

we deduce that $\varphi \in \mathcal{M}_f(x_0)$ and $D^2\varphi(x_0) = M \in \mathcal{J}^2_f(x_0)$. $\square$

Corollary 4 (Hessian with minimum element)**.** *Let U be an open convex subset of $\mathbb{R}^n$, $x_0 \in U$ and $f : U \to \mathbb{R}$ be a $C^{1,1}$ function. If $\mathcal{H}f(x_0)$ has a minimum element, then ($\mathcal{Q}$) holds.*

In the one-dimensional case ($n = 1$), since $\mathcal{H}^o f(x_0) = \mathrm{co}\,\mathcal{H}f(x_0)$ is a segment, the subset $\mathcal{H}f(x_0)$ has a minimum thus the previous result applies.

Corollary 5 (One-dimensional case)**.** *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a $C^{1,1}$ function and $x_0 \in I$. Then for some $\varphi \in \mathcal{M}_f(x_0)$ we have*

$$\varphi''(x_0) = \min \mathcal{H}f(x_0).$$

The above corollary might lead to the erroneous conclusion that the only element of the Clarke Hessian that can be represented by a subjet is its minimum (see also the example given by (5) in the introduction). In fact this is true for piecewise C^2 functions (in this case for some $\delta > 0$ the restrictions of f onto $[x_0, x_0 + \delta)$ and $(x_0 - \delta, x_0]$ are C^2 and $\mathcal{H}f(x_0)$ contains at most two elements) but fails for $C^{1,1}$ functions with oscillating second derivatives, as the following example shows:

Example 6 (Non-uniqueness of solution of ($\mathcal{Q}$) in $\mathbb{R}$)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be the following $C^{1,1}$ function:

$$f(x) = \int_0^x g(s)ds \qquad \text{with} \qquad g(s) = \begin{cases} s^2 \sin(\frac{1}{s}) & \text{if } s \neq 0 \\ 0 & \text{if } s = 0. \end{cases} \tag{12}$$

Note that $f''(0) = 0$ (but f is not C^2 around 0) and $\mathcal{H}f(0) = [-1, 1]$. Since f admits a second–order Taylor series around $x = 0$, we deduce using Lemma 25 that there exists $\varphi \in \mathcal{M}_f(0)$ with $\varphi''(0) = 0$. It follows easily that

$$\mathcal{H}f(0) \cap \mathcal{J}_f^2(0) = [-1, 0]\,,$$

showing that the element of $\mathcal{H}f(x_0)$ that satisfies ($\mathcal{Q}$) need not be unique. □

A slight modification of the above example provides an example in $\mathbb{R}^n$ which proves that the existence of a (directional) minimum element of the generalized Hessian is not a necessary condition for ($\mathcal{Q}$) to hold.

Example 7 (($\mathcal{Q}$) holds but $\mathcal{H}f(x_0)$ has no directional minima)**.** Let $F : \mathbb{R}^n \to \mathbb{R}$ be the following function ($n \geq 2$):

$$F(x) = f(\|x\|), \qquad x \in \mathbb{R}^n$$

where f is given by (12). Then it can be easily verified that F is twice differentiable everywhere, with

$$D^2F(x) = \begin{cases} g'(\|x\|)\, u\, u^\top + (g(\|x\|)/\|x\|)\ (Id_n - u\, u^\top), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

where $u := x/\|x\|$, Id_n denotes the $n \times n$-identity matrix and $u^\top$ stands for the transposed of the (column) vector u. It follows easily that F is $C^{1,1}$ (but not C^2 around 0) and

$$\mathcal{H}F(0) = \{t\, v\, v^\top : t \in [-1, 1]\,, \|v\| = 1\}.$$

Moreover, for $\|v\| = 1$ we have

$$K_{F,0}(u) = \{t\, u\, u^\top : t \in [-1, 1]\}\,.$$

We first remark that $\mathcal{H}F(0)$ does not possess a directional minimum element. Indeed, whenever w is a unit vector normal to v, each element of $\mathcal{H}F(0)$ of the form $A = t\,v\,v^\top$ is never a minorant of $B = -w\,w^\top \in K_{F,0}(w)$ in the direction of w.

Consider now some $\varphi \in \mathcal{M}_f(0)$ with $\varphi''(0) = 0$ (see Example 6); we define $\Phi := (\varphi \circ \|.\|)$ and we observe that Φ is of class C^2, with

$$D^2\Phi(x) = \begin{cases} \varphi''(\|x\|)\,u\,u^\top + (\varphi'(\|x\|)/\|x\|)\,(Id_n - u\,u^\top), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Thus $\Phi \in \mathcal{M}_F(0)$ and $D^2\Phi(0) = 0$, yielding

$$0 \in \mathcal{H}F(0) \cap \mathcal{J}^2_F(0).$$

We also remark that the null matrix is not the unique solution. We can easily characterize all solutions of $(\mathcal{Q})$: a solution $t\,v\,v^\top$ satisfies $t \in [-1, 0]$. Conversely, if $t \in [-1, 0)$ we set $\Phi(x) = \varphi(\|x\|) + \frac{1}{2}\,t\,\langle v, x\rangle^2$, and observe that $D^2\Phi(0) = t\,v\,v^\top$. Thus,

$$\mathcal{H}F(0) \cap \mathcal{J}^2_F(0) = \{t\,v\,v^\top : t \in [-1, 0]\,, \|v\| = 1\}.$$

$\square$

We provide a further example to illustrate Theorem 3 and the fact that directional minima of $\mathcal{H}f(x_0)$, when they do exist, may not be unique.

Example 8 (Non-uniqueness of directional minima)**.** Consider the following regions in the first quadrant of $\mathbb{R}^2$

$$\begin{aligned} \Omega_1 &:= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 \in (0, \tfrac{1}{2}x_1)\right\} \\ \Omega_2 &:= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 \in (\tfrac{1}{2}x_1, 2x_1)\right\} \\ \Omega_3 &:= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 \in (2x_1, +\infty)\right\}, \end{aligned}$$

and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x_1, x_2) = \begin{cases} \dfrac{3x_2^2}{2} & \text{if } (x_1, x_2) \in \operatorname{cl}\Omega_1 \\ 2x_1x_2 - \dfrac{x_1^2 + x_2^2}{2} & \text{if } (x_1, x_2) \in \operatorname{cl}\Omega_2 \\ \dfrac{3x_1^2}{2} & \text{if } (x_1, x_2) \in \operatorname{cl}\Omega_3 \end{cases} \tag{13}$$

and $f(x_1, x_2) = f(|x_1|, |x_2|)$ for $(x_1, x_2) \in \mathbb{R}^2$. Then we obtain

$$\mathcal{H}f(0, 0) = \left\{ \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} \right\}.$$

The first two matrices are bigger than the last two, which are not mutually comparable. Moreover one easily checks that each of the two matrices

$$\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} \tag{14}$$

is a directional minimum so they are elements of $\mathcal{H}f(0,0) \cap \mathcal{J}_f^2(0,0)$. Also $\varphi \equiv 0 \in \mathcal{M}_f(0,0)$ and $0 = D^2\varphi(0,0) \in \mathcal{J}_f^2(0,0)$ is not comparable with either of the two elements in (14) although from the point of view of optimization φ is the most interesting test function since f has a minimum point at $(0,0)$. □

We shall now present an example of a piecewise quadratic function in $\mathbb{R}^2$ (whose Hessian at a given point x_0 has no directional minima and) for which $(\mathcal{Q})$ has a negative answer.

Example 9 (Failure of $(\mathcal{Q})$ for a $C^{1,1}$-function in $\mathbb{R}^2$ **).** There exists a (piecewise quadratic) $C^{1,1}$ function $f : \mathbb{R}^2 \to \mathbb{R}$ such that for some $\bar{x} \in \mathbb{R}^2$,

$$\mathcal{H}^o f(\bar{x}) \cap \mathcal{J}_f^2(\bar{x}) = \emptyset. \tag{15}$$

(Construction of the counterexample.) Let us consider the following six open subsets of $\mathbb{R}^2$.

$$\begin{aligned}
&\Omega_1 := \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 \in (0, \tfrac{1}{2}x_1)\right\} && \Omega_4 := (-\infty,0) \times (0,+\infty) \\
&\Omega_2 := \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 \in (\tfrac{1}{2}x_1, 2x_1)\right\} && \Omega_5 := (-\infty,0) \times (-\infty,0) \\
&\Omega_3 := \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 \in (2x_1, +\infty)\right\} && \Omega_6 := (0,+\infty) \times (-\infty,0).
\end{aligned}$$

Observe that $\mathbb{R}^2 = \bigcup_{i=1}^6 \operatorname{cl}\Omega_i$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x_1,x_2) = \begin{cases}
\dfrac{3x_2^2}{2} & \text{if } (x_1,x_2) \in \operatorname{cl}\Omega_1 \quad | \quad \dfrac{x_1^2}{4} \quad \text{if } (x_1,x_2) \in \operatorname{cl}\Omega_4 \\[2ex]
2x_1x_2 - \dfrac{x_1^2+x_2^2}{2} & \text{if } (x_1,x_2) \in \operatorname{cl}\Omega_2 \quad | \quad \dfrac{x_1^2+x_2^2}{4} \quad \text{if } (x_1,x_2) \in \operatorname{cl}\Omega_5 \\[2ex]
\dfrac{3x_1^2}{2} & \text{if } (x_1,x_2) \in \operatorname{cl}\Omega_3 \quad | \quad \dfrac{x_2^2}{4} \quad \text{if } (x_1,x_2) \in \operatorname{cl}\Omega_6.
\end{cases} \tag{16}$$

It is straightforward to see that f is well–defined and continuous. On each of the six open subsets Ω_i the function f coincides with a quadratic function. One can easily verify that f is differentiable at each $(x_1,x_2) \in \mathbb{R}^2$ and the derivative function Df is Lipschitz continuous of constant $K := 3$ in $\mathbb{R}^2$ (the spectral radius of each Hessian matrix $D^2f(x_1,x_2)$ is less or equal to 3 for all $(x_1,x_2) \in \bigcup_{i=1}^6 \Omega_i$). Let $\bar{x} = (0,0)$ and note that $f(0,0) = 0$ and $Df(0,0) = (0,0)$, while the generalized Hessian $\mathcal{H}f(0,0)$ consists of the following six elements (corresponding to each of the six quadratic forms in the definition of f):

$$\mathcal{H}f(0,0) := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix} \right\}.$$

Let us show that none of these matrices can be the second derivative of a minorant $\varphi \in \mathcal{M}_f(0,0)$. Indeed, the five diagonal matrices can be easily excluded: if for some $\varphi \in \mathcal{M}_f(0,0)$ one has

$$D^2\varphi(0,0) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}_+$ and at least one of them is (strictly) positive, then since $\varphi(0,0) = f(0,0) = 0$ and $D\varphi(0,0) = Df(0,0) = (0,0)$, the second–order Taylor series of φ around $(0,0)$ would yield

that $f(x_1, x_2) \geq 2^{-1}(\lambda_1 x_1^2 + \lambda_2 x_2^2) + o(x_1, x_2)$, where $\|(x_1, x_2)\|^{-2} o(x_1, x_2) \to 0$. A contradiction arises by considering points either of the form $(t, 0)$ or of the form $(0, t)$, for $|t|$ sufficiently small. The second matrix can also be excluded, since it would correspond to a minorant $\varphi(x_1, x_2) = 2x_1x_2 - 2^{-1}(x_1^2 + x_2^2) + o(x_1, x_2)$ which would fail to be majorated by f on the line $\{(t, t) \in \mathbb{R}^2 : t < 0\}$. For essentially the same reason, no convex combination of the matrices in $\mathcal{H}f(0, 0)$ is representable by a subjet. Thus (15) holds true and the construction is complete.
Notice that $f \geq 0$ and it has a minimum at the origin, in particular $\varphi \equiv 0 \in \mathcal{M}_f(0, 0)$ so that the null matrix is in $\mathcal{J}_f^2(0, 0)$ and it is a maximal element of this set, but it is not an element of $\mathcal{H}^o f(0, 0)$. Nonetheless, $\mathcal{H}f(0, 0)$ contains an element with strictly negative eigenvalue. Indeed the set $\mathcal{H}f(0, 0)$ seems to contain too much information from the point of view of nonsmooth analysis, as it contradicts standard second-order necessary conditions for local minima. □

As already stated in the introduction, the cone relaxation (*iv*) of condition (*iii*) of ($\mathcal{Q}$) always yields a positive answer. This is the aim of the following remark.

Remark 10 (Cone relaxation)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be a $C^{1,1}$ function and $x_0 \in U$. Then for any $M \in \mathcal{J}_f^2(x_0)$ there exists $A \in \mathcal{H}f(x_0)$ such that $A \succeq M$.

Proof. Let us recall that every $C^{1,1}$-function, being lower-C^2 ([5, Remark 3.3] *e.g.*), is semiconvex. Thus the above statement is a simple consequence of Jensen's Lemma (see [4, Appendix]). Let us recall this argument for completeness. Fix any $\varphi \in \mathcal{M}_f(x_0)$ and set $M := D^2\varphi(x_0)$. Then the function $f - \varphi$ attains a local minimum at x_0. Moreover, modifying φ if necessary (but keeping invariant its second derivative at x_0), we may assume that the above minimum is strict and we can construct two sequences $\{x_n\}_n$ and $\{p_n\}_n$ in $\mathbb{R}^n$ with $\{x_n\} \to x_0$ such that f is twice differentiable at x_n, and such that the function

$$x \longmapsto f_n(x) := f(x) - \varphi(x) - \langle p_n, x \rangle$$

has a local minimum at x_n for all $n \in \mathbb{N}$. Applying the second–order optimality condition for the function f_n at x_n we deduce that $D^2 f(x_n) \succeq D^2\varphi(x_n)$ and passing to the limit (for a subsequence) we obtain

$$A := \lim_{x_n \to x_0} D^2 f(x_n) \succeq D^2\varphi(x_0) = M.$$

The proof is complete. □

3 The case of piecewise C^2 functions

Comparing with Theorem 3, Example 9 reveals an important difference in case the dimension of the space is more that one, namely the lack of a total ordering in the elements of $\mathcal{H}f(x_0)$. In particular, neither the minimum element $\min \mathcal{H}f(x_0)$ nor the directional minima of $\mathcal{H}f(x_0)$ necessarily exist, so there is no *a priori* natural candidate to fulfill with ($\mathcal{Q}$). In what follows, we seek for conditions guaranteeing the existence of a minimum (thus of a second-order subjet). In the sequel, the term hypersurface refers to a C^2 submanifold of $\mathbb{R}^n$ of codimension 1.

Definition 11 (C^{1,1} functions that are piecewise C^2)**.** A $C^{1,1}$ function $f : U \to \mathbb{R}$ ($U \subset \mathbb{R}^n$ open) is called piecewise C^2 near $x_0 \in U$ if for some $r > 0$ with $B(x_0, r) \subset U$ there exist finitely many hypersurfaces $\{\Gamma_j\}_j$ such that

(i) $x_0 \in \overline{\Gamma}_j$, for all j ;

(ii) $B(x_0, r) \setminus \cup_j \overline{\Gamma}_j$ has a finite number of connected components $\Omega_1, \ldots, \Omega_k$ with $x_0 \in \overline{\Omega}_i$;

(iii) $f \in C^2(\Omega_i)$ and $x \mapsto D^2 f(x)$ is uniformly continuous on Ω_i, for all $i \in \{1, \ldots, k\}$.

Remark 12. (a) Condition (ii) of Definition 11 is automatically fulfilled whenever the hypersurfaces Γ_j are analytic (or semialgebraic) manifolds.
(b) As a consequence of Definition 11 each function $D^2 f|_{\Omega_i}$ can be continuously extended to $\overline{\Omega}_i$ in a unique manner, yielding that the generalized Hessian $\mathcal{H}f(x_0)$ consists of the (not necessarily distinct) matrices $\{A_1, \ldots, A_k\}$, each of which is obtained as a (unique) limit coming from each region Ω_i, that is,

$$A_i := \lim_{\substack{x \to x_0 \\ x \in \Omega_i}} D^2 f(x) \quad \text{for } i \in \{1, \ldots, k\}.$$

$\square$

Example 13 (Particular case: $k = 2$)**.** Let $f \in C^{1,1}(U)$ and assume that for some $r > 0$ with $B(x_0, r) \subset U \subset \mathbb{R}^n$ there exist two open, nonempty, connected sets Ω^+, Ω^- and a hypersurface $\Gamma \ni x_0$ such that $\Omega^+ \cup \Gamma$ and $\Omega^- \cup \Gamma$ are manifolds with boundary, $f \in C^2(\Omega^+ \cup \Gamma) \cup C^2(\Omega^- \cup \Gamma)$ and for all $x \in B(x_0, r)$ one and only one of the following occurs:

$$x \in \Omega^+, \quad x \in \Omega^-, \quad x \in \Gamma.$$

In this case

$$\mathcal{H}f(x_0) = \{D^2 f(x_0^+), D^2 f(x_0^-)\},$$

where $D^2 f(x_0^+), D^2 f(x_0^-)$ are the limits at x_0 of Hessians of f obtained from each side of the hypersurface Γ.

In the situation of the previous example the generalized Hessian at x_0 has a minimum element, providing thus (in view of Theorem 3) a positive answer for ($\mathcal{Q}$).

Proposition 14 (Co-bordered regions give rise to comparable Hessians)**.** *Let $f \in C^{1,1}(U)$ be as in* Example 13. *Then the matrices $D^2 f(x_0^+)$ and $D^2 f(x_0^-)$ are comparable in the order of symmetric matrices. In particular $\mathcal{H}f(x_0) = \{D^2 f(x_0^+), D^2 f(x_0^-)\}$ has a minimum element.*
If the matrices $D^2 f(x_0^+), D^2 f(x_0^-)$ have a common orthonormal base of eigenvectors $\{v_1, \ldots, v_n\}$ with eigenvalues $\{\lambda_1^+, \ldots, \lambda_n^+\}$ and respectively $\{\lambda_1^-, \ldots, \lambda_n^-\}$, and a unit normal vector $n(x_0)$ to $T_{x_0}\Gamma$ is not one of the eigenvectors, then f is twice differentiable at x_0.

Proof. Let $v \in T_{x_0}\Gamma$ the tangent space, and $\gamma : (-1, 1) \to \Gamma$ be a smooth curve with $\gamma(0) = x_0$, $\dot{\gamma}(0) = v$. Then

$$\frac{d}{dt} Df(\gamma(t))\bigg|_{t=0} = D^2 f(x_0^+)v = D^2 f(x_0^-)v.$$

Since the order of symmetric matrices is an intrinsic fact, if we choose as a base for $\mathbb{R}^N$ the set $\{v_1, \ldots, v_{n-1}, n(x_0)\}$ where the first $n-1$ vectors are a base of $T_{x_0}\Gamma$ and $n(x_0)$ is a normal unit vector of Γ at x_0, then the only possible unequal element of the two matrices $D^2 f(x_0^+), D^2 f(x_0^-)$ is the element of place $n \times n$, i.e. $n(x_0) \cdot D^2 f(x_0^{\pm}) n(x_0)$, or $n(x_0) \cdot D^2 f(x_0^-) n(x_0)$, respectively. Thus the two matrices are comparable.

In the assumptions of the second part of the statement, we can find a unit tangent vector $\overline{\tau} \in T_{x_0}\Gamma$ such that $\overline{\tau} = \sum_{i=1}^{n} \alpha_i v_i$ and $\alpha_i \neq 0$ for all i. Then from the first part of the proof we get that

$$\lambda_i^- \alpha_i = D^2 f(x_0^-)\overline{\tau} \cdot v_i = D^2 f(x_0^+)\overline{\tau} \cdot v_i = \lambda_i^+ \alpha_i,$$

for all indices i. Thus $\lambda_i^- = \lambda_i^+$ for all i and the matrices $D^2 f(x_0^-), D^2 f(x_0^+)$ are identical. $\square$.

Using the above proposition we obtain the following result.

Corollary 15. *Let $f \in C^{1,1}(U)$ be a piecewise C^2 function, $x_0 \in U$. Under the notation of* Definition 11, *we set $\mathcal{H}f(x_0) = \{A_1, \ldots, A_k\}$. Then for every $i, j \in \{1, \ldots, k\}$ such that $x_0 \in \overline{\Gamma}_\ell \subset \overline{\Omega}_i \cap \overline{\Omega}_j$, the matrices A_i and A_j are comparable.*

Proof. Let $\{x_n\}_n \subset \Gamma_\ell$ be such that $x_n \to x_0$. Denoting by $D^2 f(x_n^i)$ (respectively, $D^2 f(x_n^j)$) the limit of the Hessians of f at x_n obtained from the region Ω_i (respectively, Ω_j) we have by Proposition 14 that either $D^2 f(x_n^i) \succeq D^2 f(x_n^j)$ or $D^2 f(x_n^i) \preceq D^2 f(x_n^j)$. Taking a subsequence if necessary, we may assume that the order is fixed throughout the sequence (say, $D^2 f(x_n^i) \succeq D^2 f(x_n^j)$ for all $n \geq 1$), which then passes to the limit as $n \to \infty$ yielding $A_i \succeq A_j$. $\square$

The following example shows that the generalized Hessian $\mathcal{H}f(\bar{x})$ of a piecewise C^2 function $f \in C^{1,1}(U)$ might not have a minimum element, even if all generalized Hessians are diagonal matrices.

Example 16 (Diagonal Hessians with no minimum)**.** There exists a $C^{1,1}$ function $f : \mathbb{R}^2 \to \mathbb{R}$ which is piecewise C^2 around $\bar{x} = (0,0)$ and for which all matrices of $\mathcal{H}f(\bar{x})$ are diagonal, but there is no minimum element.

(Construction of the counterexample.) Let us consider the following five open subsets of $\mathbb{R}^2$.

$$\Omega_1 := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 \in (-x_1^{1/3}, x_1^3)\right\} \qquad \Omega_4 := (-\infty, 0) \times (-\infty, 0)$$

$$\Omega_2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > x_1^3\} \qquad \Omega_5 := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 < -x_1^{1/3})\right\}.$$

$$\Omega_3 := (-\infty, 0) \times (0, +\infty)$$

Observe that $\mathbb{R}^2 = \bigcup_{i=1}^5 \operatorname{cl}\Omega_i$ and define the function $f : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$f(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) \in \operatorname{cl}\Omega_1 \quad | \quad -\frac{1}{2}x_2^6 - x_1 x_2^3 \quad \text{if } (x_1, x_2) \in \operatorname{cl}\Omega_4 \\ -\frac{1}{2}(-x_1^3 + x_2)^2 & \text{if } (x_1, x_2) \in \operatorname{cl}\Omega_2 \quad | \quad -\frac{1}{2}(x_1 + x_2^3)^2 \quad \text{if } (x_1, x_2) \in \operatorname{cl}\Omega_5. \\ -\frac{1}{2}x_2^2 & \text{if } (x_1, x_2) \in \operatorname{cl}\Omega_3. \quad | \end{cases}$$

It is easily verified that f is $C^{1,1}$, piecewise C^2 around $\bar{x} := (0,0)$ and its generalized Hessian $\mathcal{H}f(\bar{x})$ consists of the following three diagonal matrices

$$\mathcal{H}f(\bar{x}) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Then $\mathcal{H}f(\bar{x})$ does not have a minimum element. Nevertheless, one can check that for an appropriate choice of $\varepsilon(r)$ (which tends to 0 as $r \to 0$) the function $\varphi(x) = -\frac{1}{2}x_2^2 + \varepsilon(\|x\|)\|x\|^2$ belongs to $\mathcal{M}_f(\bar{x})$ and

$$D^2\varphi(\bar{x}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{H}f(\bar{x}). \tag{17}$$

Thus $(\mathcal{Q})$ is here satisfied. Notice that the above matrix is the unique element of $\mathcal{H}f(\bar{x})$ that satisfies $(\mathcal{Q})$ and that it is a directional minimum of $\mathcal{H}f(\bar{x})$. □

In the above example we remark that there exist two hypersurfaces (one-dimensional manifolds) separating the regions Ω_1, Ω_5 and Ω_4 having the same tangent $(0, -1)$. This is somehow the cause of the non-existence of a minimum element. In fact in the 2-dimensional setting of the previous example, a qualification condition that avoids the above situation, ensures the existence of a minimum element. In what follows, we denote by $\tau = (\tau_1, \tau_2)$ the unit vector that generates the tangent space of the hypersurface (smooth curve) Γ at $\bar{x}$. (In case that $\bar{x} \in \overline{\Gamma} \setminus \Gamma$, we assume that the limit of the tangents of Γ at x as $x \to \bar{x}$ exists (this is the case, if $\overline{\Gamma}$ is a manifold with boundary), and we take τ equal to this limit.) We also denote by $E := \{(1,0), (0,1), (-1,0), (0,-1)\}$ the four tangents corresponding to the cartesian semi-axes.

Theorem 17 (Diagonal matrices in $\mathbb{R}^2$ with a qualification condition)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $C^{1,1}$ function which is piecewise C^2 around the origin $\bar{x} = (0,0)$. Assume that all elements of the generalized Hessian $\mathcal{H}f(\bar{x}) = \{A_1, \ldots, A_k\}$ are diagonal matrices, and that for any $\tau \in E$ there is at most one hypersurface* (*smooth curve*) *Γ of* Definition 11 *whose tangent vector $\tau = (\tau_1, \tau_2)$ at $\bar{x}$ belongs to E. Then $\mathcal{H}f(\bar{x})$ has a minimum element (thus $(\mathcal{Q})$ holds).*

Proof. Let $\Omega_1, \ldots, \Omega_k$ be the finite regions (open connected sets) of $\bar{x}$ that give rise to the generalized Hessians $\mathcal{H}f(\bar{x}) = \{A_1, \ldots, A_k\}$ (*cf.* Definition 11), where

$$A_i := \lim_{\substack{x \to \bar{x} \\ x \in \Omega_i}} D^2 f(x) = \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix}.$$

Using Corollary 15 and reordering the set $\{1, \ldots, k\}$ if necessary, we may assume that as the index i moves progressively in the set $\{1, \ldots, k\}$ the matrices A_i and A_{i+1} are comparable (i is taken modulo k). By Proposition 14 the matrices A_i and A_{i+1} are equal, whenever the tangent of the common boundary Γ of Ω_i and Ω_{i+1} is not parallel to one the axes. Thus, a change between A_i and A_{i+1} may occur whenever the (normalized) tangent of the common boundary of Ω_i and Ω_{i+1} belongs to E, *i.e.* $\tau_1\tau_2 = 0$. In particular, if $\tau_1 = 0$, then $a_i \neq a_{i+1}$ and $b_i = b_{i+1}$, while if $\tau_2 = 0$, then $a_i = a_{i+1}$ and $b_i \neq b_{i+1}$. Now evoking the qualification condition we deduce that each of the aforementioned changes either appears twice or does not appear at all. The conclusion follows easily. □

As an easy consequence we obtain the following result.

Corollary 18. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $C^{1,1}$ function that is piecewise C^2 around the origin $\bar{x} = (0,0)$. Assume that all matrices in $\mathcal{H}f(\bar{x})$ are simultaneously diagonalizable and that distinct hypersurfaces Γ^ℓ give rise to distinct tangents τ^ℓ at $\bar{x}$. Then $\mathcal{H}f(\bar{x})$ possesses a minimum element (and $(\mathcal{Q})$ holds).*

Remark 19. Note that if all separating curves of Definition 11 are line segments (locally around x_0), then the above qualification condition is automatically satisfied.

Let us finally note, for completeness, that the existence of a minimum element of the generalized Hessian $\mathcal{H}f(\bar{x})$ does not imply the simultaneous diagonalizability of its elements.

Example 20 (Hessian with minimum element and non-commutative elements)**.** There exists a $C^{1,1}$ function $f : \mathbb{R}^2 \to \mathbb{R}$ which is piecewise C^2 around $\bar{x} = (0,0)$, such that the generalized Hessian $\mathcal{H}f(\bar{x})$ has a minimum element but its elements are not simultaneously diagonalizable matrices.

(Construction of the counterexample.) Let us consider the following six open subsets of $\mathbb{R}^2$.

$$\Omega_1 := \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 \in (0, \tfrac{1}{2}x_1)\right\} \qquad \Omega_4 := (-\infty,0) \times (0,+\infty)$$

$$\Omega_2 := \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 \in (\tfrac{1}{2}x_1, 2x_1)\right\} \qquad \Omega_5 := (-\infty,0) \times (-\infty,0)$$

$$\Omega_3 := \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 > 0,\ x_2 > 2x_1\right\} \qquad \Omega_6 := (0,+\infty) \times (-\infty,0).$$

Observe that $\mathbb{R}^2 = \bigcup_{i=1}^6 \operatorname{cl}\Omega_i$ and define the function $f : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$f(x_1,x_2) = \begin{cases} \frac{3}{2}x_2^2 & \text{if } (x_1,x_2) \in \operatorname{cl}\Omega_1 \quad | \quad -3x_1^2 \quad \text{if } (x_1,x_2) \in \operatorname{cl}\Omega_4 \\ 2x_1x_2 - \frac{1}{2}(x_1^2 + x_2^2) & \text{if } (x_1,x_2) \in \operatorname{cl}\Omega_2 \quad | \quad -3x_1^2 - 3x_2^2 \quad \text{if } (x_1,x_2) \in \operatorname{cl}\Omega_5 \\ \frac{3}{2}x_1^2 & \text{if } (x_1,x_2) \in \operatorname{cl}\Omega_3 \quad | \quad -3x_2^2 \quad \text{if } (x_1,x_2) \in \operatorname{cl}\Omega_6. \end{cases}$$

It is easily verified that f is $C^{1,1}$, piecewise C^2 around $\bar{x}$ and that

$$\mathcal{H}f(0,0) := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -6 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix} \right\}.$$

Then $\mathcal{H}f(\bar{x})$ has a minimum element (coming out from the region Ω_5) but its elements are not simultaneously diagonalizable. □

4 Characterization of directional minima

Let us now come back to the situation of Theorem 3 and the sufficiency part of its condition, that is, the existence of directional minima of the generalized Hessian $\mathcal{H}f(x_0)$. In view of Example 7 the existence of directional minima is clearly not a necessary condition. Thus, in order to obtain a characterization an additional assumption is required. Such an assumption is provided by the following definition.

Definition 21 (Semi-C^2 regularity)**.** Let U be an open subset of $\mathbb{R}^n$. A $C^{1,1}$ function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is called upper semi-C^2 at $x_0 \in U$ if for every $X \in \mathcal{H}f(x_0)$ and $v \in \mathcal{T}_{f,x_0}(X)$, there exists a sequence $\{x_n\}_n \subset U \diagdown \{x_0\}$ with $x_n \to x_0$ and $\frac{x_n - x_0}{\|x_n - x_0\|} \to v$ such that

$$f(x_n) \leq f(x_0) + Df(x_0)(x_n - x_0) + \frac{1}{2}\langle X(x_n - x_0), x_n - x_0\rangle + o(\|x_n - x_0\|^2). \tag{18}$$

Remark 22. It can be easily deduced from the forthcoming result (Theorem 23) that the function $f:\mathbb{R}\to\mathbb{R}$ of Example 6 and the function $F:\mathbb{R}^n\to\mathbb{R}$ of Example 7 do not satisfy the above regularity condition. On the other hand, all other functions considered in the examples of this work are upper semi-C^2. This is a consequence of the fact that piecewise $\mathcal{C}^2$ functions (Definition 11) are upper semi-C^2 (in fact semi-$\mathcal{C}^2$, a function f being called semi-C^2, if (18) holds with equality). To see this, note that for any $A\in\mathcal{H}f(x_0)$, the cone $\mathbb{R}_+\mathcal{T}_{f,x_0}(A)$ can be described as a finite union of tangent cones at x_0 of some $\overline{\Omega_i}$ (those giving $A=\lim D^2f(x)$ with $x\to x_0$ in Ω_i).

We are now ready to state and prove the following result.

Theorem 23 (Characterization of directional minima of $\mathcal{H}f(x_0)$)**.** *Let $U\subset\mathbb{R}^n$ be a nonempty open set and let $f:U\to\mathbb{R}$ be a $C^{1,1}$ function which is upper semi-C^2 at x_0. Then $M\in\mathcal{H}f(x_0)$ is a directional minimum of $\mathcal{H}f(x_0)$ if and only if there exists $\varphi\in\mathcal{M}_f(x_0)$ such that $D^2\varphi(x_0)=M$.*

Proof. In view of Theorem 3 we only need to prove the sufficiency part. To this end, let $M\in\mathcal{H}f(x_0)\cap\mathcal{J}^2_f(x_0)$ and let φ be a corresponding minorant of f at x_0 so that

$$f(x)\geq\varphi(x)=f(x_0)+Df(x_0)(x-x_0)+\frac{1}{2}\langle M(x-x_0),x-x_0\rangle+o(\|x-x_0\|^2).$$

Let $X\in\mathcal{H}f(x_0)$ and $v\in\mathcal{T}_{f,x_0}(X)$. Then for some sequence $\{x_n\}_n\subset U$ with $x_n\to x_0$ and $\frac{x_n-x_0}{\|x_n-x_0\|}\to v$, relation (18) holds. Combining with the above we deduce

$$\langle (X-M)(x_n-x_0),x_n-x_0\rangle\geq o(\|x_n-x_0\|^2).$$

Dividing by $\|x_n-x_0\|^2$ and taking the limit as $x_n\to x_0$ we deduce $\langle (X-M)v,v\rangle\geq 0$ which proves the assertion. □

We complement the previous result with the following remark.

Remark 24 (Jensen's lemma revisited)**.** Let $\varphi\in\mathcal{M}_f(x_0)$ and $A=D^2\varphi(x_0)$. Then

$$f(x)\geq\varphi(x)=f(x_0)+Df(x_0)(x-x_0)+\frac{1}{2}\langle A(x-x_0),x-x_0\rangle+o(\|x-x_0\|^2).$$

Set $x=x_0+tv$ where $\|v\|=1$ and $t>0$ and apply the nonsmooth second-order Taylor expansion ([7, Theorem 3.2]) to deduce

$$\frac{1}{2}\langle Av,v\rangle\,+\frac{o(t^2)}{t^2}\leq\frac{1}{t^2}\,\left(f(x)-f(x_0)-Df(x_0)\,(x-x_0)\right)\in\frac{1}{2}\langle\mathcal{H}^o f(]x_0,x[)v,v\rangle\,.$$

The above holds for any sequence $\{t_n\}_n$ with $t_n\to 0$. Taking a subsequence if necessary, and passing to the limit as $n\to\infty$ we deduce using the upper semicontinuity of $\mathcal{H}f$ that *for some* $B\in K_{f,x_0}(v)$ we have

$$\langle Av,v\rangle\;\leq\;\langle Bv,v\rangle\,.$$

Notice that Example 2 shows that the above inequality might not be true *for all* $B\in K_{f,x_0}(v)$. However whenever the set

$$\{\langle Bv,v\rangle\,:\,B\in K_{f,x_0}(v)\}$$

is a singleton, we directly conclude that A is a directional minimum and $(\mathcal{Q})$ holds. □

5 Appendix: a side lemma

In this section we state and prove the following standard lemma which has been used in the text.

Lemma 25 (Second–order regularization from above)**.** *Let* $\varepsilon : (0,\infty) \to \mathbb{R}$ *be a function satisfying* $\lim_{t\to 0} \varepsilon(t) = 0$. *Then there exists* $\delta > 0$ *and a* C^2 *function* $\psi : [0,\delta) \to [0,\infty)$ *such that*

$$\psi(0) = \psi'(0) = \psi''(0) = 0$$

and

$$\psi(t) \geq \varepsilon(t)\, t^2 \qquad \text{for all} \quad t \in [0,\delta)\,.$$

Proof. By [17, Lemma 3.7] (see also [4]) there exists $\delta_1 > 0$ and a C^1 function $\alpha : [0,\delta_1) \to [0,\infty)$ such that $\alpha(0) = \alpha'(0) = 0$ and $\alpha(t) \geq t\varepsilon(t)$ for $t \in (0,\delta_1)$. Since $\lim_{t\to 0} \alpha'(t) = \alpha'(0) = 0$ applying again [17, Lemma 3.7] to the function α' we obtain $0 < \delta < \delta_1$ and a C^1 function $\beta : [0,\delta) \to [0,\infty)$ such that $\beta(0) = \beta'(0) = 0$ and $\beta(t) \geq t\alpha'(t)$ for $t \in [0,\delta)$. Let us set $\gamma(t) = \alpha(t) + \beta(t)$ for all $t \in [0,\delta)$. Obviously $\gamma \in C^1([0,\delta))$ thus the function

$$t \longmapsto \psi(t) := \int_0^t \gamma(s)ds$$

is of class C^2 on $[0,\delta)$. Since

$$\gamma(s) = \alpha(s) + \beta(s) \geq \alpha(s) + s\alpha'(s) \geq (s\alpha(s))', \quad \text{for all } s \in [0,\delta)\,,$$

we deduce that $\psi(t) \geq \int_0^t (s\alpha(s))'ds = t\alpha(t)$ for $t \in [0,\delta)$. Thus $\psi(t) = t^2\varepsilon(t)$ for $t \in [0,\delta)$ and the conclusion follows. □

Acknowledgment. Part of this work has been made during a research visit of the second author to the University of Pau (June 2007, July 2008 and June 2009). The second author thanks his hosts for hospitality.

References

[1] Borwein, J. & Zhu, Q., Techniques of variational analysis, CMS Books in Mathematics **20**, (Springer, 2005).

[2] Clarke, F. H., *Optimization and nonsmooth analysis* (2^{nd} edition), Classics in Applied Mathematics **5**, SIAM, (Philadelphia, 1990).

[3] Crandall, M., Evans, L. & Lions, P.-L., Some properties of viscosity solutions of Hamilton–Jacobi equations, *Trans. Amer. Math. Soc.* **282** (1984), 487–502.

[4] Crandall, M., Ishii, H. & Lions, P.-L., User's guide to viscosity solutions of second order to fully nonlinear partial differential equations, *Bull. Amer. Math. Soc.* **27** (1992), 1–67.

[5] DANIILIDIS, A. & MALICK, J., Filling the gap between lower-C^1 and lower-C^2 functions, *J. Convex Anal.* **12** (2005), 315–329.

[6] DEVILLE, R., GODEFROY, G. & ZIZLER, V., *Smoothness and renormings in Banach spaces*, Pitman Monographies and Surveys in Pure and Applied Mathematics **64**, Pitman, (London, 1993).

[7] HIRIART-URRUTY, J.-B., STRODIOT, J.-J. & NGUYEN, V., Generalized Hessian matrix and second-order optimality conditions for problems with $\mathrm{C}^{1,1}$ data, *Appl. Math. Optim.* **11** (1984), 43–56.

[8] IMBERT, C., Support functions of the Clarke generalized Jacobian and of its plenary hull, *Nonlinear Anal.* **49** (2002), 1111–1125.

[9] IOFFE, A., Fuzzy principles and characterization of trustworthiness, *Set-Valued Anal.* **6** (1998), 265–276.

[10] IOFFE, A. & PENOT, J.-P., Limiting sub-Hessians, limiting subjets and their calculus, *Trans. Amer. Math. Soc.* **349** (1997), 789–807.

[11] JEYAKUMAR, V. & LUC, D. T., Approximate Jacobian matrices for nonsmooth continuous maps and C^1-optimization, *SIAM J. Control Optim.* **36** (1998), 1815–1832.

[12] JEYAKUMAR, V. & YANG, X. Q., Approximate generalized Hessians and Taylor's expansions for continuously Gâteaux differentiable functions, *Nonlinear Anal.* **36** (1999), 353–368.

[13] LEBOURG, G., Generic differentiability of Lipschitzian functions, *Trans. Amer. Math. Soc.* **256** (1979), 125–144.

[14] MALICK, J. & SENDOV, H., Clarke generalized Jacobian of the projection onto the cone of positive semidefinite matrices, *Set-Valued Anal.* **14** (2006), 273–293.

[15] PÁLES, Z. & ZEIDAN, V., Generalized Hessian for $\mathrm{C}^{1,1}$ functions in infinite-dimensional normed spaces, *Math. Programming* **74** (1996), 59–78.

[16] ROCKAFELLAR, R.T. & WETS, R., *Variational Analysis*, Grundlehren der Mathematischen Wissenschaften, Vol. **317**, (Springer, 1998).

[17] SPINGARN, J., Submonotone subdifferentials of Lipschitz functions, *Trans. Amer. Math. Soc.* **264** (1981), 77–89.

[18] ZHU, Q., The equivalence of several basic theorems for subdifferentials, *Set-Valued Anal.* **6** (1998), 171–185.

Luc Barbet

Laboratoire de Mathématiques et de leurs Applications, UMR CNRS 5142
Université de Pau et des Pays de l'Adour
Avenue de l'Université - BP 1155, 64013 Pau, France

E-mail: `luc.barbet@univ-pau.fr`

Aris Daniilidis

Departament de Matemàtiques, C1/308
Universitat Autònoma de Barcelona
E-08193 Bellaterra, Spain

E-mail: arisd@mat.uab.es
http://mat.uab.es/~arisd

Research supported by the MEC Grant No. MTM2008-06695-C03-03 (Spain).

Pierpaolo Soravia

Dipartimento di Matematica Pura e Applicata
Università di Padova
via Trieste 63, 35121 Padova, Italy

E-mail: soravia@math.unipd.it
http://www.math.unipd.it/~soravia

Research supported by the MIUR-Prin Grant “Metodi di viscosità, metrici e di teoria del controllo in equazioni alle derivate parziali nonlineari” (Italy).