Weak convergence for the stochastic heat equation driven by Gaussian white noise

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Abstract

In this paper, we consider a quasi-linear stochastic heat equation on $[0,1]$, with Dirichlet boundary conditions and controlled by the space-time white noise. We formally replace the random perturbation by a family of noisy inputs depending on a parameter $n \in \mathbb{N}$ such that approximate the white noise in some sense. Then, we provide sufficient conditions ensuring that the real-valued *mild* solution of the SPDE perturbed by this family of noises converges in law, in the space $\mathcal{C}([0,T] \times [0,1])$ of continuous functions, to the solution of the white noise driven SPDE. Making use of a suitable continuous functional of the stochastic convolution term, we show that it suffices to tackle the linear problem. For this, we prove that the corresponding family of laws is tight and we identify the limit law by showing the convergence of the finite dimensional distributions. We have also considered two particular families of noises to that our result applies. The first one involves a Poisson process in the plane and has been motivated by a one-dimensional result of Stroock, which states that the family of processes $n \int_0^t (-1)^{N(n^2 s)} ds$, where N is a standard Poisson process, converges in law to a Brownian motion. The second one is constructed in terms of the kernels associated to the extension of Donsker's theorem to the plane.

Keywords: stochastic heat equation; white noise; weak convergence; two-parameter Poisson process; Donsker kernels.

AMS subject classification: 60B12; 60G60; 60H15.

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1 Introduction

In the almost last three decades, there have been enormous advances in the study of random field solutions to stochastic partial differential equations (SPDEs) driven by general Brownian noises. The starting point of this theory was the seminal work by Walsh [36], and most of the research developed thereafter has been mainly focused on the analysis of heat and wave equations perturbed by Gaussian white noises in time with a fairly general spatial correlation (see, for instance, [2, 9, 11, 13, 27]). Notice also that some effort has been made to deal with SPDEs driven by fractional type noises (see, for instance, [19, 26, 29, 33]).

Indeed, the motivation to consider these type of models in the above mentioned references has sometimes put together theoretical mathematical aspects and applications to some real situations. Let us mention that, for instance, different type of SPDEs provide suitable models in the study of growth population, some climate and oceanographical phenomenons, or some applications to mathematical finance (see [14], [21], [1], [7], respectively).

However, real noisy inputs are only approximately white and Gaussian, and what one usually does is to justify somehow that one can approximate the randomness acting on the system by a Gaussian white noise. This fact has been illustrated by Walsh in [35], where a parabolic SPDE has been considered in order to model a discontinuous neurophysiological phenomenon. The noise considered in this article is determined by a Poisson point process and the author shows that, whenever the number of jumps increases and their size decreases, it approximates the so-called space-time white noise in the sense of convergence of the finite dimensional distributions. Then, the author proves that the solutions of the PDEs perturbed by these discrete noises converge in law (in the sense of finite dimensional distribution convergence) to the solution of the PDE perturbed by the space-time white noise.

Let us now consider the following one-dimensional quasi-linear stochastic heat equation:

$$\frac{\partial U}{\partial t}(t,x) - \frac{\partial^2 U}{\partial x^2}(t,x) = b(U(t,x)) + \dot{W}(t,x), \quad (t,x) \in [0,T] \times [0,1], \tag{1}$$

where $T > 0$ stands for a fixed time horizon, $b : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function and $\dot{W}$ is the formal notation for the space-time white noise. We impose some initial condition and boundary conditions of Dirichlet type, that is:

$$U(0,x) = u_0(x),\ x \in [0,1],$$

$$U(t,0) = U(t,1) = 0,\ t \in [0,T],$$

where $u_0 : [0,1] \to \mathbb{R}$ is a continuous function. The random field solution to Equation (1) will be denoted by $U = \{U(t,x),\ (t,x) \in [0,T] \times [0,1]\}$ and it is interpreted in the *mild* sense. More precisely, let $\{W(t,x),\ (t,x) \in [0,T] \times [0,1]\}$ denote a Brownian sheet on $[0,T] \times [0,1]$, which we suppose to be defined in some probability space $(\Omega, \mathcal{F}, P)$. For $0 \le t \le T$, let $\mathcal{F}_t$ be the σ-field generated by the random variables $\{W(s,x),\ (s,x) \in [0,t] \times [0,1]\}$, which can be conveniently completed, so that the resulting filtration $\{\mathcal{F}_t,\ t \ge$

0} satisfies the usual conditions. Then, a process U is a solution of (1) if it is $\mathcal{F}_t$-adapted and the following stochastic integral equation is satisfied:

$$U(t,x) = \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t\int_0^1 G_{t-s}(x,y)\, b(U(s,y))dyds + \int_0^t\int_0^1 G_{t-s}(x,y)W(ds,dy), \quad a.s. \tag{2}$$

for all $(t,x) \in (0,T] \times (0,1)$, where G denotes the Green function associated to the heat equation in $[0,1]$ with Dirichlet boundary conditions. We should mention that the stochastic integral in the right-hand side of Equation (2) is a Wiener integral, which can be understood either in the sense of Walsh [36] or in the framework of Da Prato and Zabczyk [12]. Besides, existence, uniqueness and pathwise continuity of the solution of (2) are a consequence of [36, Theorem 3.5].

The aim of our work is to prove that the mild solution of (1) –which is given by the solution of (2)– can be approximated in law, in the space $\mathcal{C}([0,T]\times[0,1])$ of continuous functions, by the solution of

$$\frac{\partial U_n}{\partial t}(t,x) - \frac{\partial^2 U_n}{\partial x^2}(t,x) = b(U_n(t,x)) + \theta_n(t,x), \qquad (t,x) \in [0,T]\times[0,1], \tag{3}$$

with initial condition u_0 and Dirichlet boundary conditions, where $n \in \mathbb{N}$. In this equation, θ_n will be a noisy input that approximates the white noise $\dot{W}$ in the following sense:

Hypothesis 1.1 *The finite dimensional distributions of the processes*

$$\zeta_n(t,x) = \int_0^t \int_0^x \theta_n(s,y)dyds, \quad (t,x) \in [0,T]\times[0,1],$$

converge in law to those of the Brownian sheet

Observe that, if the processes θ_n have square integrable paths, then the mild form of Equation (3) is given by:

$$U_n(t,x) = \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t\int_0^1 G_{t-s}(x,y)\, b(U_n(s,y))dyds + \int_0^t\int_0^1 G_{t-s}(x,y)\theta_n(s,y)dyds. \tag{4}$$

Standard arguments yield existence and uniqueness of solution for Equation (4) and, furthermore, as it will be detailed later on (see Section 3), the solution U_n has continuous trajectories a.s.

In order to state the main result of the paper, let us consider the following hypothesis which, as it will be made explicit in the sequel, will play an essential role:

Hypothesis 1.2 *For some $q \in [2,3)$, there exists a positive constant C such that, for any $f \in L^q([0,T]\times[0,1])$, it holds:*

$$E\left(\int_0^T \int_0^1 f(t,x)\theta_n(t,x)\, dxdt\right)^2 \le C_q \left(\int_0^T \int_0^1 |f(t,x)|^q\, dxdt\right)^{\frac{2}{q}}.$$

Hypothesis 1.3 *There exist $m > 8$ and a positive constant C such that the following is satisfied: for all s_0, $s_0' \in [0,T]$ and x_0, $x_0' \in [0,1]$ satisfying $0 < s_0 < s_0' < 2s_0$ and $0 < x_0 < x_0' < 2x_0$, and for any $f \in L^2([0,T] \times [0,1])$, it holds:*

$$\sup_{n\geq 1} E\left|\int_{s_0}^{s_0'} \int_{x_0}^{x_0'} f(s,y)\,\theta_n(s,y)dyds\right|^m \leq C\left(\int_{s_0}^{s_0'} \int_{x_0}^{x_0'} f(s,y)^2\,dyds\right)^{\frac{m}{2}}.$$

We remark that, in Hypothesis 1.2, the restriction on the parameter q will be due to the integrability properties of the Green function G. On the other hand, in the condition $s_0' < 2s_0$ (resp. $x_0' < 2x_0$) of Hypothesis 1.3, the number 2 could be replaced by any $k > 1$. We are now in position to state our main result:

Theorem 1.4 *Let $\{\theta_n(t,x),\ (t,x) \in [0,T] \times [0,1]\}$, $n \in \mathbb{N}$, be a family of stochastic processes such that $\theta_n \in L^2([0,T] \times [0,1])$ a.s., and such that Hypothesis 1.1, 1.2 and 1.3 are satisfied. Moreover, assume that $u_0 : [0,1] \to \mathbb{R}$ is continuous and $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz.*

Then, the family of stochastic processes $\{U_n,\ n \geq 1\}$ defined as the mild solutions of Equation (3) converges in law, in the space $\mathcal{C}([0,T] \times [0,1])$, to the mild solution U of Equation (1).

Let us point out that, as we will see in Section 3, Theorem 1.4 will be almost an immediate consequence of the analogous result when taking null initial condition and nonlinear term (see Theorem 3.5). Thus, the essential part of the paper will be concerned to prove the convergence in law, in the space $\mathcal{C}([0,T] \times [0,1])$, of the solution of

$$\frac{\partial X_n}{\partial t}(t,x) - \frac{\partial^2 X_n}{\partial x^2}(t,x) = \theta_n(t,x), \quad (t,x) \in [0,T] \times [0,1], \tag{5}$$

with vanishing initial data and Dirichlet boundary conditions, towards the solution of

$$\frac{\partial X}{\partial t}(t,x) - \frac{\partial^2 X}{\partial x^2}(t,x) = \dot{W}(t,x), \quad (t,x) \in [0,T] \times [0,1]. \tag{6}$$

Observe that the mild solution of Equations (5) and (6) can be explicitly written as, respectively,

$$X_n(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y)\,\theta_n(s,y)\,dyds,\ (t,x) \in [0,T] \times [0,1], \tag{7}$$

and

$$X(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y)W(ds,dy),\ (t,x) \in [0,T] \times [0,1], \tag{8}$$

where the latter defines a centered Gaussian process.

An important part of the work is also devoted to check that two interesting particular families of noises verify the hypotheses of Theorem 1.4. More precisely, consider the following processes:

1. The *Kac-Stroock processes* on the plane:

$$\theta_n(t,x) = n\sqrt{tx}\,(-1)^{N_n(t,x)}, \tag{9}$$

where $N_n(t,x) := N(\sqrt{n}t, \sqrt{n}x)$, and $\{N(t,x),\ (t,x) \in [0,T] \times [0,1]\}$ is a standard Poisson process in the plane.

2. The *Donsker kernels*: Let $\{Z_k,\ k \in \mathbb{N}^2\}$ be an independent family of identically distributed and centered random variables, with $E(Z_k^2) = 1$ for all $k \in \mathbb{N}^2$, and such that $E(|Z_k|^m) < +\infty$ for all $k \in \mathbb{N}^2$ and some sufficiently large $m \in \mathbb{N}$. For any $n \in \mathbb{N}$, we define the kernels

$$\theta_n(t,x) = n \sum_{k=(k^1,k^2)\in\mathbb{N}^2} Z_k \cdot \mathbf{1}_{[k^1-1,k^1)\times[k^2-1,k^2)}(tn, xn),\ (t,x) \in [0,T] \times [0,1]. \tag{10}$$

In the case where θ_n are the Kac-Stroock processes, it has been proved in [5] that the family of processes

$$\zeta_n(t,x) = \int_0^t \int_0^x \theta_n(s,y)dsdy, \quad n \in \mathbb{N},$$

converge in law, in the space of continuous functions $\mathcal{C}([0,1]^2)$, to the Brownian sheet. This result has been inspired by its one-dimensional counterpart, which is due to Stroock [31] and states that the family of processes

$$Y_\varepsilon(t) = \frac{1}{\varepsilon} \int_0^t (-1)^{N(\frac{s}{\varepsilon^2})} ds,\ t \in [0,1],\ \varepsilon > 0,$$

where N stands for a standard Poisson process, converges in law in $\mathcal{C}([0,1])$, as ε tends to 0, to the standard Brownian motion. Moreover, it is worth mentioning that Kac (see [22]) already considered this kind of processes in order to write the solution of the telegrapher's equation in terms of a Poisson process.

On the other hand, when θ_n are the Donsker kernels, the convergence in law, in the space of continuous functions, of the processes

$$\zeta_n(t,x) = \int_0^t \int_0^x \theta_n(s,y)dsdy, \quad n \in \mathbb{N},$$

to the Brownian sheet is a consequence of the extension of Donsker's theorem to the plane (see, for instance, [37]).

We should mention at this point that the motivation behind our results has also been considered by Manthey in [24] and [25]. Indeed, in the former paper, the author considers Equation (5) with a family of correlated noises $\{\theta_n,\ n \in \mathbb{N}\}$ whose integral processes

$$\int_0^t \int_0^x \theta_n(s,y)\,dyds,$$

converge in law (in the sense of finite dimensional distribution convergence) to the Brownian sheet. Then, sufficient conditions on the noise processes are specified under which the solution X_n of (5) converges in law, in the sense of the finite dimensional distribution convergence, to the solution of (6). Moreover, it has also been proved that, whenever the noisy processes are Gaussian, the convergence in law holds in the space of continuous functions too; these results have been extended to the quasi-linear equation (3) in [25]. In this sense, let us mention that, in an Appendix and for the sake of completeness, we have added a brief explanation of Manthey's method and showed that his results do not apply to the examples of noisy inputs that we are considering in the paper.

Let us also remark that recently there has been an increasing interest in the study of weak approximation for several classes of SPDEs (see [15, 16]). In these references, the methods for obtaining the corresponding approximation sequences are based on discretisation schemes for the differential operator driving the equation, and the rate of convergence of the weak approximations is analysed. Hence, this latter framework differs significantly from the setting that we have described above. On the other hand, we notice that weak convergence for some classes of SPDEs driven by the Donsker kernels have been considered in the literature; namely, a reduced hyperbolic equation on $\mathbb{R}_+^2$ –which is essentially equivalent to a one-dimensional stochastic wave equation– has been considered in [8, 17], while in [32], the author deals with a stochastic elliptic equation with non-linear drift. Furthermore, in [34], weak convergence of Wong-Zakai approximations for stochastic evolution equations driven by a finite-dimensional Wiener process has been studied. Eventually, it is worth commenting that other type of problems concerning SPDEs driven by Poisson-type noises have been considered e.g. in [18, 20, 23, 28, 30].

The paper is organised as follows. In Section 2, we will present some preliminaries on Equation (1), its linear form (6) and some general results on weak convergence. In Section 3, we prove the results of convergence for equations (6) and (1), so that we end up with the proof of Theorem 1.4. The proof of the fact that the Kac-Stroock processes satisfy the hypotheses of Theorem 1.4 will be carried out in Section 4, while the analysis in the case of the Donsker kernels will be performed at Section 5. Finally, we add an Appendix where we give the proof of Lemma 2.3 and relate our results with those of Manthey ([24], [25]).

2 Preliminaries

As it has been explained in the Introduction, we are concerned with the *mild* solution of the formally-written quasi-linear stochastic heat equation (1). That is, we consider a real-valued stochastic process $\{U(t,x),\ (t,x)\in[0,T]\times[0,1]\}$, which we assume to be adapted with respect to the natural filtration generated by the Brownian sheet on $[0,T]\times[0,1]$, such that the following integral equation is satisfied (see (2)): for all $(t,x)\in[0,T]\times[0,1]$,

$$\begin{aligned} U(t,x) = \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t\int_0^1 G_{t-s}(x,y)\, b(U(s,y))dyds \\ + \int_0^t\int_0^1 G_{t-s}(x,y)W(ds,dy), \quad a.s., \end{aligned} \tag{11}$$

where we recall that $G_t(x,y)$, $(t,x,y)\in\mathbb{R}_+\times(0,1)^2$, denotes the Green function associated to the heat equation on $[0,1]$ with Dirichlet boundary conditions. Explicit formulas for

G are well-known, namely:

$$G_t(x,y) = \frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{+\infty} \left(e^{-\frac{(x-y-2n)^2}{4t}} - e^{-\frac{(x+y-2n)^2}{4t}} \right)$$

or

$$G_t(x,y) = 2\sum_{n=1}^{\infty} \sin(n\pi x)\sin(n\pi y) e^{-n^2\pi^2 t}.$$

Moreover, it holds that

$$0 \le G_t(x,y) \le \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{4t}},\ t>0,\ x,y \in [0,1].$$

We have already commented in the Introduction that, in order to prove Theorem 1.4, we will restrict our analysis to the linear version of Equation (1), which is given by (6). Hence, let us consider for the moment $X = \{X(t,x),\ (t,x) \in [0,T] \times [0,1]\}$ to be the mild solution of Equation (6) with vanishing initial conditions and Dirichlet boundary conditions. This can be explicitly written as (8). Notice that, for any $(t,x) \in (0,T] \times (0,1)$, $X(t,x)$ defines a centered Gaussian random variable with variance

$$E(X(t,x)^2) = \int_0^t \int_0^1 G_{t-s}(x,y)^2 dyds.$$

Indeed, by (iii) in Lemma 2.1 below, it holds that $E(X(t,x)^2) \le Ct^{\frac{1}{2}}$, where the constant $C > 0$ does not depend on x.

In the sequel, we will make use of the following result, which is a quotation of [3, Lemma B.1]:

Lemma 2.1 *(i) Let $\alpha \in (\frac{3}{2}, 3)$. Then, for all $t \in [0,T]$ and $x, y \in [0,1]$,*

$$\int_0^t \int_0^1 |G_{t-s}(x,z) - G_{t-s}(y,z)|^\alpha dzds \le C|x-y|^{3-\alpha}.$$

(ii) Let $\alpha \in (1,3)$. Then, for all $s,t \in [0,T]$ such that $s \le t$ and $x \in [0,1]$,

$$\int_0^s \int_0^1 |G_{t-r}(x,y) - G_{s-r}(x,y)|^\alpha dydr \le C(t-s)^{\frac{3-\alpha}{2}}.$$

(iii) Under the same hypothesis as (ii),

$$\int_s^t \int_0^1 |G_{t-r}(x,y)|^\alpha dydr \le C(t-s)^{\frac{3-\alpha}{2}}.$$

Let us recall that we aim to prove that the process X can be approximated in law, in the space $\mathcal{C}([0,T] \times [0,1])$, by the family of stochastic processes

$$X_n(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y)\theta_n(s,y)\ dyds,\ (t,x) \in [0,T] \times [0,1],\ n \ge 1, \qquad (12)$$

where the processes θ_n satisfy certain conditions.

In order to prove this convergence in law, we will make use of the following two general results. The first one (Theorem 2.2) is a tightness criterium on the plane that generalizes a well-known theorem of Billingsley; it can be found in [38, Proposition 2.3], where it is proved that the hypotheses considered in the result are stronger than those of the commonly-used criterium of Centsov [10]. The second one (Lemma 2.3) will be used to prove the convergence of the finite dimensional distributions of X_n; though it can be found around in the literature, we have not been able to find an explicit proof, so that, for the sake of completeness, we will sketch it in the Appendix.

Theorem 2.2 *Let $\{X_n,\, n \in \mathbb{N}\}$ be a family of random variables taking values in $\mathcal{C}([0,T]\times[0,1])$. The family of the laws of $\{X_n,\, n \in \mathbb{N}\}$ is tight if there exist $p', p > 0$, $\delta > 2$ and a constant C such that*

$$\sup_{n\geq 1} E|X_n(0,0)|^{p'} < \infty$$

and, for every $t, t' \in [0,T]$ and $x, x' \in [0,1]$,

$$\sup_{n\geq 1} E\left|X_n(t',x') - X_n(t,x)\right|^p \leq C\left(|x'-x| + |t'-t|\right)^{\delta}.$$

Lemma 2.3 *Let $(F, \|\cdot\|)$ be a normed space and $\{J^n,\ n \in \mathbb{N}\}$ and J linear maps defined on F and taking values in the space $L^0(\Omega)$ of almost surely finite random variables. Assume that there exists a positive constant C such that, for any $f \in F$,*

$$\sup_{n\geq 1} E|J^n(f)| \leq C\|f\| \quad \textit{and} \tag{13}$$

$$E|J(f)| \leq C\|f\|, \tag{14}$$

and that, for some dense subspace D of F, it holds that $J^n(f)$ converges in law to $J(f)$, as n tends to infinity, for all $f \in D$.

Then, the sequence of random variables $\{J^n(f),\, n \in \mathbb{N}\}$ converges in law to $J(f)$, for any $f \in F$.

Eventually, for any real function X defined on $\mathbb{R}^2_+$, and $(t,x),(t',x') \in \mathbb{R}^2_+$ such that $t \leq t'$ and $x \leq x'$, we will use the notation $\Delta_{t,x}X(t',x')$ for the increment of X over the rectangle $(t,t'] \times (x,x']$:

$$\Delta_{t,x}X(t',x') = X(t',x') - X(t,x') - X(t',x) + X(t,x).$$

3 Proof of the general result

This section is devoted to prove Theorem 1.4. For this, as we have already mentioned, it is convenient to consider, first, the linear equation (6) together with its mild solution (8).

The first step consists in establishing sufficient conditions for a family of processes $\{\theta_n,\, n \in \mathbb{N}\}$ in order that the approximation processes X_n (see (12)) converge, in the sense of finite dimensional distributions, to X, the solution of (8):

$$X(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y)W(ds,dy). \tag{15}$$

Proposition 3.1 *Let $\{\theta_n(t,x),\ (t,x)\in[0,T]\times[0,1]\}$, $n\in\mathbb{N}$, be a family of stochastic processes such that $\theta_n\in L^2([0,T]\times[0,1])$ a.s. and such that Hypothesis 1.1 and 1.2 are satisfied.*

Then, the finite dimensional distributions of the processes X_n given by (12) converge, as n tends to infinity, to those of the process defined by (15).

Proof: We will apply Lemma 2.3 to the following setting: let $q\in[2,3)$ as in Hypothesis 1.2 and consider the normed space $(F:=L^q([0,T]\times[0,1]),\|\cdot\|_q)$, where $\|\cdot\|_q$ denotes the standard norm in $L^q([0,T]\times[0,1])$. Set

$$J^n(f):=\int_0^T\int_0^1 f(s,y)\theta_n(s,y)\,dyds,\quad\text{and}$$

$$J(f):=\int_0^T\int_0^1 f(s,y)W(ds,dy),\ f\in F.$$

Then, J^n and J define linear applications on F and, by Hypothesis 1.2, it holds that

$$\sup_{n\geq 1}E|J^n(f)|\leq C\|f\|_q,$$

for all $f\in L^q([0,T]\times[0,1])$. The isometry of the Wiener integral gives also that

$$E|J(f)|\leq C\|f\|_q,$$

for all $f\in L^q([0,T]\times[0,1])$. Moreover, the set D of elementary functions of the form

$$f(t,x)=\sum_{i=0}^{k-1} f_i\,\mathbf{1}_{(t_i,t_{i+1}]}(t)\mathbf{1}_{(x_i,x_{i+1}]}(x),\tag{16}$$

with $k\geq 1$, $f_i\in\mathbb{R}$, $0=t_0<t_1<\cdots<t_k=T$ and $0=x_0<x_1<\cdots<x_k=1$, is dense in $(F,\|\cdot\|_q)$.

On the other hand, the finite dimensional distributions of X_n converge to those of X if, and only if, for all $m\geq 1$, $a_1,\ldots,a_m\in\mathbb{R}$, $(s_1,y_1),\ldots,(s_m,y_m)\in[0,T]\times[0,1]$, the following convergence in law holds:

$$\sum_{j=1}^m a_jX_n(s_j,y_j)\xrightarrow[n\to\infty]{\mathcal{L}}\sum_{j=1}^m a_jX(s_j,y_j).\tag{17}$$

This is equivalent to have that $J^n(K)=\int_0^T\int_0^1 K(s,y)\theta_n(s,y)\,dyds$ converges in law, as n tends to infinity, to $\int_0^T\int_0^1 K(s,y)W(ds,dy)$, where

$$K(s,y):=\sum_{j=1}^m a_j\mathbf{1}_{[0,s_j]}(s)G_{s_j-s}(y_j,y).$$

By Lemma 2.1 (iii), the function K belongs to $L^q([0,T]\times[0,1])$. Hence, owing to Lemma 2.3, in order to obtain the convergence (17), it suffices to prove that $J^n(f)$ converges in law to $J(f)=\int_0^T\int_0^1 f(s,y)W(ds,dy)$, for every elementary function f of the form (16). In fact, if f is such a function, observe that we have

$$J^n(f)=\sum_{i=0}^{k-1} f_i \int_{t_i}^{t_{i+1}}\int_{x_i}^{x_{i+1}} \theta_n(s,y)\,dyds,$$

and this random variable converges in law, as n tends to infinity, to

$$\sum_{i=0}^{k-1} f_i \int_{t_i}^{t_{i+1}}\int_{x_i}^{x_{i+1}} W(ds,dy)=\int_0^T\int_0^1 f(s,y)W(ds,dy),$$

because the finite dimensional distributions of ζ_n converge to those of the Brownian sheet. □

Let us now provide sufficient conditions on θ_n in order that the family of laws of the processes X_n is tight in $\mathcal{C}([0,T]\times[0,1])$.

Proposition 3.2 *Let $\{\theta_n(t,x),\ (t,x)\in[0,T]\times[0,1]\}$, $n\in\mathbb{N}$, be a family of stochastic processes such that $\theta_n\in L^2([0,T]\times[0,1])$ a.s. Suppose that Hypothesis 1.3 is satisfied.*

Then, the process X_n defined in (12) possesses a version with continuous paths and the family of the laws of $\{X_n,\ n\in\mathbb{N}\}$ is tight in $\mathcal{C}([0,T]\times[0,1])$.

Proof: It suffices to prove that

$$\sup_{n\geq 1} E\left[X_n(t',x')-X_n(t,x)\right]^m \leq C[|x'-x|^{m\alpha}+|t'-t|^{\frac{m\alpha}{2}}], \qquad (18)$$

for all $\alpha\in(0,\frac{1}{2})$, $t,t'\in[0,T]$ and $x,x'\in[0,1]$. Indeed, if $m>8$, then it can be found $\alpha\in(0,\frac{1}{2})$ such that $m\frac{\alpha}{2}>2$ and we obtain the existence of a continuous version of each X_n from Kolmogorov's continuity criterium in the plane. Furthermore, by Theorem 2.2, we also obtain the tightness of the laws of X_n in $\mathcal{C}([0,T]\times[0,1])$.

Set $H(t,x;s,y):=\mathbf{1}_{[0,t]}(s)G_{t-s}(x,y)$. We will need to estimate the moment of order m, for some $m>8$, of the quantity

$$X_n(t',x')-X_n(t,x)=\int_0^T\int_0^1 [H(t',x';s,y)-H(t,x;s,y)]\theta_n(s,y)\,dyds,$$

for $t,t'\in[0,T]$ and $x,x'\in[0,1]$. Moreover, the right-hand side of the above equality can be written in the form $\Delta_{0,0}Y_n(T,1)$, where the process Y_n, which indeed depends on t,t',x,x', is defined by

$$Y_n(s_0,x_0):=\int_0^{s_0}\int_0^{x_0} [H(t',x';s,y)-H(t,x;s,y)]\theta_n(s,y)\,dyds,\ (s_0,x_0)\in[0,T]\times[0,1].$$

Hence, inequality (18) is equivalent to prove that

$$E(\Delta_{0,0}Y_n(T,1))^m \leq C[|x'-x|^{m\alpha}+|t'-t|^{\frac{m\alpha}{2}}],$$

for all $\alpha \in (0, \frac{1}{2})$ and $n \geq 1$. By [6, Lemma 3.2] (in the statement of this lemma, it is supposed that m is an even integer number, but this assumption is not used in its proof), it suffices to prove that there exist $\gamma > 0$ and $C > 0$ such that, for all $s_0, s_0' \in [0,T]$ and $x_0, x_0' \in [0,1]$ satisfying $0 < s_0 < s_0' < 2s_0$ and $0 < x_0 < x_0' < 2x_0$, then

$$\sup_{n\geq 1} E(\Delta_{s_0,x_0} Y_n(s_0', x_0'))^m \leq C \left[|t'-t|^{m\alpha} + |x'-x|^{\frac{m\alpha}{2}} \right] (s_0'-s_0)^{m\gamma} (x_0'-x_0)^{m\gamma}. \tag{19}$$

By Hypothesis 1.3 for the particular case of $f(s,y) = H(t',x';s,y) - H(t,x;s,y)$, we obtain

$$\begin{aligned} &\sup_{n\geq 1} E(\Delta_{s_0,x_0} Y_n(s_0', x_0'))^m \\ &\qquad \leq C \left(\int_0^T \int_0^1 \mathbf{1}_{[s_0,s_0']}(s) \mathbf{1}_{[x_0,x_0']}(y) |H(t',x';s,y) - H(t,x;s,y)|^2 \, dyds \right)^{\frac{m}{2}}. \end{aligned}$$

Let $p \in (1, \frac{3}{2})$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, by Hölder's inequality and the definition of H,

$$\begin{aligned} &\sup_{n\geq 1} E(\Delta_{s_0,x_0} Y_n(s_0', x_0'))^m \\ &\leq C \left(\int_0^T \int_0^1 \mathbf{1}_{[s_0,s_0']}(s) \mathbf{1}_{[x_0,x_0']}(y) \, dyds \right)^{\frac{m}{2q}} \left(\int_0^T \int_0^1 |H(t',x';s,y) - H(t,x;s,y)|^{2p} \, dyds \right)^{\frac{m}{2p}} \\ &\leq C (x_0'-x_0)^{\frac{m}{2q}} (s_0'-s_0)^{\frac{m}{2q}} \\ &\quad \times \left(\int_0^t \int_0^1 |G_{t'-s}(x',y) - G_{t-s}(x,y)|^{2p} \, dyds + \int_t^{t'} \int_0^1 |G_{t'-s}(x',y)|^{2p} \, dyds \right)^{\frac{m}{2p}}. \end{aligned} \tag{20}$$

By Lemma 2.1, the last term in the right-hand side of (20) can be bounded, up to some constant, by

$$\left(|x-x'|^{3-2p} + |t-t'|^{\frac{3-2p}{2}} \right)^{\frac{m}{2p}} \leq C \left(|x-x'|^{\frac{m(3-2p)}{2p}} + |t-t'|^{\frac{m(3-2p)}{4p}} \right).$$

Therefore, if we plug this bound in (20) and we take $\alpha = \frac{3-2p}{2p}$ and $\gamma = \frac{1}{2q}$, then we have proved (19), because $p \in (1, \frac{3}{2})$ is arbitrary. □

Remark 3.3 *As it can be deduced from the first part of the proof of Proposition 3.2, the restriction $m > 8$ has to be considered in order to be able to apply Theorem 2.2 and Kolmogorov's continuity criterium.*

As a consequence of Propositions 3.1 and 3.2, we can state the following result on convergence in law for the processes X_n:

Theorem 3.4 *Let $\{\theta_n(t,x),\ (t,x) \in [0,T] \times [0,1]\}$, $n \in \mathbb{N}$, be a family of stochastic processes such that $\theta_n \in L^2([0,T] \times [0,1])$ a.s. Assume that Hypothesis 1.1, 1.2 and 1.3 are satisfied.*

Then, the family of stochastic processes $\{X_n,\ n \geq 1\}$ defined in (12) converges in law, as n tends to infinity in the space $\mathcal{C}([0,T] \times [0,1])$, to the Gaussian process X given by (15).

We can eventually extend the above result to the quasi-linear Equation (1), so that we end up with the proof of Theorem 1.4. This will be an immediate consequence of the above theorem and the next general result:

Theorem 3.5 *Let* $\{\theta_n(t,x),\ (t,x) \in [0,T] \times [0,1]\}$*,* $n \in \mathbb{N}$*, be a family of stochastic processes such that* $\theta_n \in L^2([0,T]\times[0,1])$ *a.s. Assume that* $u_0 : [0,1] \to \mathbb{R}$ *is a continuous function and* $b : \mathbb{R} \to \mathbb{R}$ *is Lipschitz. Moreover, suppose that the family of stochastic processes* $\{X_n,\ n \geq 1\}$ *defined in (12) converges in law, as n tends to infinity in the space* $\mathcal{C}([0,T] \times [0,1])$*, to the Gaussian process* X *given by (15).*

Then, the family of stochastic processes $\{U_n,\ n \geq 1\}$ *defined as the mild solutions of Equation (3) converges in law, in the space* $\mathcal{C}([0,T] \times [0,1])$*, to the mild solution* U *of Equation (1).*

Proof: Let us first recall that we denote by $U = \{U(t,x),\ (t,x) \in [0,T] \times [0,1]\}$ the unique mild solution of Equation (1), which means that U fulfils

$$U(t,x) = \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t\int_0^1 G_{t-s}(x,y)\, b(U(s,y))dyds + \int_0^t\int_0^1 G_{t-s}(x,y)W(ds,dy), \quad a.s.$$

The approximation sequence is denoted by $\{U_n,\ n \in \mathbb{N}\}$, where $U_n = \{U_n(t,x),\ (t,x) \in [0,T] \times [0,1]\}$ is a stochastic process satisfying

$$U_n(t,x) = \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t\int_0^1 G_{t-s}(x,y)\, b(U_n(s,y))dyds + \int_0^t\int_0^1 G_{t-s}(x,y)\theta_n(s,y)dyds, \quad a.s.$$

where the noisy input θ_n has square integrable paths, a.s.

Using the properties of the Green function (see Lemma 2.1), the fact that $\theta_n \in L^2([0,T] \times [0,1])$ a.s., together with a Gronwall-type argument, we obtain that U_n has continuous paths a.s., for all $n \in \mathbb{N}$.

Next, for each continuous function $\eta : [0,T] \times [0,1] \longrightarrow \mathbb{R}$, consider the following (deterministic) integral equation:

$$z_\eta(t,s) = \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t\int_0^1 G_{t-s}(x,y)\, b(z_\eta(s,y))dyds + \eta(t,x).$$

As before, by the properties of G and the assumptions on u_0 and b, it can be checked that this equation possesses a unique continuous solution.

Now, we will prove that the map

$$\psi : \mathcal{C}([0,T] \times [0,1]) \longrightarrow \mathcal{C}([0,T] \times [0,1])$$

$$\eta \longrightarrow z_\eta$$

is continuous with respect to the usual topology on this space. Indeed, given $\eta_1,\ \eta_2 \in \mathcal{C}([0,T]\times[0,1])$, we have that

$$\begin{aligned} &|z_{\eta_1}(t,x) - z_{\eta_2}(t,x)| \\ &\qquad \le \int_0^t\int_0^1 G_{t-s}(x,y)\ \left|b(z_{\eta_1}(s,y)) - b(z_{\eta_2}(s,y))\right| dyds + |\eta_1(t,x)-\eta_2(t,x)| \\ &\qquad \le L\int_0^t\int_0^1 G_{t-s}(x,y)\ \left|z_{\eta_1}(s,y) - z_{\eta_2}(s,y)\right| dyds + |\eta_1(t,x)-\eta_2(t,x)|, \end{aligned} \tag{21}$$

where L is the Lipschitz constant of the function b.

For a given $f \in \mathcal{C}([0,T]\times[0,1])$, we introduce the following norms:

$$\|f\|_t = \max_{s\in[0,\,t],\, x\in[0,\,1]} |f(s,x)|.$$

By using this notation, we deduce that inequality (21) implies that, for any $t\in[0,T]$,

$$\|z_{\eta_1} - z_{\eta_2}\|_t \le L\int_0^t \overline{G}(t-s)\ \|z_{\eta_1} - z_{\eta_2}\|_s\, ds + \|\eta_1-\eta_2\|_{{}_T},$$

where

$$\overline{G}(s) := \sup_{x\in[0,\,1]}\int_0^1 G_s(x,y)dy \le \sup_{x\in[0,\,1]}\int_0^1 \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-y)^2}{4s}} dy \le C.$$

Applying now Gronwall's lemma, we obtain that there exists a finite constant $A>0$ such that

$$\|z_{\eta_1} - z_{\eta_2}\|_{{}_T} \le A\,\|\eta_1-\eta_2\|_{{}_T},$$

and, therefore, the map ψ is continuous.

Consider now

$$X_n(t,x) = \int_0^t\int_0^1 G_{t-s}(x,y)\theta_n(s,y)dyds$$

and

$$X(t,x) = \int_0^t\int_0^1 G_{t-s}(x,y)W(ds,dy).$$

By hypothesis, we have that X_n converges in law in $\mathcal{C}([0,T]\times[0,1])$ to X, as n goes to infinity. On the other hand, we have

$$U_n = \psi(X_n) \quad \text{and} \quad U = \psi(X),$$

and hence the continuity of ψ implies the convergence in law of U_n to U in $\mathcal{C}([0,T]\times[0,1])$. □

4 Convergence in law for the Kac-Stroock processes

This section is devoted to prove that the hypotheses of Theorem 1.4 are satisfied in the case where the approximation family is defined in terms of the Kac-Stroock process θ_n set up in (9). That is,

$$X_n(t,x) = n\int_0^t \int_0^1 G_{t-s}(x,y)\sqrt{sy}(-1)^{N_n(s,y)}\,dyds. \tag{22}$$

First, we notice that Hypothesis 1.1 has been proved in [5].

The following proposition states that Hypothesis 1.2 is satisfied in this particular situation.

Proposition 4.1 *Let θ_n be the Kac-Strock processes. Then, for all $p>1$, there exists a positive constant C_p such that*

$$E\left(\int_0^T\int_0^1 f(t,x)\theta_n(t,x)\,dxdt\right)^2 \le C_p\left(\int_0^T\int_0^1 |f(t,x)|^{2p}\,dxdt\right)^{\frac{1}{p}}, \tag{23}$$

for any $f\in L^{2p}([0,T]\times[0,1])$ and all $n\ge 1$.

The proof of this proposition is based on the following technical lemma:

Lemma 4.2 *Let $f\in L^2([0,T]\times[0,1])$ and $\alpha\ge 1$. Then, for any $u,u'\in(0,1)$ satisfying that $0<u<u'\le 2^\alpha u$,*

$$E\left(\int_0^T\int_u^{u'} f(t,x)\theta_n(t,x)\,dxdt\right)^2 \le \frac{3}{4}\left(2^{\alpha+1}-1\right)\int_0^T\int_u^{u'} f^2(t,x)\,dxdt,$$

for all $n\ge1$.

Proof: First, we observe that

$$\begin{aligned}E\left(\int_0^T\int_u^{u'} f(t,x)\theta_n(t,x)\,dxdt\right)^2 =&2n^2\int_0^T\int_u^{u'}\int_0^T\int_u^{u'} f(t_1,x_1)f(t_2,x_2)\sqrt{t_1t_2x_1x_2}\\ &\times E\left[(-1)^{N_n(t_1,x_1)+N_n(t_2,x_2)}\right]\mathbf{1}_{\{t_1\le t_2\}}dx_2dt_2dx_1dt_1.\end{aligned} \tag{24}$$

The expectation appearing in (24) can be computed as it has been done in the proof of [6, Lemma 3.1] (see also [5, Lemma 3.2]). More precisely, one writes the sum $N_n(t_1,x_1)+N_n(t_2,x_2)$ as a suitable sum of rectangular increments of N_n and applies that, if Z has a Poisson distribution with parameter λ, then $E\left[(-1)^Z\right]=\exp(-2\lambda)$. Hence, the term in the right-hand side of (24) admits a decomposition of the form I_1+I_2, where

$$\begin{aligned}I_1 =&2n^2\int_0^T\int_u^{u'}\int_0^T\int_u^{u'} f(t_1,x_1)f(t_2,x_2)\sqrt{t_1t_2x_1x_2}\\ &\times\exp\left\{-2n[(t_2-t_1)x_2+(x_2-x_1)t_1]\right\}\mathbf{1}_{\{t_1\le t_2\}}\mathbf{1}_{\{x_1\le x_2\}}dx_2dt_2dx_1dt_1,\end{aligned}$$

$$I_2 = 2n^2 \int_0^T \int_u^{u'} \int_0^T \int_u^{u'} f(t_1,x_1) f(t_2,x_2) \sqrt{t_1 t_2 x_1 x_2}$$
$$\times \exp\{-2n[(t_2-t_1)x_2 + (x_1-x_2)t_1]\} \mathbf{1}_{\{t_1 \le t_2\}} \mathbf{1}_{\{x_2 \le x_1\}} dx_2 dt_2 dx_1 dt_1.$$

Let us apply the inequality $ab \le \frac{1}{2}(a^2+b^2)$, $a, b \in \mathbb{R}$, so that we have $I_1 \le I_{11} + I_{12}$, where the latter terms are defined by

$$I_{11} = n^2 \int_0^T \int_u^{u'} \int_0^T \int_u^{u'} f^2(t_1,x_1)\, t_1 x_1$$
$$\times \exp\{-2n[(t_2-t_1)x_2 + (x_2-x_1)t_1]\} \mathbf{1}_{\{t_1 \le t_2\}} \mathbf{1}_{\{x_1 \le x_2\}} dx_2 dt_2 dx_1 dt_1,$$

$$I_{12} = n^2 \int_0^T \int_u^{u'} \int_0^T \int_u^{u'} f^2(t_2,x_2)\, t_2 x_2$$
$$\times \exp\{-2n[(t_2-t_1)x_2 + (x_2-x_1)t_1]\} \mathbf{1}_{\{t_1 \le t_2\}} \mathbf{1}_{\{x_1 \le x_2\}} dx_2 dt_2 dx_1 dt_1.$$

In order to deal with the term I_{11}, we will use the fact that $\exp\{-2n(t_2-t_1)x_2\} \le \exp\{-2n(t_2-t_1)x_1\}$, for $x_1 \le x_2$, and then integrate with respect to t_2, x_2. Thus

$$\begin{aligned} I_{11} &\le n^2 \int_0^T \int_u^{u'} \int_0^T \int_u^{u'} f^2(t_1,x_1)\, t_1 x_1 \\ &\quad \times \exp\{-2n[(t_2-t_1)x_1 + (x_2-x_1)t_1]\} \mathbf{1}_{\{t_1 \le t_2\}} \mathbf{1}_{\{x_1 \le x_2\}} dx_2 dt_2 dx_1 dt_1 \\ &\le \frac{1}{4} \int_0^T \int_u^{u'} f^2(t_1,x_1)\, dx_1 dt_1. \end{aligned} \tag{25}$$

Concerning the term I_{12}, we use similar arguments as before and, moreover, we apply the fact that, for $x_1, x_2 \in [u, u')$, then $x_2 < 2^\alpha x_1$. Hence

$$\begin{aligned} I_{12} &\le n^2 \int_0^T \int_u^{u'} \int_0^T \int_u^{u'} f^2(t_2,x_2)\, t_2 x_2 \\ &\quad \times \exp\{-2n[(t_2-t_1)x_1 + (x_2-x_1)t_2]\} \mathbf{1}_{\{t_1 \le t_2\}} \mathbf{1}_{\{x_1 \le x_2\}} dx_2 dt_2 dx_1 dt_1 \\ &\le 2^\alpha n^2 \int_0^T \int_u^{u'} \int_0^T \int_u^{u'} f^2(t_2,x_2)\, t_2 x_1 \\ &\quad \times \exp\{-2n[(t_2-t_1)x_1 + (x_2-x_1)t_2]\} \mathbf{1}_{\{t_1 \le t_2\}} \mathbf{1}_{\{x_1 \le x_2\}} dx_2 dt_2 dx_1 dt_1 \\ &\le 2^{\alpha-2} \int_0^T \int_u^{u'} f^2(t_2,x_2)\, dx_2 dt_2. \end{aligned} \tag{26}$$

The analysis of the term I_2 is slightly more involved. Namely, notice first that $I_2 \le I_{21} + I_{22}$, where

$$I_{21} = n^2 \int_0^T \int_u^{u'} \int_0^T \int_u^{u'} f^2(t_1,x_1)\, t_1 x_1$$
$$\times \exp\{-2n[(t_2-t_1)x_2 + (x_1-x_2)t_1]\} \mathbf{1}_{\{t_1 \le t_2\}} \mathbf{1}_{\{x_2 \le x_1\}} dx_2 dt_2 dx_1 dt_1,$$

$$I_{22} = n^2 \int_0^T \int_u^{u'} \int_0^T \int_u^{u'} f^2(t_2,x_2)\, t_2 x_2 \\ \times \exp\{-2n[(t_2-t_1)x_2 + (x_1-x_2)t_1]\}\, \mathbf{1}_{\{t_1\le t_2\}} \mathbf{1}_{\{x_2\le x_1\}} dx_2 dt_2 dx_1 dt_1.$$

For the term I_{12}, we simply use that, by hypothesis, $x_1 \le 2^\alpha x_2$, and we integrate with respect to t_2, x_2, so that we end up with

$$\begin{aligned} I_{21} &\le 2^\alpha n^2 \int_0^T \int_u^{u'} \int_0^T \int_u^{u'} f^2(t_1,x_1)\, t_1 x_2 \\ &\qquad \times \exp\{-2n[(t_2-t_1)x_2 + (x_1-x_2)t_1]\}\, \mathbf{1}_{\{t_1\le t_2\}} \mathbf{1}_{\{x_2\le x_1\}} dx_2 dt_2 dx_1 dt_1 \\ &\le 2^{\alpha-2} \int_0^T \int_u^{u'} f^2(t_1,x_1)\, dx_1 dt_1. \end{aligned} \tag{27}$$

The term I_{22} is much more delicate. Namely, taking into account the integration's region in I_{22} as well as the fact that $x_1 - x_2 \le (2^\alpha - 1)x_2$ (because $x_1 \le 2^\alpha x_2$), it holds

$$\begin{aligned} 2(t_2-t_1)x_2 + 2(x_1-x_2)t_1 &\ge (t_2-t_1)x_2 + \frac{1}{2^\alpha-1}(t_2-t_1)(x_1-x_2) + \frac{1}{2^\alpha-1}(x_1-x_2)t_1 \\ &= (t_2-t_1)x_2 + \frac{1}{2^\alpha-1}(x_1-x_2)t_2. \end{aligned}$$

Therefore,

$$\begin{aligned} I_{22} &\le n^2 \int_0^T \int_u^{u'} \int_0^T \int_u^{u'} f^2(t_2,x_2)\, t_2 x_2 \\ &\qquad \times \exp\left\{-n[(t_2-t_1)x_2 + \frac{1}{2^\alpha-1}(x_1-x_2)t_2]\right\} \mathbf{1}_{\{t_1\le t_2\}} \mathbf{1}_{\{x_2\le x_1\}} dx_2 dt_2 dx_1 dt_1 \\ &\le (2^\alpha - 1) \int_0^T \int_u^{u'} f^2(t_2,x_2)\, dx_2 dt_2, \end{aligned} \tag{28}$$

where the latter expression has been obtained after integrating with respect to t_1, x_1.

We conclude the proof by putting together (25)-(28). □

Proof of Proposition 4.1: Let us consider the following dyadic-type partition of $(0,1]$:

$$(0,1] = \bigcup_{k=0}^{\infty} (a_{k+1}, a_k],$$

with $a_k = \frac{1}{2^{k\alpha}}$, for some $\alpha \ge 1$. In particular, observe that $a_k - a_{k+1} = \frac{2^\alpha - 1}{2^{(k+1)\alpha}}$ and we are in position to apply Lemma 4.2: for all $k \ge 0$,

$$E\left(\int_0^T \int_{a_{k+1}}^{a_k} f(t,x)\theta_n(t,x)\, dxdt\right)^2 \le \frac{3}{4}(2^{\alpha+1}-1) \int_0^T \int_{a_{k+1}}^{a_k} f(t,x)^2\, dxdt.$$

Therefore, we have the following estimations:

$$
\begin{aligned}
E\left(\int_0^T\int_0^1 f(t,x)\theta_n(t,x)\,dxdt\right)^2 &= E\left(\sum_{k=0}^{\infty}\int_0^T\int_{a_{k+1}}^{a_k} f(t,x)\theta_n(t,x)\,dxdt\right)^2 \\
&\le \sum_{k=0}^{\infty} 2^{k+1} E\left(\int_0^T\int_{a_{k+1}}^{a_k} f(t,x)\theta_n(t,x)\,dxdt\right)^2 \\
&\le \frac{3}{4}(2^{\alpha+1}-1)\sum_{k=0}^{\infty} 2^{k+1}\int_0^T\int_{a_{k+1}}^{a_k} f(t,x)^2\,dxdt. \qquad (29)
\end{aligned}
$$

Let $p,q>1$ be such that $\frac{1}{p}+\frac{1}{q}=1$. Then, applying Hölder's inequality, the last term of (29) can be bounded by

$$
\begin{aligned}
&\frac{3}{4}(2^{\alpha+1}-1)\sum_{k=0}^{\infty} 2^{k+1}\left(\int_0^T\int_{a_{k+1}}^{a_k} |f(t,x)|^{2p}\,dxdt\right)^{\frac{1}{p}}(a_k-a_{k+1})^{\frac{1}{q}} \\
&\quad\le \frac{3}{4}(2^{\alpha+1}-1)\left(\int_0^T\int_0^1 |f(t,x)|^{2p}\,dxdt\right)^{\frac{1}{p}}\sum_{k=0}^{\infty} 2^{k+1}\frac{(2^{\alpha}-1)^{\frac{1}{q}}}{2^{(k+1)\frac{\alpha}{q}}} \\
&\quad\le \frac{3}{4}(2^{\alpha+1}-1)(2^{\alpha}-1)^{\frac{1}{q}}\left(\int_0^T\int_0^1 |f(t,x)|^{2p}\,dxdt\right)^{\frac{1}{p}}\sum_{k=0}^{\infty}\frac{1}{2^{(k+1)\left(\frac{\alpha}{q}-1\right)}} \qquad (30)
\end{aligned}
$$

and this series is convergent whenever we take α such that $\alpha>q$. Hence, expression (30) may be bounded by

$$
\frac{3}{2}(2^{\alpha+1}-1)\frac{(2^{\alpha}-1)^{\frac{1}{q}}}{2^{\frac{\alpha}{q}-2}}\left(\int_0^T\int_0^1 |f(t,x)|^{2p}\,dxdt\right)^{\frac{1}{p}},
$$

which implies that the proof is complete. □

Remark 4.3 *It is worth noticing that, in the statement of Proposition 4.1, we have not been able to obtain the validity of the result for $p=1$. Indeed, as it can be deduced from its proof, the constant C_p in (23) blows up when $p\to 1$ (because $q\to\infty$, so $\alpha\to\infty$).*

By Proposition 3.1, a consequence of Proposition 4.1 is that the finite dimensional distributions of X_n (see (22)) converge, as n tends to infinity, to those of

$$
X(t,x)=\int_0^t\int_0^1 G_{t-s}(x,y)W(ds,dy).
$$

In order to prove that Theorem 1.4 applies for the Kac-Stroock processes, it only remains to verify that Hypothesis 1.3 is satisfied. In fact, this is given by the following result:

Proposition 4.4 *Let θ_n be the Kac-Stroock kernels. Then, for any even $m \in \mathbb{N}$, there exists a positive constant C_m such that, for all s_0, $s_0' \in [0,T]$ and x_0, $x_0' \in [0,1]$ satisfying $0 < s_0 < s_0' < 2s_0$ and $0 < x_0 < x_0' < 2x_0$, we have that*

$$\sup_{n\geq 1} E\left(\int_{s_0}^{s_0'}\int_{x_0}^{x_0'} f(s,y)\,\theta_n(s,y)dyds\right)^m \leq C_m\left(\int_{s_0}^{s_0'}\int_{x_0}^{x_0'} f(s,y)^2\,dyds\right)^{\frac{m}{2}},$$

for any $f \in L^2([0,T]\times[0,1])$.

Proof: To begin with, define

$$Z_n(s_0,x_0) := \int_0^{s_0}\int_0^{x_0} f(s,y)\theta_n(s,y)\,dyds$$

and observe that we can apply the same arguments as in the proof of [6, Lemma 3.3] (see p. 324 therein) in order to obtain the following estimate:

$$\begin{aligned} E(\Delta_{s_0,x_0}Z_n(s_0',x_0'))^m \leq & m!n^m\int_{[0,T]^m\times[0,1]^m}\prod_{i=1}^m\left(\mathbf{1}_{[s_0,s_0']}(s_i)\mathbf{1}_{[x_0,x_0']}(y_i)f(s_i,y_i)\sqrt{s_iy_i}\right)\\ &\times\exp\left\{-n[(s_m-s_{m-1})y_{(m-1)}+\cdots+(s_2-s_1)y_{(1)}]\right\}\\ &\times\exp\left\{-n[(y_{(m)}-y_{(m-1)})s_{m-1}+\cdots+(y_{(2)}-y_{(1)})s_1]\right\}\\ &\times\mathbf{1}_{\{s_1\leq\cdots\leq s_m\}}ds_1\cdots ds_m dy_1\cdots dy_m, \end{aligned}$$

where $y_{(1)},\dots,y_{(m)}$ denote the variables $y_1,\dots,y_m$ ordered increasingly. Hence

$$\begin{aligned} E(\Delta_{s_0,x_0}Z_n(s_0',x_0'))^m \leq & 2^m(s_0x_0)^{\frac{m}{2}}m!n^m\int_{[0,T]^m\times[0,1]^m}\prod_{i=1}^m\left(\mathbf{1}_{[s_0,s_0']}(s_i)\mathbf{1}_{[x_0,x_0']}(y_i)f(s_i,y_i)\right)\\ &\times\exp\left\{-nx_0[(s_m-s_{m-1})+\cdots+(s_2-s_1)]\right\}\\ &\times\exp\left\{-ns_0[(y_{(m)}-y_{(m-1)})+\cdots+(y_{(2)}-y_{(1)})]\right\}\\ &\times\mathbf{1}_{\{s_1\leq\cdots\leq s_m\}}ds_1\cdots ds_m dy_1\cdots dy_m. \end{aligned} \tag{31}$$

Notice that in (31) we have not been able to order the variables $y_1,\dots,y_m$, because neither the function $(s,y)\mapsto f(s,y)$ factorizes nor $(y_1\dots,y_m)\mapsto f(s_1,y_1)\cdots f(s_m,y_m)$ is symmetric. However, the fact that the variables s_i are ordered determines $\frac{m}{2}$ couples (s_1,s_2), $(s_3,s_4)\dots,(s_{m-1},s_m)$, such that the second element in each couple is greater than or equal to the first one. Concerning the variables y_i, we also have $\frac{m}{2}$ couples $(y_{(1)},y_{(2)}),\dots,(y_{(m-1)},y_{(m)})$ satisfying the same property.

The key point of the proof relies in factorizing the product in the first part of the right-hand side of (31) into two convenient products:

$$\prod_{j=1}^{\frac{m}{2}}\left(\mathbf{1}_{[s_0,s_0']}(s_{i_j})\mathbf{1}_{[x_0,x_0']}(y_{i_j})f(s_{i_j},y_{i_j})\right)\prod_{k=1}^{\frac{m}{2}}\left(\mathbf{1}_{[s_0,s_0']}(s_{r_k})\mathbf{1}_{[x_0,x_0']}(y_{r_k})f(s_{r_k},y_{r_k})\right),$$

where $\mathcal{I}=\{i_j,\,j=1,\dots,\frac{m}{2}\}$ and $\mathcal{R}=\{r_k,\,k=1,\dots,\frac{m}{2}\}$ are two disjoint subsequences of $\{1,\dots,m\}$. In particular, it holds that $\mathcal{I}\uplus\mathcal{R}=\{1,\dots,m\}$. These subsequences will be chosen using the following rule: any couple (s_i, s_{i+1}) will contain an element of the form s_{i_j} and one of the form s_{r_k}, and any couple $(y_{(i)}, y_{(i+1)})$ will contain an element of the form y_{i_j} and one of the form y_{r_k}. For this, we will split the m elements $f(s_1, y_1), \ldots, f(s_m, y_m)$ in two groups of $\frac{m}{2}$ elements:

$$A = \{f(s_{i_1}, y_{i_1}), \ldots, f(s_{i_{\frac{m}{2}}}, y_{i_{\frac{m}{2}}})\},$$

$$B = \{f(s_{r_1}, y_{r_1}), \ldots, f(s_{r_{\frac{m}{2}}}, y_{r_{\frac{m}{2}}})\}.$$

In order to determine the elements of each group, and such that the above condition is satisfied, we proceed by an iterative method: we will start with an element of A and we will associate to it an element of B satisfying what we want; then, to the latter element of B we will associate a suitable element of A, and so on. More precisely, we start, say, with $f(s_{i_1}, y_{i_1}) = f(s_1, y_1)$. Then, if at any step of the iteration procedure we have an element $f(s_{i_j}, y_{i_j}) \in A$, we will associate to it an element $f(s_{r_k}, y_{r_k}) \in B$ in such a way that $\{s_{i_j}, s_{r_k}\}$ forms one of the couples (s_i, s_{i+1}). On the other hand, if at any step of the iteration procedure we have an element $f(s_{r_k}, y_{r_k}) \in B$, then we will associate to it $f(s_{i_j}, y_{i_j}) \in A$ such that $\{y_{i_j}, y_{r_k}\}$ determines one of the couples $(y_{(i)}, y_{(i+1)})$. The only thing that remains to be clarified is what we are going to do in case that, at some step, we end up with and element of A or B which has already appeared before. In this case, we do not take the latter element, but another one which has not been chosen by now.

Let us illustrate the above-described procedure by considering a particular example: let $m = 8$ and assume that we fix $y_1, \ldots, y_8$ in such a way that

$$y_8 < y_5 < y_4 < y_7 < y_1 < y_6 < y_2 < y_3,$$

that is:

$$y_{(1)} = y_8,\ y_{(2)} = y_5,\ y_{(3)} = y_4,\ y_{(4)} = y_7,$$

$$y_{(5)} = y_1,\ y_{(6)} = y_6,\ y_{(7)} = y_2,\ y_{(8)} = y_3.$$

Recall that we assume that $s_1 \leq \cdots \leq s_8$. We start with $f(s_1, y_1) \in A$. Then, the iteration sequence will be the following:

$$f(s_1, y_1) \longrightarrow f(s_2, y_2) \longrightarrow f(s_3, y_3) \longrightarrow f(s_4, y_4)$$

$$\longrightarrow f(s_7, y_7) \longrightarrow f(s_8, y_8) \longrightarrow f(s_5, y_5) \longrightarrow f(s_6, y_6)$$

Thus, $A = \{f(s_1, y_1), f(s_3, y_3), f(s_7, y_7), f(s_5, y_5)\}$ and $B = \{f(s_2, y_2), f(s_4, y_4), f(s_8, y_8), f(s_6, y_6)\}$. In particular, any couple (s_i, s_{i+1}) (resp. $(y_{(i)}, y_{(i+1)})$) contains one s (resp. y) of the group A and one of B.

We can now come back to the analysis of the right-hand side of (31) and we can use the above detailed procedure to estimate it by $2^{m-1}(s_0 x_0)^{\frac{m}{2}} m!\,(J_1 + J_2)$, with

$$\begin{aligned} J_1 = n^m \int_{[0,T]^m \times [0,1]^m} & \prod_{i_j \in \mathcal{I}} \left(\mathbf{1}_{[s_0, s_0']}(s_{i_j}) \mathbf{1}_{[x_0, x_0']}(y_{i_j}) f(s_{i_j}, y_{i_j})^2 \right) \\ & \times \exp\left\{-n x_0 [(s_m - s_{m-1}) + \cdots + (s_2 - s_1)]\right\} \\ & \times \exp\left\{-n s_0 [(y_{(m)} - y_{(m-1)}) + \cdots + (y_{(2)} - y_{(1)})]\right\} \\ & \times \mathbf{1}_{\{s_1 \leq \cdots \leq s_m\}} ds_1 \cdots ds_m dy_1 \cdots dy_m. \end{aligned}$$

$$
\begin{aligned}
J_2 = n^m \int_{[0,T]^m\times[0,1]^m} & \prod_{r_k\in\mathcal{R}} \left(\mathbf{1}_{[s_0,s_0']}(s_{r_k})\mathbf{1}_{[x_0,x_0']}(y_{r_k})f(s_{r_k},y_{r_k})^2\right) \\
& \times \exp\left\{-nx_0[(s_m-s_{m-1})+\cdots+(s_2-s_1)]\right\} \\
& \times \exp\left\{-ns_0[(y_{(m)}-y_{(m-1)})+\cdots+(y_{(2)}-y_{(1)})]\right\} \\
& \times \mathbf{1}_{\{s_1\le\cdots\le s_m\}} ds_1\cdots ds_m dy_1\cdots dy_m.
\end{aligned}
$$

We will only deal with the term J_1, since J_2 can be treated using exactly the same arguments. The idea is to integrate in J_1 with respect to s_{r_k}, y_{r_k}, with $r_k \in \mathcal{R}$, for $k = 1,\dots,\frac{m}{2}$. Recall that the variables s_{r_k} (resp. y_{r_k}) have been chosen in such a way that they only appear once in each couple (s_i, s_{i+1}) (resp. $(y_{(i)}, y_{(i+1)})$). Observe that we have, for any $k = 1,\dots,\frac{m}{2}$,

$$
\int_{s_0}^{s_0'} \exp\left\{-nx_0(s_{r_k}-s_i)\right\} \mathbf{1}_{\{s_i\le s_{r_k}\}} ds_{r_k} \le C\frac{1}{n}
$$

or

$$
\int_{s_0}^{s_0'} \exp\left\{-nx_0(s_{i+1}-s_{r_k})\right\} \mathbf{1}_{\{s_{r_k}\le s_{i+1}\}} ds_{r_k} \le C\frac{1}{n},
$$

for some s_i and s_{i+1}, depending on which position occupies s_{r_k} in the corresponding couple. For the integrals with respect to y_{r_k} one obtains the same type of bound. Therefore,

$$
\begin{aligned}
J_1 &\le C_m \int_{[0,T]^{\frac{m}{2}}\times[0,1]^{\frac{m}{2}}} \prod_{j=1}^{\frac{m}{2}} \left(\mathbf{1}_{[s_0,s_0']}(s_{i_j})\mathbf{1}_{[x_0,x_0']}(y_{i_j})f(s_{i_j},y_{i_j})^2\right) ds_{i_1}\cdots ds_{i_{\frac{m}{2}}} dy_{i_1}\cdots dy_{i_{\frac{m}{2}}} \\
&= C_m \left(\int_0^T\int_0^1 \mathbf{1}_{[s_0,s_0']}(s)\mathbf{1}_{[x_0,x_0']}(y)f(s,y)^2\,dyds\right)^{\frac{m}{2}}. \qquad (32)
\end{aligned}
$$

As it has been mentioned, one can use the same arguments to get the same upper bound for J_2. Hence, the right-hand side of (31) can be estimated by (32), and this concludes the proof.

□

5 Convergence in law for the Donsker kernels

In this section, we aim to prove that the hypothesis of Theorem 1.4 are satisfied in the case where the approximation sequence is constructed in terms of the Donsker kernels. Namely, we consider $\{Z_k,\, k\in\mathbb{N}^2\}$ an independent family of identically distributed and centered random variables, with $E(Z_k^2)=1$ for all $k\in\mathbb{N}^2$, and such that $E(|Z_k|^m)<+\infty$ for all $k\in\mathbb{N}^2$, and some even number $m\ge 10$. Then, for all $n\ge 1$ and $(t,x)\in[0,T]\times[0,1]$, we define the kernels

$$
\theta_n(t,x) = n \sum_{k=(k^1,k^2)\in\mathbb{N}^2} Z_k\, \mathbf{1}_{[k^1-1,k^1)\times[k^2-1,k^2)}(tn,xn).
$$

Let us remind that the approximation sequence is given by

$$X_n(t,x) = \int_0^t \int_0^1 G_{t-s}(x,y)\theta_n(s,y)dyds,\ (t,x) \in [0,T]\times[0,1]. \tag{33}$$

Recall that Hypothesis 1.1 is a consequence of the extension of Donsker's theorem to the plane (see, for instance, [37]). On the other hand, we have the following result:

Lemma 5.1 *Let θ_n be the above defined Donsker kernels. Then, there exists a positive constant C_m such that, for any $f \in L^2([0,T]\times[0,1])$, we have*

$$E\left(\int_0^T \int_0^1 f(t,x)\theta_n(t,x)dxdt\right)^m \le C_m \left(\int_0^T \int_0^1 f^2(t,x)\,dxdt\right)^{\frac{m}{2}}, \tag{34}$$

for all $n \ge 1$.

Remark 5.2 *Notice that, taking into account that $m \ge 10$, inequality (34) implies both Hypothesis 1.2 and 1.3, so that the hypotheses of Theorem 1.4 are satisfied for the Donsker kernels.*

Proof of Lemma 5.1:

First, we observe that we can write

$$\begin{aligned} & E\left(\int_0^T \int_0^1 f(t,x)\theta_n(t,x)dxdt\right)^m \qquad (35) \\ = & \int_{[0,T]^m\times[0,1]^m} f(t_1,x_1)\cdots f(t_m,x_m) E\left[\prod_{j=1}^m \theta_n(t_j,x_j)\right] dt_1\cdots dt_m dx_1\cdots dx_m \end{aligned}$$

By definition of θ_n,

$$\begin{aligned} & E\left[\prod_{j=1}^m \theta_n(t_j,x_j)\right] \\ = \ & n^m E\left[\prod_{j=1}^m \left(\sum_{k=(k^1,k^2)\in\mathbb{N}^2} Z_k\, \mathbf{1}_{[k^1-1,k^1)}(t_j n)\mathbf{1}_{[k^2-1,k^2)}(x_j n)\right)\right] \\ = \ & n^m \sum_{k_1,\dots,k_m\in\mathbb{N}^2} E(Z_{k_1}\cdots Z_{k_m}) \prod_{j=1}^m \left(\mathbf{1}_{[k_j^1-1,k_j^1)}(t_j n)\mathbf{1}_{[k_j^2-1,k_j^2)}(x_j n)\right). \end{aligned}$$

Notice that, by hypothesis, $E(Z_{k_1}\cdots Z_{k_m}) = 0$ if, for some $j \in \{1,\dots,m\}$, we have that $k_j \neq k_l$ for all $l \in \{1,\dots,m\}\setminus\{j\}$; that is, if some variable Z_{k_j} appears only once in the product $Z_{k_1}\cdots Z_{k_m}$.

On the other hand, since $E(|Z_k|^m) < \infty$ for all $k \in \mathbb{N}^2$, then $E(Z_{k_1}\cdots Z_{k_m})$ is bounded for all $k_1,\dots,k_m \in \mathbb{N}^2$. Hence,

$$E\left[\prod_{j=1}^m \theta_n(t_j,x_j)\right] \le n^m C_m \sum_{(k_1,\dots,k_m)\in A^m} \prod_{j=1}^m \left(\mathbf{1}_{[k_j^1-1,k_j^1)}(t_j n)\mathbf{1}_{[k_j^2-1,k_j^2)}(x_j n)\right),$$

with

$$A^m = \{(k_1, \dots, k_m) \in \mathbb{N}^{2m};\ \text{ for all } l \in \{1, \dots, m\}, k_l = k_j \text{ for some } j \in \{1, \dots, m\} \setminus \{l\}\}.$$

Notice that we have the following estimation:

$$\sum_{(k_1,\dots,k_m) \in A^m} \prod_{j=1}^{m} \left(\mathbf{1}_{[k_j^1 - 1, k_j^1)}(t_j n) \mathbf{1}_{[k_j^2 - 1, k_j^2)}(x_j n) \right) \le \mathbf{1}_{D^m}(t_1, \dots, t_m; x_1, \dots, x_m),$$

where D^m denotes the set of $(t_1, \dots, t_m; x_1, \dots, x_m) \in [0, T]^m \times [0, 1]^m$ satisfying the following property: for all $l \in \{1, \dots, m\}$, there exists $j \in \{1, \dots, m\} \setminus \{l\}$ such that $|t_j - t_l| < \frac{1}{n}$ and $|x_j - x_l| < \frac{1}{n}$ and, moreover, if there is some $r \neq j, l$ verifying $|t_l - t_r| < \frac{1}{n}$ and $|x_l - x_r| < \frac{1}{n}$, then $|t_j - t_r| < \frac{1}{n}$ and $|x_j - x_r| < \frac{1}{n}$.

Next, observe that we can bound $I_{D^m}(t_1, \dots, t_m; x_1, \dots, x_m)$ by a finite sum of products of indicators, where in each product of indicators there appear all the m variables $t_1, \dots, t_m$ and all the m variables $x_1, \dots, x_m$, but each indicator concerns only two or three of them. Moreover, each variable only appears in one of the indicators of each product and, whenever we have some indicator concerning two variables t_j and t_l, (respectively three variables t_j, t_l and t_r), we have the same indicator for the variables x_j and x_l, (respectively for the variables x_j, x_l and x_r). Therefore, expression (35) can be bounded by a finite sum of products of the following two kinds of terms:

(i) For some $l, j \in \{1, \dots, m\}$ such that $l \neq j$,

$$C_m n^2 \int_{[0,T]^2 \times [0,1]^2} |f(t_l, x_l)||f(t_j, x_j)| \mathbf{1}_{[0, \frac{1}{n})}(|t_j - t_l|) \mathbf{1}_{[0, \frac{1}{n})}(|x_j - x_l|) dt_j dt_l dx_j dx_l. \quad (36)$$

(ii) For some $l, j, r \in \{1, \dots, m\}$ such that $l \neq j$, $l \neq r$ and $r \neq j$,

$$\begin{aligned} C_m n^3 \int_{[0,T]^3 \times [0,1]^3} & |f(t_l, x_l)||f(t_j, x_j)||f(t_r, x_r)| \\ & \times \mathbf{1}_{[0, \frac{1}{n})}(|t_j - t_l|) \mathbf{1}_{[0, \frac{1}{n})}(|t_l - t_r|) \mathbf{1}_{[0, \frac{1}{n})}(|t_j - t_r|) \\ & \times \mathbf{1}_{[0, \frac{1}{n})}(|x_j - x_l|) \mathbf{1}_{[0, \frac{1}{n})}(|x_l - x_r|) \mathbf{1}_{[0, \frac{1}{n})}(|x_j - x_r|) dt_j dt_l dt_r dx_j dx_l dx_r. \end{aligned}$$

Then, it turns out that, in order to conclude the proof, it suffices to bound the first type of term (i) by $C_m \int_0^T \int_0^1 f^2(t, x)\, dx dt$ and the second one (ii) by $C_m \left(\int_0^T \int_0^1 f^2(t, x)\, dx dt \right)^{\frac{3}{2}}$.

Let us use first the fact that, for all $a, b \in \mathbb{R}$, $2ab \le a^2 + b^2$, so that a term of the form (36) can be bounded, up to some constant, by

$$\begin{aligned} & C_m n^2 \int_{[0,T]^2 \times [0,1]^2} f^2(t_l, x_l) \mathbf{1}_{[0, \frac{1}{n})}(|t_j - t_l|) \mathbf{1}_{[0, \frac{1}{n})}(|x_j - x_l|) dt_j dt_l dx_j dx_l \\ \le \ & C_m \int_0^T \int_0^1 f^2(t, x)\, dx dt. \end{aligned}$$

On the other hand, using that for all $a,b,c\in\mathbb{R}^+$, $2abc\le(ab^2+ac^2)$, we can study the terms of type (ii) in the following way:

$$\begin{aligned}
& C_m n^3\int_{[0,T]^3\times[0,1]^3}|f(t_l,x_l)||f(t_j,x_j)||f(t_r,x_r)| \\
& \times\mathbf{1}_{[0,\frac{1}{n})}(|t_j-t_l|)\mathbf{1}_{[0,\frac{1}{n})}(|t_l-t_r|)\mathbf{1}_{[0,\frac{1}{n})}(|t_j-t_r|) \\
& \times\mathbf{1}_{[0,\frac{1}{n})}(|x_j-x_l|)\mathbf{1}_{[0,\frac{1}{n})}(|x_l-x_r|)\mathbf{1}_{[0,\frac{1}{n})}(|x_j-x_r|)dt_jdt_ldt_rdx_jdx_ldx_r \\
\le\ & C_m n^3\int_{[0,T]^3\times[0,1]^3}|f(t_l,x_l)|f^2(t_j,x_j) \\
& \times\mathbf{1}_{[0,\frac{1}{n})}(|t_j-t_l|)\mathbf{1}_{[0,\frac{1}{n})}(|t_l-t_r|)\mathbf{1}_{[0,\frac{1}{n})}(|t_j-t_r|) \\
& \times\mathbf{1}_{[0,\frac{1}{n})}(|x_j-x_l|)\mathbf{1}_{[0,\frac{1}{n})}(|x_l-x_r|)\mathbf{1}_{[0,\frac{1}{n})}(|x_j-x_r|)dt_jdt_ldt_rdx_jdx_ldx_r \\
\le\ & C_m n\int_{[0,T]^2\times[0,1]^2}|f(t_l,x_l)|f^2(t_j,x_j)\,\mathbf{1}_{[0,\frac{1}{n})}(|t_j-t_l|)\mathbf{1}_{[0,\frac{1}{n})}(|x_j-x_l|)dt_jdt_ldx_jdx_l \\
=\ & C_m n\int_0^T\int_0^1|f(t_l,x_l)|\left(\int_0^T\int_0^1 f^2(t_j,x_j)\,\mathbf{1}_{[0,\frac{1}{n})}(|t_j-t_l|)\mathbf{1}_{[0,\frac{1}{n})}(|x_j-x_l|)dt_jdx_j\right)dt_ldx_l.
\end{aligned}$$

At this point, we apply Cauchy-Schwarz inequality, so that the latter expression can be estimated by

$$\begin{aligned}
& C_m n\left(\int_0^T\int_0^1 f^2(t_l,x_l)\,dt_ldx_l\right)^{\frac{1}{2}} \\
& \times\left(\int_0^T\int_0^1\left(\int_0^T\int_0^1 f^2(t_j,x_j)\mathbf{1}_{[0,\frac{1}{n})}(|t_j-t_l|)\mathbf{1}_{[0,\frac{1}{n})}(|x_j-x_l|)dt_jdx_j\right)^2dt_ldx_l\right)^{\frac{1}{2}} \\
=\ & C_m n\left(\int_0^T\int_0^1 f^2(t_l,x_l)\,dt_ldx_l\right)^{\frac{1}{2}} \\
& \times\left(\int_{[0,T]^3\times[0,1]^3}f^2(t_j,x_j)f(t_p,x_p)^2\mathbf{1}_{[0,\frac{1}{n})}(|t_j-t_l|)\mathbf{1}_{[0,\frac{1}{n})}(|x_j-x_l|)\right. \\
& \left.\times\mathbf{1}_{[0,\frac{1}{n})}(|t_p-t_l|)\mathbf{1}_{[0,\frac{1}{n})}(|x_p-x_l|)dt_jdt_pdt_ldx_jdx_pdx_l\right)^{\frac{1}{2}} \\
\le\ & C_m\left(\int_0^T\int_0^1 f^2(t,x)dxdt\right)^{\frac{3}{2}}.
\end{aligned}$$

This finishes the proof of the lemma. □

A Appendix

In this appendix, we give a sketch of the proof of Lemma 2.3 and discuss the relation between our results and those of Manthey in [24] (see also [25]).

Proof of Lemma 2.3: As we have already pointed out, we will only give the main lines of the proof.

Let $f \in E$ and $h \in \mathcal{C}^1(\mathbb{R})$ having a bounded derivative. We aim to prove that, for any $\eta > 0$, it holds

$$|E[h(J^n(f))] - E[h(J(f))]| < \eta, \tag{37}$$

for sufficiently big n. For this, the idea is to consider an element g in D which is close to f with respect to the norm $\|\cdot\|$. Then, one splits the left-hand side of (37) in several terms, which can be easily treated using the following facts:

1. When f is replaced by g, we have that the left-hand side of (37) converges to zero, by hypothesis.

2. One keeps control of the remaining terms using that h defines a Lipschitz function and that (13) and (14) hold.

□

Relation with Manthey results

In [24], the author considers the family of processes $\{X_n,\, n \in \mathbb{N}\}$ such that each X_n is the mild solution of the equation

$$\frac{\partial X_n}{\partial t}(t,x) - \frac{\partial^2 X_n}{\partial x^2}(t,x) = \theta_n(t,x), \quad (t,x) \in [0,T] \times [0,1],$$

with null initial condition and Dirichlet boundary conditions. The processes θ_n are correlated noises satisfying the following conditions:

(i) For all $(t,x) \in [0,T] \times [0,1]$,

$$\int_0^t \int_0^x \theta_n(s,y)^2 dyds < \infty, \ a.s.$$

(ii) For each $m \in \mathbb{N}$ and $(t_1,x_1), \dots, (t_m,x_m) \in [0,T] \times [0,1]$, the random vector

$$\left(\int_0^{t_1} \int_0^{x_1} \theta_n(s,y) dyds, \dots, \int_0^{t_m} \int_0^{x_m} \theta_n(s,y) dy \right)$$

converges weakly to $(W(t_1,x_1), \dots, W(t_m,x_m))$, where we recall that $\{W(t,x),\ (t,x) \in [0,T] \times [0,1]\}$ denotes a Brownian sheet.

(iii) For all $(t,x) \in [0,T] \times [0,1]$, $E[\theta_n(t,x)] = 0$.

(iv) There exists $n_0 \in \mathbb{N}$ such that

$$\sup_{\substack{n \geq n_0 \\ (t,x) \in [0,T] \times [0,1]}} \int_0^t \int_0^x \left| E\big[\theta_n(s,y)\theta_n(t,x)\big] \right| dyds < \infty.$$

Under these hypotheses, it has been proved that X_n converges weakly, in the sense of the convergence of finite dimensional distributions, to the process X which is the mild solution of

$$\frac{\partial X}{\partial t}(t,x) - \frac{\partial^2 X}{\partial x^2}(t,x) = \dot{W}(t,x), \quad (t,x) \in [0,T] \times [0,1].$$

Furthermore, it is showed that, if the processes θ_n are Gaussian, the convergence also holds in $\mathcal{C}([0,T] \times [0,1])$. These results are extended to the quasi-linear equation (2) in [25].

First, it is worth pointing out that one can easily see that condition (iii) is not essential in the proof. Moreover, in Manthey's result, condition (iv) is stated in a weaker form, though we believe that, in his proof, the statement which has been used is indeed condition (iv) as stated above (see the last inequality in p. 163 of [24]).

Secondly, one can easily see that condition (iv) stated above implies Hypothesis 1.2 with $q = 2$. Therefore, the hypotheses assumed in Proposition 3.1 (which assures the convergence of the finite dimensional distributions) are weaker than (i)-(iv).

Eventually, processes θ_n given by the Kac-Stroock processes and the Donsker kernels are not Gaussian so that, if conditions (i)-(iv) were satisfied, using Manthey's result only convergence of the finite dimensional distributions could be obtained. In fact, it is straightforward to check that the Donsker kernels satisfy these conditions, but condition (iv) fails for the Kac-Stroock processes. This is proved in the following lemma:

Lemma A.1 *Assume that* $\{\theta_n(t,x),\ (t,x) \in [0,T] \times [0,1]\}$*,* $n \geq 1$*, is the Kac-Stroock process (9). Then, the family* $\{\theta_n,\ n \in \mathbb{N}\}$ *does not satisfy condition (iv) above.*

Proof: We will show that, when

$$\theta_n(s,y) = n\sqrt{sy}\,(-1)^{N_n(s,y)},$$

then the quantity

$$\int_0^T \int_0^1 |E[\theta_n(s,y)\theta_n(t,x)]|\; dyds$$

is not uniformly bounded in n, t and x. Indeed, it holds

$$\int_0^T \int_0^1 |E(\theta_n(s,y)\theta_n(t,x))|\, dyds = \int_0^T \int_0^1 n^2\sqrt{sytx}\, E\left[(-1)^{N_n(s,y)+N_n(t,x)}\right] dyds. \quad (38)$$

Owing to the proof of [6, Lemma 3.1] (see also [5, Lemma 3.2]), we have that

$$\begin{aligned} E\left[(-1)^{N_n(s,y)+N_n(t,x)}\right] &= e^{-2n[(t-s)x+(x-y)s]}\,\mathbf{1}_{\{s\leq t, y\leq x\}} + e^{-2n[(t-s)x+(y-x)s]}\,\mathbf{1}_{\{s\leq t, y\geq x\}} \\ &\quad + e^{-2n[(s-t)y+(x-y)t]}\,\mathbf{1}_{\{s\geq t, y\leq x\}} + e^{-2n[(s-t)y+(y-x)t]}\,\mathbf{1}_{\{s\geq t, y\geq x\}}. \end{aligned}$$

Then, expression (38) is the sum of four positive integrals. It is clear that one of them is given by

$$I(n,t,x) = \int_0^t \int_x^1 n^2\sqrt{sytx}\, e^{-2n[(t-s)x+(y-x)s]} dyds.$$

We will check that this integral is not uniformly bounded. In fact,

$$\begin{aligned}\sup_{n,t,x} I(n,t,x) &\geq \sup_n I\left(n,T,\frac{1}{n}\right)\\ &= \sup_n \sqrt{T}\int_0^T\int_{\frac{1}{n}}^1 n^2\frac{\sqrt{sy}}{\sqrt{n}}\,e^{-2n[(T-s)\frac{1}{n}+(y-\frac{1}{n})s]}dyds\\ &= \sqrt{T}e^{-2T}\sup_n\int_0^T\int_{\frac{1}{n}}^1 n^{\frac{3}{2}}\sqrt{sy}\,e^{4s}e^{-2nys}dyds\\ &= \sqrt{T}e^{-2T}\sup_n\int_0^T\int_1^n \sqrt{sz}\,e^{4s}e^{-2zs}dzds\\ &= \sqrt{T}e^{-2T}\int_0^T\int_1^\infty \sqrt{sz}\,e^{4s}e^{-2zs}dzds. \qquad (39)\end{aligned}$$

Let us apply the change of coordinates $v = sz$, for any fixed s, and then Fubini Theorem in the last integral of (39), so that we end up with

$$\sup_{n,t,x} I(n,t,x) \geq \sqrt{T}e^{-2T}\int_0^{+\infty}\sqrt{v}\,e^{-2v}\left(\int_0^{v\wedge T}\frac{1}{s}\,e^{4s}ds\right)dv,$$

and the latter is clearly divergent. This fact concludes the proof. □

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