Duality in spaces of finite linear combinations of atoms

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Abstract

In this note we describe the dual and the completion of the space of finite linear combinations of (p,∞)-atoms, $0 < p \leq 1$. As an application, we show an extension result for operators uniformly bounded on (p,∞)-atoms, $0 < p < 1$, whose analogue for $p = 1$ is known to be false. Let $0 < p < 1$ and let T be a linear operator defined on the space of finite linear combinations of (p,∞)-atoms, $0 < p < 1$, which takes values in a Banach space B. If T is uniformly bounded on (p,∞)-atoms, then T extends to a bounded operator from $H^p(\mathbb{R}^n)$ into B.

1 Introduction

For each $0 < p \leq 1$ consider the space F^p of finite linear combinations of (p,∞)-atoms, endowed with its natural norm

$$\|f\|_{F^p} = \inf\{{\sum_j}' |\lambda_j| : f = {\sum_j}' \lambda_j a_j\,,\ a_j \text{ a } (p,\infty)\text{-atom}\,,\ \lambda_j \in \mathbb{C}\}\,, \tag{1}$$

where $\sum'$ denotes a finite sum. Recall that a is a (p,∞)-atom if a is a measurable function supported on a ball B, satisfying the cancellation condition

$$\int a(x) x^\alpha \, dx = 0, \quad |\alpha| \leq n(\frac{1}{p} - 1),$$

and the size condition

$$|a| \leq \frac{1}{|B|^{\frac{1}{p}}}.$$

The space F^p is clearly contained in $H^p = H^p(\mathbb{R}^n)$, the standard real Hardy space on $\mathbb{R}^n$. The elements of H^p are the distributions that admit an atomic decomposition, $f = \sum_{j=1}^\infty \lambda_j a_j$, converging in the sense of distributions, for some (p,∞)-atoms a_j and scalars λ_j with $\sum_{j=1}^\infty |\lambda_j|^p < \infty$ (for $p = 1$, $H^1 \subset L^1$ and atomic sums converge in the L^1-norm). In [MTW] Meyer,Taibleson and Weiss observed that the F^p-norm is not comparable to the H^p-norm on F^p . Recently, it was shown in [B] that the Meyer-Taibleson-Weiss result leads to the following conclusion in the case $p = 1$: there exists a bounded linear functional on F^1 which does not extend to a bounded linear functional on H^1. In other words, there is a linear operator which is uniformly bounded on $(1, \infty)$-atoms but does not extend to a bounded linear operator on H^1.

In this paper we describe the structure of the completion $\widetilde{F^p}$ of F^p, $0 < p \leq 1$, and of its dual space. We show in particular that, when $p < 1$, F^p and H^p have the same dual, and therefore no example like the one in [B] can be exhibited for $p < 1$. An immediate consequence of this is that if $0 < p < 1$ and the linear operator

$$T : F^p \to B, \tag{2}$$

maps F^p into a Banach space B satisfying the inequality

$$\|T(a)\|_B \leq C\,,$$

for some positive constant C and all (p, ∞)-atoms, then T extends to a bounded linear operator from H^p into B. The argument proceeds by duality as follows. Take any u in the dual B^* of B. Since $u \circ T \in (F^p)^* = (H^p)^*$,

$$|u(T(f))| \leq C\,\|u\|\,\|f\|_{H^p}\,,$$

and so, by the dual expression of the norm in a Banach space,

$$\|T(f)\|_B \leq C\,\|f\|_{H^p}\,.$$

We prove the following facts about $\widetilde{F^p}$, $0 < p \leq 1$.

(i) The closed subspace $\widetilde{F^{p,c}}$ of $\widetilde{F^p}$ spanned by the continuous (p, ∞)-atoms is isomorphic to H^p as a Banach space, and $\widetilde{F^p}$ splits as the direct sum of $\widetilde{F^{p,c}}$ and a *non-trivial* complementary closed subspace N^p.

(ii) Every element ξ of $\widetilde{F^p}$ admits an atomic decomposition

$$\xi = \sum_{j=1}^{\infty} \lambda_j a_j\,,$$

for (p, ∞)-atoms a_j and scalars λ_j with $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Moreover, the $\widetilde{F^p}$-norm of ξ is equivalent to its atomic norm

$$\inf\left\{ \Big(\sum_{j=1}^{\infty} |\lambda_j|^p \Big)^{\frac{1}{p}} : \sum_{j=1}^{\infty} \lambda_j a_j = \xi \text{ in } \widetilde{F^p} \right\}.$$

(iii) If an atomic sum $\sum_{j=1}^{\infty} \lambda_j a_j$, with λ_j and a_j as above, converges to 0 in $\widetilde{F^p}$, it also converges to 0 in H^p, but not viceversa. In fact, N^p consists of those elements of $\widetilde{F^p}$ that are represented by atomic sums converging to 0 in H^p.

In other words, H^p and $\widetilde{F^p}$ are both quotients of the space of "formal series" of (p,∞)-atoms with ℓ^p coefficients, but the equivalence relation defining $\widetilde{F^p}$ is finer than that defining H^p.

So, the reason why $(F^1)^*$ is strictly larger than $(H^1)^*$ is that it is the direct sum of $(\widetilde{F^{1,c}})^* = (H^1)^*$ and $(N^1)^*$. Notice that $(N^1)^*$ is non-trivial, as the dual of the non-trivial Banach space N^1. On the other hand, it turns out that $(N^p)^*$ is trivial for $p < 1$.

To describe our results we need to introduce some notation and recall some basic classical facts in the theory of Banach algebras (see Section 3 for details).

Denote by $L^\infty_0(\mathbb{R}^n)$ the space of bounded measurable functions on $\mathbb{R}^n$ vanishing at infinity. Then $L^\infty_0(\mathbb{R}^n)$ is a commutative Banach algebra without unit, and its maximal ideal space is a locally compact, non-compact space, which we call $\widehat{\mathbb{R}^n}$.

By the Gelfand-Naimark theorem, the Gelfand transform establishes an isometric isomorphism between $L^\infty_0(\mathbb{R}^n)$ and the algebra $C_0(\widehat{\mathbb{R}^n})$ of all continuous functions on $\widehat{\mathbb{R}^n}$ vanishing at ∞. On the other hand, $C_0(\mathbb{R}^n)$ is a closed subalgebra of $L^\infty_0(\mathbb{R}^n)$, and its maximal ideal space is $\mathbb{R}^n$. This embedding induces a continuous projection π from $\widehat{\mathbb{R}^n}$ onto $\mathbb{R}^n$.

In a similar way, given any ball B in $\mathbb{R}^n$, the maximal ideal space of $L^\infty(B)$ is a compact space $\widehat{B}$, endowed with a projection π_B onto $\bar{B}$ induced by the inclusion of $C(\bar{B})$ in $L^\infty(B)$. Moreover, $L^\infty(B) \cong C(\widehat{B})$, again by the Gelfand-Naimark theorem.

The restriction map $f \mapsto f_{|B}$ from $L^\infty_0(\mathbb{R}^n)$ to $L^\infty(B)$ induces a natural embedding $\iota_B : \widehat{B} \to \widehat{\mathbb{R}^n}$, which is compatible with the projections π and π_B, in the sense that

$$\pi_B = \pi \circ \iota_B \ .$$

Similar embeddings $\iota_{B,B'} : \widehat{B'} \to \widehat{B}$ exist for pairs of balls B, B' with $B' \subset B$, with the same compatibility with respect to the corresponding projections.

Denote by m the Lebesgue measure on $\mathbb{R}^n$. The continuous linear functional $f \mapsto \int f\,dm$ on $L^\infty(B)$ is represented by a positive Borel measure $\widehat{m}_B$ on $\widehat{B}$, that is,

$$\int f\,dm = \int \hat{f}\,d\widehat{m}_B, \quad f \in L^\infty(B) \ . \tag{3}$$

If B is contained in a second ball B', then the restriction of $\widehat{m}_{B'}$ to $\widehat{B}$ is precisely $\widehat{m}_B$ and thus we can define a positive Borel measure $\widehat{m}$ globally on $\widehat{\mathbb{R}^n}$ by requiring that its restriction to $\widehat{B}$ be $\widehat{m}_B$ for each ball B.

We can now state our main result.

Theorem.

(A) *Let ℓ be a bounded linear functional on F^1. Then there exist a function $b \in BMO(\mathbb{R}^n)$ and a Radon measure μ on $\widehat{\mathbb{R}^n}$, singular with respect to $\widehat{m}$, satisfying*
$$|\mu|(\widehat{B}) \leq C\, m(B), \quad \text{for each ball} \quad B, \tag{4}$$
such that
$$\ell(f) = \int f\, b\, dm + \int \hat{f}\, d\mu, \quad f \in F^1. \tag{5}$$
Conversely, if b and μ are as above, then the identity (5) *defines a bounded linear functional on F^1 and*
$$\|\ell\|_{(F^1)^*} \cong \|b\|_{BMO} + \sup_B \frac{|\mu|(\widehat{B})}{m(B)} .$$

(B) *Each bounded linear functional on F^p, $0 < p < 1$, extends uniquely to a bounded linear functional on $H^p(\mathbb{R}^n)$. Thus $(F^p)^* = H^p(\mathbb{R}^n)^*$, $0 < p < 1$.*

It is clear that relation (5) determines the function b and the measure μ uniquely. Therefore $(F^1)^*$ differs from $(H^1)^* = BMO$ by the presence of the complementary subspace S of singular measures satisfying (4). We will show that S is non-trivial; in fact, the Meyer, Taibleson and Weiss argument may be interpreted as the construction of a non-zero measure in S. The decomposition of $(F^1)^*$ as $BMO \oplus S$ is the dual counterpart of the decomposition of $\widetilde{F^1}$ as $\widetilde{F^{1,c}} \oplus N^1$, although S and BMO do not coincide with the annihilators of $\widetilde{F^{1,c}}$ and N^1 respectively.

The nature of the elements of N^p, including $p = 1$, is somehow mysterious. It is not clear at all to us if they can be represented by concrete analytic objects.

Section 2 contains the discussion of the completion of F^p and a constructive argument which proves the non-triviality of N^p. In Section 3 we prove the Theorem. We also give an example of a non-zero singular measure satisfying (4).

We remark here that a variation of the main argument in the proof of the Theorem provides an alternative proof of some results in [MSV] and [YZ]) on the equivalence of the finite and infinite atomic norms of $(1, q)$-atoms, $q < \infty$, and on extension of bounded operators defined on finite linear combinations of (p, q)-atoms with $1 < q < \infty$.

2 The completion of F^p

Let $F^{p,c}$ stand for the subspace of H^p consisting of finite linear combinations of continuous (p, ∞) atoms. A surprising recent result in [MSV] states that the H^p and the F^p norms are equivalent on $F^{p,c}$, $0 < p \leq 1$. Indeed, the result is proved in [MSV] only for $p = 1$, but, as suggested in Remark 3.2 there, the same argument extends to the case $0 < p < 1$.

More precisely, we can quote Lemma 3.1 and Remark 3.2 in [MSV] as follows.

Lemma 1. *The following norms are equivalent on $F^{p,c}$:*

(a) *the H^p-norm;*

(b) *the F^p-norm* (1)*;*

(c) *the $F^{p,c}$-norm*

$$\|f\|_{F^{p,c}} = \inf\{\sum_j{}' |\lambda_j| : f = \sum_j{}' \lambda_j a_j\,,\ a_j \text{ a continuous } (p,\infty)\text{-atom}\,,\ \lambda_j \in \mathbb{C}\}\,.$$

Since $F^{p,c}$ is dense in H^p, the natural inclusion of $F^{p,c}$ in F^p extends uniquely to a continuous linear operator T from H^p to $\widetilde{F^p}$. By Lemma 1, T maps H^p isomorphically onto the closure $\widetilde{F^{p,c}}$ of $F^{p,c}$ in $\widetilde{F^p}$. Notice that, again by Lemma 1, $\widetilde{F^{p,c}}$ is the completion of $F^{p,c}$ endowed either with the norm $\|\cdot\|_{F^{p,c}}$ or with the norm inherited from F^p.

On the other hand, the inclusion of F^p (endowed with its natural norm) into H^p is continuous, and it extends to a continuous linear operator U from $\widetilde{F^p}$ to H^p. We then have the diagram

$$H^p \xrightarrow{T} \widetilde{F^p} \xrightarrow{U} H^p\,,$$

with $U \circ T$ being the identity map. In particular U is surjective. Set $P = T \circ U$, so that P is a projection, that is, $P^2 = P$. The kernel of P is the kernel of U, which we denote by N^p, and the kernel of $I - P$ is $T(H^p) = \widetilde{F^{p,c}}$. Hence we get the topological direct sum decomposition

$$\widetilde{F^p} = \widetilde{F^{p,c}} \oplus N^p\,.$$

Notice that N^p is non-trivial, since otherwise the H^p and the F^p norms would be comparable on F^p.

To better understand the space $\widetilde{F^p}$ we prove now the following.

Proposition. *Given any sequence of (p,∞) atoms a_j and any ℓ^p-sequence of scalars λ_j, the series $\sum_{j=1}^\infty \lambda_j\, a_j$ converges in $\widetilde{F^p}$ to an element ξ such that $\|\xi\|^p_{\widetilde{F^p}} \le \sum_{j=1}^\infty |\lambda_j|^p$. Conversely, each $\xi \in \widetilde{F^p}$ can be written as*

$$\xi = \sum_{j=1}^\infty \lambda_j\, a_j\,, \tag{6}$$

where each a_j is a (p,∞) atom and the sum is convergent in $\widetilde{F^p}$. Moreover,

$$\|\xi\|^p_{\widetilde{F^p}} = \inf\left\{\sum_{j=1}^\infty |\lambda_j|^p\right\}\,, \tag{7}$$

where the infimum is taken over all decompositions (6) *of ξ.*

Proof. Let ξ be an element of $\widetilde{F^p}$. To prove (6), express ξ as the limit in $\widetilde{F^p}$ of a sequence S_k of elements of F^p. Given $\epsilon > 0$, we may assume that $\|S_1\|^p_{F^p} < (1+\epsilon)\|\xi\|^p_{\widetilde{F^p}}$ and that $\|S_k - S_{k+1}\|^p_{F^p} < \epsilon^k \, \|\xi\|^p_{\widetilde{F^p}}$. Thus

$$\xi = \lim_{k\to\infty} S_1 + (S_2 - S_1) + \cdots + (S_k - S_{k-1}) \, .$$

Set

$$S_1 = \sum_{j=1}^{N_1} \lambda_j \, a_j \, ,$$

where the above expression of has been chosen so that

$$\sum_{j=1}^{N_1} |\lambda_j|^p < (1+\epsilon)\|\xi\|^p_{\widetilde{F^p}} \, .$$

Similarly, set

$$S_\ell - S_{\ell-1} = \sum_{j=N_{\ell-1}+1}^{N_\ell} \lambda_j \, a_j \, , \quad \ell \geq 2 \, ,$$

with

$$\sum_{j=N_{\ell-1}+1}^{N_\ell} |\lambda_j|^p < \epsilon^\ell \|\xi\|^p_{\widetilde{F^p}} \, .$$

Then $\sum_{j=1}^{\infty} |\lambda_j|^p < (1-\epsilon)^{-1}\|\xi\|^p_{\widetilde{F^p}}$ and the partial sums $\xi_m = \sum_{j=1}^{m} \lambda_j a_j$ form a Cauchy sequence in F^p. This shows that (6) holds.

Notice also that, for each $\xi \in \widetilde{F^p}$, the inequality $\|\xi\|^p_{\widetilde{F^p}} \leq \inf\{\sum_{j=1}^{\infty} |\lambda_j|^p\}$, where the infimum is taken over all possible expressions (6), is due to the fact that $\|\cdot\|^p_{\widetilde{F^p}}$ satisfies the triangle inequality. □

The atomic decomposition of elements of $\widetilde{F^p}$ given above provides an explicit description of the operator U.

Corollary. *Let $\xi \in \widetilde{F^p}$ be represented by the sum* (6)*. Then $U(\xi)$ is the sum of the same series in H^p.*

We end this section by providing a constructive proof of the non triviality of N^p. Let us first describe the Meyer, Taibleson and Weiss construction as presented in [B]. Let B denote the open ball centered at the origin with radius 1. Take a sequence of open disjoint balls B_j , $j \geq 1$, such that $\cup_j B_j$ is dense in B . Notice that we may also choose the B_j so that the Lebesgue measure of their union $\sum_{j\geq 1} |B_j|$ is as small as we wish. As shown in [B] , for each j there exists a (non-continuous) (p, ∞) atom a_j supported on B_j with the property that $|a_j| \geq c\,|B_j|^{-\frac{1}{p}}$, where c is a small positive constant depending only on n. Thus, setting

$$f = \sum_{j\geq 1} |B_j|^{\frac{1}{p}} \, a_j \, , \tag{8}$$

we get $|f| \geq c$ on $\cup B_j$. From that is not difficult to conclude (see [B]) that

$$\|f\|_{F^p} \geq c\,|B|^{\frac{1}{p}} \,.$$

On the other hand, we clearly have $\|f\|_{H^p}^p \leq \sum_{j\geq 1} |B_j|$, so that the ratio between H^p-norm and F^p-norm can be made as small as we wish.

We can now construct a sequence $\{f_m\}$ in F^p satisfying

$$\|f_m\|_{F^p}^p \geq c^p\,|B|\,,$$

$$\|f_m - f_{m+1}\|_{F^p}^p \leq\, 2^p \frac{|B|}{2^m}\,,$$

$$\|f_m\|_{H^p}^p \leq \frac{|B|}{2^m}\,.$$

The first two conditions imply that $\{f_m\}$ has a non-zero limit $\xi \in \widetilde{F^p}$, whereas the third implies that $Uf_m = f_m$ tends to 0 in H^p. Hence $\xi \in N^p$.

The functions f_m have the form (8), precisely

$$f_m = \sum_{j\geq 1} |B_j^m|^{\frac{1}{p}}\, a_j^m\,, \tag{9}$$

where, for each m, $\{B_j^m\}_j$ is a disjoint family of balls contained in B with dense union and small total measure, and each a_j^m is a (p,∞)-atom with $|a_j^m| \geq c\,|B_j^m|^{-\frac{1}{p}}$.

The first function f_1 can be any function as in (8) with, say, $\sum_{j\geq 1} |B_j^1| < |B|/2$. We then construct inductively f_{m+1} from f_m as follows.

Take N so large that $\sum_{j>N} |B_j^m| < (1/4)\sum_{j\geq 1} |B_j^m|$. Inside each $B_j^m\,, 1 \leq j \leq N$, we take open disjoint balls $B'_{j\,l}\,, l \geq 1$, such that $\cup_{l\geq 1} B'_{j\,l}$ is dense in B_j^m and $\sum_{l\geq 1} |B'_{j\,l}| < |B_j^m|/4$.

Then

$$\sum_{j=1}^{N}\sum_{l\geq 1} |B'_{j\,l}| + \sum_{j>N} |B_j^m| \leq \frac{1}{2}\sum_{j\geq 1} |B_j^m|\ . \tag{10}$$

Let $a'_{j\,l}$ be a (p,∞) atom supported on $B'_{j\,l}$ with $|a'_{j\,l}| \geq c\,|B'_{j\,l}|^{-\frac{1}{p}}$. Set

$$f_{m+1} = \sum_{j=1}^{N}\sum_{l\geq 1} |B'_{j\,l}|^{\frac{1}{p}}\, a'_{j\,l} + \sum_{j>N} |B_j^m|^{\frac{1}{p}}\, a_j^m\ .$$

Since $|f_{m+1}| \geq c$ on an open dense subset of B, $\|f_{m+1}\|_{F^p}^p \geq c^p\,|B|$. Moreover,

$$f_m - f_{m+1} = \sum_{j=1}^{N}\Big(|B_j^m|^{\frac{1}{p}}\, a_j^m - \sum_{l\geq 1} |B'_{j\,l}|^{\frac{1}{p}}\, a'_{j\,l}\Big)\,.$$

For each j, the function

$$|B_j^m|^{\frac{1}{p}}\, a_j^m - \sum_{l\geq 1} |B'_{j\,l}|^{\frac{1}{p}}\, a'_{j\,l}$$

is supported on B_j^m and its absolute value is not greater than 2. Hence

$$\|f_m - f_{m+1}\|_{F^p}^p \leq \sum_{j=1}^{N} 2^p\, |B_j^m|\,.$$

We relabel now the balls in such a way that $\{B_j^{m+1}\}_{j\geq 1} = \{B_j^m\}_{j>N} \cup \{B'_{j\,l}\}_{j\leq N\,,l\geq 1}$, and rename the atoms in f_{m+1} as a_j^{m+1} accordingly. Then, inductively from (10),

$$\sum_{j\geq 1} |B_j^m| \leq 2^{-m}|B|$$

for every m, and the required estimates can be easily verified.

3 Proof of the Theorem

We start by proving, for the reader's sake, a few statements made (explicitly or not) in the last part of the introduction concerning the Gelfand spectrum $\widehat{\mathbb{R}^n}$ and its projection π on $\mathbb{R}^n$.

The first statement we want to prove is that π is in fact well defined. Given ϕ in $\widehat{\mathbb{R}^n}$, i.e., a nontrivial multiplicative functional on $L_0^\infty(\mathbb{R}^n)$, it is clear that its restriction to $C_0(\mathbb{R}^n)$ is also multiplicative. We must show that this restriction is evaluation at some point $x = \pi(\phi)$ of $\mathbb{R}^n$, or, equivalently, that it is not identically zero.

Since $L_0^\infty(\mathbb{R}^n)$ is a C^*-algebra, it is symmetric, so that $\phi(\bar{f}) = \overline{\phi(f)}$ for every f. Therefore, $f \geq 0$ implies that $\phi(f) \geq 0$, so that ϕ is monotonic on real-valued functions. If ϕ vanishes identically on $C_0(\mathbb{R}^n)$, it also vanishes on characteristic functions of compact sets. By linearity and continuity, this would be a contradiction.

The second statement is that the mapping π is surjective. We know that to each $\phi \in \widehat{\mathbb{R}^n}$ we can associate a point $\pi(\phi)$ in $\mathbb{R}^n$. Given $y \in \mathbb{R}^n$, we can define a translate $\tau_y\phi \in \widehat{\mathbb{R}^n}$ by

$$\tau_y\phi(f) = \phi\big(f(\cdot + y)\big)\ . \tag{11}$$

It is quite clear that $\pi(\tau_y\phi) = \pi(\phi) + y$. Since $\widehat{\mathbb{R}^n}$ is nonempty, π is surjective.

The last statement which remained unproved in the introduction is that $\widehat{\mathbb{R}^n}$ is the union of the $\widehat{B}$ over all balls B. This is a direct consequence of (ii) in the following lemma.

Lemma 2. *Let B be an open ball in $\mathbb{R}^n$. Then*

(i)
$$\widehat{B} = \{\phi \in \widehat{\mathbb{R}^n} : \phi(\chi_B) = 1\} = \operatorname{supp} \widehat{\chi_B} \ , \tag{12}$$
where $\widehat{f}$ stands for the Gelfand transform of $f \in L_0^\infty(\mathbb{R}^n)$.

(ii)
$$\pi^{-1}(B) \subset \widehat{B} \subset \pi^{-1}(\overline{B}) \ .$$

Proof. To prove (12) notice that $\phi(\chi_B)$ is either 0 or 1 by the multiplicative property.

If $\phi(\chi_B) = 1$, then $\phi(f) = \phi(f\chi_B)$, $f \in L_0^\infty(\mathbb{R}^n)$, which means that ϕ factors through a character of $L^\infty(B)$. Thus $\phi \in \widehat{B}$. The argument can be reversed, so (12) is proved.

Assume now that for some $\phi \in \widehat{\mathbb{R}^n}$ we have $\pi(\phi) \in B$. Let f be a continuous function on $\mathbb{R}^n$, with $f(\pi(\phi)) = 1$ and compact support contained in B. Then $f\chi_B = f$ and so
$$1 = \phi(f) = \phi(f)\phi(\chi_B) = \phi(\chi_B).$$
Then $\phi \in \widehat{B}$ because of (12).

If $\pi(\phi)$ is not in $\overline{B}$, then there is a continuous function f on $\mathbb{R}^n$, with $f(\pi(\phi)) = 1$ and compact support in $\mathbb{R}^n \setminus \overline{B}$. Thus $f\chi_B = 0$ and so $\phi(\chi_B) = 0$, that is, ϕ is not in $\widehat{B}$. $\square$

We turn now to the proof of the Theorem. We begin by discussing the converse statement in part (A) of the Theorem. Obviously, given $b \in BMO$, the linear functional $f \mapsto \int fb\,dm$ is bounded on $\widetilde{F^1}$ with a norm controlled from above by the BMO-norm of b. On the other hand, restriction of the functional to $\widetilde{F^{1,c}}$ gives a control from below by the same BMO-norm.

We first remark that (12) clearly implies that, given $f \in L_0^\infty(\mathbb{R}^n)$, the support of f is contained in B if and only if the support of $\widehat{f}$ is contained in $\widehat{B}$.

Let μ be a Radon measure on $\widehat{\mathbb{R}^n}$ satisfying (4). For each $(1,\infty)$-atom a supported on a ball B one has
$$\Big|\int \widehat{a}\,d\mu\Big| \le \|a\|_\infty\,|\mu|(\widehat{B}) \le \frac{|\mu|(\widehat{B})}{m(B)} < C \ .$$
Hence μ determines a bounded linear functional on $\widetilde{F^1}$.

Assume now that ℓ is a bounded linear functional on F^1. Fix a ball B and let $L_0^\infty(B)$ stand for the set of functions in $L^\infty(B)$ with zero integral. Given $f \in L_0^\infty(B)$,
$$\frac{1}{m(B)}\frac{f}{\|f\|_\infty}$$
is a $(1,\infty)$-atom. Thus
$$|\ell(f)| \le \|\ell\|\,\|f\|_\infty\, m(B), \quad f \in L_0^\infty(B). \tag{13}$$

The restriction of ℓ to $L_0^\infty(B)$ extends to a bounded linear functional on $L^\infty(B) = C(\widehat{B})$. Thus there exists a measure ν_B on $\widehat{B}$ such that

$$\ell(f) = \int \widehat{f}\, d\nu_B, \quad f \in L_0^\infty(B). \tag{14}$$

If $f \in L^\infty(B)$, then clearly $\widehat{f}_{\widehat{B}} = f_B$, where g_E stands for the mean of the function g on the set E with respect to the underlying measure ($\widehat{m}$ or m in the case at hand). Then

$$\begin{aligned}
\ell(f - f_B) &= \int_{\widehat{B}} (\widehat{f} - \widehat{f}_{\widehat{B}})\, d\nu_B \\
&= \int_{\widehat{B}} (\widehat{f} - \widehat{f}_{\widehat{B}})\, (d\nu_B - \nu_B(\widehat{B}) \frac{\widehat{m}}{\widehat{m}(\widehat{B})}) \\
&= \int_{\widehat{B}} \widehat{f}\, (d\nu_B - \frac{\nu_B(\widehat{B})}{\widehat{m}(\widehat{B})} d\widehat{m}),
\end{aligned} \tag{15}$$

for each $f \in L^\infty(B)$. Therefore, if ν_B represents ℓ on $L_0^\infty(B)$, that is, if (14) holds, then $d\nu_B - \nu_B(\widehat{B})\frac{\widehat{m}}{\widehat{m}(\widehat{B})}$ is uniquely determined.

Let B_N stand for the open ball with center at the origin and radius N, $N = 1, 2, ...$ Take any measure ν_1 on $\widehat{B_1}$ that represents ℓ on $L_0^\infty(B_1)$. Each other such measure differs from ν_1 by a constant multiple of $\chi_{\widehat{B_1}}\, \widehat{m}$. By the preceding remark applied to B_N there exists a unique measure ν_N on $\widehat{B_N}$ which represents ℓ on $L_0^\infty(B_N)$ and $\nu_N(B_1) = \nu_1(B_1)$. Clearly ν_N restricted to $\widehat{B_{N-1}}$ is precisely ν_{N-1}. Therefore we can define a measure ν on $\mathbb{R}^n$ by requiring that ν restricted to $\widehat{B_N}$ be ν_N.

Given any ball B take N such that $B \subset B_N$. Since the restriction of ν to $\widehat{B_N}$ represents ℓ on $L_0^\infty(B_N)$, which contains $L_0^\infty(B)$, the restriction of ν to $\widehat{B}$ represents ℓ on $L_0^\infty(B)$ as well. By (15)

$$\Big| \int_{\widehat{B}} \widehat{f}(d\nu - \nu(\widehat{B}) \frac{\widehat{m}}{\widehat{m}(\widehat{B})}) \Big| \le 2\, \|\ell\|\, \|f\|_\infty\, m(B), \quad f \in L^\infty(B),$$

or

$$\Big\| d\nu - \nu(\widehat{B}) \frac{\widehat{m}}{\widehat{m}(\widehat{B})} \Big\|_{\widehat{B}} \le 2\, \|\ell\|\, m(B). \tag{16}$$

Let us now consider the Radon-Nikodym decomposition of ν

$$\nu = g\, d\widehat{m} + \mu,$$

where $g \in L^1_{\mathrm{loc}}(\widehat{m})$ and μ is singular with respect to $\widehat{m}$. By (16)

$$|\mu|(\widehat{B}) \le 2\, \|\ell\|\, m(B). \tag{17}$$

and

$$\int_{\widehat{B}} \Big| g - g_{\widehat{B}} - \frac{\mu(\widehat{B})}{\widehat{m}(\widehat{B})} \Big| \, d\widehat{m} \le 2\, \|\ell\| \, m(B). \tag{18}$$

We are left with the task of finding the BMO-function b.

Combining (17) and (18) we readily get

$$\int_{\widehat{B}} |g - g_{\widehat{B}}| \, d\widehat{m} \le 4\, \|\ell\| \, m(B). \tag{19}$$

We need a Lemma.

Lemma 3. *For each function $g \in L^1_{\text{loc}}(\widehat{m})$ there exists a unique function $f \in L^1_{\text{loc}}(m)$ with the property that for each ball B,*

$$\int g\, \widehat{\varphi} \, d\widehat{m} = \int f \, \varphi \, dm, \quad \varphi \in L^\infty(B).$$

Such f satisfies

$$\int_{\widehat{B}} |g - g_{\widehat{B}}| \, d\widehat{m} = \int_B |f - f_B| \, dm,$$

for each ball B.

Once the lemma is proved we complete the proof of part (A) of the Theorem by just calling b the function f associated with g in Lemma 2. Inequality (19) tells us that $b \in BMO(\mathbb{R}^n)$ and that its $BMO(\mathbb{R}^n)$ norm is not greater than $4\,\|\ell\|$.

Proof of Lemma 3. We will show that for each ball B the Gelfand transform, which is an isometry between $L^\infty(B)$ and $C(\widehat{B})$, extends to an isometry between $L^1(B, m)$ and $L^1(\widehat{B}, \widehat{m})$. This immediately provides a further extension of the Gelfand transform to a topological isomorphism between $L^1_{\text{loc}}(m)$ and $L^1_{\text{loc}}(\widehat{m})$.

We begin by showing that, for each ball B in $\mathbb{R}^n$ and every $f \ge 0$ in $L^\infty(B)$,

$$\int_{\widehat{B}} \widehat{f} \, d\widehat{m} = \int_B f \, dm, \ . \tag{20}$$

This follows from

$$\begin{aligned} \int_B f\, dm &= \sup_{\varphi} \int_B f \, \varphi \, dm \\ &= \sup_{\varphi} \int_{\widehat{B}} \widehat{f}\, \widehat{\varphi} \, d\widehat{m} \\ &= \int_{\widehat{B}} \widehat{f} \, d\widehat{m}, \end{aligned}$$

where the supremum is taken on the closed unit ball of $L^\infty(B)$.

By linearity, (20) provides an extension of the Gelfand transform to a topological isomorphisms $f \to \widehat{f}$ of $L^1_{\text{loc}}(m)$ onto $L^1_{\text{loc}}(\widehat{m})$. Given $g \in L^1_{\text{loc}}(\widehat{m})$ take $f \in L^1_{\text{loc}}(m)$ with $g = \widehat{f}$. The first identity in the statement of Lemma 2 follows by approximating $f \in L^1(B, m)$ by functions in $L^\infty(B)$ and the second follows from (20). $\square$

Before proving part (B) of the Theorem we give an explicit example, modeled on the Meyer-Taibleson-Weiss argument, of a non-zero measure which is singular with respect to $\widehat{m}$ and satisfies (4).

Take an open set U of $\mathbb{R}^n$, $U \subset B_0 = \{x : |x| \le 1\}$, such that U is dense in B_0 and $m(U) < m(B_0)$. Then the compact set $E = B_0 \setminus U$ has positive Lebesgue measure. Set $V = \pi^{-1}(U)$, so that $V \subset \widehat{B_0}$ by Lemma 1. Then $U \subset \pi(\overline{V})$ and so $\pi(\overline{V}) = B_0$, because U is dense in B_0. Hence $\pi(\partial V) = E$. Now, the boundary of each open set in $\widehat{B_0}$ has zero $\widehat{m}$ measure ([R, p. 286]). Therefore $\widehat{m}(\partial V) = 0$ but $m(\pi(\partial V)) = m(E) > 0$. Identify $C(E)$ to the subspace S of continuous functions on ∂V of the form $f \circ \pi$, $f \in C(E)$. The bounded linear functional on S defined by $f \to \int f\, dm$ extends by Hahn-Banach to a bounded linear functional on $C(\partial V)$ with the same norm. Thus there exists a positive measure μ on ∂V such that

$$\int (f \circ \pi)\, d\mu = \int f\, dm, \quad f \in C(E).$$

If B is an open ball, then by Lemma 2

$$\mu(\widehat{B}) \le \mu(\pi^{-1}(\overline{B})) = m(\overline{B} \cap E) \le m(B),$$

and condition (4) is satisfied.

Proof of (B) of the Theorem. The argument is analogous to the proof of part (A), except for minor technical details. If $0 < p < 1$, then, as we will see, the singular measure μ vanishes and so we will conclude that $(F^p)^* = H^p(\mathbb{R}^n)^*$.

Let ℓ be a bounded linear functional on F^p, $0 < p < 1$. Let d be the integer part of $n(\frac{1}{p} - 1)$. Given a ball B let $L^\infty_d(B)$ stand for the set of functions $f \in L^\infty(B)$ such that

$$\int f(x)\, x^\alpha\, dx = 0, \quad |\alpha| \le d.$$

For each $f \in L^\infty_d(B)$,

$$\frac{1}{m(B)^{\frac{1}{p}}} \frac{f}{\|f\|_\infty}$$

is a (p, ∞) atom and so

$$|\ell(f)| \le \|\ell\|\, \|f\|_\infty\, m(B)^{\frac{1}{p}}, \quad f \in L^\infty_d(B). \tag{21}$$

For each $f \in L^\infty(B)$ let $P_B(f)$ be (the restriction to B of) the unique polynomial of degree not greater than d such that

$$\int f(x)\, x^\alpha\, dx = \int_B P_B(f)(x)\, x^\alpha\, dx, \quad |\alpha| \le d.$$

Since $P_B(f)$ is the orthogonal projection (in $L^2(B)$) of f into the subspace of polynomials of degree not greater than d,

$$\|P_B(f)\|_2 \leq \|f\|_2 \leq \|f\|_\infty,$$

where the L^2 norms are taken with respect to the normalized Lebesgue measure on B. We want now to compare the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ on the space $P_d(B)$ of restrictions to B of polynomials of degree not greater than d. After appropriate translation and dilation we may assume that B has center 0 and radius 1. Since $P_d(B)$ is finite dimensional, there is a constant $C(d,n)$, depending only on d and n, such that

$$\|P\|_\infty \leq C(d,n)\,\|P\|_2, \quad P \in P_d(B),$$

and so

$$\|P_B(f)\|_\infty \leq C(d,n)\,\|f\|_\infty, \quad f \in L^\infty(B).$$

Therefore by (21)

$$|\ell(f - P_B(f))| \leq (1 + C(d,n))\,\|\ell\|\,\|f\|_\infty\, m(B)^{\frac{1}{p}}, \quad f \in L^\infty(B). \tag{22}$$

By (21) there is a measure ν_B on $\widehat{B}$ such that

$$\ell(f) = \int \widehat{f}\, d\nu_B, \quad f \in L^\infty_d(B). \tag{23}$$

Given a measure ν on $\widehat{B}$ there is a unique polynomial $P_B(\nu) \in P_d(B)$ such that

$$\int_{\widehat{B}} \big(\pi(\phi)\big)^\alpha\, d\nu(\phi) = \int_B P_B(\nu)(x)\, x^\alpha\, dx, \quad |\alpha| \leq d\ .$$

Therefore, by (23),

$$\begin{aligned}
\ell(f - P_B(f)) &= \int_{\widehat{B}} (\widehat{f} - \widehat{P_B(f)})\, d\nu_B \\
&= \int_{\widehat{B}} (\widehat{f} - \widehat{P_B(f)})\, (d\nu_B - \widehat{P_B(\nu_B)}\widehat{m}) \qquad (24) \\
&= \int_{\widehat{B}} \widehat{f}\, (d\nu_B - \widehat{P_B(\nu_B)}\widehat{m}),
\end{aligned}$$

for each $f \in L^\infty(B)$. Hence the measure $d\nu_B - \widehat{P_B(\nu_B)}\widehat{m}$ is determined by ℓ.

As before, with B_N denoting the ball of radius N centered at the origin, we fix a measure ν_1 on $\widehat{B_1}$ that represents ℓ on $L^\infty_d(B_1)$ and then take the unique measure ν_N on $\widehat{B_N}$ which represents ℓ on $L^\infty_d(B_N)$ and such that $P_{B_1}(\nu_N) = P_{B_1}(\nu_1)$. Then ν_N restricted to B_{N-1} is ν_{N-1} and so we can define a measure ν on $\widehat{\mathbb{R}^n}$ by requiring that ν restricted to B_N be ν_N.

Given any ball B, take N such that $B \subset B_N$. Then the restriction of ν to $L^\infty_d(B)$ is ℓ and so, by (22) and (24),

$$\Big| \int_{\widehat{B}} \widehat{f}\,(d\nu - \widehat{P_B(\nu)}\widehat{m}) \Big| \le C\, \|f\|_\infty\, m(B)^{\frac{1}{p}}, \quad f \in L^\infty(B).$$

Hence

$$|d\nu - \widehat{P_B(\nu)}|(\widehat{B}) \le C\, m(B)^{\frac{1}{p}}. \tag{25}$$

Consider now the Radon-Nikodym decomposition of ν,

$$\nu = g\, d\widehat{m} + \mu,$$

with μ singular with respect to $\widehat{m}$. We get, by (25) and Lemma 2,

$$|\mu|(\pi^{-1}(B)) \le |\mu|(\widehat{B}) \le C\, m(B)^{\frac{1}{p}},$$

for each open ball B. Since $0 < p < 1$, we readily conclude that $\mu = 0$.

Take now $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with $g = \widehat{f}$. Then

$$\int_B |f - P_B(f)|\, dm \le C\, m(B)^{\frac{1}{p}},$$

which is precisely the condition that guarantees that f determines a bounded linear functional on $H^p(\mathbb{R}^n)$ ([TW]). Thus ℓ is a bounded linear functional on $H^p(\mathbb{R}^n)$ and the proof is complete. □

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