The calculation of the L^2-norm of the index of a plane curve and related formulae

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Abstract

In this paper we provide formulae to calculate the L^2-norm of the index function of a rectifiable closed curve in the complex plane. Some applications to isoperimetric inequalities are given. The main tool used is the decomposition of any rectifiable closed curve in a sequence of Jordan curves plus some curves with null index functions and an exceptional set.

1 Introduction and notation

By a parameterized curve (for simplicity a curve) in the complex plane $\mathbb{C}$ we mean a continuous map $\gamma : [a, b] \to \mathbb{C}$. We put $\gamma^* = \gamma([a, b])$ for the trace of γ and the curve γ will be closed when $\gamma(a) = \gamma(b)$. Occasionally, in this case, we will assume (and we will do it without changing the notation) that γ is defined on the boundary, here denoted by $\mathbb{T}$, of the unit disc of $\mathbb{C}$. When γ^* is homeomorphic to $\mathbb{T}$ we say that γ^* is a Jordan curve and let $D(\gamma)$ stand for the bounded domain limited by γ. The curve γ is rectifiable when γ is of bounded variation on $[a, b]$. Then we consider on $[a, b]$ the Lebesgue-Stieltjes measure $d\gamma$ and its corresponding total variation measure $|d\gamma|$ (see [16, p. 173]). We write $\Lambda(\gamma) = |d\gamma|([a, b])$ for the length of γ. If K is a compact subset of $\mathbb{C}$ we denote by $\widehat{K}$ the polynomial convex hull of K.

For the remaining notation we will refer to [5] but there is an issue which was not left clear enough there. Here dz denote the complex measure, associated to γ, defined on $\mathbb{C}$ by $\int f\, dz = \int_a^b f(\gamma(t))\, d\gamma(t)$, if $f \in C(\mathbb{C})$, and $|dz|$ will be the arc length measure defined by $\int f\, |dz| = \int_a^b f(\gamma(t))\, |d\gamma|(t)$. However in this paper, because it deals mainly with involved curves, the total variation measure of dz plays an important role. This measure is not currently used in the literature and even there is not an appropriate notation for it. We will denote it by $\| dz \|$ and then we put $V(dz) = \| dz \| (\gamma^*)$. With this notation one has

$$\Big| \int_\gamma f(z)\, dz \Big| \le \int_\gamma |f(z)| \, \| dz \|, \quad f \in C(\gamma^*). \tag{1.1}$$

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We remark that $\shortparallel dz \shortparallel$ is, in general, different from $|dz|$. For example take $\gamma(t)=e^{i|t|}$, $t\in[-\pi,2\pi]$. Then $V(dz)=\pi$ but $\Lambda(dz)=3\pi$. The measure $\shortparallel dz \shortparallel$ takes into consideration the multiple points of γ^*, each of them with its corresponding orientation. Therefore the inequality $\shortparallel dz \shortparallel \leq |dz|$ holds for each rectifiable curve and the equality $\shortparallel dz \shortparallel = |dz|$ holds if γ is one to one. For better understanding we mention that $\shortparallel dz \shortparallel = h d\Lambda^1$, where Λ^1 is the one-dimensional Hausdorff measure and $h(s)=|\sum_{\{t:\gamma(t)=s\}} \frac{\gamma'(t)}{|\gamma'(t)|}|$ for enough regular γ. Occasionally, if there is possibility of confusion, we write dz_γ instead of dz. Also we need to consider the measures $dz_{\gamma_{|E}}$ and $\shortparallel dz_{\gamma_{|E}} \shortparallel$ defined on $\gamma(E)$ by $\gamma_{|E}$, the restriction of γ to each measurable subset E of $[a,b]$. It is worthy to note that $\shortparallel dz_{\gamma_{|F}} \shortparallel \leq |dz_{\gamma_{|F}}|$ for each closed subset $F\subset[a,b]$.

A key object of study in this paper is the index function $\text{Ind}(\gamma,\cdot)$ of a closed curve γ on $\mathbb{C}$. See [3, Chap. IV] for index's properties. In this paper we deal mainly with rectifiable curves. When γ is rectifiable the index can be computed by

$$\text{Ind}(\gamma,w)=\frac{1}{2\pi i}\int_\gamma \frac{1}{z-w}\,dz,\quad w\notin\gamma^*. \tag{1.2}$$

Let $g\in L^1_{\text{loc}}(\mathbb{C},m)$ be compactly supported, where m is the planar Lebesgue measure. We denote by $C(g)$ the Cauchy transform of the measure $g\,m$, that means

$$C(g)(w)=\int_{\mathbb{C}}\frac{g(z)}{z-w}\,dm(z),$$

which is defined for m-almost all $w\in\mathbb{C}$. The regularity properties of this transform are well known and we refer to [7, Chap. II] for information on it. We will denote by C^γ the Cauchy transform of $\text{Ind}(\gamma,\cdot)$.

Our goal in this paper is to provide integral formulae where the function index appears and where the integrals are taken with respect to m. Several of them are obtained here, for example (3.5). The main formula in our work is given by Theorem 6. In particular it says:

$$\int_{\mathbb{C}} F(z,z)\,\text{Ind}(\gamma,z)^2\,dm(z)=\frac{1}{4\pi}\int_{\gamma\times\gamma}F(z,w)\frac{\overline{z}-\overline{w}}{z-w}\,dzdw, \tag{1.3}$$

where F is an holomorphic function on $\widehat{\gamma^*}\times\widehat{\gamma^*}$. Topping in [18] has proved the special case of (1.3) when $F=1$ and γ is a Jordan curve. This is the unique reference that we know of that deals with this kind of formulae.

Taking $F=1$ on (1.3) and using (1.1) we obtain the new inequality

$$\|\text{Ind}(\gamma,\cdot)\|_2^2\leq\frac{1}{4\pi}V(\gamma)^2, \tag{1.4}$$

where the norm is that of the space $L^2(\mathbb{C},m)$.

The inequality (1.4) is a generalization of the classical isoperimetric inequality for Jordan domains and also it is an appropriate extension of

$$\|\text{Ind}(\gamma,\cdot)\|_2^2\leq\frac{1}{4\pi}\Lambda(\gamma)^2. \tag{1.5}$$

We identify (1.4) (respectively (1.5)) as the isoperimetric index inequality with variation (respectively with length). The reason for which (1.4) is better than (1.5) is the following. Let us say that two closed curves γ_1 and γ_2 are equivalent when $\mathrm{Ind}(\gamma_1,\cdot)=\mathrm{Ind}(\gamma_2,\cdot)$ a. e. with respect to m. The inequality (1.5) is not compatible with this equivalence relation. The use of the $V(dz)$ instead of $\Lambda(\gamma)$ solves this problem because both terms in (1.4) vanish simultaneously by [5, Cor. 4(a)].

We refer to [13, 15, 21, 18] for proofs of the isoperimetric inequality on $\mathbb{C}$. Some comments on the history of (1.5) are in order. As far as we know the inequality (1.5), under differentiability hypotheses on γ, first appeared in [1] where a guide for a proof is given. The fact that $\mathrm{Ind}(\gamma,\cdot)\in L^2(\mathbb{C},m)$ is implicit in [6, p. 167] since $\mathrm{Ind}(\gamma,\cdot)$ is a function of bounded variation on $\mathbb{C}$. In fact, by using an approximation argument and the Gagliardo-Nirenberg-Sobolev theorem [6, p. 138], with sharp constant, one gets $\|\mathrm{Ind}(\gamma,\cdot)\|_2^2\le\Lambda(\gamma)^2$. In [5] a geometric proof of (1.5) was provided which uses the isoperimetric inequality for Jordan domains limited by polygonal curves. With the aim to complete this topic we have written the section 3, which is devoted to present a self-contained proof of (1.5) (shorter than one in [5]) and to compare it with the work of Topping in [18]. It ends analysing when the equality in (1.4) or (1.5) holds.

If one tries to prove directly (1.3) using (1.2) some problems appear. This fact is shown in section 2. Therein two formulae for computing the left-hand side of (1.3), but only for $F=1$, are obtained. However the hypothesis of γ having finite logarithmic energy is essentially needed: this is the content of Theorem 1. As a matter of fact these formulae depend on the kernels

$$K_j(z,w)=k_j(z-w),\ j=1,2$$

$$k_1(z)=\frac{\overline{z}}{z},\ k_2(z)=\log\frac{1}{|z|},z\neq 0,\ \text{and}\, k_1(0)=1,\ k_2(0)=+\infty. \tag{1.6}$$

In this paper log denotes the principal branch of the logarithm defined on $\mathbb{C}\backslash(-\infty,0]$. We note that k_1/π and $-2k_2/\pi$ are the fundamental solutions of the elliptic operators $\overline{\partial}^2$ and $\partial\overline{\partial}$, respectively (see [2, 19]).

We obtain (1.3) as a consequence of a result on the structure of rectifiable curves which is Theorem 4, a key point in our research. Section 4 is completely devoted to prove it. Clearly it says much more that really is needed to prove (1.3). However the ideas around it, the definitions and its proof can be of some interest in the geometric measure theory. It is not a complete statement but it is a step towards a better knowledge of the complexity of the rectifiable curves. Roughly speaking it says that if γ is a rectifiable curve defined on $\mathbb{T}$, then $\mathbb{T}=\bigcup(E_n\cup F_n\cup S_n)\cup S$ where $\gamma_n=\gamma_{|\,E_n}$ is a Jordan loop, $\gamma_{|\,F_n},\gamma_{|\,S_n}$ are closed curves with vanishing index function and $\gamma(S)$ is small in some sense; in particular it says that $dz_\gamma=\sum dz_{\gamma_n}$.

In section 5 we prove some formulae, one of them is (1.3). The surprising fact that C^γ is continuous everywhere is obtained. We remind the reader that Cauchy's transform of a compactly supported function in $L^2(\mathbb{C})$ is in $VMO(\mathbb{C})$, (see [10]). Finally section 6 makes evident that the kernel K_2 is not adequate to obtain the main results of the previous sections. Therefore the present research shows, once more (see for example [4, 14, 19]), the fact that the use of K_1 gives better results than using K_2.

2 Formulae for the case of finite energy

Let γ be a rectifiable closed curve in $\mathbb{C}$. In order to compute $\|\mathrm{Ind}(\gamma,\cdot)\|_2^2$ we can evaluate the integrand, using the fact that $\mathrm{Ind}(\gamma,\cdot)$ takes only integer values, in two different ways:

$$|\mathrm{Ind}(\gamma,\tau)|^2 = \frac{1}{4\pi^2}\int_{\gamma\times\gamma}\frac{dz\,d\overline{w}}{(z-\tau)(\overline{w}-\overline{\tau})}, \quad \tau\in\mathbb{C}\setminus\gamma^*, \tag{2.1}$$

$$|\mathrm{Ind}(\gamma,\tau)|^2 = -\frac{1}{4\pi^2}\int_{\gamma\times\gamma}\frac{dz\,dw}{(z-\tau)(w-\tau)}, \quad \tau\in\mathbb{C}\setminus\gamma^*. \tag{2.2}$$

The integrals in (2.1) and (2.2) look very similar and one could expect that the results that can be derived from them will be equivalent. This is not the case because, as this paper shows, the consequences obtained from (2.1) and (2.2) are quite different.

Lemma 1. *Let $R>0$ and $z,w\in\mathbb{C}$, $z\neq w$. Then*

$$(a)\qquad \int_E \frac{dm(\tau)}{|z-\tau|\,|w-\tau|} \le \frac{4[\pi m(E)]^{\frac12}}{|z-w|}, \tag{2.3}$$

$$(b)\qquad \frac{1}{\pi}\int_{D(0,R)}\frac{dm(\tau)}{(z-\tau)(\overline{w}-\overline{\tau})} = 2K_2(z,w)+\log(R^2-z\overline{w}), \tag{2.4}$$

$$(c)\qquad \frac{1}{\pi}\int_{D(0,R)}\frac{dm(\tau)}{(z-\tau)(w-\tau)} = -K_1(z,w), \tag{2.5}$$

where $E\subset\mathbb{C}$ is a measurable set in the item (a) and the equations (2.4) *and* (2.5) *hold for $z,w\in D(0,R)$.*

Proof. (a) The proof follows from the estimates

$$\frac{|z-w|}{|z-\tau||w-\tau|} \le \frac{1}{|z-\tau|}+\frac{1}{|w-\tau|},$$

$$\int_E\frac{dm(\tau)}{|w-\tau|} \le 2[\pi m(E)]^{\frac12}, \quad w\in\mathbb{C}, \tag{2.6}$$

if E is a measurable set (see [2, p. 150]).

(b) The integrals in (2.4) and (2.5) are well defined by (2.3). Assume first that $|w|\le|z|$. Taking polar coordinates $\tau=re^{i\theta}$ and writing, for each r, the integral with respect to $d\theta$ as a complex line integral over $C_r(t)=re^{it}$, $t\in[0,2\pi]$, one gets

$$\frac{1}{\pi}\int_{D(0,R)}\frac{dm(\tau)}{(z-\tau)(\overline{w}-\overline{\tau})} = \Big(\int_0^{|w|}+\int_{|w|}^{|z|}+\int_{|z|}^{R}\Big)J(r)\,dr, \tag{2.7}$$

where

$$J(r)=\frac{1}{\pi i}\int_{C_r}F_r(\zeta)\,d\zeta \quad\text{with}\quad F_r(\zeta)=\frac{r}{(z-\zeta)(\overline{w}\zeta-r^2)}.$$

Now we compute each $J(r)$, for $0<r<R$, by using the Residue theorem. Taking into account the poles of F_r in $D(0,r)$, for each r in

the intervals appearing in the decomposition (2.7), and computing the corresponding residues one gets that (2.7) equals

$$\left(-\int_0^{|w|} + \int_{|z|}^{R}\right) \frac{2r}{r^2 - z\overline{w}}\, dr = L(0) - L(|w|) - L(|z|) + L(R). \tag{2.8}$$

In (2.8) L denotes the function $L(x) = \log(x^2 - z\overline{w})$, which is well defined on $[0, |w|] \cup [|z|, R]$ because $z \neq w$. By (2.8), the proof of (2.4) will finish if we prove

$$L(0) + \log|z - w|^2 = L(|w|) + L(|z|). \tag{2.9}$$

The identity (2.9) follows from $-z\overline{w}|z-w|^2 = (|w|^2 - z\overline{w})(|z|^2 - z\overline{w})$ and from the fact that $\log z_1 + \log z_2 = \log(z_1 z_2)$ whenever $z_1, z_2, z_1 z_2$ lie simultaneously in $\{s : \operatorname{Im} s \geq 0\}$ or $\{s : \operatorname{Im} s \leq 0\}$, which is the case we are dealing with. The case $|w| > |z|$ can be reduced to the previous one by taking complex conjugation in both sides of (2.4).

(c) A short way to prove it is to write the integral in (2.5) as a difference of two terms and then apply the formula

$$\frac{1}{\pi}\int_{D(0,R)} \frac{dm(\tau)}{w - \tau} = \begin{cases} \min\{|w|^2, R^2\}/w, & \text{if} \quad w \neq 0 \\ 0, & \text{if} \quad w = 0. \end{cases} \tag{2.10}$$

□

When integrating (2.1) and (2.2) with respect to m we will need to apply Fubini's theorem. For this purpose we will use the following estimate which is more accurate than (2.3) when $E = D(0,R)$.

Lemma 2. *Let $R > 0$ and $z \neq w$ with $|z|, |w| \leq R$, $|z - w| \leq R$. Then*

$$\int_{D(0,R)} \frac{dm(\tau)}{|z-\tau||w-\tau|} \leq 8\pi + 2\pi\log 2 + 2\pi \log\frac{R}{|z-w|}. \tag{2.11}$$

Proof. Put $\rho = |z - w|$ and denote by I the integral in the left-hand side of (2.11). Performing the change of variables $\tau = z + (z - w)s$, one gets

$$I \leq \int_{D(0,2)} \frac{dm(s)}{|s||1-s|} + \int_{D(0,2R/\rho)\setminus D(0,2)} \frac{dm(s)}{|s||1-s|}.$$

Changing to polar coordinates in the second integral and using (2.3), we obtain

$$I \leq 8\pi + 2\pi \int_2^{\frac{2R}{\rho}} \frac{dr}{r-1} \leq 8\pi + 2\pi\log\frac{2R}{\rho}.$$

□

Remark 1. *The integral in* (2.11) *cannot be evaluated in terms of elementary functions and has been considered several times. For example [7, p. 77] gives* (2.11) *with different constants. Also the next formula due to Ahlfors, [11, p. 131] gives the asymptotic behavior of such an integral*

$$\int_{D(0,R)} \frac{dm(\tau)}{|z-\tau||w-\tau|} = 2\pi\log\frac{R}{|z-w|} + 4\pi\log 2 + \delta(z,w,R), \tag{2.12}$$

where $\delta(z,w,R) \to 0$ as $R \to +\infty$, uniformly when z, w remain bounded.

Going into the proof of (2.12) *we can see that the following estimates hold for* $|z|, |w| \leq R/4$ *with* $z \neq w$*,*

$$\Big|\int_{D(0,R)} \frac{dm(\tau)}{|z-\tau||w-\tau|} - 2\pi \log \frac{R}{|z-w|}\Big| \leq 8\pi \log 2. \tag{2.13}$$

Therefore (2.13) *shows that the elementary upper estimate in* (2.11) *has the right growth.*

Let $\Delta = \{(z,z) : z \in \gamma^*\}$ be the diagonal of $\gamma^* \times \gamma^*$. In formulae (2.1) and (2.2) one can have $(z,w) \in \Delta$, while in Lemma 1 the hypothesis $(z,w) \notin \Delta$ is needed. The next lemma avoids this difficulty.

Lemma 3. *Let* γ *be a rectifiable curve. Then*

(a) the measure dz *has no atoms,*

(b) the set Δ *has zero measure with respect to* $|dz|\,|dw|$*.*

In the sequel a role will be played by the logarithmic energy $E(\|dz\|)$ of the measure $\|dz\|$, corresponding to a rectifiable curve γ, defined by

$$E(\|dz\|) = \int_{\gamma\times\gamma} K_2(z,w) \ \|dz\,dw\| \ .$$

We note that $-\infty < E(\|dz_\gamma\|) \leq +\infty$ for each γ. See [8, p. 79] and [20, p. 54] for more information about the energy of a positive measure.

Theorem 1. *Let* γ *be a rectifiable closed curve such that* $E(\|dz\|) < +\infty$*. Then*

$$(a) \qquad \|Ind(\gamma,\cdot)\|_2^2 = \frac{1}{2\pi}\int_{\gamma\times\gamma} \log \frac{1}{|z-w|}\, dz\, d\overline{w}.$$

$$(b) \qquad \|Ind(\gamma,\cdot)\|_2^2 = \frac{1}{4\pi}\int_{\gamma\times\gamma} \frac{\overline{z}-\overline{w}}{z-w}\, dz\, dw.$$

Proof. Chose $R > 0$ so large that $\gamma^* \subset D(0,R)$ and integrate both terms of the equalities (2.1) and (2.2) with respect to $dm(\tau)$ on $D(0,R)$. Now Lemma 2, Lemma 3 and the assumption $E(\|dz\|) < +\infty$ imply

$$\int_{D(0,R)} \Big(\int_{\gamma\times\gamma} \frac{\|dz\,dw\|}{|z-\tau||w-\tau|}\Big)\, dm(\tau) < +\infty. \tag{2.14}$$

So the use of Fubini's theorem is justified and we can change the order of integration and apply Lemma 1. This gives directly (b) and for (a) the extra term $\int_{\gamma\times\gamma} \log(R^2 - z\overline{w})\, dz\, d\overline{w}$ appears, but it vanishes by Fubini's and Cauchy's theorems. □

The arguments used in this section are not appropriate to prove (1.3) for all rectifiable curves. The sufficient condition (2.14) was applied and it turns out that it is equivalent to $E(\|dz\|) < +\infty$, by (2.13). Without this assumption it seems impossible to interchange the integrals of the function $1/(z-\tau)(w-\tau)$ with respect to the measures $dm(\tau)$ and $dz\,dw$, so the use of Fubini's theorem was essential. Here a natural question appears: *Are there rectifiable curves* γ *for which* $E(\|dz\|_\gamma) = +\infty$*?* The answer is yes and it is even possible to find such a γ to be Jordan (see section 6). The existence of these curves shows that the item (a) cannot go further. However the equality in (b) has a different character because both sides always have a meaning. In section 5 formula (1.3) is proved. Consequently Theorem 1(b) holds for every rectifiable closed curve.

3 Isoperimetric index inequalities

Topping in [18, Lem. 1] gives a proof of the isoperimetric inequality for Jordan domain with regular boundary. In the words of the author: "we believe to be shorter than any previously known proof". It consists in proving directly (1.3) for $F = 1$ and γ^* being a regular Jordan curve.

In this section we prove the isoperimetric index inequality with length, using some ideas and results of section 2. It is the shortest proof that we know. However, instead of going directly, we do it by steps. These steps provide more information, even in the case of Topping's situation.

The next lemma will be used in section 5. The particular case $F = 1$ is well-known [4, 19] and it is enough for proving Proposition 1.

Lemma 4. *Let γ be a rectifiable curve and let $A \neq \emptyset$ be a subset of $\mathbb{C}$. Assume that $F : \gamma^* \times A \longrightarrow \mathbb{C}$ is a continuous function. Then the function $w \to \int_\gamma F(z,w)K_1(z,w)\,dz$ is continuous on A.*

Proof. Since F is bounded on compact sets, the continuity follows from Lemma 3(a) and the dominated convergence theorem. □

If $g \in L^\infty(\mathbb{C}, m)$ with compact support, then $C(g)$ belongs to the Zymund class, so in particular it is continuous. Here we give a proof of this fact that is more direct than the standard one; see [2, p. 154].

Lemma 5. *Let $g \in L^\infty(\mathbb{C}, m)$ be compactly supported. Then $C(g)$ is continuous.*

Proof. The hypothesis and (2.6) imply that $C(g)$ is defined for all $w \in \mathbb{C}$. By Lemma 2 one has, for some big enough $R > 0$ and $z \neq w$,

$$|C(g)(w) - C(g)(z)| \leq c|w-z|\Big(1 + \log\frac{R}{|z-w|}\Big) \to 0 \ \text{ as } \ |w-z| \to 0.$$

□

Proposition 1. *Let γ be a rectifiable closed curve. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{z}-\overline{w}}{z-w}\,dz = \frac{1}{\pi}\int_{\mathbb{C}} \frac{Ind(\gamma,\tau)}{\tau - w}\,dm(\tau), \quad w \notin \gamma^*. \tag{3.1}$$

If moreover $Ind(\gamma,\cdot)$ is bounded, then

$$\frac{1}{2\pi i}\int_\gamma K_1(z,w)\,dz = \frac{1}{\pi}\int_{\mathbb{C}} \frac{Ind(\gamma,\tau)}{\tau - w}\,dm(\tau), \quad w \in \mathbb{C}. \tag{3.2}$$

Proof. Fix $w \notin \gamma^*$ and take $R > 0$ so large that $\gamma^* \cup \{w\} \subset D(0, R)$. By integration, with respect to dz, of both sides of (2.5) we get

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{z} - \overline{w}}{z - w}\, dz = \frac{i}{2\pi^2}\int_\gamma \Big(\int_{D(0,R)} \frac{dm(\tau)}{(z-\tau)(w-\tau)}\Big)\, dz. \tag{3.3}$$

By (2.3) the function $z \to \int_{D(0,R)} \dfrac{dm(\tau)}{|z-\tau||w-\tau|}$ is bounded when $z \in \gamma^*$. Then Fubini's theorem can be applied and allows us to change the order of integration in (3.3) and to obtain that the function $\tau \to \dfrac{\mathrm{Ind}(\gamma, \tau)}{\tau - w}$ is integrable with respect to m for each $w \notin \gamma^*$ and (3.1) holds.

With the additional assumption on γ, Lemma 4 and Lemma 5 tell us that we have two continuous functions on $\mathbb{C}$ that are equal on $\mathbb{C} \setminus \gamma^*$ by (3.1) and so (3.2) holds. □

Theorem 2. *Let γ be a rectifiable closed curve. Then*

$$\|Ind(\gamma, \cdot)\|_2^2 \le \frac{1}{4\pi}\Lambda(\gamma)^2,$$

i.e. (1.5) *holds.*

Proof. Integrating both sides of (3.2) with respect to dw and reversing the order of integration, we obtain that

$$4\pi\|\mathrm{Ind}(\gamma, \cdot)\|_2^2 = \int_{\gamma\times\gamma} K_1(z, w)\, dzdw = \Big|\int_{\gamma\times\gamma} K_1(z, w)\, dzdw\Big| \le \Lambda(\gamma)^2 \tag{3.4}$$

holds for each rectifiable closed curve γ with $\mathrm{Ind}(\gamma, \cdot)$ bounded. Let (P_n) be a sequence of polygonal closed curves converging uniformly on $[a, b]$ to γ, as in [5, Lem. 4]. So, using Fatou's lemma and (3.4), one has

$$\|\mathrm{Ind}(\gamma, \cdot)\|_2^2 \le \liminf \|\mathrm{Ind}(P_n, \cdot)\|_2^2 \le \frac{1}{4\pi}\liminf \Lambda(P_n)^2 = \frac{\Lambda(\gamma)^2}{4\pi}.$$

□

Theorem 3. *Let U be an open set and let f be a holomorphic function defined on U. Assume that γ is a rectifiable closed curve which is homologous to zero respect to U and such that $Ind(\gamma, \cdot)$ is bounded. Then*

$$\frac{1}{4\pi}\int_{(f\circ\gamma)\times(f\circ\gamma)} K_1(z, w)\, dz\, dw = \int_{\mathbb{C}} Ind(\gamma, \cdot)(Ind(f\circ\gamma, \cdot)\circ f)|f'|^2\, dm. \tag{3.5}$$

Proof. Let V be an open connected set with $\gamma^* \subset V \subset \overline{V} \subset U$ such that $K := \overline{V}$ is a compact set and γ is homologous to zero with respect to V. We can assume that f is non constant on K. Let $w \in f(K) \setminus f(\gamma^*)$ be fixed and let $w_j \in K$, $1 \le j \le r$ with $f(w_j) = w$. Take $\varepsilon > 0$ small enough such that $\overline{D(w_j, \varepsilon)} \subset V$, $\overline{D(w_j, \varepsilon)} \cap \gamma^* = \emptyset$ for each j. Consider the cycle $\gamma_\varepsilon := \gamma - \sum_{j=1}^r \mathrm{Ind}(\gamma, w_j)C_j$ with $C_j(t) = w_j + \varepsilon e^{it}$, $t \in [0, 2\pi], 1 \le j \le r$. Then γ_ε is homologous to zero with respect to V. Applying Green's formula with multiplicities [5, p. 107], we obtain

$$\frac{1}{2i}\int_{\gamma_\varepsilon} \frac{\overline{f(\zeta)} - \overline{w}}{f(\zeta) - w} f'(\zeta)\, d\zeta = \int_K \frac{|f'(\tau)|^2}{f(\tau) - w}\mathrm{Ind}(\gamma_\varepsilon, \tau)\, dm(\tau). \tag{3.6}$$

We want to study the possibility of taking $\varepsilon \to 0$ in both sides of (3.6). In the left-hand side there is no problem since the integrand is continuous. For the other one we argue as follow. The hypothesis on γ, the Area theorem [6, p. 96] and (2.6) give

$$\int_K \frac{|f'(\tau)|^2}{|f(\tau)-w|}|\mathrm{Ind}(\gamma,\tau)|\,dm(\tau) \le c\int_{f(K)} \frac{n(f,\tau')}{|\tau'-w|}\,dm(\tau') \le c,\; w\in f(K), \tag{3.7}$$

where $n(f,\tau')$ is the number of points in $f^{-1}(\tau')\cap K$. Moreover by using the same arguments one has

$$\int_{D(w_j,\varepsilon)} \frac{|f'(\tau)|^2}{|f(\tau)-w|}\,dm(\tau) \le c\,[m(f(D(w_j,\varepsilon)))]^{1/2}, \quad 1\le j\le r.$$

Now we claim

$$\frac{1}{2i}\int_\gamma \frac{\overline{f(\zeta)}-\overline{w}}{f(\zeta)-w} f'(\zeta)\,d\zeta = \int_K \frac{|f'(\tau)|^2}{f(\tau)-w}\mathrm{Ind}(\gamma,\tau)\,dm(\tau),\; w\in f(K). \tag{3.8}$$

That (3.8) holds for $w\in f(K)\setminus f(\gamma^*)$ is just obtained by letting $\varepsilon\to 0$ in (3.6) which is correct by the arguments just given. The right-hand side of (3.8) is a well defined function of w by (3.7). Moreover it is a continuous function. This is proved by a similar argument to the one used in Lemma 5 that consists in estimating its modulus of continuity by applying the Area theorem and (2.3). Then (3.8) is obtained arguing as at the end of the proof of Proposition 1.

Finally integrate both sides of (3.8) with respect to $(1/2\pi i)dw_{f\circ\gamma}$ and interchange the order of the integrals. Here the use of Fubini's theorem is legitimate by (3.7) and hence (3.5) is obtained. □

The Theorem in [21, p. 534] (for $p=1$) gives

$$4\pi\int_{\mathbb{D}} |g(z)|^2\,dm(z) \le \Big(\int_{\partial\mathbb{D}} |g(e^{it})|\,dt\Big)^2, \quad g\in H^1.$$

In [13, Th. 4] the above estimate is proved for the case that $g=f'$ and g is defined on a domain limited by analytic Jordan curves. Theorem 3 can be adapted, with a similar proof, to cover this situation. Therefore, as a consequence of Theorem 3 and the Argument Principle, we obtain the following corollary which is closely related to these results. Also it must be compared with [18, Lem. 2].

Corollary 1. *Let $\gamma=\gamma_1+\cdots+\gamma_n$ be a rectifiable cycle. Assume that $\gamma_j^*\cap\gamma_k^*=\emptyset$ for $j\ne k$. Let U be an open set such that $\gamma^*\subset U$ and γ is homologous to zero with respect to U. Let f be a holomorphic function on $\overline{U}$. Then*

$$\int_{\mathbb{C}} |Ind(\gamma,\tau)|\,|f'(\tau)|^2\,dm(\tau) \le \frac{1}{4\pi}\Big(\sum_{j=1}^n V(f\circ\gamma_j)\Big)^2.$$

Now we analyze when the equality in (1.4) holds. On the basis of the obtained result we do the same for (1.5). Here we need to assume that (1.3) holds for all rectifiable curves with $F=1$.

Proposition 2. *Let γ be a rectifiable closed curve and assume that dz_γ is not the null measure. Then the equality in* (1.4) *holds if and only if there exist a circle $C(a,R)$ and $Z \subset \mathbb{C}$ such that*

$$\gamma^* = C(a,R) \cup Z, \quad C(a,R) \cap Z = \emptyset \text{ and } \shortparallel dz \shortparallel (Z) = 0. \tag{3.9}$$

Proof. For the sufficiency let $E := \gamma^{-1}(C(a,R))$ and define

$$F(w) := \frac{1}{2\pi i}\int_E \frac{d\gamma(t)}{\gamma(t)-w}, \quad w \notin \gamma(E).$$

The function F is continuous on $\mathbb{C} \setminus \gamma(E)$ and by (3.9) one has $F(w) = \operatorname{Ind}(\gamma, w)$ if $w \notin \gamma^*$, so F takes only integer values and $F(w) = 0$ if $w \notin \overline{D(a,R)}$. The fact that $dz \neq 0$ and [5, Cor. 4(a)] imply that there is $w_0 \notin \gamma^*$ with $F(w_0) = n \neq 0$, $n \in \mathbb{Z}$; so $w \in D(a,R)$ and F is constant on $D(a,R)$. Then the left-hand side of (1.4) equals to $\pi R^2 n^2$. To evaluate $V(dz)$ put $C_n(t) = a + Re^{int}$, $t \in [0, 2\pi]$. By [5, Cor. 4(b)] one gets $\int_{C_n} g\,dz = \int_\gamma g\,dz$ for each $g \in C(\gamma^*)$. So $\shortparallel dz \shortparallel (\gamma^*) = \int_{C_n} \shortparallel dz \shortparallel = 2\pi R n$ and the equality (1.4) holds.

For the necessity, let $dz = \rho(z) \shortparallel dz \shortparallel$ be the polar decomposition of dz on γ^*, [16, p. 124]. Using (1.3) with $F = 1$ and the conditions under which the absolute value of an integral equals the integral of the absolute value of its integrand, we see that equality in (1.4) holds if and only if there is $\alpha \in \mathbb{C}$ such that

$$k_1(z,w)\rho(z)\rho(w) = \alpha,\ |\alpha| = 1, \text{ for } dzdw \text{ a. a. } (z,w) \in \gamma^* \times \gamma^*. \tag{3.10}$$

A key point is to show that $\alpha = 1$. By properties of rectifiable curves and (3.10), we can choose $z_0 \in \gamma^*$ such that: (i) There exists $t \in \gamma^{-1}(z_0)$ such that $\gamma'(t) \neq 0$ exists and $\rho(z_0) = \gamma'(t)/|\gamma'(t)|$. (ii) The set $L_{z_0} := \{w \in \gamma^* \setminus \{z_0\} : k_1(z_0, w)\rho(z_0)\rho(w) = \alpha\}$ has full measure. (iii) z_0 is an accumulation point for a closed set $E \subset L_{z_0}$ for which $\rho_{|E}$ is continuous (by Lusin's theorem). Now taking a sequence $w_n \in E$ we see, by (i) and (iii), that $k_1(z_0, w_n)\rho(z_0)\rho(w_n) \to 1$ as $n \to \infty$, and hence $\alpha = 1$ in (3.10). For simplicity, and without changing the notation, we may suppose that $z_0 = 0$, $\rho(z_0) = 1$. Since $dz \neq 0$ we choose $w_0 \notin \mathbb{R}$, $w_0 \in L_0$ with L_{w_0} having full measure. Applying twice (3.10) (recall that $\alpha = 1$), we obtain

$$\frac{\overline{w} - \overline{w_0}}{w - w_0}\,\frac{w_0}{\overline{w_0}}\,\frac{w}{\overline{w}} = 1, \quad \text{if } w \in L_{w_0} \cap L_0\,.$$

The above equality is $|w|^2(w_0 - \overline{w_0}) = |w_0|^2(w - \overline{w})$. This means that w belongs to the circle C that is tangent to 1 at the origin and meets the point w_0, so $L_{w_0} \cap L_0 \subset C$ and $\shortparallel dz \shortparallel (\gamma^* \setminus C) = 0$. Since $dz_\gamma \neq 0$ one has $C \subset \gamma^*$ and defining $Z = \gamma^* \setminus C$ then (3.9) holds. □

Proposition 3. *(a) Let $a \in \mathbb{C}$, $R > 0$, $n \in \mathbb{Z}$ and let $\phi : [0,1] \to \mathbb{R}$ be a continuous increasing function with $\phi(1) = \phi(0) + 1$. If the curve γ is defined by $\gamma(t) = a + Re^{2\pi i n\phi(t)}$, $t \in [0,1]$, then the equality in* (1.5) *holds.*

(b) Conversely, if $\gamma : [0,1] \longrightarrow \mathbb{C}$ is a rectifiable closed curve such that equality in (1.5) *holds, then γ is of the form described in (a).*

Proof. The following fact is true: If $\gamma(t) = a + Re^{i\phi(t)}$ for some continuous real valued function ϕ, then $\Lambda(\gamma) = RV(\phi)$, where $V(\phi)$ is the total variation of ϕ. Using this fact the item (a) follows immediately.

In proving (b) we assume that γ is not constant. If the equality holds in (1.5) then there is equality in (1.4) and so there are $C(a, R)$ and Z satisfying (3.9) and $V(dz_\gamma) = \Lambda(\gamma)$. Since $\shortparallel dz_{\gamma_{|F}} \shortparallel \leq |dz_{\gamma_{|F}}|$, with $F = \gamma^{-1}(C(a, R))$, one obtains $|dz|(Z) = 0$. The local connectivity of Z and the connectedness of γ^* implies $Z = \emptyset$. The curve $t \to (\gamma(t) - a)/Re^{2\pi int}$ has zero index with respect to 0 if $n = \mathrm{Ind}(\gamma, a)$. Therefore there exists a real function ψ, defined on $[0, 1]$, with $\psi(0) = \psi(1)$, such that $(\gamma(t) - a)/Re^{2\pi int} = e^{2\pi i\psi(t)}$. The function that we are looking for is $\phi(t) = t + \frac{\psi(t)}{n}$. By the assumption and the definition of ϕ, one gets $V(\phi) = \phi(1) - \phi(0) = 1$ and this assertion on ϕ implies that ϕ is increasing. $\square$

4 A structural theorem

Let $E \neq \emptyset$ be a closed subset of $\mathbb{T}$. Then $\mathbb{T} \setminus E = \cup_n I_n$, where the open arcs I_n are the connected components of $\mathbb{T} \setminus E$ and n runs in a subset $N(E) \subset \mathbb{N}$. If $N(E) \neq \emptyset$, we write $I_n = I(a_n, b_n)$, for each $n \in N(E)$, where a_n is the beginning point and b_n is the endpoint of I_n assuming $\mathbb{T}$ is endowed with the positive orientation.

Definition 1. *Let $\gamma : \mathbb{T} \to \mathbb{C}$ be a curve.*

(a) A closed subcurve of γ is given by a pair $(E, \tilde{\gamma}_E)$ where

(i) $E \neq \emptyset$ is a perfect subset of $\mathbb{T}$,

(ii) $\gamma(a_n) = \gamma(b_n)$, for each connected component $I(a_n, b_n)$, $n \in N(E)$ of $\mathbb{T} \setminus E$,

(iii) $\tilde{\gamma}_E$ is the closed curve defined by $\tilde{\gamma}_E(\zeta) = \gamma(\zeta)$ if $\zeta \in E$ and $\tilde{\gamma}_E(\zeta) = \gamma(a_n)$ if $\zeta \in I(a_n, b_n)$ for some $n \in N(E)$.

(b) A Jordan loop of γ is a closed subcurve $(E, \tilde{\gamma}_E)$ such that $\tilde{\gamma}_E$ is one to one on $\mathbb{T} \setminus \cup_n [a_n, b_n)$.

The following remark justify the needed properties on E in Definition 1.

Remark 2. *For every perfect set $\emptyset \neq E \subset \mathbb{T}$ one can find a closed curve γ such that $(E, \tilde{\gamma}_E)$ is a non-constant subcurve of γ.*

Also it is interesting to mention that for a closed curve γ it is not geometrically evident how to distinguish, among all Jordan curves included on γ^*, those that are $\gamma(E)$ for $(E, \tilde{\gamma}_E)$ a Jordan loop of γ.

Definition 2. *Let γ be a closed curve.*

(a) We say that γ is non-null if there is a closed subcurve $(E, \tilde{\gamma}_E)$ of γ and a point $z \notin \tilde{\gamma}_E(\mathbb{T})$ such that Ind$(\tilde{\gamma}_E, z) \neq 0$. Otherwise we say that γ is a null curve.

(b) A subcurve $(E, \tilde{\gamma}_E)$ is a root subcurve of γ provided that $(E, \tilde{\gamma}_E)$ is a non-constant null curve, E is a closed interval and there is no null subcurve $(F, \tilde{\gamma}_F)$ of γ with F being an interval, $E \subset F$ and $E \neq F$.

Recall that $d_H(E,F) = \max\{\sup_{x\in E} d(x,F), \sup_{y\in F} d(y,E)\}$ defines the Hausdorff distance on the space of non-empty compact subsets of $\mathbb{T}$, with $d(z,w) = |z-w|$, $z,w \in \mathbb{T}$. As it is known [12, p. 47] this space becomes compact with this metric.

We need the following technical result.

Lemma 6. *Let γ be a closed curve and let $(E_n, \tilde\gamma_{E_n})$ be a sequence of closed subcurves of γ. Assume that the sequence (E_n) converges to E as $n \to \infty$ with respect to d_H. Then*

(a) E satisfies (ii) of the Definition 1,

(b) $\tilde\gamma_{E_n} \to \tilde\gamma_E$ uniformly on $\mathbb{T}$ as $n \to \infty$.

Proof. Let us fix a connected component $I(a_0,b_0)$ of $\mathbb{T} \setminus E$ and let a_M be the middle point of $I(a_0,b_0)$. Let $\delta_n = |a_0 - b_0|/n, \quad n \geq 3$. By the hypothesis there exist

$$k_n \in \mathbb{N},\ a'_n, b'_n \in E_{k_n} \text{ such that } |a_0 - a'_n| < \delta_n,\ |b_0 - b'_n| < \delta_n, \tag{4.1}$$

$$E_{k_n} \subset \{t \in \mathbb{T} : d(t,E) < \delta_n\}. \tag{4.2}$$

By (4.2) there exists a connected component of $\mathbb{T} \setminus E_{k_n}$ that contains a_M, let it be $I(a_n,b_n)$, with $a_n, b_n \in E_{k_n}$. By (4.1), (4.2) and Definition 1 one has $|a_0 - a_n| < \delta_n$, $|b_0 - b_n| < \delta_n$ and $\gamma(a_n) = \gamma(b_n)$. Letting $n \to \infty$ we conclude $\gamma(a_0) = \gamma(b_0)$ and that proves (a).

To prove (b) let us fix δ and take $n_0 \in \mathbb{N}$ such that

$$E \subset \{s : d(s,E_n) < \delta\}, \quad E_n \subset \{s : d(s,E) < \delta\} \quad \text{if} \quad n \geq n_0. \tag{4.3}$$

For $n \geq n_0$ fixed we will prove

$$|\tilde\gamma_{E_n}(t) - \tilde\gamma_E(t)| \leq \omega(\gamma,\delta), \quad t \in \mathbb{T}, \tag{4.4}$$

where $\omega(\gamma,\cdot)$ is the modulus of continuity of γ. The uniform continuity of γ on $\mathbb{T}$ and (4.4) give (b).

According to the position of t with respect to E and E_n there are four cases. We prove (4.4) in two of them since the other cases can handed in a similar way.

Case 1: $t \in E$ and $t \notin E_n$. Let $I(a_n,b_n)$ be the connected component of $\mathbb{T} \setminus E_n$ that contains t. Then $d(t,E_n) = \min\{|t-a_n|, |t-b_n|\}$. Thus $d(t,E_n) < \delta$ by (4.3) and one gets

$$|\tilde\gamma_{E_n}(t) - \tilde\gamma_E(t)| = |\gamma(t) - \gamma(a_n)| = |\gamma(t) - \gamma(b_n)| \leq \omega(\gamma,\delta).$$

Case 2: $t \notin E$ and $t \notin E_n$. Let $I(a_n,b_n)$ be as in Case 1 and similarly let $I(a,b) \subset \mathbb{T} \setminus E$ containing t. By the symmetry in (4.3) we note that it is enough to prove (4.4) in the case that a_n is between a and t and $|a - a_n| \leq |a_n - b|$. Then, by (4.3), one gets $|a_n - a| = d(a_n,E) < \delta$ and so $|\tilde\gamma_{E_n}(t) - \tilde\gamma_E(t)| = |\gamma(a_n) - \gamma(a)| \leq \omega(\gamma,\delta)$. □

The following result gives the existence of Jordan loops for each non-null closed curve.

Proposition 4. *Let γ be a closed curve and let $z_0 \in \widehat{\gamma^*} \setminus \gamma^*$. Then there exists a Jordan loop $(E, \tilde\gamma_E)$ of γ such that $z_0 \in D(\tilde\gamma_E)$.*

Proof. If γ is a Jordan curve, then the result is clear. Otherwise, let Γ_{z_0} be the set of all closed subcurves $(E, \tilde{\gamma}_E)$ of γ satisfying $\mathrm{Ind}(\tilde{\gamma}_E, z_0) \neq 0$. We first prove that $\Gamma_{z_0} \neq \emptyset$. It is enough to consider the case that $\mathrm{Ind}(\gamma, z_0) = 0$. If $g(t) = \log(\gamma(e^{it}) - z_0)$ is a continuous branch of the logarithm defined for $t \in [0, 2\pi]$, then g is a closed curve. We prove the following claim.

Claim: There exist $k \in \mathbb{Z}, k \neq 0$, and $s, t \in [0, 2\pi]$ such that $\mathrm{Re}\ g(s) = \mathrm{Re}\ g(t)$ and $\mathrm{Im}\ g(s) - \mathrm{Im}\ g(t) = 2k\pi$.

If the claim is not true then the exponential function would be one to one on g^*. This means that there is a continuous branch of $\log(z - z_0)$ defined on γ^*. So, by the Eilenberg separation theorem [12, p. 544], z_0 and ∞ would not be separated by γ^*. This contradicts that $z_0 \in \widehat{\gamma^*} \setminus \gamma^*$, and therefore the claim is proved.

Since $\gamma(e^{it}) = \gamma(e^{is})$ and $k \neq 0$ we can take the restriction of γ to any one of the two arcs in $\mathbb{T}$ joining e^{is} with e^{it} as an element of Γ_{z_0}, so $\Gamma_{z_0} \neq \emptyset$.

The subcurve just found is not, in general, a Jordan loop. To find the required loop we introduce in Γ_{z_0} the partial order given by

$$(E, \tilde{\gamma}_E) \leq (F, \tilde{\gamma}_F) \quad \text{if and only if} \quad E \subset F. \tag{4.5}$$

With the aim of applying Zorn's lemma (see [9, p. 142]) let us show that each chain of Γ_{z_0} has a lower bound. To see this let $(E_\alpha, \tilde{\gamma}_\alpha), \alpha \in \Omega$ be the elements of a chain. Let $E = \bigcap_\alpha E_\alpha$, so E is a non-empty compact set. Using first Lindelof's theorem [22, p. 4] and later the chain condition we can choose a sequence $(E_{\alpha(n)})$ such that $E_{\alpha(n+1)} \subset E_{\alpha(n)}$ and $E = \bigcap_n E_{\alpha(n)}$. This implies $d_H(E_{\alpha(n)}, E) \to 0$ and then $(E, \tilde{\gamma}_E)$ satisfies (a) and (b) of Lemma 6. Hence $\mathrm{Ind}(\tilde{\gamma}_E, z_0) = \lim_{n\to\infty} \mathrm{Ind}(\tilde{\gamma}_{\alpha(n)}, z_0) \neq 0$.

Now we consider E_c the set of all condensation points of E (see [22, p. 53]). Observe that $E_c \neq \emptyset$. According to Cantor-Bendixson theorem E_c is a perfect set and $E = E_c \cup M$, $E_c \cap M = \emptyset$ where M has at most countably many points. So $(E_c, \tilde{\gamma}_{E_c})$ is a lower bound in Γ_{z_0} of the given chain provided it is a subcurve. To prove this, let (a, b) be a connected component of $\mathbb{T} \setminus E_c$ which is not a component of $\mathbb{T} \setminus E$. Since $M \cap (a, b)$ is relatively closed in (a, b) one has $(a, b) \setminus M = \cup_j (c_j, d_j)$, where each (c_j, d_j), $j \geq 2$ is a component of $\mathbb{T} \setminus E$. Since $\gamma(c_j) = \gamma(d_j)$ by (ii) of Definition 1 we can define a continuous function γ_1 on (a, b) such that $\gamma_1(t) = \gamma(c_j)$, $t \in [c_j, d_j]$. Then $\gamma_1((a, b)) \subset \cup_j \{\gamma(c_j)\}$. This implies that γ_1 is constant; therefore $\gamma(a) = \gamma(b)$ and E_c satisfies (ii) of Definition 1.

By Zorn's lemma there is a minimal element $(E_0, \tilde{\gamma}_{E_0}) \in \Gamma_{z_0}$. We will prove that $(E_0, \tilde{\gamma}_{E_0})$ is a Jordan loop. We know that $E_0 \neq \mathbb{T}$ because Γ_{z_0} has at least two elements. Then we define the relation $t \sim s$ if and only if $t = s$ or $t, s \in \overline{J}$, where J is a connected component of $\mathbb{T} \setminus E_0$. This is an equivalence relation since E_0 is a perfect set. Furthermore the topological space $\mathbb{T}/\sim$ is homeomorphic to $\mathbb{T}$. Let ρ be the factorization map of $\tilde{\gamma}_{E_0}$ to $\mathbb{T}/\sim$. Since $\rho(\mathbb{T}/\sim) = \tilde{\gamma}_{E_0}(\mathbb{T})$ we need only to prove that ρ is one to one. Assume that this is not the case. Then we could find $t_1, t_2 \in \mathbb{T}$ with t_1 not equivalent to t_2, such that $\tilde{\gamma}_{E_0 \cap \overline{I(t_1, t_2)}} \in \Gamma_{z_0}$. The fact that $E_0 \cap \overline{I(t_1, t_2)} \neq E_0$ contradicts the minimal property of $(E_0, \tilde{\gamma}_{E_0})$. □

Next corollary provides information about null curves.

Corollary 2. *Let γ be a closed curve.*

(a) If γ is a null curve, then $\mathbb{C} \setminus \gamma^*$ is connected.

(b) Assume that γ^* has no interior points. Then $\mathbb{C} \setminus \gamma^*$ is connected if and only if γ is a null curve.

(c) Suppose that γ is rectifiable; then $\shortparallel dz \shortparallel_\gamma = 0$ whenever γ is a null curve.

We denote by $\mathrm{diam}(L)$ the diameter of a subset L of $\mathbb{C}$. Now the following conjecture arises in a natural way.

Conjecture 1. *Let γ be a non-null rectifiable closed curve on $\mathbb{C}$. Then there exists a Jordan loop $(E, \tilde{\gamma}_E)$ of γ such that*

$$diam(\gamma(E)) = \sup\{diam(\gamma(F)) : (F, \tilde{\gamma}_F) \text{ is a Jordan loop of } \gamma\}.$$

Conjecture 1 holds in all examples that we have considered, but we have not been able to prove it. It is worthwhile mentioning that, in the above conjecture, the use of diameter is essential. The reason is that the assertion that consists in replacing in the conjecture the diameter with the length is false.

Now we prove the next result bypassing the conjecture.

Theorem 4. *Let γ be a rectifiable closed non-null curve.*

(i) There exists a family $(E^{(n)}, \tilde{\gamma}_{E^{(n)}})$ of Jordan loops of γ with $1 \leq n \leq n_0 \leq \infty$, such that for each n, n', $n' \neq n$, the set $E^{(n)} \cap E^{(n')}$ contains at most two points.

If $\mathbb{T} \neq \cup E_n$, then $\mathbb{T} = \bigcup_{n,m,p}(E^{(n)} \cup F^{(m)} \cup S^{(p)}) \cup S$, where each set (if it exists) has the following properties:

(ii) Each $(F^{(m)}, \tilde{\gamma}_{F^{(m)}})$ is a null subcurve for $1 \leq m \leq m_0 \leq \infty$; $F^{(m)} \cap F^{(m')} = \emptyset$ if $m \neq m'$; each set $F^{(m)}$ is a finite union of intervals of $\mathbb{T}$ and for each m, $F^{(m)} \cap E^{(n)} = \emptyset$ except for finite number of n for which $F^{(m)} \cap E^{(n)}$ contains at most two points. Moreover all extreme points of the intervals of each $F^{(m)}$ are included in $\cup E^{(n)}$.

(iii) Each $(S^{(p)}, \tilde{\gamma}_{S^{(p)}})$ is a non-constant null subcurve for $1 \leq p \leq p_0 \leq \infty$; $S^{(p_1)} \cap S^{(p_2)} = S^{(p_1)} \cap F^{(m)} = \emptyset$ for all m, $p_1 \neq p_2$ and $S^{(p)} \cap E^{(n)}$ contains at most two points for all n, p.

(iv) $S \cap (E^{(n)} \cup F^{(m)} \cup S^{(p)}) = \emptyset$ for each n, m, p and the set $\gamma(S)$ has zero measure with respect to $\shortparallel dz_{\gamma_{|W}} \shortparallel$, where $W = \gamma^{-1}(Q) \cap S$, for each measurable set $Q \subset \gamma(S)$.

(v) Let $B = \{q \in \gamma^* :$ there exists a sequence of Jordan curves $(\gamma(E^{(n_k)}))$ such that $d_H(\gamma(E^{(n_k)}), \{q\}) \to 0$ as $n_k \to \infty\}$. Then $\gamma(S) \subset B$ and $B \cap \gamma(S^{(p)}) \neq \emptyset$ for each p, $1 \leq p \leq p_0$.

Proof. In order to define the desired subcurves of γ and the appropriate subsets of $\mathbb{T}$ we make a recurrence process as follows. To begin with let

$$0 < d = \sup\{\mathrm{diam}(\gamma(E)) : (E, \tilde{\gamma}_E) \text{ is a Jordan loop of } \gamma\}$$

and take $(E(-), \tilde{\gamma}_{E(-)})$ to be a Jordan loop of γ such that d is equal to $\mathrm{diam}(\gamma(E(-)))$ if it exists or otherwise such that $\mathrm{diam}(\gamma(E(-))) > d/2$. Now we write $I(j_1)$ for the connected components of $\mathbb{T} \setminus E(-)$ where j_1 runs in a set $N(E(-))$. We define $N^*(E(-))$ as the subset of $N(E(-))$ that contains all j_1 for which $\tilde{\gamma}_{\overline{I(j_1)}}$ is a non-null subcurve of γ. For each $j_1 \notin N^*(E(-))$ we put $F(j_1) = \overline{I(j_1)}$.

Assume now that for each finite sequence of length n, $\tau(n) = j_1, \ldots, j_n$, with $n \geq 1$, the sets $I(\tau(n)), E(\tau(n-1)), N^*(E(\tau(n-1)))$ and $F(\tau(n))$ are already well defined. From now on we understand that $E(\tau(0)) = E(-)$ and $\overline{I(\tau(0))} = \overline{I(-)} = \mathbb{T}$.

For each $\tau(n) = j_1, \ldots, j_n$ with $j_n \in N^*(E(\tau(n-1)))$, $n \geq 1$ we consider

$$0 < d(\tau(n)) = \sup\{\mathrm{diam}(\gamma(E)) : (E, \tilde{\gamma}_E) \text{ is a Jordan loop of } \tilde{\gamma}_{\overline{I(\tau(n))}}\}$$

and we take a Jordan loop $(E(\tau(n)), \tilde{\gamma}_{E(\tau(n))})$ (with the same characteristics as before) such that

$$d(\tau(n)) \geq \mathrm{diam}(\gamma(E(\tau(n))) > d(\tau(n))/2. \tag{4.6}$$

We mention here that we are thinking that each closed subcurve $\tilde{\gamma}_{\overline{I(\tau'(n))}}$ is defined on $\mathbb{T}$ in order to apply our procedure. This means that each interval $\overline{I(j_1)}$ becomes $\mathbb{T}$ by identification of the two extreme points of $\overline{I(j_1)}$ and since each $\overline{I(j_1, \ldots, j_n, j_{n+1})}$ is a subset of $I(j_1, \ldots, j_n)$, the identification of $\overline{I(j_1, \ldots, j_n, j_{n+1})}$ with $\mathbb{T}$ is compatible with the previous identification of $\overline{I(j_1, \ldots, j_n)}$ with $\mathbb{T}$. Furthermore we need to consider the sets $E(j_1, \ldots, j_n)$ and $I(j_1, \ldots, j_n)$ defined on the original $\mathbb{T}$ and so we must reverse the aforementioned identifications, but paying attention to the fact that each $E(\tau(n))$ has to be a perfect set. Doing this it is clear that each set $\overline{I(j_1, \ldots, j_n)}$ becomes a finite union of intervals of $\mathbb{T}$.

Let us continue by writing $I(j_1, \ldots, j_n, j_{n+1})$ for each connected components of $\overline{I(j_1, \ldots, j_n)} \setminus E(j_1, \ldots, j_n)$ with $j_{n+1} \in N(E(j_1, \ldots, j_n))$ and $N^*(E(j_1, \ldots, j_n))$ for the set of j_{n+1} for which $\tilde{\gamma}_{\overline{I(j_1, \ldots, j_{n+1})}}$ is a non-null subcurve of γ. We also put $F(\tau(n+1)) = \overline{I(\tau(n+1))}$ if $\tau(n+1) = j_1, \ldots, j_{n+1}$ and $j_{n+1} \notin N^*(E(j_1, \ldots, j_n))$.

The subsets of $\mathbb{T}$ that we have chosen satisfy, for each $n \geq 2$,

$$\mathbb{T} = E(-) \cup E_{n-1} \cup F_{n-1} \cup \bigcup_{j_1, \ldots, j_n} \overline{I(j_1, \ldots, j_n)}, \tag{4.7}$$

where E_{n-1} is equal to the union of the sets $E(j_1, \ldots, j_k)$ for all indices $1 \leq k \leq n-1$, $j_1 \in N^*(E(-)), j_2 \in N^*(E(j_1)), \ldots, j_k \in N^*(E(j_1, \ldots, j_{k-1}))$; F_{n-1} equals to the union of the sets $F(j_1, \ldots, j_k)$ for all indices $1 \leq k \leq n-1$, $j_1 \in N^*(E(-)), j_2 \in N^*(E(j_1)), \ldots, j_k \notin N^*(E(j_1, \ldots, j_{k-1}))$ and the last union in (4.7) is extended over all indices $j_1 \in N^*(E(-)), j_2 \in N^*(E(j_1)), \ldots, j_n \in N(E(j_1, \ldots, j_{n-1}))$. Now two cases are possible.

Case 1: There is $n \in \mathbb{N}$ such that $N^*(E(j_1, \ldots, j_n)) = \emptyset$ for all $j_1, \ldots, j_n$. Then we say that the process stops at the level n and hence

$$\overline{I(j_1, \ldots, j_n)} = E(j_1, \ldots, j_n) \cup \bigcup_{j_{n+1}} F(j_1, \ldots, j_n, j_{n+1}). \tag{4.8}$$

Case 2: For each n there exist $j_1, j_2, \dots, j_n$ for which $N^*(E(j_1, \dots, j_n))$ is nonempty. Then the process can be continued indefinitely and we define the exceptional set

$$R = \bigcap_{n=1}^{\infty} \bigcup_{j_1 \in N^*(E(-)), \dots, j_n \in N^*(E(j_1, \dots, j_{n-1}))} \overline{I(j_1, \dots, j_n)}. \tag{4.9}$$

It can happen that $R = \emptyset$.

In both cases we take as $E^{(n)}$, $1 \le n \le n_0 \le \infty$ the sequence of the sets $E(j_1, \dots, j_k)$ and $F^{(m)}, 1 \le m \le m_0 \le \infty$, as $F(j_1, \dots, j_k)$ for each $j_1, \dots, j_k$ corresponding to the indices that appear in (4.7).

In order to prove the second assertion in (i) let $\tau(n) = j_1, \dots, j_n$ and $\tau'(k) = i_1, \dots, i_k$, $k \le n$. By the construction one has

$$\overline{I(\tau(n))} \cap \overline{I(\tau'(k))} \neq \emptyset \Rightarrow i_1 = j_1, \dots, i_k = j_k \text{ and } I(\tau(n)) \subset I(\tau'(k)). \tag{4.10}$$

If $E(\tau(n)) \cap E(\tau'(k)) \neq \emptyset$, then (4.10) implies $k < n$ and $i_r = j_r$ for each $1 \le r \le k$. Consequently $E(j_1, \dots, j_n) \subset \overline{I(i_1, \dots, i_k)}$. Moreover each $\overline{I(i_1, \dots, i_k)}$ is a finite union of at most k intervals on $\mathbb{T}$, where each pair of extreme points that are identified in the same point when we think the curve $\tilde{\gamma}_{\overline{I(i_1, \dots, i_k)}}$ defined on $\mathbb{T}$. Then $E(\tau(n)) \cap E(\tau'(k)) \subset \overline{I(i_1, i_2, \dots, i_{k+1})} \cap E(\tau'(k))$ and this set has two points.

The proofs of the analogous statements for $F^{(m)}$ are similar.

When the process has ended the proof finishes by (4.8). In Case 2 we continue the argument assuming that $R \neq \emptyset$. From (4.7) and (4.9) we get

$$\operatorname{Ind}(\tilde{\gamma}_{\overline{I(\tau)}}, w) = \sum_{E(\sigma) \subset \overline{I(\tau)}} \operatorname{Ind}(\tilde{\gamma}_{E(\sigma)}, w) + \frac{1}{2\pi i} \int_{I(\tau) \cap R} \frac{d\gamma(\zeta)}{\gamma(\zeta) - w}, \; w \notin \gamma^* \tag{4.11}$$

for any $I(\tau)$ that appears in (4.9).

Let fix now a set $I(\tau(n))$ (for $\tau(n) = j_1, \dots, j_n$) that appears in (4.9). We will show that

$$\int_{I(\tau(n)) \cap R} \frac{d\gamma(\zeta)}{\gamma(\zeta) - w} = 0 \quad \text{for all} \quad w \notin \gamma^*. \tag{4.12}$$

Assume that (4.12) fails for some $w_0 \notin \gamma^*$. Since

$$0 \neq \int_{I(\tau(n)) \cap R} \frac{d\gamma(\zeta)}{\gamma(\zeta) - w_0} = \sum_{j_{n+1}} \int_{I(\tau(n), j_{n+1})} \frac{d\gamma(\zeta)}{\gamma(\zeta) - w_0} \tag{4.13}$$

we can find an index j_{n+1} such that the corresponding integral in the right hand side of (4.13) is not zero. Put $J_1 = \overline{I(\tau(n))}$ and $J_2 = \overline{I(\tau(n), j_{n+1})}$, for this j_{n+1}. By a recurrence process, using in each step (4.13) and (4.11), we find a sequence $J_1 \supset J_2 \supset \cdots \supset J_k \supset \cdots$ with $J_k = \overline{I(\tau(n + k - 1))}$, with the additional property that

$$\Big| \int_{J_k \cap R} \frac{d\gamma(\zeta)}{\gamma(\zeta) - w_0} \Big| \ge 2\pi. \tag{4.14}$$

Taking $J = \cap_k J_k$ we get a closed curve $\gamma_{|J}$ that, by (4.14) and the relative position of J, satisfies

$$0 \neq \int_{J\cap R} \frac{d\gamma(\zeta)}{\gamma(\zeta) - w_0} = \int_J \frac{d\gamma(\zeta)}{\gamma(\zeta) - w_0}.$$

So $\gamma_{|J}$ is a non-null closed curve and, by Proposition 4, there exists $(E_0, \tilde{\gamma}_{E_0})$ a Jordan loop of $\gamma_{|J}$. Using the fact that $E_0 \subset I(\tau(n+k))$, (4.6), the property (i) of the loops $(E^{(n)})$ and the estimate $2\text{diam}(\rho^*) \leq \Lambda(\rho)$, for a rectifiable closed curve ρ, we get

$$0 < \text{diam}(\gamma(E_0)) \leq d(\tau(n+k)) \leq 2\text{diam}(\gamma(E(\tau(n+k))) \to 0$$

as $k \to \infty$, which is a contradiction. So (4.12) is proved.

Now consider all decreasing sequences for which each term is a set $\overline{I(\tau(n))}$ that appears in (4.9) (here decreasing means that $\tau(n+1) = \tau(n), j_{n+1}$, for each n). The sets $S^{(p)}$, where p runs in a set of indices A, are exactly the limit sets of such sequences for which $\gamma(S^{(p)})$ is not a singleton. In that case $S^{(p)}$ is a perfect set and $(S^{(p)}, \tilde{\gamma}_{S^{(p)}})$ is a subcurve of γ. Since each $\gamma_{|S^{(p)}}$ has positive length and $S^{(p)} \cap S^{(q)} = \emptyset$ for $p \neq q$ one concludes that there are at most countably many of them and consequently we may assume that $A \subset \mathbb{N}$.

To prove that each $(S^{(p)}, \tilde{\gamma}_{S^{(p)}})$ is a null curve let us assume that this is not true for some p. Then there exists $\tau(n)$ for each n, such that $S^{(p)} \subset \overline{I(\tau(n))}$. Using again Proposition 4 we could find a Jordan loop $(L, \tilde{\gamma}_L)$ with $L \subset S^{(p)}$ and so

$$0 < \text{diam}(\gamma(L)) \leq 2\text{diam}(\gamma(E(\tau(n)))) \to 0$$

as $n \to \infty$, which gives a contradiction. The remaining assertions in (iii) are clear.

Now define $S = \mathbb{T} \setminus \cup_{n,m,p}(E^{(n)} \cup F^{(m)} \cup S^{(p)})$. To prove the second part of (iv) let $W \subset S$ satisfying the hypothesis of (iv). According to the definition of S one has

$$W = \bigcap_{n=1}^{\infty} \bigcup_{\overline{I(\sigma(n))} \cap W \neq \emptyset} \left(\overline{I(\sigma(n))} \cap R\right) \setminus \bigcup_p S^{(p)}. \tag{4.15}$$

Using (4.12), (4.15) and the fact that each $S^{(p)}$ is a null curve, one obtains

$$\int_W \frac{d\gamma(t)}{\gamma(t) - w} = 0 \quad \text{for all} \quad w \notin \gamma^*. \tag{4.16}$$

Now (4.16) exactly means that $C(\mu)(w) = 0$ for all $w \notin \gamma^*$ provided that $\mu = dz_{\gamma_{|W}}$. This implies, by [7, Chap. II, Th. 1.2], that $\| dz_{\gamma_{|W}} \| (\gamma(S)) = 0$, which gives (iv).

To prove the first part of (v) let $q = \gamma(t)$, $t \in S$. Then $t \in R$ and $t \notin S^{(p)}$ for each p. This implies the existence of a sequence $(\tau(k_n))$ such that $t \in I(\tau(k_n))$ and $I(\tau(k_{n-1}) \supset I(\tau(k_n))$ and $\tilde{\gamma}_{\overline{I(\tau(k_n))}}$ converges uniformly to a constant curve, as $k_n \to \infty$. In each $\overline{I(\tau(k_n))}$ there are two points $a(\tau(k_n)), b(\tau(k_n))$ that are extreme points of $\overline{I(\tau(k_n))}$ and that belong to $E(\tau(k_{n-1}))$. Since $\mathrm{diam}\gamma(E(\tau(k_{n-1}))) \to 0$ and $\gamma(a(\tau(k_n))) \to q$ as $n \to \infty$ one has that $\gamma(E(\tau(k_n))$ tends to $\{q\}$ in the Hausdorff topology.

To prove the second part of (v) let $(I(\tau(k_n)))$ be the sequence that defines $S^{(p)}$, for a fixed p. Select a point $t_n \in E(\tau(k_n))$ for each $\tau(k_n)$ and take a partial sequence of (t_n) that converges to t_0. Then $t_0 \in S^{(p)}$ and $\gamma(t_0)$ has the desired property. □

We note that the conclusion (iv) of Theorem 4 does not hold if one replaces the measures $\shortparallel dz_{\gamma_{|W}} \shortparallel$ with the measures $|dz_{\gamma_{|W}}|$ and that the hypothesis on W is essential. These assertions can be justified by means of the forthcoming example.

Example 3. Let K be a linear Cantor set on $[0, 1]$ with positive length. To fix the ideas the reader can take the Cantor set constructed as the usual 1/3-Cantor set but now using 1/4 instead of 1/3. Let $f \geq 0$ to be a C^∞-function that vanishes exactly on K. To begin with we define the curve $\gamma : [0, 2] \longrightarrow \mathbb{C}$ as follows:

$$\gamma(t) = \begin{cases} t + if(t), & \text{if} \quad t \in [0, 1] \\ (2 - t) - if(2 - t), & \text{if} \quad t \in [1, 2]. \end{cases}$$

Now we enlarge the curve γ to obtain a new curve γ_1 in the following manner. Put a root subcurve (with length $1/2^n$) starting and ending at $\gamma(t_n)$ where t_n is the center of each interval of $[0, 1] \setminus K$ and do the same but now starting and ending at points $\alpha_n \in K$, $n \geq 1$ that are not end points of some interval of $[0, 1] \setminus K$. This define a closed curve γ_1 which has the properties: $|dz|(\gamma_1(S)) > 0$; there is $W \subset S$ for which $\shortparallel dz_{\gamma_{|W}} \shortparallel$ is not zero and $n_0 = m_0 = p_0 = \infty$.

Remark 3. *It seems clear that the set $\gamma(S)$ that appears in Theorem 4 is very small. This is the case in all examples that we have considered. It will be interesting to specify in which sense is $\gamma(S)$ small.*

5 General formulae

By Theorem 2 one has $\mathrm{Ind}(\gamma, \cdot) \in L^2$ but in general $\mathrm{Ind}(\gamma, \cdot) \notin L^{2+\varepsilon}(\mathbb{C}, m)$ for $\varepsilon > 0$, see [5, Ex. 2]). Then we know that $C^\gamma \in VMO$. Therefore some specific arguments are needed to prove the finiteness and the continuity of C^γ at each point of $\mathbb{C}$. We provide them in Proposition 5 and Theorem 5 respectively.

Proposition 5. *Let γ be a rectifiable closed curve. Then*

$$\int_{\mathbb{C}} \frac{|Ind(\gamma, \tau)|}{|\tau - w|} \, dm(\tau) \leq \Lambda(\gamma) \quad \text{for each} \quad w \in \mathbb{C}. \tag{5.1}$$

Proof. Arguing as in the proof of Theorem 2 we see that it is enough to prove (5.1) for a polygonal curve. If P is such a curve, according to [5, p. 118-119] there exist $\gamma_1, \gamma_2 \ldots, \gamma_n$ Jordan loops of P, such that

$$\mathrm{Ind}(P, \cdot) = \sum_{j=1}^{n} \mathrm{Ind}(\gamma_j, \cdot), \quad \sum_{j=1}^{n} \Lambda(\gamma_j) \leq \Lambda(P). \tag{5.2}$$

Then, by (5.2), (2.6) and (1.5), one has

$$\int_{\mathbb{C}} \frac{|\mathrm{Ind}(P,\tau)|}{|\tau - w|}\, dm(\tau) \le \sum_{j=1}^{n} \int_{D(\gamma_j)} \frac{1}{|\tau - w|}\, dm(\tau) \le \Lambda(P).$$

□

Before stating the continuity of C^γ, we see how this transform looks by computing it for a particular curve which has unbounded index.

Example 1. Let $C_n(t) = r_n e^{it}$, $t \in [0, 2\pi]$ where (r_n), $n \ge 1$, is a decreasing sequence of positive numbers with $\sum_n r_n < +\infty$. First describe C_1, after that the line segment from r_1 to r_2 and then C_2 and so on. Finally join the origin with the point r_1. Let us denote by ρ this rectifiable curve. Then, taking into account (2.10) and putting $r_0 = +\infty$, the continuous function C^ρ is given by

$$C^\rho(w) = \begin{cases} -\pi n \overline{w} - \frac{\pi}{w} \sum_{k=n+1}^{\infty} r_k^2 & \text{if} \quad r_{n+1} \le |w| < r_n,\ n \ge 0, \\ 0 & \text{if} \quad w = 0. \end{cases}$$

Theorem 5. *Let γ be a rectifiable closed curve, let $\emptyset \ne A \subset \mathbb{C}$ and let $X = \widehat{\gamma^*}$. Assume that $F : X \times A \to \mathbb{C}$ is a continuous function. Then the functions defined on A by*

(a) $w \to C_F(\gamma, w) = \int_{\mathbb{C}} \frac{F(\tau,w)}{\tau - w} Ind(\gamma,\tau)\, dm(\tau)$,

(b) $w \to N_F(\gamma, w) = \int_{\mathbb{C}} \frac{F(\tau,w)}{|\tau - w|} Ind(\gamma,\tau)\, dm(\tau)$

are continuous.

Proof. Proposition 5 and the boundedness of F tell us that both functions are finite for each $w \in A$. We only prove the continuity of $N_F(\gamma, \cdot)$ since the continuity of $C_F(\gamma, \cdot)$ can be proved in a similar way. Let now (γ_j) be the Jordan loops given in (i) of Theorem 4 for $1 \le j \le n_0$ and put $\gamma_j = 0$ for $j > n_0$. For a fixed compact subset $L \subset A$ we claim that

$$N_F(\gamma, w) = \sum_{j=1}^{\infty} N_F(\gamma_j, w), \text{ absolutely and uniformly on } w \in L. \quad (5.3)$$

The uniform convergence of the series in (5.3) is a consequence of Proposition 5 and Theorem 4 that give

$$\sum_{j=1}^{\infty} |N_F(\gamma_j, w)| \le \sup_{(\tau,w)\in X\times L} |F(\tau, w)| \sum_{j=1}^{\infty} \int_X \frac{|\mathrm{Ind}(\gamma_j,\tau)|}{|\tau - w|}\, dm(\tau) \le c\Lambda(\gamma).$$

The equality, for each $w \in L$, in (5.3) comes now from Theorem 4 which says that $dz_\gamma = \sum_j dz_{\gamma_j}$, and the dominated convergence theorem. According to (5.3) the proof will be finished if we show that $N_F(\gamma, \cdot)$ is continuous when γ has bounded index. For this purpose let us fix $w_0 \in A$

and let $w_n \in A \setminus \{w_0\}$, $|w_n - w_0| \to 0$. Then one gets

$$
\begin{aligned}
|N_F(\gamma, w_n) - N_F(\gamma, w_0)| &\le \Big| \int_X \frac{F(\tau, w_n) - F(\tau, w_0)}{|\tau - w_n|} \mathrm{Ind}(\gamma, \tau)\, dm(\tau) \Big| \\
&+ \Big| \int_X F(\tau, w_0) \frac{|\tau - w_0| - |\tau - w_n|}{|\tau - w_n||\tau - w_0|} \mathrm{Ind}(\gamma, \tau)\, dm(\tau) \Big| \\
&\le c_1 \sup_{\tau \in X} |F(\tau, w_n) - F(\tau, w_0)| + c_2 |w_n - w_0| (1 + |\log |w_n - w_0||) \to 0,
\end{aligned}
$$

as $n \to \infty$, where we have used (2.6), the boundedness and the uniform continuity of F on compact sets and Lemma 2. □

Now the following generalization of Proposition 1 is obtained.

Proposition 6. *Let $U \subset \mathbb{C}$ be an open set and let γ be a rectifiable closed curve that is homologous to zero with respect to U. Assume that F is a C^1 complex-valued function defined on $U \times U$. Then the equality*

$$
\int_\gamma F(z, w) \frac{\overline{z} - \overline{w}}{z - w}\, dz = 2i \int_{\mathbb{C}} \Big(\frac{\partial F}{\partial \overline{z}}(\tau, w) \frac{\overline{\tau} - \overline{w}}{\tau - w} + \frac{F(\tau, w)}{\tau - w} \Big) Ind(\gamma, \tau)\, dm(\tau), \tag{5.4}
$$

holds for each $w \in U$.

Proof. In the case that $w \in U \setminus \gamma^*$ equality (5.4) follows by a similar argument to one used in the proof of (3.9) in [5]. Note that there does not appear any term without the integration sign, because of the boundedness of the function $F \cdot K_1$.

Now we observe that each term in (5.4) is a continuous function on U. For the left-hand side term and for the second term in the right-hand side these are, respectively, the assertions of Lemma 4 and Theorem 5. The continuity of the remaining term follows because its integrand is dominated by $|\mathrm{Ind}(\gamma, \cdot)|$. □

We can now state the main formula of this paper.

Theorem 6. *Let U be an open set in $\mathbb{C}$ and let γ_1, γ_2 be two rectifiable closed curves that are homologous to zero with respect to U. Assume that $F : U \times U \longrightarrow \mathbb{C}$ is of class C^2. Then*

$$
\begin{aligned}
\pi \int_{\mathbb{C}} F(\tau, \tau) Ind(\gamma_1, \tau) Ind(\gamma_2, \tau)\, dm(\tau) &= \frac{1}{4} \int_{\gamma_1 \times \gamma_2} F(z, w) K_1(z, w)\, dz\, dw \\
&+ \int_{\mathbb{C}^2} \frac{\partial^2 F}{\partial \overline{w} \partial \overline{z}}(\tau, \sigma) K_1(\tau, \sigma) Ind(\gamma_1, \tau) Ind(\gamma_2, \sigma)\, dm(\tau, \sigma) \\
&+ \int_{\mathbb{C}^2} \Big(\frac{\partial F}{\partial \overline{w}}(\tau, \sigma) - \frac{\partial F}{\partial \overline{z}}(\tau, \sigma) \Big) \frac{1}{\tau - \sigma} Ind(\gamma_1, \tau) Ind(\gamma_2, \sigma)\, dm(\tau, \sigma).
\end{aligned} \tag{5.5}
$$

Proof. We present a sketch of the proof following three steps. To begin with write (5.4) for the curve γ_1 and for each $w \in U$. Secondly integrate the obtained formula with respect to $-dw/4\pi$ on γ_2 and commute in each term the order of integration using, when it is necessary, Proposition 5. Doing this one gets

$$
\begin{aligned}
-\frac{1}{4\pi}\int_{\gamma_1\times\gamma_2} & F(z,w)K_1(z,w)\,dz\,dw \\
&= \int_{\mathbb{C}}\Big[\frac{1}{2\pi i}\int_{\gamma_2}\frac{\partial F}{\partial \overline{z}}(\tau,w)K_1(\tau,w)\,dw\Big]\mathrm{Ind}(\gamma_1,\tau)\,dm(\tau) \\
&\qquad + \int_{\mathbb{C}}\Big[\frac{1}{2\pi i}\int_{\gamma_2}\frac{F(\tau,w)}{\tau-w}\,dw\Big]\mathrm{Ind}(\gamma_1,\tau)\,dm(\tau). \qquad (5.6)
\end{aligned}
$$

Finally one needs to transform the two terms in the right-hand side of (5.6) in the following way. In the first one use (5.4) applied to the inner integral with respect to the curve γ_2 and to the function $\overline{\partial}F$. In the second one use the Cauchy-Green's formula with multiciplities [5, p. 110] applied to the curve γ_2 and to the function $F(\tau,\cdot)$ for each τ. With these replacements the formula (5.6) becomes (5.5). □

Now the formula (1.3) is a consequence of Theorem 6.

6 The logarithmic potential of dz

In this section we analyze what difficulties appear if one tries to replace the kernel K_1 with the kernel K_2 in some of the results of sections 3 and 5. The conclusion is that the corresponding analogous to such results for the kernel K_2 are not true. This follows as a consequence of the existence of rectifiable closed curves ρ (even Jordan curves) for which the logarithmic energy of $\| dz_\rho \|$ is infinity.

Let us denote by $P_{\log}[\mu](w) = \int K_2(z,w)\,d\mu(z)$, the logarithmic potential of a complex measure μ, defined for all w for which $P_{\log}[|\mu|](w)$ is finite (see [8, p. 79] and [20, p. 54]). Note that the energy of μ is $E(\mu) = \int P_{\log}[\mu](w)\,d\mu(w)$. The analogous to the first part of Proposition 1 is given by the next result.

Proposition 7. *Let γ be a rectifiable closed curve. Then*

$$
P_{\log}[dz_\gamma](w) = -i\int_{\mathbb{C}}\frac{Ind(\gamma,\tau)}{\overline{\tau}-\overline{w}}\,dm(\tau), \quad w \notin \gamma^*. \qquad (6.1)
$$

Proof. Follow the pattern of the proof of Proposition 1 but now integrating both sides of (2.4) with respect to dz_γ and later evaluating the integrals by using Cauchy's and Fubini's theorems. □

Now we can ask if (6.1) also holds for each $w \in \gamma^*$ analogously to (5.4) with $F = 1$. But this result fails since the right-hand side of (6.1) is a continuous function on $\mathbb{C}$ for any γ by Theorem 5 while Lemma 7 shows that there are curves ρ for which $P_{\log}[\| dz_\rho \|](0) = +\infty$, with $0 \in \gamma^*$. So a corresponding result to Proposition 6 for the kernel K_2 is not true.

Lemma 7. *Let ρ be as in the Example 1 but taking now, for each n, $r_n = (n+2)^{-1}\log^{-3/2}(n+2)$. Then $P_{\log}[\| dz_\rho \|](0) = E(\| dz_\rho \|) = +\infty$.*

Proof. By the definition one has

$$\frac{1}{2\pi}P_{\log}[\shortparallel dz \shortparallel_\rho](0) = \sum_{n=1}^{\infty} r_n \log\frac{1}{r_n} \geq \sum_{n=3}^{\infty}\frac{1}{n\log^{1/2} n} = +\infty.$$

Using the formula [17, p. 102]

$$\int_{C_r} \log|z-w|\,|dz| = 2\pi r\log(\max\{r, |w|\}),\ r>0,\ w\in\mathbb{C},$$

we obtain

$$\frac{1}{2\pi}P_{\log}[\shortparallel dz \shortparallel_\rho](w) = \sum_{n<m} r_n \log\frac{1}{r_n} + (\log\frac{1}{r_m})\sum_{n\geq m} r_n,\ |w| = r_m. \quad (6.2)$$

From (6.2) we see that $E(\shortparallel dz \shortparallel_\rho) \geq 4\pi^2 \sum_{m=1}^{\infty} r_m(\log\frac{1}{r_m})\sum_{n\geq m} r_n = +\infty$, since we have, for each $m\geq 1$,

$$r_m(\log\frac{1}{r_m})\sum_{n\geq m} r_m \geq r_m(\log\frac{1}{r_m})\int_{m+2}^{\infty}\frac{1}{x\log^{\frac{3}{2}} x}\,dx \geq \frac{1}{(m+2)\log(m+3)}.$$

□

The curve ρ in Lemma 7 shows that (6.1) does not hold at the point $w=0$. Also, for each curve ρ with infinite logarithmic energy, the right-hand side of equality (a) in Theorem 1 cannot be interpreted as a Lebesgue integral and it fails for those ρ. Perhaps one can think that these kind of curves are not good enough to satisfy (6.1) for all w, neither Theorem 1(a), because of the fact that their index functions are not bounded. But the boundedness of index function plays no role as the following result shows. Consequently the analogous to (3.2) for the kernel K_2 also fails.

Proposition 8. *There exists a rectifiable Jordan curve σ such that the logarithmic energy of $|dz_\sigma|$ is infinity.*

Proof. Denote now by γ the curve in Lemma 7. For each decreasing sequence $\delta = (\delta_n)$ with $\lim\delta_n = 0$ we define the set γ^δ as

$$\gamma^\delta = \bigcup_{n=1}^{\infty} r_{2n-1}e^{i[0,\delta_n]} \cup r_{2n}e^{i[0,\delta_n]} \cup r_{2n}e^{i[\pi,\pi+\delta_n]} \cup r_{2n+1}e^{i[\pi,\pi+\delta_n]},$$

where $e^{iA} = \cup_{t\in A}\{e^{it}\}$ if $A\subset\mathbb{R}$.

Denoting by α_n the right-hand side of (6.2) and using it one gets

$$\int_{\gamma^\delta} P_{\log}[\shortparallel dz_\gamma \shortparallel](w)\,|dw| = 2\pi\sum_{n=1}^{\infty}(\alpha_{2n-1}r_{2n-1} + \alpha_{2n}r_{2n})\delta_n < +\infty \quad (6.3)$$

if we take, for example, $(\delta_n) = (1/2n)$, which is enough for our purpose.

The Jordan curve that we have in mind will contain essentially $\gamma^*\setminus\gamma^\delta$. A direct way to define it is the following. Let

$$S_n = \{z : \log r_{2n} \leq \mathrm{Re}\ z \leq \log r_{2n-1},\ \delta_n \leq \mathrm{Im}\ z \leq 2\pi\},$$
$$P_n = \{z : \log r_{2n+1} \leq \mathrm{Re}\ z \leq \log r_{2n},\ \pi \leq \mathrm{Im}\ z \leq \pi+\delta_n\}.$$

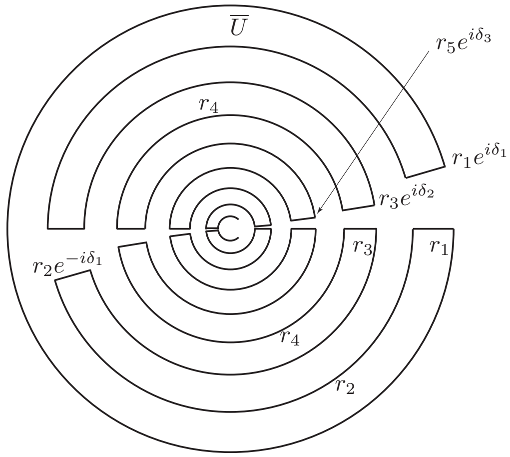

Figure 1: The Jordan domain $\overline{U}$

Put $D = \cup_{n=1}^{\infty}(S_n \cup P_n)$ and $\overline{U} = \exp(D)$. Now U is a Jordan domain and we take as σ a parameterization of ∂U, (see Figure 1).

The inclusion $\sigma^* \supset \gamma^* \setminus ([0, r_1] \cup \gamma^\delta)$, the fact that $\shortparallel dz_\gamma \shortparallel ([0, r_1]) = 0$ and $\operatorname{diam}(\sigma^*) \leq 1$ imply

$$E(|dz_\sigma|) \geq \int_{\gamma^* \setminus \gamma^\delta} \int_{\gamma^* \setminus \gamma^\delta} \log \frac{1}{|z - w|} \shortparallel dz_\gamma \shortparallel \shortparallel dz_\gamma \shortparallel$$
$$\geq E(\shortparallel dz_\gamma \shortparallel) - 2 \int_{\gamma^\delta} P_{\log}[\shortparallel dz_\gamma \shortparallel](w)\, |dw| = +\infty$$

by Lemma 7 and (6.3). □

Taking into account the results in this section, it may be of interest to provide a wide class of curves for which Theorem 1(a) can be applied.

Definition 3. *We say that a rectifiable curve γ has linear growth (with respect to $\shortparallel dz \shortparallel$) if there exists $M > 0$ such that*

$$\shortparallel dz \shortparallel (\overline{D(w, r)}) \leq Mr, \quad w \in \mathbb{C},\ r > 0. \tag{6.4}$$

We recall that γ is a C^1-regular curve if $\gamma \in C^1([a, b])$ and $\gamma'(t) \neq 0$ for every $t \in [a, b]$. For these two kind of curves the following result holds.

Proposition 9. *If γ is a rectifiable curve with linear growth, then:*

- *(a) The functions $P_{\log}[\shortparallel dz_\gamma \shortparallel]$ and $P_{\log}[dz_\gamma]$ are well defined and continuous on $\mathbb{C}$.*
- *(b) One has $E[\shortparallel dz_\gamma \shortparallel] < +\infty$ and so Theorem 1(a) applies.*

If γ be a C^1-regular closed curve, then

- *(c) γ has linear growth.*
- *(d) Equality (6.1) holds also for each $w \in \gamma^*$.*
- *(e) The family of subcurves defined in item (ii) in Theorem 4 does not contain any root subcurve.*

We only give some hints about the proof. We compute the first potential, evaluated at each point w with $d(w,\gamma^*) \leq 2$, by using the distribution function $m(w,r) = \shortparallel dz \shortparallel (\overline{D(w,r)})$. Using (6.4) we obtain

$$\begin{aligned} P_{\log}[\shortparallel dz \shortparallel](w) &= \int_0^R \log \frac{1}{r}\, dm(w,r) \\ &= \shortparallel dz \shortparallel (\gamma^*) \log \frac{1}{R} + \int_0^R \frac{m(w,r)}{r}\, dr \leq M', \end{aligned}$$

where $R = 2 + \operatorname{diam}(\gamma^*)$ and M' is a constant depending on γ. This gives that the potential functions in item (a) are well defined and item (b) holds. To prove the continuity in item (a) it is enough to use the previous computations and the fact that given a sequence (w_n) with $w_n \to w_0$ then $m(w_n, s) \to m(w_0, s)$ as $n \to \infty$ for dr-almost all point s. To prove item (c) the key point is to see that γ is locally bilipschitz. Since $\operatorname{Ind}(\gamma, \cdot)$ is a bounded function, then item (d) follows from (6.1) and from the continuity given by (a) and (5). The assertion in item (e) is due to the fact that in some non-cut point [22, p. 54] of the image of each root subcurve there is not a tangent to γ.

Finally we mention that there are C^1-regular curves γ having, with the notations of Theorem 4, $n_0 = m_0 = p_0 = +\infty$ and $|dz_\gamma|(\gamma(S)) > 0$.

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