On the asymptotics of visible elements and homogeneous equations in surface groups

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Abstract

Let F be a group whose abelianization is $\mathbb{Z}^k$, $k \geq 2$. An element of F is visible if its image in the abelianization is visible, that is, the greatest common divisor of its coordinates is 1.

We compute densities of visible elements in some strongly Markov groups, which include free and surface groups. We use this result to show that the probability of a homogeneous equation in a surface group to have solutions is neither 0 nor 1, as the length of the right- and left-hand side of the equation go to infinity.

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1 Introduction

Let F be a group whose abelianization is $\mathbb{Z}^k$, with $k \geq 2$. An element of F is called *visible* if its image in the abelianization is visible, that is, the greatest common divisor of its coordinates is 1. Let Σ be a compact connected orientable surface of genus r, $r \geq 2$. If Σ has no boundary, then a presentation for the fundamental group of Σ, which we call the *surface group of genus* r, is $\langle a_1, b_1, \ldots, a_r, b_r \mid [a_1, b_1] \cdots [a_r, b_r] \rangle$. If Σ has boundary, then the fundamental group of Σ is simply the free group of finite rank.

In this paper we compute the density of visible elements in a large class of groups containing the surface groups of compact connected orientable surfaces, with or without boundary. We thus extend the results of [5], where the density of visible elements in the free group of rank two was computed. Since the limits we obtain are different from 0 and from 1, this shows that visible elements form a set of *intermediate* density in a large class of groups. Intermediate density of sets in groups has been displayed for the first time in [5], and this tends to be a relatively rare behaviour for many combinatorial and algebraic properties encountered in group theory. Most of the properties studied in the literature ([6]) turned out to be *negligible* or *generic*, that is, with density equal to 0 or 1, respectively.

A consequence of our results is the fact that the solvability of homogeneous equations in the class of groups that we study is also a non-negligible and non-generic property. Let G be a finitely generated group, A a fixed generating set, and $X = \{X_1, \ldots, X_n\}$, $n \geq 1$, a set of variables. An *equation* in variables $X_1, \dots, X_n$ with coefficients $g_1, \dots, g_{m+1}$ in G is a formal expression given by

$$g_1 X_{i_1}^{\varepsilon_1} g_2 X_{i_2}^{\varepsilon_2} \dots X_{i_m}^{\varepsilon_m} g_{m+1} = 1,$$

where $m \geq 1$, $\varepsilon_j \in \{1, -1\}$ for all $1 \leq j \leq m$, and $i_j \in \{1, \dots, n\}$. An equation is *homogeneous* if the variables are on the left-hand side of the equation and the constants are on the right-hand side of the equation:

$$X_{i_1}^{\varepsilon_1} X_{i_2}^{\varepsilon_2} \dots X_{i_m}^{\varepsilon_m} = w, \tag{1.1}$$

where $w \in G$. We say that the equation (1.1) is *a homogeneous equation of type* $(m, |w|_A)$ or an $(m, |w|_A)$*-homogeneous equation.*

We will be interested in the asymptotic behavior of $(m, |w|_A)$-homogeneous equations when G is a surface or a free group, and m and $|w|_A$ go to infinity. Our study of the asymptotics of homogeneous equations was motivated by two related questions: firstly, how often does a homogeneous equation in a free or surface group have solutions, and secondly, how likely is it, for two random words u and v in the group to have that v is an endomorphic image of u? The second question was partly inspired by the work of Kapovich, Schupp and Shpilrain ([6]). They show that the probability of two elements u and v in F_k to be in the same automorphic orbit is 0 as the lengths of u and v go to infinity. The following paragraph clarifies the relation between the two questions.

Suppose that $z(X_1, \dots, X_n)$ is the word in $X_1, \dots, X_n$ representing the left-hand side of (1.1), i.e. $z(X_1, \dots, X_n) = X_{i_1}^{\varepsilon_1} X_{i_2}^{\varepsilon_2} \dots X_{i_m}^{\varepsilon_m}$. Let F_n be the free group of rank n on generators $x_1, \dots, x_n$. Notice that the equation (1.1) has solutions if and only if there exists an homomorphism $\phi \colon F_n \to G$ such that $\phi(z(x_1, \dots, x_n)) = w$, where z is written in the generators $x_1, \dots, x_n$. The following ratios quantify the pairs of elements of the form (z, w).

Definitions 1.1. Let F, G be groups finitely generated by finite sets A and B, respectively.

1. The (s,t)*-mapping ratio* $e_\rho(F, G, s, t)$ is the ratio of the pairs of elements $(f, g) \in F \times G$ such that $|f|_A \leq s$, $|g|_B \leq t$ and with the property that g is the homomorphic image of f, among all pairs $(f, g) \in F \times G$ with $|f|_A \leq s$, $|g|_B \leq t$, that is,

$$e_\rho(F, G, s, t) = \frac{\sharp\{(f,g) \in F \times G : |f|_A \leq s, |g|_B \leq s, \phi(f) = g \text{ for some } \phi \in \mathrm{Hom}(F, G)\}}{\sharp\{(f,g) \in F \times G : |f|_A \leq s, |g|_B \leq s\}}.$$

2. The *spherical* (s,t)*-mapping ratio* $e_\gamma(F, G, s, t)$ is the ratio of the pairs of elements $(f, g) \in F \times G$ such that $|f|_A = s$, $|g|_B = t$ and with the property that g is the homomorphic image of f among all pairs $(f, g) \in F \times G$ with $|f|_A = s$, $|g|_B = t$, that is,

$$e_\gamma(F, G, s, t) = \frac{\sharp\{(f,g) \in F \times G : |f|_A = s, |g|_B = s, \phi(f) = g \text{ for some } \phi \in \mathrm{Hom}(F, G)\}}{\sharp\{(f,g) \in F \times G : |f|_A = s, |g|_B = s\}}.$$

In Section 3 we will study the asymptotic behavior of the (s,t)-mapping ratio $e_\rho(F, G, s, t)$ for F and G free-abelian groups. For some homogeneous equations in groups with (finite rank) free-abelian abelianization, one can decide whether they have solutions or not by looking at the equations in the abelianization. The computation of the asymptotic behavior of this ratio is based on the densities of visible elements in a free-abelian group.

In Section 4 we study the densities of visible elements in free and surface groups (Corollary 4.10). We obtain one of our main results which relates the densities of visible points in surface and free groups with the densities in the abelianization. In fact, our results hold for a wider class of strongly Markov groups, as discussed in Appendix A.

In Section 5 we study the asymptotic behavior of the spherical (s,t)-mapping ratio $e_\gamma(F,G,s,t)$ when F,G are free or surface groups. As a corollary, we obtain that the probability of an (s,t)-homogeneous equation in a surface group to be solvable is neither 0 nor 1, as s,t go to infinity.

We were informed that the asymptotic behavior of homogeneous equations is also being studied in a work in progress by B. Gilman, A. Miasnikov and V. Romankov.

2 Notation

We will use the following notation for sequences: a sequence is a mapping $v\colon I \to X$ from an interval I of $\mathbb{Z}$ to a set X. We will write v_I for the sequence $v\colon I \to X$ and for $i \in I$, $v_i \coloneqq v(i)$.

Let i, $j \in \mathbb{Z}$. We write $[i{\uparrow}j] \coloneqq \{k \in \mathbb{Z} : k \geq i \text{ and } k \leq j\}$, $]{-\infty}{\uparrow}j] \coloneqq \{k \in \mathbb{Z} : k \leq j\}$, and $[i{\uparrow}\infty[\ \coloneqq \{k \in \mathbb{Z} : k \geq i\}$ to denote intervals in $\mathbb{Z}$ indexing a sequence.

Definitions 2.1. Let F be a finitely generable group, and let A be a finite generating set of F. If $w \in F$, then $|w|_A$ denotes the length of the shortest word in $A^{\pm 1}$ representing w.

1. Let $S \subset F$ and $n \geq 0$. Then
$$\rho(n,S) = \sharp\{x \in S : |x|_A \leq n\},$$
and
$$\gamma(n,S) = \sharp\{x \in S : |x|_A = n\}$$
denote the cardinality of the ball and sphere of radius n in F, respectively.

2. Let $S \subset F$. The *asymptotic density* of S in F is
$$\bar{\rho}_A(S) = \limsup_{n\to\infty} \frac{\rho(n,S)}{\rho(n,F)}.$$
If the limit exists, then we denote it by $\rho_A(S)$ and we call it the *strict asymptotic density.*

3. Let $S \subset F$. The *spherical density* of S in F is
$$\bar{\gamma}_A(S) = \limsup_{n\to\infty} \frac{\gamma(n,S)}{\gamma(n,F)}.$$
If the limit exists, then we denote it by $\gamma_A(S)$ and we call it the *strict spherical density.*

4. Let $S \subset F$. The *annular density* of S in F is

$$\bar{\sigma}_A(S) = \limsup_{n\to\infty} \frac{1}{2}\left(\frac{\sharp\{x \in S : |x|_A = n-1\}}{\sharp\{x \in F : |x|_A = n-1\}} + \frac{\sharp\{x \in S : |x|_A = n\}}{\sharp\{x \in F : |x|_A = n\}}\right)$$

If the limit exists, then we denote it by $\sigma_A(S)$ and we call it the *strict annular density.*

Let $r \geq 1$ be an integer. A length function $l_p\colon \mathbb{Z}^r \to \mathbb{R}$ will denote the restriction to $\mathbb{Z}^r$ of the $||.||_p$-norm from $\mathbb{R}^r$ for some $1 \leq p \leq \infty$.

We can define the asymptotic and annular densities in a similar way in $\mathbb{Z}^r$ by changing $|x|_A$ with $l_p(x)$ and denoting the limits by $\bar{\rho}_p(S)$ and $\bar{\sigma}_p(S)$.

Definitions 2.2. For a nonzero element $z \in \mathbb{Z}^r$ we denote by $\gcd(z)$ the greatest common divisor of its coordinates. If $z = (0, \dots, 0) \in \mathbb{Z}^r$ we set $\gcd(z) = \infty$. Note that gcd is invariant under the action of $\mathrm{Aut}(\mathbb{Z}^r) = SL(r, \mathbb{Z})$. Hence, for all $z \in \mathbb{Z}^r$, $\gcd(z)$ does not depend on the basis of $\mathbb{Z}^r$.

An element of $z \in \mathbb{Z}^r$ is called *visible* if $\gcd(z) = 1$. If $\gcd(z) = t$, then we call the element *t-visible.*

We denote by F_{ab} the abelianization of the group F, that is, $F_{\mathrm{ab}} = F/[F, F]$. Suppose that F_{ab} is a free-abelian group of finite rank and let $\mathrm{ab}\colon F \to F_{\mathrm{ab}}$ be the abelianization map. We say that an element $f \in F$ is *visible* (resp. t-visible) if $\mathrm{ab}(f)$ is visible (resp. t-visible) in F_{ab}.

3 Densities of visible elements in $\mathbb{Z}^r$

Let $r \geq 2$ be an integer and let U_t denote the set of all t-visible elements in $\mathbb{Z}^r$. For a complex number k, recall that the Riemann zeta function is given by

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}, \qquad \mathfrak{Re}(k) > 1.$$

A classical result in number theory provides the value for the strict asymptotic density of t-visible elements in $\mathbb{Z}^r$.

Proposition 3.1. *For any integer $t \geq 1$ we have*

$$\rho_\infty(U_t) = \frac{1}{t^r \zeta(r)}.$$

□

The following lemma shows that homomorphisms between groups with free-abelian abelianization (of finite rank) send t-visible elements to tm-visible elements, where t, m are positive integers. The second part of the lemma shows that a visible element in a group can be mapped to any element in the image via a homomorphism.

Lemma 3.2. *Let F, G be groups whose abelianization is free-abelian of finite rank. Let $f \in F$.*

(i). *Let* $\phi : F \to G$ *be a group homomorphism. Then* $\gcd(\mathrm{ab}(\phi(f)))$ *is a multiple of* $\gcd(\mathrm{ab}(f))$*. In particular, if* $\gcd(\mathrm{ab}(f)) = \infty$*, then* $\gcd(\mathrm{ab}(\phi(f))) = \infty$.

(ii). *If, moreover* $\gcd(\mathrm{ab}(f)) = 1$*, then for any element* g *in* G *there exists an homomorphism* $\phi\colon F \to G$ *such that* $\phi(f) = g$.

Proof. Let n be the rank of F_{ab} and let $\{e_1, \dots, e_n\}$ be a basis of F_{ab}. For $f \in F_{\mathrm{ab}}$, we denote by $(f)_i$ the ith coordinate of f with respect the basis. That is, $f = (f)_1 e_1 + \cdots + (f)_n e_n$.

(i). Let $g = \phi(f)$. Then $(\mathrm{ab}(g))_j = \sum_{i=1}^{n} (\mathrm{ab}(f))_i (\phi(e_i))_j$.

Thus each $(\mathrm{ab}(g))_j$ is a multiple of $\gcd(\mathrm{ab}(f))$, since each $(\mathrm{ab}(f))_i$ is a multiple of $\gcd(\mathrm{ab}(f))$.

(ii). Since $\gcd(\mathrm{ab}(f)) = 1$, then $\gcd((\mathrm{ab}(f))_1, \dots, (\mathrm{ab}(f))_n) = 1$ and therefore there exist integers $p_1, \dots, p_n$ such that $\sum_{i=1}^{n} (\mathrm{ab}(f))_i p_i = 1$. Consider the homomorphism $\psi_1\colon F_{\mathrm{ab}} \to \langle x \mid \quad \rangle$ which sends e_i to x^{p_i} for all $1 \le i \le n$. It follows that $\psi_1(\mathrm{ab}(f)) = x$. Let $\psi_2\colon \langle x \mid \quad \rangle \to G$ be any homomorphism sending x to g. This shows that the composition of ab, ψ_1 and ψ_2 produces a homomorphism $\phi\colon F \to G$ such that $\phi(f) = g$.

□

Corollary 3.3. *Let* $\mathbb{Z}^n$ *and* $\mathbb{Z}^k$ *be the free abelian groups of ranks* n *and* k*, respectively. Then the following inequalities hold:*

$$\frac{1}{\zeta(n)} \le \liminf_{s\to\infty, t\to\infty} e_\rho(\mathbb{Z}^n, \mathbb{Z}^k, s, t), \tag{3.1}$$

$$\limsup_{s\to\infty, t\to\infty} e_\rho(\mathbb{Z}^n, \mathbb{Z}^k, s, t) \le 1 - \frac{1}{\zeta(k)} \left(1 - \frac{1}{\zeta(n)}\right). \tag{3.2}$$

Proof. Let $e_{\mathrm{ab}}(s,t) := e_\rho(\mathbb{Z}^n, \mathbb{Z}^k, s, t)$.

By Lemma 3.2(ii)

$$e_{\mathrm{ab}}(s,t) \ge \frac{\{(u,v) \in \mathbb{Z}^n \times \mathbb{Z}^k : |u| \le s, |v| \le t, \gcd(u) = 1\}}{\rho(s, \mathbb{Z}^n)\rho(t, \mathbb{Z}^k)} = \frac{\{u \in \mathbb{Z}^n : |u| \le s, \gcd(u) = 1\}}{\rho(s, \mathbb{Z}^n)}$$

Taking limits, we obtain (3.1) by Proposition 3.1.

By Lemma 3.2(i)

$$\begin{aligned} e_{\mathrm{ab}}(s,t) &\le 1 - \frac{\{(u,v) \in \mathbb{Z}^n \times \mathbb{Z}^k : |u| \le s, |v| \le t, \gcd(u) \ne 1, \gcd(v) = 1\}}{\rho(s, \mathbb{Z}^n)\rho(t, \mathbb{Z}^k)} \\ &= 1 - \left(1 - \frac{\{u \in \mathbb{Z}^n : |u| \le s, \gcd(u) = 1\}}{\rho(s, \mathbb{Z}^n)}\right) \frac{\{v \in \mathbb{Z}^k : |v| \le t, \gcd(v) = 1\}}{\rho(t, \mathbb{Z}^k)}. \end{aligned}$$

Taking limits, we obtain (3.2) by Proposition 3.1. □

One of the key ingredients needed to extend the previous result to surface groups as the ones mentioned in the Introduction is determining the asymptotic density of elements of even length in $\mathbb{Z}^k$. This was done in [5, Proposition 3.6] for $k=2$, and we now compute the value for a general k.

Proposition 3.4. *Let $k \geq 2$, and let U_1^{ev} denote the visible elements of even length in $\mathbb{Z}^k$. Then*

$$\rho_\infty(U_1^{ev}) = \frac{2^{k-1}-1}{2^k-1}\rho_\infty(U_1) = \frac{2^{k-1}-1}{(2^k-1)\zeta(k)}.$$

Proof. Let n be a positive integer. For $X_1, \dots, X_k \in \{\mathcal{A}, \mathcal{O}, \mathcal{E}\}$ we denote by $X_1X_2\dots X_k(n)$ the number of all $z=(z_1,\dots,z_k) \in U_1$ such that $0 \leq z_i \leq n$ and the parity of z_i is X_i. Here $\mathcal{A}$ stands for "any", $\mathcal{E}$ stands for "even" and $\mathcal{O}$ stands for "odd".

We will use the convention $\underbrace{X\dots X}_{\text{k times}} = X^k$, for any $X \in \{\mathcal{A}, \mathcal{O}, \mathcal{E}\}$ and $k \geq 1$.

Note that $X_1X_2\dots X_k(n) = X_{s(1)}X_{s(2)}\dots X_{s(k)}(n)$, for any permutation s of $\{1,\dots,k\}$, and that $\mathcal{E}^k(n)=0$ for any $k, n \geq 1$.

The total number of elements in U_1 of length $\leq n$ is

$$\mathcal{A}^k(n) = \sum_{i=0}^{k}\binom{k}{i}\mathcal{E}^{k-i}\mathcal{O}^i(n). \tag{3.3}$$

Let $U_1^{ev}(n)$ be the set of elements of U_1^{ev} of length at most n. Then

$$|U_1^{ev}(n)| = \sum_{i=1}^{\left[\frac{k}{2}\right]}\binom{k}{2i}\mathcal{E}^{k-2i}\mathcal{O}^{2i}(n). \tag{3.4}$$

We claim that:

$$\mathcal{E}^{k-i}\mathcal{O}^i(n) = \mathcal{O}^k(n) + o(n^k) \text{ for all } 1 \leq i \leq k. \tag{3.5}$$

Assume first that (3.5) holds. From (3.4) and (3.5) we get

$$|U_1^{ev}(n)| = \sum_{i=1}^{\left[\frac{k}{2}\right]}\binom{k}{2i}\mathcal{O}^k(n) + o(n^k),$$

and since $\sum_{i=1}^{\left[\frac{k}{2}\right]}\binom{k}{2i} = 2^{k-1}-1$, we get that

$$|U_1^{ev}(n)| = (2^{k-1}-1)\mathcal{O}^k(n) + o(n^k).$$

From (3.3) and (3.5) we get

$$\mathcal{O}^k(n)(2^k-1) = \mathcal{A}^k(n) + o(n^k),$$

and hence

$$|U_1^{ev}(n)| = \frac{2^{k-1}-1}{2^k-1}\mathcal{A}^k(n) + o(n^k).$$

Since $\rho_\infty(U_1) = \frac{1}{\zeta(k)}$, we get that

$$\begin{aligned}\rho_\infty(U_1^{ev}) &= \limsup_{n\to\infty} \frac{|U_1^{ev}(n)|}{(2n)^k} \\ &= \lim_{n\to\infty} \frac{\frac{2^{k-1}-1}{2^k-1}\mathcal{A}^k(n) + o(n^k)}{(2n)^k} \\ &= \frac{2^{k-1}-1}{2^k-1}\rho_\infty(U_1) = \frac{2^{k-1}-1}{(2^k-1)\zeta(k)}.\end{aligned}$$

This completes the proof of the proposition. We now show (3.5). Notice first that

$$\mathcal{O}^i\mathcal{E}^{k-i-1}\mathcal{A}(n) = \mathcal{O}^i\mathcal{E}^{k-i}(n) + \mathcal{O}^{i+1}\mathcal{E}^{k-i-1}(n).$$

Hence it is enough to show

$$\mathcal{O}^i\mathcal{E}^{k-i-1}\mathcal{A}(n) = 2\mathcal{O}^i\mathcal{E}^{k-i}(n) + o(n^k) \text{ for all } 1 \le i \le k. \tag{3.6}$$

Let μ denote the Möbius function and recall that $\sum_{d|n}\mu(d)$ is equal to 1, if $n = 1$ and 0 otherwise. Hence

$$\mathcal{O}^i\mathcal{E}^{k-i-1}\mathcal{A}(n) = \sum_{\substack{x_j\le n,\, 2\nmid x_j\\ j=1,\dots,i}} \sum_{\substack{x_j\le n,\, 2|x_j\\ j=i+1,\dots,k-1}} \sum_{x_k\le n} \sum_{d|\gcd(x_1,\dots,x_k)} \mu(d)$$

and

$$\mathcal{O}^i\mathcal{E}^{k-i}(n) = \sum_{\substack{x_j\le n,\, 2\nmid x_j\\ j=1,\dots,i}} \sum_{\substack{x_j\le n,\, 2|x_j\\ j=i+1,\dots,k}} \sum_{d|\gcd(x_1,\dots,x_k)} \mu(d).$$

Now we switch the order in the summation. We rearrange the terms depending on $d \mid \gcd(x_1,\dots,x_k)$, writing $x_i = y_i d$. Since there is an odd coordinate, $2 \nmid d$. We obtain that

$$\mathcal{O}^i\mathcal{E}^{k-i-1}\mathcal{A}(n) = \sum_{2\nmid d}\mu(d) \sum_{\substack{y_j\le n/d,\, 2\nmid y_j\\ j=1,\dots,i}} \sum_{\substack{y_j\le n/d,\, 2|y_j\\ j=i+1,\dots,k-1}} \sum_{y_k\le n/d} 1$$

and

$$\mathcal{O}^i\mathcal{E}^{k-i}(n) = \sum_{2\nmid d}\mu(d) \sum_{\substack{y_j\le n/d,\, 2\nmid y_j\\ j=1,\dots,i}} \sum_{\substack{y_j\le n/d,\, 2|y_j\\ j=i+1,\dots,k}} 1.$$

Hence $\mathcal{O}^i\mathcal{E}^{k-i-1}\mathcal{A}(n) - 2\mathcal{O}^i\mathcal{E}^{k-i}(n)$ is equal to

$$\sum_{2\nmid d}\mu(d) \sum_{\substack{y_j\le n/d,\, 2\nmid y_j\\ j=1,\dots,i}} \sum_{\substack{y_j\le n/d,\, 2|y_j\\ j=i+1,\dots,k-1}} \left(\left[\frac{n}{d}\right] - 2\left[\frac{n}{2d}\right]\right). \tag{3.7}$$

The term in parenthesis is either 0 or 1, and it is always 0 for $d > n$. Thus the asymptotic behavior of (3.7) is of type

$$\begin{aligned} O(\sum_{d\leq n} \sum_{\substack{y_j\leq n/d \\ j=1,\dots,k-1}} 1) \subseteq & O(\sum_{d=1}^{n} (n/d)^{k-1}) \\ =& O(n^{k-1}\left(\frac{1}{k-2} - \frac{1}{(k-2)n^{k-2}}\right)) \\ =& O(n^{k-1}) \subset o(n^k) \end{aligned}$$

□

4 Densities of visible elements in surface groups

The main result of this section allows us to compute densities of visible elements in some strongly Markov groups and in particular in free and surface groups. This result is based on Theorem A.7 of Sharp. To avoid additional definitions, we will use the following technical hypothesis:

Hypothesis 4.1. Let F be a group generated by a finite set A such that $F_{\mathrm{ab}} \cong \mathbb{Z}^r$. We denote by $\mathrm{ab} : F \to \mathbb{Z}^r$ the abelianization map. Suppose that there exists a symmetric positive definite real matrix D such that

$$\lim_{n\to\infty} \left| (\det D)^{1/2} n^{r/2} \left(\frac{\gamma_A(n, \mathrm{ab}^{-1}(\alpha))}{\gamma_A(n, F)} + \frac{\gamma_A(n+1, \mathrm{ab}^{-1}(\alpha))}{\gamma_A(n+1, F)} \right) - \frac{2}{(2\pi)^{r/2}} e^{-\langle \alpha, D^{-1}\alpha\rangle/2n} \right| = 0, \tag{4.1}$$

uniformly in $\alpha \in \mathbb{Z}^r$.

Appendix A contains a description of those groups which satisfy the above hypothesis.

In order to compute the densities of visible elements in free and surface groups, we need to fix some notation.

Notation 4.2. For $k \geqslant 2$, we denote by F_k either the free group of rank k or the surface group of genus k.

If F_k is a free group, then it has a presentation of the form

$$\langle a_1, \dots, a_k \mid \quad \rangle$$

and we let $A = \{a_1, \dots, a_k\}^{\pm 1}$.

If F_k is a surface group, then it has presentation of the form

$$\langle a_1, b_1, \dots, a_k, b_k \mid [a_1, b_1] \cdots [a_k, b_k] \rangle,$$

and we let $A = \{a_1, b_1, \dots, a_k, b_k\}^{\pm 1}$.

Let r denote the rank of the abelianization of F_k, that is $r = k$ if F_k is a free group, and $r = 2k$ if F_k is a surface group.

Remark 4.3. Let F, A and r as in Notation 4.2. Then by [13, Theorem 1, Theorem 4], F and A satisfy Hypothesis 4.1.

Definition 4.4. Let G_r be the set of all $M \in SL(r,\mathbb{Z})$ such that $M = I_r$ in $SL(r,\mathbb{Z}/2\mathbb{Z})$. Then G_r is a finite-index subgroup of $SL(r,\mathbb{Z}/2\mathbb{Z})$.

Proposition 4.5. *[5, Proposition 3.3.] Let $S \subseteq \mathbb{Z}^r$ be a G_r-invariant subset such that $\delta = \rho_\infty(S)$ exists. Let $\Omega \subseteq \mathbb{R}^r$ be a nice bounded open set and let*

$$\mu_{t,S}(\Omega) := \frac{\sharp(S \cap t\Omega)}{t^r}.$$

Then we have

$$\lim_{t\to\infty} \mu_{t,S}(\Omega) = \delta\lambda(\Omega), \tag{4.2}$$

where λ is the Lebesgue measure.

Although [5] indicates that the proof is similar to that of [5, Proposition 2.3], we include here a proof for Proposition 4.5 for the sake of completeness.

Proof. Each $\mu_{t,S}$ can be regarded as a measure on $\mathbb{R}^r$. We prove the result by showing that $\mu_{t,S}$ weakly converge to $\delta\lambda$ as $t \to \infty$.

By Helly's theorem there exists a sequence $t_{[1\uparrow\infty[}$ with $\lim_{i\to\infty} t_i = \infty$ such that the sequence $\mu_{t_1,S}$, $\mu_{t_2,S}$, ... is weakly convergent to some limiting measure. We now identify this measure by showing that for every convergent subsequence of $\mu_{t_i,S}$ the limiting measure is equal to $\delta\lambda$.

Indeed, we assume that $\eta = t_{[1\uparrow\infty[}$ is a sequence with $\lim_{i\to\infty} t_i = \infty$ such that the sequence $\mu_{t_i,S}$ converges to the limiting measure $\mu_\eta = \lim_{i\to\infty} \mu_{t_i,S}$. Every $\mu_{t_i,S}$ is invariant with respect to the G_r-action on $\mathbb{R}^r$. Therefore, the limiting measure μ_η is also G_r-invariant. Moreover, the measures $\mu_{t,S}$ are dominated by the measures λ_t defined as $\lambda_t(\Omega) = \frac{\sharp(\mathbb{Z}^r \cap t\Omega)}{t^r}$.

It is well known that if $\Omega \subseteq \mathbb{R}^r$ is a nice bounded open set, then the measures λ_t converge to the Lebesgue measure λ. It follows that μ_η is absolutely continuous with respect to λ. It is also known that the natural action of G_r on $\mathbb{R}^r$ is ergodic with respect to λ (see [14] for the proof of ergodicity). Therefore μ_η is a constant multiple $c\lambda$ of λ. The constant c can be computed for a set such as the open unit ball B in the $\|\cdot\|_\infty$ norm on $\mathbb{R}^r$ defining the length function l on $\mathbb{Z}^r$. By assumption we know that

$$\lim_{t\to\infty} \frac{\sharp\{z \in \mathbb{Z}^r : z \in S \cap tB\}}{\sharp\{z \in \mathbb{Z}^r : z \in tB\}} = \delta.$$

We also have

$$\lim_{t\to\infty} \frac{\sharp\{z \in \mathbb{Z}^r : z \in tB\}}{t^r} = \lambda(B)$$

and hence

$$\lim_{t\to\infty} \frac{\sharp\{z \in \mathbb{Z}^r : z \in S \cap tB\}}{t^r} = \delta\lambda(B).$$

Therefore $c = \delta$ and $\mu_\eta = \delta\lambda$. The above argument shows in fact that every convergent subsequence of $\mu_{t,S}$ converges to $\delta\lambda$ and $\lim_{t\to\infty} \mu_{t,S} = \delta\lambda$. □

Definition 4.6. Let F be a group generated by the finite set A and such that $F_{\mathrm{ab}} \cong \mathbb{Z}^r$.

For an integer $n \geqslant 1$ and a point $x \in \mathbb{R}^r$, let p_n be given by

$$p_n(x) = \frac{1}{2}\left(\frac{\gamma_A(n-1, \{g \in F : \mathrm{ab}(g) = x\sqrt{n}\})}{\gamma_A(n-1, F)} + \frac{\gamma_A(n, \{g \in F : \mathrm{ab}(g) = x\sqrt{n}\})}{\gamma_A(n, F)}\right). \tag{4.3}$$

This is a distribution supported on finitely many points of $\frac{1}{\sqrt{n}}\,\mathbb{Z}^r$.

We need the following results from [9, 12] about the sequence of distributions p_n.

In our context, the statement of the local limit theorem obtained by Rivin in [9] and, via different methods, by Sharp [12] is the following:

Proposition 4.7. *([9, 12, 5]) Let F, A, r as in Hypothesis 4.1 with $r \geqslant 2$. Then:*

(a) The sequence of distributions p_n converges weakly to a normal distribution $\mathfrak{N}$ with density $\mathfrak{n}$.

(b) We have

$$\sup_{x \in \mathbb{Z}^r/\sqrt{n}} |n^{r/2} p_n(x) - \mathfrak{n}(x)| \longrightarrow 0, \text{ as } n \to \infty. \tag{4.4}$$

(c) For $c > 0$, let $\overline{\Omega_c} := \{x \in \mathbb{Z}^r/\sqrt{n} : \|x\| \geqslant c\}$. Then

$$\lim_{c \to \infty} \sum_{x \in \overline{\Omega_c}} p_n(x) = 0. \tag{4.5}$$

Proof. By using the *Continuity Theorem* (see Theorem 2, p. 508 in [4]) and [9, Theorem 5.1], we have that the sequence of distributions p_n converges weakly to the density of the centered normal distribution with covariance matrix D. This proves (a).

By (4.1) and some easy computations, we have that when $n \to \infty$ and $\alpha = x\sqrt{n}$ the following hold.

$$\begin{aligned}
&\left| n^{r/2} p_n(x) - \frac{1}{(2\pi)^r (\det D)^{1/2}} e^{-\langle x, D^{-1} x\rangle/2} \right| \\
&= \frac{1}{2(\det D)^{1/2}} \Bigg| (\det D)^{1/2} n^{r/2} \\
&\qquad \left(\frac{\gamma_A(n-1, \mathrm{ab}^{-1}(x\sqrt{n}))}{\gamma_A(n-1, F)} + \frac{\gamma_A(n, \mathrm{ab}^{-1}(x\sqrt{n}))}{\gamma_A(n, F)}\right) - \frac{2}{(2\pi)^r} e^{-\langle \frac{\alpha}{\sqrt{n}}, \frac{D^{-1}\alpha}{\sqrt{n}}\rangle/2} \Bigg| \\
&= \frac{1}{2(\det D)^{1/2}} \Bigg| (\det D)^{1/2} n^{r/2} \\
&\qquad \left(\frac{\gamma_A(n-1, \mathrm{ab}^{-1}(\alpha))}{\gamma_A(n-1, F)} + \frac{\gamma_A(n, \mathrm{ab}^{-1}(\alpha))}{\gamma_A(n, F)}\right) - \frac{2}{(2\pi)^r} e^{-\langle \frac{\alpha}{\sqrt{n}}, \frac{D^{-1}\alpha}{\sqrt{n}}\rangle/2} \Bigg| \\
&= \frac{1}{2(\det D)^{1/2}} \Bigg| (\det D)^{1/2} n^{r/2} \\
&\qquad \left(\frac{\gamma_A(n-1, \mathrm{ab}^{-1}(\alpha))}{\gamma_A(n-1, F)} + \frac{\gamma_A(n, \mathrm{ab}^{-1}(\alpha))}{\gamma_A(n, F)}\right) - \frac{2}{(2\pi)^r} e^{-\langle \alpha, D^{-1}\alpha\rangle/2n} \Bigg| \longrightarrow 0,
\end{aligned}$$

We have thus obtained (b).

We now prove (c).

For $c > 0$, denote $\Omega_c = \{x \in \mathbb{Z}^r/\sqrt{n},\ \|x\| < c\}$, the complement of $\overline{\Omega_c}$. Then, by Proposition 4.5 and (b), we have that

$$\begin{aligned}\lim_{c\to\infty} \sum_{x\in\overline{\Omega_c}} p_n(x) &= 1 - \lim_{c\to\infty} \sum_{x\in\Omega_c} p_n(x) \\ &= 1 - \int_{x\in\Omega_\infty} n(x) d\lambda(x) = 0.\end{aligned}$$

□

Theorem 4.8. *Let $\Omega \subseteq \mathbb{R}^r$ be a nice bounded open set. Let $S \subseteq \mathbb{Z}^r$ be a G_r-invariant subset such that $\delta = \rho_\infty(S)$ exists. Then*

$$\lim_{n\to\infty} \sum_{x\in S\cap\sqrt{n}\Omega} p_n(x/\sqrt{n}) = \delta\mathfrak{N}(\Omega).$$

Proof. Note that the proof is the same as that of Theorem 3.4 in [5]. The only difference lies in the use of Proposition 4.5.

We have

$$\begin{aligned}\sum_{x\in\mathbb{Z}_t^r\cap\sqrt{n}\Omega} p_n(x/\sqrt{n}) &= \sum_{y\in\frac{1}{\sqrt{n}}S\cap\Omega} p_n(y) \\ &= n^{-r/2} \sum_{y\in\frac{1}{\sqrt{n}}S\cap\Omega} \mathfrak{n}(y) \\ &+ n^{-r/2} \sum_{y\in\frac{1}{\sqrt{n}}S\cap\Omega} (n^{r/2}p_n(y) - \mathfrak{n}(y)).\end{aligned}$$

The local limit theorem of Proposition 4.7(b) tells us that, as $n \to \infty$, each summand $n^{-r/2}p_n(y) - \mathfrak{n}(y)$ of the sum in the last line of the equation above converges to zero and hence so does their Cesaro mean. Proposition 4.5 implies that, as $n \to \infty$, the sum $n^{-r/2}\sum_{y\in\frac{1}{\sqrt{n}}S\cap\Omega}\mathfrak{n}(y)$ converges to

$$\delta\int_\Omega \mathfrak{n} d\lambda = \delta\mathfrak{N}(\Omega).$$

□

We obtain the main result of this section by basically following [5, Theorem A]. Our theorem provides the formula for the spherical density of visible elements in groups that satisfy Hypothesis 4.1, which include free groups of all finite ranks and surface groups.

Theorem 4.9. *Let F, A, r be as in Hypothesis A.5, or more generally, as in Hypothesis 4.1.*

Let $S \subseteq \mathbb{Z}^r$ be a G_r-invariant subset and $\tilde{S} = \mathrm{ab}^{-1}(S)$.

(i). *The strict annular density $\sigma_A(\tilde{S})$ exists and, moreover, $\sigma_A(\tilde{S}) = \rho_\infty(S)$.*

(ii). *Let U_1 denote the set of visible elements in $\mathbb{Z}^r$ and $V_1 = \mathrm{ab}^{-1}(U_1)$ denote the visible elements in F. Let U_1^{ev} denote the set of visible elements of $\mathbb{Z}^r$ of even length. If $\mathrm{ab}^{-1}(U_1^{ev}) = \{v \in V_1 : |v|_A$ is even$\}$ then*

$$\begin{array}{rcccc} \lim_{m\to\infty} \dfrac{\gamma_A(2m, V_1)}{\gamma_A(2m, F)} & = & \rho_\infty(U_1^{ev}) & = & \dfrac{2^k - 2}{(2^k-1)\zeta(k)}, \\ \lim_{m\to\infty} \dfrac{\gamma_A(2m-1, V_1)}{\gamma_A(2m-1, F)} & = & \rho_\infty(U_1) - \rho_\infty(U_1^{ev}) & = & \dfrac{2^k}{(2^k-1)\zeta(k)}. \end{array}$$

Proof. For $c > 0$ denote $\Omega_c := \{x \in \mathbb{R}^r : \|x\| < c\}$ and let $\overline{\Omega_c}$ be the complement of Ω_c. Then

$$\lim_{c\to\infty} \mathfrak{N}(\Omega_c) = 1 \tag{4.6}$$

Let $\epsilon > 0$ be arbitrary. By (4.6) and Proposition 4.7(c) we can choose $c > 0$ such that

$$|\mathfrak{N}(\Omega_c) - 1| \le \epsilon/3$$

and such that

$$\lim_{n\to\infty} \sum_{x\overline{\Omega_c}} p_n(x) \le \epsilon/6.$$

Let S be a G_r-invariant subset of $\mathbb{Z}^r$. By Theorem 4.8, and the above formula there is some $n_0 \ge 1$ such that for all $n \ge n_0$ we have

$$\left| \sum_{x\in S\cap\sqrt{n}\Omega_c} p_n(x/\sqrt{n}) - \rho_\infty(S)\mathfrak{N}(\Omega_c) \right| \le \epsilon/3,$$

and

$$\sum_{x\in\overline{\Omega_c}} p_n(x) \le \epsilon/3.$$

Let

$$Q(n) := \frac{\gamma_A(n-1, \mathrm{ab}^{-1}(S))}{2\gamma_A(n-1, F)} + \frac{\gamma_A(n, \mathrm{ab}^{-1}(S))}{2\gamma_A(n, F)}.$$

For $n \ge n_0$ we let

$$\begin{aligned} Q(n) = & \\ & \left(\frac{\sharp\{g \in F : \mathrm{ab}(g) \in S, |g|_A = n-1 \text{ and } \|\mathrm{ab}\, g\| < c\sqrt{n}\}}{2\gamma_A(n-1, F)} \right. \\ & \left. + \frac{\sharp\{g \in F : \mathrm{ab}(g) \in S, |g|_A = n \text{ and } \|\mathrm{ab}\, g\| < c\sqrt{n}\}}{2\gamma_A(n-1, F)} \right) \\ & + \left(\frac{\sharp\{g \in F : \mathrm{ab}(g) \in S, |g|_A = n-1 \text{ and } \|\mathrm{ab}\, g\| \ge c\sqrt{n}\}}{2\gamma_A(n-1, F)} \right. \\ & \left. + \frac{\sharp\{g \in F : \mathrm{ab}(g) \in S, |g|_A = n \text{ and } \|\mathrm{ab}\, g\| \ge c\sqrt{n}\}}{2\gamma_A(n-1, F)} \right) \\ & = \sum_{x\in S\cap\sqrt{n}\Omega_c} p_n(x/\sqrt{n}) + \sum_{x\in S\cap(\mathbb{R}^r - \sqrt{n}\Omega_c)} p_n(x/\sqrt{n}). \end{aligned}$$

In the last line of the above equation, the first sum differs from $\rho_\infty(S)\mathfrak{N}(\Omega_c)$ by at most $\epsilon/3$ since $n \geq n_0$, and the second sum is $\leq \epsilon/3$ by the choice of c and n_0.

Therefore, again by the choice of c, we have $|Q(n) - \rho_\infty(S)| \leq \epsilon$. Since ϵ is arbitrary, this proves (i).

We now prove (ii). First notice that since U_1 is $SL(r,\mathbb{Z})$-invariant, it is also G_r-invariant. We check that U_1^{ev} is G_r-invariant as well. Let $u \in \mathbb{Z}$. Then $u \in U_1^{ev}$ if and only if $\sum_{1\leq i\leq r}(u)_i \mod 2 = 0$ and $\gcd(u) = 1$. Let $M \in G_r$. As $M \in SL(r,\mathbb{Z})$, $\gcd(Mu) = \gcd(u) = 1$. Also, as $M = I_r$ in $SL(r, \mathbb{Z}/2\mathbb{Z})$,

$$\sum_{1\leq i\leq r}(Mu)_i \mod 2 = \sum_{1\leq i\leq r}(u)_i \mod 2 = 0.$$

Hence, U_1^{ev} is G_r-invariant.

We now take $S = U_1^{ev}$, for $n \geq 2$ even. Then

$$\begin{aligned} Q(n) &= \frac{\gamma_A(n-1, \mathrm{ab}^{-1}(U_1^{ev}))}{2\gamma_A(n-1,F)} + \frac{\gamma_A(n, \mathrm{ab}^{-1}(U_1^{ev}))}{2\gamma_A(n,F)} \\ &= \frac{\gamma_A(n, \mathrm{ab}^{-1}(U_1^{ev}))}{2\gamma_A(n,F)}. \end{aligned}$$

The latter equality follows from the fact that $\mathrm{ab}^{-1}(U_1^{ev}) = \{v \in V_1 : |v|_A \text{ is even}\}$.

By (i),

$$2\lim_{m\to\infty} Q(2m) = 2\lim_{m\to\infty}\frac{\gamma_A(2m, V_1)}{\gamma_A(2m,F)} = 2\rho_\infty(U_1^{ev}).$$

Thus $\lim_{m\to\infty}\dfrac{\gamma_A(2m-1,V_1)}{\gamma_A(2m-1,F)} = \rho_\infty(U_1) - 2\rho_\infty(U_1^{ev})$. By Proposition 3.1 and Proposition 3.4, we obtain the desired results. □

We now focus on surface and free groups.

Corollary 4.10. *Let $k \geq 2$ and let F_k be a free group of rank k or a surface group of genus k. Let A and r as in Notation 4.2. Then*

(i). $\lim_{m\to\infty}\dfrac{\gamma_A(2m,V_1)}{\gamma_A(2m,F)} = \dfrac{2^r - 2}{(2^r-1)\zeta(r)}$.

(ii). $\lim_{m\to\infty}\dfrac{\gamma_A(2m-1,V_1)}{\gamma_A(2m-1,F)} = \dfrac{2^r}{(2^r-1)\zeta(r)}$.

Proof. By Remark 4.3, F_k, A and r satisfy the Hypothesis of Theorem 4.9. It only remains to show that $\mathrm{ab}^{-1}(U_1^{ev}) = \{v \in V_1 : |v|_A \text{ is even}\}$. Let f be an element of F_k such that $\mathrm{ab}(f) = 0 \in \mathbb{Z}^r$. Then any word representing w has the same number of a and a^{-1} and thus it has even length.

Since ab maps elements of A to unit vectors, for $u \in U_1^{ev}$ there exists $v \in \mathrm{ab}^{-1}(U_1^{ev})$ of even length. If $\mathrm{ab}(v) = \mathrm{ab}(v')$, then $\mathrm{ab}(v'v^{-1}) = 0$. Hence $v'v^{-1}$ has even length, and so does v'. Thus Theorem 4.9(ii) applies. □

5 Asymptotic behavior of homogeneous equations in surface groups

We now study the asymptotic behavior of $e_\gamma(F_n, F_k, s, t)$ when F_n and F_k are surface or free groups.

Theorem 5.1. *Let F_k and F_n be free or surface groups and let A, B be their respective generating sets, as in Notation 4.2. Let $r(k)$ and $r(n)$ denote the ranks of the abelianization of F_k and F_n, respectively. Let $\epsilon, \delta \in \{0, 1\}$. Then the following inequalities hold:*

$$\frac{2^{r(n)} - 2(1-\varepsilon)}{(2^{r(n)}-1)\zeta(r(n))} \leq \liminf_{s\to\infty, t\to\infty} e_\gamma(F_n, F_k, 2s+\varepsilon, 2t+\delta),$$

$$\limsup_{s\to\infty, t\to\infty} e_\gamma(F_n, F_k, 2s+\varepsilon, 2t+\delta) \leq 1 - \frac{2^{r(k)} - 2(1-\delta)}{(2^{r(k)}-1)\zeta(r(k))}\left(1 - \frac{2^{r(n)} - 2(1-\varepsilon)}{(2^{r(n)}-1)\zeta(r(n))}\right).$$

Proof. Let V_t and W_t denote the sets of t-visible elements in F_n and F_k, respectively. Let

$$E(s,t) = \{(u,v) \in F_n \times F_k : |u| = s, |v| = t, \phi(u) = v \text{ for some } \phi \in \mathrm{Hom}(F_n, F_k)\}.$$

Then $e_\gamma(F_n, F_k, s, t) = \frac{|E(s,t)|}{\gamma_B(s,F_n)\gamma_A(t,F_k)}$.

By Lemma 3.2 we have the following inequalities:

$$\gamma_B(s, W_1)\gamma_A(t, F_k) \leq |E(s,t)| \leq \gamma_B(s, F_n)\gamma_A(t, F_k) - \sum_{r\neq 1}\gamma_B(s, W_r)\gamma_A(t, V_1).$$

The left inequality holds because every element v in F_k is the homomorphic image of a visible element in F_n. The right inequality holds because no visible element in F_k is the homomorphic image of an r-visible element in F_n, if $r \neq 1$.

By dividing both sides by $\gamma_B(s, F_n)\gamma_A(t, F_k)$, we get

$$\frac{\gamma_B(s, W_1)}{\gamma_B(s, F_n)} \leq e_\gamma(F_n, F_k, s, t) \leq 1 - \frac{\sum_{r\neq 1}\gamma_B(s, W_r)\gamma_A(t, V_1)}{\gamma_B(s, F_n)\gamma_A(t, F_k)} = f(s,t),$$

where

$$f(s,t) = 1 - \frac{\gamma_A(t, V_1)}{\gamma_A(t, F_k)}\frac{\gamma_B(s, F_n) - \gamma_B(s, W_1)}{\gamma_B(s, F_n)}.$$

Let us use $\beta_{m,k}$ to denote the limits, which depend on the parity of m and the rank of the abelianization of F_n and F_k, found in Corollary 4.10. That is, $\beta_{m,k} = \frac{2^{r(k)}-2}{(2^{r(k)}-1)\zeta(r(k))}$ if m is even, and $\beta_{m,k} = \frac{2^{r(k)}}{(2^{r(k)}-1)\zeta(r(k))}$ if m is odd. In order to simplify the exposition we will abuse the fact that $\beta_{m,k}$ depends on the parity of m and for the next paragraph ignore the parities of s and t.

Then

$$\lim_{s\to\infty, t\to\infty} f(s,t) = 1 - \beta_{t,k}(1 - \beta_{s,n}),$$

and we get the following inequalities

$$\beta_{s,n} \leq \liminf_{s\to\infty,t\to\infty} e_\gamma(F_n, F_k, s, t) \leq \limsup_{s\to\infty,t\to\infty} e_\gamma(F_n, F_k, s, t) \leq 1 - \beta_{t,k}(1-\beta_{s,n}). \tag{5.1}$$

Now taking into account the parities of s and t we get the inequalities in the statement of the theorem. $\square$

Thus the probability of an (s,t)-homogeneous equation to be solvable is neither 0 nor 1 as s, t go to infinity.

Corollary 5.2. *Let G be a surface group of genus $g \geq 2$ or a free group of rank ≥ 2.*
Let

$$A(s,t) = \frac{\sharp\{\textit{solvable } (s,t)\textit{-homogeneous equations on } G \textit{ in } n \textit{ variables}\}}{\sharp\{(s,t)\textit{-homogeneous equations on } G \textit{ in } n \textit{ variables}\}}.$$

Then

$$0 < \liminf_{s\to\infty,t\to\infty} A(s,t) \leq \limsup_{s\to\infty,t\to\infty} A(s,t) < 1.$$

Similarly, for a fixed orientable surface Σ, the probability of a closed curve in Σ to be the image of another closed curve in Σ by a continuous map is neither 0 nor 1, as the curves get more and more "complicated."

Corollary 5.3. *Let Σ be an orientable closed surface of genus $k \geq 2$. We fix a presentation for $\pi_1(\Sigma)$, $\langle a_1, b_1, \ldots, a_k, b_k \mid [a_1, b_1] \cdots [a_k, b_k]\rangle$. For a closed curve γ in Σ we denote by $[\gamma]$ the image of γ in $\pi_1(S)$ and by $|[\gamma]|$ the length of $[\gamma]$ with respect to $\{a_1, b_1, \ldots, a_k, b_k\}$.*
We say that γ_2 is the image of γ_1, if it is the image of γ_1 under a continuous map $S \to S$.
Let

$$B(s,t) = \frac{\sharp\{([\gamma_1], [\gamma_2]) \in \pi_1(S)^2,\ (|[\gamma_1]|, |[\gamma_2]|) = (s,t) \textit{ with } \gamma_2 \textit{ the image of } \gamma_1\}}{\sharp\{([\gamma_1], [\gamma_2]) \in \pi_1(S)^2,\ (|[\gamma_1]|, |[\gamma_2]|) = (s,t)\}}.$$

Then

$$0 < \liminf_{s\to\infty,t\to\infty} B(s,t) \leq \limsup_{s\to\infty,t\to\infty} B(s,t) < 1.$$

A Appendix: The hypothesis of Theorem 4.9

We will show in this Appendix that Theorem 4.9 holds not only for free and surface groups, but also for a number of strongly Markov groups.

Definition A.1. We say that a finitely generated group G is *strongly Markov* if for every finite symmetric generating set S we can find:

(i). a finite oriented graph $\Gamma = (\Gamma, V, E, \iota, \tau)$ consisting of vertices V and edges E, where one can write the edges as elements of $V \times V$, and if $e = (u, v)$, the incidence functions are given by $\iota e = u$ and $\tau e = v$;

(ii). a distinguished vertex $* \in V$, with no edges terminating at $*$;

(iii). a labeling map $\rho \colon E \to S$;

such that

(a). there is a bijection between finite paths in the graph Γ starting at $\star$ and passing through the consecutive edges $e_1, \ldots, e_n$, and elements $g \in G$ given by the correspondence $g = \rho(e_1) \cdots \rho(e_n)$ (where the empty path corresponds to the identity element);

(b). the word length $|g|$ is equal to the path length n.

In particular, this condition is satisfied by all (Gromov) hyperbolic groups (see, for instance, [2] and [3]).

Example A.2. As free groups are hyperbolic groups, they are strongly Markov, and one can easily describe the graph in Definition A.1. Let F be the free group of rank two freely generated by $\{a, b\}$. Then a symmetric generating set of F is given by $S = \{a, a^{-1}, b, b^{-1}\}$. It is clear that the labeled graph in Figure 1 satisfies the desired conditions.

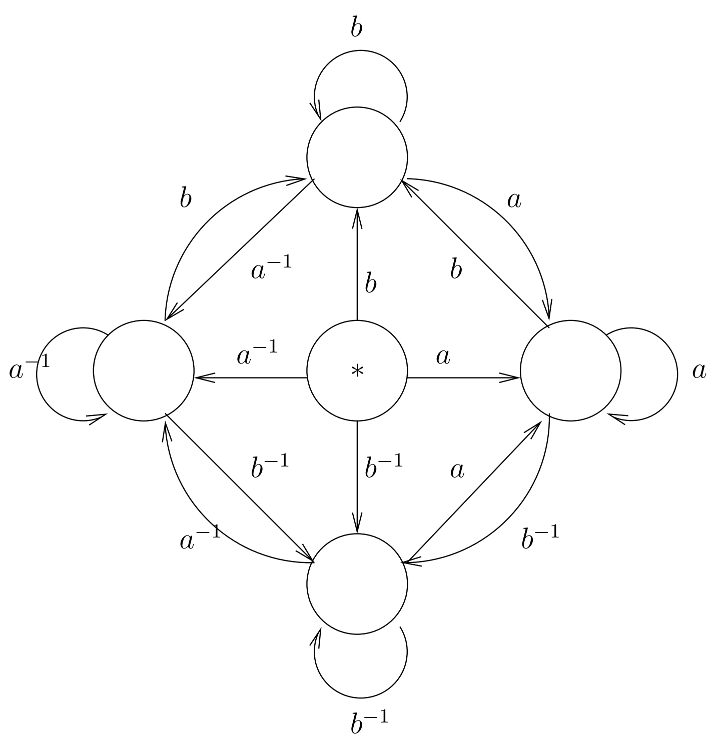

Figure 1: The graph of F.

By the following proposition the fundamental groups of surfaces are also always strongly Markov.

Proposition A.3. *(see [2]) The fundamental group of a compact manifold admitting a hyperbolic structure is strongly Markov.*

This result was originally proved in 1984 by Cannon [2] and a particular nice account can be found in [3].

A.1 The shift space

Let M be an $l \times l$ matrix with entries in $\{0, 1\}$ and define the associated shift space X_M by

$$X_M = \{x_{[0\uparrow\infty[} \text{ sequence in } \{0, 1, \ldots, l-1\} : M(x_n, x_{n+1}) = 1,\ \forall n \in [0 \uparrow \infty[\}.$$

The subshift of finite type $\sigma\colon X_M \longrightarrow X_M$ is defined by $\sigma(x_{[0\uparrow\infty[}) = x_{[1\uparrow\infty[}$.

The matrix M can be viewed as the incidence matrix of an oriented graph Γ with l vertices, and an edge from vertex i to vertex j if $M(i, j) = 1$. From this point of view, X_M can be thought of as the set of infinite paths in Γ that always follow the orientation of the edges.

The matrix M is *aperiodic* if there exists $n > 0$ such that M^n has all its entries > 0. This is equivalent to the map σ being topologically mixing. From now on, we assume the matrix M is aperiodic. Then, by the Perron-Frobenius Theorem, an aperiodic matrix M will have a simple positive eigenvalue $\lambda > 1$ which is strictly maximal in modulus.

Let $\mathcal{M}$ be the set of σ-invariant probability measures on X_M.

Set
$$\mathrm{Fix}_n = \{x_{[0\uparrow\infty[} \in X_M : \sigma^n x_{[0\uparrow\infty[} = x_{[0\uparrow\infty[}\}.$$
It is well-known that $\sharp\,\mathrm{Fix}_n = \text{trace } M^n \sim \lambda^n$, as $n \to \infty$. We want to study the asymptotics of certain subsets of Fix_n.

Fix a function $f\colon X_M \longrightarrow \mathbb{Z}^r$ such that $f(x_{[0\uparrow\infty[})$ depends on only finitely many coordinates of x. Without loss of generality, we may suppose that $f(x_{[0\uparrow\infty[})$ depends only on the first two coordinates, that is, $f(x_{[0\uparrow\infty[}) = f(x_0, x_1)$. Write

$$f[n](x_{[0\uparrow\infty[}) = f(x_{[0\uparrow\infty[}) + f(\sigma x_{[0\uparrow\infty[}) + \cdots + f(\sigma^{n-1} x_{[0\uparrow\infty[}).$$

For $\alpha \in \mathbb{Z}^r$, consider the following subset of Fix_n:

$$\{x_{[0\uparrow\infty[} \in \mathrm{Fix}_n : f[n](x_{[0\uparrow\infty[}) = \alpha\}.$$

We now study the asymptotics of the cardinality of this set as n and α vary.

Following the work of Sharp (see [12]), we will assume that f satisfies the following two natural conditions:

(H1) The set $\cup_{n=1}^{\infty}\{f[n](x_{[0\uparrow\infty[}) \in \mathbb{Z}^r : x_{[0\uparrow\infty[} \in \mathrm{Fix}_n\}$ generates $\mathbb{Z}^r$, and

(H2) $\int f \, d\mu = 0$, where μ is the measure of maximal entropy.

We define the subgroup of $\mathbb{Z}^r$

$$\Delta_f = \left\langle \bigcup_{n=1}^{\infty} \{f[n](x_{[0\uparrow\infty[}) - f[n](y_{[0\uparrow\infty[}) \in \mathbb{Z}^r : x_{[0\uparrow\infty[}, y_{[0\uparrow\infty[} \in \mathrm{Fix}_n\} \right\rangle \leqslant \mathbb{Z}^r.$$

Choose $x_{[0\uparrow\infty[} \in \mathrm{Fix}_n$ and $y_{[0\uparrow\infty[} \in \mathrm{Fix}_{n+1}$ (for some fixed n) and set $c_f = f[n+1](x_{[0\uparrow\infty[}) - f[n](y_{[0\uparrow\infty[})$. Then $\Delta_f + c_f$ is well-defined and $\mathbb{Z}^r/\Delta_f$ is the cyclic group generated by $\Delta_f + c_f$ (see[8]). Furthermore, conditions (H1) and (H2) ensure that $\mathbb{Z}^r/\Delta_f$ is finite and we can write $d = |\mathbb{Z}^r/\Delta_f|$ (see [7]).

In the paper of Sharp (see [12]) there are some local limit theorems for the function f with respect to the periodic points of $\sigma\colon X_M \longrightarrow X_M$. More precisely, we have the following result:

Theorem A.4. *(Corollary 2.1, [12]) Let M be an aperiodic square matrix. Suppose that $f\colon X_M \longrightarrow \mathbb{Z}^r$ is a function that depends on finitely many coordinates and satisfies the conditions (A1) and (A2). Let $d = |\mathbb{Z}^r/\Delta_f|$. Then there exists a positive definite real matrix D such that*

$$\lim_{n\to\infty} \Big| \sum_{j=0}^{d-1} \frac{(\det D)^{1/2} n^{r/2}}{\sharp\,\mathrm{Fix}_{n+j}} \sharp\{x_{[0\uparrow\infty[} \in \mathrm{Fix}_{n+j} : f[n+j](x_{[0\uparrow\infty[}) = \alpha\} - \frac{d}{(2\pi)^{r/2}} e^{-\langle\alpha, D^{-1}\alpha\rangle/2n} \Big| = 0, \tag{A.1}$$

uniformly in $\alpha \in \mathbb{Z}^r$.

We now prove that Hypothesis A.5 implies Hypothesis 4.1, as shown below by Theorem A.7.

Hypothesis A.5. Let F be a strongly Markov group with abelianization $F/[F,F]$ isomorphic to $\mathbb{Z}^r$, A be a finite symmetric generating set of F, and $\mathrm{ab}\colon F \to \mathbb{Z}^r$ be the abelianization map.

Let Γ and ρ be a graph and a labeling associated to the group F with respect the generating set A as in Definition A.1. Let M be the incidence matrix of Γ, and assume that M', the matrix obtained from M by suppressing the row and column associated to $*$, is aperiodic. Suppose that $f\colon X_M \to \mathbb{Z}^r$ satisfies the following two conditions

$$\Gamma_f \coloneqq \langle \cup_{n=1}^{\infty} \{f[n](x_{[0\uparrow\infty[}) : x_{[0\uparrow\infty[} \in \mathrm{Fix}_n\}\rangle \text{ is equal to } \mathbb{Z}^r \tag{A.2}$$

and $d = |\mathbb{Z}^r/\Delta_f| = 2$.

Example A.6. Let F_2 be the free group of rank 2, as in Example A.2, with the graph Γ and labeling ρ described therein. Let M' be the adjacency matrix of the graph Γ with the row and column containing $*$ suppressed. It is not hard to check that $f : X_M \to \mathbb{Z}^2$, defined by $f(x_{[0\uparrow\infty[}) = \mathrm{ab}(\rho((x_0, x_1)))$, satisfies Hypothesis A.5. Here $\rho((x_0, x_1))$ is the label of the edge that goes from the vertex x_0 to x_1.

Similarly to Theorem A.4, Sharp obtains the following local limit theorem for F in [12].

Theorem A.7. *(Theorem 3, [12]) Let F, A, r as in Hypothesis A.5. Then there exists a symmetric positive definite real matrix D such that*

$$\lim_{n\to\infty} \left| (\det D)^{1/2} n^{r/2} \left(\frac{\gamma_A(n, \mathrm{ab}^{-1}(\alpha))}{\gamma_A(n, F)} + \frac{\gamma_A(n+1, \mathrm{ab}^{-1}(\alpha))}{\gamma_A(n+1, F)} \right) - \frac{2}{(2\pi)^{r/2}} e^{-\langle\alpha, D^{-1}\alpha\rangle/2n} \right| = 0, \tag{A.3}$$

uniformly in $\alpha \in \mathbb{Z}^r$.

Sketch of the proof. We consider the matrix M and the function f of Hypothesis A.5.

Since the matrix M' is aperiodic, the function f induces a function $f_{M'}\colon X_{M'} \to \mathbb{Z}^r$. By (A.2), it can be shown that $f_{M'}$ satisfies (H1). By arguments of symmetry one can also show that $f_{M'}$ satisfies (H2).

Hence we can apply Theorem A.4 with $d = 2$.

The arguments in Section 4 and Section 5 of [12] show that we can replace

$$\frac{\sharp\{x_{[0\uparrow\infty[} \in \mathrm{Fix}_{n+j} : f[n+j](x_{[0\uparrow\infty[})\}}{\sharp\,\mathrm{Fix}_{n+j}}$$

by $\dfrac{\gamma_A(n, \mathrm{ab}^{-1}(\alpha))}{\gamma_A(n, F)}$. □

Remark A.8. It can be also shown that surface groups with the Notation 4.2, F_k, $A^{\pm 1}$ and r satisfy Hypothesis A.5.

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