A MASS-TRANSPORTATION APPROACH TO A ONE DIMENSIONAL FLUID MECHANICS MODEL WITH NONLOCAL VELOCITY

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ABSTRACT. We consider a one dimensional transport model with nonlocal velocity given by the Hilbert transform and develop a global well-posedness theory of probability measure solutions. Both the viscous and non-viscous cases are analyzed. Both in original and in self-similar variables, we express the corresponding equations as gradient flows with respect to a free energy functional including a singular logarithmic interaction potential. Existence, uniqueness, self-similar asymptotic behavior and inviscid limit of solutions are obtained in the space $\mathcal{P}_2(\mathbb{R})$ of probability measures with finite second moments, without any smallness condition. Our results are based on the abstract gradient flow theory developed in [2]. An important byproduct of our results is that there is a unique, up to invariance and translations, global in time self-similar solution with initial data in $\mathcal{P}_2(\mathbb{R})$, which was already obtained in [17, 6] by different methods. Moreover, this self-similar solution attracts all the dynamics in self-similar variables. The crucial monotonicity property of the transport between measures in one dimension allows to show that the singular logarithmic potential energy is displacement convex. We also extend the results to gradient flow equations with negative power-law locally integrable interaction potentials.

Keywords: Gradients flows, Optimal transport, Asymptotic Behavior, Inviscid Limit

1. INTRODUCTION

In this work, we are interested in developing a well-posedness theory of measure solutions to the equation

$$(1.1) \qquad \begin{cases} u_t + (H(u)u)_x = 0 \\ u(x,0) = u_0(x) \end{cases} ,$$

with general nonnegative initial Borel measures u_0. Here, the term $H(u)$ denotes the classical Hilbert transform

$$H(u) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{u(z)}{x-z} \, dz \, .$$

Since the equation is of transport nature and in divergence form, we expect sign preservation and mass conservation. Therefore, we will restrict our attention to probability measures as initial data. This equation is nothing else than a $1D$-dimensional continuity equation in which the velocity field is given by the Hilbert transform and it has been proposed as a simplified model in fluid mechanics [13] and in dislocation dynamics [6] as we will discuss in next subsection. This equation can be formally considered a particular example of the theory of gradient flows in the space of probability measures [2] as it will be further elaborated in subsection 1.2.

The main aim of this work is to show that unique measure solutions of gradient-flow type can be constructed for the problem

$$(1.2) \qquad \begin{cases} u_t + (H(u)u)_x = \kappa u_{xx} \\ u(x,0) = u_0(x) \end{cases} ,$$

with $\kappa \geq 0$ and $u_0 \in \mathcal{P}_2(\mathbb{R})$ the set of probability measures on the real line with finite second moments. Moreover, the solutions will continuously depend on both the initial data u_0 and the viscosity parameter $\kappa \geq 0$. The main tools of this construction are the variational schemes based on optimal transportation theory originated for the seminal work [24].

Moreover, we will be able to characterize the large time behavior of the solutions. In fact, we show that suitable scaled equations related to (1.1) and (1.2) have unique stationary solutions fixed by the normalization of the mass. Furthermore, we are able to show that the solutions constructed converge for large times to these stationary solutions exponentially fast in some transport distance. These stationary solutions correspond to self-similar solutions for the original equations.

This manuscript is organized as follows. In the next two subsections we recall the main results already obtained in the literature and the origin of these models. On the other hand, we introduce some basic notations and definitions about optimal mass transportation theory essential to our construction. Section 2 is devoted to introduce self-similar variables and rewrite our problem in the form of a gradient flow in the space of probability measures of a free energy functional. Key properties of the functionals involved are shown in subsections 2.1 and 2.2. Finally, we state and prove our existence, asymptotic behavior and inviscid limit results in Section 3.

1.1. **Motivation: Fluid and Fracture Mechanics.** One of the motivations to analyze these equations arose from the mathematical fluid mechanics literature. In fact, it appears as a simplified one dimensional model [3] mimicking the structure of the $3D$-Navier-Stokes equations and the $2D$-quasi-geostrophic equations [13]:

$$(1.3) \qquad \begin{cases} u_t + (\theta \cdot \nabla)u = 0 \\ \theta = \nabla^\perp \phi, \;\; u = -(-\triangle)^{\frac{1}{2}}\phi \\ u(x,0) = u_0(x), \end{cases}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$ and $u(x,t)$ represents the air temperature. Since $\mathrm{div}(R^\perp u) = 0$, rewriting the system (1.3) in terms of the Riesz transform given by

$$R_j(u)(x,t) = \frac{1}{2\pi}P.V. \int_{\mathbb{R}^2} \frac{(x_j - y_j)}{|x-y|^3}\, u(y,t)\, dy$$

the result is

$$(1.4) \qquad \begin{cases} u_t + \mathrm{div}\left[\left(R^\perp u\right)u\right] = 0 \\ u(x,0) = u_0(x), \end{cases}$$

Considering $u(x,t)$ with $(x,t) \in [0,\infty) \times \mathbb{R}$ and replacing the Riesz transform in (1.4) by the Hilbert transform in one dimension leads to (1.1) or (1.2) with diffusion, see [14] for previous related works and simplified models. Mathematical fluid mechanics arguments have been used to analyze existence and uniqueness, finite time blow-up of smooth solutions, and other issues, see [27, 13, 15, 16, 18, 25] and the references therein related to these equations and other nonconservative variants.

More precisely in our case, sign-changing periodic C^1-solutions of (1.1) blow up in finite time, in the sense that its C^1-norm diverges in finite time as shown in [13]. On the other hand, global existence and uniqueness of smooth solutions for the Cauchy problem on the whole real line is proved in [12] for strictly positive initial data for (1.1) and for general nonnegative initial data for (1.2). The same authors show that for non negative touching-down initial data the Cauchy problem for (1.1) is locally well-posed for smooth solutions and that solutions do blow up in finite time in the C^1 norm.

Apart from the structural similarities, the equations (1.4) and (1.1) have different properties. For instance, while the first one has a Hamiltonian structure, the second one being one dimensional can be considered rather as a gradient flow as we will discuss in next subsection.

The other source of motivation to analyze equations (1.1) and (1.2) comes from dislocation dynamics in crystals [20, 21, 22, 17]. Here, the unknown u represents the number density of fractures per unit length in the material. The existence of explicit self-similar solutions and the convergence

towards them was studied in [17, 6] showing that nonnegative solutions play an important role in the large time asymptotics of (1.2) and related problems. In fact, we will give a characterization of the self-similar solution as the minimizer of a free energy functional intimately related to its gradient flow structure. In this way, we will show that the solution does really converge in suitable scaling and in transport distances to the self-similar profile.

1.2. **Gradient Flows for Probability Measures.** Let us remind some basic facts about optimal mass transport, which will be useful to our study of solutions of the Cauchy problem (1.2). For more details we refer the reader to [30, 2]. Let us denote by $\mathcal{P}(\mathbb{R}^d)$ the space of probability measures on $\mathbb{R}^d$. We start reminding the definition of push forward of a measure $\rho \in \mathcal{P}(\mathbb{R}^d)$.

Definition 1.1. *Let ρ be a probability measure on $\mathbb{R}^d$ and let $T : \mathbb{R}^d \to \mathbb{R}$ be a Borel map. The push forward $T_\sharp \rho \in \mathcal{P}(\mathbb{R}^d)$ of ρ through T is defined by $T_\sharp \rho(I) := \rho(T^{-1}(I))$ for any Borel subset $I \subset \mathbb{R}$. The measures ρ and $T_\sharp \rho$ satisfies*

$$\int_{\mathbb{R}^d} f(T(x))d\rho(x) = \int_{\mathbb{R}^d} f(y)dT_\sharp \rho(y)\,,$$

for every bounded or positive continuous function f.

Let $\rho, \mu \in \mathcal{P}(\mathbb{R}^d)$ and $T_\rho^\mu : \mathbb{R}^d \to \mathbb{R}^d$ such that $T_{\rho\,\sharp}^\mu \rho = \mu$. The map T_ρ^μ is called a transport map between the probability measure ρ and μ. We also recall the notion of transport plan between two probability measures.

Definition 1.2. *Given two measures ρ and μ of $\mathcal{P}(\mathbb{R}^d)$ the set of transport plans between them is defined by*

$$\Gamma(\rho, \mu) := \left\{ \gamma \in \mathcal{P}(\mathbb{R} \times \mathbb{R}) : \pi_\sharp^1 \gamma = \rho, \pi_\sharp^2 \gamma = \mu \right\},$$

where $\pi^i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, 2$ are the projections onto the first and second coordinate: $\pi^1(x, y) = x$, $\pi^2(x, y) = y$. In other words, transport plans are those having marginals ρ and μ.

Our aim is to study solutions of the Cauchy problem (1.1) and (1.2) in an appropriate subspace of $\mathcal{P}(\mathbb{R})$ endowed with a transport distance, the so-called euclidean Wasserstein distance. Consider the set

$$\mathcal{P}_2(\mathbb{R}^d) = \left\{ \rho \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 \, d\rho(x) < \infty \right\}.$$

The euclidean Wasserstein distance is defined on $\mathcal{P}_2(\mathbb{R}^d)$ as:

Definition 1.3. *For any probability measure $\rho, \mu \in \mathcal{P}_2(\mathbb{R}^d)$ the euclidean Wasserstein distance between them is defined by*

$$d_2(\rho, \mu) := \min \left\{ \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^2 d\gamma(x, y) \right)^{\frac{1}{2}} : \gamma \in \Gamma(\rho, \mu) \right\}.$$

We denote by $\Gamma_0(\rho, \mu)$ the set of optimal plans, i.e., the subset of $\Gamma(\rho, \mu)$ where the minimum is attained, i.e,

$$\Gamma_0(\rho, \mu) = \left\{ \gamma \in \Gamma(\rho, \mu) : \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^2 d\gamma(x, y) = d_2^2(\rho, \mu) \right\}.$$

The space $\mathcal{P}_2(\mathbb{R}^d)$ endowed with d_2 becomes a complete metric space. The convergence in d_2 is equivalent to weak-$*$ convergence as measures together with convergence of the second moments, see [30, Theorem 7.12]. We will denote by $\mathcal{P}_2^{ac}(\mathbb{R}^d)$ the subset of probability measures with absolutely continuous densities with respect to Lebesgue measure and finite second moments. It is well-known that for any $\rho, \mu \in \mathcal{P}_2^{ac}(\mathbb{R}^d)$, the minimum in the definition of d_2 is achieved by a plan defined by an optimal map, i.e., by a plan defined by $\gamma = (1_{\mathbb{R}^d} \times T_\rho^\mu)_{\#} \rho$.

Let us remark that the Wasserstein distance in one dimension can be easily characterized since the optimal transport map, if exists, in one dimension is always a monotone non decreasing function.

In fact, as shown in [30, Theorem 2.18], the optimal plan in one dimension is independent of the cost and given in terms of the distribution functions associated to the probability measures and their pseudo-inverses. In fact, one can show that:

Lemma 1.4. *Given $\rho, \mu \in \mathcal{P}_2^{ac}(\mathbb{R})$, the optimal transport map T_ρ^μ in $\mathbb{R}$ for d_2 between them is essentially increasing, i.e., T_ρ^μ is increasing except in a ρ-null set.*

Proof.- Indeed, we can use the distribution function $F_\rho(x) := \rho((-\infty, x))$, and define its pseudo-inverse of F_ρ by the formula

$$F_\rho^{-1}(s) := \sup \left\{ x \in \mathbb{R} : F_\rho(x) \leq s \right\}, \quad \text{for } s \in [0,1].$$

In one dimension, the optimal map for d_2, and for general costs, is given by $T_\rho^\mu = F_\mu^{-1} \circ F_\rho$ satisfying obviously $T_\rho^\mu(s_1) \leq T_\rho^\mu(s_2)$ for all $0 \leq s_1 \leq s_2 \leq 1$. Thus, the optimal transport map T_ρ^μ is nondecreasing. Since $\rho \in \mathcal{P}^{ac}(\mathbb{R})$, T_ρ^μ is an injective function except in a ρ-null set (see [2, Remark 6.2.11]). Therefore it follows at once that T_ρ^μ is increasing except in a ρ-null set, i.e., it is essentially increasing. ◇

Following the seminal ideas for the porous medium equation in [28] and the linear Fokker-Planck equation in [24], a theory of gradient flows in the space of probability measures $(\mathcal{P}_2(\mathbb{R}^d), d_2)$ has been fruitfully applied to general class of equations in the last decade [2, 10, 9, 1]. These equations are continuity equations where the velocity field is given by the gradient of the variational derivative of an energy functional. More precisely, they are of the form

$$(1.5) \qquad \frac{\partial \rho}{\partial t} = \operatorname{div}\left(\rho \nabla \frac{\delta \mathcal{E}}{\delta \rho} \right), \qquad \text{in} \qquad (0, +\infty) \times \mathbb{R}^d,$$

where the free energy functional $\mathcal{E}$ is given by

$$(1.6) \qquad \mathcal{E}[\rho] := \int_{\mathbb{R}^d} U(\rho(x)) \, dx + \int_{\mathbb{R}^d} \rho(x) \, V(x) \, dx + \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} W(x-y) \, \rho(x) \, \rho(y) \, dx \, dy$$

under the basic assumptions $U : \mathbb{R}^+ \to \mathbb{R}$ is a density of internal energy, $V : \mathbb{R}^d \to \mathbb{R}$ is a confinement potential and $W : \mathbb{R}^d \to \mathbb{R}$ is a symmetric interaction potential. The internal energy U should satisfy the following dilation condition, introduced in McCann [26]

$$(1.7) \qquad \lambda \longmapsto \lambda^d U(\lambda^{-d}) \qquad \text{is convex non-increasing on } \mathbb{R}^+.$$

The most classical case of application, as it is for our case, is $U(s) = s \log s$, which identifies the internal energy with Boltzmann's entropy.

We can check that, at least formally, our equations of interest (1.1) and (1.2) are of the form (1.5) in $d = 1$ defined by the functional (1.6) with the choices: $U = V = 0$, and $W(x) = -\frac{1}{\pi} \log |x|$; and $U(s) = \kappa \, s \log s$, $V = 0$, and $W(x) = -\frac{1}{\pi} \log |x|$, respectively. However, the theory developed in [2] is not directly applicable to (1.1) and (1.2) for two reasons: this theory uses a convexity property of the functional $\mathcal{E}$ that we will discuss next and the potentials V, W have to be smooth functions while we deal with the singular potential $-\frac{1}{\pi} \log |x|$ with an apriori unclear convexity properties.

The needed notion of convexity for functionals on measures was introduced in [26] and named displacement convexity. This notion provides functionals of the form (1.6) with a natural convexity structure allowing to show that the variational scheme introduced in [24] is convergent under smoothness and convexity assumptions of the confining and interaction potentials V, W and the convexity property of the internal energy in (1.7), see [2] for precise statements. Let us define

$$(1.8) \qquad \tilde{W}(x) = \begin{cases} -\frac{1}{\pi} \log |x| & \text{for } x \neq 0 \\ +\infty & \text{at } x = 0 \end{cases}.$$

In our case, we will show that the interaction functional $\mathcal{E}$ with $U = V = 0$ and $W = \tilde{W}$ given by (1.8) in one dimension is indeed displacement convex. The intuition behind this is that $\tilde{W}(x)$ is

clearly convex for $x \geq 0$ with the definition above and the optimal map between two measures is essentially increasing as shown in Lemma 1.4. Therefore, when transporting measures we only "see" the convex part of $\tilde{W}(x)$.

On the other hand, this convexity will allow us to avoid the singularity too. In plain words, we will show that this interaction potential is extremely repulsive in one dimension producing that any initial measure is instantaneously regularized to an absolutely continuous measure for all $t > 0$. This behavior is very interesting compared to fully attractive potentials. In fact, equation (1.5) has been studied in $d = 1$ with the displacement concave attractive potential $W(x) = \frac{1}{\pi} \log |x|$, $U(s) = \kappa s \log s$ and $V = 0$, the so-called one dimensional version of the Patlak-Keller-Segel model, in [7]. There, it is shown that the variational scheme in [24] converges to a weak solution of the equation in case the diffusion κ does not go below certain critical value. Let us point out that nonpositive solutions to (1.2) corresponds easily to nonnegative solutions to this 1D-PKS model via reflection, cf. [12].

Other related works for fully attractive potentials may lead to finite time blow-up, in the sense of finite time aggregation to Delta Dirac points, see [4, 8]. For fully repulsive potentials like the one we consider here, we are aware about the recent work in [5] dealing with the asymptotic behavior of $L^1 \cap L^\infty$-solutions in dimensions $d \geq 2$ for the Newtonian potential from a more fluid mechanics perspective.

2. Free Energy Properties

With the purpose in mind to give a well-posedness theory for probability measure solutions to (1.1) and (1.2), we should keep in mind that we are also interested in the asymptotic behaviour of the solutions. For both reasons, it is obvious that a deep preliminary study of the minimization and convexity properties of the free energy functionals involved has to be performed.

In order to find self-similar solutions to (1.2), we will need to rescale variables, as usually done in nonlinear diffusion equations [11] to translate possible self-similar solutions onto stationary solutions. The rescaled equations can also be considered gradient flows of certain free energy functionals which are uniformly 1-convex functionals in the sense of displacement convexity. These are the objectives of this section.

2.1. Self-similar variables and gradient flow structure.
We introduce the following self-similar variables

$$(2.1) \qquad \begin{cases} y = x(1 + 2t)^{-\frac{1}{2}}, & \text{for all } t > 0 \text{ and } x \in \mathbb{R} \\ \tau = \dfrac{1}{2} \log(1 + 2t) \end{cases}$$

Now, observe that if $u(x, t)$ is a solution of system (1.2), then the function

$$\rho(y, \tau) = (2t + 1)^{\frac{1}{2}} u(x, t)$$

with (y, τ) defined by (2.1) satisfies the equation

$$(2.2) \qquad \partial_\tau \rho = \partial_y (y\rho) + \kappa \partial_{yy} \rho - (H(\rho)\rho)_y .$$

Then, we can write the system (1.2) in the new variables as

$$(2.3) \qquad \begin{cases} \partial_\tau \rho = \partial_y(y\rho) + \kappa \partial_{yy}\rho - (H(\rho)\rho)_y, & \forall \tau > 0 \text{ and } y \in \mathbb{R} \\ \rho(y, 0) = u_0 & \forall y \in \mathbb{R} \end{cases} .$$

Equation (2.3) has a gradient flow structure in the sense of subsection 1.2, i.e, we can rewrite it in the following form

$$(2.4) \qquad \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial y} \left[\rho \frac{\partial}{\partial y} \left(\kappa \log \rho - \frac{1}{\pi} \log |y| * \rho + \frac{y^2}{2} \right) \right],$$

where we have replaced the letter τ by t again. From now on, we identify the time dependent probability measure $\rho(\cdot, t) = \rho_t$ with its density with respect to Lebesgue and we use the notation $d\rho_t = d\rho(x, t) = \rho(x, t)\, dx$.

Let us begin by introducing a precise definition of a free energy functional $E_{\kappa,\alpha}$ on the space of probability measures $\mathcal{P}_2(\mathbb{R})$. We define $E_{\kappa,\alpha} : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ as

$$(2.5) \qquad E_{\kappa,\alpha}[\rho] = \begin{cases} \kappa\,\mathcal{U}[\rho] + \alpha\,\mathcal{V}[\rho] + \mathcal{W}[\rho] & \text{for } \rho \in \mathcal{P}_2^{ac}(\mathbb{R}) \\ +\infty & \text{otherwise} \end{cases},$$

with $\alpha, \kappa \geq 0$ and where for $\rho \in \mathcal{P}_2(\mathbb{R})$

$$\mathcal{U}[\rho] := \begin{cases} \displaystyle\int_{\mathbb{R}} \rho(x) \log \rho(x)\, dx & \text{for } \rho \in \mathcal{P}_2^{ac}(\mathbb{R}) \\ +\infty & \text{otherwise} \end{cases}, \qquad \mathcal{V}[\rho] := \int_{\mathbb{R}} \frac{x^2}{2} \rho(x)\, dx,$$

$$\text{and} \qquad \mathcal{W}[\rho] := \frac{1}{2} \int_{\mathbb{R}^2} \tilde{W}(x - y)\rho(x)\rho(y)\, dx\, dy,$$

where $\tilde{W}$ is given by (1.8). We can now identify that (2.3) or (2.4), (1.1) and (1.2) belong to the class of equations (1.5) with the choices $W(x) = \tilde{W}(x)$, $U(s) = \kappa\, s \log s$ and $V = \alpha\frac{x^2}{2}$ with different values for κ and α. Thus, they are formal gradient flows of the corresponding free energy functionals $E_{\kappa,\alpha}[\rho]$. It can be easily checked that the functional $E_{\kappa,\alpha}$ is formally a Lyapunov functional for the equation (2.2), i.e,

$$\frac{d}{dt} E_{\kappa,\alpha}[\rho(t)] = -\mathcal{L}_{\kappa,\alpha}[\rho(t)] \leq 0$$

where

$$\mathcal{L}_{\kappa,\alpha}[\rho] := \int_{\mathbb{R}} \left(\left(\kappa \log \rho(x) + \alpha \frac{x^2}{2} - \int_{\mathbb{R}} \tilde{W}(x - y)\rho(y) dy \right)_x \right)^2 \rho(x)\, dx.$$

Let us remark that the functional $\mathcal{W}$ is also known as the logarithmic energy of ρ as introduced and deeply analyzed in [29] (see also [31, 32]). The next lemma shows the lower semi-continuity of the functionals $\mathcal{U}(\rho)$, $\mathcal{V}(\rho)$, and $\mathcal{W}(\rho)$, and as a consequence, of the functional $E_{\kappa,\alpha}$.

Lemma 2.1. *The functionals $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are lower semi-continuous in $\mathcal{P}_2(\mathbb{R})$ with respect to d_2. Moreover, the functionals $\mathcal{U}$ and $E_{0,\alpha}$ with $\alpha > 0$ are weak-$*$ lower semi-continuous in $\mathcal{P}_2(\mathbb{R})$.*

Proof. The weak-$*$ lower semi-continuity of $\mathcal{U}$ is proven in [26, Lemma 3.4], which implies the d_2 lower semicontinuity. The weak-$*$ lower semi-continuity of $\mathcal{V}$ is straightforward from properties of weak-$*$ sequences and it is trivially continuous for the d_2 topology.

Before starting the proof for $\mathcal{W}$, let us comment that it is essentially contained in [26, Lemma 3.6], although the author deals with a more regular interaction potential $W(x)$. The proof is inspired from arguments in [29, Theorem 1.3]. Here, we included it for completeness. Let us consider the functional

$$E_{0,\alpha}[\rho] := \alpha \mathcal{V}(\rho) + \mathcal{W}(\rho) = \int_{\mathbb{R}^2} R(x, y)\, \rho(x)\rho(y)\, dx\, dy,$$

with $\alpha > 0$ and

$$R(x, y) := \begin{cases} -\dfrac{1}{2\pi} \log\left(|x - y| e^{-\frac{\alpha\pi(x^2 + y^2)}{2}} \right) & \text{for } x \neq y \\ +\infty & \text{for } x = y \end{cases}.$$

Since the function $R(x, y) \to \infty$ as $|(x, y)| \to \infty$ and as $|x - y| \to 0$, it is obviously smooth except at the diagonal and bounded from below, then it can be approximated pointwise by an increasing

sequence of functions $R_k(x, y) \in C_0^\infty(\mathbb{R} \times \mathbb{R})$ as $k \to \infty$. If $\rho_n \to \rho$ weak-$*$ as measures, then certainly the product measure $\rho_n \times \rho_n$ converges to $\rho \times \rho$ weak-$*$ as measures in $\mathbb{R} \times \mathbb{R}$. Now, define

$$E_{0,\alpha}^k[\rho] := \int_{\mathbb{R}^2} R_k(x, y)\, \rho(x)\rho(y)\, dx\, dy\,,$$

and note that $E_{0,\alpha}^k[\rho_n] \le E_{0,\alpha}[\rho_n]$, for all $n \in \mathbb{N}$. Then, due to the weak-$*$ convergence, we get

$$E_{0,\alpha}^k[\rho] = \lim_{n \to \infty} E_{0,\alpha}^k[\rho_n] \le \liminf_{n \to \infty} E_{0,\alpha}[\rho_n],$$

for fixed $k \in \mathbb{N}$. On the other hand, by monotone convergence $E_{0,\alpha}^k[\rho] \to E_{0,\alpha}[\rho]$ as $k \to \infty$ and we obtain $E_{0,\alpha}[\rho] \le \liminf_{n \to \infty} E_{0,\alpha}[\rho_n]$. The remaining statements follow from the continuity of $\mathcal{V}$ in the d_2 topology. $\diamond$

Remark 2.2. *Let us note that the domain of the functional $\mathcal{W}$ consists only of absolutely continuous measures with respect to Lebesgue $D(\mathcal{W}) \subset \mathcal{P}_2^{ac}(\mathbb{R})$. This is a consequence of the definition of $\tilde{W}$ and the fact that given a measure μ with atomic or singular part in its Lebesgue decomposition, then $\mu \times \mu$ will charge the diagonal with positive measure. Notice that $\mathcal{P}_2 \cap L^1 \cap L^\infty(\mathbb{R}) \subset D(\mathcal{W})$.*

2.2. **Minimizing the inviscid free energy functional.** The aim of this section is to make a summary about how to show the existence of a unique minimum among all probability measures in $\mathcal{P}_2(\mathbb{R})$ to the free energy functional $P[\rho] := E_{0,1}[\rho]$. By Lemma 2.1, we already know that P is weak-* lower semi-continuous, and then, in order to ensure the existence of a minimum, we only need to show that the functional is bounded from below. Uniqueness, compact support, characterization and the explicit form of the minimum of this functional were studied in relation to the logarithmic capacity of sets, free probability and connections to random matrices in [29, 23, 31, 32]. We will prove some of them for the sake of the reader in the next proposition.

Proposition 2.3. [29, 23] *Let $\vartheta := \inf \{P[\rho]; \rho \in \mathcal{P}_2(\mathbb{R})\}$. Then:*

i) *ϑ is finite.*
ii) *There is a unique $\bar{\rho} \in \mathcal{P}_2^{ac}(\mathbb{R})$ such that $P[\bar{\rho}] = \vartheta$ with compact support.*
iii) *Moreover, one can characterize $\bar{\rho}$ as the unique measure in $\mathcal{P}_2^{ac}(\mathbb{R})$ satisfying that*

$$\frac{x^2}{2} - \int_{\mathbb{R}} \log|x - y|\bar{\rho}(y)\, dy \ge C_{\bar{\rho}}$$

a.e. $x \in \mathbb{R}$ with equality on $supp(\bar{\rho})$ and with

$$C_{\bar{\rho}} := 2\vartheta - \int_{\mathbb{R}} \frac{x^2}{2}\bar{\rho}(x)\, dx\,.$$

iv) *Furthermore, the minimum can be explicitly computed by using the previous characterization and is given by the semicircle law, i.e., $\bar{\rho}$ is the absolutely continuous measure with respect to Lebesgue with density given by*

$$\bar{\rho}(x)\, dx = \frac{1}{\pi}\sqrt{(2 - x^2)_+}\, dx\,.$$

Proof. *Part i):* We first show that $P[\rho] > 0$ for all $\rho \in \mathcal{P}_2^{ac}(\mathbb{R})$ implying that $\vartheta > -\infty$. For this, observe that, for all $(x, y) \in \mathbb{R}^2$,

$$0 \le |x - y|e^{-\frac{(x^2+y^2)}{2}} \le (|x| + |y|)e^{-\frac{(|x|+|y|)^2}{4}} \le \sup_{r \ge 0} re^{-\frac{r^2}{4}} = \left(\frac{2}{e}\right)^{1/2} < 1.$$

Thus,

$$-\log\left(|x - y|e^{-\frac{(x^2+y^2)}{2}}\right) \ge \frac{1}{2}\log(e/2) > 0$$

and

$$(2.6) \qquad P[\rho] = \frac{1}{2} \int_{\mathbb{R}^2} -\log\left(|x-y|e^{-\frac{(x^2+y^2)}{2}}\right) d\rho(x)d\rho(y)$$

$$\geq \int_{\mathbb{R}^2} \frac{1}{2}\log(e/2)d\rho(x)d\rho(y) = \frac{1}{2}\log(e/2) > 0\,.$$

Therefore, $\vartheta > 0$. Choose $\rho_{12} = dx|_{(1,2)} \in \mathcal{P}_2^{ac}(\mathbb{R})$, where dx denotes the Lebesgue measure. Observe that $-\log|x-y| \geq 0$ on $(1,2) \times (1,2)$. Noting that $\sup\{|2-y|, |1-y|\} \leq 1$ when $y \in (1,2)$, then we obtain from Tonelli theorem that

$$-\int_{\mathbb{R}^2} \log|x-y|\,d(\rho_{12} \times \rho_{12}) = -\int_{(1,2)}\left(\int_{(1,2)} \log|x-y|dx\right)dy \leq -\int_{(1,2)}\left(\int_{(0,1)} \log z dz\right)dy < \infty\,.$$

We conclude that $\vartheta < +\infty$.

Part ii): The proof follows closely [29, Theorem I.1.3]. We start by showing that given a probability measure, we can always construct another compactly supported measure that lowers the energy. First, we observe that for a sequence $(x_n, y_n)_{n=1}^\infty$ with $\lim_{n\to\infty}(|x_n|+|y_n|) = \infty$, we have

$$0 \leq |x_n - y_n|e^{-\frac{(x_n^2+y_n^2)}{2}} \leq (|x_n|+|y_n|)\,e^{-\frac{(|x_n|+|y_n|)^2}{4}} \to 0, \quad \text{as } n \to \infty,$$

and then

$$\lim_{n\to\infty} \log\left[|x_n - y_n|e^{-\frac{(x_n^2+y_n^2)}{2}}\right]^{-1} = +\infty.$$

Therefore, there exists sufficiently small $\varepsilon > 0$, such that

$$(2.7) \qquad -\log|x-y|e^{-\frac{(x^2+y^2)}{2}} > 2(\vartheta+1) \quad \text{if } (x,y) \notin \Sigma_\varepsilon \times \Sigma_\varepsilon,$$

with $\Sigma_\varepsilon := \{x \in \mathbb{R};\ e^{-x^2/2} \geq \varepsilon\}$.

Next, we claim that if $\rho \in \mathcal{P}(\mathbb{R})$, with $\operatorname{supp}(\rho) \cap (\mathbb{R}\backslash\Sigma_\varepsilon) \neq \emptyset$ and $P(\rho) < \vartheta + 1$, then there exists a $\tilde\rho \in \mathcal{P}(\Sigma_\varepsilon)$ such that $P(\tilde\rho) < P(\rho)$. Note this implies that there exists $\varepsilon > 0$, such that

$$(2.8) \qquad \vartheta = \inf\{P(\rho); \rho \in \mathcal{P}(\Sigma_\varepsilon)\}\,.$$

Thus $P(\rho) = \vartheta$ is possible only for measures ρ with support in Σ_ε.

Now, observe that (2.7) and (2.6) together with $P(\rho) < \vartheta + 1$ implies $\rho(\Sigma_\varepsilon) > 0$. This allows us to define

$$\tilde\rho = \frac{\rho|_{\Sigma_\varepsilon}}{\rho(\Sigma_\varepsilon)}\,.$$

Moreover, we have

$$P[\rho] = \frac{1}{2}\left(\int_{\Sigma_\varepsilon \times \Sigma_\varepsilon} -\log\left[|x-y|e^{-\frac{x^2+y^2}{2}}\right]\rho(x)\rho(y)dxdy\right)$$

$$+ \frac{1}{2}\left(\int_{(\Sigma_\varepsilon \times \Sigma_\varepsilon)^c} -\log\left[|x-y|e^{-\frac{x^2+y^2}{2}}\right]\rho(x)\rho(y)dxdy\right)$$

$$> \rho(\Sigma_\varepsilon)^2\left(\frac{1}{2}\int_{\Sigma_\varepsilon \times \Sigma_\varepsilon} -\log\left[|x-y|e^{-\frac{x^2+y^2}{2}}\right]\tilde\rho(x)\tilde\rho(y)dxdy\right)$$

$$+ \frac{1}{2}\int_{\mathbb{R}^2\backslash\Sigma_\varepsilon \times \Sigma_\varepsilon} 2(\vartheta+1)\rho(x)\rho(y)dxdy$$

$$= P[\tilde\rho]\rho(\Sigma_\varepsilon)^2 + (\vartheta+1)(1 - \rho(\Sigma_\varepsilon)^2).$$

$$> P[\tilde\rho]\rho(\Sigma_\varepsilon)^2 + P[\rho](1 - \rho(\Sigma_\varepsilon)^2),$$

since $P(\rho) < \vartheta + 1$. Hence, $P[\rho] > P[\tilde\rho]$ and the claim follows.

As a consequence of (2.8), if ρ is a minimum for P, then ρ has compact support in Σ_ε, and thus $\rho \in \mathcal{P}_2(\mathbb{R})$.

A standard argument in calculus of variations now shows that the minimum is attained in the set $\mathcal{P}_2(\mathbb{R})$. By definition of ϑ and (2.8), there exists a minimizing sequence, i.e., $\{\rho_n\} \subseteq \mathcal{P}_2(\mathbb{R})$ with $P[\rho_n] \to \vartheta$ as $n \to \infty$ with $\operatorname{supp}(\rho_n) \subset \Sigma_\varepsilon$ for all $n \in \mathbb{N}$. Note that each ρ_n has support in the compact Σ_ε, and then we have that the minimizing sequence of measures is tight in the weak convergence of measures. Therefore, we can select from $\{\rho_n\}_{n\in\mathbb{N}}$ a weak* convergent subsequence and without loss of generality, we can assume that $\{\rho_n\}_{n\in\mathbb{N}}$ itself converges to $\rho \in \mathcal{P}_2(\mathbb{R})$ in the weak* topology of measures and in the d_2 sense. Therefore, from weak$-*$ semi-continuity of P, we have

$$\vartheta \leq P[\rho] \leq \lim_{n\to\infty} \inf P[\rho_n] = \vartheta$$

and thus $P[\rho] = \vartheta$. The absolutely continuity of the minimum is a direct consequence of the Remark 2.2 since $D(\mathcal{W}) \subset \mathcal{P}_2^{ac}(\mathbb{R})$.

The uniqueness of the minimum is proven in [29, Theorem I.1.3]. However, in our context this will be clear later on from convexity properties, so we postpone this discussion.

Parts iii) and iv): The characterization of the minimum is due to the Euler-Lagrange equations of the variational problem with the mass constraint. However, since the minimum has compact support, then one obtains this variational inequality outside its support. Moreover, this characterization allows to find explicitly the minimum given by the absolutely continuous measure with density defining the semicircle law in (iv). We refer for all details to [29, Theorem I.1.3 and Theorem IV.5.1] and [23] since it is a well-known fact in free probability and logarithmic capacity. ◇

Remark 2.4. *An important consequence of this result is that there exists a unique compactly supported stationary solution of problem (2.3) in $\mathcal{P}_2(\mathbb{R})$ explicitly given by the semicircle law [23] or the Barenblatt-Pattle profile for $m = 3$ of nonlinear diffusions [11]. Therefore, using the self-similar change of variables (2.1), the problem (1.2) admits a unique, up to invariance and translations, global in time self-similar solution with initial data in $\mathcal{P}_2(\mathbb{R})$. This is already obtained and studied in [17, 6].*

2.3. **The viscous case $\kappa > 0$.** In this subsection, we are concerned with the study of the functional $E_{\kappa,\alpha}$ for $\kappa > 0$. Our intent is to show that functional reaches a unique minimum point on $\mathcal{P}_2^{ac}(\mathbb{R})$. As we already discussed before, a suitable notion of convexity of the functional in the set of measures will be very important in this case. Next, we recall the definition of convexity along generalized geodesics of a functional $E : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathbb{R}$.

Definition 2.5. *[2] A generalized geodesic connecting ρ to μ (with base in ν and induced by γ) is a curve of the type $g_t = \left(\pi_t^{2\to3}\right) \sharp \gamma$, $t \in [0,1]$, where $\gamma \in \Gamma(\nu, \rho, \mu)$, $\pi_\sharp^{1,2}\gamma \in \Gamma_0(\nu, \rho)$, $\pi_\sharp^{1,3}\gamma \in \Gamma_0(\nu, \mu)$, and $\pi_t^{2\to3} = (1-t)\pi_2 + t\pi_3$.*

In particular, when dealing with absolutely continuous measures $\rho, \mu, \nu \in \mathcal{P}_2^{ac}(\mathbb{R})$ and with $\rho = \nu$, $g_t := [(1-t)Id + tT_\rho^\mu]_\sharp \rho$ is a generalized geodesic connecting ρ to μ. In this case, we call g_t the displacement interpolation between ρ and μ.

Definition 2.6. *[2] A functional $E : \mathcal{P}_2(\mathbb{R}) \to (-\infty, +\infty]$ is $\lambda-$convex along generalized geodesics (a.g.g. by shorten) if for every $\nu, \rho, \mu \in D(E) := \{\mu \in \mathcal{P}_2(\mathbb{R}); E[\mu] < \infty\} \subset \mathcal{P}_2(\mathbb{R})$ and for every generalized geodesic g_t connecting ρ to μ induced by a plan $\gamma \in \Gamma(\nu, \rho, \mu)$, the following inequality holds:*

$$E[g_t] \leq (1-t)E[\rho] + tE[\mu] - \frac{\lambda}{2}t(1-t)\, d_\gamma^2(\rho, \mu),$$

where

$$d_\gamma^2(\rho, \mu) := \int_{\mathbb{R}^3} |x_3 - x_2|^2\, d\gamma(x_1, x_2, x_3) \geq d_2^2(\rho, \mu)\,.$$

If g_t is the displacement interpolation and $\lambda = 0$, we say that the functional E is displacement convex as originally introduced in [26].

We readily apply these notions of convexity to our functional $E_{\kappa,\alpha}$.

Proposition 2.7. *Let $\kappa, \alpha \geq 0$. The functional $E_{\kappa,\alpha}$ defined by (2.5) is $\alpha-$convex along generalized geodesics.*

Proof. Following the notation in (2.5), we can reduce ourselves to show that the functional $\mathcal{W}$ is convex along generalized geodesics in $D(E_{\kappa,\alpha}) \subset \mathcal{P}_2^{ac}(\mathbb{R})$. In fact, it is well-known that $\mathcal{U}$ is convex ($\lambda = 0$) along generalized geodesics, and that the functional $\mathcal{V}$ is $1-$ convex along generalized geodesics in $D(E_{\kappa,\alpha}) \subset \mathcal{P}_2^{ac}(\mathbb{R})$, see [2, 26] for details.

Let $\rho, \mu, \nu \in D(E_{\kappa,\alpha}) \subset \mathcal{P}_2^{ac}(\mathbb{R})$ and g_t be a generalized geodesic connecting ρ to μ with base point ν. As we are working on the real line $\mathbb{R}$, we can express the generalized geodesics as

$$g_t = ((1-t)T_\nu^\rho + tT_\nu^\mu)_\# \nu,$$

where T_ν^ρ and T_ν^μ are the optimal transport between ν and ρ, and ν and μ respectively, with the properties in Lemma 1.4.

Let us observe that by definition of g_t and its absolute continuity with respect to Lebesgue, we get

$$\mathcal{W}[g_t] = \int_{\mathbb{R}^2} -\log(|x - y|) \, d(g_t \times g_t)$$

$$= \int_{\mathbb{R}^2} -\log(|(1-t)(T_\nu^\rho(x) - T_\nu^\rho(y)) + t(T_\nu^\mu(x) - T_\nu^\mu(y))|) \, d(\nu \times \nu)$$

$$\leq (1-t) \int_{\mathbb{R}^2} -\log(|(T_\nu^\rho(x) - T_\nu^\rho(y))|) \, d(\nu \times \nu) + t \int_{\mathbb{R}^2} -\log(|(T_\nu^\mu(x) - T_\nu^\mu(y))|) \, d(\nu \times \nu)$$

$$= (1-t)\mathcal{W}[\rho] + t\mathcal{W}[\mu],$$

where the last step follows from convexity of the function $-\log|x|$. This last step is fully rigorous provided the following claim (C) holds: there exists a ν-null set A such that

$$(1-t)(T_\nu^\rho(x) - T_\nu^\rho(y)) + t(T_\nu^\mu(x) - T_\nu^\mu(y)) \neq 0,$$

for all $t \in [0, 1]$, $x, y \in A^c$ and $x \neq y$. In other words, the interpolation map reaches the logarithmic singularity only if $x = y$ or in a $\nu \times \nu$-null set.

In order to prove this claim, we remind that the optimal transport on the real line between two measures in $\mathcal{P}_2^{ac}(\mathbb{R})$ is essentially increasing, see Lemma 1.4. Now, let A be a ν-null set such that T_ν^μ and T_ν^ρ are increasing in A^c. If $x, y \in A^c$ and $x \neq y$ then let us show that

$$(1-t)(T_\nu^\rho(x) - T_\nu^\rho(y)) + t(T_\nu^\mu(x) - T_\nu^\mu(y)) \neq 0, \ \forall t \in [0, 1].$$

To prove this, suppose that $\exists \, t_0 \in (0, 1]$, $x, y \in A^c$ and $x \neq y$ such that

$$(2.9) \qquad\qquad (1-t_0)(T_\nu^\rho(x) - T_\nu^\rho(y)) + t_0(T_\nu^\mu(x) - T_\nu^\mu(y)) = 0\,,$$

then we deduce

$$\frac{(T_\nu^\mu(x) - T_\nu^\mu(y))}{(T_\nu^\rho(x) - T_\nu^\rho(y))} = \frac{t_0 - 1}{t_0} \leq 0,$$

which provides a contradiction, because the optimal transport maps T_ν^μ and T_ν^ρ are increasing in A^c. For the case $t_0 = 0$ in (2.9), we have $T_\nu^\rho(x) - T_\nu^\rho(y) = 0$ which yields $x = y$ or $x, y \in A$, because of essentially injectivity of T_ν^ρ. This finally shows the claim (C), and thus that $\mathcal{W}[g_t]$ is a convex function in $t \in [0, 1]$ for all generalized geodesics corresponding to absolutely continuous measures, which gives by definition the convexity of $\mathcal{W}$ along generalized geodesics in $D(E_{\kappa,\alpha})$.   ◇

Proposition 2.8. *Let $\kappa \geq 0$, $\alpha > 0$, and $\vartheta_{\kappa,\alpha} := \inf\{E_{\kappa,\alpha}(\rho); \rho \in \mathcal{P}_2(\mathbb{R})\}$. Then:*

    i) *$\vartheta_{\kappa,\alpha}$ is finite.*

ii) *There is a unique $\bar{\rho}_{\kappa,\alpha} \in \mathcal{P}_2^{ac}(\mathbb{R})$ such that $E_{\kappa,\alpha}[\bar{\rho}_{\kappa,\alpha}] = \vartheta_{\kappa,\alpha}$.*

Proof. This results is proven for $\kappa = 0$ in Proposition 2.3, except the uniqueness part. Let us assume from now that $\kappa > 0$.

Part i): Recalling the definition of the functional $E_{\kappa,\alpha}$ in terms of $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ we split

$$E_{\kappa,\alpha} = E_{0,\alpha/2} + \kappa\mathcal{U} + \frac{\alpha}{2}\mathcal{V}.$$

It is straightforward to use Jensen's inequality to show that

$$\kappa\mathcal{U}[\rho] + \frac{\alpha}{2}\mathcal{V}[\rho] = \kappa\int_{\mathbb{R}} \frac{\rho}{e^{-\delta x^2/2}} \log\left(\frac{\rho}{e^{-\delta x^2/2}}\right)e^{-\delta x^2/2}dx \geq \frac{\kappa}{2}(\log\delta - \log(2\pi)), \ \forall\rho \in \mathcal{P}_2^{ac}(\mathbb{R})$$

with $\delta = \alpha/2\kappa$. Proceeding analogously to the proof of *Part i)* of Proposition 2.3, we obtain

$$0 \leq |x-y|e^{-\alpha\frac{(x^2+y^2)}{4}} \leq (|x|+|y|)e^{-\alpha\frac{(|x|+|y|)^2}{8}} \leq \sup_{r>0} re^{-\alpha\frac{r^2}{8}} = \left(\frac{4}{\alpha e}\right)^{1/2},$$

and

$$(2.10) \qquad E_{0,\alpha/2}[\rho] = \int_{\mathbb{R}^2} -\log\left(|x-y|e^{-\alpha\frac{(x^2+y^2)}{4}}\right)d\rho(x)d\rho(y) \geq -\frac{1}{2}\log\left(\frac{4}{\alpha e}\right).$$

Therefore, the functional $E_{\kappa,\alpha}$ is bounded from below, and so $\vartheta_{\kappa,\alpha} > -\infty$. Since the domain $D(E_{\kappa,\alpha}) \neq \varnothing$, there is $\rho \in \mathcal{P}_2(\mathbb{R})$ such that $E_{\kappa,\alpha}(\rho) < \infty$, and therefore the infimum $\vartheta_{\kappa,\alpha}$ is finite.

Part ii): From Lemma 2.1, $E_{\kappa,\alpha}$ is weak-$*$ semi-continuous in $\mathcal{P}_2^{ac}(\mathbb{R})$, which shows that the infimum is achieved at some point $\bar{\rho} \in \mathcal{P}_2^{ac}(\mathbb{R})$. In fact, it is easy to check based on the same arguments for *Part i) and* Proposition 2.3 that any minimizing sequence is weakly compact in $L^1(\mathbb{R})$, see similar arguments in [7], since $\kappa > 0$.

The uniqueness claim for $\kappa \geq 0$ follows from the strict displacement convexity of $E_{\kappa,\alpha}$. Indeed, let ρ_1 and ρ_2 be two different minimum in $\mathcal{P}_2^{ac}(\mathbb{R})$ to $E_{\kappa,\alpha}$ and consider $\rho_{\frac{1}{2}}$ the displacement interpolation between ρ_1 and ρ_2 at $t = 1/2$. By the α-displacement convexity of $E_{\kappa,\alpha}$, we have

$$\vartheta_{\kappa,\alpha} \leq E_{\kappa,\alpha}[\rho_{\frac{1}{2}}] < \frac{1}{2}E_{\kappa,\alpha}[\rho_1] + \frac{1}{2}E_{\kappa,\alpha}[\rho_2] = \vartheta_{\kappa,\alpha},$$

which provides a contradiction. Therefore, there exists a unique minimum of $E_{\kappa,\alpha}$.   ◇

3. Well-posedness, asymptotic behavior and inviscid limit

As pointed out in the introduction we will obtain solutions for (2.3) as the limit of a Euler approximation scheme in probability space $\mathcal{P}_2(\mathbb{R})$. More precisely, consider a step time $\tau > 0$ and a initial data $\rho_0 \in \mathcal{P}_2(\mathbb{R})$. We define, for a fixed μ, the functional $\mathcal{I}(\tau, \mu, .) : \mathcal{P}_2(\mathbb{R}) \to (-\infty, \infty]$ as

$$\mathcal{I}(\tau, \mu; \rho) := \frac{1}{2\tau}d_2^2(\mu, \rho) + E[\rho].$$

Formally, we define the following recursive sequence $(\rho_\tau^n)_{n\in\mathbb{N}}$:

$$(3.1) \qquad\qquad\qquad \rho_\tau^0 := \rho_0$$

$$(3.2) \qquad\qquad\qquad \rho_\tau^n = \min_{\rho\in\mathcal{P}_2(\mathbb{R})} I(\tau, \rho_\tau^{n-1}; \rho), \quad n \in \mathbb{N},$$

which can be seen as the discrete approximate Euler solution to the gradient flux equation

$$\frac{\partial\rho}{\partial t} = -\nabla E[\rho], \quad t > 0$$

in the metric space $(\mathcal{P}_2(\mathbb{R}), d_2)$. More precisely, one calls a *discrete solution*, the curve ρ_t^τ obtained as the time interpolation of the discrete scheme (3.1)-(3.2) connecting every pair $(\rho_\tau^{n-1}, \rho_\tau^n)$ with a velocity constant geodesic in $t \in [(n-1)\tau, n\tau)$, see [2].

3.1. **Gradient flows.** Below we remember the definition of a gradient flow solution.

Definition 3.1. *We say that a map $\rho_t \in AC_{loc}^2((0, \infty); \mathcal{P}_2(\mathbb{R}))$ is a solution of the gradient flow equation*

$$-v_t \in \partial E(\rho_t), \quad t > 0,$$

if $-v_t \in Tan_{\rho_t}\mathcal{P}_2(\mathbb{R})$ belongs to the subdifferential of E at ρ_t, a.e. $t > 0$.

It is known that ρ_t being a gradient flow in $\mathcal{P}_2(\mathbb{R})$ is equivalent to the existence of a velocity vector field $-v_t \in Tan_{\rho_t}\mathcal{P}_2(\mathbb{R}) \cap \partial E(\rho_t)$ a.e. $t > 0$, such that $\|v_t\|_{L_{\rho_t}^2(\mathbb{R})} \in L_{loc}^2(0, \infty)$ and the continuity equation holds in the distribution sense:

$$(3.3) \qquad \frac{\partial(\rho_t)}{\partial t} + \nabla \cdot (\rho_t v_t) = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty).$$

The next theorem ensures the existence of a gradient flow solution for the free energy functional $E_{\kappa,\alpha}$ as in (2.5).

Theorem 3.2. *Let $\kappa, \alpha \geq 0$, $\rho_0 \in \mathcal{P}_2(\mathbb{R})$ and the functional $E_{\kappa,\alpha}$. The following assertions hold:*
  (1) *(**Existence and Uniqueness**) The discrete solution ρ_t^τ converges locally uniformly to a locally Lipschitz curve $\rho_t := S_t[\rho_0]$ in $\mathcal{P}_2(\mathbb{R})$ which is the unique gradient flow of $E_{\kappa,\alpha}$ with $\lim_{t\to0+}\rho_t = \rho_0$. Moreover, the curve lies in $\mathcal{P}_2^{ac}(\mathbb{R})$, for all $t > 0$.*
  (2) *(**Contractive semigroup**) The map $t \mapsto S_t[\rho_0]$ for all $\alpha \geq 0$ is a α-contracting semigroup on $\mathcal{P}_2(\mathbb{R})$, i.e.*
$$d_2(S_t[\rho_0], S_t[\mu_0]) \leq e^{-\alpha t} d_2(\rho_0, \mu_0) \qquad \text{for all } \rho_0, \mu_0 \in \mathcal{P}_2(\mathbb{R}).$$
  (3) *(**Asymptotic behavior**) Let $\alpha > 0$ and let us denote by $\bar{\rho}_{\kappa,\alpha}$ the unique minimum of $E_{\kappa,\alpha}$. Then for all $0 < t_0 < t < \infty$, we have*
$$d_2(\rho_t, \overline{\rho}_{\kappa,\alpha}) \leq e^{-\alpha(t-t_0)} d_2(\rho_{t_0}, \overline{\rho}_{\kappa,\alpha})$$
  and
$$E_{\kappa,\alpha}[\rho_t] - E_{\kappa,\alpha}[\overline{\rho}_{\kappa,\alpha}] \leq e^{-2\alpha(t-t_0)}(E_{\kappa,\alpha}[\rho_{t_0}] - E_{\kappa,\alpha}[\overline{\rho}_{\kappa,\alpha}]).$$
  (4) *(**Free Energy identity**) The solution $\rho_t := S_t[\rho_0]$ is a curve of maximal slope and it satisfies the identity:*
$$E_{\kappa,\alpha}[\rho_t] = E_{\kappa,\alpha}[\rho_s] + \int_s^t \int_{\mathbb{R}} |v_\tau(x)|^2 dx d\tau$$
  for all $0 \leq s \leq t$, where $v_t \in L_{loc}^2(0, \infty; L_{\rho_t}^2(\mathbb{R}))$ is the associated velocity field satisfying (3.3) in Definition 3.1.

Proof. First notice that $\mathcal{P}_2 \cap L^1 \cap L^\infty \subset D(E_{\kappa,\alpha}) \subset \mathcal{P}_2^{ac}(\mathbb{R})$ by Remark 2.2 and $\overline{D(E_{\kappa,\alpha})} = \mathcal{P}_2(\mathbb{R})$. Let us start with the case $\alpha > 0$. Collecting the results obtained through previous sections, we have that the functional $E_{\kappa,\alpha} : \mathcal{P}_2(\mathbb{R}) \to (-\infty, +\infty]$ is a proper, l.s.c., coercive functional and $\alpha-$convex along generalized geodesics. Moreover, $\mathcal{I}(\mu, \tau; \rho)$ admits at least a minimum point μ_τ, for all $\tau \in (0, \tau_*)$ and $\mu \in P_2(\mathbb{R})$. The minimum $\mu_\tau \in P_2^{ac}(\mathbb{R})$ for $\kappa > 0$, because minimizing sequences are weakly compact in $L^1(\mathbb{R})$ as in Proposition 2.8 and by the Remark 2.2 if $\kappa = 0$. Therefore, all the statements result directly from the general theory of gradient flows developed in [2, Theorem 11.2.1]. In case $\alpha = 0$, we deal with plain convex functionals along generalized geodesics and the same results apply. However, we need to be careful with the coercivity and the existence of minimizers for the one-step variational scheme since we lack of a direct confinement. This is easily provided by the following observation using the triangular inequality and

$$(3.4) \quad \mathcal{I}(\tau, \mu; \rho) := \frac{1}{2\tau}d_2^2(\mu, \rho) + E_{\kappa,0}[\rho] \geq \frac{1}{4\tau}d_2^2(\rho, \delta_0) - \frac{1}{2\tau}d_2^2(\delta_0, \mu) + E_{\kappa,0}[\rho] = -\frac{1}{\tau}\mathcal{V}[\mu] + E_{\kappa,2/\tau}[\rho]$$

for all $\mu, \rho \in \mathcal{P}_2(\mathbb{R})$ and all $\tau > 0$. Therefore, this implies the boundedness from below, the existence of minimizers of the one-step variational scheme, and the coercivity in the case of $\alpha = 0$. Again the results of [2, Theorem 11.2.1] apply directly. $\diamond$

Remark 3.3. *As a consequence of previous theorem, we have shown the global-in-time well-posedness for the Cauchy problem for general measures in $\mathcal{P}_2(\mathbb{R})$ as initial data for equations (1.1) and (1.2) and their self-similar counterparts (2.3). Moreover, we have shown the convergence towards self-similarity in the sense expressed in the third part of Theorem 3.2. Note that the gradient flows obtained in Theorem 3.2 for the functionals $E_{\kappa,0}$ and $E_{\kappa,1}$ are equivalent through the change of variables (2.1). Finally, note that the evolution is defined in a unique way for any initial data in $\mathcal{P}_2(\mathbb{R})$. However, the evolution flow regularizes instantaneously since it belongs to $\mathcal{P}_2^{ac}(\mathbb{R})$ for all $t > 0$. This is the precise mathematical statement showing that the repulsive logarithmic interaction potential in one dimension is "very repulsive".*

Remark 3.4. *(Power-law potentials) Consider the power-law interaction potential $W(x) = |x|^{-\beta}$, for $0 < \beta < 1$, and its natural extension*

$$(3.5) \qquad \tilde{W}_\beta(x) = \begin{cases} |x|^{-\beta} & \text{for } x \neq 0 \\ +\infty & \text{at } x = 0 \end{cases}.$$

Let the free energy functional $E_{\beta,\kappa,\alpha} : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ be defined as in (2.5) with $\tilde{W}_\beta$ instead of $\tilde{W}$. Noting that $|x|^{-\beta}$ is locally integral when $0 < \beta < 1$, similar arguments as in Propositions 2.3 and 2.8 give that the infimum $\vartheta_{\beta,\kappa,\alpha}$ is finite and unique. Moreover, $\mathcal{P}_2 \cap L^1 \cap L^\infty(\mathbb{R}) \subset D(E_{\beta,\kappa,\alpha}) \subset \mathcal{P}_2^{ac}(\mathbb{R})$ and $\overline{D(E_{\beta,\kappa,\alpha})} = \mathcal{P}_2(\mathbb{R})$. Since the function $|x|^{-\beta}$ is convex, we have that $E_{\beta,\kappa,\alpha}$ is $\alpha-$convex along generalized geodesics. Thus, the results on gradient flows given by Theorem 3.2 also hold true for (3.5) and we can take initial measure $\rho_0 \in \mathcal{P}_2(\mathbb{R})$.

For range $\beta \geq 1$, we have $D(\mathcal{W}) = \{0\}$ and $\overline{D(E_{\beta,\kappa,\alpha})} = \{0\}$ and then the theory trivializes. In particular, if $\rho \in D(\mathcal{W})$ were a continuous function and $K = \{x \in \mathbb{R} : \rho(x) > \frac{1}{n}\}$ is a positive measure set, for some $n \in \mathbb{N}$, then

$$\infty = \frac{1}{n^2} \int_{K \times K} \frac{1}{|x - y|^\beta} dx dy \leq \mathcal{W}(\rho) < \infty,$$

which gives a contradiction. Therefore, the only continuous function in the domain is $\rho = 0$.

Theorem 3.5. *(Inviscid limit) Let us consider the functionals $E_{\kappa,\alpha}$ and $E_{0,\alpha}$ with $\alpha \geq 0$, corresponding to viscosity $\kappa > 0$ and $\kappa = 0$ respectively, and assume that $\rho_0 \in D(E_{\epsilon_0,\alpha})$ with $\epsilon_0 > 0$. If $\rho_\kappa(t), \rho(t)$ are the corresponding gradient flow solutions in $\mathcal{P}_2(\mathbb{R})$ with initial data ρ_0, then*

$$\rho_\kappa(t) \to \rho(t) \text{ in } \mathcal{P}_2(\mathbb{R})$$

locally uniformly in $[0, \infty)$, as $\kappa \to 0^+$.

Proof. In view of the stability property of [2, Theorem 12.2.1], we need only to verify (in a neighborhood of $\kappa = 0$) the equicoercivity of the family of functionals $\{E_{\kappa,\alpha}\}_{\kappa \geq 0}$ and the uniform boundedness at ρ_0. More precisely, we need to show

$$\sup_{\kappa \in (0, \epsilon_0)} E_{\kappa,\alpha}[\rho_0] < \infty \quad \text{and} \quad \inf_{\kappa \in (0, \epsilon_0), \, \rho \in \mathcal{P}_2(\mathbb{R})} \frac{1}{2\tau} d_2^2(\mu, \rho) + E_{\kappa,\alpha}[\rho] > -\infty,$$

for some $\tau > 0$ and $\mu \in \mathcal{P}_2(\mathbb{R})$ and $\epsilon_0 > 0$. Firstly, observe that $\rho_0 \in D(E_{\epsilon_0,\alpha})$ implies $\mathcal{U}[\rho_0] < \infty$, $\mathcal{V}[\rho_0] < \infty$ and $\mathcal{W}[\rho_0] < \infty$. It follows from (2.5) that $\rho_0 \in D(E_{\kappa,\alpha})$ for all $\kappa \in (0, \infty)$. Also,

$$\sup_{\kappa \in (0, \epsilon_0)} E_{\kappa,\alpha}[\rho_0] \leq \max\{0, \epsilon_0 \mathcal{U}[\rho_0]\} + \alpha \mathcal{V}[\rho_0] + \mathcal{W}[\rho_0] < \infty.$$

In order to conclude the proof, it remains verify the equicoercivity. By using (3.4), we observe

$$\frac{1}{2\tau}d_2^2(\mu,\rho) + E_{\kappa,\alpha}[\rho] \geq \frac{1}{2\tau}d_2^2(\mu,\rho) + E_{\kappa,0}[\rho] \geq -\frac{1}{\tau}\mathcal{V}[\mu] + E_{\kappa,1/2\tau}[\rho]$$

for all $\alpha \geq 0$. Let us split the functional $E_{\kappa,1/\tau}$ as

$$E_{\kappa,1/2\tau}[\rho] = \kappa\mathcal{U}[\rho] + \frac{1}{2\tau}\mathcal{V}[\rho] + \mathcal{W}[\rho] = E_{0,1/4\tau}[\rho] + \kappa\mathcal{U}[\rho] + \frac{1}{4\tau}\mathcal{V}[\rho].$$

Let us now remark that $\kappa\mathcal{U} + \frac{\alpha}{2}\mathcal{V}$ is bounded from below. Note that it is the relative logarithmic entropy functional leading to the classical linear Fokker-Planck equation whose minimum is a Gaussian $M(x)$ determined by

$$M(x) = \left(4\pi\frac{\kappa}{\alpha}\right)^{-1/2}\exp\left(-\frac{\alpha x^2}{4\kappa}\right).$$

Therefore, we get

$$(3.6) \qquad \kappa\mathcal{U}[\rho] + \frac{\alpha}{2}\mathcal{V}[\rho] \geq \kappa\mathcal{U}[M] + \frac{\alpha}{2}\mathcal{V}[M] = -\frac{1}{2}\log\left(4\pi\frac{\kappa}{\alpha}\right) \geq -\frac{1}{2}\log\left(4\pi\frac{\epsilon_0}{\alpha}\right),$$

for $\kappa \in (0,\epsilon_0)$. Due to Proposition 2.8 and (2.10), $E_{0,1/4\tau}$ is also bounded from below by $\vartheta_{0,1/4\tau}$. Using (3.6) with $\alpha = \frac{1}{2\tau}$, we conclude

$$\inf_{\kappa\in(0,\epsilon_0),\ \rho\in\mathcal{P}_2(\mathbb{R})}\left(\frac{1}{2\tau}d_2^2(\mu,\rho) + E_{\kappa,\alpha}[\rho]\right) \geq -\frac{1}{\tau}\mathcal{V}[\mu] - \frac{1}{2}\log\left(8\pi\epsilon_0\tau\right) + \vartheta_{0,1/4\tau}.$$

$$\diamond$$

3.2. Solutions in sense of distributions.
An important point is to know whether the gradient flow solutions are solutions in the sense of distributions. Firstly, let us define the notion of weak solution which we deal with. We say that a measure ρ_t is a weak solution to equation (2.3), with initial condition ρ_0, if for all $\varphi \in C_0^\infty(\mathbb{R})$

$$(3.7) \qquad \frac{d}{dt}\int_{\mathbb{R}}\varphi(x)d\rho_t = \kappa\int_{\mathbb{R}}\varphi''(x)d\rho_t - \int_{\mathbb{R}}\varphi'(x)xd\rho_t + \frac{1}{2}\int_{\mathbb{R}^2}\frac{\varphi'(x) - \varphi'(y)}{x-y}d(\rho_t\times\rho_t)$$

in the distributional sense in $(0,\infty)$ with $\rho_t \rightharpoonup \rho_0$ weakly-$*$ as measures.

In order to obtain a connection between gradient flows and weak solutions, we need to describe, see [2], the minimal selection of the subdifferential of $E = E_{\kappa,1}$, that is the set $\partial^\circ E_{\kappa,1}(\rho_t)$. For that matter we need to consider the following functional: for each fixed $\rho \in \mathcal{P}_2(\mathbb{R})$ we define $L_\rho : C_0^1(\mathbb{R}) \to \mathbb{R}$ as

(3.8)

$$L_\rho(\varphi) = \lim_{\delta\to 0^+}\int_{|x-y|\geq\delta}\frac{1}{x-y}\varphi(x)d(\rho\times\rho) = \lim_{\delta\to 0^+}\frac{1}{2}\int_{|x-y|\geq\delta}\frac{\varphi(x) - \varphi(y)}{x-y}d(\rho\times\rho) < \infty, \forall\varphi \in C_0^1(\mathbb{R}).$$

It is straightforward to check that

$$(3.9) \qquad |L_\rho(\varphi)| \leq \frac{1}{2}\left\|\varphi'\right\|_\infty\int_{\mathbb{R}^2}d(\rho\times\rho) = \frac{1}{2}\left\|\varphi'\right\|_\infty,$$

and therefore $L_\rho \in (C_0^1(\mathbb{R}))^*$. So, there exists $\mu \in \mathcal{M}(\mathbb{R})$ and a constant c_0 such that (see [19, p.225])

$$L_\rho(\varphi) = \int_{\mathbb{R}}\varphi'd\mu + c_0\varphi(0) \quad \forall\varphi \in C_0^1(\mathbb{R}).$$

From (3.9), we can also see that $c_0 = 0$ and we obtain the following representation to L_ρ:

$$(3.10) \qquad L_\rho(\varphi) = \int_{\mathbb{R}}\varphi'd\mu, \ \forall\varphi \in C_0^1(\mathbb{R}).$$

Lemma 3.6. *Let $\kappa \geq 0$ and μ as mentioned above. If a measure $\rho \in D(E_{\kappa,\alpha}) \subset \mathcal{P}_2(\mathbb{R})$ belongs to $D(|\partial E_{\kappa,\alpha}|)$ then we have $(\kappa\rho + \mu) \in W^{1,1}_{\text{loc}}(\mathbb{R})$ and*

$$(3.11) \qquad \rho\omega = \partial_x(\kappa\rho + \mu) + \alpha\rho x \quad \text{for some } \omega \in L^2_\rho(\mathbb{R}).$$

In this case the vector ω defined by (3.11) is the minimal selection in $\partial E_{\kappa,\alpha}[\rho]$, i.e. $\omega = \partial^\circ E_{\kappa,\alpha}[\rho]$.

Proof. For each compactly supported smooth test function $\varphi \in C^\infty_0(\mathbb{R})$, let us consider the map $\psi_\varepsilon := Id + \varepsilon\varphi$. It is easy check that $\psi_{\varepsilon\#}\rho \in D(E_{\kappa,\alpha})$, when $\rho \in D(E_{\kappa,\alpha})$ and for $\varepsilon > 0$ small enough. Since $\rho \in D(|\partial E_{\kappa,\alpha}|)$ and from the definition of metric slope $|\partial E_{\kappa,\alpha}|[\rho]$, see [2], we have

$$|A_1(\varphi) + A_2(\varphi) + A_3(\varphi)| := \left| \kappa \lim_{\varepsilon\to 0} \frac{\mathcal{U}[\psi_{\varepsilon\#}\rho] - \mathcal{U}[\rho]}{\varepsilon} + \alpha \lim_{\varepsilon\to 0} \frac{\mathcal{V}[\psi_{\varepsilon\#}\rho] - \mathcal{V}[\rho]}{\varepsilon} + \lim_{\varepsilon\to 0} \frac{\mathcal{W}[\psi_{\varepsilon\#}\rho] - \mathcal{W}[\rho]}{\varepsilon} \right|$$

$$\leq |\partial E_{\kappa,\alpha}|[\rho] \lim_{\varepsilon\to 0} \frac{d_2(\psi_{\varepsilon\#}\rho, \rho)}{\varepsilon} < \infty.$$

The terms A_1 and A_2 can be exactly treated as in [2, Chapter 11] and one obtains

$$A_1(\varphi) = -\kappa \int_{\mathbb{R}} \varphi' \, d\rho \quad \text{and} \quad A_2(\varphi) = \alpha \int_{\mathbb{R}} x\varphi \, d\rho.$$

Now, we deal with the term A_3. Notice that the map

$$Q(\varepsilon, x, y) = \frac{1}{2} \frac{-\log|(x - y + \varepsilon(\varphi(x) - \varphi(y)))| - (-\log|x - y|)}{\varepsilon}$$

is nondecreasing in $\varepsilon > 0$, for fixed $x, y \neq 0$. As $A_1(\varphi)$ and $A_2(\varphi)$ are finite, then $A_3(\varphi)$ is also finite. By the monotone convergence theorem, we have

$$A_3(\varphi) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^2} Q(\varepsilon, x, y) d(\rho \times \rho) = -\frac{1}{2} \int_{\mathbb{R}^2} \frac{\varphi(x) - \varphi(y)}{x - y} d(\rho \times \rho)$$

$$= -L_\rho(\varphi) = -\int_{\mathbb{R}} \varphi' \, d\mu < \infty.$$

Notice that the second integral above is not a singular integral because $\varphi \in C^\infty_0(\mathbb{R})$. Observing that

$$\lim_{\varepsilon\to 0} \frac{d_2(\psi_{\varepsilon\#}\rho, \rho)}{\varepsilon} \leq \|\varphi\|_{L^2_\rho(\mathbb{R})},$$

we get

$$(A_1 + A_2 + A_3)(\varphi) = \int_{\mathbb{R}} -\kappa\varphi'(x) d\rho + \alpha \int_{\mathbb{R}} x\varphi(x) d\rho - L_\rho(\varphi)$$

$$\geq -|\partial E_{\kappa,\alpha}|[\rho] \lim_{\varepsilon\to 0} \frac{d_2(\psi_{\varepsilon\#}\rho, \rho)}{\varepsilon} \geq -|\partial E_{\kappa,\alpha}|[\rho] \|\varphi\|_{L^2_\rho(\mathbb{R})}.$$

Changing φ by $-\varphi$, we finally obtain

$$\left| \int_{\mathbb{R}} (-\kappa\varphi' + \alpha x\varphi) d\rho - L_\rho(\varphi) \right| \leq |\partial E_{\kappa,\alpha}|[\rho] \|\varphi\|_{L^2_\rho(\mathbb{R})}.$$

So, there exists $\omega \in L^2_\rho(\mathbb{R})$ with $\|\omega\|_{L^2_\rho(\mathbb{R})} \leq |\partial E_{\kappa,\alpha}|[\rho]$ such that

$$\int_{\mathbb{R}} \omega\varphi \, d\rho = (A_1 + A_2 + A_3)(\varphi)$$

$$(3.12) \qquad = -\left(\kappa \int_{\mathbb{R}} \varphi' \, d\rho + L_\rho(\varphi) \right) + \alpha \int_{\mathbb{R}} x\varphi \, d\rho, \, \forall \varphi \in C^\infty_0(\mathbb{R}).$$

Thus $\omega \in \partial E_{\kappa,\alpha}[\rho]$ and ω is the minimal selection in $\partial E_{\kappa,\alpha}[\rho]$, i.e. $\omega = \partial^{\circ} E_{\kappa,\alpha}[\rho]$. Finally, let us characterize ω. Since $\rho \in \mathcal{P}_2(\mathbb{R})$ implies that $\psi[\varphi] = \int_{\mathbb{R}} x\varphi \, d\rho$ is bounded in $L^2_\rho(\mathbb{R})$ (with norm at most $\int_{\mathbb{R}} x^2 d\rho$), we get

$$| < \partial_x(\kappa\rho + \mu), \varphi > | = \left| \int_{\mathbb{R}} \varphi' \, d(\kappa\rho + \mu) \right| \leq \left(|\partial E_{\kappa,\alpha}|(\rho) + \alpha \int_{\mathbb{R}} x^2 d\rho \right) \|\varphi\|_{L^2_\rho(\mathbb{R})}$$

$$\leq \left(|\partial E_{\kappa,\alpha}|(\rho) + \alpha \int_{\mathbb{R}} x^2 d\rho \right) \|\varphi\|_\infty .$$

Therefore, $\partial_x(\kappa\rho + \mu) \in \mathcal{M}(\mathbb{R})$, i.e. $(\kappa\rho + \mu) \in BV(\mathbb{R})$. Integration by parts holds:

$$\left| \int_{\mathbb{R}} \varphi(x) d(\partial_x(\kappa\rho + \mu)) \right| \leq \left(|\partial E_{\kappa,\alpha}|(\rho) + \alpha \int_{\mathbb{R}} x^2 d\rho \right) \|\varphi\|_{L^2_\rho(\mathbb{R})} ,$$

which implies $\partial_x(\kappa\rho + \mu) \in L^2_\rho(\mathbb{R}) \cap \mathcal{M}(\mathbb{R})$ and then $(\kappa\rho + \mu) \in W^{1,1}_{loc}(\mathbb{R})$. Finally, coming back to (3.12), we obtain the following expression for ω, the element of minimal norm in the subdifferential of $E_{\kappa,\alpha}$: $\rho\omega = \partial_x(\kappa\rho + \mu) + \alpha x\rho$.   ◇

The next theorem gives a connection between gradient flows and the notion of weak solution (3.7).

Theorem 3.7. (Distributional solution) *Let μ_t correspond to ρ_t through (3.8). For every $\rho_0 \in \mathcal{P}_2(\mathbb{R})$ and every $\kappa, \alpha \geq 0$, the gradient flow ρ_t in $\mathcal{P}_2(\mathbb{R})$ of the functional $E_{\kappa,\alpha}$ is a distributional solution of the equation*

$$(3.13) \qquad \frac{\partial \rho_t}{\partial t} = \frac{\partial}{\partial x}(\rho_t \omega_t) = \frac{\partial}{\partial x}\left[\rho_t \left(\frac{\partial_x(\kappa\rho_t + \mu_t)}{\rho_t} + \alpha x \right) \right],$$

satisfying $\rho(t) \to \rho_0$ as $t \to 0^+$, $\rho_t \in L^1_{\mathrm{loc}}((0, +\infty); W^{1,1}_{\mathrm{loc}}(\mathbb{R}))$, and

$$\left\| \frac{\partial}{\partial x}\left(\frac{\partial_x(\kappa\rho_t + \mu_t)}{\rho_t} + \alpha x \right) \right\|_{L^2(\rho_t;\mathbb{R})} \in L^2_{\mathrm{loc}}(0, +\infty).$$

Proof. Since $\omega_t = \partial^{\circ} E_{\kappa,\alpha}(\rho_t)$ and ρ_t is the gradient flow of $E_{\kappa,\alpha}$, it follows from Lemma 3.6 that (3.13) is satisfied by ρ_t with the additional conditions found in the statement of the theorem.

Now, observe that ρ_t satisfies (3.13) is equivalent to ρ_t satisfies

$$\frac{d}{dt} \int_{\mathbb{R}} \varphi(x) d\rho_t = -\int_{\mathbb{R}} \varphi'(x) d(\partial_x(\kappa\rho_t + \mu_t)) - \alpha \int_{\mathbb{R}} \varphi'(x) x d\rho_t$$

$$= \int_{\mathbb{R}} \varphi''(x) d(\kappa\rho_t + \mu_t) - \alpha \int_{\mathbb{R}} \varphi'(x) x d\rho_t$$

$$= \kappa \int_{\mathbb{R}} \varphi''(x) d\rho_t - \alpha \int_{\mathbb{R}} \varphi'(x) x d\rho_t + \int_{\mathbb{R}} \varphi''(x) d\mu_t$$

$$= \kappa \int_{\mathbb{R}} \varphi''(x) d\rho_t - \alpha \int_{\mathbb{R}} \varphi'(x) x d\rho_t + \frac{1}{2} \int_{\mathbb{R}^2} \frac{\varphi'(x) - \varphi'(y)}{x - y} d(\rho_t \times \rho_t),$$

in the sense of distributions on $t \in (0, \infty)$ and for all $\varphi \in C^\infty_0(\mathbb{R})$.   ◇

Remark 3.8. *Although in the general theory in [2] the previous result is a characterization of gradient flow solutions, we do not know how to get the converse in the characterization of the element of minimal norm in Lemma 3.6 since we do not how to show that μ is absolutely continuous with respect to ρ. This implies that we do not know how to show that distributional solutions with the properties written in Theorem 3.7 are gradient flow solutions.*

Remark 3.9. *(Power-law potential) In the case of potential $W(x) = |x|^{-\beta}$ with $0 < \beta < 1$, we also have that the associated gradient flows (see Remark 3.4) is a solution in sense of distributions as in*

(3.7) *or* (3.13). *Instead of* (3.8), *in this time the operator* $L_\rho(\varphi)$ *is given by*

$$L_\rho(\varphi) = \lim_{\delta \to 0^+} \int_{|x-y| \geq \delta} -\beta \frac{(x-y)}{|x-y|^{\beta+2}} \varphi(x) d(\rho \times \rho)$$

$$= \lim_{\delta \to 0^+} \frac{1}{2} \int_{|x-y| \geq \delta} -\beta \frac{1}{|x-y|^\beta} \frac{\varphi(x) - \varphi(y)}{x-y} d(\rho \times \rho) < \infty,$$

for $\varphi \in C_0^1(\mathbb{R})$. *Thus, since* $\rho \in D(E_{\beta,\kappa,\alpha})$,

$$|L_\rho(\varphi)| \leq \frac{\beta}{2} \left\| \varphi' \right\|_\infty \int_{\mathbb{R}^2} \frac{1}{|x-y|^\beta} d(\rho \times \rho) = C \mathcal{W}[\rho] \left\| \varphi' \right\|_\infty ,$$

for all $\varphi \in C_0^1(\mathbb{R})$. *Therefore* $L_\rho \in (C_0^1(\mathbb{R}))^*$ *and, similarly to* (3.10), $L_\rho(\varphi) = \int_{\mathbb{R}} \varphi' d\mu$, *for* $\varphi \in C_0^1(\mathbb{R})$.

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