Some alternative methods for hydrodynamic closures to dissipative kinetic models

M. Bisi[1], J.A. Carrillo[2], and G. Spiga[1]

[1] Dipartimento di Matematica, Università di Parma, Viale G. P. Usberti 53/A, 43124 Parma, Italia
[2] ICREA (Institució Catalana de Recerca i Estudis Avançats) and Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Spain

Abstract. Two different strategies for deriving hydrodynamic equations for dissipative kinetic models are presented and discussed. The homogeneity scaling approach, not very well-known in the physical literature, is expanded and applied to different show-cases in several applications. Then, we show that this strategy may fail, as it occurs for a thermalized granular gas in a host medium, in which case the problem can be dealt with by resorting to classical moment closure methods.

1 Introduction

In this paper, we review several strategies to get hydrodynamic equations for dissipative kinetic models being the hard-spheres inelastic Boltzmann equation with thermostats our main example. The hydrodynamic equations for the inelastic Boltzmann collision equation have been object of ample interest in the scientific community of rapid granular flows and granular gases.

Most of the by-now classical works in the physical literature deal with the standard closure procedure using the balance equations for mass, momentum and energy and closing these equations by different arguments. These different hydrodynamic systems are obtained based on several equations of state, different orders on the expansion parameters or the inelasticity parameter and different type of particles. We refer to a selected list of classical references [48, 49,43,17,39–41,46,50] and the reviews [42,18] for complete references. These works derive the equations without no mechanism of energy supply into the system.

In fact, the use and application of these hydrodynamic systems outside their supposed limit of validity have been reported in several works dealing with vibrating granular flows [12,16,15], the shock formation around obstacles [52], the homogeneous clustering instability [53,23] and the pattern formation due to Faraday instability in 2d granular layers [24].

In this work, we want to report about another point of view in deriving hydrodynamic equations for dissipative systems not so spread in the physical literature when energy-gain mechanisms, i.e., thermostats, are present. The strategy can be summarized as follows. The hydrodynamic equations obtained should only consider the exact macroscopic equations relevant to the conserved quantities, i.e., mass and mean velocity. The closure procedure should be performed in such a way that the second moments of the unknown distribution are approximated by the ones of the equilibrium distributions in the homogeneous setting. The steady states of the homogeneous case can be expressed due to homogeneity scaling from the basic stationary state with unit mass and zero mean velocity. This coincides with the same principles applied to the classical elastic Boltzmann equation.

The objective of the second section is then to demonstrate this strategy to several dissipative kinetic models with thermostats appearing in econophysics, animal collective motion, simplified granular models and the inelastic Boltzmann equation with stochastic thermostat. Finally, the last section is devoted to the Maxwellian model [13] of granular gases in a host medium, where momentum and energy exchange by collisions with the background are described by a linear inelastic Boltzmann operator. In this case, as it will be shown, the homogeneity scaling strategy fails. However, the goal of deriving fluid–dynamic equations for the hydrodynamic variables may be achieved by resorting to a classical tool like a moment closure technique.

2 Closures by homogeneity scaling

In this section, we review and expand an argument based on homogeneity of the stationary states to get the Euler-type hydrodynamic equations associated to several dissipative kinetic equations. As already mentioned in the introduction, in this strategy we try to get equations for the evolution of the conserved quantities of the system closing them by using the stationary states of the homogeneous regime. As classically done for the standard Boltzmann equation for rarefied gases, we then exploit the homogeneity of the solutions with respect to the conserved quantities to close the system at the Euler level.

This approach was introduced in [5], see also [19], for the one dimensional inelastic kinetic model with random thermostat proposed in [4]. We will give below a review of this result in any dimension. We will show the details of this strategy in several examples of dissipative kinetic systems appearing in economy, collective behavior of individual agents and granular media models.

2.1 Wealth distribution models

Recently, there have been a whole trend of research in the modelling of the formation of wealth distribution curves in terms of statistical mechanics. Actually, the idea that these curves are the steady results of infinitely many binary transactions leads to models based on Boltzmann type equations [47,34,26,27,36,37]. This field of Econophysics has been quite fruitful in the last few years, see [28,54,38] and the references therein. In several works [14,30] a first step to deduce macroscopic balance laws has been the derivation of Fokker-Planck equations from the Boltzmann equation using certain expansion in terms of small “inelasticity”. The homogeneity scaling argument allows to overcome this first step and to obtain the hydrodynamics starting directly from the Boltzmann equation. Let us demonstrate this approach in the case of a particular model in these wealth distribution type models.

In [35], the authors introduce a kinetic model of conservative economy, in which the density of wealth depends also on the propensity to invest. They are led to study the evolution of a distribution function $f = f(x, w, t)$ depending on the propensity $x \in [0, 1]$, on the wealth $w \in \mathbb{R}^+$ and on time $t \in \mathbb{R}^+$. The evolution is governed by a non–homogeneous Boltzmann–like equation,

$$\frac{\partial f}{\partial t} + \Psi(x, w)\,\frac{\partial f}{\partial x} = Q(f, f)\,, \tag{1}$$

where Ψ is the law of variation of the propensity to invest, and $Q(f, f)$ is a collision operator describing the effects of the trade. The law Ψ is assumed for simplicity linearly dependent on w,

$$\Psi(x, w) = (w - \chi\,\bar{w})\,\mu(x)\,,$$

where χ is a positive constant and $\bar{w}$ represents a suitable fixed value of the wealth. As concerns the collision operator, we consider only trades that can be modelled using the following scheme: when two agents encounter in a trade, their pre-trade wealths w, w_* change into the post-trade wealths w', w'_* according to the rule

$$\begin{aligned} w' &= p_1\,w + q_1\,w_*\,,\\ w'_* &= p_2\,w + q_2\,w_*\,, \end{aligned}$$

where the interaction coefficients p_i and q_i are suitable non-negative random variables such that the mean of both p_1+q_1 and p_2+q_2 holds 1 (in this sense, the model is conservative). In particular, the microscopic exchange rule considered in [30,35,20] is

$$\begin{aligned} w' &= (1-\gamma)w + \gamma w_* + \eta\, w, \\ w'_* &= (1-\gamma)w_* + \gamma w + \eta_* w_*, \end{aligned} \tag{2}$$

where $\gamma \in (0, 1/2)$ is a fixed transaction parameter, and η, η_* are random variables characterized by the same distribution with zero mean, variance σ^2 and such that $\eta, \eta_* \geq -(1-\gamma)$. Consequently, in a suitable Maxwellian setting in which each trade generates non–negative outputs so that the collision kernel does not need indicator functions depending on (w, w_*), the weak form of $Q(f,f)$ may be cast as

$$\int_0^\infty \varphi(w)\, Q(f,f)(w)\, dw = \frac{1}{2}\left\langle \int_{\mathbb{R}_2^+} \Big[\varphi(w') + \varphi(w'_*) - \varphi(w) - \varphi(w_*)\Big] f(w) f(w_*)\, dw\, dw_* \right\rangle$$

where $\langle h \rangle$ is the expectation of h with respect to the random variable η.

If $\rho(x,t)$ denotes the local density of agents with propensity x at time t, and $m(x,t)$ the local mean wealth

$$\rho(x,t) = \int_0^\infty f(x,w,t)\, dw\,, \qquad\qquad m(x,t) = \frac{1}{\rho(x,t)} \int_0^\infty w\, f(x,w,t)\, dw\,,$$

the corresponding evolution equations read as

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mu(x)\, \dfrac{\partial}{\partial x}\Big(\rho\, m - \chi\, \bar{w}\, \rho\Big) = 0 \\[2ex] \dfrac{\partial (\rho m)}{\partial t} + \mu(x)\, \dfrac{\partial}{\partial x}\left[\displaystyle\int_0^\infty w^2\, f(x,w,t)\, dw - \chi\, \bar{w}\, \rho\, m\right] = 0\,. \end{cases} \tag{3}$$

In order to close this system in a suitable hydrodynamic regime, we have to express $\int w^2 f\, dw$ in terms of the conserved quantities ρ and m. The most basic closure, at the Euler accuracy, consists in replacing $\int w^2 f\, dw$ by the second order moment corresponding to the equilibrium state with density ρ and mean m.

Our next objective is to write the stationary states in a suitable form. Let $g_\infty(x,w)$ be steady state of the Boltzmann equation (1) with $\rho_g \equiv 1$ and $m_g \equiv 1$. Due to the homogeneity, it is not difficult to check that the distribution

$$f_\infty = \frac{\rho}{m}\, g_\infty\left(\frac{w}{m}\right)$$

is again steady state for the model, with density ρ and mean m. Therefore, higher order moments of all steady states (with general density and mean wealth) are amenable to the ones corresponding to the state g_∞ as

$$M_k = \int_0^\infty w^k f_\infty(w) dw = \frac{\rho}{m} \int_0^\infty w^k g_\infty\left(\frac{w}{m}\right) dw = \rho\, m^k \bar{M}_k \quad \text{where} \quad \bar{M}_k = \int_0^\infty w^k g_\infty(w) dw.$$

Information on $\bar{M}_k$ (hence on M_k) may be derived taking into account that all moments of collision operator vanish at the equilibrium state:

$$\int_0^\infty w^n Q(g_\infty, g_\infty)\, dw = 0\,, \qquad \forall n \in \mathbb{N}\,.$$

This yields

$$\int_0^\infty w^n Q(g_\infty, g_\infty)\, dw = \frac{1}{2}\left\langle \int_{\mathbb{R}_+^2} \Big((w')^n + (w'_*)^n - w^n - w_*^n\Big) g_\infty(w) g_\infty(w_*)\, dw\, dw_* \right\rangle$$
$$= \frac{1}{2}\left\langle \int_{\mathbb{R}_+^2} \left[\sum_{k=0}^{n} \binom{n}{k} \Big(p_1^k q_1^{n-k} + p_2^k q_2^{n-k}\Big) w^k w_*^{n-k} - w^n - w_*^n\right] g_\infty(w) g_\infty(w_*)\, dw\, dw_* \right\rangle$$
$$= \frac{1}{2}\left\langle \int_{\mathbb{R}_+^2} \left[(p_1^n + p_2^n - 1)\, w^n + (q_1^n + q_2^n - 1)\, w_*^n\right] g_\infty(w) g_\infty(w_*)\, dw\, dw_* \right\rangle +$$
$$\frac{1}{2}\left\langle \int_{\mathbb{R}_+^2} \left[\sum_{k=1}^{n-1} \binom{n}{k} \Big(p_1^k q_1^{n-k} + p_2^k q_2^{n-k}\Big) w^k w_*^{n-k}\right] g_\infty(w) g_\infty(w_*)\, dw\, dw_* \right\rangle = 0\,,$$

from which we have

$$\Big(2 - \langle p_1^n + p_2^n + q_1^n + q_2^n\rangle\Big)\bar{M}_n = \sum_{k=1}^{n-1} \binom{n}{k} \langle p_1^k q_1^{n-k} + p_2^k q_2^{n-k}\rangle \bar{M}_k \bar{M}_{n-k}\,. \tag{4}$$

Thus, even if the equilibrium state is not known in explicit form, in the Maxwellian frame it has been possible to derive a formula (of recursive type) for its moments.

In particular, coming back to the model discussed in [35] (with p_i and q_i according to the rules (2)), we have, just by definition, $\bar{M}_0 = 1$ and $\bar{M}_1 = 1$; then, since

$$\binom{2}{1} = 2\,, \qquad \langle p_1 q_1 + p_2 q_2\rangle = 2\gamma(1-\gamma), \qquad \langle p_1^2 q_1^2 + p_2^2 q_2^2\rangle = 2\sigma^2 - 4\gamma(1-\gamma) + 2\,,$$

from formula (4) we get

$$\bar{M}_2 = \frac{2\,\gamma(1-\gamma)}{2\,\gamma(1-\gamma) - \sigma^2}\,,$$

therefore the moment of order 2 is well defined only if $\sigma^2 < 2\,\gamma(1-\gamma)$. Under this assumption, we may close at the Euler level the macroscopic equations (3) by putting

$$\int_0^\infty w^2 f(w)\, dw = \frac{2\,\gamma(1-\gamma)}{2\,\gamma(1-\gamma) - \sigma^2}\ \rho\, m^2\,.$$

This is exactly the same result achieved in [35], with $\lambda = \sigma^2/(\gamma(1-\gamma))$. We have shown that this hydrodynamic closure does not hold only in the continuous trading limit $\gamma \to 0$, $\sigma \to 0$, $\sigma^2/\gamma \to \lambda$, and that it may be obtained directly from the Boltzmann equation, with no need of resorting to an asymptotic Fokker–Planck–type equation [30,20].

If we add to the system (3) an evolution equation for the second order moment, we need a suitable expression for M_3 to achieve the closure. In order to determine if $\bar{M}_3$ is finite, we have to evaluate $\langle p_1^3 + p_2^3 + q_1^3 + q_2^3\rangle$, but this is not possible without information on $\langle \eta^3\rangle$, $\langle \eta_*^3\rangle$. For instance, in the particular case $\eta, \eta_* \equiv 0$ (thus $\sigma^2 = 0$, $\bar{M}_2 = 1$), we have

$$p_1^3 + p_2^3 + q_1^3 + q_2^3 = 2(1 - 3\,\gamma + 3\,\gamma^2) > 0\,,$$

and in analogous way we may evaluate higher order moments.

2.2 Simplified inelastic Granular Models

Several one dimensional models for inelastic interactions have been introduced in the literature in order to capture part of the features of the full inelastic Boltzmann kinetic equation [1,3–5]. These models can be seen as a one dimensional reduction of simplified granular media models studied in [22,55]. In this case, the whole inelasticity of the system is modeled by an effective first-order convolution like operator in the velocity variable leading to concentration of velocity due to the loss of kinetic energy. More exactly, the models look like

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \lambda \nabla_{\mathbf{v}} \cdot (\nabla_{\mathbf{v}} F f) + \sigma \Delta_{\mathbf{v}} f \tag{5}$$

with

$$F(\mathbf{x}, \mathbf{v}, t) = \frac{1}{\gamma + 2} \int_{\mathbb{R}^3} |\mathbf{v} - \mathbf{w}|^{\gamma+2} f(\mathbf{x}, \mathbf{w}, t)\, d\mathbf{w}.$$

Here, λ is the inelasticity parameter, σ is a stochastic thermostat to avoid complete cooling of the system and γ is the parameter related to mimick the different collision frequencies ($\gamma = 1$ for hard-spheres). These models are obtained as small inelasticity expansions in the weak form of Boltzmann inelastic operators, see [55]. Define as usual the particle density and the mean velocity by

$$\rho(\mathbf{x}, t) = \int_{\mathbb{R}^3} f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v} \qquad \text{and} \qquad \rho\mathbf{u}(\mathbf{x}, t) = \int_{\mathbb{R}^3} \mathbf{v} f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v}\ .$$

Integrating (5) over $\mathbf{v}$ we find the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0. \tag{6}$$

Multiplying (5) by $\mathbf{v}$ and integrating over $\mathbf{v}$ we obtain

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla_{\mathbf{x}} \cdot \mathbf{P} \tag{7}$$

where $\mathbf{P}$ is the pressure tensor given by the fluctuations of the velocity, that is,

$$\mathbf{P}(\mathbf{x}, t) = \int_{\mathbb{R}^3} \left[(\mathbf{v} - \mathbf{u}(\mathbf{x}, t)) \otimes (\mathbf{v} - \mathbf{u}(\mathbf{x}, t)) \right] f\, d\mathbf{v}\ .$$

We again realize that due to the scaling homogeneity of the problem we get that the family of stationary solutions can be written in terms of the steady state with mass 1 and zero mean velocity g_∞ as

$$f_\infty(\mathbf{v}) = \rho^{1+\frac{3}{2+\gamma}} g_\infty(\rho^{\frac{1}{2+\gamma}} (\mathbf{v} - \mathbf{u})), \tag{8}$$

for any value of the conserved quantities by the collision operator, i.e., density ρ and mean velocity $\mathbf{u}$. Moreover, the stationary states for (5) are known to be isotropic, therefore the pressure tensor can be approximated in our case by its expression computed from (8) giving

$$P_{ij}(\mathbf{x}, t) = \delta_{ij} \int_{\mathbb{R}^3} (v_i - u_i)(v_j - u_j) f_\infty\, d\mathbf{v} = \delta_{ij}\ A_\infty\, \rho^{1-\frac{2}{2+\gamma}} \tag{9}$$

and A_∞ is given by

$$A_\infty = \frac{1}{3} \int_{\mathbb{R}^3} |\mathbf{v}|^2 g_\infty\, d\mathbf{v},$$

the second moment of g_∞. Therefore the system (6)-(7) with the pressure given by (9) is the hydrodynamical system associated to this simplified inelastic model. This leads to the isentropic Euler equation with strange exponents below 1.

2.3 Collective Behavior Models

In recent years, the modelling of the complex behavior of highly-organize social animals as insects, birds or fishes has been analysed in terms of statistical mechanics tools. The formation of large scale patterns or self-organization is not yet understood and models based on simple rules as attraction, repulsion and orientation mechanisms have been proposed, see [51, 31, 29]. In this spirit kinetic modelling is one of the basic modelling techniques [45, 21].

Here, we concentrate on a particular kinetic model based on the Cucker-Smale orientation algorithm [32, 33] proposed as an ingredient in the explanation of the formation of flocks for birds. This model including noise [44] looks like

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \nabla_{\mathbf{v}} \cdot [\xi(f)(\mathbf{x},\mathbf{v},t) f(\mathbf{x},\mathbf{v},t)] + \sigma \Delta_{\mathbf{v}} f$$

where

$$\xi(f)(\mathbf{x},\mathbf{v},t) = \int_{\mathbb{R}^{2d}} \frac{\mathbf{v}-\mathbf{w}}{\left(1+|\mathbf{x}-\mathbf{y}|^2\right)^{\beta/2}} f(\mathbf{y},\mathbf{w},t)\, d\mathbf{y}\, d\mathbf{w},$$

where $d = 2, 3$. The particle model behind this kinetic equation reads

$$\begin{cases} \dfrac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \\ \dfrac{d\mathbf{v}_i}{dt} = \displaystyle\sum_{j=1}^{N_p} m_j H(|\mathbf{x}_i - \mathbf{x}_j|)\,(\mathbf{v}_j - \mathbf{v}_i) + \Gamma(t), \end{cases}$$

where $\Gamma(t)$ is the brownian motion with zero mean and strength $\sqrt{2\sigma}$ and

$$H(\mathbf{x}) = \frac{1}{\left(1+|\mathbf{x}|^2\right)^{\beta/2}}, \qquad \mathbf{x} \in \mathbb{R}^d,$$

is called the communication rate which measures the influence distance of each individual agent. Let us consider the case in which $\beta > d$ and thus the function H is integrable, let us denote by c_β its integral. In this case, it is easy to see that stationary homogeneous steady state exist since they satisfy

$$\nabla_{\mathbf{v}} \cdot [\xi_s(f)(\mathbf{v}) f(\mathbf{v})] + \sigma \Delta_{\mathbf{v}} f = 0$$

with $\xi_s(f)(\mathbf{v}) = c_\beta \nabla_{\mathbf{v}} \frac{|\mathbf{v}|^2}{2} * f$. Therefore, they are a particular case of the homogeneous cases of previous subsection and thus the steady states are given by $f_\infty(\mathbf{v}) = \rho^{\frac{5}{2}} g_\infty(\rho^{\frac{1}{2}}(\mathbf{v}-\mathbf{u}))$, from which, one can again obtain the hydrodynamic system associated obtaining the pressure-less Euler equations:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0, \\ \rho\, \dfrac{\partial \mathbf{u}}{\partial t} + \rho\,(\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} = 0. \end{cases}$$

These pressure-less hydrodynamic systems appear naturally in these models [29, 21].

2.4 Inelastic Boltzmann Models

In this final subsection, we aim at deriving hydrodynamic equations for a thermalized granular medium where the particle interaction is modelled by the full inelastic Boltzmann operator. Let $f(\mathbf{x},\mathbf{v},t)$ denote the distribution function. Let us recall that number density ρ, mass velocity $\mathbf{u}$, granular temperature T, viscous stress $\mathbf{p}$ and heat flux $\mathbf{q}$ may be derived as suitable moments of $f(\mathbf{v})$ in this way:

$$\rho = \int f(\mathbf{v})\, d\mathbf{v}\,, \qquad \rho\,\mathbf{u} = \int \mathbf{v}\, f(\mathbf{v})\, d\mathbf{v}\,, \qquad 3\,\rho T = m \int c^2 f(\mathbf{v})\, d\mathbf{v} \qquad (\mathbf{c} = \mathbf{v} - \mathbf{u})\,,$$

$$p_{ij} = m \int \left(c_i c_j - \frac{1}{3} c^2\, \delta_{ij} \right) f(\mathbf{v})\, d\mathbf{v}\,, \qquad q_i = \frac{1}{2}\, m \int c_i\, c^2 f(\mathbf{v})\, d\mathbf{v}\,.$$

As usual in kinetic theory, collisions between grains themselves are modelled by a quadratic inelastic Boltzmann operator for hard-spheres that reads, in weak formulation, as

$$\langle \varphi, Q(f,f)\rangle = \frac{1}{4\pi} \iiint |\mathbf{v}-\mathbf{w}| \Big[\varphi(\mathbf{v}') - \varphi(\mathbf{v})\Big] f(\mathbf{v}) f(\mathbf{w})\, d\mathbf{v}\, d\mathbf{w}\, d\hat{\sigma} \tag{10}$$

where $\langle \cdot, \cdot \rangle$ is the usual dual product, and the post–collision velocity $\mathbf{v}'$ is given by

$$\mathbf{v}' = \frac{3-e}{4}\,\mathbf{v} + \frac{1+e}{4}\,\mathbf{w} + \frac{1+e}{4}\, g\,\hat{\sigma}\,,$$

with $g = |\mathbf{v}-\mathbf{w}|$, and e is a positive restitution coefficient (less than 1). We refer to [18,56] and the references therein for deeper discussion of the models.

In order to have again non trivial stationary states we consider the stochastic thermostat as in previous sections. Then, as usual the linear diffusion term avoids the granular temperature to go to 0 in the Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} = Q(f,f) + F\,\Delta_{\mathbf{v}} f\,.$$

This model again preserves both number density and mass velocity, and the relevant conservation equations are

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x_i}\big(\rho\, u_i\big) = 0\,, \\[2ex] \rho\dfrac{\partial u_i}{\partial t} + \rho u_j \dfrac{\partial u_i}{\partial x_j} + \dfrac{1}{m}\,\dfrac{\partial \rho T}{\partial x_i} + \dfrac{1}{m}\,\dfrac{\partial p_{ij}}{\partial x_j} = 0\,. \end{cases}$$

The closure at the Euler level requires the evaluation of temperature and viscous stress at the steady state. In the three-dimensional case however it is well known that the equilibrium is isotropic ($p_{ij} = 0$), thus only temperature has to be actually evaluated. Let g_∞ be a steady state with $\rho_g = 1$ and $\mathbf{u}_g = \mathbf{0}$ and, as in previous subsections, let us look for $\alpha > 0$ such that

$$f_\infty = \rho^{3\alpha+1} g_\infty(\rho^\alpha(\mathbf{v}-\mathbf{u})) \tag{11}$$

is steady state with density ρ and mean $\mathbf{u}$. We have $\langle Q(g_\infty, g_\infty), \varphi\rangle = \rho^{\alpha-2}\langle Q(f_\infty, f_\infty), \bar{\varphi}\rangle$, and $\langle \Delta_{\mathbf{v}} g_\infty, \varphi\rangle = \rho^{-1-2\alpha}\langle \Delta_{\mathbf{v}} f_\infty, \bar{\varphi}\rangle$, (where $\bar{\varphi}(\mathbf{v}) = \varphi(\rho^\alpha \mathbf{v})$), so that by imposing

$$\langle \bar{\varphi}, Q(f_\infty, f_\infty)\rangle + F\,\langle \bar{\varphi}, \Delta_{\mathbf{v}} f_\infty\rangle = 0$$

for all $\bar{\varphi}$, we get $\alpha = \frac{1}{3}$, hence

$$f_\infty = \rho^2 g_\infty(\rho^{\frac{1}{3}}(\mathbf{v}-\mathbf{u}))\,.$$

Isotropic moments of f_∞ are then related to the corresponding ones of g_∞ by

$$\int |\mathbf{v}-\mathbf{u}|^{2\,k} f_\infty(\mathbf{v})\, d\mathbf{v} = \rho^{1-\frac{2\,k}{3}} \int |\mathbf{w}|^{2\,k} g_\infty(\mathbf{w})\, d\mathbf{w}\,.$$

In the particular case $k = 1$ we get $\rho (T_\infty)_f = \rho^{\frac{1}{3}}\,(T_\infty)_g$. If we want to compute the exact value of the temperature for the steady state with unit density and zero mean velocity, we cannot do it explicitly for the hard-spheres collision kernel.

Assuming a suitable pseudo–Maxwellian approximation is adopted, as in [13,6,7,2,9,25], then this computation can be made explicit. In this approximation, the relative speed appearing in the collision kernel is replaced by $B\sqrt{T}$, and from now on we shall denote $\tilde{Q}(f,f)$ the operator (10) with this approximation. Simple computations provide

$$\frac{1}{3}\,F \int v^2\,\Delta_{\mathbf{v}} g_\infty\, d\mathbf{v} = 2\,F$$

while, skipping details

$$\frac{1}{3}\int v^2 \tilde{Q}(g_\infty, g_\infty)\, d\mathbf{v} = -\, B\sqrt{(T_\infty)_g}\, \frac{1-e^2}{4}\, (T_\infty)_g \,.$$

Imposing the equilibrium condition, we find the equilibrium temperature

$$(T_\infty)_g = \left(\frac{8F}{B(1-e^2)}\right)^{\frac{2}{3}} .$$

Then Euler closure is achieved by replacing ρT in (2.4) by

$$\rho (T_\infty)_f = \left[\left(\frac{8F}{B(1-e^2)}\right)^2 \rho\right]^{\frac{1}{3}} .$$

3 Hydrodynamic equations by means of the moments of the steady state

Another interesting topic dealt with in several papers in recent literature is the evolution of a granular powder in a host medium, considered as a fixed background [11,8]. It is modelled by adding to the inelastic Boltzmann equation a linear operator $L(f)$ describing the scattering with the background:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} = Q(f,f) + L(f) \,. \tag{12}$$

The linear operator may be cast, in weak form, as

$$\langle \varphi, L(f)\rangle = \frac{1}{4\pi\,\lambda}\iiint |\mathbf{v}-\mathbf{w}|\Big[\varphi(\mathbf{v}'_L) - \varphi(\mathbf{v})\Big] f(\mathbf{v}) f_B(\mathbf{w})\, d\mathbf{v}\, d\mathbf{w}\, d\hat{\sigma} \tag{13}$$

where f_B is the distribution of the host medium (not necessarily a Maxwellian distribution),

$$\mathbf{v}'_L = \mathbf{v} - \alpha\, \frac{1+e_B}{2}\Big(\mathbf{v} - \mathbf{w} - g\,\hat{\sigma}\Big) \,,$$

the post–collision velocity (the restitution coefficient e_B may be different from e), and $\alpha = m_B/(m+m_B)$ the mass ratio factor.

As in the final part of the previous section, we will adopt the pseudo–Maxwellian approximation for the binary collision operator, as in [13,6,7,2,2,25]. Also in the linear operator we adopt the pseudo–Maxwellian assumption, setting in the kernel $|\mathbf{v}-\mathbf{w}| = B_b\,(\sqrt{T})^\gamma$ (with $\gamma \geq 0$), and $\tilde{L}(f)$ shall denote (13) with this approximation.

The most interesting case seems to be the option $\gamma = 1$, for which in both collision operators the relative speed is approximated by the thermal speed. However, let us note that in this case it is not possible to proceed as above, recovering the moments of all steady states in terms of the ones of the equilibrium with number density equal to 1. In fact, let us look for the suitable α such that $f_\infty(\mathbf{v})$ of the form (11) is a steady state for (12). Since

$$\langle \varphi, \tilde{Q}(g_\infty, g_\infty)\rangle = \rho^{\alpha-2}\,\langle \bar{\varphi}, \tilde{Q}(f_\infty, f_\infty)\rangle$$

and

$$\langle \varphi, \tilde{L}(g_\infty)\rangle = \rho^{\alpha\,\gamma-1}\,\langle \bar{\varphi}, \tilde{L}(f_\infty)\rangle \,,$$

by imposing the equilibrium we get

$$\alpha = \frac{1}{1-\gamma} \,,$$

that unfortunately is not defined for $\gamma = 1$, and moreover is positive only for $\gamma < 1$.

We have therefore to find another strategy to achieve closed sets of hydrodynamic equations corresponding to the physical option $\gamma = 1$. It is well known that for a granular material in a host medium the unique macroscopic quantity that is conserved during the evolution is the number density:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0\,, \tag{14}$$

and to close this equation it suffices to express the mass velocity $\mathbf{u}$ in terms of n and of the background parameters. Taking the weak forms (10) and (13) (with the Maxwellian approximation) corresponding to the weight function $\varphi(\mathbf{v}) = \mathbf{v}$, we get

$$\langle \mathbf{v}, \tilde{Q}(f,f)\rangle = 0\,, \qquad \langle \mathbf{v}, \tilde{L}(f)\rangle = -\,\alpha\,\frac{1+e_B}{2\,\lambda}\,B_b\sqrt{T}\,\rho^B \rho(\mathbf{u}-\mathbf{u}^B)\,,$$

therefore at the equilibrium we have $\mathbf{u} = \mathbf{u}^B$. Equation (14) is then, at the Euler level, a quite trivial drift–diffusion equation at the background mean velocity. Thus, we will apply a typical closure method for hydrodynamic equations to the balance laws of macroscopic moments up to the temperature equation.

Macroscopic equations for ρ, $\mathbf{u}$, T may be cast in convective form as

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\big(\rho\, u_i\big) = 0\,,\\
&\rho\frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{m}\,\frac{\partial \rho T}{\partial x_i} + \frac{1}{m}\,\frac{\partial p_{ij}}{\partial x_j} = -\,\alpha\,\frac{1+e_B}{2\,\lambda}\,B_b\sqrt{T}\,\rho^B\rho(\mathbf{u}-\mathbf{u}^B)\,,\\
&\frac{3}{2}\,\rho\frac{\partial T}{\partial t} + \frac{3}{2}\,\rho u_i \frac{\partial T}{\partial x_i} + \rho T\frac{\partial u_i}{\partial x_i} + p_{ij}\,\frac{\partial u_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i} = \langle \frac{1}{2}\,mc^2, \tilde{Q}(f,f)\rangle + \langle \frac{1}{2}\,mc^2, \tilde{L}(f)\rangle
\end{aligned} \tag{15}$$

where only the first equation represents a conservation law, and the complete set is not closed because of the presence, in the streaming part, of the viscous stress $\mathbf{p}$ and of the heat flux $\mathbf{q}$.

We may close the system by replacing p_{ij} and q_i by their equilibrium values, obtained imposing that

$$\left\langle \left(c_i c_i - \frac{1}{3}\,c^2\delta_{ij}\right), \tilde{Q}(f_\infty, f_\infty)\right\rangle + \left\langle \left(c_i c_i - \frac{1}{3}\,c^2\delta_{ij}\right), \tilde{L}(f_\infty)\right\rangle = 0 \tag{16}$$

$$\langle c_i\, c^2, \tilde{Q}(f_\infty, f_\infty)\rangle + \langle c_i\, c^2, \tilde{L}(f_\infty)\rangle = 0\,. \tag{17}$$

The evaluation of the required collision moments involve long, even if quite standard, calculations, that for brevity we do not report in detail. We remark only that all integrals over the unit vector $\hat{\sigma}$ are amenable to the following ones:

$$\int_{S^2} d\hat{\sigma} = 4\,\pi\,, \qquad \int_{S^2} \sigma_i\, d\hat{\sigma} = 0\,, \qquad \int_{S^2} \sigma_i\,\sigma_j\, d\hat{\sigma} = \frac{4}{3}\,\pi\,\delta_{ij}\,.$$

The second order moment of the quadratic operator results in

$$m\langle c_i\, c_j, \tilde{Q}(f,f)\rangle = -\,\frac{(3-e)(1+e)}{8}\,B\sqrt{T}\,\rho\, p_{ij} - \frac{1-e^2}{4}\,B\sqrt{T}\,\rho^2\, T\,\delta_{ij}\,,$$

and consequently, since the viscous stress tensor is traceless,

$$m\langle c^2, \tilde{Q}(f,f)\rangle = -\,\frac{3}{4}\,(1-e^2)\,B\sqrt{T}\,\rho^2\, T\,,$$

so that

$$m\left\langle \left(c_i c_j - \frac{1}{3}\,c^2\delta_{ij}\right), \tilde{Q}(f,f)\right\rangle = -\,\frac{(3-e)(1+e)}{8}\,B\sqrt{T}\,\rho\, p_{ij}\,.$$

As concerns the third order moment, the final result is

$$\frac{1}{2}\,m\langle c_i\,c^2,\tilde{Q}(f,f)\rangle = -\,\frac{(1+e)(11-7e)}{24}\,B\,\sqrt{T}\,\rho\,q_i\,.$$

A bit more involved (and also new in the literature, to our knowledge), is the evaluation of contributions corresponding to the linear inelastic operator. They shall contain also the background macroscopic moments, including its viscous stress and heat flux, since here we are not assuming the host medium accommodated at a Maxwellian shape. Skipping all intermediate details, we have

$$\begin{aligned} m\langle c_i\,c_j,\tilde{L}(f)\rangle = &-\,\alpha\,(1+e_B)\frac{B_b}{\lambda}\,\sqrt{T}\Big(\rho^B p_{ij} + \rho\rho^B T\,\delta_{ij}\Big) + \alpha^2\,\frac{(1+e_B)^2}{4}\,\frac{B_b}{\lambda}\,\sqrt{T}\Big[\rho^B p_{ij} \\ &+\,\rho\rho^B T\,\delta_{ij} + \frac{1-\alpha}{\alpha}\,\rho p^B_{ij} + \frac{1-\alpha}{\alpha}\,\rho\rho^B T^B\,\delta_{ij} + \rho\rho^B\left(T + \frac{1-\alpha}{\alpha}\,T^B\right)\delta_{ij} \\ &+\frac{1}{3}\,m\rho\rho^B(\mathbf{u}-\mathbf{u}^B)^2\delta_{ij} + m\rho\rho^B(u_i-u_i^B)(u_j-u_j^B)\Big]\,, \end{aligned}$$

and consequently

$$m\langle c^2,\tilde{L}(f)\rangle = -\,\alpha\,(1+e_B)\frac{B_b}{\lambda}\,\sqrt{T}\,\rho\rho^B\left[3T - \alpha\,\frac{1+e_B}{2}\left(3T + 3\,\frac{1-\alpha}{\alpha}\,T^B + m(\mathbf{u}-\mathbf{u}^B)^2\right)\right]$$

so that

$$\begin{aligned} m\left\langle\left(c_ic_j - \frac{1}{3}\,c^2\delta_{ij}\right),\tilde{L}(f)\right\rangle = &-\,\alpha\,(1+e_B)\frac{B_b}{\lambda}\,\sqrt{T}\left\{\rho^B p_{ij} - \alpha\,\frac{1+e_B}{4}\left[\rho^B p_{ij} + \frac{1-\alpha}{\alpha}\,\rho p^B_{ij}\right.\right. \\ &\left.\left.+\rho\rho^B\left(m(u_i-u_i^B)(u_j-u_j^B) - \frac{1}{3}\,m(\mathbf{u}-\mathbf{u}^B)^2\delta_{ij}\right)\right]\right\}. \end{aligned}$$

The third order moment results in

$$\begin{aligned} \frac{1}{2}\,m\langle c_i\,c^2,\tilde{L}(f)\rangle = \frac{B_b}{\lambda}\,\sqrt{T}\,\alpha(1+e_B)\Bigg\{&\left[-\,\frac{3}{2} + \frac{7}{6}\,\alpha(1+e_B) - \frac{1}{3}\,\alpha^2(1+e_B)^2\right]\rho^B q_i \\ &+\,\frac{1}{3}\,\alpha^2(1+e_B)^2\,\frac{m}{m^B}\,\rho q^B_i - \frac{1}{3}\,\alpha^2(1+e_B)^2\frac{m}{m^B}\,\rho p^B_{ij}(u_j-u_j^B) \\ &+\left[-\,\frac{1}{2} + \frac{11}{12}\,\alpha(1+e_B) - \frac{1}{3}\,\alpha^2(1+e_B)^2\right]\rho^B p_{ij}(u_j-u_j^B) \\ &+\left[-\,\frac{5}{4} + \frac{5}{3}\,\alpha(1+e_B) - \frac{5}{6}\,\alpha^2(1+e_B)^2\right]\rho\rho^B T(u_i-u_i^B) \\ &-\,\frac{5}{6}\,\alpha^2(1+e_B)^2\frac{m}{m^B}\,\rho\rho^B T^B(u_i-u_i^B) \\ &-\frac{1}{6}\,\alpha^2(1+e_B)^2 m\rho\rho^B(\mathbf{u}-\mathbf{u}^B)^2(u_i-u_i^B)\Bigg\}. \end{aligned}$$

First of all, notice that collision contribution appearing in the third of (15) involves only the moments ρ, $\mathbf{u}$, T:

$$\begin{aligned} \left\langle\frac{1}{2}\,mc^2,\tilde{Q}(f,f)\right\rangle + \left\langle\frac{1}{2}\,mc^2,\tilde{L}(f)\right\rangle = &-\,\frac{3}{8}\,(1-e^2)\,B\,\sqrt{T}\,\rho^2\,T \\ &-\,\frac{1}{2}\,\alpha\,(1+e_B)\frac{B_b}{\lambda}\,\sqrt{T}\,\rho\rho^B\left[3T - \alpha\,\frac{1+e_B}{2}\left(3T + 3\,\frac{1-\alpha}{\alpha}\,T^B + m(\mathbf{u}-\mathbf{u}^B)^2\right)\right]. \end{aligned}$$

Bearing in mind that $\mathbf{u}_\infty = \mathbf{u}^B$ at the equilibrium state, it follows that the steady state temperature is given by

$$T_\infty = \frac{(1-\alpha)\,\dfrac{1+e_B}{2}}{1-\alpha\,\dfrac{1+e_B}{2} + \dfrac{B\,\lambda}{B_b}\,\dfrac{1}{4\alpha}\left(\dfrac{1-e^2}{1+e_B}\right)\dfrac{\rho}{\rho^B}}\; T^B\,,$$

that in the particular case $e = 1$ (elastic quadratic operator) reproduces the result of [10].

By imposing now the constraints (16) we get closed expressions for p_{ij} and q_i in terms of ρ, $\mathbf{u}$, T and of the background parameters:

$$\begin{aligned}&\left[\frac{(3-e)(1+e)}{8}\,B\rho + \alpha(1+e_B)\frac{B_b}{\lambda}\left(1-\alpha\,\frac{1+e_B}{4}\right)\rho^B\right]p_{ij} = \\ &\qquad \alpha\left(\frac{1+e_B}{2}\right)^4\frac{B_b}{\lambda}\,\rho\left\{(1-\alpha)p^B_{ij} + \alpha\,m\rho^B\left[(u_i-u^B_i)(u_j-u^B_j) - \frac{1}{3}(\mathbf{u}-\mathbf{u}^B)^2\delta_{ij}\right]\right\}\end{aligned} \tag{18}$$

and

$$\begin{aligned}&\left\{\frac{(1+e)(11-7e)}{24}\,B\rho + \frac{B_b}{\lambda}\,\alpha(1+e_B)\left[\frac{1}{3}\,\alpha^2(1+e_B)^2 - \frac{7}{6}\,\alpha(1+e_B) + \frac{3}{2}\right]\rho^B\right\}q_i = \\ &\qquad \frac{B_b}{\lambda}\,\alpha^3(1+e_B)^3\left[\frac{1}{3}\,\frac{m}{m^B}\,\rho q^B_i - \frac{1}{3}\,\frac{m}{m^B}\,\rho p^B_{ij}(u_j-u^B_j) - \frac{5}{6}\,\frac{m}{m^B}\,\rho\rho^B T^B(u_i-u^B_i)\right. \\ &\qquad \left. -\frac{1}{6}\,m\rho\rho^B(u_i-u^B_i)(\mathbf{u}-\mathbf{u}^B)^2\right] + \frac{B_b}{\lambda}\,\alpha(1+e_B)\left\{\left[-\frac{1}{3}\alpha^2(1+e_B)^2 + \frac{11}{12}\,\alpha(1+e_B)\right.\right. \\ &\qquad \left.\left. -\frac{1}{2}\right]\rho^B p_{ij}(u_j-u^B_j) + \left[-\frac{5}{6}\,\alpha^2(1+e_B)^2 + \frac{5}{3}\,\alpha(1+e_B) - \frac{5}{4}\right]\rho\rho^B T(u_i-u^B_i)\right\}.\end{aligned} \tag{19}$$

By inserting (18) and (19) into the streaming part of equations (15), we obtain an approximated closed set of evolution equations for ρ, $\mathbf{u}$, T.

Unfortunately expressions (18) and (19) are quite complicated, but become more manageable for some particular asymptotic scalings. Let ε be the relevant pertinent small parameter.

In conditions not far from thermodynamical equilibrium (for both gas and background), with ε standing for Knudsen number, it seems reasonable to suppose that $\mathbf{u} - \mathbf{u}^B = O(\varepsilon)$, and that simultaneously $\mathbf{p}^B = O(\varepsilon)$ and $\mathbf{q}^B = O(\varepsilon)$. Under these assumptions also p_{ij} and q_i are $O(\varepsilon)$ and, neglecting $O(\varepsilon^2)$ terms, equations (18) and (19) reduce to:

$$\begin{aligned}&\left[\frac{(3-e)(1+e)}{8}\,B\rho + \alpha(1+e_B)\frac{B_b}{\lambda}\left(1-\alpha\,\frac{1+e_B}{4}\right)\rho^B\right]p_{ij} = \alpha\left(\frac{1+e_B}{2}\right)^4\frac{B_b}{\lambda}\,(1-\alpha)\,\rho\,p^B_{ij}\,, \\ &\left\{\frac{(1+e)(11-7e)}{24}\,B\rho + \frac{B_b}{\lambda}\,\alpha(1+e_B)\left[\frac{1}{3}\,\alpha^2(1+e_B)^2 - \frac{7}{6}\,\alpha(1+e_B) + \frac{3}{2}\right]\rho^B\right\}q_i = \\ &\qquad\qquad \frac{B_b}{\lambda}\,\alpha^3(1+e_B)^3\,\frac{1}{3}\,\frac{m}{m^B}\,\rho q^B_i - \frac{B_b}{\lambda}\,\alpha^3(1+e_B)^3\,\frac{5}{6}\,\frac{m}{m^B}\,\rho\rho^B T^B(u_i-u^B_i) \\ &\qquad\qquad + \frac{B_b}{\lambda}\,\alpha(1+e_B)\left[-\frac{5}{6}\,\alpha^2(1+e_B)^2 + \frac{5}{3}\,\alpha(1+e_B) - \frac{5}{4}\right]\rho\rho^B T(u_i-u^B_i)\,.\end{aligned}$$

Another simplification may arise from the fact that, in physical problems like the evolution of a granular powder in the atmosphere, grains mass is much higher than field particle mass, namely $m \gg m^B$, so that we may assume α itself as small parameter. Therefore the constitutive equations for p_{ij} and q_i take, at first order accuracy in $\varepsilon = \alpha$, very simple expressions:

$$\left[\frac{(3-e)(1+e)}{8}\,B\rho + \alpha(1+e_B)\frac{B_b}{\lambda}\rho^B\right]p_{ij} = \alpha\left(\frac{1+e_B}{2}\right)^4\frac{B_b}{\lambda}\,\rho\,p^B_{ij}\,,$$

$$\left[\frac{(1+e)(11-7e)}{24}\,B\rho + \frac{3}{2}\frac{B_b}{\lambda}\,\alpha(1+e_B)\,\rho^B\right] q_i = -\,\frac{5}{4}\,\frac{B_b}{\lambda}\,\alpha(1+e_B)\,\rho\rho^B T(u_i - u_i^B)\,.$$

Though these last two truncations are much in the spirit of the Chapman–Enskog expansion (leading to Navier–Stokes–type equations), the achieved constitutive equations express p_{ij} and q_i in terms of the background properties and of the fundamental fields, not of their gradients.

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