A NUMERICAL ESTIMATE OF THE REGULARITY OF A FAMILY OF STRANGE NON–CHAOTIC ATTRACTORS

LLUÍS ALSEDÀ I SOLER
JOSEP MARIA MONDELO GONZÁLEZ
AND
DAVID ROMERO I SÀNCHEZ.

Abstract. We estimate numerically the regularities of a two parameter family of Strange Non–Chaotic Attractors related with one of the models studied in [GOPY84] (see also [Kel96]). To estimate these regularities we use wavelet analysis in the spirit of [dlLP02] together with some ad-hoc techniques that we develop to overcome the theoretical difficulties that arise in the application of the method to the particular family that we consider. These difficulties are mainly due to the facts that we do not have an explicit formula for the attractor and it is discontinuous almost everywhere for some values of the parameters.

1. Introduction

In [GOPY84], a quasiperiodically forced skew product on the cylinder was studied and the term *Strange Non–Chaotic Attractor (SNA)* was coined to denote the attractor found in this system. The strangeness of the attractor refers to complicated geometry. Indeed, the attractor obtained in [GOPY84] (as shown by Keller in [Kel96]), is the graph of an upper semi-continuous function from the circle to $\mathbb{R}$ in the *pinched case* (that is, when there exists a fibre whose image is degenerate to a point), whereas in the non pinched one the attractor is the graph of a map with the same regularity as the skew product (see also [Sta99] and [Sta97]).

A classical tool to measure the degree of differentiability of a function (and to define certain functional spaces) is the study of the asymptotic decay of the Fourier Transform. However, this is a tool that is not well adapted to analyse highly irregular functions. In that case the use of wavelet analysis seems a better technique.

In [dlLP02], the authors make numerical implementations of wavelet analysis to estimate the *positive* regularity of invariant objects which are graphs of functions in appropriate spaces. However, due to the complexity of the SNAs described above we need to consider the possibility that these objects have zero or even negative regularity (see [Coh03]). Hence, the techniques of [dlLP02] need to be extended to this case. To this end, we develop ad-hoc techniques to overcome the theoretical difficulties of the objects we study.

This paper is organized as follows. In Section 2 we state the problem. In particular we state Keller Theorem and we remark crucial aspects of its proof that will be used later. Section 3 is devoted to review the notion of regularity through Besov functional spaces and discuss it by means of a particular simple example. In

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Besov spaces the regularity can be any real number (in contrast to Hölder regularity defined only for positive regularities). In Section 4 we recall some facts about the theory of wavelet bases and in Section 5 we review the relation between the regularity and the wavelet coefficients of a function. Section 6 is devoted to present and test an algorithm to estimate regularities numerically. In Section 7, this algorithm is adapted to compute the regularity of the attractors of the family that we consider. Finally, in Section 8, the results of this computation are presented and discussed.

2. Statement of the problem

We want to estimate the regularities of a family of Strange Non-Chaotic Attractors that occur in quasiperiodically forced skew products on the cylinder. More precisely, these systems are defined on the Cartesian product of the circle $\mathbb{S}^1 = \mathbb{R} \backslash \mathbb{Z}$ and $\mathbb{R}^+ = [0, \infty)$, and are of the type

$$\begin{pmatrix} \theta_{n+1} \\ x_{n+1} \end{pmatrix} = \mathfrak{F}(\theta_n, x_n) = \begin{pmatrix} R_\omega(\theta_n) \\ f(x_n) g(\theta_n) \end{pmatrix}, \tag{1}$$

where $(\theta_n, x_n) \in \mathbb{S}^1 \times \mathbb{R}^+$, $R_\omega(\theta_n) = \theta_n + \omega \pmod 1$ and $\omega \in \mathbb{R} \backslash \mathbb{Q}$. On the second component, the map $g \colon \mathbb{S}^1 \longrightarrow [0, \infty)$ is continuous (hence bounded—for example $|\cos(2\pi\theta)|$) and the map $f \colon [0, \infty) \longrightarrow [0, \infty)$ is $\mathcal{C}^1$, bounded, increasing, strictly concave and such that $f(0) = 0$ (e.g. $\tanh(x)\big|_{\mathbb{R}^+}$). Observe that, since $f(0) = 0$, the circle $x \equiv 0$ is invariant.

Recall that the vertical Lyapunov Exponent at a point (θ_0, x_0) is defined by

$$\limsup_{n\to\infty} \frac{1}{n} \log \left\| \begin{pmatrix} 1 & 0 \\ \dfrac{\partial x_n}{\partial \theta} & \dfrac{\partial x_n}{\partial x} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = \limsup_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x} \right|.$$

Therefore, by using Birkhoff Ergodic Theorem, it can be shown that the Lyapunov Exponent at $x \equiv 0$ is

$$\sigma(f, g) := \int_{\mathbb{S}^1} \log \left| \left. \frac{\partial f(x) g(\theta)}{\partial x} \right|_{x=0} \right| d\theta = \log(f'(0)) + \int_{\mathbb{S}^1} \log |g(\theta)| \, d\theta.$$

When $\sigma(f, g)$ is positive, $x \equiv 0$ is a repellor for System (1). Moreover, since f and g are bounded, infinity is also a repellor and the system must have an attractor different from $x \equiv 0$. These attractors, which are the objects that we want to study, are typically very complicated.

We are going to restrict ourselves to the study of a particular subfamily of Model (1), which is

$$\begin{pmatrix} \theta_{n+1} \\ x_{n+1} \end{pmatrix} = \mathfrak{F}(\theta_n, x_n) = \begin{pmatrix} R_\omega(\theta_n) \\ 2\sigma \tanh(x) \cdot (\varepsilon + |\cos(2\pi\theta)|) \end{pmatrix}, \tag{2}$$

with $\omega = \frac{1+\sqrt{5}}{2}$, $\sigma > 0$ and $\varepsilon \geq 0$. Apart from the parameter ε, it is the natural restriction to $\mathbb{R}^+$ of the system considered in [GOPY84] (see Figure 1, where a graph of the attractor of this system with $\sigma = 1.5$ and $\varepsilon = 0$ is shown). In this case, the Lyapunov Exponent $\sigma(f, g)$ at $x \equiv 0$ is precisely $\log(\sigma)$. Hence, the interesting case (for us) occurs when $\sigma > 1$.

The attractor of System (1) and its dynamics is described by the following theorem:

Theorem 2.1 (G. Keller [Kel96])**.** *There exists an upper semi continuous function $\varphi \colon \mathbb{S}^1 \longrightarrow \mathbb{R}^+$ whose graph is invariant under System (1) and satisfies*

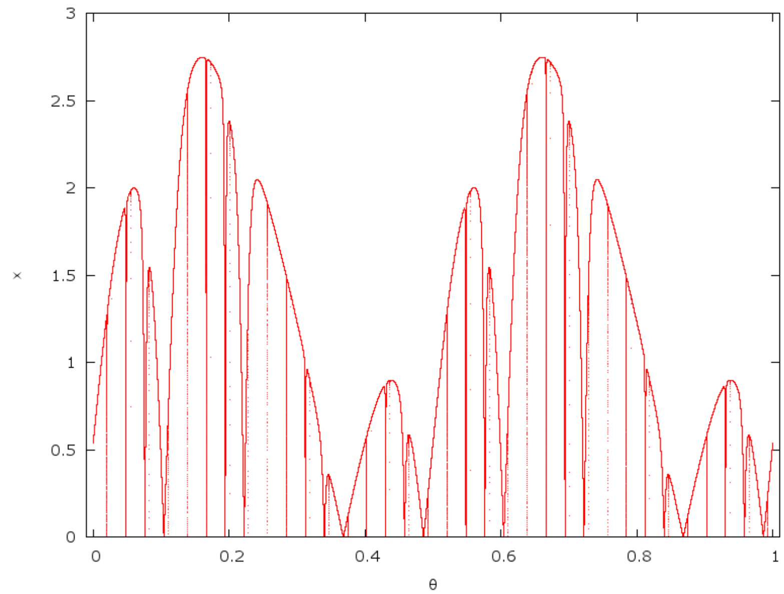

FIGURE 1. The attractor of System (2) for $\sigma = 1.5$ and $\varepsilon = 0$. Notice the abrupt changes in the graph of the attractor.

(a) The Lebesgue measure on the circle, lifted to the graph of φ *is a Sinai-Ruelle-Bowen measure (that is,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\mathfrak{F}^k(\theta, x)) = \int_{\mathbb{S}^1} f(\theta, \varphi(\theta))\ d\theta$$

for every $f \in \mathcal{C}^0(\mathbb{S}^1 \times \mathbb{R}^+, \mathbb{R})$ *and Lebesgue almost every* $(\theta, x) \in \mathbb{S}^1 \times \mathbb{R}^+$*),*

(b) if $\sigma(f, g) \le 0$ *then* $\varphi \equiv 0$*,*

(c) if $\sigma(f, g) > 0$ *then* $\varphi(\theta) > 0$ *for almost every* θ*,*

(d) if $\sigma(f, g) > 0$ *and* g *vanishes at some point then the set* $\{\theta \in \mathbb{S}^1 : \varphi(\theta) > 0\}$ *is meager and* φ *is almost everywhere discontinuous,*

(e) if $\sigma(f, g) > 0$ *and* $g > 0$ *then* φ *is positive and continuous; if* $g \in \mathcal{C}^1$ *then so is* φ*,*

(f) if $\sigma(f, g) \ne 0$ *then* $|x_n - \varphi(\theta_n)| \to 0$ *exponentially fast for almost every* θ *and every* $x > 0$*.*

Observe that, when $\sigma(f, g) > 0$ and g vanishes at some point (i.e. when the system is *pinched*), it follows from statements (c,d) that φ is discontinuous almost everywhere. In the particular case of System (2), the pinching condition implies that $\varepsilon = 0$ and, since $|\cos(2\pi\theta)|$ vanishes for $\theta \in \{\frac{1}{4}, \frac{3}{4}\}$, it follows that the set

$$\left\{(\tfrac{i}{4} + n\omega \pmod 1, 0)\ :\ n \in \mathbb{N},\ i \in \{1, 3\}\right\} \tag{3}$$

is both a subset of the attractor and is dense (and invariant) in $x \equiv 0$.

The proof of the above theorem is based on the iteration of the *Transfer Operator* of the system. Since in Section 6 we will use this construction let us briefly explain it. Let $\mathscr{P}$ be the space of all functions (not necessarily continuous) from $\mathbb{S}^1$ to $\mathbb{R}$. If we look for a functional version of the System (1) in the space $\mathscr{P}$ one can define the *Transfer Operator* $\mathfrak{T}\colon \mathscr{P} \longrightarrow \mathscr{P}$ as

$$\mathfrak{T}(\varphi)(\theta) = f(\varphi(R_\omega^{-1}(\theta))) \cdot g(R_\omega^{-1}(\theta)).$$

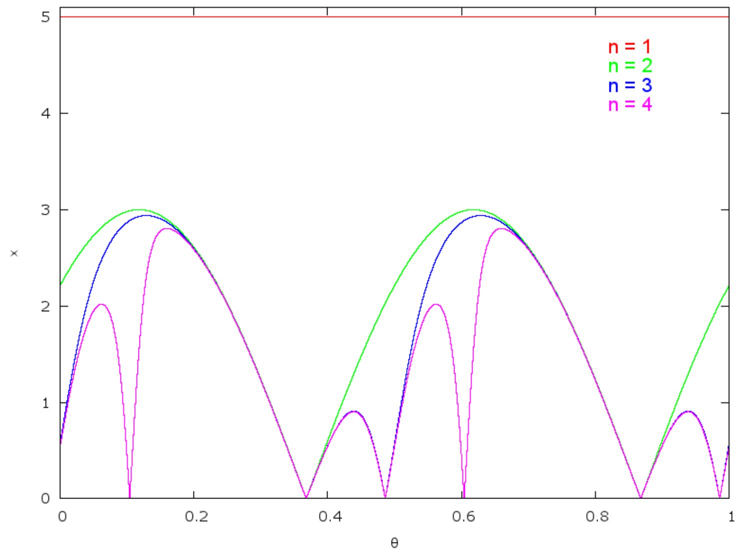

FIGURE 2. The constant function $c = 5$ and three iterations of the Transfer Operator $\mathfrak{T}$ for System (2) with $\sigma = 1.5$ and $\varepsilon = 0$. The function c is plotted in red, $\mathfrak{T}(c)$ in green, $\mathfrak{T}^2(c)$ in blue and $\mathfrak{T}^3(c)$ in magenta.

Remark 2.2. From the above definition we obtain

$$\mathfrak{T}(\varphi)(\theta) = \pi_x \left(\mathfrak{F}(R_\omega^{-1}(\theta), \varphi(R_\omega^{-1}(\theta)))\right)$$

where $\pi_x \colon \mathbb{S}^1 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ denotes the projection with respect to the second component. ∎

Notice that the graph of a function $\varphi \colon \mathbb{S}^1 \longrightarrow \mathbb{R}$ is invariant for the System (1) if and only if $\mathfrak{T}(\varphi) = \varphi$.

To obtain the map φ from Theorem 2.1, Keller takes a sufficiently large constant function $\varphi_0 = c$ (with $c > (\sup_{x\in\mathbb{R}} f(x)) \left(\max_{\theta\in[0,1]} g(\theta)\right)$) and iterates it under the transfer operator $\mathfrak{T}$ (see Figure 2). In such a way he gets, since the map f is monotone, a non-increasing sequence of continuous maps $\varphi_k = \mathfrak{T}(\varphi_{k-1}) = \mathfrak{T}^k(c)$. Then,

$$\varphi := \lim_{k\to\infty} \varphi_k = \inf_{k\to\infty} \varphi_k.$$

This idea will be a key point in our algorithm for the estimation of the regularity of the two parameter family of Strange Non–Chaotic Attractors given by System (2).

3. Defining regularity through Besov spaces

In this section we will describe, in two steps, the functional spaces that define the notion of regularity. As we will see, in estimating the regularity of the attractor of System (2), we have to deal with non-positive regularities. The framework to define these regularity values is given by the Besov spaces (see [Tri83, BL76]). First, following [Tri83], we start by defining Besov spaces on the real line. Next we will recall the extension of such definition to $\mathbb{S}^1$.

3.1. The spaces $\mathscr{B}^s_{\infty,\infty}(\mathbb{R})$.

The space of all real valued rapidly decreasing infinitely differentiable functions is called the *(real) Schwartz space* and it is denoted by $\mathcal{S}(\mathbb{R})$. The topological dual of $\mathcal{S}(\mathbb{R})$ is the space of *tempered distributions* which it is denoted by $\mathcal{S}'(\mathbb{R})$. For $f \in \mathcal{S}'(\mathbb{R})$, $\widehat{f}(\xi)$ denotes the *Fourier transform* of f and $f^\vee(x)$ stands for the *inverse Fourier transform*.

Let $\varphi_0 \in \mathcal{S}(\mathbb{R})$ be such that

$$\varphi_0(x) := \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 3/2 \end{cases}$$

and set

$$\varphi_j(x) := \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$$

for $j \in \mathbb{N}$. It is not difficult to show that, independently of the choice of φ_0, $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}$. Each of the families $\{\varphi_j\}_{j=0}^{\infty}$ is called a *Dyadic Resolution of Unity* in $\mathbb{R}$.

Definition 3.1. Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be a Dyadic Resolution of Unity and $s \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R})$ we define the quasi-norm

$$\|f\|_{\infty,\infty,\varphi,s} = \sup_{j\ge 0} 2^{js} \left(\sup_{x\in\mathbb{R}} \left| (\varphi_j \widehat{f})^{\vee}(x) \right| \right).$$

Then, we define the *Besov Spaces* by

$$\mathscr{B}^s_{\infty,\infty}(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) : \ \|f\|_{\infty,\infty,\varphi,s} < \infty \right\}.$$

■

As it can be seen in [Tri83], the spaces $\mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ are, in fact, independent of the chosen dyadic resolution of unity φ. Therefore, we can remove the subscript φ from $\|f\|_{\infty,\infty,\varphi,s}$. So, in what follows we will write $\|f\|_{\infty,\infty,s}$ instead of $\|f\|_{\infty,\infty,\varphi,s}$. The spaces $\mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ are a particular case of the Generalized Besov Spaces $\mathscr{B}^s_{p,q}(\mathbb{R})$ defined also, for example, in [Tri83] and one has the inclusion property. That is, if $s < s'$, then $\mathscr{B}^{s'}_{p,q}(\mathbb{R}) \subset \mathscr{B}^s_{p,q}(\mathbb{R})$.

For $s > 0$, the spaces $\mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ coincide with the Hölder (or Lipschitz) spaces and it is natural to extend the notion of regularity to $s \le 0$ through $\mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ in the above way (we refer to [Tri83, Ste70] for a more complete explanation).

Definition 3.2. We say that a map f *has regularity* $s \in \mathbb{R}$ if $f \in \mathscr{B}^s_{\infty,\infty}(\mathbb{R})$. ■

The following two examples help to clarify this regularity notion.

- Any continuous non-differentiable function belongs to $\mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ with $s \in (0,1)$. In particular, the *Weierstraß function*

$$\mathfrak{W}_{A,B}(x) := \sum_{n=1}^{\infty} A^n \sin(B^n x), \tag{4}$$

  where $A, B \in \mathbb{R}$ are such that $B^{-1} < A < 1 < B$, has regularity $-\log_B(A)$; that is $\mathfrak{W}_{A,B} \in \mathscr{B}^{-\log_B(A)}_{\infty,\infty}(\mathbb{R})$.
- $\delta(x) \in \mathscr{B}^{-1}_{\infty,\infty}(\mathbb{R})$ where $\delta(x)$ stands for Dirac's delta.

3.2. **The Besov spaces on $\mathbb{S}^1$.** In this section we will extend the Besov spaces to $\mathbb{S}^1$. To do it we follow [BL76, Tri92]. Indeed, given $f \in \mathcal{S}'(\mathbb{S}^1)$ (the space of tempered distributions on $\mathbb{S}^1$) it is known that

$$f = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{inx}.$$

Definition 3.3. Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be a dyadic resolution of unity (on $\mathbb{R}$). We define the *Besov Spaces* on $\mathbb{S}^1$ by

$$\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1) := \left\{ f \in \mathcal{S}'(\mathbb{S}^1) : \ \|f\|_{\infty,\infty,s} < \infty \right\}$$

where

$$\|f\|_{\infty,\infty,s} = \sup_{j\geq 0} 2^{js}\left(\sup_{x\in\mathbb{R}}\left|\sum_{n\in\mathbb{Z}}\varphi_j(n)\widehat{f}(n)e^{inx}\right|\right)$$

is a quasi-norm for the quasi-Banach space $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$. ∎

As in Definition 3.2 we say that a circle map f *has regularity* $s \in \mathbb{R}$ if $f \in \mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$.

The following lemma shows that the regularity of a circle map coincides with the regularity of its real extension which we define as follows. Given $f \in \mathcal{S}'(\mathbb{S}^1)$ there exists a unique $f^{\text{PER}} \in \mathcal{S}'(\mathbb{R})$ such that f^{PER} is 1-periodic and the restriction of f^{PER} over $[0, 1)$ coincides with f (such an f^{PER} can be defined as $f(\{\cdot\})$, where $\{\cdot\}$ denotes the fractional part function). This lemma is usually omitted and used implicitly but we include here for completeness.

Lemma 3.4. *For every $f \in \mathcal{S}'(\mathbb{S}^1)$ it follows that $f^{\text{PER}} \in \mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ if and only if $f \in \mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$.*

Proof. Since f^{PER} is 1-periodic and $f^{\text{PER}}\big|_{[0,1]} = f$,

$$\widehat{f^{\text{PER}}}(n) = \int_0^1 f^{\text{PER}}(x)e^{-inx}\,dx = \int_0^1 f(x)e^{-inx}\,dx = \widehat{f}(n).$$

Hence,

$$\begin{aligned}\sup_{x\in\mathbb{R}}\left|\sum_{n\in\mathbb{Z}}\varphi_j(n)\widehat{f}(n)e^{inx}\right| &= \sup_{x\in\mathbb{R}}\left|\sum_{n\in\mathbb{Z}}\varphi_j(n)\widehat{f^{\text{PER}}}(n)e^{inx}\right| \\ &= \sup_{x\in\mathbb{R}}\left|\sum_{n\in\mathbb{Z}}(\varphi_j\widehat{f^{\text{PER}}})(n)e^{inx}\right| = \sup_{x\in\mathbb{R}}\left|(\varphi_j\widehat{f^{\text{PER}}})^{\vee}\right|.\end{aligned}$$

That is, $\|f^{\text{PER}}\|_{\infty,\infty,s} = \|f\|_{\infty,\infty,s}$ and, hence, $f^{\text{PER}} \in \mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ if and only if $f \in \mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$. □

4. Wavelets and wavelet bases in $\mathscr{L}^2(\mathbb{R})$

It is well known that, using Fourier Analysis, a function can be approximated by finite sums of trigonometric polynomials. In a similar way, we want to approximate a function from $\mathscr{L}^2(\mathbb{R})$ by finite wavelet expansions of the type

$$a_0 + \sum_{j=-J}^{0}\sum_{n=0}^{2^{-j}-1}\langle f, \psi_{j,n}\rangle\,\psi_{j,n},$$

where a_0 is a constant and $\psi_{j,n}$ are members of the wavelet basis.

Orthonormal wavelet bases can be constructed in a natural way with the help of the Multiresolution Analysis. We refer the reader to [Mal98, HW96] for more detailed and comprehensive expositions.

Definition 4.1. A sequence of closed subspaces $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ of $\mathscr{L}^2(\mathbb{R})$ is a *Multiresolution Analysis* (or simply a *MRA*) if it satisfies the following six properties:

(a) $\{0\} \subset \cdots \subset \mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1} \subset \cdots \subset \mathscr{L}^2(\mathbb{R})$.
(b) $\{0\} = \bigcap_{j\in\mathbb{Z}}\mathcal{V}_j$.
(c) $\operatorname{clos}\left(\bigcup_{j\in\mathbb{Z}}\mathcal{V}_j\right) = \mathscr{L}^2(\mathbb{R})$.
(d) There exists a function ϕ whose integer translates, $\phi(x-n)$, form an orthonormal bases of $\mathcal{V}_0$. Such function is called the *scaling function.*
(e) For each $j \in \mathbb{Z}$ it follows that $f(x) \in \mathcal{V}_j$ if and only if $f(x - 2^j n) \in \mathcal{V}_j$ for each $n \in \mathbb{Z}$.

(f) For each $j \in \mathbb{Z}$ it follows that $f(x) \in \mathcal{V}_j$ if and only if $f(x/2) \in \mathcal{V}_{j+1}$.

■

If we fix an MRA, we know that $\mathcal{V}_j \subset \mathcal{V}_{j-1}$, for every j, and that $\mathcal{V}_j$ has an orthonormal basis $\{\phi_{j,n}\}_{n\in\mathbb{Z}}$, for every j, where

$$\phi_{j,n}(x) = 2^{-j/2}\phi\left(\frac{x-2^j n}{2^j}\right).$$

Now define the subspace $\mathcal{W}_j$ as the orthogonal complement of $\mathcal{V}_j$ on $\mathcal{V}_{j-1}$. That is,

$$\mathcal{V}_{j-1} = \mathcal{W}_j \oplus \mathcal{V}_j. \tag{5}$$

Therefore, by the inclusion of the spaces $\mathcal{V}_j$ we have

$$\mathscr{L}^2(\mathbb{R}) = \operatorname{clos}\left(\bigoplus_{j\in\mathbb{Z}} \mathcal{W}_j\right) = \operatorname{clos}\left(\mathcal{V}_0 \oplus \bigoplus_{j=-\infty}^{0} \mathcal{W}_j\right). \tag{6}$$

Now, define the *mother wavelet* $\psi \in \mathcal{W}_0$ to be the function whose Fourier transform is

$$\widehat{\psi}(\xi) = \frac{1}{\sqrt{2}} e^{-i\xi}\widehat{h}^*(\xi+\pi)\widehat{\phi}(\xi) \tag{7}$$

where $\widehat{h}^*(\xi)$ is the complex conjugate of

$$\widehat{h}(\xi) = \sum_{n\in\mathbb{Z}} h[n]e^{-in\xi}$$

and $h[n] = \left\langle \frac{1}{\sqrt{2}}\phi\left(\frac{x}{2}\right), \phi(x-n)\right\rangle$. The following theorem allows to obtain the wavelet basis from the scaling function:

Theorem 4.2 (Mallat, Meyer [Mal98]). *The mother wavelet given by Equation* (7) *verifies that, for each integer j, the family $\{\psi_{j,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{W}_j$, where:*

$$\psi_{j,n}(x) = 2^{-j/2}\psi\left(\frac{x-2^j n}{2^j}\right).$$

As a consequence, the family $\{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}\times\mathbb{Z}}$ is an orthonormal basis of $\mathscr{L}^2(\mathbb{R})$.

Thus, taking into account (6) and the above theorem, every map $f \in \mathscr{L}^2(\mathbb{R})$ can be written as

$$\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \langle f, \psi_{j,n}\rangle \psi_{j,n} = \sum_{n\in\mathbb{Z}} \langle f, \phi_{0,n}\rangle \phi_{0,n}(x) + \sum_{j=-\infty}^{0}\sum_{n\in\mathbb{Z}} \langle f, \psi_{j,n}\rangle \psi_{j,n}.$$

Moreover, for every $f \in \mathscr{L}^2(\mathbb{R})$

$$\sum_{n=0}^{2^J-1} \langle f, \phi_{-J,n}\rangle \phi_{-J,n}, \tag{8}$$

is a projection of f into the finite-dimensional subspace of $\mathcal{V}_{-J} \subset \mathscr{L}^2(\mathbb{R})$ generated by $\phi_{-J,0}, \phi_{-J,1}, \ldots, \phi_{-J,2^J-1}$. Furthermore, this projection is a good approximation of f provided that J is large enough.

As we have said we want to approximate maps from $f \in \mathscr{L}^2(\mathbb{R})$ by finite wavelet expansions. To do this we will use as initial data the approximation of f given by (8) and perform an iterative procedure known as the *Fast Wavelet Transform (FWT)*, that allows to rewrite (8) as

$$\sum_{n=0}^{2^J-1} \langle f, \phi_{-J,n}\rangle \phi_{-J,n} = \langle f, \phi_{0,0}\rangle \phi_{0,0} + \sum_{j=-J}^{0}\sum_{n=0}^{2^j-1} \langle f, \psi_{j,n}\rangle \psi_{j,n}.$$

In order to give a formulae for the FWT, denote

$$a_j[n] := \langle f, \phi_{j,n} \rangle \quad \text{and} \quad d_j[n] := \langle f, \psi_{j,n} \rangle .$$

Then, starting with the coefficients $a_{-J}[n]$ given by (8), in view of the decomposition of the spaces $\mathcal{V}_{-j}$ (with $j > 0$) given by (5), we can iteratively use the following decomposition formulas from [Mal98] which define the FWT:

$$\text{(9)} \quad \begin{cases} \displaystyle\sum_{n=0}^{2^j-1} a_{-j}[n]\phi_{-j,n} = \sum_{n=0}^{2^{j-1}-1} a_{-j+1}[n]\phi_{-j+1,n} + \sum_{n=0}^{2^{j-1}-1} d_{-j+1}[n]\psi_{-j+1,n}, \text{ with} \\ \displaystyle a_{j+1}[p] = \sum_{n\in\mathbb{N}} h[n-2p]a_j[n] \quad \text{and} \quad d_{j+1}[p] = \sum_{n\in\mathbb{N}} g[n-2p]a_j[n]; \\ \text{and } g[p] = (-1)^{1-p}h[p] \end{cases}$$

for every $p \in \mathbb{Z}$. As a final outcome of this iterative procedure we obtain

$$f \sim \sum_{n=0}^{2^J-1} a_{-J}[n]\phi_{-J,n} = a_0 + \sum_{j=-J}^{0} \sum_{n=0}^{2^j-1} d_j[n]\psi_{j,n},$$

as we wanted.

One remaining problem is left. Namely, to find a good estimate of the initial coefficients $a_{-J}[n] = \langle f, \phi_{-J,n} \rangle$. In the literature there is a lot of discussion on how to compute these coefficients, but a simple customary approach is to use the following estimate (see, for instance, [Fra99, Lemma 5.54] and its proof):

Lemma 4.3. *Assume that f verifies $|\langle f, \phi_{j,n} \rangle| < \infty$ for every $j, n \in \mathbb{Z} \times \mathbb{Z}$ and*

$$|f(x) - f(y)| \le C_1 |x-y|^\alpha \;\; \textit{with } \alpha \in (0,1]$$

for all real numbers x, y and a constant $C_1 < \infty$. Suppose that the scaling function ϕ from an MRA $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ is such that

$$\phi \in \mathscr{L}^1(\mathbb{R}), \;\; \widehat{\phi}(0) = \int_{\mathbb{R}} \phi(x)\, dx = 1 \;\textit{and}\; \int_{\mathbb{R}} |x|^\alpha\, \phi(x)\, dx < C_2.$$

Then, for every $j, n \in \mathbb{Z} \times \mathbb{Z}$,

$$\left| \langle f, \phi_{j,n} \rangle - 2^{-j/2} f(2^{-j}n) \right| < C_1 C_2 2^{-j\left(\alpha + \frac{1}{2}\right)}.$$

As a corollary of this lemma we see that if f is Lipschitz, then

$$a_{-J}[n] \approx 2^{-J/2} f(2^{-J}n).$$

5. Wavelets and regularity

In Section 3 we have recalled the notion of the regularity of a function through the spaces $\mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ and $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$. Also, we have introduced the wavelet expansions of a given function in $\mathscr{L}^2(\mathbb{R})$. Next, we want to show the relationship between this notion of regularity and the wavelet coefficients. The main tool for this will be the Daubechies wavelets, because they are orthonormal bases on $\mathscr{L}^2(\mathbb{R})$ (see [Mal98]) and, depending on the number of vanishing moments, they are well adapted to the functional spaces $\mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ (see [HW96, Tri06]).

Definition 5.1. Let ψ be a wavelet from a MRA $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$. We say that ψ has *k-vanishing moments* if the integer k is the minimum non-negative integer such that

$$\int_{\mathbb{R}} x^p \psi(x)\, dx = 0 \text{ for } 0 \le p < k.$$

∎

Daubechies wavelets are a family of wavelets with compact support that has an element with k vanishing moments for each $k \geq 1$ (see [Mal98] for a definition and construction). From [HW96, Coh03, Tri06] we can state the following theorem, where $\operatorname{sign}(\cdot)$ stands for the sign function.

Theorem 5.2. *Let $s \in \mathbb{R} \setminus \{0\}$ and let ψ be a Daubechies wavelet with more than $\max(s, 5/2 - s)$ vanishing moments. Then $f \in \mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ if and only if*

$$\sup_{n \in \mathbb{Z}} |\langle f, \psi_{j,n} \rangle| \leq C 2^{(s+\operatorname{sign}(s)/2)j}$$

for all $j \leq 0$ and some $C > 0$.

Remark 5.3. Since $\mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ satisfy the inclusion property described above one can derive the case $s = 0$ from the above theorem. Indeed, $f \in \mathscr{B}^0_{\infty,\infty}(\mathbb{R})$ if and only if

$$C 2^{-|j|/2} \leq \sup_{n \in \mathbb{Z}} |\langle f, \psi_{j,n} \rangle| \leq C 2^{|j|/2}$$

for all $j \leq 0$ and some $C > 0$. ■

Remark 5.4. In view of Lemma 3.4, to estimate the regularity of an $f \in \mathcal{S}'(\mathbb{S}^1)$ it is enough to use Theorem 5.2 and Remark 5.3 for f^{PER}. ■

Remark 5.5. Theorem 5.2 does not determine the value of the regularity s (it just gives an upper bound of it). However, from [Mal98] we know that when f is self-similar then so it is the wavelet transform (see [Mal98] for a definition of self-similar function and wavelet transform). Moreover, the coefficients $\sup_{n \in \mathbb{Z}} |\langle f, \psi_{j,n} \rangle|$ should decay exponentially with respect to j. That is,

$$s_j := \log_2 \left(\sup_{n \in \mathbb{Z}} |\langle f, \psi_{j,n} \rangle| \right) = j(s + \operatorname{sign}(s)/2) + \log_2(C).$$

This tells us that, in this case, to compute the value of regularity s we can make a linear regression to estimate the slope of the graph of the pairs (j, s_j) and get the correct value of s from this slope. The Pearson correlation coefficient controls the degree of linear correlation between the variables j and s_j and, hence, it is a test of the self-similarity of f (and so, of the validity of the estimated s). ■

6. An Algorithm to estimate regularities on $\mathscr{L}^\infty(\mathbb{R})$

In [dlLP02], numerical implementations of wavelet analysis to estimate the *positive* regularity of conjugacies between critical circle maps are done. Due to Theorem 5.2 and Remark 5.3, we can generalize such techniques to any value (positive or not) of the regularity measured in terms of the Besov Spaces $\mathscr{B}^s_{\infty,\infty}(\mathbb{R})$. The algorithm described below explains how to implement this generalization.

Assume that a function $f \in \mathscr{L}^\infty(\mathbb{S}^1)$ is given. Hence, one has

$$|\langle f, \psi_{j,n} \rangle| < \infty$$

for all $j, n \in \mathbb{Z} \times \mathbb{Z}$. Let ϕ be the scale function of a Daubechies wavelet with k vanishing moments and assume that there exists a positive integer J such that

$$\sum_{n=0}^{2^J - 1} \langle f^{\text{PER}}, \phi_{-J,n} \rangle \, \phi_{-J,n}$$

gives a sufficiently good approximation of f in a certain $\mathcal{V}_{-J} \subset \mathscr{L}^2(\mathbb{R})$. Then we can perform the following steps:

Step 1. Compute $a^{\text{PER}}_{-J}[n] := \langle f^{\text{PER}}, \phi_{-J,n} \rangle$ for $0 \leq n \leq 2^J - 1$. This can be done by using Lemma 4.3.

Step 2. Calculate the coefficients

$$d_j^{\mathrm{PER}}[n] = \langle f^{\mathrm{PER}}, \psi_{j,n} \rangle$$

for $j = 0, \dots, -J$ and $0 \le n \le 2^{-\mathrm{j}} - 1$. This is done starting with $a_{-J}^{\mathrm{PER}}[n]$ and using recursively Equation (9).

Step 3. By using the coefficients $d_j^{\mathrm{PER}}[n]$ from Step 2, calculate

$$s_j = \log_2 \left(\sup_{0 \le n \le 2^{-j}-1} \left| d_j^{\mathrm{PER}}[n] \right| \right)$$

for $j = 0, \dots, -J$.

Step 4. Make a linear regression to estimate the slope $\widetilde{s}$ of the graph of the pairs (j, s_j) with $j = 0, \dots, -J$. Then we set

$$s = \mathrm{R}(\widetilde{s}) = \begin{cases} \widetilde{s} + \frac{1}{2} \operatorname{sign}(\frac{1}{2} - \widetilde{s}) & \text{if } |\widetilde{s}| > \frac{1}{2}, \\ 0 & \text{otherwise}, \end{cases}$$

if there is evidence of linear correlation between the variables j and s_j.

If $k > \max(s, 5/2 - s)$ then, by Theorem 5.2 and Remarks 5.3 and 5.5, $f \in \mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ and, hence, f has regularity s. Otherwise we need to repeat the algorithm with a Daubechies wavelet having a larger value of k.

To test the quality of this algorithm we will try it with the Weierstraß function since we have a *closed* expression for it and we have an analytic formula for its regularity in terms of its parameters. This idea is borrowed from [dlLP02], but since we use more data than [dlLP02] we reproduce the example.

Example 6.1. From Section 4 we know that $\mathfrak{W}_{A,B} \in \mathscr{B}^{-\log_B(A)}_{\infty,\infty}(\mathbb{R})$. To test the algorithm we fix the parameter $B = 2$ and we take $A \in [0.56745, 0.86475]$. Hence, $\mathfrak{W}_{A,2} \in \mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ with $s = -\log_2(A) \in [0.2051\dots, 0.8174\dots]$. Then, observe that

$$1 < \max\left(s, \frac{5}{2} - s\right) = \frac{5}{2} - s < 3.$$

Therefore the above algorithm is valid in this case only for Daubechies Wavelets with $k \ge 3$ vanishing moments.

To perform the above algorithm we take $J = 30$ (that is, we use a sample of the graph of $\mathfrak{W}_{A,2}$ of 2^{30} points). To carry out Step 1, by Lemma 4.3, we can estimate

$$a_{-J}[n] \approx 2^{-J/2} W_{A,2}(2^{-J} n).$$

Then, after executing Steps 2–4 of the algorithm we obtain the results depicted in Figure 3. We want to remark that the best numerical estimate of the regularity of $\mathfrak{W}_{A,2}(x)$ with $A \in [0.56\cdots, 0.86\cdots]$ computed with a Daubechies wavelet of 10 vanishing moments is obtained for $A = 0.86\cdots$ (that is, when the regularity is closer to zero). The fact that we have to work with the Daubechies wavelet of 10 vanishing moments can be explained as follows. Daubechies Wavelets with higher vanishing moments have bigger domain and regularity (see [Mal98]) and, hence, they are less adapted to approximate the Weierstraß function, which has highly concentrated oscillations. It turns out that the value of 10 vanishing moments is the best adapted (in the sense that minimises the error) to the Weierstraß function for the range of parameters considered.

We also want to remark that all the computed Pearson correlation coefficients are bigger than 0.999. This agrees with the fact that the Weierstraß function is self-similar. Then, the coefficients $d_j[n]$, (as pointed out in Remark 5.5) must be on a straight line. It turns out that the Daubechies wavelet with 10 vanishing moments

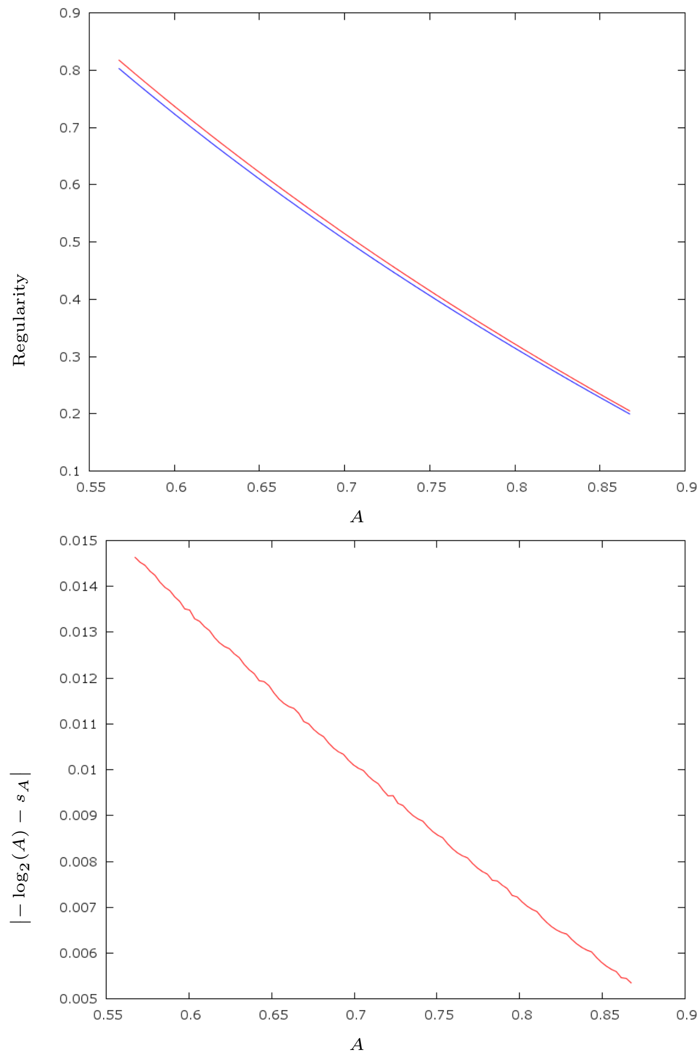

FIGURE 3. On the first picture the theoretical and estimated regularity of $\mathfrak{W}_{A,2}$ with $A \in [0.56\cdots, 0.86\cdots]$ are shown. The theoretical curve is plotted in blue and the *numerical* one in red. The estimated regularity is computed with a Daubechies Wavelet with 10 vanishing moments. On the second picture the *Error* function $|-\log_2(A) - s_A|$ is plotted (here s_A denotes the estimated regularity of $\mathfrak{W}_{A,2}$). Notice that the error is decreasing as the regularity gets closer to zero.

also maximizes globally the computed Pearson correlation coefficients in the range of parameters that we consider. ∎

7. THE REGULARITY OF THE ATTRACTORS OF SYSTEM (2)

As it has been already said in Section 2, we aim at estimating the regularity of the attractor of System (2). By Keller Theorem, this attractor is the graph of an upper semi continuous function $\varphi\colon \mathbb{S}^1 \longrightarrow \mathbb{R}^+$. Thus we have to use the algorithm described in the previous section applied to the function φ^{PER} (see Lemma 3.4).

However notice that, according to Theorem 2.1(d) (and the fact that the circle $x \equiv 0$ is invariant), when $\sigma > 1$, the map φ is discontinuous almost everywhere and the corresponding attractor is called *strange*. Therefore, φ^{PER} is also discontinuous almost everywhere and we are *not* allowed to apply verbatim the algorithm from the previous section.

In the rest of this section we will describe how to solve this problem in the implementation of the algorithm from the previous section.

First, since we do not have an explicit formula for φ we will use Theorem 2.1(f) and the transfer operator to get a sufficiently good numerical approximation of this function. Indeed, by Theorem 2.1(f), for almost every $\theta_0 \in \mathbb{S}^1$, any $x_0 > 0$ and any $\varepsilon > 0$ there exists N_0 such that for every $n \geq N_0$ we have:

$$|x_n - \varphi(\theta_n)| < \varepsilon$$

where $(\theta_n, x_n) = \mathfrak{F}^n(\theta_0, x_0)$. Moreover, the points (θ_n, x_n) with $n \in \{N_0, N_0 + 1, \dots, N_0 + 2^J - 1\}$ approximate exponentially fast the points $(\theta_n, \varphi(\theta_n))$ from $\operatorname{graph}(\varphi)$. Therefore,

$$\big\{(\theta_n, x_n) : n = N_0, N_0 + 1, \dots, N_0 + 2^J - 1\big\} \tag{10}$$

is an approximate mesh of $\operatorname{graph}(\varphi)$ provided that J is large enough. To fix the mesh we choose a random point θ_0 and we fix some $x_0 > \sup_{x\in\mathbb{R}} \tanh(x) = 1$. However, this approximate mesh has two problems to be used in our computations:

Problem (1) as we will see, we need a mesh of the graph of φ at dyadic points of the form $i2^{-J}$ for $i = 0, 1, \dots, 2^J - 1$,

Problem (2) we can not use Lemma 4.3 to estimate the initial coefficients $a_{-J}[n]$ since our map φ is discontinuous almost everywhere (and, hence, not Lipschitz).

7.1. **A solution to Problem (1): a $\mathcal{C}^2$ homeomorphism.** As we have said, we need a mesh of the graph of φ at dyadic points of the form $\theta_i = i2^{-J}$ for $i = 0, 1, \dots, 2^J - 1$ but, clearly, if we obtain the points (θ_n, x_n) just as iterates of a single point by $\mathfrak{F}$ this condition is not satisfied. A solution to this problem is the following. First we relabel the points $\{(\theta_n, x_n)\}_{n=N_0}^{N_0+2^J-1}$ to a sequence $\{(\widetilde{\theta}_i, z_i)\}_{i=0}^{2^J-1}$ so that

$$0 \leq \widetilde{\theta}_0 < \widetilde{\theta}_1 < \cdots < \widetilde{\theta}_{2^J-1} < 1$$

(we do this simply by sorting the data (10) with respect to the first coordinate; see Remark 7.3). In particular if $n \in \{N_0, N_0 + 1, \dots, N_0 + 2^J - 1\}$ and $i = i(n) \in \{0, 1, \dots, 2^J - 1\}$ is such that $\widetilde{\theta}_i = \theta_n$, then $z_i = x_n$.

Now we consider a $\mathcal{C}^m$ diffeomorphism $h\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that $h\left(i2^{-J}\right) = \widetilde{\theta}_i$ for $i = 0, \dots, 2^J - 1$, $h(1) = \widetilde{\theta}_0 + 1$ and m is large enough (to be determined later). Such map h can be obtained by taking h to be, for instance, a spline regular enough in the intervals $\left[i2^{-J}, (i+1)2^{-J}\right]$ for $i = 0, \dots, 2^J - 1$.

Now consider the map $\varrho = \varphi \circ h$. Clearly $\left(i2^{-J}, z_i\right)$ is now a mesh based at the dyadic points that approximates $\operatorname{graph}(\varrho)$, as we wanted. Thus, *we will use the list of pairs* $\left(i2^{-J}, z_i\right)$ *to estimate the regularity of* ϱ.

Remark 7.1. The map ϱ has the following dynamical interpretation. Consider the homeomorphism $H\colon \mathbb{S}^1 \times \mathbb{R}^+ \longrightarrow \mathbb{S}^1 \times \mathbb{R}^+$ defined by $H(\theta, x) = (h(\theta), x)$. Then, one can check that, $\operatorname{graph}(\varrho)$ is the attractor of the dynamical system

$$\left(H^{-1} \circ \mathfrak{F} \circ H\right)(\theta, x) = (h^{-1}(R_\omega(h(\theta))), f(x)g(h(\theta))),$$

which is conjugate to System (2). ∎

To obtain the regularity of φ we need to relate the regularities of ϱ and φ in terms of the Besov Spaces $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$. To do this we use the fact that h is a $\mathcal{C}^2$ diffeomorphism and that a diffeomorphism is an isomorphic mapping of $\mathscr{B}^s_{\infty,\infty}(\mathbb{R})$ onto itself (we refer the reader to Section 4.3 from [Tri92] for a more detailed explanation). More precisely,

Proposition 7.2. *Let $f \in \mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$ with $s \in \mathbb{R}$ and let $h\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be a $\mathcal{C}^m$ diffeomorphism with m big enough. Then $f \circ h$ belongs to $\mathscr{B}^s_{\infty,\infty}(\mathbb{S}^1)$.*

Thus, the regularity of φ and ϱ coincide and we can estimate the regularity of ϱ by using the mesh $\left(i2^{-J}, z_i\right)$ for $i = 0, \ldots, 2^J - 1$. To this end, as in Section 4, we introduce the following notation for the wavelet coefficients of ϱ^{PER}:

$$a_j^{\text{PER}}[n] := \langle \varrho^{\text{PER}}, \phi_{j,n} \rangle \quad \text{and} \quad d_j^{\text{PER}}[n] := \langle \varrho^{\text{PER}}, \psi_{j,n} \rangle$$

for $j, n \in \mathbb{Z}$. Then, we will use Formulae (9) compute the coefficients $d_j^{\text{PER}}[n]$.

Remark 7.3. The process of sorting the data of an array of 2^{30} points from $\mathbb{S}^1 \times \mathbb{R}^+$ (stored as pairs of `double` variables in `C`) turns to be the bottleneck of the whole algorithm (and the most time consuming task of the whole program). Moreover, even the process of computing and filling the array with the initial mesh of the function φ already spends a "visible" amount of CPU time. Indeed, the iteration, storing and sorting process (with a standard sort algorithm like Heapsort) of this data spends about 2200 CPU seconds, with a remarkable variability which depends on the initial sorting of the data, in a computer with a Xeon processor at 3 GHz and 32 Gb of RAM memory. In order to reduce the time elapsed in the sorting process we use the following trick based on the fact that the dynamical system generating the θ_i data is the irrational rotation R_ω. In this case we know that the Lebesgue measure is the unique ergodic measure of R_ω and, hence, its averaged spatial distribution is uniform and it is controlled approximately by the Birkhoff Ergodic Theorem applied to the Lebesgue measure. Indeed, we have

$$\sharp\left(\left\{\theta, R_\omega(\theta), \ldots, R_\omega^{k-1}(\theta)\right\} \cap \left[\tfrac{i}{N}, \tfrac{i+1}{N}\right)\right) \approx \frac{k}{N}$$

for k large enough and for every $i \in \{0, 1, \ldots, N-1\}$. The interpretation of this equation is that the statement

$$\sharp\left(\left\{\theta_{N_0}, \theta_{N_0+1}, \ldots, \theta_{N_0+2^J-1}\right\} \cap \left[\tfrac{i}{2^J}, \tfrac{i+1}{2^J}\right)\right) = 1 \tag{11}$$

holds with high frequency for J large enough (observe that in this case we have $\{\theta_{N_0}, \theta_{N_0+1}, \ldots, \theta_{N_0+2^J-1}\} = \left\{\theta_{N_0}, R_\omega(\theta_{N_0}), \ldots, R_\omega^{2^J-1}(\theta_{N_0})\right\}$). Moreover, when (11) holds, we have $i = \left\lfloor 2^J \theta_l \right\rfloor$, where θ_l is the unique element from the set $\{\theta_{N_0}, \theta_{N_0+1}, \ldots, \theta_{N_0+2^J-1}\} \cap \left[\frac{i}{2^J}, \frac{i+1}{2^J}\right)$ and $\lfloor \cdot \rfloor$ denotes the integer part function. This observation gives a good "hash function" and the following efficient algorithm to store and sort the data $\{(\theta_n, x_n)\}_{n=N_0}^{N_0+2^J-1}$. First, for $n = N_0, N_0{+}1, \ldots N_0{+}2^J{-}1$ we compute the point $(\theta_n, x_n) = \mathfrak{F}(\theta_{n-1}, x_{n-1})$. Then, we store it in the position $i = \left\lfloor 2^J \theta_n \right\rfloor$ of the array data, if this slot is free. Otherwise, we store the point (θ_n, x_n) in a free position $j = j(i)$ of the array data such that $|j - i|$ is minimal. According to the above observations this will happen with low frequency and the array data will be almost sorted. Moreover, the positions of the array data which are not sorted are close the place where they should be when the array is sorted. This is the situation when the direct insertion sorting algorithm can be used with very good results. This means that we are using a method of order $\mathcal{O}(2^J + d)$ where d is the number of insertions (which are very low due to the way we have stored all data) instead of a method of order $\mathcal{O}(J2^J)$ as the Heapsort algorithm.

With this trick, the iteration, storing and sorting process lasts about 300 CPU seconds, almost without variability, which clearly improves the efficiency of the program. ∎

7.2. **A solution to Problem (2): calculating the coefficients $a_{-J}^{\text{PER}}[n]$ of ϱ^{PER}.** When ϱ is regular enough, Lemma 4.3 gives $2^{-J/2}\varrho\left(\frac{n}{2^J}\right)$ as an estimate for the coefficients $a_{-J}^{\text{PER}}[n]$. But, as we have pointed out, φ (and hence ϱ^{PER}) is discontinuous almost everywhere and the above estimate of $a_{-J}^{\mathrm{PER}}[n]$ is, a priori, not valid. However, as we will see, the element $z_n \approx \varrho\left(n2^{-J}\right)$ from our data give indeed a good estimate for $a_{-J}^{\mathrm{PER}}[n]$ because our mesh is based at the dyadic points $n2^{-J}$.

As it has been already said in Section 2, φ is the pointwise limit of a non-increasing sequence of continuous (and, hence, uniformly continuous) functions $\varphi_k \colon \mathbb{S}^1 \longrightarrow \mathbb{R}^+$ defined by

$$\varphi_0(\theta) = c \qquad \text{and} \qquad \varphi_{k+1}(\theta) = \mathfrak{T}(\varphi_k)(\theta)$$

for every $\theta \in \mathbb{S}^1$ and $c > \sup_{x\in\mathbb{R}} \tanh(x)$. Consequently, $\varrho(\theta) = \lim_{k\to\infty} \varphi_k(h(\theta))$ for every θ.

Remark 7.4. If we take $x_0 = c = \varphi_0(\theta_0)$ then $x_k = \varphi_k(\theta_k)$ for every $k \geq 1$. To see it notice that, from the definition of the points (θ_n, x_n) and $\mathfrak{F}$, we get

$$\theta_k = R_\omega(\theta_{k-1}) \qquad \text{and} \qquad x_k = \pi_x(\mathfrak{F}(\theta_{k-1}, x_{k-1}))$$

for every $k \geq 1$. Assume that $x_{k-1} = \varphi_{k-1}(\theta_{k-1})$ fore some $k \geq 0$. Then, by Remark 2.2,

$$\begin{aligned} x_k &= \pi_x(\mathfrak{F}(\theta_{k-1}, x_{k-1})) = \pi_x(\mathfrak{F}(\theta_{k-1}, \varphi_{k-1}(\theta_{k-1}))) \\ &= \mathfrak{T}(\varphi_{k-1})(R_\omega(\theta_{k-1})) = \varphi_k(\theta_k). \end{aligned}$$

■

Since the scaling function ϕ of a Daubechies Wavelet is continuous, so is $\phi_{-J,n}$ for each n. Hence, from the definition of the coefficients $a_{-J}^{\mathrm{PER}}[n]$ and the Dominated Convergence Theorem we have:

$$\begin{aligned} a_{-J}^{\mathrm{PER}}[n] &= \int_{\mathrm{supp}(\phi_{-J,n})} (\varphi \circ h)^{\mathrm{PER}}(\theta)\phi_{-J,n}(\theta)\, d\theta \\ &= \lim_{k\to\infty} \int_{\mathrm{supp}(\phi_{-J,n})} (\varphi_k \circ h)^{\mathrm{PER}}(\theta)\phi_{-J,n}(\theta)\, d\theta \\ &= \lim_{k\to\infty} a_{-J}^{k,\mathrm{PER}}[n], \end{aligned}$$

where $a_{-J}^{k,\mathrm{PER}}[n] := \left\langle (\varphi_k \circ h)^{\mathrm{PER}}, \phi_{-J,n} \right\rangle$. From the proof of the Dominated Convergence Theorem, it can be shown that $a_{-J}^{k,\mathrm{PER}}[n]$ converge exponentially fast to $a_{-J}^{\mathrm{PER}}[n]$. Therefore, if k is large enough, by Lemma 4.3 we have

$$a_{-J}^{\mathrm{PER}}[n] \sim a_{-J}^{k,\mathrm{PER}}[n] \approx 2^{-J/2}(\varphi_k \circ h)^{\mathrm{PER}}(n2^{-J}) = 2^{-J/2}\varphi_k(h(n2^{-J}))$$

for $n = 0, \ldots, 2^J - 1$ (where $\sim$ means exponentially close).

From the definition of h it follows that, given $n \in \{0, 1, \ldots, 2^J - 1\}$, there exists $k \in \{N_0, N_0 + 1, \ldots, N_0 + 2^J - 1\}$ such that $h\left(n2^{-J}\right) = \widetilde{\theta}_n = \theta_k$. Therefore, by Remark 7.4,

$$\varphi_k\left(h\left(n2^{-J}\right)\right) = \varphi_k(\theta_k) = x_k = z_n.$$

Hence, if N_0 is large enough,

$$a_{-J}^{\mathrm{PER}}[n] \approx 2^{-J/2}\varphi_k(h(n2^{-J})) = 2^{-J/2} z_n$$

for $n = 0, \ldots, 2^J - 1$. This gives the necessary approximation of the coefficients $a_{-J}^{\mathrm{PER}}[n]$ to initialize the algorithm.

7.3. **A summary on the implementation of the algorithm on System (2).**
Step 1.1. Fix $\sigma > 1$, choose a random $\theta_0 \in [0, 1)$ and $x_0 > 1$ and, by using the recurrence $(\theta_n, x_n) = \mathfrak{F}(\theta_{n-1}, x_{n-1})$, generate the data

$$\left\{(\theta_n, x_n) : n = N_0, N_0 + 1, \dots, N_0 + 2^J - 1\right\},$$

with $N_0 = 10^5$ and $J = 30$.

Step 1.2. Sort the above data to obtain a sequence $\{(\widetilde{\theta}_n, z_n)\}_{n=0}^{2^J-1}$ so that

$$0 \leq \widetilde{\theta}_0 < \widetilde{\theta}_1 < \cdots < \widetilde{\theta}_{2^J-1} < 1,$$

and delete the concrete values of the points $\widetilde{\theta}_n$. This defines a map ϱ with the same regularity that the map φ such that $\varrho\left(n2^{-J}\right) \approx z_n$ for $n = 0, \dots, 2^J - 1$.

Step 1.3. Set $a_{-J}^{\text{PER}}[n] \approx 2^{-J/2} z_n$ for $n = 0, \dots, 2^J - 1$.

Now, Steps 2–4 of the algorithm from Section 6 remain unaltered. As a result we get an estimate of the regularity of the (strange) attractor of System (2) for the chosen value of σ and ε.

8. Conclusions and results

We have used the above algorithm with System (2) for $\sigma \in [1, 2]$ and $\varepsilon = \max(0, (\sigma - 1.5)^2)$. With this parametrization we have $\varepsilon = 0$ (that is, the system is pinched) if and only if $\sigma \leq 1.5$.

In Figure 4 we plot the estimated regularities of System (2) as a function of σ with the above parametrization. By taking into account the precision observed in the test performed for the Weierstraß function (see Example 4) it seems safe to affirm that the regularity of Keller-GOPY attractor when $\sigma \in [1, 2]$ and $\varepsilon = \max(0, (\sigma - 1.5)^2)$ is in the interval $[0, c)$ with $0 < c < 1$. In particular, for the pinched case (i.e. $\sigma \leq 1.5$), we observe that the regularity is equal to zero. In this figure one can clearly appreciate three regions with different qualitative behaviour. One of them corresponds to the pinched case (i.e. $\sigma \in [1, 1.5)$) and the other two to the non-pinched one: $\sigma \in [1.5, \widetilde{\sigma})$ and $\sigma \in [\widetilde{\sigma}, 2]$ with $\widetilde{\sigma} \approx 1.527$. In what follows we discuss in detail these three regions.

Non pinched case: $\sigma \in [\widetilde{\sigma}, 2]$. In this region we have $\varepsilon = (\sigma - 1.5)^2 \gtrapprox 7.29 \times 10^{-4}$ and hence we are far from the pinched case. The function φ whose graph is the attractor is continuous but not differentiable (see [Kel96, Sta99, Sta97]). Moreover, since we are far from the pinched case, φ is rather well behaved since still we have lack of differentiability in few points (see Figure 5). This is confirmed by the estimated regularities that, not surprisingly, are in the interval $(0, 1)$ and "far" from zero: $\mathrm{R}(\widetilde{s}) \in [0.6822, 0.9669]$.

Approaching the pinched case: $\sigma \in [1.5, \widetilde{\sigma})$. In this region, since for $\sigma = 1.5$ we are already in the pinched case, the function becomes more irregular (see Figure (6)). Therefore, the regularity falls to zero abruptly and we have big differences in regularity between parametrically close attractors.

The pinched case: $\sigma \in [1, 1.5)$. In this case, according to 2.1, the attractor is pinched and, hence, discontinuous almost everywhere (see Figure 1). Then it is not surprising that the regularity is equal to zero for the whole range of parameters.

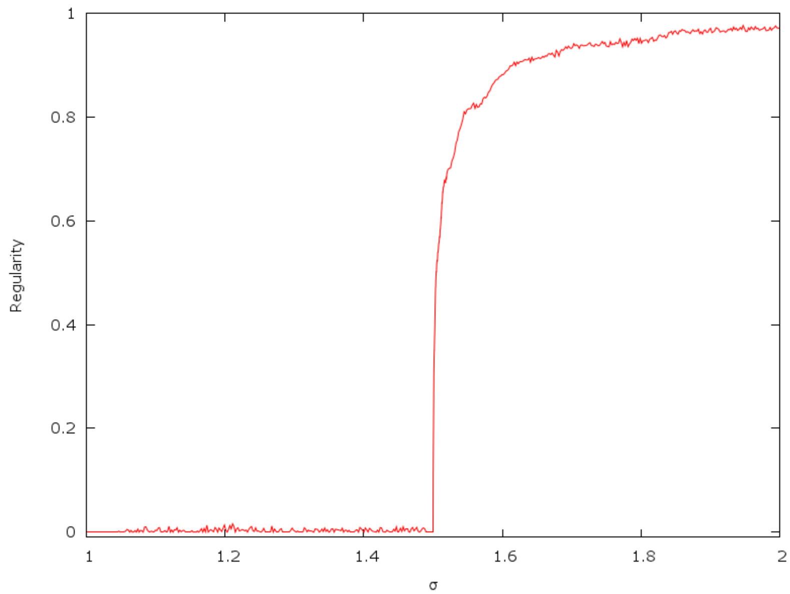

FIGURE 4. The estimate of the regularity $\mathrm{R}(\widetilde{s})$ of the (strange) attractor of System (2) for $\sigma \in [1,2]$ and $\varepsilon = \max(0,(\sigma-1.5)^2)$. The results are obtained by using a sample of 2^{30} points (that is, $J=30$), a transient $N_0 = 10^5$ and the Daubechies Wavelet with 16 vanishing moments. For this number of vanishing moments we obtain the minimum variance of Pearson correlation coefficient.

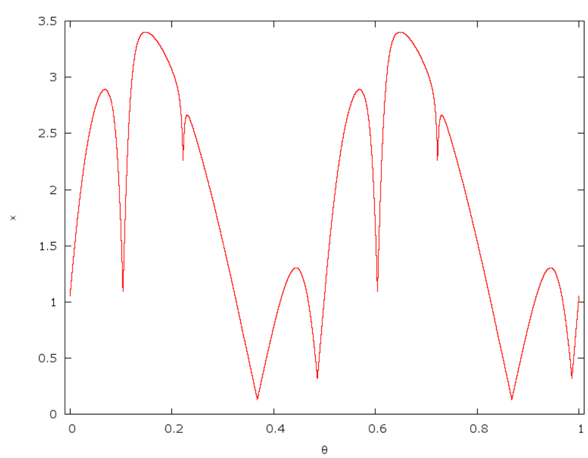

FIGURE 5. The attractor of System (2) for $\sigma = 1.699219$ (and $\varepsilon = 0.039688$). In this case $\mathrm{R}(\widetilde{s}) = 0.91431$.

REFERENCES

[BL76] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[Coh03] Albert Cohen. *Numerical analysis of wavelet methods*, volume 32 of *Studies in Mathematics and its Applications.* North-Holland Publishing Co., Amsterdam, 2003.

[dlLP02] Rafael de la Llave and Nikola P. Petrov. Regularity of conjugacies between critical circle maps: an experimental study. *Experiment. Math.*, 11(2):219–241, 2002.

[Fra99] Michael W. Frazier. *An introduction to wavelets through linear algebra.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1999.

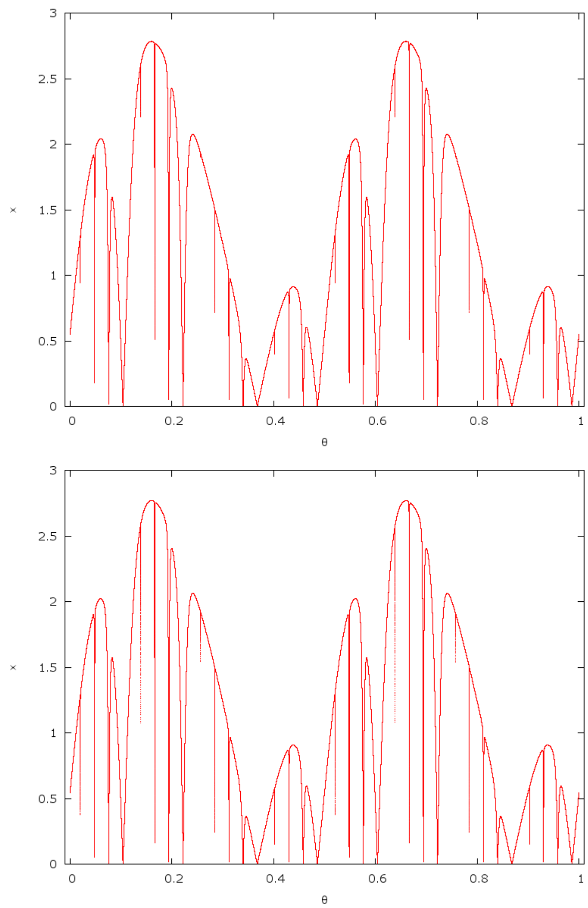

FIGURE 6. On the first picture it is shown the attractor of System (2) for $\sigma = 1.513672$ (and $\varepsilon = 0.000187$). The regularity of this attractor is $\mathrm{R}(\widetilde{s}) = 0.6266$. On the second picture the parameter are $\sigma = 1.507812$ ($\varepsilon = 0.000061$) and $\mathrm{R}(\widetilde{s}) = 0.4951$. Notice the *big* difference between the regularities of the two *parametrically close attractors.*

[GOPY84] Celso Grebogi, Edward Ott, Steven Pelikan, and James A. Yorke. Strange attractors that are not chaotic. *Phys. D*, 13(1-2):261–268, 1984.

[HW96] Eugenio Hernández and Guido Weiss. *A first course on wavelets*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1996. With a foreword by Yves Meyer.

[Kel96] Gerhard Keller. A note on strange nonchaotic attractors. *Fund. Math.*, 151(2):139–148, 1996.

[Mal98] Stéphane Mallat. *A wavelet tour of signal processing*. Academic Press Inc., San Diego, CA, 1998.

[Sta97] Jaroslav Stark. Invariant graphs for forced systems. *Phys. D*, 109(1-2):163–179, 1997. Physics and dynamics between chaos, order, and noise (Berlin, 1996).

[Sta99] Jaroslav Stark. Regularity of invariant graphs for forced systems. *Ergodic Theory Dynam. Systems*, 19(1):155–199, 1999.

[Ste70] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[Tri83] Hans Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1983.

[Tri92] Hans Triebel. *Theory of function spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.

[Tri06] Hans Triebel. *Theory of function spaces. III*, volume 100 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2006.

Departament de Matemàtiques, Edifici Cc, Universitat Autònoma de Barcelona, 08913 Cerdanyola del Vallès, Barcelona, Spain

E-mail address: alseda@mat.uab.cat, jmm@mat.uab.cat, dromero@mat.uab.cat