Examples of Armendariz Rings

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Abstract

We construct various examples of Armendariz and related rings by reviewing and extending some results concerning the structure of nil(R). In particular, we give some examples of Armendariz rings related to power series rings and an example of an n-Armendariz ring for all $n \geq 1$ which is not Armendariz.

1 Introduction

A ring R (associative with unit) is said to be *Armendariz* if the product of two polynomials in $R[x]$ is zero if and only if the product of their coefficients is zero. More precisely, if $f(x) = a_0 + \cdots + a_n x^n$ and $g(x) = b_0 + \cdots + b_m x^m \in R[x]$ such that $f(x)g(x) = 0$, then $a_i b_j = 0$ for all $i = 0, \ldots, n$ and $j = 0, \ldots, m$. We will refer to this as the *Armendariz condition*.

This definition was given by Rege and Chhawchharia in [19] using the name Armendariz since E.P. Armendariz had proved in [5] that reduced rings satisfied this condition. In the mentioned paper, Armendariz used this condition to prove that for a reduced ring R the polynomial ring $R[x]$ is Baer if and only if R is a Baer ring, and that the same is true for pp-rings. Then, after their introduction in [19], Armendariz rings have appeared in many results concerning certain types of annihilator conditions in a ring R being preserved under the pass to the polynomial ring $R[x]$.

In the following theorem we summarize some results in this directions by M. Rege, A. Chhawchharia, N.K. Kim, Y. Lee, Y. Hirano, C.Y. Hong and T.K. Kwak.

*Work supported by MEC-DGESIC (Spain) through grant MTM2008-0621-C02-01, and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya through grant 2009 SGR 1389

Theorem 1. *Let R be an Armendariz ring. Then*

(1) R is semicommutative if and only if $R[x]$ is semicommutative [19].

(2) R is Baer if and only if $R[x]$ is Baer [13].

(3) R is a pp-ring if and only if $R[x]$ is a pp-ring [13].

(4) R satisfies right (left) acc$\perp$ if and only if $R[x]$ satisfies right (left) acc$\perp$ [8].

(5) R is reversible if and only if $R[x]$ is reversible [14].

(6) R is right zip $\Rightarrow$ $R[x]$ is right zip [9].

In fact, Hirano in [8, Proposition 3.1] gives a very nice characterization of Armendariz rings through a bijection between the sets of annihilators of subsets of R and subsets of $R[x]$. Recently, several types of generalizations of Armendariz rings have been introduced for some of which variations of the previous results are also valid. We will review some of the known examples of Armendariz rings and provide some new examples.

From [5], the first non-trivial example of an Armendariz ring should be a reduced ring. Rege and Chhawchharia proved in [19] that $\mathbb{Z}/(m)$, and more generally R/I where R is a DIP, is an Armendariz ring. Kim and Lee, in [13], prove that if R is a reduced ring, the subring S of the triangular matrix ring $T_3(R)$ whose diagonal terms are equal, is an Armendariz ring. A more elaborated example is given by Huh Lee and Smocktunowicz in [10]; if K is a field, $R = K\langle a, b, c \mid cc = ac = crc = 0 \ \forall r \in A\rangle$, where $A = \langle a, b, c\rangle$, is an Armendariz ring.

Other Armendariz rings can be obtained through typical ring constructions: Subrings of Armendariz rings are clearly Armendariz. Rege and Chhawchharia studied conditions for which trivial extensions $T(R, R/I)$ of a reduced ring R are Armendariz and in [15] Lee and Zhou characterize when trivial extensions of Armendariz rings are Armendariz. Also if R is a ring and e a central idempotent, then it is easy to see that R is Armendariz if and only if eR and $(1-e)R$ are Armendariz and hence $S \times T$ is an Armendariz ring if and only if S and T are Armendariz rings. Anderson and Camillo, in [2], prove that the polynomial ring $R[x]$ over an Armendariz ring R is Armendariz. They also prove the following result

Theorem 2 (Anderson, Camillo[2])**.** *Let R be a ring and $n \geq 2$, then $R[x]/(x^n)$ is Armendariz if and only if R is a reduced ring.*

This construction can be seen as *universally adjoining a nilpotent commutative element* to the ring R, and, in the case of K-algebras and through coproducts of rings, the corresponding non commutative version is proved in [3]

Theorem 3 (Antoine [3])**.** *Let K be a field, S a K-algebra and $n \geq 2$. Then $S *_K K\langle x \mid x^n = 0\rangle$ is Armendariz if and only if S is a domain with $\mathcal{U}(S) = K^*$.*

We will use this constructions to provide examples of Armendariz rings by reviewing some of the situations in which they appear.

2 Examples of Armendariz rings

2.1 Nilpotent elements in Armendariz rings

We observe from the previous examples that the set of nilpotent elements in an Armendariz ring has an important role (see [3]). If R is a ring, we denote this set by $\mathrm{nil}(R)$. In the examples listed above the set of nilpotent elements is very well determined and forms an ideal. Also, all of the ring constructions mentioned (subrings, trivial extensions, polynomial rings, products of rings, ...) preserve the property of $\mathrm{nil}(R)$ being an ideal except for Theorem 3. Hence, in [3], the following example is given:

Let K be a field and $R = K\langle a, b \mid b^2 = 0\rangle$. It is clear that $S \cong K[a] *_K K\langle b \mid b^2 = 0\rangle$ and satisfies the conditions of Theorem 3. Therefore R is an Armendariz ring, but b is nilpotent and neither ab, nor ba are nilpotent.

Anyway, the set of nilpotent elements forms a subring without unit, and this is true not only in the case of R Armendariz, but in many of the generalizations of this condition. Recall that a ring R is 1-Armendariz[1] if linear polynomials satisfy the Armendariz condition. That is

$$(a_0 + a_1 x) \cdot (b_0 + b_1 x) = 0 \ \Rightarrow a_0 b_1 = a_1 b_0 = 0.$$

Anderson and Camillo, proved that for von-Neuman regular rings, this condition is equivalent to the Armendariz condition and, in fact, to the ring being reduced. Lee and Wong in [16] extend some known results to 1-Armendariz rings and provide an example of a 1-Armendariz ring which is not Armendariz,

$$\mathbb{F}_3[a, b]/(a^3, a^2 b^2, b^3), \tag{1}$$

[1]These rings are also called Weak-Armendariz (see [16] for example) but we will use this notation for coherence with n-Armendariz rings and because the term *weak-Armendariz* is also used for another definition which we will see later.

where we can see that $(a+bx)^3 = (a+bx)(a^2+2abx+b^2x^2) = 0$, whereas $aba \neq 0$.

To deal with $\mathrm{nil}(R)$ in the general case of a 1-Armendariz ring, we prove the following lemma which is essentially a generalization of [10, Lemma 7].

Lemma 1. *Let R be a 1-Armendariz ring. If $a, b, c \in R$ such that $ac = 0$ and b is nilpotent, then $abc = 0$.*

Proof. We claim that, for any $r \geq 1$, $ab^{2^r}c = 0$ implies $ab^{2^{r-1}}c = 0$. Then, if $b^n = 0$ for some $n \geq 1$, taking r such that $2^r > n$ we have $ab^{2^r}c = 0$. Thus, using induction in $r \geq 1$, we obtain $abc = 0$.

To prove the claim, it is enough to consider the following product in $R[x]$

$$(a + ab^{2^{r-1}}x) \cdot (c - b^{2^{r-1}}cx) = 0,$$

which implies $ab^{2^{r-1}}c = 0$, since R is 1-Armendariz. □

Using the previous Lemma, if $a, b \in R$ are nilpotent elements in an Armendariz ring R, say $a^n = b^m = 0$, then $a^{n-r_1}ba^{r_1} = 0$ for any $r_1 = 0, \ldots, n$. Now, given $0 \leq r_2 \leq r_1$, we have $a^{n-r_1}ba^{r_1-r_2}ba^{r_2} = 0$. Repeating this process, any word in $\{a, b\}$ containing at least n occurrences of a is 0. The same can be done for b and hence, it is clear that

$$(ab)^n = (a-b)^{n+m} = 0.$$

Therefore we have proved the following:

Corollary 1. *If R is a 1-Armendariz ring, then nil(R) forms a subring without unit of R.*

This condition is also satisfied by another generalization of Armendariz rings, namely Nil-Armendariz rings (see [3]). A ring R is *Nil-Armendariz* if $f(x)g(x) \in \mathrm{nil}(R)[x]$ implies $a_ib_j \in \mathrm{nil}(R)$. This definition is motivated by the following fact, which is easy to prove since reduced rings are Armendariz.

Proposition 1. *Let R be a ring such that nil$(R) \trianglelefteq R$. Then, if the product of two polynomials has nilpotent coefficients, then the product of their coefficients is also nilpotent.*

A similar definition is given by Liu and Zhao in [17]. A ring R is *weak-Armendariz* if whenever the product of two polynomials is zero, the product of their coefficients is nilpotent. It is clear that Nil-Armendariz rings are Weak-Armendariz, and it can be seen that Armendariz rings are Nil-Armendariz.

The Armendariz condition can be extended to the product of more than two polynomials, [2, Proposition 1]. This way of extending the Armendariz condition to a product of more than two polynomials is not true for the case of 1-Armendariz rings, as we can see from the above example of Lee and Wong (1), and is not known for Weak-Armendariz rings. This tool is what makes Nil Armendariz rings more suitable than weak-Armendariz rings.

By observing that any commutative ring is Nil-Armendariz (since $\mathrm{nil}(R) \trianglelefteq R$ and Proposition 1), for any field K, $K[x,y]/(x^2,y^2)$ is an example of Nil-Armendariz ring which is not 1-Armendariz. Hence we have the following diagram where the question marks are unknown examples.

$$\begin{tikzcd}
\text{Armendariz} \arrow[r] \arrow[d] & \text{Nil Armendariz} \arrow[r, shift left] \arrow[d, shift left] & \text{Weak-Armendariz} \arrow[l, shift left, "?"] \\
\text{1-Armendariz} \arrow[r] & \mathrm{nil}(R) \text{ is a subrng} \arrow[u, shift left, "?"] &
\end{tikzcd}$$

Armendariz rings have a tight relation with annihilator conditions. We extend the previous diagram to a more elaborate one. Recall that a ring R is *reversible* if $ab=0$ implies $ba=0$. R is semicommutative if $ab=0$ implies $aRb=0$. It is not difficult to see that reduced rings are reversible, reversible rings are semicommutative, and that for a semicommutative rings R, we have $\mathrm{nil}(R) \trianglelefteq R$.

It can be seen using the previous results that the following list of examples correspond to the ones marked in the table.

①,❶ A field K and a non commutative skew field D respectively.

②,❷ $K[x]/(x^2)$ and $D[x]/(x^2)$.

③,❸ $R_1 = \mathbb{F}_3[a,b,c]/(a^2b^2, a^3, b^3)$ and $R_1 \times D$.

④,❹ $R_2 = K[x,y]/(x^2,y^2)$ and $D[x,y]/(x^2,y^2)$.

⑤,❺ $R_3 = K\langle x, y \mid xy = 0\rangle$ (see [3]), and $R_1 \times R_3$ for the 1-Armendariz but not Armendariz case.

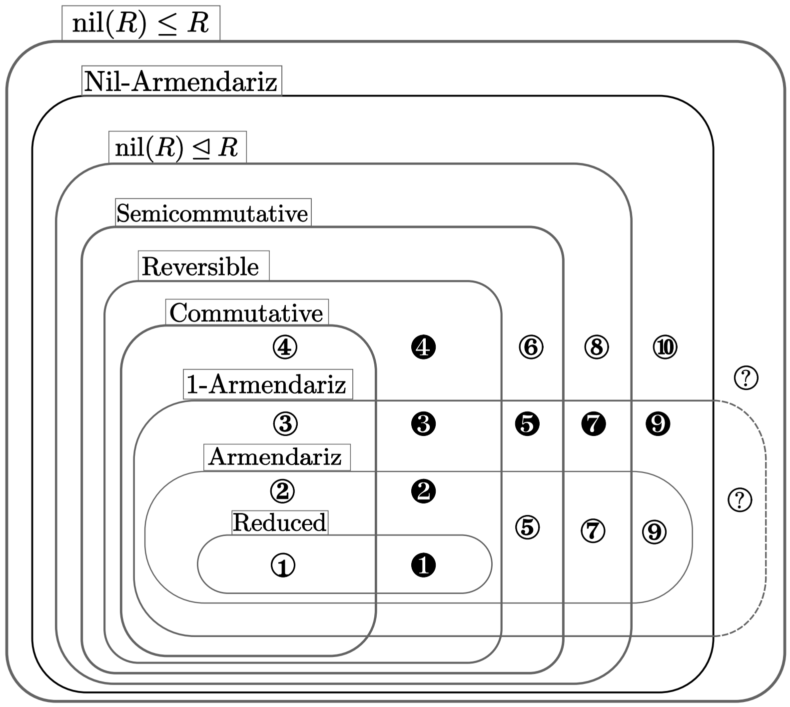
⑥ $R_2 \times R_3$.

⑦,❼ $R_4 = K\langle a,b,c \mid cc = ac = crc = 0 \ \forall r \in A\rangle$, where $A = \langle a,b,c\rangle$, (see [10]) and then $R_4 \times R_1$.

⑧ $T_2(K)$ the upper triangular matrix ring.

⑨,❾ $R_5 = K\langle x, y \mid x^2 = 0\rangle$ and then $R_5 \times R_1$.

⑩ $T_2(R_5)$.



Power Series rings

There are a couple of questions related to the power series ring. One is whether or not the Armendariz condition is preserved in power series rings, and the other is the class of rings for which the Armendariz condition is satisfied for products of power series.

Rege and Bhuphang in [18] give an example of a commutative Armendariz ring R whose power series ring is not Armendariz. The ring is constructed using a non trivial Prfer domain D, and considering the Nagata extension $R = T(D, Q(D)/D)$, which is Armendariz, but whose power series ring is not even 1-Armendariz. Using Theorem 3 we will give a new example.

Example 1. *$R = K\langle a, b \mid b^2 = 0\rangle$ is an Armendariz ring such that the power series ring $R[[x]]$ is not Nil-Armendariz nor 1-Armendariz.*

Proof. R is Armendariz by Theorem 3. Now, let $S = R[[x]]$ be the power series ring with coefficients in R.

Let $u = (1-ax) \in S$. u is a unit in S with $u^{-1} = (1+ax+a^2x^2+\dots) \in S$, and $f = ubu^{-1}$ is such that $f^2 = 0$.

One can see that in the polynomial ring $S[Y]$, $(b + bfY) \cdot (b - fbY) = 0$ but $bfb \neq 0$. But observing that $b, f \in \mathrm{nil}(S)$ whereas bf is not nilpotent, we obtain that $\mathrm{nil}(S)$ is not a subring of S and hence S is not Nil-Armendariz nor 1-Armendariz.

Observe that for any $n \geq 1$, $(bf)^n$ is the product

$$\underbrace{b(1-ax)b(1+ax+a^2x^2+\dots)}_{=bf} b(1-ax)\cdots \underbrace{b(1-ax)b(1+ax+a^2x^2+\dots)}_{=bf}.$$

Since $b^2 = 0$, and $(ba)^n \neq 0$, it is clear by the above expression that the least non zero coefficient of x is obtained by taking a b and an a alternatively in each factor. Therefore, the coefficient of x^{2n} in the above expression is $(-1)^n(baba)^n \neq 0$. Hence $(bf)^n \neq 0$ for all $n \geq 1$ and $\mathrm{nil}(S)$ is not a subring of S. $\square$

Kim, Lee and Lee in [12], define *powerserieswise Armendariz rings* as rings such that $(\sum_{i\geq 0} a_i x^i)(\sum_{i\geq 0} b_i x^i) = 0$ imply $a_i b_j = 0$ for all $i, j \geq 0$.

Powerserieswise Armendariz rings are clearly Armendariz rings, but Armendariz rings need not be Powerserieswise Armendariz as they show by giving two examples. One is the cited above example by Huh Lee and Smocktunowicz in [10]. The other one, by Hamnn and Swan [7], $S = K\langle a_i \ (i \geq 0) \mid a_i a_j a_k = 0 \ (i, j, k \geq 0)\rangle$ where K is a field.

Again, the ring obtained using Theorem 3 gives a new example.

Example 2. *Let K be a field and $R = K\langle a, b \mid b^2 = 0\rangle$. As noted above, R is Armendariz, but*

$$(b - bax) \cdot (b + abx + a^2bx^2 + a^3bx^3 + \dots) = 0$$

whereas $bab \neq 0$. And hence, R is not Powerserieswise Armendariz.

In fact, in [12] it is proved that if R is powerserieswise Armendariz, then $\mathrm{nil}(R)$ coincides with the upper and lower nilradical, and in particular is an ideal.

Armendariz rings and n-Armendariz Rings

Rege and Buhphang, in [18], introduced the following generalization of Armendariz rings

Definition 1 (Rege Buhphang)**.** *A commutative ring R is n-Armendariz if $(a_0 + a_1x)(b_0 + b_1x + \cdots + b_nx^n) = 0$ in $R[x]$ implies $a_ib_j = 0$.*

Rege and Buhphang give an example of a 1-Armendariz ring which is not 2-Armendariz using the extension $\mathbb{Q}(\sqrt[3]{2})$ to build a P_2-closed domain which is not P_3-closed. The same example can be generalized using different extensions. For example, given a prime $q > 0$, we will consider the Galois field $\mathbb{F}_{2^q}$ to give an example of a $q-2$-Armendariz ring which is not $q-1$-Armendariz.

Example 3. *This example is based on [18, Remark 4]. Recall that a subring $R \subseteq D$ is P_n closed if, for all $\beta \in D$ which is a root of a monic polynomial of degree n in $R[x]$, $\beta \in R$.*

Let $K = \mathbb{F}_{2^q} = \mathbb{F}_2(\alpha)$ with α a primitive element of the field. Let $A = K[x]$, $R = \mathbb{F}_2 + xK[x]$ and $D = K(x)$. It is clear that D is the field of fractions of A and of R. Since α is a root of a degree q monic irreducible polynomial with coefficients in $\mathbb{F}_2$, $\alpha \in D$ can be viewed as a root of a degree q monic polynomial in $R[y]$. But $\alpha \notin R$ and hence $R \subseteq D$ is not P_q-closed.

Suppose $u \in D$ is a root of a degree $n < q$ monic polynomial $y^n + f_{n-1}y^{n-1} + \cdots + f_1 y + f_0$ with $f_i \in R$. Since $f_i \in A = K[x]$ which is integrally closed in $K(x)$, $u \in A$. If we evaluate the previous polynomial in $x = 0$ and let $\beta = u(0)$, $c_i = f_i(0) \in \mathbb{F}_2$, we have

$$\beta^n + c_{n-1}\beta^{n-1} + \cdots + c_1\beta + c_0 = 0.$$

Hence $r = [\mathbb{F}_2(\beta) : \mathbb{F}_2] \leq n$, but since $\mathbb{F}_2(\beta) \subseteq K$, r should divide q. Since q is prime, we have $[\mathbb{F}_2(\beta) : \mathbb{F}_2] = 1$, and hence $u(0) = \beta \in \mathbb{F}_2$. Therefore $u \in R$ and we have $R \subseteq D$ is P_{n-1}-closed.

Now, using [18, Theorem and Proposition 2.5], the Nagata extension or trivial extension $T(D, D/R)$ is $(q-2)$-Armendariz but not $q-1$-Armendariz.

This way we can construct, for arbitrarily large n, a ring R which is n-Armendariz but not Armendariz.

Observe that n-Armendariz rings can be called $\{1, n\}$-Armendariz meaning the Armendariz condition is true for polynomials f and g of degrees 1 and n respectively, and then define $\{m, n\}$-Armendariz rings in the natural way. Also for the non commutative case, we could define the same for an ordered pair (m, n), but we will only consider commutative rings.

Using this terminology, we prove that there exists a ring R which is $\{1, n\}$-Armendariz for all $n \geq 1$ but which is not $\{2, 2\}$-Armendariz, and thus it is not Armendariz.

Let K be a field with $\mathrm{char}(K) \neq 2$, and let R be the K-algebra with commuting generators

$$A = \{a_0, a_1, a_2, b_0, b_1, b_2\},$$

and with relations

1. $a_0b_0 = a_2b_2 = 0$.
2. $a_0b_1 + a_1b_0 = 0$.
3. $a_1b_2 + a_2b_1 = 0$.
4. $a_0b_2 + a_1b_1 + a_2b_0 = 0$.
5. $x_1x_2x_3 = 0$ where $x_1, x_2, x_3 \in A$.

R is graded by the length of the monomials since all the relations are homogeneous in this degree. Therefore, if we denote the grading by $R = R_0 \oplus R_1 \oplus R_2$, elements can be written uniquely as $\alpha = \alpha_0 + \alpha_1 + \alpha_2$ where $\alpha_i \in R_i$. Hence, except where noted, given $\alpha \in R$, α_i will denote the corresponding part in this decomposition.

In R we also have an $\{a, b\}$-grading. We can define this grading through $d(a_i) = 0$ and $d(b_j) = 1$ and observing that the relations are homogeneous in this degree. Hence we can write $R = R^{(0)} \oplus R^{(1)} \oplus R^{(2)}$ the corresponding decomposition.

Combining this two gradings, observe that $R_1 \cap R^{(2)} = \{0\}$ and hence given $\alpha \in R_1$ we can write it uniquely as $\alpha = \alpha_a + \alpha_b$, where $\alpha_a \in R^{(0)}$ and $\alpha_b \in R^{(1)}$ with the subindexs labeled for obvious reasons.

By the relations it is clear that $R_0 = K$ and that A is a K-basis for R_1. Now, using the Diamond Lemma, it can be proved that the following is a K-basis for for R_2,

$$\mathcal{C} = \{a_0b_1, a_0b_2, a_1b_1, a_1b_2, \text{ and } a_ia_j, b_ib_j \text{ such that } i, j = 0, 1, 2\}.$$

We will use this gradings and K-basis to study the zero divisors of R

Lemma 2. *Let $\alpha, \beta \in R \setminus \{0\}$. Then $\alpha\beta = 0$ if and only if $\alpha_0 = \beta_0 = 0$ and $\alpha_1\beta_1 = 0$.*

Proof. We have

$$\alpha\beta = (\alpha_0\beta_0) + (\alpha_0\beta_1 + \alpha_1\beta_0) + (\alpha_0\beta_2 + \alpha_1\beta_1 + \alpha_2\beta_0)$$

Where each of the parenthesis corresponds to each component in R_i. Let α_i, β_j be the least non zero degree components in α, β respectively. Then $\alpha_i\beta_j = (\alpha\beta)_{i+j}$, the corresponding $(i+j)$-th component. Since $\alpha\beta = 0$, each of its components is zero and hence $\alpha_i\beta_j = 0$. But $R_0 = K$ which is a a field, and $\alpha_i, \beta_j \neq 0$. Hence, $i, j > 0$ and the result is clear. $\square$

Lemma 3. *Let $\alpha, \beta \in R_1 \setminus \{0\}$ such that $\alpha\beta = 0$. Then we are in one of the following cases*

1. $\alpha = \lambda a_0$, $\beta = \mu b_0$ *for some* $\lambda, \mu \in K^*$.
2. $\alpha = \lambda a_2$, $\beta = \mu b_2$ *for some* $\lambda, \mu \in K^*$.
3. $\alpha = \lambda(a_0 + \gamma a_1 + \gamma^2 a_2)$, $\beta = \mu(b_0 + \gamma b_1 + \gamma^2 b_2)$ *for some* $\lambda, \mu, \gamma \in K^*$.

And the symmetric cases 1',2',3' interchanging a_i's and b_j's.

Proof. If we write $\alpha = \alpha_a + \alpha_b$, $\beta = \beta_a + \beta_b$ with $\alpha_a, \beta_a \in R^{(0)}$ and $\alpha_b, \beta_b \in R^{(1)}$ as in the $\{a, b\}$-grading decomposition, then

$$\alpha\beta = (\alpha_a\beta_a) + (\alpha_a\beta_b + \alpha_b\beta_a) + (\alpha_b\beta_b) = 0.$$

Since each component in $R^{(i)}$ should be zero, $\alpha_a\beta_a = \alpha_b\beta_b = 0$. The relations in R make it clear that if we multiply two non zero elements within $R_1 \cap R^{(0)}$ or $R_1 \cap R^{(1)}$ we obtain a nonzero element.

Thus, since $\alpha, \beta \neq 0$, we will have $\alpha = \alpha_a, \beta = \beta_b$, or $\alpha = \alpha_b, \beta = \alpha_a$. We consider the first case and write

$$\alpha = \lambda_0 a_0 + \lambda_1 a_1 + \lambda_2 a_2, \text{ and } \beta = \mu_0 b_0 + \mu_1 b_1 + \mu_2 b_2,$$

where $\lambda_i, \mu_j \in K$. The other case is proved symmetrically and leads to cases *1',2', 3'*.

Now, $\alpha\beta = 0$ gives

$$\begin{aligned} 0 = \alpha\beta = \quad & \lambda_0\mu_0 a_0 b_0 + \\ & \lambda_0\mu_1 a_0 b_1 + \lambda_1\mu_0 a_1 b_0 + \\ & \lambda_0\mu_2 a_0 b_2 + \lambda_1\mu_1 a_1 b_1 + \lambda_2\mu_0 a_2 b_0 + \\ & \lambda_2\mu_1 a_2 b_1 + \lambda_1\mu_2 a_1 b_2 + \\ & \lambda_2\mu_2 a_2 b_2 \end{aligned} \tag{2}$$

Applying the relations to each of the lines in (2) to write them in terms of the K-basis $\mathcal{C}$, and equaling the coefficients to zero, we obtain the following equations

(a) $\lambda_0\mu_1 - \lambda_1\mu_0 = 0.$

(b) $\lambda_0\mu_2 - \lambda_2\mu_0 = 0.$

(c) $\lambda_1\mu_1 - \lambda_2\mu_0 = 0.$

(d) $\lambda_2\mu_1 - \lambda_1\mu_2 = 0.$

Now we consider the following cases which will lead to cases *1, 2* or *3* in the Lemma,

(i) $\lambda_1 = \lambda_2 = 0$, $\lambda_0 \neq 0$.

(ii) $\lambda_0 = \lambda_2 = 0$, $\lambda_1 \neq 0$.

(iii) $\lambda_0 = \lambda_1 = 0$, $\lambda_2 \neq 0$.

(iv) $\lambda_0 = 0$, $\lambda_1, \lambda_2 \neq 0$.

(v) $\lambda_1 = 0$, $\lambda_0, \lambda_2 \neq 0$.

(vi) $\lambda_2 = 0$, $\lambda_0, \lambda_1 \neq 0$.

(vii) $\lambda_0, \lambda_1, \lambda_2 \neq 0$.

Case (i) By (a), (b) we have $\mu_1 = \mu_2 = 0$. Hence $\alpha = \lambda_0 a_0$ and $\beta = \mu_0 b_0$.

Case (ii) By (a), (c) and (d) we have $\mu_i = 0$ and hence $\beta = 0$. A contradiction.

Case (iii) By (b), (d) we have $\mu_0 = \mu_1 = 0$. Hence $\alpha = \lambda_2 a_2$ and $\beta = \mu_2 b_2$.

Case (iv) By (a) we have $\mu_0 = 0$. Now by (c) $\mu_1 = 0$ and now by (d) $\mu_2 = 0$ which implies $\beta = 0$, a contradiction.

Cases (v),(vi) Similar to the previous case, this both lead to a contradiction.

Case (v) By arguing symmetrically for the μ_i as in the previous cases, we see that $\mu_0, \mu_1, \mu_2 \neq 0$. Hence, multiplying by a nonzero constant we suppose that $\lambda_0 = \mu_0 = 1$. Now we obtain the following equations

$$\begin{aligned} \mu_1 - \lambda_1 &= 0, & \mu_2 - \lambda_2 &= 0, \\ \lambda_2\mu_1 - \lambda_1\mu_2 &= 0, & \lambda_1\mu_1 - \lambda_2 &= 0. \end{aligned}$$

Therefore, $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2$ and $\lambda_2 = \lambda_1\mu_1 = \lambda_1^2$. Hence, in the general case we will have, for some λ, μ and $\gamma \in K \setminus \{0\}$,

$$\alpha = \lambda(a_0 + \gamma a_1 + \gamma^2 a_2),$$

$$\beta = \mu(b_0 + \gamma b_1 + \gamma^2 b_2).$$

Since this gives $\alpha\beta = 0$ we are done.

□

In view of this zero divisors we denote, for every $\gamma \in K^*$,

$$c_\gamma = a_0 + \gamma a_1 + \gamma^2 a_2, \quad d_\gamma = b_0 + \gamma b_1 + \gamma^2 b_2.$$

Lemma 4. *Let R be as above, and let $D \subset R$ be the set of zero divisors in R. Then,*

1. $a_0 R \cap b_0 R = a_0 b_1 K = a_1 b_0 K$.
2. $a_2 R \cap b_2 R = a_1 b_2 K = a_2 b_1 K$.
3. $c_\gamma R \cap d_\gamma D = c_\gamma D \cap d_\gamma R = \{0\}$ *for all* $\gamma \in K^*$.

Proof. (i) and (ii) are clear by looking at a K-basis for R_2. Let us prove that $c_\gamma R \cap d_\gamma D = \{0\}$. Let $r, s \in R \setminus \{0\}$ such that $c_\gamma r = d_\gamma s$. It is clear that if we write $r = r_0 + r_1 + r_2$ and $s = s_0 + s_1 + s_2$ then $r_0 = s_0 = 0$ and $c_\gamma r_1 = c_\gamma r = d_\gamma s = d_\gamma s_1$. Hence we assume, $r, s \in R_1$. Now, let us consider the $\{a, b\}$-grading and write $r = r_a + r_b$ and $s = s_a + s_b$. We have

$$c_\gamma r_a + c_\gamma r_b = d_\gamma s_a + d_\gamma s_b.$$

The $R^{(0)}$ part in the above equation gives $c_\gamma r_a = 0$. Since $c_\gamma, r_a \in R_1 \cap R^{(0)}$ and $c_\gamma \neq 0$, $r_a = 0$. Similarly looking at the $R^{(2)}$ part, we obtain $s_b = 0$. Therefore we can write $r = \mu_0 b_0 + \mu_1 b_1 + \mu_2 b_2$ and $s = \lambda_0 a_0 + \lambda_1 a_1 + \lambda_2 a_2$.

By expanding the following equation,

$$c_\gamma(\mu_0 b_0 + \mu_1 b_1 + \mu_2 b_2) = d_\gamma(\lambda_0 a_0 + \lambda_1 a_1 + \lambda_2 a_2),$$

and applying the relations to write it in terms of the K-basis for R_2, we obtain the following equations after identifying the coefficients

$$\mu_1 - \gamma\mu_0 = \lambda_0\gamma - \lambda_1, \tag{3}$$
$$\mu_2 - \gamma^2\mu_0 = \lambda_0\gamma^2 - \lambda_2, \tag{4}$$
$$\gamma\mu_1 - \gamma^2\mu_0 = \lambda_1\gamma - \lambda_2, \tag{5}$$
$$\gamma\mu_2 - \gamma^2\mu_1 = \lambda_1\gamma^2 - \lambda_2\gamma. \tag{6}$$

We compute the equations $(4) - (5) - \gamma(3)$ and $\gamma(3) - (5)$, to obtain

$$\gamma^2\mu_0 - 2\gamma\mu_1 + \mu_2 = 0, \tag{7}$$
$$\gamma^2\lambda_0 - 2\gamma\lambda_1 + \lambda_2 = 0. \tag{8}$$

From equation (8), since $\gamma \neq 0$ and K has characteristic different from 2, if two $\lambda_i = 0$, then the third is also zero. Hence, if $s \in D \setminus \{0\}$, by looking at the form of the zero divisors in Lemma 3, s has the form $s = \delta c_\lambda$ for some $\delta, \lambda \in K^*$. We suppose, without loss of generality, $\delta = 1$ and thus $\lambda_0 = 1$, $\lambda_1 = \lambda$ and $\lambda_2 = \lambda^2$. If we substitute this in equation (8), we have $(\gamma - \lambda)^2 = 0$ and hence $\gamma = \lambda$. Therefore $d_\gamma s = d_\gamma c_\gamma = 0$.

In case $r \in D \setminus \{0\}$ we can argue the same way through equation (7), to prove $r = \delta' d_\gamma$ and hence $c_\gamma r = 0$.

∎

Lemma 5. *R is n-Armendariz for all $n \geq 1$.*

Proof. Let $f(x) = \alpha_0 + \alpha_1 x$, $g(x) = \beta_0 + \beta_1 x + \cdots + \beta_n x^n \in R[x]$ such that $f(x)g(x) = 0$. We want to see that $\alpha_i\beta_j = 0$.

If either α_0 or β_0 are 0 the result is trivial. Also, suppose $\beta_n \neq 0$ and factoring out x^k conveniently, we may suppose $\beta_0 \neq 0$. By a similar argument as that in Lemma 2 for the grading in $R[x]$ given by $(R[x])_i = R_i[x]$, we may suppose $\alpha_i, \beta_j \in R_1$.

Now we have

$$\alpha_0\beta_0 = \alpha_1\beta_n = 0 \quad \text{and} \quad \alpha_0\beta_i + \alpha_1\beta_{i-1} = 0 \text{ for } i = 1, \ldots, n.$$

For the product $\alpha_0\beta_0$ we will consider the cases *1,2* and *3* in lemma 3, and leave cases *1',2'* and *3'* which are proved symmetrically. Also, by multiplying by certain nonzero elements in K, we consider the following cases

1. $\alpha_0 = a_0, \beta_0 = b_0$ *2.* $\alpha_0 = a_2, \beta_0 = b_2$ *3.* $\alpha_0 = c_\gamma, \beta_0 = d_\gamma,\ \gamma \in K^*$

1. In this case we have $a_0\beta_1 + b_0\alpha_1 = 0$. Since $\alpha_1\beta_n = 0$, by Lemma 3 and the fact that $a_0R \cap b_0R = a_0b_1K = a_1b_0K$, the only possibility is $\alpha_1 = \lambda a_0$ which implies $\beta_1 = \mu_1 b_0$. Now by $a_0\beta_i + \lambda a_0\beta_{i-1} = 0$ we can see inductively that $\beta_i = \mu_i b_0$ and hence, that $\alpha_i\beta_j = 0$.

2. This case is similar to the previous, using that $a_2R \cap b_2R = a_1b_2K = a_2b_1K$.

3. We have $c_\gamma\beta_1 = -d_\gamma\alpha_1$. Since $\alpha_1\beta_n = 0$, α_1 is a zero divisor. Now by Lemma 3 and the fact that $c_\gamma R \cap d_\gamma D = \{0\}$, we have that $\alpha_1 = \lambda c_\gamma$ and hence $\beta_1 = \mu_1 d_\gamma$. Inductively we see that $\beta_i = \mu_i d_\gamma$ for some $\mu_i \in K$. Hence, $\alpha_i\beta_j = 0$.

□

Example 4. *R is an n-Armendariz ring for all $n \geq 0$ but it is not Armendariz.*

Proof. As seen before, R is n-Armendariz for all $n \geq 0$, but

$$(a_0 + a_1x + a_2x^2) \cdot (b_0 + b_1x + b_2x^2) = 0,$$

whereas $a_0b_1 \neq 0$, and thus it is not Armendariz (it is not $\{2, 2\}$-Armendariz). □

2.2 Polynomial rings over nil-Armendariz rings

As we have seen in Theorem 1, the Armendariz condition is useful for taking properties related to annihilators from the ring to its polynomial ring. In the case of Nil-Armendariz rings, it is natural to ask wether or not this properties are also preserved. But we can see from the following examples, that in the general case there is a Nil-Armendariz counterexample to each of the assertions in the Theorem.

Recall that reversible and semicommutative rings are Nil-Armendariz. Thus, since there exist examples of semicommutative or reversible rings such that the polynomial ring is not (see [10, Example 2] and [14, Example 2.1] respectively), these are also Nil-Armendariz counterexamples to Theorem 1, (1) and (4).

Recall that a ring R is *Baer* if all right annihilators of subsets of R are generated by idempotents, and that a ring is a *right (left) pp-ring* (for principally projective) provided all right (left) annihilators of elements of R are generated by an idempotent. Also recall that whereas the Baer condition is left/right symmetric, this is not the case for pp-rings, and a ring is a *pp-ring* if it is a right and left pp-ring.

If K is a field, $R = T_2(K)$, the ring of upper triangular matrices over K, is a Nil-Armendariz ring since $\mathrm{nil}(R) \trianglelefteq R$. Now, the right and left annihilators of subsets of R are $\{0\}$, R, $R\left(\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & a \\ 0 & 1 \end{smallmatrix}\right) R$, which are all generated by idempotents. Hence R is both a Baer and a *pp*-ring.

Now observe that in $R[x]$

$$\mathrm{r.ann}_{R[x]}\left\{\begin{pmatrix} -x & 1 \\ 0 & 0 \end{pmatrix}\right\} = \begin{pmatrix} 0 & a(x) \\ 0 & xa(x) \end{pmatrix} R[x]$$

which is not generated by an idempotent (idempotents have 0 or 1 along the diagonal). Hence we can give Nil-Armendariz counterexamples to Theorem 1, (2) and (3).

Example 5. *Given a field k, $R = T_2(k)$ is a Nil-Armendariz, Baer and pp-ring, such that the polynomial ring $R[x]$ is not a Baer nor a pp-ring.*

Following [8], if an Armendariz ring has the ascending chain condition on annihilators, so has the polynomial ring. But even commutative Goldie rings need not satisfy so. JW. Kerr in [11] for the case of $K = \mathbb{F}_2$ and Antoine and Ced in [4] for a finite field $K = \mathbb{F}_{p^n}$, give examples of commutative Goldie K-algebras, such that the polynomial ring does not satisfy the ascending chain condition on annihilators. Since commutative rings are Nil-Armendariz by

Proposition 1, this examples are Nil-Armendariz counterexamples to Theorem 1, (4).

F. Cedó in [6] provides an example of a Zip ring for which the polynomial ring is not zip. By carefully studying this example, we will prove that $\mathrm{nil}(R) \trianglelefteq R$ and hence R is Nil-Armendariz.

Example 6 (Cedó [6])**.** *Let K be a field. Let R be the K-algebra with generators $A = \{a_{0,n}, a_{1,n}, b_{1,n}, b_{2,n}, a_\infty, a_\lambda \mid n \geq 0, \lambda \in K\}$ and relations*

1. $a_{0,i}b_{1,j} = a_{0,i}b_{2,j} = a_{1,i}b_{1,j}$ *$(j \geq i \geq 0)$,*
2. $a_{1,i}b_{2,j} = 0$ *$(j \geq i \geq 0)$,*
3. $a_{1,i}a_\infty = (a_{0,i} + \lambda a_{1,i})a_\lambda = 0$ *$(i \geq 0, \lambda \in K)$,*
4. $a_\infty x = a_\lambda x = b_{1,i}x = b_{2,i}x = 0$ *$(i \geq 0, \lambda \in K, x \in A)$.*

In [6], a K-basis for R is given which is split in disjoint subsets as $A_0 \cup A_1 \cup A_2 \cup A_3$ where

$$A_0 = \{1\}, \quad A_1 = \{a_{l_1,i_1} \cdots a_{l_n,i_n} \mid n \geq 1, i_\eta \geq 0, l_\eta \in \{0,1\}\},$$

$$A_2 = \{a_{l_1,i_1} \cdots a_{l_n,i_n} a_\mu \mid n \geq 0, i_\eta \geq 0, l_\eta \in \{0,1\}, l_n = 0, \mu \in K \cup \{\infty\}\} \text{ and}$$

$$A_3 = \{a_{l_1,i_1} \cdots a_{l_n,i_n} b_{k,j} \mid n \geq 0, i_\eta \geq 0, l_\eta \in \{0,1\}, k \in \{1,2\}, \text{ and if } n > 0 \text{ and } j \geq i_n, l_n = 0 \text{ and } k = 1\}.$$

Given $\alpha \in R$, α can be written uniquely as $\alpha = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$ where $\alpha_i \in span_K(A_i)$. If α is nilpotent, it is clear by the relations that $\alpha_0 = 0$. Also, we see that $A_2^2 = A_3^2 = A_2A_1 = A_2A_3 = A_3A_1 = A_3A_2 = 0$ and hence, if $\alpha \in A$, with $\alpha_0 = 0$, we have

$$\alpha^2 = \alpha_1^2 + \underbrace{\alpha_1\alpha_2}_{\in span_K(A_2)} + \underbrace{\alpha_1\alpha_3}_{\in span_K(A_3)}$$

and for $n > 1$

$$\alpha^n = \alpha_1^n + \underbrace{\alpha_1^{n-1}\alpha_2}_{\in span_K(A_2)} + \underbrace{\alpha_1^{n-1}\alpha_3}_{\in span_K(A_3)} .$$

By looking at the set of basis elements in A_1, it is clear that $\alpha_1^n = 0$ if and only if $\alpha_1 = 0$. Observe that if $\alpha_0 = \alpha_1 = 0$ then $\alpha^2 = 0$. Hence, α is nilpotent if and only if $\alpha_0 = \alpha_1 = 0$, and we have $nil(R) = span(A_2 \cup A_3)$. Since $A_1A_2 \subseteq A_2$ and $A_1A_3 \subseteq A_3$, we see that this set forms a two sided ideal and by Proposition 1, R is Nil-Armendariz.

Ced in [6] proves that R is a right zip ring and $R[x]$ is not. Hence there is a Nil-Armendariz counterexample to Theorem 1 (6).

Therefore we have counterexamples to all possible generalizations of Theorem 1 to Nil-Armendariz rings. Moreover, one could ask whether or not the Nil-Armendariz condition itself passes to the polynomial ring. This question is addressed in [3], where, by relating it to a question of Amitsur ([1]), is true in the case of K-algebras over a non denumerable field but fails in general using counterexamples provided by A. Smocktunowicz in [20].

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