PURELY INFINITE CORONA ALGEBRAS OF SIMPLE C*-ALGEBRAS

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Dedicated to the memory of Patricia Ann Kucerovsky (1965–2008)

ABSTRACT. In this paper we study the problem of when the corona algebra of a non-unital C*-algebra is purely infinite. A complete answer is obtained for stabilisations of simple and unital algebras that have enough comparison of positive elements. Our result relates the pure infiniteness condition (from its strongest to weakest forms) to the geometry of the tracial simplex of the algebra, and to the behaviour of corona projections, despite the fact that there is no real rank zero condition.

INTRODUCTION

The notion of pure infiniteness, in the simple case, can be traced back to Cuntz (see [5]). He defined a simple C*-algebra to be purely infinite if it is infinite dimensional and, whenever a is a non-zero element, one can find x and y with $xay = 1$. This was extended by Kirchberg and Rørdam to the non-simple setting (see [10]): a C*-algebra A is purely infinite provided that A has no characters and, whenever a and b are positive elements with $a \in \overline{AbA}$, then there is a sequence (v_n) in A with $a = \lim\limits_{n\to\infty} v_n b v_n^*$.

Quite possibly the first (and most natural) example of a purely infinite (simple) corona algebra is constituted by the Calkin algebra. This naturally raises the question of studying which corona algebras of simple C*-algebras are purely infinite. In the real rank zero situation (i.e. when we can produce projections on demand), this was pursued by Lin and Zhang. For example, Zhang showed that if A has real rank zero and $\mathcal{M}(A)/A$ is simple, then it is indeed purely infinite simple (see [27]). In [28], Zhang also showed that if A is σ-unital and has real rank zero, then the corona algebra $\mathcal{M}(A\otimes\mathcal{K})/(A\otimes\mathcal{K})$ is simple if and only if A is either simple purely infinite or elementary. Not much later, in [16] (see also [22]), the same result was proved without assuming real rank zero.

A more general result was made available by Lin, who in [17] proved that, for a simple, non-unital, non-elementary σ-unital algebra A without any real rank zero condition, $\mathcal{M}(A)/A$ being simple is equivalent to $\mathcal{M}(A)/A$ being purely infinite simple. This was also shown to be equivalent to a condition satisfied by the base algebra, termed continuous scale, and expressed in terms of Cuntz comparison of positive elements (see below for the precise definitions).

The first and third-named authors analysed, in [14], conditions under which corona algebras of simple C*-algebras with real rank zero, stable rank one and weakly unperforated K_0 group are purely infinite. Those conditions were expressible almost solely in terms of

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the state space of the K_0 group. In the stable case, they amount to the existence of finitely many extremal (quasi)-traces, and in turn, to the existence of finitely many corona ideals. In a somewhat different direction, the first and second-named authors studied the pure infiniteness of a stable, separable algebra of real rank zero in terms of appropriate properties of a so-called AF-skeleton (see [19]).

In the current paper we deal with a much more general class of algebras, which need not have real rank zero (in general, they will not), yet we can still compare elements by means of the values of appropriate states. The comparison theory we are referring to is the already mentioned Cuntz comparison, and we shall assume our algebras have strict comparison of positive elements. This is a regularity property that has a strong K-theoretic flavour and supports the (philosophical) point of view that Cuntz comparison may be regarded as rank comparison. A particularly important class of algebras that enjoy strict comparison consists of those separable, nuclear, $\mathcal{Z}$-stable simple C^*-algebras (see [24]) as well as simple AH-algebras with slow dimension growth ([25]). Here, $\mathcal{Z}$ is the Jiang-Su algebra, a simple, nuclear, separable and unital C^*-algebra that has the same K-Theory as the complex numbers, yet it is infinite dimensional ([9]). It has become in recent years one of the most prominent examples of simple, separable, and nuclear C^*-algebras, which has deep connections with the classification program (see [7] and the references therein).

A C^*-algebra is termed $\mathcal{Z}$-stable if it absorbs $\mathcal{Z}$ tensorially. (We remind the reader here that $\mathcal{Z}$ itself is $\mathcal{Z}$-stable.) There is evidence that strict comparison and $\mathcal{Z}$-stability might prove to be equivalent (for simple, nuclear, unital and separable algebras); this has in fact has been verified in a number of instances (see, e.g. [26]).

Our results in this paper can be succinctly summarized in the following

Theorem A. *Let A be a simple unital finite C^*-algebra which is either exact and $\mathcal{Z}$-stable or an AH-algebra with slow dimension growth. Then, the following conditions are equivalent:*

(i) $\mathcal{M}(A\otimes\mathcal{K})/(A\otimes\mathcal{K})$ *is strongly purely infinite,*
(ii) $\mathcal{M}(A\otimes\mathcal{K})/(A\otimes\mathcal{K})$ *is purely infinite,*
(iii) $\mathcal{M}(A\otimes\mathcal{K})/(A\otimes\mathcal{K})$ *is weakly purely infinite,*
(iv) *All projections in* $\mathcal{M}(A\otimes\mathcal{K})/(A\otimes\mathcal{K})$ *are properly infinite,*
(v) *A has finitely many extremal traces,*
(vi) $\mathcal{M}(A\otimes\mathcal{K})/(A\otimes\mathcal{K})$ *has finitely many ideals.*

In order to prove Theorem A above, we need to establish a number of intermediate results, some of which may well be of independent interest, so this is the reason why we shall postpone its proof to the end of the paper.

It is eerie to have that projections determine pure infiniteness for algebras whose multipliers (hence also themselves) will not have real rank zero. This raises the natural question of whether the corona algebras of such algebras do have real rank zero. Despite the positive evidence, a result in this direction has not yet come by, even for the corona algebra of $\mathcal{Z}\otimes\mathcal{K}$.

In outline the paper is as follows. Section 1 is mainly devoted to reminding the reader of Cuntz comparison and the various representation theorems of the Cuntz semigroup (that have been established in [3] and [2]). These were written basically under some different hypotheses; our reformulation holds with the same proofs and has been noticed by a number of researchers. We also prove assorted useful lemmas that shall be used in the sequel.

The notions of pure infiniteness (in their various guises) are recalled in Section 2, where we prove that they are equivalent (beyond the simple case), when the algebra has finitely many ideals. This, as a consequence, yields an easy computation of the Cuntz semigroup of a purely infinite algebra with finite ideal lattice (which contains the purely infinite simple case, where the semigroup is degenerate, only consisting of $\{0,\infty\}$). Sections 3 and 4 constitute the core of the paper. In there we prove necessary conditions for purely infinite corona (described in Proposition 3.2), and sufficient conditions (Theorem 4.3).

1. CUNTZ COMPARISON

The Cuntz semigroup. For a C*-algebra A, denote as usual $\mathrm{M}_\infty(A)$ the algebraic limit of the direct system $(\mathrm{M}_n(A),\phi_n)$, where $\phi_n\colon \mathrm{M}_n(A)\to\mathrm{M}_{n+1}(A)$ is given by

$$a\mapsto\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}.$$

If a and b are positive elements in $\mathrm{M}_\infty(A)$, we write $a\oplus b$ to denote the element $\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}$, again a positive element in $\mathrm{M}_\infty(A)$.

Given $a,b\in\mathrm{M}_\infty(A)_+$, we say that a is *Cuntz subequivalent* to b, in symbols $a\precsim b$, if there is a sequence $(v_n)_{n=1}^\infty$ of elements of $\mathrm{M}_\infty(A)$ such that

$$\|v_nbv_n^*-a\|\overset{n\to\infty}{\longrightarrow}0.$$

We say that a and b are *Cuntz equivalent*, in symbols $a\sim b$, if $a\precsim b$ and $b\precsim a$. This defines an equivalence relation on $M_\infty(A)$ and the equivalence class of a is denoted by $\langle a\rangle$. The Cuntz semigroup of A is the object

$$\mathrm{W}(A):=\mathrm{M}_\infty(A)_+/\sim\,,$$

which is a positively partially ordered Abelian monoid with addition

$$\langle a\rangle+\langle b\rangle=\langle a\oplus b\rangle$$

and order

$$\langle a\rangle\leq\langle b\rangle\Leftrightarrow a\precsim b\,.$$

Given a in $\mathrm{M}_\infty(A)_+$ and $\epsilon>0$, we denote

$$(a-\varepsilon)_+=f(a)\in C^*(a)\,,\ \text{where}\ f(t)=\max\{0,t-\epsilon\}\,.$$

We shall use the following three technical facts, which are proved in, for example [23] and [10]:

Let A be a C*-algebra, and let $a,b\in A$.

(a) If $a\precsim b$ then for any $\varepsilon>0$, there is a $\delta>0$ and an element r in A such that $(a-\varepsilon)_+=r\,(b-\delta)_+\,r^*$. In particular $(a-\varepsilon)_+\precsim(b-\delta)_+$.
(b) If $(a-\varepsilon)_+\precsim b$ for any $\varepsilon>0$, then $a\precsim b$.
(c) If $\|a-b\|<\epsilon$, then $(a-\epsilon)_+\precsim b$.

Lemma 1.1. *Let A be a C*-algebra, and let a,b be positive elements in A. If $b\precsim(a-\varepsilon)_+$, then we have $(b-\varepsilon)_+=rf(a)r^*$, where f is any function that is equal to 1 on $[\varepsilon,\|a\|]$, and the norm of r is less than $\|b\|^{1/2}$.*

Proof. From the hypothesis, there is an element c with $(b-\varepsilon)_+ = c\,(a-\varepsilon)_+\,c^*$. Notice that, by construction, $f(a)$ acts as the unit from the right upon $r := c\,(a-\varepsilon)_+^{1/2}$. Thus we may write $(b-\varepsilon)_+ = rf(a)r^*$. A straightforward use of the C*-equation ensures that the norm of r is as claimed. □

Lemma 1.2. *Let A be a separable C*-algebra. Let $a \in A_+$ and $b \in \mathcal{M}(A)$. Let (b_n) be an approximate unit for $\overline{bAb}$. If $a \precsim b$, then for every $\varepsilon > 0$ there is $n \geq 1$ with $(a-\varepsilon)_+ \precsim b_n$.*

Proof. Given $\varepsilon > 0$, we know that there is x in $\mathcal{M}(A)$ with $(a-\varepsilon/2)_+^{1/2} = xb^2x^*$. Hence

$$(a-\varepsilon/2)_+ = (a-\varepsilon/2)_+^{1/4} x b^2 x^* (a-\varepsilon/2)_+^{1/4}\,,$$

so by changing notation we have $(a-\varepsilon/2)_+ = xb^2x^*$ with x in A.

Put $y = bx^*xb$, which is Cuntz equivalent to $(a-\varepsilon/2)_+$.

Therefore $(a-\varepsilon)_+ = ((a-\varepsilon/2)_+ - \varepsilon/2)_+ \precsim (y-\delta)_+$ for some $\delta > 0$. With this constant, find $n \geq 1$ such that

$$\|y - b_n y b_n\| < \delta\,.$$

Hence $(y-\delta)_+ \precsim b_n y b_n \sim y^{1/2} b_n^2 y^{1/2} \precsim b_n$. □

The function representation of the Cuntz semigroup. If A is a unital C*-algebra and $\tau \in \mathrm{T}(A)$ is a normalized trace, one defines

$$d_\tau \colon M_\infty(A)_+ \to \mathbb{R}^+$$

by $d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n})$, which is lower semicontinuous and depends only the Cuntz class of a.

One moreover has that d_τ defines a (normalized) state on $\mathrm{W}(A)$, which is referred to as a *lower semicontinuous function* (as opposed to the set of all normalized states on $\mathrm{W}(A)$, the so-called *dimension functions*). As costumary, we denote the convex set of all dimension functions by $\mathrm{DF}(A)$ and the subset of those lower semicontinuous dimension functions by $\mathrm{LDF}(A)$. It was proved in [1] that $\mathrm{LDF}(A)$ is a face of $\mathrm{DF}(A)$, usually dense (as shown in [3, Theorem 6.4]). The correspondence $\tau \mapsto d_\tau$ defines an affine bijection $\mathrm{T}(A) \to \mathrm{LDF}(A)$, generally not continuous (see [1, Theorem II.2.2]).

The interesting property is when comparison of lower semicontinuous states determines the order of the semigroup. This has attracted a good deal of attention in recent years, and has been termed *strict comparison*: $a \precsim b$ whenever $d(a) < d(b)$ for all $d \in \mathrm{LDF}(A)$.

Denote by $\mathrm{V}(A)$ the projection semigroup of A and $\mathrm{LAff}_b(\mathrm{T}(A))^{++}$ the semigroup of bounded, real-valued lower semicontinuous functions on $\mathrm{T}(A)$ which are strictly positive. Then

$$\mathrm{V}(A) \sqcup \mathrm{LAff}_b(\mathrm{T}(A))^{++}$$

is a semigroup with addition extending the usual operations and moreover $x + f = \widehat{x} + f$, where $x \in \mathrm{V}(A)$, $f \in \mathrm{LAff}_b(\mathrm{T}(A))^{++}$ and $\widehat{x}(\tau) = \tau(x)$.

This semigroup can also be ordered by taking the usual (algebraic) order on $\mathrm{V}(A)$, the pointwise ordering on $\mathrm{LAff}_b(\mathrm{T}(A))^{++}$, and by further setting for $x \in \mathrm{V}(A)$ and $f \in \mathrm{LAff}_b(\mathrm{T}(A))^{++}$:

(i) $x \leq f$ if $\widehat{x}(\tau) < f(\tau)$ for all $\tau \in \mathrm{T}(A)$, and
(ii) $f \leq x$ if $f(\tau) \leq \widehat{x}(\tau)$ for all $\tau \in \mathrm{T}(A)$.

We will say that a partially ordered semigroup $(M, \leq)$ is *weakly divisible* if for any x in M and $n \in \mathbb{N}$, there is y in M with $ny \leq x \leq (n+1)y$. It is known that C*-algebras that absorb $\mathcal{Z}$ tensorially are weakly divisible, in the sense that $\mathrm{W}(A)$ is weakly divisible, but this has not appeared explicitly in the literature. We offer a short argument below:

Lemma 1.3. *Let A be a (unital) $\mathcal{Z}$-stable* C*-*algebra. Then* $\mathrm{W}(A)$ *is weakly divisible.*

Proof. First note that since matrices over a $\mathcal{Z}$-stable algebra are also $\mathcal{Z}$-stable, we may assume that our given element $x \in \mathrm{W}(A)$ has a representative coming from A. Next, it was shown in [20, Lemma 3.4] that any element of A is Cuntz equivalent to an element of the form $a \otimes 1_{\mathcal{Z}}$.

Now, there is by [24, Lemma 4.2] a sequence of elements (e_n) in $\mathcal{Z}$ with

$$n\langle e_n\rangle \leq \langle 1_{\mathcal{Z}}\rangle \leq (n+1)\langle e_n\rangle\,,$$

and then [24, Lemma 4.1] implies that

$$n\langle a\otimes e_n\rangle \leq \langle a\otimes 1_{\mathcal{Z}}\rangle \leq (n+1)\langle a\otimes e_n\rangle\,,$$

from which the proof is complete. □

We shall make use of the following results

Theorem 1.4. ([20, Theorem 4.4], [3, Corollary 5.7]) *Let A be a simple, unital, finite C*-algebra with strict comparison. Then, the map*

$$\phi\colon \mathrm{W}(A) \to \mathrm{V}(A) \sqcup \mathrm{LAff}_b(\mathrm{T}(A))^{++}\,,$$

defined as $\phi(\langle p\rangle) = [p]$ if p is a projection and $\phi(\langle x\rangle)(\tau) = d_\tau(x)$ if x is not equivalent to a projection, is an order embedding. If $\mathrm{W}(A)$ is weakly divisible or A is an AH-algebra with slow dimension growth, then $\iota := \phi_{|\mathrm{W}(A)_+}$ is surjective, hence ϕ is an isomorphism.

This was subsequently applied to compute the Cuntz semigroup of a stabilisation of a unital C*-algebra. Denote $\mathrm{SAff}(\mathrm{T}(A))$ the semigroup of affine functions defined on $\mathrm{T}(A)$ that are pointwise suprema of increasing sequences of continuous, affine and strictly positive functions on $\mathrm{T}(A)$. If A is separable, then, because of metrizability, $\mathrm{SAff}(\mathrm{T}(A))$ is just the set of all strictly positive, affine lower semicontinuous functions (possibly unbounded) defined on $\mathrm{T}(A)$.

Theorem 1.5. ([2, Theorem 2.5])*Let A be a simple, unital exact and tracial C*-algebra with strict comparison. Then, there is an order-isomorphism*

$$\phi\colon \mathrm{W}(A\otimes\mathcal{K}) \to \mathrm{V}(A) \sqcup \mathrm{SAff}(\mathrm{T}(A))\,,$$

whenever ι as in 1.4 is surjective.

It is instructive to recall how the isomorphism above can be constructed, since for one thing we shall use this below. Let (e_n) be an approximate unit for $\mathcal{K}$ consisting of an increasing sequence of projections with $\mathrm{rank}(e_n) = n$, and put $P_n = 1 \otimes e_n$. Then, if $a \in (A\otimes\mathcal{K})_{++}$, put

$$\phi(\langle a\rangle)(\tau) = \sup_n d_\tau(P_n a P_n)\,,$$

and if p is a projection then $\phi(\langle p\rangle) = [p]$ via the corresponding identification of $\mathrm{V}(A\otimes\mathcal{K})$ with $\mathrm{V}(A)$. One can also regard the isomorphism of Theorem 1.5 as a completion via suprema of the isomorphism in Theorem 1.4.

For a any C*-algebra A, let us denote $K(A)$ the Pedersen ideal of A, that is, the minimal dense ideal of A.

Lemma 1.6. *Let A be a simple, unital exact and tracial C*-algebra with strict comparison. Let $a \in K(A \otimes \mathcal{K})_+$. Then $\phi(\langle a \rangle)$ is a bounded function.*

Proof. Take a projection e in $A \otimes \mathcal{K}$, e.g. $e = 1 \otimes e_1$, where e_1 is a rank one projection in $\mathcal{K}$. Then, for any trace τ in $\mathrm{T}(A)$, we have

$$\phi(\langle e \rangle) = \sup_n d_\tau(P_n e P_n) = \sup d_\tau(1 \otimes e_1) = \tau(1 \otimes e_1) = 1\,.$$

Then, since A is simple, $A \otimes \mathcal{K}$ is also simple and $K(A \otimes \mathcal{K})$ is algebraically simple, from which it follows that $K(A \otimes \mathcal{K}) = K(A \otimes \mathcal{K})eK(A \otimes \mathcal{K})$. Then $a = \sum_{i=1}^n a_i e b_i$ for some elements a_i, b_i in $K(A \otimes \mathcal{K})$, whence $a \precsim ne$. From this, $\phi(\langle a \rangle) \leq n\phi(\langle e \rangle) = n$. □

2. Pure infiniteness

In this section we recall the definitions of three notions of pure infiniteness (see [10] and [11])

Definition 2.1. *Let A be a C*-algebra. We say that*

(i) *A is* strongly purely infinite *if for every positive element*

$$\begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \in M_2(A)_+\,,$$

and every $\varepsilon > 0$, there are d_1, d_2 in A such that

$$\left\| \begin{pmatrix} d_1^* & 0 \\ 0 & d_2^* \end{pmatrix} \begin{pmatrix} a & x^* \\ x & b \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\| \leq \varepsilon$$

(ii) *A is* purely infinite *if every non-zero positive element a is properly infinite, i.e. $a \oplus a \precsim a$.*

(iii) *A is* weakly purely infinite *if there is an n such that every non-zero positive element a in A satisfies that $a \oplus \cdots \oplus a = a \otimes 1_n$ is properly infinite.*

Recall that a C*-algebra has *Property IP* if projections separate ideals (equivalently, if ideals are generated by projections). It is shown in [18, Proposition 2.11] that, in the separable case, zero-dimensional primitive ideal space and the purely infinite property imply property IP. We now show, using similar methods, that finite dimensional primitive ideal space and the purely infinite property imply property IP even without separability, and it then follows that such an algebra is strongly purely infinite.

We first record the following observation, that will be used tacitly whenever convenient.

Proposition 2.2. *If a C*-algebra has finitely many ideals, then for any positive element a, there exists an $\varepsilon > 0$ such that $b \leq a$ generates the same ideal as a whenever $\|a - b\| < \varepsilon$.*

Proof. Clearly b is at least contained in the ideal generated by a. Consider the union of the set of ideals that do not contain a. This is a closed set, and in particular has open complement. Thus, there is a neighbourhood of a contained in the complement, and this gives us the ε that we need. □

Proposition 2.3. *Let A be a purely infinite C*-algebra with finitely many ideals. Then A has property IP.*

Proof. We notice that each ideal is necessarily singly generated (as an ideal). This can be seen by induction on the number of ideals. It is clear in the simple case, and for the induction step, we may suppose that the result is proven for all C*-algebras with less than n ideals. Thus in

$$0 \to I_0 \to I \to I/I_0 \to 0$$

the algebras on the end are singly generated as ideals, and lifting the generator for the quotient to I and adding to it the generator for I_0 we have a single generator for I.

Suppose that I is generated by a. By Proposition 2.2, it is also generated by $(a-\varepsilon)_+$ for some $\varepsilon > 0$. Then the argument in the implication (i) $\implies$ (ii) of [18, Proposition 2.7] applies (without needing to assume separability) to show that I is actually generated by a projection. □

The following result then holds without needing to assume separability.

Proposition 2.4. *For a C*-algebra with finitely many ideals, the properties of being purely infinite, strongly purely infinite and weakly purely infinite are all equivalent. Moreover, the algebra has the IP property.*

Proof. The above proposition shows that we have property IP. Then Proposition 2.14 in [18] shows that the three properties of interest are equivalent. □

Corollary 2.5. *Let A be a C*-algebra with finitely many ideals. If A is purely infinite, then $(\mathrm{W}(A),+)$ is order-isomorphic to $\{2^n,\cup\}$ where 2^n is the Boolean algebra of subsets of an n-element set.*

Proof. Let $M = \{I \mid I \text{ is a closed ideal of } A\}$, which is an abelian semigroup under usual addition of ideals, and partially ordered by inclusion. Given $\langle a\rangle$ in $\mathrm{W}(A)$, we have $a \in M_n(A)$ for some n, and the ideal generated by a has the form $M_n(I)$, for a (unique) ideal I in M. If $a \precsim b$, clearly a belongs to the ideal generated by b, whence this defines an order-preserving map $\varphi\colon \mathrm{W}(A) \to M$. Notice this is a semigroup homomorphism as if $a \perp b$, then the ideal generated by $a+b$ equals the sum of the ideals generated by a and b.

Since A is purely infinite, φ is in fact an order-embedding. And if the ideal lattice is finite, then φ is surjective, since as we have shown every ideal will be singly generated. If n is the number of ideals, then it is clear that M is order-isomorphic to $\{2^n,\cup\}$. □

3. NECESSARY CONDITIONS

Lemma 3.1. *Let A be a separable C*-algebra, and let (a_n) be an increasing sequence with limit a (in the norm topology). Then $\langle a\rangle = \sup\langle a_n\rangle$ in $\mathrm{W}(A)$ and there is a sequence $\varepsilon_n > 0$ decreasing to zero with $\langle a\rangle = \sup\langle (a_n-\varepsilon_n)_+\rangle$.*

Proof. Since $a_n \leq a_{n+1}$, we have $a_n \precsim a_{n+1}$. Given $\varepsilon > 0$, there is n_0 with $\|a-a_n\| < \varepsilon$ if $n \geq n_0$. Then $\langle (a-\varepsilon)_+\rangle \leq \langle a_n\rangle \leq \sup\langle a_n\rangle$. From this it follows that

$$\langle a\rangle = \sup_{\varepsilon>0}\langle (a-\varepsilon)_+\rangle \leq \sup\langle a_n\rangle \leq \langle a\rangle\,.$$

For the second part, choose $\varepsilon_n < 1/n$ inductively. Pick $\varepsilon_1 < 1$. Since $a_1 \precsim a_2$, there is $\delta > 0$ with $(a_1-\varepsilon_1)_+ \precsim (a_2-\delta)_+$. Let $\varepsilon_2 < \min\{\varepsilon_1,\delta,1/2\}$, so that $(a_2-\delta)_+ \leq (a_2-\varepsilon_2)_+$. We obtain in this way a sequence (ε_n) decreasing to zero and $(a_n-\varepsilon_n)_+ \precsim (a_{n+1}-\varepsilon_{n+1})_+$. Note that

$$\|a-(a_n-\varepsilon_n)_+\| \leq \|a-a_n\| + \|a_n-(a_n-\varepsilon_n)_+\| \leq \|a-a_n\| + \varepsilon_n \to 0$$

as $n\to\infty$, so $\lim_n (a_n-\varepsilon_n)_+ = a$. As in the first part of the proof, given $\varepsilon>0$ there is n with $(a-\varepsilon)_+ \precsim (a_n-\varepsilon_n)_+$, and it follows that

$$\langle (a-\varepsilon)_+\rangle \leq \sup\langle (a_n-\varepsilon_n)_+\rangle \leq \sup\langle a_n\rangle = \langle a\rangle\,.$$

□

Proposition 3.2. *Let A be a unital, separable C*-algebra with stable rank one and assume that A is moreover simple, exact with strict comparison and weakly divisible $\mathrm{W}(A)$.*

Suppose that for each $k\geq 1$ and every non-zero projection p in $M_k(\mathcal{M}(A\otimes\mathcal{K}))\setminus M_k(A\otimes\mathcal{K})$, there is m such that the image of $p\otimes 1_m$ in the corona is properly infinite. Then A has finitely many extremal traces.

Proof. We argue by contradiction, so assume there is a sequence of distinct extremal traces $\{\tau_n\}$ in $\mathrm{T}(A)$.

Define a sequence of continuous, affine, strictly positive functions (f_i) on $\mathrm{T}(A)$ as follows. Define $f_1\in \mathrm{Aff}(\mathrm{T}(A))^{++}$ such that $f_1(\tau_1)=1$ (by using, e.g. [8, Theorem 11.14]). Likewise, define f_2' as $f_2'(\tau_1)=1$ and $f_2'(\tau_2)=2$ and use [8, Corollary 11.16] to find $f_2\in\mathrm{Aff}(\mathrm{T}(A))^{++}$ such that $f_1, f_2'\leq f_2$ and $f_2(\tau_i)=\max\{f_1(\tau_i), f_2'(\tau_i)\}$, for $i=1,2$. We thus see that $f_1\leq f_2$, $f_2(\tau_1)\geq 1$ and $f_2(\tau_2)\geq 2$. Continuing in this way we get an increasing sequence (f_n) such that $f_i(\tau_j)\geq j$ for $1\leq j\leq i$. Let $f=\sup_i f_i$. Then $f\in\mathrm{SAff}(\mathrm{T}(A))$ and $f(\tau_i)\geq i$, hence $(f(\tau_i))$ is an unbounded sequence. By Theorem 1.5 (see also Theorem 1.4), there is x in $\mathrm{W}(A\otimes\mathcal{K})$ such that f represents x.

Since A has stable rank one, it follows from the results in [4] that there is a semigroup isomorphism

$$\varphi\colon \mathrm{V}(\mathcal{M}(A\otimes\mathcal{K}))\to \mathrm{W}(A\otimes\mathcal{K})\,.$$

(Notice though that this is *not* an ordered semigroup isomorphism, since the order in the Cuntz semigroup is not the algebraic order.)

Thus, we can find a projection p in $M_k(\mathcal{M}(A\otimes\mathcal{K}))$ (for some k) such that $\varphi([p])=x$. Let

$$\pi\colon \mathcal{M}(A\otimes\mathcal{K})\to\mathcal{M}(A\otimes\mathcal{K})/A\otimes\mathcal{K}$$

denote the quotient map, and observe that $\pi(p)\neq 0$.

Put $q:=p\otimes 1_m$, where $m\in\mathbb{N}$. We claim now that $\pi(q)$ cannot be properly infinite, and this will contradict our assumption. Suppose then that $\pi(q)\oplus\pi(q)\precsim\pi(q)$. There is then an element a in $M_m(M_k(A\otimes\mathcal{K}))=M_{km}(A\otimes\mathcal{K})$ such that, upstairs, we have

$$q\oplus q\precsim q\oplus a\,.$$

Since, for $\varepsilon<1$, $q\sim(q-\varepsilon)_+$, there is $\delta>0$ such that $q\oplus q\precsim q\oplus(a-\delta)_+$, and $(a-\delta)_+\in K(M_{km}(A\otimes\mathcal{K}))_+$, hence without loss of generality we have $a\in K(M_{km}(A\otimes\mathcal{K}))_+ = M_{km}(K(A\otimes\mathcal{K}))_+$.

Next, let (u_n) be an approximate unit for $pM_k(A\otimes\mathcal{K})p$, and define

$$v_n=\frac{1}{2^{n-1}}(u_n-u_{n-1})+\frac{1}{2^{n-2}}(u_{n-1}-u_{n-2})+\cdots+\frac{1}{2}(u_2-u_1)+u_1\,.$$

Then $v_n \precsim p$ and $\lim_{n\to\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{2^n} u_n$, which we shall denote by u. By Lemma 3.1, there is a sequence of strictly positive real numbers (ϵ_n) decreasing to zero such that $\langle (v_n - \varepsilon_n)_+ \rangle$ has $\langle u \rangle$ as supremum.

Notice that, for each n, we have $u_n \precsim u$, and in fact $\varphi([p]) = \langle u \rangle = x$ (and $\varphi([q]) = \varphi([p \otimes 1_m]) = mx$).

Now, if (w_n) is an approximate unit for $aM_{km}(A \otimes \mathcal{K})a$, clearly $(u_n \otimes 1_m) \oplus w_n$ is an approximate unit for $((u \otimes 1_m) \oplus a) M_{2km}(A \otimes \mathcal{K})((u \otimes 1_m) \oplus a)$.

For each n, apply Lemma 1.2 to ε_n so there is l with

$$((v_n - \varepsilon_n)_+ \otimes 1_m) \oplus ((v_n - \varepsilon_n)_+ \otimes 1_m) \precsim (u_l \otimes 1_m) \oplus w_l \precsim (u \otimes 1_m) \oplus a \,.$$

Next, applying ι from Theorems 1.4 and 1.5 and using Lemma 1.6 so that $\iota(\langle a \rangle) \leq M$ for some positive constant M, we get

$$2m\iota(\langle (v_n - \varepsilon_n)_+ \rangle) \leq m\iota(\langle u \rangle) + \iota(\langle a \rangle) \leq mf + M \,.$$

Since ι preserves suprema, it follows that

$$2mf = 2m\iota(\langle u \rangle) = 2m\iota(\sup \langle v_n - \varepsilon_n)_+ \rangle) = \sup 2m\iota(\langle (v_n - \varepsilon_n)_+ \rangle) \leq mf + M \,.$$

This implies that $mf(\tau_i) \leq M$ for any i, a contradiction. □

Theorem 3.3. *Let A be a unital, separable, simple, exact C*-algebra with strict comparison and such that $\mathrm{W}(A)$ is weakly divisible. If $\mathcal{M}(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ is weakly purely infinite, then A has finitely many extremal traces.*

Proof. This follows from Proposition 3.2 together with the fact that weak pure infiniteness is stable under matrix formation (see [11, Proposition 4.5]). □

4. Sufficient conditions

Lemma 4.1. *Let $\sum a_n$ and $\sum b_n$ be sums of projections converging strictly in the multipliers of some σ-unital C*-algebra A. Suppose also that the $\{a_n\}$ are pairwise orthogonal, and that the $\{b_n\}$ are pairwise orthogonal. Suppose now that there is a sequence m_n in the unit ball of the multipliers $\mathcal{M}(A)$. Then $\sum a_n m_n b_n$ converges strictly.*

Proof. Let s be a strictly positive element of the algebra. It is enough to show that $\|\sum_l^m s(a_n m_n b_n)\|$ and $\|\sum_l^m (a_n m_n b_n)s\|$ both have the Cauchy property: namely, given $\varepsilon > 0$, there is an N making both of these sums less than ε in norm for all $m > n > N$. But

$$\left\| \sum_l^m s(a_n m_n b_n) \right\| = \left\| \sum_l^m s(a_n m_n b_n b_n^* m_n^* a_n) s \right\|^{1/2}$$

and of course $\sum_l^m s(a_n m_n b_n b_n^* m_n^* a_n)s \leq s\left(\sum_l^m a_n\right)s$. It is clear that this sum will have the Cauchy property, since the a_n converge strictly. The other sum of interest can be estimated similarly. □

Let (p_n) be a sequence of pairwise orthogonal projections such that $\sum_n p_n$ converges strictly. We say that an element is *diagonal with respect to* (p_n) if it is of the form $\sum p_n m_n p_n$ for some sequence of elements (m_n) of the multipliers. Recall that if τ is a normalized trace on a unital C*-algebra A, then it extends to a normalized trace τ' of $\mathcal{M}(A\otimes\mathcal{K})$ by $\tau'(a)=\sup_n \tau((\sum_{l=1}^n p_l)a(\sum_{l=1}^n p_l))$.

For the reader's convenience, we briefly summarize some facts (used in the proof of the next results) about the ideal structure of the multipliers of $A\otimes\mathcal{K}$ where A is a simple, unital C*-algebra with strict comparison and finitely many extremal traces, following [22] (see also [15] for the AF-algebras case). There are finitely many such ideals, and each proper one is an intersection of maximal ideals. The maximal ideals of $\mathcal{M}(A\otimes\mathcal{K})$ are exactly the ones given by extension of an extremal trace τ to the multiplier algebra, in the following form

$$I_\tau=\{a\in\mathcal{M}(A\otimes\mathcal{K})\mid \tau(a^*a)<\infty\}^- .$$

Lemma 4.2. *Let A be a unital C*-algebra, let $\tau\in \mathrm{T}(A)$ and d_τ be the corresponding lower semi-continuous dimension function. Then, for any $a\in\mathcal{M}(A\otimes\mathcal{K})$, the following conditions are equivalent:*

(i) $a\in(I_\tau)_+$
(ii) $\tau((a-\varepsilon)_+)<\infty$ *for any* ε,
(iii) $d_\tau((a-\varepsilon)_+)<\infty$ *for any* ε.

Proof. (i) $\Longrightarrow$ (ii). If $a\in(I_\tau)_+$, then for any ε there is a contraction d and a positive x (in $\mathcal{M}(A\otimes\mathcal{K})$) with $(a-\varepsilon)_+=dxd^*$ and $\tau(x)<\infty$, by [11, Lemma 2.2] from which we infer that $\tau((a-\varepsilon)_+)<\infty$.

(ii) $\Longrightarrow$ (iii). Notice that, by the functional calculus and for any given $\varepsilon>0$, there exists $C>0$ such that for all $n\geq 1$,

$$(a-\varepsilon)_+^{1/n}\leq C\left(a-\frac{1}{2}\varepsilon\right)_+ .$$

Thus, $d_\tau((a-\varepsilon)_+)\leq C\tau(\left(a-\frac{1}{2}\varepsilon\right)_+)$, and (iii) follows.

Finally, assuming (iii) we see that $\tau((a-\varepsilon)_+)$ is finite, so $(a-\varepsilon)_+\in I_\tau$ and so is a, being the limit of $(a-\varepsilon)_+$ as ε goes to zero. □

Theorem 4.3. *Let A be a unital, simple, separable, exact C*-algebra with strict comparison and finitely many extremal traces. Let $\Lambda=\sum a_n$ be a diagonal element in $\mathcal{M}(A\otimes\mathcal{K})$ with respect to a sequence of pairwise orthogonal projections (q_n) in $A\otimes\mathcal{K}$ such that $\sum_n q_n=1_{\mathcal{M}(A\otimes\mathcal{K})}$, where the sum converges in the strict topology. Then Λ has an image which is either zero or properly infinite in $\mathcal{M}(A\otimes\mathcal{K})/A\otimes\mathcal{K}$.*

Proof. If the a_n form a norm-convergent series, then Λ is in A and there is nothing to prove. If this is not the case, then by passing to blocks (and replacing the q_n by a subsequence) we may assume that the norms of the a_n are bounded above and below by some positive constants.

By Proposition 2.2 and the finiteness of the ideal lattice, there is an $\varepsilon>0$ such that $(\Lambda-2\varepsilon)_+$ generates the same ideal as Λ, and so that $(a_i-2\varepsilon)_+$ is not zero for any i (and also $(\Lambda-\delta)_+$ generates the same ideal as Λ whenever $\delta<2\varepsilon$).

If the the set of traces is empty, then the corona is simple, whence also purely infinite simple and in this case there is nothing to prove. Otherwise, let $\{\tau_1,\ldots,\tau_m\}$ and $\{\tau_1',\ldots,\tau_k'\}$ the extremal traces such that the d_{τ_i} are infinite upon $(\Lambda-\varepsilon)_+$ and the $d_{\tau_j'}$ are finite upon $(\Lambda-\varepsilon)_+$. The case in which one of these two sets is empty is not excluded. To ease the notation, put $d_i = d_{\tau_i}$ and $\phi_j = d_{\tau'_j}$.

By stability, the BDF sum $\Lambda+\Lambda$ is equivalent to a diagonal element over the algebra A. Let (e_n) denote the approximate unit of projections associated to $\Lambda+\Lambda$ and let b_n denote the corresponding elements (which are of course equivalent to the BDF sum a_n+a_n). We now construct two sequences of integers m_i and n_i inductively, also constructing one positive element p during the first step. Let $m_0=1$. The first step chooses n_0, p, and m_1, in that order. By the norm convergence of $\sum_{k=1}^{\infty}\phi_i((a_k-\varepsilon)_+)$, choose n_0 so

$$\sum_{k=n_0+1}^{\infty}\phi_i((b_k-\varepsilon)_+) < \phi_i((a_{m_0}-\varepsilon)_+) \text{ for all } i\,. \tag{4.1}$$

We choose p such that $\phi_i(p) > \sum_{k=1}^{n_0}\phi_i(b_k)$, and we may also for convenience suppose that p is orthogonal to all the a_i (by one more BDF sum). By the divergence of $\sum_{k=1}^{\infty} d_i((a_k-\varepsilon)_+)$, choose m_1 so that

$$\sum_{k=1}^{n_0} d_i((b_k-\varepsilon)_+) < \sum_{k=n_0+1}^{m_1-1} d_i((a_k-\varepsilon)_+)\,. \tag{4.2}$$

In the next step of the induction, we choose $n_1\ge m_1$, and then $m_2\ge n_1$,

$$\sum_{k=n_1+1}^{\infty}\phi_i((b_k-\varepsilon)_+) < \phi_i((a_{m_1}-\varepsilon)_+) \tag{4.3}$$

$$\sum_{k=n_0+1}^{n_1}\phi_i((b_k-\varepsilon)_+) < \phi_i((a_{m_0}-\varepsilon)_+) \tag{4.4}$$

$$\sum_{k=n_0+1}^{n_1} d_i((b_k-\varepsilon)_+) < \sum_{k=n_1+1}^{m_2-1} d_i((a_k-\varepsilon)_+) \tag{4.5}$$

The first equation above is obtained by choosing n_1 appropriately, using the fact $\sum_{k=1}^{\infty}\phi_i((a_k-\varepsilon)_+)$ converges in norm, the second equation follows immediately from equation 4.1, and then the third one is obtained by choosing m_2 using the fact that $\sum_{k=1}^{\infty} d_i((a_k-\varepsilon)_+)$ diverges.

From the first step of the induction, we deduce that $\sum_{k=1}^{n_0}(b_k-\varepsilon)_+$ is majorized, on all dimension functions, by $p+\sum_{k=n_0+1}^{m_1-1}(a_k-\varepsilon)_+$. The comparison properties then give that

$$\sum_{k=1}^{n_0}(b_k-\varepsilon)_+ \precsim p+\sum_{k=n_0+1}^{m_1-1}(a_k-\varepsilon)_+\,. \tag{4.6}$$

From the second step of the induction, we deduce that $\sum_{n_0+1}^{n_1}(b_k-\varepsilon)_+$ is majorized, on all dimension functions, by $(a_{m_0}-\varepsilon)_+ + \sum_{k=n_1+1}^{m_2-1}(a_k-\varepsilon)_+$. The induction step is the same as step 2, but with generalized indices. Therefore, we conclude that for $\ell=0,1,2,\cdots$ we have

$$\sum_{k=n_\ell+1}^{n_{\ell+1}}(b_k-\varepsilon)_+ \precsim (a_{m_\ell}-\varepsilon)_+ + \sum_{k=n_{\ell+1}+1}^{m_{\ell+2}-1}(a_k-\varepsilon)_+\,. \tag{4.7}$$

All terms on the right in 4.7 are mutually orthogonal, since $m_\ell < n_\ell$. Comparing the right hand sides of 4.7 for different choices of ℓ, we note that a_{m_ℓ} is orthogonal to any block of the form $\displaystyle\sum_{k=n_{\ell'+1}+1}^{m_{\ell'+2}-1} (a_k - \varepsilon)_+$, the indices having been chosen so that the a_{m_ℓ} fall into the gaps between these blocks.

Applying Lemma 1.1 to 4.6 and 4.7 we have

$$\sum_{k=1}^{n_0} (b_k - 2\varepsilon)_+ = r_p f\left(p + \sum_{k=n_0+1}^{m_1-1} a_k\right) r_p^* \tag{4.8}$$

$$\sum_{k=n_\ell+1}^{n_{\ell+1}} (b_k - 2\varepsilon)_+ = r_\ell f\left(a_{m_\ell} + \sum_{k=n_{\ell+1}+1}^{m_{\ell+2}-1} a_k\right) r_\ell^* \tag{4.9}$$

where $f(\lambda)$ is a function that acts as the unit upon $(\lambda - \varepsilon)_+$. Note that the set of elements $\{r_p\} \cup \{r_\ell\}_\ell$ is uniformly bounded (also by Lemma 1.1.

From the proof of Lemma 1.1, we notice that $q_{m_\ell} + \sum_{k=n_{\ell+1}+1}^{m_{\ell+2}-1} q_k$ acts as a unit on the right for r_ℓ, and that $\sum_{k=n_\ell+1}^{n_{\ell+1}} e_k$ acts as a unit on the left for r_ℓ (where (e_n) is the chosen approximate unit consisting of projections associated to the diagonal element $\Lambda + \Lambda$). On the other hand, $p + \sum_{k=n_0+1}^{m_1-1} q_k$ and $\sum_{k=1}^{n_0} e_k$ act as units from the right and left, respectively, for r_p.

Define $r := r_p + \sum_{\ell=0}^{\infty} r_\ell$, which is a strictly convergent sum in view of the previous considerations and using Lemma 4.1. Moreover, our remarks show that

$$r\left(p + \sum_{k=n_0+1}^{m_1-1} a_k\right) = r_p\left(p + \sum_{k=n_0+1}^{m_1-1} a_k\right)$$

and

$$r\left(a_{m_\ell} + \sum_{k=n_{\ell+1}+1}^{m_{\ell+2}-1} a_k\right) = r_l\left(a_{m_\ell} + \sum_{k=n_{\ell+1}+1}^{m_{\ell+2}-1} a_k\right).$$

Taking thus into account that $f(t) = g(t)t$ for some bounded, positive function g, we have (using 4.8 and 4.9), that

$$\begin{aligned}((\Lambda + \Lambda) - 2\varepsilon)_+ &= \sum_{k=1}^{n_0} (b_k - 2\varepsilon)_+ + \sum_{l \geq 0} \sum_{k=n_\ell+1}^{n_{\ell+1}} (b_k - 2\varepsilon)_+ \\ &= r_p f\left(p + \sum_{k=n_0+1}^{m_1-1} a_k\right) r_p^* + \sum_\ell r_\ell f\left(a_{m_\ell} + \sum_{k=n_{\ell+1}+1}^{m_{\ell+2}-1} a_k\right) r_\ell^* \\ &= r f(p + \Lambda) r^* .\end{aligned}$$

Let $\pi\colon \mathcal{M}(A\otimes\mathcal{K}) \to \mathcal{M}(A\otimes\mathcal{K})/(A\otimes\mathcal{K})$ be the natural quotient map. Since p is in $A\otimes\mathcal{K}$, when we pass to the corona we have $((\pi(\Lambda) + \pi(\Lambda)) - 2\varepsilon)_+ = \pi(r)f(\pi(\Lambda))\pi(r)^*$, for any sufficiently small ε. Taking again into account that $f(\Lambda)$ is a multiple of Λ, and since $\epsilon > 0$ is arbitrarily small, we have that Λ is purely infinite in the corona. □

Proposition 4.4. *Let A be a unital, simple, separable, exact* C*-*algebra with strict comparison of positive elements, and suppose that A moreover has finitely many extremal traces. Given any positive element $a \in \mathcal{M}(A\otimes\mathcal{K})/(A\otimes\mathcal{K})$, there is a diagonal element $\Lambda \in \mathcal{M}(A)$ such that $\Lambda \leq a$ in the corona, and Λ generates the same corona ideal as a.*

Proof. We may as well suppose a is a positive element of the multipliers (by lifting the given a). We now use an argument that appears in the proof of [6, Theorem 3.1] and also in [13, Lemma 2.6], that we outline here for completeness.

Let (e_n) be an approximate unit of projections for $A \otimes \mathcal{K}$ (with $e_0 = 0$). Passing to a subsequence if necessary, we may assume that

$$\sum_{n=1}^{\infty} \|(1 - e_{n+1})a^{1/2}(e_n - e_{n-1})\| < \infty\,.$$

Write $f_n = e_n - e_{n-1}$. Then put

$$y_i = \sum_{n=0}^{\infty}(f_{3n+1+i} + f_{3n+i} + f_{3n+i-1})a^{1/2}f_{3n+i}\,,$$

for $i = 0, 1, 2$, which yields $a^{1/2} = y_0 + y_1 + y_2 + b$, and $b \in A \otimes \mathcal{K}$. The construction gives, moreover, that $y_i y_j^* = 0$ whenever $i \neq j$. Taking this into account, and squaring this decomposition, we obtain that

$$a = y_0y_0^* + y_1y_1^* + y_2y_2^* + b'\,,$$

for some $b' \in A \otimes \mathcal{K}$. Put $x_i = y_iy_i^*$, and note that x_i is diagonal with respect to $g_n^i := f_{3n+1+i} + f_{3n+i} + f_{3n+i-1}$. Our Proposition 2.2 and the fact that there are finitely many multiplier ideals ([22]) shows that we are free to replace each x_i by a slightly smaller element $(x_i - \varepsilon)_+$ that is in the Pedersen ideal. We suppose this has been done, and denote $x_0 + x_1 + x_2$ by x.

Let $\{\tau_1, \ldots, \tau_k\}$ be the set of extremal traces where the corresponding lower semicontinuous functions (evaluated at x) $d_{\tau_i}(x)$ are infinite. We will construct a diagonal element Λ in the Pedersen ideal that is infinite on this same set of traces, and is majorized by x. It will then follow from the fact that all multiplier ideals come from tracial ideals (using again [22]) that x and Λ generate the same multiplier ideal, and hence that a and Λ generate the same corona ideal.

The first observation is that whenever $d_{\tau_j}(x_i)$ is infinite, there is some block b within x_i such that $d_{\tau_j}\left(\sum_{n=k}^{\ell} g_n^i x_i g_n^i\right) > 1$, and the second observation is that if we have already chosen such blocks $b_1, b_2, b_3, ..., b_N$ we can always make sure that b is orthogonal to $b_1, b_2, b_3, ..., b_N$ by making k sufficiently large (this follows from the fact that g_n^i is orthogonal to g_m^j for all i and j if $|m - n| > 3$).

We may choose mutually orthogonal blocks $b_1, b_2, b_3, ...,$ such that $d_1(b_1) > 1$, $d_2(b_2) > 1$, etc. This defines a diagonal element $\Lambda := \sum b_i$ (diagonal with respect to some subsequence of the (e_n)) where $d_j(b_i) > 1$ whenever $i = j$ modulo k. Clearly $\Lambda \leq x_1 + x_2 + x_3 = x$ so x majorizes Λ, and the tracial properties ensure that Λ and x generate the same ideal. □

Theorem 4.5. *Let A be a unital, simple, separable, exact* C*-*algebra with strict comparison of positive elements, and assume that A has finitely many extremal traces. Then $\mathcal{M}(A\otimes\mathcal{K})/(A\otimes\mathcal{K})$*

has finitely many ideals and is purely infinite. Hence it is also strongly and weakly purely infinite and every hereditary subalgebra has the IP property.

Proof. That the corona has finitely many ideals follows from our assumption on strict comparison and the results in [22, Section 4]. Therefore we only need to prove pure infiniteness of the corona as the strongly and weakly purely infinite properties, as well as the IP property for hereditary subalgebras will follow from Proposition 2.4.

Choose a (non-zero) positive element a in the corona. Then, Proposition 4.4 gives a diagonal element $\Lambda \leq a$ that generates the same ideal as a in the corona (and hence it is also non-zero). By Theorem 4.3, the element Λ is properly infinite. But since Λ is full in the ideal generated by a, we have $a \precsim n\Lambda$, which by the properly infinite property implies that $a \precsim \Lambda$. But of course $\Lambda \precsim a$ since a majorizes Λ. Hence a, being Cuntz equivalent to a properly infinite element, is itself properly infinite. □

Proof of Theorem A: That (i) $\implies$ (ii) $\implies$ (iii) follows from [11, Proposition 5.4] and [10, Theorem 4.16], respectively (and holds for any C*-algebra).

(iii) $\implies$ (v) is proved in Theorem 3.3.

(v) $\implies$ (i) is proved in Theorem 4.5.

(iv) $\implies$ (v) follows from Proposition 3.2

(v) $\implies$ (vi) follows from [22, Theorem 4.4].

(vi) $\implies$ (v) was noticed in, e.g. one of the conditions in [14, Theorem 3.6] (which in fact assumes real rank zero, but that is not needed for that part of the proof). □

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