Continuity in the Hurst parameter of the law of the Wiener integral with respect to the fractional Brownian motion

Maria Jolis Noèlia Viles
Departament de Matemàtiques, Universitat Autònoma de Barcelona
08193-Bellaterra, Barcelona, Spain.
E-mail addresses: mjolis@mat.uab.cat, nviles@mat.uab.cat

Abstract

We prove the convergence in law, in the space of continuous functions $\mathscr{C}([0,T])$, of the Wiener integral of a deterministic function f with respect to the fractional Brownian motion with Hurst parameter H to the Wiener integral of f with respect to the fractional Brownian motion with parameter H_0, when H tends to $H_0 \in (0,1/2]$.

1 Introduction

Consider the laws in $\mathscr{C}([0,T])$ of the family of processes $\{B^H,\, H \in (0,1)\}$, where each $B^H = \{B_t^H,\, t \in [0,T]\}$ is a fractional Brownian motion with Hurst parameter H. It is easily seen that these laws converge weakly to that of B^{H_0}, when H tends to $H_0 \in (0,1)$ (for a proof of this result, see the Introduction of [8]).

It is an interesting question to study if some important functionals of the fractional Brownian motion conserve this property. That is, we ask if the law of the functional of B^H remains near to that of the same functional of B^{H_0}, when H is near to H_0. It is worth to mention that this kind of results justify the use of $B^{\hat{H}}$ as a model in applied situations where the true value of the Hurst parameter is unknown and $\hat{H}$ is some estimation of it.

In the previous works [7] and [8], we have considered this problem for the functionals given by the multiple fractional integrals with $H \in (1/2,1)$ and the local time, respectively.

In [7], as a first and easy step to prove the weak convergence of the multiple Itô-Wiener and Stratonovich integrals, we have proved the convergence in law, in $\mathscr{C}([0,T])$, of the family of Wiener integrals of a deterministic function with respect to B^H with $H \in (\frac{1}{2},1)$, when $H \to H_0$.

In this note we study the weak convergence of the Wiener integrals when $H \in (0,\frac{1}{2})$. We do not consider the problem of the convergence of multiple integrals, because the domains and the norms involved in the multiple integrals are much more complicated in this case (see [4], for instance).

We will prove in our main result (see Theorem 3.5) that for f belonging to the domain of the Wiener integral with respect to $B^{H'}$, with $0 < H' < H_0$, the Wiener integral of f with respect to B^H, with $H \in [H', 1/2]$ converges in law to the Wiener integral of f with respect to B^{H_0}.

We have organized the paper as follows. In Section 2 we give some preliminaries about the Wiener integral with respect to the fractional Brownian motion with Hurst parameter $H < 1/2$, following [4]. Using the characterization of the domain of the integral given in this last paper, we prove in Section 3 the result about the convergence in law of the Wiener integral of a function with respect to B^H, in $\mathscr{C}([0,T])$, when $H \to H_0$.

MSC 2000: 60B12, 60H05, 60G15.
Key words and phrases: Fractional Brownian motion, Wiener integral, convergence in law.

2 Preliminaries

Let $B^H = \{B_t^H,\, t \in [0,T]\}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. That is, B^H is a centered Gaussian process with covariance function given by

$$R_H(s,t) = E[B_s^H B_t^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Next, we introduce the Wiener integral with respect to B^H and, from now on, we only consider the case $H < 1/2$, although H_0 can be also equal to $1/2$. Consider the space of simple functions $\mathcal{S}$ in [0,T] of the form

$$f = \sum_{i=1}^{N} f_i \mathbf{1}_{[a_i,b_i)}, \quad \text{where} \quad [a_i,b_i) \subset [0,T].$$

The Wiener integral of f as above with respect to the fractional Brownian motion B^H can be defined in the natural way as

$$I_1^H(f) = \sum_{i=1}^{N} f_i (B_{b_i}^H - B_{a_i}^H).$$

One can easily see that $I_1^H(f)$ does not depend on the particular representation of f as a simple function and that I_1^H is a linear map from $\mathcal{S}$ into a subspace of $L^2(\Omega)$.

For any $f,\ g \in \mathcal{S}$ one can define the following scalar product

$$\Psi_H(f,g) = E[I_1^H(f) I_1^H(g)].$$

The Wiener integral can be extended in a standard way to the space $\mathcal{L}^H$ that is the completion of $\mathcal{S}$ with respect to the inner product Ψ_H. This completion is called *the domain of the Wiener integral*. This space is well-known and can be characterized in terms of fractional derivatives (see, for instance, [2], [5] or [9]).

Nevertheless, along this paper we will use another characterization of the domain proved in [4] because it is related in a more direct form with the usual fractional Sobolev spaces. This characterization is based in the following result:

Lemma 2.1 *(Lemma 2.1., [4]) For all $f,\ g \in \mathcal{S}$, we have that*

$$\begin{aligned}\Psi_H(f,g) &= \frac{1}{2}H(1-2H)\int_0^T\int_0^T \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{2-2H}}dxdy \\ &+ H\int_0^T f(x)g(x)\left[\frac{1}{x^{1-2H}} + \frac{1}{(T-x)^{1-2H}}\right]dx.\end{aligned}$$

It can be given also a compact form of the above equality as follows (see [4]):

$$\Psi_H(f,g) = \frac{1}{2}H(1-2H)\iint_{\mathbb{R}^2} \frac{(\bar{f}(x)-\bar{f}(y))(\bar{g}(x)-\bar{g}(y))}{|x-y|^{2-2H}}dxdy, \tag{1}$$

where

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \in [0,T] \\ 0, & \text{otherwise.}\end{cases}$$

For all measurable $f:[0,T]\to\mathbb{R}$, we can introduce the following, possibly infinite, quantity:

$$\|f\|_{H}=\left[\frac{1}{2}H(1-2H)\int_0^T\int_0^T\frac{(f(x)-f(y))^2}{|x-y|^{2-2H}}dxdy\right.$$
$$\left.+\,H\int_0^T f(x)^2\left(\frac{1}{x^{1-2H}}+\frac{1}{(T-x)^{1-2H}}\right)dx\right]^{1/2}. \tag{2}$$

In the following theorem (see Theorem 2.5, [4]) the characterization of the space $\mathcal{L}^H$ is given.

Theorem 2.2 *(Theorem 2.5, [4]) The domain of the Wiener integral I_1^H is given by*

$$\mathcal{L}^H=\{f\in L^2([0,T]):\ \|f\|_{H}<+\infty\}.$$

If we provide this space with the scalar product

$$\Psi_{H}(f,g)=\frac{1}{2}H(1-2H)\int_0^T\int_0^T\frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{2-2H}}dxdy$$
$$+\,H\int_0^T f(x)g(x)\left[\frac{1}{x^{1-2H}}+\frac{1}{(T-x)^{1-2H}}\right]dx, \tag{3}$$

then the Wiener integral I_1^H is an isometry between $\mathcal{L}^H$ and a closed subspace of $L^2(\Omega)$.

We recall also here the definition of the Sobolev spaces of fractionary order on an interval $[0,T]$. Let $\alpha\in\mathbb{R}$ and $p\in(1,+\infty)$, for any measurable $f:[0,T]\longrightarrow\mathbb{R}$ define the following (possibly infinite) quantity

$$\|f\|_{\alpha,p}=\left(\iint_{[0,T]^2}\frac{|f(x)-f(y)|^p}{|x-y|^{1+\alpha p}}dxdy\right)^{1/p}. \tag{4}$$

For $\alpha>0$, the Sobolev space $W^{\alpha,p}([0,T])$ is defined as

$$W^{\alpha,p}([0,T])=\{f:\|f\|_{L^p([0,T]}+\|f\|_{\alpha,p}<+\infty\}. \tag{5}$$

This space provided with the norm $\|f\|_{L^p([0,T])}+\|f\|_{\alpha,p}$ is a Banach space.
When $p=2$, $W^{\alpha,2}([0,T])$ is a Hilbert space with scalar product defined by

$$\langle f,g\rangle_{W^{\alpha,2}([0,T])}=\iint_{[0,T]^2}\frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{1+2\alpha}}dxdy+\langle f,g\rangle_{L^2([0,T])}.$$

In the following section we will work with the fractionary Sobolev spaces $W^{1/2-H,2}([0,T])$ of order $\alpha=\frac{1}{2}-H$, that is,

$$W^{1/2-H,2}([0,T])=\left\{f\in L^2([0,T]):\iint_{[0,T]^2}\frac{(f(x)-f(y))^2}{|x-y|^{2-2H}}dxdy<+\infty\right\}. \tag{6}$$

Remark 2.3 Owing Theorem 2.2, we can give the following characterization of the domain of the Wiener integral with respect to the fractional Brownian motion when $H\in(0,1/2)$:

$$\mathcal{L}^H=\{f\in W^{1/2-H,2}([0,T]):\ \int_0^T f(x)^2\left(\frac{1}{x^{1-2H}}+\frac{1}{(T-x)^{1-2H}}\right)dx<\infty\}$$

and, using the compact expression of Ψ_H given in (1), we have also this other characterization of the space $\mathcal{L}_T^H$:

$$\mathcal{L}^H = \{f \in L^2([0,T]) : \ \bar{f} \in W^{1/2-H,2}(\mathbb{R})\},$$

where $W^{1/2-H,2}(\mathbb{R})$ is the Sobolev space defined by

$$W^{1/2-H,2}(\mathbb{R}) = \Big\{ f \in L^2(\mathbb{R}) : \iint_{\mathbb{R}^2} \frac{(f(x)-f(y))^2}{|x-y|^{2-2H}} dxdy < +\infty \Big\}.$$

We will need also the following Sobolev Embedding Theorem (see Theorem 5.4, [1]).

Theorem 2.4 *(Sobolev Embedding Theorem) Suppose that $0 < \alpha < \frac{1}{p}$ with $p \in (1, +\infty)$. Then,*

$$W^{\alpha,p}([0,T]) \hookrightarrow L^q([0,T]),$$

where $p \leqslant q \leqslant \frac{p}{1-\alpha p}$.

Applying the above Sobolev Embedding Theorem with $p = 2$, $\alpha = \frac{1}{2} - H$ and $q = \frac{p}{1-\alpha p} = \frac{1}{H}$, we obtain that

$$W^{1/2-H,2}([0,T]) \hookrightarrow L^{1/H}([0,T]),$$

and, as a consequence,

$$\mathcal{L}^H \subset L^{1/H}([0,T]).$$

3 Convergence in law of the Wiener integral

3.1 Results in order to obtain Tightness

Along this section, we will denote by

$$\mathcal{I}_1^H(f) = \{I_1^H(f\, 1_{[0,t]}),\, t \in [0,T]\},$$

the Wiener integral of f with respect to B^H as a process.

Next, we state some technical results in order to prove the tightness of the family of laws of $\{\mathcal{I}_1^H(f)\}_H$ for H belonging to a neighborhood of H_0. The first one, is a very simple inequality.

Lemma 3.1 *Let $T,\ k > 0$ positive real numbers, then for any $x \in \mathbb{R}$ such that $|x| \leqslant T$ and $0 < \alpha_1 < \alpha_2 \leqslant k$, there exists a constant $C_{T,k}$ (only depending on T and k) for which we have*

$$|x|^{\alpha_2} \leqslant C_{T,k} |x|^{\alpha_1}.$$

The following proposition is the main result of this section since it provides an estimation of the second order moment of the increments $\mathcal{I}_1^H(f)$ that will give the tightness of the family of laws of the Wiener integral processes for H in a neighborhood of H_0.

Proposition 3.2 *Let $f \in \mathcal{L}^{H'}$ with $H' \in (0, \frac{1}{2})$. Then, for $H_1 \in (H', \frac{1}{2})$, there exists a positive constant $C = C_{H',H_1,f,T}$, only depending on $H',\ H_1,\ f$ and T, such that*

$$\sup_{H \in [H_1, \frac{1}{2})} E|I_1^H(f\boldsymbol{1}_{[s,t]})|^2 \leqslant C(t-s)^{2(H_1-H')}, \tag{7}$$

for all $0 \leqslant s \leqslant t \leqslant T$.

Proof: Let $H \in [H_1, \frac{1}{2})$. In order to simplify the notations, and taking into account that we are only interested on the possible dependence on H of the constants, from now on we will use C to denote all the constants in this proof, except those that depend on H.
Using expression (2) we have that for all $s \leqslant t$

$$\begin{aligned} E|I_1^H(f\mathbf{1}_{[s,t]})|^2 &= \|f\mathbf{1}_{[s,t]}\|_H^2 \\ &= \frac{1}{2}H(1-2H)\left(\int_s^t\int_s^t \frac{(f(x)-f(y))^2}{|x-y|^{2-2H}}dxdy + \int_s^t\int_{[0,T]\setminus[s,t]}\frac{f(y)^2}{|x-y|^{2-2H}}dxdy \right. \\ &\quad \left. + \int_{[0,T]\setminus[s,t]}\int_s^t \frac{f(x)^2}{|x-y|^{2-2H}}dxdy\right) + H\int_s^t f(x)^2\left[\frac{1}{x^{1-2H}} + \frac{1}{(T-x)^{1-2H}}\right]dx. \end{aligned} \tag{8}$$

Applying Hölder's inequality with measure $\mu(dx,dy) = (f(x)-f(y))^2 dx\, dy$, taking $p = \frac{1-H'}{1-H} > 1$ and q its conjugate, we majorize the integral of the first term of (8) in the following way:

$$\begin{aligned} \int_s^t\int_s^t \frac{(f(x)-f(y))^2}{|x-y|^{2-2H}}dxdy &\leqslant \left(\int_s^t\int_s^t \frac{(f(x)-f(y))^2}{|x-y|^{2-2H'}}dxdy\right)^{1-H/1-H'} \\ &\quad \times \left(\int_s^t\int_s^t (f(x)-f(y))^2 dxdy\right)^{H-H'/1-H'}. \end{aligned}$$

Since $H \geqslant H_1$, we have that

$$\left(\int_s^t\int_s^t \frac{(f(x)-f(y))^2}{|x-y|^{2-2H'}}dxdy\right)^{1-H/1-H'} \leqslant \max(1, \|f\|^2_{W^{1/2-H',2}([0,T])})^{\frac{(1-H_1)}{1-H'}}.$$

Since, due to the Sobolev Embedding Theorem, $f \in L^{1/H'}([0,T])$, and applying again Hölder's inequality, with $p = \frac{1}{2H'},\ q = \frac{1}{1-2H'}$, we obtain

$$\begin{aligned} \left(\int_s^t\int_s^t f(x)^2 dxdy\right)^{\frac{H-H'}{1-H'}} &\leqslant \left(\int_s^t\left(\int_s^t |f(x)|^{1/H'}dx\right)^{2H'}(t-s)^{1-2H'}dy\right)^{\frac{H-H'}{1-H'}} \\ &= \left(\int_s^t |f(x)|^{1/H'}dx\right)^{2H'\left(\frac{H-H'}{1-H'}\right)}(t-s)^{2(H-H')} \\ &\leqslant C(t-s)^{2(H-H')}. \end{aligned}$$

In this way, we have an upper bound for the first summand of (8) of the form $C(t-s)^{2(H-H')}$.
The remaining terms can be handled easily. In fact, the second and third summands of (8) can be bounded in an analogous way. We next detail the steps of the treatment of the second one.
Computing the integral with respect to x of the second term and making some simple majorizations,

$$\begin{aligned} &\int_s^t\int_{[0,T]\setminus[s,t]}\frac{f(y)^2}{|x-y|^{2-2H}}dxdy \\ &= \frac{1}{1-2H}\left[\int_s^t f(y)^2((y-s)^{2H-1}-y^{2H-1})dy + \int_s^t f(y)^2((t-y)^{2H-1}-(T-y)^{2H-1})dy\right] \\ &\leqslant \frac{1}{1-2H}\left(\int_s^t f(y)^2(y-s)^{2H-1}dy + \int_s^t f(y)^2(t-y)^{2H-1}dy\right). \end{aligned}$$

Using again that $f \in L^{1/H'}([0,T])$, applying Hölder's inequality to each summand of the right-hand side of the last inequality and using that the exponent $\frac{2H-1}{1-2H'} > -1$, due to the fact that $H \geqslant H_1 > H'$, we have that

- $$\begin{aligned}\int_s^t f(y)^2 (y-s)^{2H-1} dy &\leqslant \left(\int_s^t |f(y)|^{1/H'} dy\right)^{2H'} \left(\int_s^t (y-s)^{2H-1/1-2H'} dy\right)^{1-2H'} \\ &\leqslant C\left(\frac{1-2H'}{2(H-H')}\right)^{1-2H'} (t-s)^{2(H-H')}.\end{aligned}$$

- $$\begin{aligned}\int_s^t f(y)^2 (t-y)^{2H-1} dy &\leqslant \left(\int_s^t |f(y)|^{1/H'} dy\right)^{2H'} \left(\int_s^t (t-y)^{2H-1/1-2H'} dy\right)^{1-2H'} \\ &\leqslant C\left(\frac{1-2H'}{2(H-H')}\right)^{1-2H'} (t-s)^{2(H-H')}.\end{aligned}$$

So, taking into account these inequalities and using that $H - H' > H_1 - H' > 0$, we obtain that

$$\frac{1}{2} H(1-2H) \int_{[0,T]\setminus[s,t]} \int_s^t \frac{f(x)^2}{|x-y|^{2-2H}} dx dy \leqslant C(t-s)^{2(H-H')}.$$

In a similar way we deal with the fourth summand of (8). Indeed, first using that $f \in L^{1/H'}([0,T])$ and then applying in a convenient form Hölder's inequality, we obtain that

$$\begin{aligned}&\int_s^t f(x)^2 \left[\frac{1}{x^{1-2H}} + \frac{1}{(T-x)^{1-2H}}\right] dx \\ &\quad\leqslant \left(\int_s^t |f(x)|^{1/H'} dx\right)^{2H'} \left(\int_s^t x^{2H-1/1-2H'} dx\right)^{1-2H'} \\ &\qquad + \left(\int_s^t |f(x)|^{1/H'} dx\right)^{2H'} \left(\int_s^t (T-x)^{2H-1/1-2H'} dx\right)^{1-2H'} \\ &\quad\leqslant C[t^{2(H-H')} - s^{2(H-H')} + (T-s)^{2(H-H')} - (T-t)^{2(H-H')}]. \end{aligned} \tag{9}$$

Since for $0 < \alpha < 1$,

$$x^\alpha - y^\alpha \leqslant (x-y)^\alpha,$$

we have that:

- $t^{2(H-H')} - s^{2(H-H')} \leqslant (t-s)^{2(H-H')}$.
- $(T-s)^{2(H-H')} - (T-t)^{2(H-H')} \leqslant (t-s)^{2(H-H')}$.

Therefore, the term (9) can be bounded by

$$C(t-s)^{2(H-H')}.$$

Finally, using the above inequalities and Lemma 3.1, we deduce that

$$\sup_{H\in[H_1,\frac{1}{2})} E|I_1^H(f\mathbf{1}_{[s,t]})|^2 \leqslant C(t-s)^{2(H-H')} \leqslant C(t-s)^{2(H_1-H')},$$

with C a positive constant.

3.2 Results for the convergence of the finite-dimensional distributions

An important tool in the proof of the convergence of the finite-dimensional distributions is the following general lemma, that we state in our particular setting.

Lemma 3.3 *Let $(G, \|\cdot\|)$ be a normed space, and let $\{\mathcal{J}^H\}_{H\in V_0}$ (where V_0 is an interval that contains H_0) be a family of linear maps defined on G and taking values in $(L^0(\Omega))^m$, the space of m-dimensional finite a.s. random vectors. Denote by $|\cdot|$ the euclidian norm in $\mathbb{R}^m$. Suppose that there exists a positive constant C such that, for any $f \in G$,*

$$(\mathsf{C}) \qquad \sup_{H\in V_0} E|\mathcal{J}^H(f)| \leqslant C\|f\|.$$

Suppose also that, for some dense subset $D \subset G$, we have that

$$\mathcal{J}^H(f) \xrightarrow{\mathscr{L}} \mathcal{J}^{H_0}(f), \qquad \textit{for all } f \in D, \textit{ when } H \to H_0.$$

Then, $\mathcal{J}^H(f) \xrightarrow{\mathscr{L}} \mathcal{J}^{H_0}(f)$, when $H \to H_0$, for all $f \in G$.

To check that our family of processes satisfies the condition (C) of the above lemma we need the inequality stated in the next lemma.

Lemma 3.4 *For any $f \in \mathcal{L}^{H'}$ and $H \in [H', \frac{1}{2}]$, there exists a constant $C_{T,H'}$ (only depending on T and H') such that*

$$\|f\|_H^2 \leqslant C_{T,H'}\|f\|_{H'}^2.$$

Proof: We distinguish the two cases $0 < H < \frac{1}{2}$ and $H = \frac{1}{2}$.

1. Case $H \in \left(0, \frac{1}{2}\right)$:

   As $0 < 1-2H \leqslant 1-2H'$ for $H \in \left[H', \frac{1}{2}\right)$, and applying Lemma 3.1 we obtain that:

   - $|x-y|^{2-2H} \geqslant C_{T,H'}|x-y|^{2-2H'}$,
   - $x^{1-2H} \geqslant C_{T,H'}x^{1-2H'}$,
   - $(T-x)^{1-2H} \geqslant C_{T,H'}(T-x)^{1-2H'}$.

   Applying these inequalities to each summand of the expression of $\|f\|_H$ we have that:

$$\begin{aligned}\|f\|_H^2 &= \frac{1}{2}H(1-2H)\int_0^T\!\!\int_0^T \frac{(f(x)-f(y))^2}{|x-y|^{2-2H}}dxdy + H\int_0^T f^2(x)\left[\frac{1}{x^{1-2H}} + \frac{1}{(T-x)^{1-2H}}\right]dx \\ &\leqslant C_{T,H'}\int_0^T\!\!\int_0^T \frac{(f(x)-f(y))^2}{|x-y|^{2-2H'}}dxdy + \frac{1}{2}C_{T,H'}\int_0^T f^2(x)\left[\frac{1}{x^{1-2H'}} + \frac{1}{(T-x)^{1-2H'}}\right]dx\end{aligned}$$

   From here, we deduce that

$$\|f\|_H^2 \leqslant C_{H',T}\|f\|_{H'}^2.$$

2. Case $H = \frac{1}{2}$:

   Given that $x = \frac{T}{2}$ is the minimum of the function

$$g(x) = \frac{1}{x^{1-2H'}} + \frac{1}{(T-x)^{1-2H'}}, \quad x \in [0,T],$$

   we have that

$$\int_0^T f^2(x) \left[\frac{2}{\left(\frac{T}{2}\right)^{1-2H'}} \right] dx \leqslant \int_0^T f^2(x) \left[\frac{1}{x^{1-2H'}} + \frac{1}{(T-x)^{1-2H'}} \right] dx$$

   or equivalently,

$$\int_0^T f^2(x) dx \leqslant \frac{1}{2} \left(\frac{T}{2} \right)^{1-2H'} \int_0^T f^2(x) \left[\frac{1}{x^{1-2H'}} + \frac{1}{(T-x)^{1-2H'}} \right] dx$$

   So, taking into account this last inequality, we can obtain easily the bound

$$\|f\|^2_{1|2} \leqslant C_{T,H'} \|f\|^2_{H'}.$$

3.3 Main result

We finally prove the convergence in law in $\mathcal{C}([0,T])$ of the processes $\{\mathcal{I}_1^H(f)\}_H$, when $H \to H_0 \in (0, \frac{1}{2}]$, for all $f \in \mathcal{L}^{H'}$, for some $H' < H_0$.

Theorem 3.5 *Let $f \in \mathcal{L}^{H'}$ with $H' < H_0$ and $H_0 \in (0, \frac{1}{2}]$. Then, the family of processes $\{\mathcal{I}_1^H(f)\}_{H \in (H', \frac{1}{2}]}$ converges in law to $\mathcal{I}_1^{H_0}(f)$, in the space $\mathscr{C}([0,T])$, when $H \to H_0$.*

Proof: The first step is to prove the existence of a continuous version of the Wiener integral $\mathcal{I}_1^H(f)$. Taking into account that $\mathcal{I}_1^H(f)$ is a centered Gaussian process, we only need to apply Proposition 3.2 and the Kolmogorov's continuity criterium.

Using again that the processes $\mathcal{I}_1^H(f)$ are centered and Gaussian, by Proposition 3.2 and Billingsley's criterium (see [3, Theorem 12.3]) we obtain the tightness of the laws of the family $\{\mathcal{I}_1^H(f),\ H \in [H_1, \frac{1}{2})\}$ in $\mathscr{C}([0,T])$, for all $0 < H' < H_1 < \frac{1}{2}$.

It only remains to show the convergence of the finite-dimensional distributions of the processes $\mathcal{I}_1^H(f)$. We prove this by applying Lemma 3.3 taking $G = \mathcal{L}_T^{H'}$ endowed with the norm $\|\cdot\|_{H'}$, and

$$\begin{aligned} \mathcal{J}^H :& G \longrightarrow (L^0(\Omega))^m \\ & f \mapsto (I_1^H(f\mathbf{1}_{[0,t_1]}), \ldots, I_1^H(f\mathbf{1}_{[0,t_m]})). \end{aligned}$$

By Lemma 3.4, the $\mathcal{J}^H$ satisfy the condition (C) of Lemma 3.3. Moreover, for $f \in \mathcal{S}$ (that is a dense subspace of G), the convergence in law of $\mathcal{I}_1^H(f)$ to $\mathcal{I}_1^{H_0}(f)$ is obtained from the fact that the integrals $I_1^H(f\mathbf{1}_{[0,t_j]})$ are linear combinations of increments of B^H, and B^H converges in law to B^{H_0}, as $H \to H_0$.

Acknowledgement
The authors are supported by the grant MEC-FEDER Ref. MTM2006-06427 from the Dirección General de Investigación, Ministerio de Educación y Ciencia, Spain.

References

[1] Adams, R. A. *Sobolev spaces*. Academic Press, New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] Alòs, E., Mazet, O. Nualart, D. Stochastic calculus with respect to Gaussian processes. *Ann. Probab.*, 29, no. 2, 766–801, 2001.

[3] Billingsley, P. *Convergence of probability measures*. John Wiley & Sons Inc., New York, 1968.

[4] Bardina, X., Jolis, M. Multiple fractional integral with Hurst parameter less than $\frac{1}{2}$. *Stochastic Process. Appl.*, 116, no. 3, 463–479, 2006.

[5] Decreusefond, L., Üstünel, A. S. Stochastic analysis of the fractional Brownian motion. *Potential Anal.*, 10, no. 2, 177–214, 1999.

[6] Jolis, M. On the Wiener integral with respect to the fractional Brownian motion on an interval. *J. Math. Anal. Appl.* 330, no. 2, 1115–1127, 2007.

[7] Jolis, M., Viles, N. Continuity with respect to the Hurst parameter of the laws of the multiple fractional integrals. *Stochastic Process. Appl.* 117, no. 9, 1189–1207, 2007.

[8] Jolis, M., Viles, N. Continuity in law with respect to the Hurst parameter of the local time of the fractional Brownian motion. *J. Theoret. Probab.* 20, no. 2, 133–152, 2007

[9] Pipiras, V., Taqqu, M. S. Are classes of deterministic integrands for fractional Brownian motion on an interval complete? *Bernoulli*, 7, no. 6, 873–897, 2001.

[10] Rudin, W. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[11] Samorodnitsky, G,, Taqqu, M.S. *Stable non-Gaussian random processes*. Stochastic Modeling. Chapman & Hall, New York, 1994. Stochastic models with infinite variance.