The Perron Method and the Non-Linear Plateau Problem

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Abstract: We describe a novel technique for solving the Plateau problem for constant curvature hypersurfaces based on recent work of Harvey and Lawson. This is illustrated by an existence theorem for hypersurfaces of constant Gaussian curvature in $\mathbb{R}^{n+1}$.

Key Words: Gaussian Curvature, Plateau Problem, Perron Method, Monge-Ampère Equation, Non-Linear Elliptic PDEs.

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1 - Introduction.

In this paper we describe a novel technique for constructing solutions to the Plateau problem for convex hypersurfaces of constant Gaussian curvature, which we illustrate through the proof of the following theorem:

Theorem 1.1

Let $\hat{K} \subseteq \mathbb{R}^{n+1}$ be a compact, strictly convex set with smooth boundary. Let Ω be an open subset of $\partial\hat{K}$ with (non-trivial) smooth boundary. Suppose there exists $k > 0$ such that the Gaussian curvature of $\hat{K}$ is everywhere at least k. Then, for all $t \in]0, k]$, there exists a convex subset $K_t \subseteq \hat{K}$ such that:

(i) $K_t \cap \partial\hat{K} = \Omega^c$; and

(ii) the boundary of K_t is smooth in the interior of K and is of constant Gaussian curvature equal to t.

Remark: This follows directly from Lemma 4.1 and Theorem 5.1 of [5].

Hypersurfaces of constant Gaussian curvature are interesting objects of study for various reasons. When $n = 2$, the Gaussian curvature is (more or less) equivalent to the intrinsic curvature of the surface, which only depends on one variable. In higher dimensions, although no such relation exists, Gaussian curvature continues to provide relatively simple PDEs that make it a good model for the study of more general non-linear notions of curvature. A tremendous literature exists devoted to the study of this problem, of which the most significant results are perhaps [3] of Caffarelli, Nirenberg and Spruck and [5] of Guan and Spruck. We refer the reader to the introduction of the paper [7] by the second author for a broader overview.

The novel technique that we introduce is a version of the Perron Method recently developed by Harvey and Lawson in [1] and [2]. This yields convex sets whose boundaries are of constant Gaussian curvature in the viscosity sense (c.f. [4]). We then show that these hypersurfaces are smooth away from their boundaries by appealling to Theorem 5.1 of [5]. The elementary nature of this proof as well as the remarkable generality of the results of [1] and [2] hint at potential generalisations. Indeed, it can easily be extended to yield (not necessarily unique) viscosity solutions in any Hadamard manifold, and the regularity result of [5] can then be applied whenever the ambient manifold is also affine flat (c.f. [7]). This is, in particular, the case for hyperbolic space.

This approach can also be extended to treat other notions of curvature. In fact, it is currently applicable to any notion of curvature which constitutes a "convex condition" in the sense of Section 2. In particular, this includes special Lagrangian curvature, which has been studied extensively by the second author in [8] and [9].

An important aspect of this technique that departs from the approach of Harvey and Lawson is its dependance on convex sets. We have introduced this essentially in order to make the problem more tractable, and it does so in two ways. The first is by eliminating complicated geometric considerations such as self intersections. This is not an issue in [1] and [2], since Harvey and Lawson are working there with functions, where the geometry is constant, as it were. The second is by allowing us to use the $C^{0,1}$ regularity properties of convex sets, and thus neatly sidestep the problem of proving regularity, which is the hardest step in Harvey and Lawson's proof and in the study of viscosity solutions in general. Naturally, however, this dependance on convex sets is very restrictive, and excludes large families of interesting curvature functions (see, for example, [3] or [6]). We nonetheless expect appropriate modifications to yield stronger results in the near future.

This paper is structured as follows:

(i) in Section 2, we define the basic notions used throughout this paper, recalling the definition of Dirichlet set as introduced by Harvey and Lawson in [1];

(ii) in Section 3, we define what it means for a convex set to be of type F. We show that this constitutes a "Perron system" in the sense that it satisfies the basic axioms required for the Perron Method; and

(iii) in Section 4, we apply the Perron Method to obtain viscosity solutions to the Plateau problem, and, appealing to Theorem 5.1 of [5], this proves Theorem 1.1.

The second author would like to thank Prof. Lawson for bringing [2] to his attention.

2 - Dirichlet Sets.

Let $\mathrm{Symm}(\mathbb{R}^n)$ denote the space of symmetric matrices over $\mathbb{R}^n$. We define $P \subseteq \mathrm{Symm}(\mathbb{R}^n)$ to be the set of all symmetric, non-negative semi-definite matrices. Thus A is an element of P if and only for all $x \in \mathbb{R}^n$:

$$\langle Ax, x\rangle \geqslant 0.$$

Trivially, P is a closed convex cone. Let F be a closed subset of $\mathrm{Symm}(\mathbb{R}^n)$. Following [1], we will say that F is a Dirichlet set if and only if:

$$F + P \subseteq F.$$

Moreover, we will say that F is invariant if and only it is preserved by conjugation by matrices in $O(n)$. In other words, F is invariant if and only if, for all $A \in F$, and for all $M \in O(n)$:

$$M^t A M \in F.$$

Finally, we will say that F defines a convex condition if and only if:

$$F \subseteq P.$$

In this paper, we are interested in invariant Dirichlet sets which define convex conditions.

Example: For $k > 0$ define F_k by:

$$F_k = \{A \in P \text{ s.t. } \mathrm{Det}(A) \geqslant k\}.$$

It is easily verified that F_k is an invariant Dirichlet set which trivially also defines a convex condition. In fact, F_k is also convex with smooth boundary. □

3 - The Perron System.

Let N be a unit normal vector field over a smooth hypersurface, Σ. In the sequel, we adopt the convention whereby the shape operator, A, of Σ satisfies:

$$A \cdot X = \nabla_X \mathsf{N}.$$

Let F be an invariant Dirichlet set. Let $X \subseteq \mathbb{R}^{n+1}$ be a compact set. We say that X is of type F if and only if, for all $p \in \partial X$, if Ω is an open subset of X^o (the interior of X) such that:

(i) $\partial\Omega$ is smooth; and

(ii) $p \in \partial\Omega$,

then the shape operator of $\partial\Omega$ at p with respect to the outward pointing normal is conjugate to a matrix in F.

Remark: The shape operator of $\partial\Omega$ is conjugate to a matrix in F if and only if its matrix with respect to an orthonormal basis for $T\partial\Omega$ lies in F. Since F is $O(n)$-invariant, this does not depend on the orthonormal basis chosen.

Remark: Observe that if F is not a Dirichlet set, then this definition is essentially empty.

Following [1] and [2], we obtain the following characterisation of subsets of type F:

Lemma 3.1

Let $X \subset \mathbb{R}^{n+1}$ be a compact set. X is not of type F if and only if there exists $p \in \partial X$, $r > 0$ and $f : B_r(p) \to \mathbb{R}$ such that:

(i) $f(p) = 0$;

(ii) $f^{-1}(]-\infty, 0]) \subseteq X \cap B_r(p)$;

(iii) $(\nabla f)(q) \neq 0$ for all $q \in B_r(p)$; and

(iv) $\mathrm{Hess}(f)|_{(\nabla f)^\perp}(q)$ is conjugate to an element of $\|\nabla f(q)\| F^c$ for all $q \in B_r(p)$.

Proof: We recall that if f is smooth and if $\nabla f \neq 0$ at a point q, then ∇f is colinear with the normal vector to the level set of f passing through q, which we denote by Σ_q. Moreover, if A_q is the shape operator of Σ_q with respect to the normal pointing in the same direction as ∇f, then:

$$A_q = \|(\nabla f)(q)\|^{-1} \mathrm{Hess}(f)|_{(\nabla f)^\perp}(q).$$

The result follows directly from these relations and the contrapositive of the definition of being of type F. $\square$

In the case of smooth boundary, we obtain:

Lemma 3.2

Let $X \subseteq \mathbb{R}^{n+1}$ be a compact set. Suppose that ∂X is smooth, then X is of type F if and only if the shape operator of ∂X with respect to the outward pointing normal is conjugate to an element of F at every point of ∂X.

Now suppose, moreover, that F defines a convex condition. The Perron method is based on the following result:

Lemma 3.3

Let $\mathcal{F}$ be a family of compact, convex sets of type F. Let X be the intersection of all members of $\mathcal{F}$. Then X is also a compact, convex set of type F.

Proof: X is trivially compact and convex. Suppose that X is not of type F. By Lemma 3.1, there exists $p \in \partial X$, $r > 0$ and $f : B_r(p) \to \mathbb{R}$ such that:

(i) $f(p) = 0$;

(ii) $f^{-1}(]-\infty, 0]) \subseteq X \cap B_r(p)$;

(iii) $(\nabla f)(q) \neq 0$ for all $q \in B_r(p)$; and

(iv) $\text{Hess}(f)|_{(\nabla f)^\perp}(q)$ is conjugate to an element of $\|\nabla f(q)\| F^c$ for all $q \in B_r(p)$.

Moreover, since F^c is open, by reducing r if necessary, f may be chosen such that there exists $\epsilon > 0$ such that, for all $q \in X^c \cap \partial B_r(p)$:

$$f(q) \geqslant \epsilon.$$

Choose $q \in B_r(p) \cap X^c$ such that $f(q) \leqslant \epsilon/2$. There exists $Y \in \mathcal{F}$ such that $q \notin Y$. However, since $X \subseteq Y$:

$$f^{-1}(]-\infty, 0]) \subseteq Y.$$

Let $p' \in B_r(p)$ be the point in the closure of $Y^c \cap B_r(p)$ realising the infimum of f over this set, and let δ be the value of this infimum. Trivially $0 \leqslant \delta \leqslant \epsilon/2$. Since, for all $q \in Y \cap \partial B_r(p)$, $f(q) \geqslant \epsilon$, p' is an interior point of $B_r(p)$, so there exists $r' > 0$ such that:

$$B_{r'}(p') \subseteq B_r(p).$$

Defining $f' : B_{r'}(p) \to \mathbb{R}$ by $f' = f - \delta$, we deduce by Lemma 3.1 that Y is not of type F. This contradicts the hypothesis on $\mathcal{F}$, and the result follows. $\square$

4 - Duality and the Viscosity Solution.

Let $\hat{K} \subseteq \mathbb{R}^{n+1}$ be a compact, strictly convex subset with smooth boundary of Gaussian curvature at least $k > 0$. Let $\Omega \subseteq \partial K$ be an open subset with smooth boundary. Let K_0 be the convex hull of Ω^c. Observe that, since Ω has smooth boundary, K_0 has non-trivial interior.

Let $X \subseteq \mathbb{R}^{n+1}$ be a compact set. We say that X is of type F' if and only if, for all $p \in \partial X$, if Ω is an open subset of X^c such that:

(i) $\partial\Omega$ is smooth; and

(ii) $p \in \partial\Omega$,

then the shape operator of $\partial\Omega$ at p with respect to the inward pointing normal is conjugate to a matrix in $\overline{F^c}$.

Remark: In the language of [1] and [2], this is the dual property to the property of being of type F. The duality becomes evident when we observe that $\tilde{F} := -\overline{F^c}$ is also an invariant Dirichlet set (c.f. [1]), and, when X is the closure of its interior, X is of type F' if and only if $\overline{X^c}$ is of type $\tilde{F}$.

For all $t > 0$, we define $F_t \subseteq \text{Symm}(\mathbb{R}^n)$ by:

$$F_t = \{A \in P \text{ s.t. } \text{Det}(A) \geqslant t\}.$$

As discussed in section 2, F_t is an invariant Dirichlet set which defines a convex condition.

Lemma 4.1

For all $t \in]0, k]$, there exists a compact, convex subset K_t of $\hat{K}$ such that:

(i) $K_0 \subseteq K_t$;

(ii) $K_t \cap \partial\hat{K} = \Omega^c$; and

(iii) K_t is of type F_t and of type F_t' over the interior of $\hat{K}$.

Remark: Thus, for all t, K_t has constant Gaussian curvature in the viscosity sense (c.f. [4]).

Proof: Choose $t \in]0, k]$. Let $\mathcal{F}$ denote the set of all convex subsets of $\hat{K}$ which contain K_0 and which are of type F_t. $\mathcal{F}$ is non-empty since $\hat{K} \in \mathcal{F}$. Let K_t be the intersection of all members of $\mathcal{F}$. Trivially, $K_0 \subseteq K_t$. Since $\hat{K}$ is strictly of type F_t, $K_t \cap \partial\hat{K} = \Omega^c$. Finally, by Lemma 3.3, K_t is of type F_t over the interior of $\hat{K}$. It thus remains to prove that K_t is of type F_t' over the interior of $\hat{K}$.

Suppose the contrary. Observe that, since K_0 has non-trivial interior, so does K_t. Moreover, $K_t \in \mathcal{F}$. By Lemma 3.1 (reversing orientation), there exists $p \in \partial K_t \cap \hat{K}$, $r > 0$ and a smooth function $f : B_r(p) \to \mathbb{R}$ such that:

(i) $f(p) = 0$;

(ii) $K_t \cap B_r(p) \subseteq f^{-1}(] - \infty, 0])$;

(iii) $(\nabla f)(q) \neq 0$ for all $q \in B_r(p)$; and

(iv) $\text{Hess}(f)|_{(\nabla f)^\perp}(q)$ is conjugate to an element of $\|(\nabla f)(q)\| F_t^o$ for all $q \in B_r(p)$, where F_t^o is the interior of F_t.

Moreover, since F_t^o is open, by reducing r if necessary, f may be chosen such that there exists $\epsilon > 0$ such that, for all $q \in K_t \cap \partial B_r(p)$:

$$f(q) \leqslant -\epsilon.$$

For all $t \in] - \epsilon, 0]$, define Σ_t by:

$$\Sigma_t = f^{-1}(\{t\}).$$

For all such t, $\partial\Sigma_t$, which lies in $\partial B_r(p)$ is a subset of K_t^c and therefore also of K_0^c. Moreover, for all such t, Σ_t is strictly convex over its interior. We recall that, since K_0 is a convex hull of a subset of $\partial\hat{K}$, ∂K_0 is locally ruled throughout the interior of $\hat{K}$. In other words, for all $p \in \partial K_0$ lying in the interior of $\hat{K}$, there exists a straight line segment, Γ, containing p in its interior and which also lies in ∂K_0. Thus, since Σ_0 lies in the closure of the complement of K_0, we deduce by the geometric maximum principal that so does Σ_t for all $t \in]-\epsilon, 0]$. In particular, if we define K_t' by:

$$K_t' = (K_t \cap f^{-1}(]-\infty, -\epsilon/2])) \cup (K_t \cap B_r(p)^c),$$

then $K_0 \subseteq K_t'$. However, K_t' is a compact, convex subset of $\hat{K}$. Moreover, being locally the intersection of two convex sets of type F_t, by Lemma 3.3, K_t is also of type F_t. In particular, it is an element of $\mathcal{F}$ which is a strict subset of K_t, which yields a contradiction. The result follows. □

We thus obtain Theorem 1.1:

Proof of Theorem 1.1: Lemma 4.1 yields convex sets whose boundaries are of constant Gaussian curvature in the viscosity sense (c.f. [1] and [2]). By Theorem 5.1 of [5], the boundaries of these sets are smooth, and this completes the proof. □

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