GAUSS-BONNET THEOREM AND CROFTON TYPE FORMULAS IN COMPLEX SPACE FORMS

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Abstract. We give an expression, in terms of the so-called Hermitian intrinsic volumes, for the measure of the set of complex r-planes intersecting a regular domain in any complex space form. Moreover, we obtain two different expressions for the Gauss-Bonnet-Chern formula in complex space forms. One of them expresses the Gauss curvature integral in terms of the Euler characteristic and some Hermitian intrinsic volumes. The other one, which is shorter, involves the measure of complex hyperplanes meeting the domain. As a tool, we obtain new variation formulas in integral geometry of complex space forms. Finally, we express the average over the complex r-planes of the total Gauss curvature of the intersection domain.

1. Introduction

In the space of constant sectional curvature k, $\mathbb{M}^n(k)$, Santaló [San04] gave the expression of the measure of the set of r-planes meeting a regular domain in terms of the mean curvature integrals. Suppose that $\mathcal{L}_r$ denotes the space of r-dimensional planes in $\mathbb{M}^n(k)$ and dL_r a density of $\mathcal{L}_r$ invariant under the isometry group of $\mathbb{M}^n(k)$. If $\Omega \subset \mathbb{M}^n(k)$ is a regular domain and $r = 2l$, then

$$\int_{\mathcal{L}_{2l}} \chi(\Omega \cap L_{2l}) dL_{2l} = c_0 \mathrm{vol}(\Omega) + \sum_{i=1}^{l} c_i k^{l-i} M_{2i-1}(\partial\Omega), \tag{1}$$

where c_i are known and depend only on n, r and i, while $M_j(\partial\Omega)$ denotes the mean curvature integral defined by

$$M_j(\partial\Omega) = \binom{n-1}{j}^{-1} \int_{\partial\Omega} \sigma_j(\mathrm{II}) dx \tag{2}$$

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where $\sigma_j(\mathrm{II})$ is the j-th symmetric elementary function of the eigenvalues of the second fundamental form. An analogous formula holds in the case of odd-dimensional planes.

In the proof of formula (1), Santaló used the Gauss-Bonnet-Chern theorem in $\mathbb{M}^n(k)$, that for n even states

(3)
$$M_{n-1}(\partial\Omega)=\frac{O_n}{2}\chi(\Omega)-kc_{n-3}M_{n-3}(\partial\Omega)-\ldots-k^{\frac{n-2}{2}}c_1M_1(\partial\Omega)-k^{\frac{n}{2}}\mathrm{vol}(\Omega),$$

and the reproductive property of the mean curvature integrals in $\mathbb{M}^n(k)$,

$$\int_{\mathcal{L}_r} M_i^{(r)}(\partial\Omega\cap L_r)dL_r = cM_i(\partial\Omega), \tag{4}$$

where $M_i^{(r)}(\partial\Omega\cap L_r)$ denotes the i-th mean curvature integral of $\partial\Omega\cap L_r$ as a hypersurface in L_r and c is known and depends only on n, r and i.

In this paper we generalize formula (1) and (3) to the space of constant holomorphic curvature 4ϵ, $\mathbb{CK}^n(\epsilon)$. The role of $\mathcal{L}_r$ in (1) will be played by $\mathcal{L}_r^{\mathbb{C}}$, the space of complex r-planes (totally geodesic complex submanifolds). For simplicity, we shall restrict to compact domains $\Omega\subset\mathbb{CK}^n(\epsilon)$ with smooth boundary, and call them *regular domains*.

The method used by Santaló cannot be applied in this situation. Mean curvature integrals are defined in a complex space form as in (2), but in the standard Hermitian space $\mathbb{C}^n$, it was not certain whether the reproductive property remains true for the mean curvature integrals when we integrate it over $\mathcal{L}_r^{\mathbb{C}}$. Moreover, in the complex projective space and in the complex hyperbolic space, an explicit Gauss-Bonnet formula is not known. Instead, we use variational arguments, as well as some facts about the theory of valuations, which we briefly describe next.

Definition 1.1. Let $\mathcal{K}(V)$ denote the family of non-empty compact convex subsets of a finite dimensional real vector space V of dimension n. A scalar valued functional $\phi:\mathcal{K}(V)\to\mathbb{C}$ is called a *valuation* if

$$\phi(A\cup B)=\phi(A)+\phi(B)-\phi(A\cap B)$$

whenever $A,B,A\cup B\in\mathcal{K}(V)$.

The space of invariant valuations under the full group of isometries of $\mathbb{R}^n$ was studied by Hadwiger [Had57], who proved that the dimension of this space is $n+1$. Mean curvature integrals, volume and Euler characteristic form a basis of this space.

On the standard Hermitian space $\mathbb{C}^n$ with its isometry group $IU(n)=\mathbb{C}^n\rtimes U(n)$, Alesker [Ale03] proved that there are more linearly independent valuations than on $\mathbb{R}^n$, the dimension of this space is $\binom{n+2}{2}$. Both the Euler characteristic, and the measure of complex planes intersecting the domain belong to this space.

Bernig and Fu [BF08], consider several valuation bases on $\mathbb{C}^n$. Here we will use the basis whose elements $\{\mu_{k,q}\}_{\{k,q\}}$ are called Hermitian intrinsic volumes. These valuations where first introduced by Park in complex space forms (cf. [Par02]). In section 2 we recall their definition.

The main results of this paper can be stated as follows.

Theorem 1.2. *Let Ω be a regular domain in $\mathbb{CK}^n(\epsilon)$. Then, for $r = 1, ..., n-1$*

$$\begin{aligned}\int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega\cap L_r)dL_r = \mathrm{vol}(G^{\mathbb{C}}_{n-1,r})\binom{n-1}{r}^{-1}(\epsilon^r(r+1)\mathrm{vol}(\Omega)+ \\ +\sum_{k=n-r}^{n-1}\epsilon^{k-(n-r)}\omega_{2n-2k}\binom{n}{k}^{-1}\cdot((k+r-n+1)\mu_{2k,k}+ \\ +\sum_{q=\max\{0,2k-n\}}^{k-1}\frac{1}{4^{k-q}}\binom{2k-2q}{k-q}\mu_{2k,q}))\end{aligned}\tag{5}$$

where dL_r denotes an invariant measure in the space of complex r-planes $\mathcal{L}_r^{\mathbb{C}}$. Moreover,

$$\begin{aligned}O_{2n-1}\chi(\Omega) = 2n(n+1)\epsilon^n\mathrm{vol}(\Omega)+ \\ +\sum_{c=0}^{n-1}\frac{\epsilon^c O_{2n-2c-1}}{\binom{n-1}{c}}\left(\sum_{q=\max\{0,2c-n\}}^{c-1}\frac{1}{4^{c-q}}\binom{2c-2q}{c-q}\mu_{2c,q}+(c+1)\mu_{2c,c}\right)\end{aligned}\tag{6}$$

where ω_i denotes the volume of the euclidean unit ball and O_i the volume of the euclidean unit sphere.

Formula (5) is a generalization of (1) in the sense that the measure of the complex r-planes meeting a regular domain is expressed as a linear combination of the volume and some other valuations related to mean curvature integrals. This answers a question posed by Naveira in [Nav05]. In case $r = 1$, formula (5) was already proved by different methods in [Aba09]. For $2r \geq n$, and $\epsilon = 0$, formula (5) was proved in [BF08].

Formula (6) generalizes to complex space forms the Gauss-Bonnet theorem (3). In complex dimensions $n = 2, 3$, formula (6) was obtained in [Par02].

Combining expressions (5) and (6) we obtain

$$\begin{aligned}M_{2n-1}(\partial\Omega) = O_{2n-1}\chi(\Omega) - 2n\epsilon\int_{\mathcal{L}^{\mathbb{C}}_{n-1}}\chi(\Omega\cap L_{n-1})dL_{n-1}- \\ -\sum_{k=1}^{n-1}\epsilon^k O_{2n-2k-1}\binom{n-1}{k}^{-1}\mu_{2k,k} - 2n\epsilon^n\mathrm{vol}(\Omega).\end{aligned}$$

This expression is similar to the following one for real space forms.

Theorem ([Sol06]). *Let S be a hypersurface bounding a compact domain Q in a real space form with sectional curvature k and dimension n. Then*

$$M_{n-1}(S) = O_{n-1}\chi(Q) - k\frac{2(n-1)}{O_{n-2}}\int_{\mathcal{L}_{n-2}}\chi(Q\cap L_{n-2})dL_{n-2}.$$

The main idea for the proof of Theorem 1.2 is to take variation along a vector field in $\mathbb{CK}^n(\epsilon)$, in both sides of equalities (5) and (6), and to compare them.

In order to obtain a first expression of the variation of $\int_{\mathcal{L}_r^{\mathbb{C}}}\chi(\partial\Omega\cap L_r)dL_r$ along a vector field in $\mathbb{CK}^n(\epsilon)$, we proceed as in [Sol06] (see Section 3.1). In $\mathbb{C}^n$, the variation of the Hermitian intrinsic volumes was obtained by Bernig and Fu [BF08]. Here we use the same method to find the generalization for $\epsilon \neq 0$ (see Section 3.2).

Using formula (5), we prove in Section 6 that the total Gauss curvature does not satisfy the reproductive property and we get in $\mathbb{C}^n$ the following expression:

$$\int_{\mathcal{L}_r^{\mathbb{C}}} M_{2r-1}(\partial\Omega\cap L_r)dL_r = 2r\omega_{2r}^2\mathrm{vol}(G_{n-1,r}^{\mathbb{C}})\binom{n-1}{r}^{-1}\binom{n}{r}^{-1}\cdot$$

$$\cdot\left(\sum_{q=\max\{0,n-2r\}}^{n-r}\frac{1}{4^{n-r-q}}\binom{2n-2r-2q}{n-r-q}\mu_{2n-2r,q}\right). \tag{7}$$

In [Aba09], it is proved that the reproductive property (4) is not satisfied by the mean curvature integral either.

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2. Hermitian intrinsic volumes on $\mathbb{CK}^n(\epsilon)$

Let $\mathbb{CK}^n(\epsilon)$ be a (simply connected) complex space form with constant holomorphic curvature 4ϵ. We denote by $T'\mathbb{CK}^n(\epsilon)$ the unit tangent bundle of $\mathbb{CK}^n(\epsilon)$.

Definition 2.1. Let Ω be a regular domain in $\mathbb{CK}^n(\epsilon)$. The unit (inner) normal bundle of $\partial\Omega$ is defined as

$$N(\Omega) = \{(p,v)\in T'\mathbb{CK}^n(\epsilon) : p\in\partial\Omega, v \text{ inner normal to } T'_p\partial\Omega\}.$$

Given a $4n-1$ form ω in $T'\mathbb{CK}^n(\epsilon)$ we may consider, for every regular domain Ω the integral over $N(\Omega)$ of the form ω. The resulting functional is called a *smooth valuation*.

Let $z\in\mathbb{CK}^n(\epsilon)$, $e_1\in T'\mathbb{CK}^n(\epsilon)$ and let $\{z;e_1,...,e_n\}$ be a moving frame defined on an open subset $U\subset T'\mathbb{CK}^n(\epsilon)$. We denote by

$\{\omega_1, \omega_2, ..., \omega_n\}$ the 1-forms on $T'\mathbb{CK}^n(\epsilon)$ defined as the dual basis of $\{e_1, ..., e_n\}$, and by $\{\omega_{ij}\}$ the connection forms of $\mathbb{CK}^n(\epsilon)$. That is, if $(\,,)$ denotes the Hermitian product on $\mathbb{CK}^n(\epsilon)$ and ∇ the Levi-Civita connection, then

$$\omega_j = (dz, e_j) \quad \text{and} \quad \omega_{jk} = (\nabla e_j, e_k) \quad \text{where} \quad j,k \in \{1, ..., n\}.$$

Thus, these forms are $\mathbb{C}$-valued. We denote

$$\begin{aligned} \omega_j &= \alpha_j + i\beta_j, \\ \omega_{jk} &= \alpha_{jk} + i\beta_{jk}. \end{aligned} \tag{8}$$

Remark 2.2. Forms α_1, β_1 and β_{11} are global forms in $T'\mathbb{CK}^n(\epsilon)$. We denote by α, β, γ the forms α_1, β_1, β_{11}, respectively. Note that α coincides with the contact form of the unit tangent bundle.

Lemma 2.3. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain. Forms α and $d\alpha$ vanish at $N(\Omega) \subset T'\mathbb{CK}^n(\epsilon)$.*

Proof. Let $V \in T_{(p,v)}N(\Omega)$. Then, $\alpha(V)_{(p,v)} = \langle d\pi(V), v\rangle = 0$ where $\pi : T'\mathbb{CK}^n(\epsilon) \to \mathbb{CK}^n(\epsilon)$ is the canonical projection.

To prove that $d\alpha$ vanishes at the unit normal bundle, we consider the inclusion of the unit normal bundle to the unit tangent bundle $i : N(\Omega) \to T'(\Omega)$ and we use that exterior differential commutes with the inclusion map to obtain

$$d\alpha|_{N(\Omega)} = (i^* \circ d)(\alpha) = (d \circ i^*)(\alpha) = d(i^*\alpha) = d(0) = 0.$$

□

Consider the following 2-forms in $T'\mathbb{CK}^n(\epsilon)$

$$\begin{aligned} \theta_0 &= -\mathrm{Im}((\nabla e_1, \nabla e_1)) = -\mathrm{Im}(\sum_{i=1}^n \omega_{1i} \otimes \overline{\omega}_{1i}) \\ \theta_1 &= -\mathrm{Im}((dz, \nabla e_1) - (\nabla e_1, dz)) \\ &= -\mathrm{Im}(\sum_{i=1}^n \omega_i \otimes \overline{\omega}_{1i} - \sum_{i=1}^n \omega_{1i} \otimes \overline{\omega}_i) \\ \theta_2 &= -\mathrm{Im}((dz, dz)) = -\mathrm{Im}(\sum_{i=1}^n \omega_i \otimes \overline{\omega}_i) \\ \theta_s &= \mathrm{Re}((dz, \nabla e_1) - (\nabla e_1, dz)) \\ &= \mathrm{Re}(\sum_{i=1}^n \omega_i \otimes \omega_{1i} - \sum_{i=1}^n \overline{\omega}_{1i} \otimes \overline{\omega}_i). \end{aligned} \tag{9}$$

This forms coincide with the invariant 2-forms, θ_0, θ_1, θ_2 and θ_s defined in $T'\mathbb{C}^n$ by Bernig and Fu [BF08, p.14]. Note that, θ_s is the symplectic form of $T\mathbb{CK}^n(\epsilon)$.

Remark 2.4. Park [Par02] defined invariant 2-forms in $T'\mathbb{CK}^n(\epsilon)$ similar to the ones in (9).

The forms $\beta_{k,q}$ and $\gamma_{k,q}$ defined in $T'\mathbb{C}^n$ in [BF08], can be extended to $T'\mathbb{CK}^n(\epsilon)$ from (9).

Definition 2.5. For positive integers $k, q \in \mathbb{N}$ with $\max\{0, k-n\} \leq q \leq \frac{k}{2} < n$, it is defined in $\Omega^{2n-1}(T'\mathbb{CK}^n(\epsilon))$

$$\beta_{k,q} := c_{n,k,q}\beta \wedge \theta_0^{n-k+q} \wedge \theta_1^{k-2q-1} \wedge \theta_2^q, \quad k \neq 2q$$

$$\gamma_{k,q} := \frac{c_{n,k,q}}{2}\gamma \wedge \theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^q, \quad n \neq k-q$$

where

$$c_{n,k,q} := \frac{1}{q!(n-k+q)!(k-2q)!\omega_{2n-k}}$$

and ω_{2n-k} denotes the volume of the $(2n-k)$-dimensional euclidean ball.

Given a regular domain $\Omega \subset \mathbb{CK}^n(\epsilon)$, we define (for $\max\{0, k-n\} \leq q \leq \frac{k}{2} < n$)

$$B^{\Omega}_{k,q}(\Omega) := \int_{N(\Omega)} \beta_{k,q} \quad (k \neq 2q), \qquad \Gamma^{\Omega}_{k,q}(\Omega) := \int_{N(\Omega)} \gamma_{k,q} \quad (n \neq k-q).$$

In $\mathbb{C}^n$, it is satisfied $B^{\Omega}_{k,q}(\Omega) = \Gamma^{\Omega}_{k,q}(\Omega)$ since $d\beta_{k,q} = d\gamma_{k,q}$. Next we recall the exterior derivative of θ_0, θ_1 and θ_2, which can be found in [BF08] when $\epsilon = 0$, or in [Par02] for general ϵ.

Lemma 2.6 ([Par02])**.** *In $T'\mathbb{CK}^n(\epsilon)$ it is satisfied*

$$\begin{aligned} d\alpha &= -\theta_s, & d\theta_0 &= -\epsilon(\alpha \wedge \theta_1 + \beta \wedge \theta_s), \\ d\beta &= \theta_1, & d\theta_1 &= 0, \\ d\gamma &= 2\theta_0 - 2\epsilon\theta_2 - 2\epsilon\alpha \wedge \beta, & d\theta_2 &= 0. \end{aligned}$$

Next, we give the relation among $\{B^{\Omega}_{k,q}(\Omega)\}$ and $\{\Gamma^{\Omega}_{k,q}(\Omega)\}$ in $\mathbb{CK}^n(\epsilon)$ which generalizes the relation in $\mathbb{C}^n$.

Proposition 2.7. *In $\mathbb{CK}^n(\epsilon)$, for any positive integers k, q such that $\max\{0, k-n\} < q < k/2 < n$ it is satisfied*

$$\Gamma^{\Omega}_{k,q}(\Omega) = B^{\Omega}_{k,q}(\Omega) - \epsilon\frac{c_{n,k,q}}{c_{n,k+2,q+1}}B^{\Omega}_{k+2,q+1}(\Omega).$$

Proof. We denote by I the ideal generated by α, $d\alpha$ and the exact forms in $N(\Omega)$. If λ, ρ are $(2n-1)$-forms in $N(\Omega)$ equal modulo I, then by Lemma 2.3

$$\int_{N(\Omega)} \lambda = \int_{N(\Omega)} \rho.$$

Thus, it is enough to prove

$$\gamma_{k,q} \equiv \beta_{k,q} - \epsilon\frac{c_{n,k,q}}{c_{n,k+2,q+1}}\beta_{k+2,q+1} \quad \mod I. \tag{10}$$

Consider the form $\eta = (\theta_s - \beta \wedge \gamma) \wedge \theta_0^{n-k+q-1}\theta_1^{k-2q-1}\theta_2^q$. As $d\eta$ is exact, we have $d\eta \equiv 0 \mod I$. On the other hand, by Lemma 2.6 it follows that modulo I

$$d\eta \equiv -\gamma\theta_0^{n-k+q-1}\theta_1^{k-2q}\theta_2^q + 2\beta\theta_0^{n-k+q}\theta_1^{k-2q-1}\theta_2^q - 2\epsilon\beta\theta_0^{n-k+q-1}\theta_1^{k-2q-1}\theta_2^{q+1}.$$

Using the definition of $\gamma_{k,q}$ and $\beta_{k,q}$ we obtain the relation in (10). $\square$

Remark 2.8. In complex dimensions $n = 2, 3$, the previous relations were found in [Par02].

In view of the previous equalities, we define (for $\max\{0, k-n\} \leq q \leq \frac{k}{2} < n$)

$$\mu_{k,q}(\Omega) := \begin{cases} B^\Omega_{k,q}(\Omega) & \text{if } k \neq 2q \\ \Gamma^\Omega_{2q,q}(\Omega) & \text{if } k = 2q. \end{cases} \tag{11}$$

Remark 2.9. In [BF08], it is proved that valuations $\{\mu_{k,q}, \mathrm{vol}\}$ where $k \in \{0, ..., n-1\}$ and $q \in \{\max\{0, k-n\}, ..., \lfloor k/2 \rfloor\}$ form a basis of invariant continuous valuations on $\mathbb{C}^n$.

3. Variation formulas

3.1. **Variation of Hermitian intrinsic volumes.** In order to study the variation on $\mathbb{CK}^n(\epsilon)$ of the Hermitian intrinsic volumes, we follow the method used by Bernig and Fu [BF08] in Corollary 2.6. First, we recall the definition of the Rumin operator, introduced in [Rum94], and the definition of the Reeb vector field in a contact manifold.

Definition 3.1. Given $\mu \in \Omega^{2n-1}(T'\mathbb{CK}^n(\epsilon))$, let α be the contact form of $T'\mathbb{CK}^n(\epsilon)$, and let $\alpha \wedge \xi \in \Omega^{2n-1}(T'\mathbb{CK}^n(\epsilon))$ be the unique form such that $d(\mu + \alpha \wedge \xi)$ is multiple of α (cf. [Rum94]). Then the Rumin operator D is defined as

$$D\mu := d(\mu + \alpha \wedge \xi).$$

Definition 3.2. Let M be a contact manifold and let α be the contact form. The Reeb vector field T is the unique vector field over M such that

$$\begin{cases} i_T\alpha = 1, \\ \mathcal{L}_T\alpha = 0. \end{cases} \tag{12}$$

If the contact manifold is the unit tangent bundle of a Riemannian manifold, then the Reeb vector field is the geodesic flow (cf. [Bla76, p. 17]).

Lemma 3.3. *In $T'\mathbb{CK}^n(\epsilon)$, it is satisfied*

$$\begin{aligned} &i_T\alpha = 1, && i_T\theta_1 = \gamma, \\ &i_T\theta_2 = \beta, && i_T\beta = i_T\gamma = i_T\theta_0 = 0. \end{aligned}$$

Proof. The first equality comes directly from the definition (cf. (12)). As T is the geodesic flow, we have $\alpha_i(T) = \beta_i(T) = 0$ and $\alpha_{1i} = \beta_{1i} = 0$, $i \in \{2, ..., n\}$. By Definition in (9), we obtain the result. □

Given a smooth valuation μ, and a vector field X with flow φ_t, we are interested in computing

$$\delta_X \mu(\Omega) := \left.\frac{d}{dt}\right|_{t=0} \mu(\varphi_t(\Omega)).$$

This can be done by means of the following result stated in [BF08].

Lemma 3.4 (Lemma 2.5 [BF08])**.** *Suppose $\Omega \subset \mathbb{R}^n$ is a regular domain, N is the inner normal field to $\partial\Omega$, X is a smooth vector field on $\mathbb{C}^n$ and μ is a smooth valuation given by a $(2n-1)$-form ρ. Then*

$$\delta_X \mu(\Omega) = \int_{N(\Omega)} \langle X, N \rangle \, i_T(D\rho)$$

where T is the Reeb vector field on $T'\mathbb{R}^n$ and $D\rho$ is the Rumin operator of ρ.

Although this result is stated and proved in $\mathbb{C}^n$, the given proof is also valid in an arbitrary Riemannian manifold.

From Lemma 2.6, we obtain the exterior differential of the forms $\beta_{k,q}$ and $\gamma_{k,q}$.

Lemma 3.5. *In $\mathbb{CK}^n(\epsilon)$*

$$d\beta_{k,q} = c_{n,k,q}(\theta_0^{n-k+q} \wedge \theta_1^{k-2q} \wedge \theta_2^q - \epsilon(n-k+q)\alpha \wedge \beta \wedge \theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^q)$$

and

$$\begin{aligned} d\gamma_{k,q} =& c_{n,k,q}(\theta_0^{n-k+q} \wedge \theta_1^{k-2q} \wedge \theta_2^q - \epsilon\theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^{q+1} \\ & - \epsilon\alpha \wedge \beta \wedge \theta_0^{n-k+q-1} \wedge \theta_1^{k-2q} \wedge \theta_2^q \\ & - \epsilon\frac{(n-k+q-1)}{2}\alpha \wedge \gamma \wedge \theta_0^{n-k+q-2} \wedge \theta_1^{k-2q+1} \wedge \theta_2^q \\ & - \epsilon\frac{(n-k+q-1)}{2}\beta \wedge \gamma \wedge \theta_s \wedge \theta_0^{n-k+q-2} \wedge \theta_1^{k-2q} \wedge \theta_2^q). \end{aligned}$$

Notation 3.6. Let Ω be a regular domain in $\mathbb{CK}^n(\epsilon)$ and let N be a normal field to $\partial\Omega$. Let X be a smooth vector field on $\mathbb{CK}^n(\epsilon)$. We denote

$$\tilde{B}_{k,q} := \tilde{B}_{k,q}(\Omega) = \int_{\partial\Omega} \langle X, N \rangle \beta_{k,q}, \quad \tilde{\Gamma}_{k,q} := \tilde{\Gamma}_{k,q}(\Omega) = \int_{\partial\Omega} \langle X, N \rangle \gamma_{k,q}.$$

The variation of the valuations $\{\mu_{k,q}\}$ on $\mathbb{C}^n$ is given in [BF08, Proposition 4.6]. The next proposition gives this variation in the previous notation.

Proposition 3.7 (Proposition 4.6 [BF08])**.**

$$\delta_X\mu_{k,q} = 2c_{n,k,q}(c_{n,k-1,q}^{-1}(k-2q)^2\tilde{\Gamma}_{k-1,q} - c_{n,k-1,q-1}^{-1}(n+q-k)q\tilde{\Gamma}_{k-1,q-1}$$
$$+c_{n,k-1,q-1}^{-1}(n+q-k+\frac{1}{2})q\tilde{B}_{k-1,q-1} - c_{n,k-1,q}^{-1}(k-2q)(k-2q-1)\tilde{B}_{k-1,q}).$$

We extend this result to $\mathbb{CK}^n(\epsilon)$ as follows.

Proposition 3.8. *Let $X \in \mathfrak{X}(\mathbb{CK}^n(\epsilon))$ and let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain. Then*

$$\delta_X B_{k,q}^{\Omega}(\Omega) = 2c_{n,k,q}(c_{n,k-1,q}^{-1}(k-2q)^2\tilde{\Gamma}_{k-1,q} - c_{n,k-1,q-1}^{-1}(n+q-k)q\tilde{\Gamma}_{k-1,q-1}$$
$$+ c_{n,k-1,q-1}^{-1}(n+q-k+\frac{1}{2})q\tilde{B}_{k-1,q-1} - c_{n,k-1,q}^{-1}(k-2q)(k-2q-1)\tilde{B}_{k-1,q}$$
$$+\epsilon(c_{n,k+1,q+1}^{-1}(k-2q)(k-2q-1)\tilde{B}_{k+1,q+1} - c_{n,k+1,q}^{-1}(n-k+q)(q+\frac{1}{2})\tilde{B}_{k+1,q}))$$

and

$$\begin{aligned}\delta_X\Gamma_{2q,q}^{\Omega}(\Omega) = &2c_{n,2q,q}\Big(-c_{n,2q-1,q-1}^{-1}(n-q)q\tilde{\Gamma}_{2q-1,q-1}\\ &+ c_{n,2q-1,q-1}^{-1}(n-q+\frac{1}{2})q\tilde{B}_{2q-1,q-1}\\ &+\epsilon(-c_{n,2q+1,q}^{-1}((n-q)(2q+\frac{3}{2}) - \frac{1}{2}(q+1))\tilde{B}_{2q+1,q}\\ &+ c_{n,2q+1,q}^{-1}(n-q-1)(q+1)\tilde{\Gamma}_{2q+1,q}\\ &- \epsilon c_{n,2q+3,q+1}^{-1}(n-q-1)(q+\frac{3}{2})\tilde{B}_{2q+3,q+1})\Big).\end{aligned}$$

Proof. The proof is based on that of Proposition 4.6 in [BF08].

Consider first $\delta_X B_{k,q}^{\Omega}(\Omega)$. Lemma 3.4 provides an expression for the variation of a valuation given by a smooth form. By this lemma and Lemma 2.3, it is enough to find the Rumin operator of the form $\beta_{k,q}$ modulo α, $d\alpha$ since both forms vanish over the unit normal fiber bundle. In the case $\epsilon = 0$ (cf. Proposition 4.6 in [BF08]) the Rumin differential of $\beta_{k,q}$ is given by $D\beta_{k,q} = d(\beta_{k,q} + \alpha\wedge\xi)$ where ξ is an invariant form fulfilling

(13)
$$\begin{aligned}\xi \equiv\ & c_{n,k,q}\beta\gamma\theta_0^{n+q-k-1}\theta_1^{k-2q-2}\theta_2^{q-1}\\ &\wedge\big((n+q-k)q\theta_1^2 - (k-2q)(k-2q-1)\theta_0\theta_2\big) \mod (\alpha, d\alpha).\end{aligned}$$

For general ϵ we take a form ξ^ϵ such that $\xi^\epsilon_{(p,v)} \equiv \xi_{(p',v')}$ when we identify $T_{(p,v)}T'\mathbb{CK}^n(\epsilon)$ and $T_{(p',v')}T'\mathbb{C}^n$, for every $(p,v) \in T'\mathbb{CK}^n(\epsilon), (p',v') \in T'\mathbb{C}^n$. Then, it is clear from Lemma 3.5 that $d(\beta_{k,q} + \alpha\wedge\xi^\epsilon) \equiv 0$ modulo α, since the term multiple of ϵ in the expression of $d\beta_{k,q}$ is also multiple of α.

By Lemma 2.6, the exterior differential ξ for any ϵ is

$$\begin{aligned}d\xi \equiv\ & c_{n,k,q}\theta_0^{n+q-k-1}\theta_1^{k-2q-2}\theta_2^{q-1}((n-k+q)q\theta_1^2 - (k-2q)(k-2q-1)\theta_0\theta_2)\\ &\wedge(\gamma\theta_1 - 2\beta\theta_0 + 2\epsilon\beta\theta_2) \qquad \mod \alpha\end{aligned}$$

and the contraction of $d\beta_{k,q}$ with respect to the field T, by Lemma 3.3, is

$$\begin{aligned} i_T d\beta_{k,q} \equiv\ & c_{n,k,q}\theta_0^{n+q-k-1}\theta_1^{k-2q-1}\theta_2^{q-1} \\ & \wedge ((k-2q)\gamma\theta_0\theta_2 + q\beta\theta_0\theta_1 - \epsilon(n-k+q)\beta\theta_1\theta_2) \qquad \bmod \alpha. \end{aligned}$$

By substituting the last expressions in $i_T D\beta_{k,q} \equiv i_T d\beta_{k,q} - d\xi$ (mod α, $d\alpha$), we get the result.

To compute $\delta_X \Gamma_{2q,q}$, note that $d\gamma_{2q,q}$ has 3 terms which are not multiple of α (cf. Lemma 3.5). Then the Rumin differential is given by $D\gamma_{2q,q} = d(\gamma_{2q,q} + \alpha \wedge (\xi + \xi_2 + \xi_3))$ where ξ corresponds, as above, to the form in (13),

$$\xi_2 \equiv\ c_{n,2q+2,q+1}(n-q-1)(q+1)\beta\gamma\theta_0^{n-q-2}\theta_2^q \quad \bmod (\alpha, d\alpha)$$

and

$$\xi_3 \equiv c_{n,2q,q}\frac{n-q-1}{2}\beta\gamma\theta_0^{n-q-2}\theta_2^q \quad \bmod (\alpha, d\alpha).$$

Indeed, as in Proposition 4.6 in [BF08], ξ cancels the first term of $d\gamma_{2q,q}$ modulo α and ξ_2 the second one. The third term is canceled by ξ_3 since it is the same multiple of $d\alpha = -\theta_s$. □

3.2. **Variation of the measure of complex r-planes intersecting a regular domain.** We denote by $\mathcal{L}_r^{\mathbb{C}}$, $r \in \{1, ..., n-1\}$ the space of complex r-planes in $\mathbb{CK}^n(\epsilon)$. Complex r-planes are totally geodesic submanifolds of complex dimension r isometric to $\mathbb{CK}^r(\epsilon)$ (cf. [Gol99, Lemma 2.2.4]). Moreover, Santaló [San52] proved the following properties of this space (as usual, J denotes the complex structure).

Lemma 3.9 ([San52]). *$\mathcal{L}_r^{\mathbb{C}}$, is a homogeneous space and*

$$\mathcal{L}_r^{\mathbb{C}} \cong U_\epsilon(n)/U_\epsilon(r) \times U(n-r)$$

where

$$U_\epsilon = \begin{cases} \mathbb{C}^n \ltimes U(n), & \text{if } \epsilon = 0, \\ U(1+n), & \text{if } \epsilon > 0, \\ U(1,n), & \text{if } \epsilon < 0. \end{cases}$$

Let $\{g; g_1, Jg_1, ..., g_n, Jg_n\}$ be a local orthonormal frame such that $\{g_1, Jg_1, ..., g_r, Jg_r\}$ generate the tangent space of a complex r-plane at g. The density of $\mathcal{L}_r^{\mathbb{C}}$ is given by

$$dL_r = \left| \bigwedge_{i=r+1}^{n} \omega_i \wedge \overline{\omega_i} \bigwedge_{\substack{i=1,\dots,r \\ j=r+1,\dots,n}} \omega_{ij} \wedge \overline{\omega_{ij}} \right| \tag{14}$$

where $\{\omega_i, \omega_{ij}\}_{\{i,j\}}$ are defined as in (8).

On $\partial\Omega$ there is a canonical vector field given by JN, and a distribution $\mathcal{D} = \langle N, JN\rangle^{\perp}$. At every point $x \in \partial\Omega$, $\mathcal{D}_x$ is the maximal complex linear subspace of $T_x\mathbb{CK}(\epsilon)$ contained in $T_x\partial\Omega$. We shall consider the bundle $G^{\mathbb{C}}_{n,r}(T\partial\Omega)$ whose fiber at every point $x \in \partial\Omega$ is the Grassmanian $G^{\mathbb{C}}_{n,r}(T_x\partial\Omega)$ of r-dimensional complex subspaces of $\mathcal{D}_x$; i.e., $G^{\mathbb{C}}_{n,r}(T\partial\Omega) = \{(x,l)|x \in \partial\Omega, l \text{ is a } J\text{-invariant } r\text{-dimensional linear subspace of } T_x\partial\Omega\}$.

Proposition 3.10. *Suppose $\Omega \subset \mathbb{CK}^n(\epsilon)$ is a regular domain, X is a smooth vector field on $\mathbb{CK}^n(\epsilon)$, ϕ_t is the flow associated to X and $\Omega_t = \phi_t(\Omega)$, then*

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}^{\mathbb{C}}_r} \chi(\Omega_t \cap L_r) dL_r = \int_{\partial\Omega} \langle \partial\phi/\partial t, N\rangle \left(\int_{G^{\mathbb{C}}_{n,r}(T_x\partial\Omega)} \sigma_{2r}(\mathrm{II}|_V) dV \right) dx$$

where N is the inner normal field and $\sigma_{2r}(\mathrm{II}|_V)$ denotes the $2r$-th symmetric elementary function of II *restricted to $V \in G^{\mathbb{C}}_{n,r}(T_x\partial\Omega)$.*

Proof. This proof is based on the one in [Sol06, Theorem 4]. For every $l \in G^{\mathbb{C}}_{n,r}(T_x\partial\Omega)$, we make the parallel translation l_t of l along $\phi_t(x)$. Recall that parallel translation preserves the complex structure (cf. [O'N83, p. 326]). Then we project l_t onto $\mathcal{D}_{\phi_t(x)}$, obtaining a complex r-plane l'_t (at least for small values of t). We define

$$\begin{array}{rccc} \gamma: & G^{\mathbb{C}}_{n,r}(T\partial\Omega) \times (-\epsilon,\epsilon) & \longrightarrow & \mathcal{L}^{\mathbb{C}}_r \\ & ((x,l),t) & \mapsto & \exp_{\phi_t(x)} l'_t. \end{array}$$

As

$$\gamma^*(dL_r) = \iota_{\partial t}(\gamma^*(dL_r))dt = \gamma_t^*(\iota_{d\gamma\partial t} dL_r)dt$$

where $\gamma_t = \gamma(\cdot, t)$, in the same way as in [Sol06], using the co-area formula and taking derivatives with respect to t, we get

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}^{\mathbb{C}}_r} \chi(\Omega_t \cap L_r) dL_r = \int_{G^{\mathbb{C}}_{n,r}(T\partial\Omega)} \mathrm{sign}\phi \, \mathrm{sign} K(L_r) \gamma_0^*(\iota_{d\phi\partial t} dL_r).$$

Suppose that $\{g; g_1, Jg_1, ..., g_n, Jg_n\}$ is a local orthonormal frame defined on $G^{\mathbb{C}}_{n,r}(T\partial\Omega_0) \times (-\epsilon,\epsilon)$ such that $g((x,l),t) = \phi(x,t)$, $\gamma = \langle g, g_1, Jg_1, ..., g_r, Jg_r\rangle \cap \mathbb{CK}^n(\epsilon)$ and $Jg_n((x,l),t) = N_t$ (N_t denotes the normal vector to $\partial\Omega_t$ at $\phi_t(x)$). Consider the curve $L_r(t)$ given by the parallel translation of L_r along the geodesic given by N, the inner normal vector to $\partial\Omega_0$. If P denotes the tangent vector to $L_r(t)$, then for $t = 0$

$$\begin{aligned} \omega_i(P) &= \langle dg(P), g_i\rangle = \langle \frac{d}{dt} g(L_r(t)), g_i\rangle = 0, \\ \omega_{\overline{n}}(P) &= \langle dg(P), N\rangle = 1, \\ \omega_{kj}(P) &= \langle dg_k(P), g_j\rangle = \langle \frac{d}{dt} g_k(L_r(t)), g_j\rangle = 0 \end{aligned}$$

where $i \in \{r+1, \overline{r+1}, ..., n-1, \overline{n-1}\}$, $j \in \{r+1, \overline{r+1}, ..., n, \overline{n}\}$, and $k \in \{1, \overline{1}, ..., r, \overline{r}\}$. By (14) and last equations we get

$$dL_r = |\omega_{\overline{n}}| \iota_P dL_r$$

since $\iota_P dL_r = |\omega_{\overline{n}}(P)| \cdot |\bigwedge_{h=r+1}^n \omega_h \wedge \overline{\omega_h} \bigwedge \omega_{ij}|$. Thus,

$$\iota_{d\gamma\partial t} dL_r = |\omega_{\overline{n}}(d\gamma\partial t)| \iota_P dL_r + |\omega_{\overline{n}}| \iota_{d\gamma\partial t} \iota_P dL_r$$

and

$$\begin{aligned} \omega_{\overline{n}}(d\gamma\partial t) &= \langle dg(d\gamma\partial t), N\rangle = \langle \frac{\partial \phi}{\partial t}, N\rangle, \\ \gamma_0^*(\omega_{\overline{n}})(v) &= \langle dg(d\gamma_0(v)), N\rangle = 0 \quad \forall v \in T_{(p,l)} G_{n,r}^{\mathbb{C}}(T\partial\Omega_0). \end{aligned}$$

So,

$$\gamma_0^*(\iota_{d\gamma\partial t} dL_r) = |\langle \frac{\partial \phi}{\partial t}, N\rangle| \gamma_0^*(\iota_P dL_r).$$

Finally, using that $\gamma_0^*(\iota_P dL_r) = |K(l)| dL_{r[x]}^{\mathbb{C}} dx$, we get the result. □

Remark 3.11. In real space forms it is known

$$\int_{G_{n-1,r}} \sigma_r(\mathrm{II}|_V) dV = \mathrm{vol}(G(n-1,r)) \sigma_r(\mathrm{II}),$$

but in complex space forms we do not know how to evaluate directly the analogous integral over the complex Grassmanian.

If $r = n-1$, then this integral can be easily evaluated in $\mathbb{CK}^n(\epsilon)$.

Corollary 3.12. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$ be a regular domain, let X be a smooth vector field on $\mathbb{CK}^n(\epsilon)$, let ϕ_t be the flow associated to X and let $\Omega_t = \phi_t(\Omega)$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}_{n-1}^{\mathbb{C}}} \chi(\Omega_t \cap L_{n-1}) dL_{n-1} = \omega_{2n-1} \tilde{B}_{1,0}(\Omega).$$

Proof. Since there is only one complex hyperplane contained in the tangent space of a hypersurface, using the following relations, we obtain the result directly from Proposition 3.10 .

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}_{n-1}^{\mathbb{C}}} \chi(\Omega_t \cap L_{n-1}) dL_{n-1} &= \int_{\partial Q_0} \langle \partial\phi/\partial t, N\rangle \int_{G_{n-1,n-1}^{\mathbb{C}}} \sigma_{2n-2}(\mathrm{II}|_V) dV dx \\ &= \int_{\partial Q_0} \langle \partial\phi/\partial t, N\rangle \sigma_{2n-2}(\mathrm{II}|_{\mathcal{D}}) dx \\ &= \int_{\partial Q_0} \langle \partial\phi/\partial t, N\rangle \frac{\beta \wedge \theta_0^{n-1}}{(n-1)!} \\ &= \frac{c_{n,1,0}^{-1}}{(n-1)!} \tilde{B}_{1,0} = \omega_{2n-1} \tilde{B}_{1,0}. \end{aligned}$$

□

4. Crofton type formulas

4.1. **In the standard Hermitian space.**

Theorem 4.1. *Let $\Omega \subset \mathbb{C}^n$, let X be a smooth vector field over $\mathbb{C}^n$, let ϕ_t be the flow associated to X and let $\Omega_t = \phi_t(\Omega)$. Then*

$$\frac{d}{dt}\bigg|_{t=0} \int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega_t \cap L_r) dL_r = \mathrm{vol}(G^{\mathbb{C}}_{n-1,r})\omega_{2r+1}(r+1)\binom{n-1}{r}^{-1}\binom{n}{r}^{-1}\cdot$$

$$\cdot\left(\sum_{q=\max\{0,n-2r\}}^{n-r-1}\binom{2q-1}{q}\frac{1}{4^{q-1}}\tilde{B}_{2n-2r-1,n-r-q}(\Omega)\right) \tag{15}$$

and

$$\int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega \cap L_r) dL_r = \mathrm{vol}(G^{\mathbb{C}}_{n-1,r})\omega_{2r}\binom{n-1}{r}^{-1}\binom{n}{r}^{-1}\cdot$$

$$\cdot\left(\sum_{q=\max\{0,n-2r\}}^{n-r}\frac{1}{4^{n-r-q}}\binom{2n-2r-2q}{n-r-q}\mu_{2n-2r,q}(\Omega)\right). \tag{16}$$

Proof. In order to simplify the following computations, we consider

$$B'^{\Omega}_{k,q}(\Omega) = c^{-1}_{n,k,q}B^{\Omega}_{k,q}(\Omega), \quad \Gamma'^{\Omega}_{k,q}(\Omega) = 2c^{-1}_{n,k,q}\Gamma^{\Omega}_{k,q}(\Omega) \tag{17}$$

and

$$\tilde{B}'_{k,q} = c^{-1}_{n,k,q}\tilde{B}_{k,q}, \quad \tilde{\Gamma}'_{k,q} = 2c^{-1}_{n,k,q}\tilde{\Gamma}_{k,q}. \tag{18}$$

The expression $\int_{\mathcal{L}_r^{\mathbb{C}}}\chi(L_r\cap\Omega)dL_r$ is a valuation on $\mathbb{C}^n$ with degree of homogeneity $2n-2r$. Thus, it can be expressed as a linear combination of the elements of a basis of valuations with the same degree of homogeneity. Then, by Remark 2.9 and (11), we have

$$\int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega \cap L_r) dL_r = \sum_{q=\max\{0,n-2r\}}^{n-r-1} C_q B'_{2n-2r,q} + D\Gamma'_{2n-2r,n-r} \tag{19}$$

for certain constants C_q, D which we wish to determine. This will be done by comparing the variation of both sides of this equality. From here on we assume $2r < n$. The case $2r \geq n$ can be treated in the same way (cf. Remark 4.2).

By Proposition 3.8, the variation of the right hand side of (19) is a linear combination of the following type

$$\sum_{q=n-2r-1}^{n-r-1} c_q \tilde{B}'_{2n-2r-1,q} + \sum_{q=n-2r}^{n-r-1} d_q \tilde{\Gamma}'_{2n-2r-1,q} \tag{20}$$

where the coefficients c_q and d_q can be expressed in terms of a linear combination with known coefficients of the variables C_q and D, that still remain unknown.

The variation of the left hand side of (19), by Proposition 3.10 is

$$\left.\frac{d}{dt}\right|_{t=0}\int_{\mathcal{L}_r^{\mathbb{C}}}\chi(\Omega_t\cap L_r)dL_r=\int_{\partial\Omega}\langle\partial\phi/\partial t,N\rangle\int_{G_{n-1,r}^{\mathbb{C}}}\sigma_{2r}(\mathrm{II}|_V)dV\,dx. \tag{21}$$

After fixing a frame, the integral $\int_{G^{\mathbb{C}}_{n-1,r}}\sigma_{2r}(\mathrm{II}|_V)dV$ is a polynomial of the second fundamental form II restricted to the distribution $\mathcal{D}=\langle N,JN\rangle^{\perp}$. When pulling-back the form $\gamma_{k,q}$ from $N(\Omega)$ to $\partial\Omega$, one gets a polynomial expression $P_{k,q}$ of degree k in the coefficients of II. Each of the monomials of $P_{k,q}$ is a minor $k\times k$ of II containing the entry $\mathrm{II}(JN,JN)$. For $q\neq q'$, all the monomials of $P_{k,q}$ are different from those of $P_{k,q'}$. Therefore, every linear combination of $\{P_{k,q}\}_q$ must contain the variable $\mathrm{II}(JN,JN)$. Since $JN\notin\mathcal{D}$, comparing the expressions of (20) and (21), it follows that $d_q=0$ for all $q\in\{n-2r,\dots,n-r-1\}$.

As c_q and d_q depend on C_q and D, we will obtain the value of c_q once we know the value of C_q and D. We will get their value from the equalities $\{d_q=0\}$. Note that this gives r equations, since q runs from $n-2r$ to $n-r-1$ in (20). As for the unknowns, we need to find r constants C_q plus the constant D in (19).

We will get an extra equation by taking $\mathrm{II}|_{\mathcal{D}}=\mathrm{Id}$ and equating (21) to (20). Then, for any pair (n,r) we have a compatible linear system since constants in (19) exist. Next we find the solution, which turns out to be unique.

Let us relate explicitly the coefficients $\{c_q\}$ and $\{d_q\}$ in (20) with C_q and D in (19). To simplify the range of the subscripts, we denote d_{n-r-a} with $a=1,\dots,r$ and c_{n-r-a} with $a=1,\dots,r+1$.

Coefficient d_{n-r-1}.From the variation of $B'_{k,q}$ in $\mathbb{C}^n$ (Proposition 3.7), the coefficient of $\tilde{\Gamma}'_{2n-2r-1,n-r-1}$ comes from the variation of $B'_{2n-2r,n-r-1}$ and $\Gamma'_{2n-2r,n-r}$. Then,

$$\begin{aligned} d_{n-r-1}&=-2r(n-r)D+(2n-2r-2(n-r-1))^2C_{n-r-1}\\ &=4C_{n-r-1}-2r(n-r)D. \end{aligned}\tag{22}$$

Coefficient $d_{n-r-a},\ a=2,...,r$. The coefficient of $\tilde{\Gamma}'_{2n-2r-1,n-r-a}$ comes from the variation of $B'_{2n-2r,n-r-a}$ and $B'_{2n-2r,n-r-a+1}$. Then,

$$\begin{aligned} d_{n-r-a}&=(2n-2r-2(n-r-a))^2C_{n-r-a}\\ &\quad-(2r+n-r-a+1-n)(n-r-a+1)C_{n-r-a+1}\\ &=4a^2C_{n-r-a}-(r-a+1)(n-r-a+1)C_{n-r-a+1}. \end{aligned}\tag{23}$$

Coefficient c_{n-r-1}. The coefficient of $\tilde{B}'_{2n-2r-1,n-r-1}$ comes from the variation of $B'_{2n-2r,n-r-1}$ and $\Gamma'_{2n-2r,n-r}$. Then,

$$\begin{aligned} c_{n-r-1}&=4(r+1/2)(n-r)D-4C_{n-r-1}\\ &=2(2r+1)(n-r)D-4C_{n-r-1}. \end{aligned}\tag{24}$$

Coefficient c_{n-r-a}, $a = 2, ..., r-2$. The coefficient of $\tilde{B}'_{2n-2r-1,n-r-a}$ comes from the variation of $B'_{2n-2r,n-r-a}$ and $B'_{2n-2r,n-r-a+1}$. Then,

$$\begin{aligned} c_{n-r-a} &= -2(2a)(2a-1)C_{n-r-a} + 2(r-a+3/2)(n-r-a+1)C_{n-r-a+1} \\ &= -4a(2a-1)C_{n-r-a} + (2r-2a+3)(n-r-a+1)C_{n-r-a+1}. \end{aligned} \tag{25}$$

Coefficient c_{n-2r-1}. The coefficient of $\tilde{B}'_{2n-2r-1,n-2r-1}$ comes from the variation of $B'_{2n-2r,n-2r}$. Then,

$$\begin{aligned} c_{n-2r-1} &= (2r-2(r+1)+3)(n-r-(r+1)+1)C_{n-2r} \\ &= (n-2r)C_{n-2r}. \end{aligned} \tag{26}$$

Now, we solve the linear system given by $\{d_{n-r-a} = 0\}$ where $a \in \{1, ..., r\}$. From equations (22) and (23) we have that the system is given by:

$$\begin{cases} r(n-r)D &= 2C_{n-r-1} \\ 4a^2 C_{n-r-a} &= (n-r-a+1)(r-a+1)C_{n-r-a+1}. \end{cases}$$

Thus,

$$\begin{aligned} C_{n-r-a} &= \frac{(n-r-a+1)\cdot ... \cdot (n-r)(r-a+1) ... \cdot r}{2\cdot 4^{a-1}a^2(a-1)^2\cdot ... \cdot 1^2} D \\ &= \frac{(n-r)!r!}{2^{2a-1}(n-r-a)!(r-a)!a!a!} D \\ &= \frac{D}{2^{2a-1}}\binom{n-r}{a}\binom{r}{a}. \end{aligned} \tag{27}$$

To obtain the value of D, we calculate $\int_{G^{\mathbb{C}}_{n-1,r}} \sigma_{2r}(p)dV$ and $\beta'_{2n-2r-1,n-r-a}$ with $\mathrm{II}|_{\mathcal{D}}(p) = \mathrm{Id}$. On the one hand, we have

$$\int_{G^{\mathbb{C}}_{n-1,r}} \sigma_{2r}(p)(\mathrm{Id}|_V)dV = \mathrm{vol}(G^{\mathbb{C}}_{n-1,r}).$$

On the other hand, if $\mathrm{II}|_{\mathcal{D}} = \mathrm{Id}$, then the connection forms satisfy $\alpha_{1i} = \omega_i$ and $\beta_{1i} = \omega_i$. Thus, $\theta_1 = 2\theta_2$ and $\theta_0 = \theta_2$ and we obtain

$$\begin{aligned} \beta'_{2n-2r-1,n-r-a}(p) &= (\beta \wedge \theta_0^{r-a+1} \wedge \theta_1^{2a-2} \wedge \theta_2^{n-r-a})(p) \\ &= (\beta \wedge \theta_2^{n-1})(p) = 2^{2a-2}(n-1)!. \end{aligned}$$

So, the equation

$$\mathrm{vol}(G^{\mathbb{C}}_{n-1,r}) = \sum_{a=1}^{r+1} c_{n-r-a} 2^{2a-2}(n-1)!$$

must be satisfied.

Equations in (24), (25) and (26) give us the relation among c_{n-r-a}, D and C_{n-r-a}. Using these relations we have

$$\begin{aligned}
\frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{(n-1)!} &= (2(2r+1)(n-r)D - 4C_{n-r-1}) \\
&\quad + \sum_{a=2}^{r} 2^{2a-2}((2r-2a+3)(n-r+a+1)C_{n-r-a+1} - 4a(2a-1)C_{n-r-a}) \\
&\quad + 2^{2r}(n-2r)C_{n-2r} \\
&= 2(2r+1)(n-r)D + 4C_{n-r-1}((2r-1)(n-r-1)-1) \\
&\quad + \sum_{a=2}^{r-1}(-2^{2a-2}4a(2a-1) + 2^{2a}(2r-2a+1)(n-r-a))C_{n-r-a} \\
&\quad + C_{n-2r}(2^{2r}(n-2r) - 2^{2r-2}4r(2r-1)) \\
&= 2(2r+1)(n-r)D + \sum_{a=1}^{r} 2^{2a} C_{n-r-a}((2r-2a+1)(n-r-a) - a(2a-1)) \\
&\overset{(27)}{=} D\left(2(2r+1)(n-r) + 2(n-r)!r!\sum_{a=1}^{r}\frac{(2r-2a+1)(n-r-a)-a(2a-1)}{(n-r-a)!(r-a)!}a!a!\right) \\
&= D\left(2(2r+1)(n-r) + 2(n-r)!r!\frac{n! - r!(n-r)!(2r+1)}{r!r!(n-r)!(n-r-1)!}\right) \\
&= D\frac{2\,n!}{r!(n-r-1)!}.
\end{aligned}$$

Thus,

$$D = \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\,n!}\binom{n-1}{r}^{-1},$$

$$C_{n-r-a} = \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{4^a n!}\binom{n-1}{r}^{-1}\binom{n-r}{a}\binom{r}{a}$$

and, for $2r < n$, we have

$$\begin{aligned}
\int_{\mathcal{L}^{\mathbb{C}}_r} \chi(\Omega \cap L_r)dL_r &= \sum_{a=1}^{r} C_{n-r-a}B'_{2n-2r,n-r-a} + D\Gamma'_{2n-2r,n-r} \\
&= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{2\,n!}\binom{n-1}{r}^{-1}\left(\sum_{a=1}^{r}\binom{n-r}{a}\binom{r}{a}2^{-2a+1}B'_{2n-2r,n-r-a} + \Gamma'_{2n-2r,n-r}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0}\int_{\mathcal{L}^{\mathbb{C}}_r}\chi(\Omega_t \cap L_r)dL_r &= (2(2r+1)(n-r)D - 4C_{n-r-1})B'_{2n-2r-1,n-r-1} \\
&+ \sum_{a=2}^{r}((2r-2a+3)(n-r+a+1)C_{n-r-a+1} - 4a(2a-1)C_{n-r-a})B'_{2n-2r-1,n-r-a} \\
&+ (n-2r)C_{n-2r}B'_{2n-2r-1,n-2r-1} \\
&= \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{n!}\binom{n-1}{r}^{-1}\left(\sum_{a=1}^{r+1}\binom{n-r}{a}\binom{r+1}{a}\frac{a}{4^{a-1}}\tilde{B}'_{2n-2r-1,n-r-a}\right).
\end{aligned}$$

Finally, we use the relation in (17) and (11) to obtain the result. □

Remark 4.2. If $2r \geq n$, then formula (15) follows directly from the relations among the different bases of valuations on $\mathbb{C}^n$ given in [BF08] and the following relation in [Ale03]

$$\int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega \cap L_r) dL_r = \frac{1}{O_{2r-1}} \int_{\mathcal{L}_r^{\mathbb{C}}} M_{2r-3}(\partial\Omega \cap L_r) dL_r = cU_{2(n-r),n-r}$$

for a certain constant c coming from the different normalizations in dL_r.

4.2. In $\mathbb{CK}^n(\epsilon)$.

Corollary 4.3. *Let $\Omega \subset \mathbb{CK}^n(\epsilon)$, let X be a smooth vector field over $\mathbb{CK}^n(\epsilon)$, let ϕ_t be the flow associated to X and let $\Omega_t = \phi_t(\Omega)$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} \int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega_t \cap L_r) dL_r = \mathrm{vol}(G_{n-1,r}^{\mathbb{C}}) \omega_{2r+1}(r+1) \binom{n-1}{r}^{-1} \binom{n}{r}^{-1} \cdot$$

$$\cdot \left(\sum_{q=\max\{0,n-2r-1\}}^{n-r-1} \binom{2q-1}{q} \frac{1}{4^{q-1}} \tilde{B}_{2n-2r-1,n-r-q}(\Omega) \right). \tag{28}$$

Proof. Comparing equation (15) and Proposition 3.10 in case $\epsilon = 0$ shows that

$$\int_{\partial\Omega} \langle X, N \rangle \left(\int_{G_{n,r}^{\mathbb{C}}} \sigma_{2r}(\mathrm{II}|V) dV \right) dx$$

equals the right hand side of equation above. By taking a field X that vanishes outside an arbitrarily small neigborhood of any $x \in \partial\Omega$, we deduce the following equality between forms

$$\left(\int_{G_{n,r}^{\mathbb{C}}} \sigma_{2r}(\mathrm{II}|V) dV \right) dx = \frac{\omega_{2r+1}}{\binom{n-1}{r}\binom{n}{r}} \cdot$$

$$\cdot \sum_{q=\max\{0,n-2r-1\}}^{n-r-1} \binom{2q-1}{q} \frac{c_{n,2n-2r-1,n-r-q}}{4^{q-1}} \beta \wedge \theta_0^{r-q+1} \wedge \theta_1^{2q-2} \wedge \theta_2^{n-r-q}$$

This equation extends obviously to $\mathbb{CK}^n(\epsilon)$ without change. Then, using Proposition 3.10 gives the result. □

Theorem 4.4. *Let Ω be a regular domain in $\mathbb{CK}^n(\epsilon)$. Then*

$$\int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega \cap L_r) dL_r = \mathrm{vol}(G_{n-1,r}^{\mathbb{C}}) \binom{n-1}{r}^{-1} (\epsilon^r (r+1)\mathrm{vol}(\Omega) +$$

$$+ \sum_{k=n-r}^{n-1} \epsilon^{k-(n-r)} \omega_{2n-2k} \binom{n}{k}^{-1} \cdot ((k+r-n+1)\mu_{2k,k} +$$

$$+ \sum_{q=\max\{0,2k-n\}}^{k-1} \frac{1}{4^{k-q}} \binom{2k-2q}{k-q} \mu_{2k,q})).$$

Proof. First, we write the formula to be proved in terms of $\{B'_{k,q}\}$ and $\{\Gamma'_{k,q}\}$ defined in (17):

$$\int_{\mathcal{L}_r^{\mathbb{C}}} \chi(\Omega \cap L_r) dL_r = \frac{\mathrm{vol}(G^{\mathbb{C}}_{n-1,r})}{n!} \binom{n-1}{r}^{-1} (\epsilon^r (r+1) n! \mathrm{vol}(\Omega) +$$

$$+ \sum_{k=n-r}^{n-1} \epsilon^{k-(n-r)} \Bigg(\sum_{q=\max\{0,2k-n\}}^{k-1} \frac{1}{4^{k-q}} \binom{n-k}{k-q} \binom{k}{q} B'_{2k,q} + \frac{k+r-n+1}{2} \Gamma'_{2k,k} \Bigg)). \tag{29}$$

We calculate the variation of both sides of the equation (29) to prove that they coincide. The variation of the left hand side is given in Corollary 4.3. To calculate the variation of the right hand side we use Proposition 3.8 which we recall in the following table. We denote by $(\delta_X B'_{k,q}, \tilde{B}'_{r,s})$ the coefficient of $\tilde{B}_{r,s}$ in the expression of $\delta_X B'_{k,q}$.

$$(\delta_X B'_{k,q}, \tilde{B}'_{r,s}) = \begin{cases} 2q(n+q-k+1/2), & \text{if } r=k-1, s=q-1 \\ -2(k-2q)(k-2q-1), & \text{if } r=k-1, s=q \\ 2\epsilon(k-2q)(k-2q-1), & \text{if } r=k+1, s=q+1 \\ -2\epsilon(n-k+q)(q+1/2), & \text{if } r=k+1, s=q \\ 0, & \text{otherwise.} \end{cases}$$

$$(\delta_X B'_{k,q}, \tilde{\Gamma}'_{r,s}) = \begin{cases} (k-2q)^2, & \text{if } r=k-1, s=q \\ -(n+q-k)q, & \text{if } r=k-1, s=q-1 \\ 0, & \text{otherwise.} \end{cases}$$

$$(\delta_X \Gamma'_{2q,q}, \tilde{B}'_{r,s}) = \begin{cases} 4q(n-q+1/2), & \text{if } r=2q-1, s=q-1 \\ 4\epsilon((q+1)/2-(n-q)(2q+3/2)), & \text{if } r=2q+1, s=q \\ 4\epsilon^2(n-q-1)(q+3/2), & \text{if } r=2q+3, s=q+1 \\ 0, & \text{otherwise.} \end{cases}$$

$$(\delta_X \Gamma'_{2q,q}, \tilde{\Gamma}'_{r,s}) = \begin{cases} -2(n-q)q, & \text{if } r=2q-1, s=q-1 \\ 2\epsilon(n-q-1)(q+1), & \text{if } r=2q+1, s=q \\ 0, & \text{otherwise.} \end{cases}$$

With these expressions, one can check that the variation of the right hand side of (29) coincides with the right hand side of (28).

Finally, we take a deformation Ω_t of Ω such that Ω_t converges to a point. Since both sides of (29) vanish in the limit, the result follows. □

5. Gauss-Bonnet theorem

Theorem 5.1. *Let Ω be a regular domain in $\mathbb{CK}^n(\epsilon)$. Then*

$$O_{2n-1}\chi(\Omega) = 2n(n+1)\epsilon^n \mathrm{vol}(\Omega) +$$

$$+ \sum_{c=0}^{n-1} \frac{O_{2n-2c-1}\epsilon^c}{\binom{n-1}{c}} \Bigg(\sum_{q=\max\{0,2c-n\}}^{c-1} \frac{1}{4^{c-q}} \binom{2c-2q}{c-q} \mu_{2c,q} + (c+1)\mu_{2c,c} \Bigg). \tag{30}$$

Proof. We proceed analogously to the proof of Theorem 4.4. In fact, the same computations of the previous proof show (in case $r=n$) that the right hand side of (30) has null variation.

If $\epsilon=0$, then we get the known Gauss-Bonnet formula in $\mathbb{C}^n$. Taking some smooth deformation of Ω to get a domain contained in a ball of arbitrarily small radius, we have that the equality is true for all $\epsilon\in\mathbb{R}$. $\square$

Theorem 5.2. *Let Ω be a regular domain in $\mathbb{CK}^n(\epsilon)$. Then*

$$\begin{aligned} M_{2n-1}(\partial\Omega) = & O_{2n-1}\chi(\Omega) - 2n\epsilon\int_{\mathcal{L}^{\mathbb{C}}_{n-1}}\chi(\partial\Omega\cap L_{n-1})dL_{n-1} - \\ & -\sum_{k=1}^{n-1}\epsilon^k O_{2n-2k-1}\binom{n-1}{k}^{-1}\mu_{2k,k} - 2n\epsilon^n\mathrm{vol}(\Omega). \end{aligned}$$

Proof. First, we use the following relation between $M_{2n-1}(\partial\Omega)$ and $\mu_{0,0}(\Omega)$

$$\begin{aligned} M_{2n-1}(\partial\Omega) &= \int_{\partial\Omega}\sigma_{2n-1}(\mathrm{II}_x)dx = \int_{\partial\Omega}\frac{\gamma\wedge\theta_0^{n-1}}{(n-1)!} \\ &= \frac{2c_{n,0,0}^{-1}}{(n-1)!}\frac{c_{n,0,0}}{2}\int_{\partial\Omega}\gamma\wedge\theta_0^{n-1} = \frac{2c_{n,0,0}^{-1}}{(n-1)!}\mu_{0,0}(\Omega) \\ &= 2n\omega_{2n}\mu_{0,0}(\Omega) = O_{2n-1}\mu_{0,0}(\Omega). \end{aligned} \tag{31}$$

Now, from Theorems 4.4 and 5.1, it follows that

$$\begin{aligned} \chi(\Omega) =& \sum_{c=0}^{n-1}\frac{\epsilon^c\, c!}{\pi^c}\left(\sum_{q=\max\{0,2c-n\}}^{n-1}\frac{1}{4^{c-q}}\binom{2c-2q}{c-q}\mu_{2c,q} + (c+1)\mu_{2k,k}\right) \\ &+\frac{\epsilon^n(n+1)!}{\pi^n}\mathrm{vol}(\Omega) \\ =& \mu_{0,0} + \sum_{c=1}^{n-1}\frac{\epsilon^c\, c!}{\pi^c}\left(\sum_{q=\max\{0,2c-n\}}^{n-1}\frac{1}{4^{c-q}}\binom{2c-2q}{c-q}\mu_{2c,q} + (c+1)\mu_{2c,c}\right) \\ &+\frac{\epsilon^n(n+1)!}{\pi^n}\mathrm{vol}(\Omega) \\ =& \mu_{0,0} + \frac{\epsilon\, n!}{\pi^n}\sum_{c=1}^{n-1}\frac{\epsilon^{c-1}\, c!\pi^{n-c}}{n!}\left(\sum_{q=\max\{0,2c-n\}}^{n-1}\frac{1}{4^{c-q}}\binom{2c-2q}{c-q}\mu_{2c,q} + c\mu_{2c,c}\right) \\ &+\frac{\epsilon\, n!}{\pi^n}\sum_{c=1}^{n-1}\frac{\epsilon^{c-1}c!\pi^{n-c}}{n!}\mu_{2c,c} + \frac{\epsilon^n(n+1)!}{\pi^n}\mathrm{vol}(\Omega) \\ =& \mu_{0,0} + \frac{\epsilon\, n!}{\pi^n}\int_{\mathcal{L}_{n-1}}\chi(\partial\Omega\cap L_{n-1})dL_{n-1} + \sum_{c=1}^{n-1}\frac{\epsilon^c\, c!}{\pi^c}\mu_{2c,c} \\ &+\left(\frac{\epsilon^n(n+1)!}{\pi^n} - \frac{\epsilon^n\, n!n}{\pi^n}\right)\mathrm{vol}(\Omega). \end{aligned}$$

□

Remark 5.3. For $n=2$ and $n=3$, the Gauss-Bonnet-Chern formula in $\mathbb{CK}^n(\epsilon)$ given in Theorem 5.1 was already stated in [Par02].

6. Total Gauss curvature integral

Theorem 6.1. *Let Ω be a regular domain in $\mathbb{C}^n$. Then*

$$\int_{\mathcal{L}_r^{\mathbb{C}}} M_{2r-1}(\partial\Omega\cap L_r)dL_r = 2r\omega_{2r}^2 \mathrm{vol}(G_{n-1,r}^{\mathbb{C}})\binom{n-1}{r}^{-1}\binom{n}{r}^{-1}\cdot$$
$$\cdot\left(\sum_{q=\max\{0,n-2r\}}^{n-r}\frac{1}{4^{n-r-q}}\binom{2n-2r-2q}{n-r-q}\mu_{2n-2r,q}\right).$$

Proof. On the one hand, by Gauss-Bonnet formula in $\mathbb{C}^n$, we have

$$\int_{\mathcal{L}_r^{\mathbb{C}}}\chi(\Omega\cap L_r)dL_r = \int_{\mathcal{L}_r^{\mathbb{C}}}\mu_{0,0}(\Omega\cap L_r)dL_r.$$

On the other hand, by Theorem 4.4 in $\mathbb{C}^n$, we have

$$\int_{\mathcal{L}_r^{\mathbb{C}}}\chi(\partial\Omega\cap L_r)dL_r = \mathrm{vol}(G_{n-1,r}^{\mathbb{C}})\binom{n-1}{r}^{-1}\omega_{2r}\binom{n}{n-r}^{-1}$$
$$\cdot\left(\sum_{q=\max\{0,n-2r\}}^{n-r}\frac{1}{4^{n-r-q}}\binom{2n-2r-2q}{n-r-q}\mu_{2n-2r,q}\right).$$

If we equate both expressions and we use the relation (31) between the total Gauss curvature and the valuation $\mu_{0,0}$, we obtain the result. □

Remark 6.2. The previous Theorem is not necessarily true in $\mathbb{CK}^n(\varepsilon)$ for $\epsilon\neq 0$. Indeed, Howard's transfer principle can not be used here since Theorem 6.1 is not local: it does not apply to general hypersurfaces, but only to the closed embedded ones.

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