# NON-EXISTENCE OF REFLECTIONLESS MEASURES FOR THE $s$-RIESZ TRANSFORM WHEN $0 < s < 1$

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Abstract. A measure $\mu$ on $\mathbb{R}^d$ is called reflectionless for the $s$-Riesz transform if the singular integral $R^s\mu(x) = \int \frac{y-x}{|y-x|^{s+1}}\, d\mu(y)$ is constant on the support of $\mu$ in some weak sense and, moreover, the operator defined by $R^s_\mu(f) = R^s(f\,\mu)$ is bounded in $L^2(\mu)$. In this paper we show that the only reflectionless measure for the $s$-Riesz transform is the zero measure when $0 < s < 1$.

## 1. Introduction and Background

Fix $d \in \mathbb{N}$ and for $0 < s < d$, consider the signed vector valued Riesz kernels

$$K^s(x) = \frac{x}{|x|^{1+s}},\ x \in \mathbb{R}^d,\ x \neq 0.$$

The $s$-Riesz transform of a real Radon measure $\mu$ is

$$R^s\mu(x) = \int K^s(y-x)d\mu(y),$$

whenever the integral makes sense. To avoid technical problems with the absolute convergence of the integral, one considers the truncated $s-$Riesz transform of $\mu$, which is defined as

$$R^s_\varepsilon\mu(x) = \int_{|y-x|>\varepsilon} K^s(y-x)d\mu(y), \quad x \in \mathbb{R}^d,\ \varepsilon > 0.$$

Given a positive Radon measure and a function $f \in L^1(\mu)$, we consider the operators $R^s_\mu(f) := R^s(fd\mu)$ and $R^s_{\mu,\varepsilon}(f) := R^s_\varepsilon(fd\mu)$. We say that $R^s_\mu$ is bounded in $L^2(\mu)$ if the truncated Riesz transforms $R^s_{\mu,\varepsilon}$ are bounded in $L^2(\mu)$ uniformly in $\varepsilon$, and we set

$$\|R^s_\mu\|_{L^2(\mu)\to L^2(\mu)} = \sup_{\varepsilon>0} \|R^s_{\mu,\varepsilon}\|_{L^2(\mu)\to L^2(\mu)}.$$

We are interested in the following conjecture: for non-integer $0 < s < d$, the $s$-Riesz transform never defines a bounded operator on a set of positive and finite $s$-Hausdorff measure. For $0 < s < 1$, this conjecture was solved in the affirmative by Prat [Pr] (see also [MPV]), and for $d-1 < s < d$ by Eiderman, Nazarov and Volberg [ENV] (relying on a previous result by Vihtilä [Vi]). In the case $0 < s < 1$, this can be proved by the symmetrization method, which was originally used by Melnikov [Me] in connection with

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the Cauchy kernel. For $0 < s < 1$ and three different points $x_1, x_2, x_3 \in \mathbb{R}^d$, consider the quantity

$$p_s(x_1, x_2, x_3) = \frac{1}{2} \sum_{\sigma} \frac{x_{\sigma(2)} - x_{\sigma(1)}}{|x_{\sigma(2)} - x_{\sigma(1)}|^{1+s}} \cdot \frac{x_{\sigma(3)} - x_{\sigma(1)}}{|x_{\sigma(3)} - x_{\sigma(1)}|^{1+s}}, \tag{1.1}$$

where the sum is taken over the six permutations of the set $\{1, 2, 3\}$ and "$\cdot$" denotes the scalar product. In [Pr, Lemma 4.2] it was proved that

$$p_s(x_1, x_2, x_3) \approx \frac{1}{\max\{|x_1 - x_2|, |x_1 - x_3|, |x_2 - x_3|\}^{2s}}.$$

As shown in [Pr, Theorem 4.4], it turns out that if $\mu$ is a finite positive Borel measure such that

$$\Theta^{s,*}(x, \mu) := \limsup_{r \to 0} \frac{\mu(B(x, r))}{(2r)^s} > 0 \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d,$$

then

$$p_s(\mu) = \iiint p_s(x_1, x_2, x_3) = +\infty.$$

If, moreover, $\mu(B(x, r)) \leq c\, r^s$ for all $x \in \mathbb{R}^d$ and all $r > 0$ (which is a necessary condition for the $L^2(\mu)$ boundedness of $R^s_\mu$ if $\mu$ has no point masses), then we have

$$\int |R^s_\varepsilon \mu(x)|^2 d\mu(x) = \frac{1}{3} \iiint_{\substack{|x_1 - x_2| > \varepsilon \\ |x_1 - x_3| > \varepsilon \\ |x_2 - x_3| > \varepsilon}} p_s(x_1, x_2, x_3) d\mu(x_1) d\mu(x_2) d\mu(x_3) + O(\|\mu\|).$$

See [Pr, Section 4.4] for more details. As a consequence, for $0 < s < 1$, we deduce that

$$\text{if } 0 < \Theta^{s,*}(x, \mu) < \infty \ \mu\text{-a.e. and } R^s_\mu \text{ is bounded in } L^2(\mu), \text{ then } \mu \text{ is the zero measure.} \tag{1.2}$$

In particular, the are no sets $E \subset \mathbb{R}^d$ with $0 < \mathcal{H}^s(E) < \infty$ such that for $\mu = \mathcal{H}^s_{|E}$ the Riesz transform $R^s_\mu$ is bounded in $L^2(\mu)$.

In the case $d - 1 < s < d$, (1.2) also holds, as well as the analogous consequence for $\mu = \mathcal{H}^s_{|E}$, because of the aforementioned works of Eiderman, Nazarov and Volberg [ENV] and Vihtilä [Vi].

On the other hand, (1.2) remains open for non-integer values $s \in (1, d - 1)$. In [JN] Jaye and Nazarov reduce this open problem to the problem of describing the reflectionless measures associated with $R^s_\mu$. One says that $\mu$ is reflectionless for the $s$-Riesz transform if $\mu$ has no point masses, $R^s_\mu$ is bounded in $L^2(\mu)$, and

$$\langle R^s \mu, \psi \rangle_\mu = 0,$$

for all Lipschitz continuous compactly supported functions $\psi$ such that $\int \psi d\mu = 0$, where $R^s \mu$ is defined modulo constants as a weak limit of the functions $R^s_\varepsilon \mu$ when $\varepsilon \to 0$. Recall that a function $\psi$ is called Lipschitz continuous if

$$\|\psi\|_{\mathrm{Lip}} = \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|\psi(x) - \psi(y)|}{|y - x|} < \infty.$$

In particular, it is shown in [JN] that if, for a given $s \in (0, d)$, the only reflectionless measure for $R^s_\mu$ is the zero measure, then (1.2) holds. Further, in the same work the authors prove that in the case $s \in (d-1,d)$ there do not exist non-zero reflectionless measures for the $s$-Riesz transform, which yields a new proof of (1.2) for this range of $s$.

The main result of this paper is the following:

**Theorem 1.1.** *There are no non-trivial reflectionless measures for the $s$-dimensional Riesz transform if $0 < s < 1$.*

Our arguments to prove this result combine the symmetrization method described above with some variational techniques which have already appeared in other previous works, such as [ENV] and [NToV].

## 2. Preliminary results

In what follows, we assume that $0 < s < 1$. Given a ball $B \subset \mathbb{R}^d$ of radius $r(B)$, we denote by $\theta_\mu^s(B)$ the average $s$-dimensional density of $\mu$ on $B$, that is

$$\theta_\mu^s(B) = \frac{\mu(B)}{r(B)^s}.$$

Also we consider the Poisson type density

$$\mathcal{P}(B) = \sum_{k\geq 0} \theta_\mu^s(2^k B) 2^{-k}.$$

Given two sets $E, F \subset \mathbb{R}^d$, we write

$$p_\mu(x, E, F) = \iint_{E\times F} p_s(x,y,z)\, d\mu(y)\, d\mu(z).$$

We say that $\mu$ has $s$-growth if there exists some constant $c$ such that

$$\mu(B(x,r)) \leq c\, r^s \qquad \text{for all } x \in \mathbb{R}^d \text{ and } r > 0. \tag{2.1}$$

It is well known that if $\mu$ has no point masses and $R_\mu^s$ is bounded in $L^2(\mu)$, then $\mu$ has $s$-growth (see [Da, Proposition 1.4, p.56]). Further, as explained in the previous section, by [Pr] $\mu$ satisfies

$$\Theta^{s,*}(x,\mu) = 0 \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{2.2}$$

As shown in [MaV] (see also [To, Chapter 8]), then one deduces that for all $f \in L^2(\mu)$,

$$\lim_{\varepsilon\to 0} R_{\mu,\varepsilon} f(x) \qquad \text{exists for } \mu\text{-a.e. } x \in \mathbb{R}^d, \tag{2.3}$$

and it coincides $\mu$-a.e. with $R_\mu f(x)$, where $R_\mu f$ is the weak limit of $R_{\mu,\varepsilon} f$ as $\varepsilon \to 0$. In particular, this fact, as well as (2.1), (2.2) and (2.3), hold if $\mu$ is reflectionless for the $s$-Riesz transform.

We need now to recall the definition of balls with thin boundaries. Given $t > 0$, a ball $B(x,r)$ is said to have $t$-thin boundary (or just thin boundary) if

$$\mu\big(\{y \in B(x,2r) : \operatorname{dist}(y, \partial B(x,r)) \leq \lambda\, r\}\big) \leq t\, \lambda\, \mu(B(x,2r))$$

for all $\lambda > 0$. The following result is well known. For the proof (with cubes instead of balls) see Lemma 9.43 of [To], for example.

**Lemma 2.1.** *Let $\mu$ be a Radon measure on $\mathbb{R}^d$. Let $t$ be some constant big enough (depending only on $d$). Let $B(x,r) \subset \mathbb{R}^d$ be any fixed ball. Then there exists $r' \in [r, 2r]$ such that the ball $B(x,r')$ has $t$-thin boundary.*

The next lemma will be also used later on.

**Lemma 2.2.** *Let $\mu$ be a Radon measure on $\mathbb{R}^d$. Let $B$ be a ball with thin-boundary and $\|a\|_\infty < \infty$. Then*

(1) $|R^s_\mu(a\chi_B)(x)| \leq C\|a\|_\infty \theta^s_\mu(2B) \qquad \forall x \in 2B \setminus B.$

(2) $|R^s_\mu(a\chi_{2B\setminus B})(x)| \leq C\|a\|_\infty \theta^s_\mu(2B) \qquad \forall x \in B.$

*Proof.* We will only show (2), since the proof of (1) follows the same reasoning. Let $r$ denote the radius of our ball $B$. For $x \in B$ we have

$$\begin{aligned} |R^s_\mu(a\chi_{2B\setminus B})(x)| &\leq \|a\|_\infty \int_{2B\setminus B} \frac{d\mu(y)}{|y-x|^s} \\ &\lesssim \|a\|_\infty \sum_{k\geq 1} \frac{\mu\left(B(x,2^{-k}r)\cap 2B\setminus B\right))}{(2^{-k}r)^s} \\ &\lesssim \|a\|_\infty \sum_{k\geq 1} 2^{-k(1-s)} \frac{\mu(2B)}{r^s} \lesssim \|a\|_\infty \theta^s_\mu(2B), \end{aligned}$$

since $\mu\Big(B(x,2^{-k}r)\cap 2B\setminus B\Big) \lesssim 2^{-k}\mu(2B)$ because of the fact that

$$B(x,2^{-k}r)\cap 2B \subset 2B \cap U_{2^{-k+1}}(\partial B)$$

and the thin-boundary property of $B$. Here $U_\delta(A)$ stands for the $\delta$-neighborhood of $A$. $\square$

**Lemma 2.3.** *Let $\mu$ be a Radon measure on $\mathbb{R}^d$ with $s$-growth such that $\lim_{r\to 0} \frac{\mu(B(x,r))}{(2r)^s} = 0$ for $\mu$-a.e. $x \in \mathbb{R}^d$. Then, for all bounded sets $E, F \subset \mathbb{R}^d$ and $\mu$-a.e. $x \in \mathbb{R}^d$,*

$$R^s_\mu\chi_E(x)\cdot R^s_\mu\chi_F(x) + R^{s\,*}_\mu\left((R^s_\mu\chi_F)\chi_E\right)(x) + R^{s\,*}_\mu\left((R^s_\mu\chi_E)\chi_F\right)(x) = p_\mu(x,E,F). \tag{2.4}$$

*In particular,*

$$|R^s_\mu\chi_E(x)|^2 + 2R^{s\,*}_\mu((R^s_\mu\chi_E)\chi_E)(x) = p_\mu(x,E,E).$$

*Proof.* Fix $\varepsilon > 0$ and $x \in \mathbb{R}^d$. Consider the sets

$$S_\varepsilon = \{(y,z) \in E\times F : |y-x| > \varepsilon,\ |z-x| > \varepsilon,\ |z-y| > \varepsilon\},$$

$$T^1_\varepsilon = \{(y,z) \in E\times F : |y-x| > \varepsilon,\ |z-y| > \varepsilon,\ |z-x| \leq \varepsilon\}$$

and

$$T^2_\varepsilon = \{(y,z) \in E\times F : |y-x| > \varepsilon,\ |z-x| > \varepsilon,\ |z-y| \leq \varepsilon\}.$$

Then, using (1.1) (arguing as in [MeV, Section 2]), one can write

$$
\begin{aligned}
&R^s_{\mu,\varepsilon}(\chi_E)(x)\cdot R^s_{\mu,\varepsilon}(\chi_F)(x) + R^{s*}_{\mu,\varepsilon}\left(R^s_{\mu,\varepsilon}(\chi_F)\chi_E\right)(x) + R^{s*}_{\mu,\varepsilon}\left(R^s_{\mu,\varepsilon}(\chi_E)\chi_F\right)(x)\\
&\qquad = \iint_{S_\varepsilon} p_s(x,y,z)\,d\mu(y)d\mu(z) + 2\iint_{T^1_\varepsilon} K(x-y)\cdot K(z-y)\,d\mu(y)d\mu(z)\\
&\qquad\quad + \iint_{T^2_\varepsilon} K(y-x)\cdot K(z-x)\,d\mu(y)d\mu(z)\\
&\qquad = p_{\mu,\varepsilon}(x,E,F) + A_\varepsilon(x) + B_\varepsilon(x),
\end{aligned}
\tag{2.5}
$$

the last equality being a definition for $p_{\mu,\varepsilon}$, $A_\varepsilon$ and $B_\varepsilon$.

Notice that as $\varepsilon$ goes to zero, by (2.2) and (2.3), the first term in (2.5) verifies

$$
\lim_{\varepsilon\to 0} R^s_{\mu,\varepsilon}(\chi_E)(x)\cdot R^s_{\mu,\varepsilon}(\chi_F)(x) = R^s_\mu(\chi_E)(x)\cdot R^s_\mu(\chi_F)(x) \qquad \text{for } \mu\text{-a.e. } x\in\mathbb{R}^d. \tag{2.6}
$$

On the other hand, it is also clear that

$$
\lim_{\varepsilon\to 0} p_{\mu,\varepsilon}(x,E,F) = p_\mu(x,E,F) \qquad \text{for all } x\in\mathbb{R}^d. \tag{2.7}
$$

Concerning $A_\varepsilon(x)$ since $|x-y|\approx|z-y|$ in the domain of integration, using the $s$-growth of $\mu$, we get

$$
\begin{aligned}
|A_\varepsilon(x)| &\le \iint_{\substack{|y-x|>\varepsilon\\ |z-x|\le\varepsilon\\ |z-y|>\varepsilon}} \frac{d\mu(y)d\mu(z)}{|x-y|^s|z-y|^s} \lesssim \int_{|y-x|>\varepsilon} \frac{\mu(B(x,\varepsilon))d\mu(y)}{|x-y|^{2s}}\\
&\lesssim \theta^s_\mu(B(x,\varepsilon)) \to 0 \quad \text{ as } \varepsilon\to 0, \text{ for } \mu\text{-a.e. } x\in\mathbb{R}^d.
\end{aligned}
\tag{2.8}
$$

Let us turn our attention to $B_\varepsilon(x)$. Using that $|x-y|\approx|x-z|$ in the domain of integration, we derive

$$
|B_\varepsilon(x)| \lesssim \int_{y\in E:|y-x|>\varepsilon} \frac{\mu(B(y,\varepsilon))}{|y-x|^{2s}}\,d\mu(y).
$$

Integrating with respect to $x$ in $\mathbb{R}^d$, applying Fubini, and using the $s$-growth of $\mu$, we get

$$
\begin{aligned}
\int |B_\varepsilon(x)|\,d\mu(x) &\lesssim \iint_{y\in E:|y-x|>\varepsilon} \frac{\mu(B(y,\varepsilon))}{|y-x|^{2s}}\,d\mu(y)\,d\mu(x)\\
&= \int_{y\in E} \mu(B(y,\varepsilon)) \int_{|y-x|>\varepsilon} \frac{1}{|y-x|^{2s}}\,d\mu(x)\,d\mu(y)\\
&\lesssim \int_{y\in E} \frac{\mu(B(y,\varepsilon))}{\varepsilon^s}\,d\mu(y) \to 0 \qquad \text{as } \varepsilon\to 0,
\end{aligned}
$$

by the dominated convergence theorem. Hence we infer that there exists a sequence $\varepsilon_k\to 0$ such that

$$
\lim_{k\to\infty} B_{\varepsilon_k}(x) = 0 \qquad \text{for } \mu\text{-a.e. } x\in\mathbb{R}^d. \tag{2.9}
$$

Finally we deal with the second and third terms on the left hand side of (2.5). We write

$$
f_\varepsilon(y) = R^s_{\mu,\varepsilon}(\chi_F)(y)\chi_E(y) + R^s_{\mu,\varepsilon}(\chi_E)(y)\chi_F(y),
$$

so that $R^{s*}_{\mu,\varepsilon} f_\varepsilon(x)$ equals the sum of the last two terms on the left hand side of (2.5). Also, we set

$$f(y) = R^s_\mu(\chi_F)(y)\chi_E(y) + R^s_\mu(\chi_E)(y)\chi_F(y).$$

We claim that for $\mu$-a.e. $x \in \mathbb{R}^d$ there exists a subsequence $\varepsilon_{k_j} \to 0$ (depending on $x$) such that

$$\lim_{j\to\infty} R^{s*}_{\mu,\varepsilon_{k_j}} f_{\varepsilon_{k_j}}(x) = R^{s*}_\mu f(x). \tag{2.10}$$

To prove this, notice that $R^s_{\mu,\varepsilon_k} f(x) \to R^s_\mu f(x)$ as $k \to \infty$ for $\mu$-a.e. $x \in \mathbb{R}^d$ and thus for such $x$ we have

$$\begin{aligned}\liminf_{k\to\infty} \big|R^{s*}_{\mu,\varepsilon_k} f_{\varepsilon_k}(x) - R^{s*}_\mu f(x)\big| &= \liminf_{k\to\infty} \big|R^{s*}_{\mu,\varepsilon_k} f_{\varepsilon_k}(x) - R^{s*}_\mu f(x) + R^{s*}_\mu f(x) - R^{s*}_{\mu,\varepsilon_k} f(x)\big| \\ &= \liminf_{k\to\infty} \big|R^{s*}_{\mu,\varepsilon_k}(f_{\varepsilon_k} - f)(x)\big|.\end{aligned}$$

Thus, for any $\lambda > 0$, by Chebyshev's inequality, Fatou's lemma, and the $L^2(\mu)$ boundedness of $R^s_\mu$,

$$\begin{aligned}\mu\Big(\Big\{x \in \mathbb{R}^d &\colon \liminf_{k\to\infty} \big|R^{s*}_{\mu,\varepsilon_k} f_{\varepsilon_k}(x) - R^{s*}_\mu f(x)\big| > \lambda\Big\}\Big) \\ &= \mu\Big(\Big\{x \in \mathbb{R}^d \colon \liminf_{k\to\infty} \big|R^{s*}_{\mu,\varepsilon_k}(f_{\varepsilon_k} - f)(x)\big| > \lambda\Big\}\Big) \\ &\le \frac{1}{\lambda^2} \int \liminf_{k\to\infty} \big|R^{s*}_{\mu,\varepsilon_k}(f_{\varepsilon_k} - f)(x)\big|^2 \, d\mu(x) \\ &\le \frac{1}{\lambda^2} \liminf_{k\to\infty} \int \big|R^{s*}_{\mu,\varepsilon_k}(f_{\varepsilon_k} - f)(x)\big|^2 \, d\mu(x) \\ &\le \frac{c}{\lambda^2} \liminf_{k\to\infty} \big\|f_{\varepsilon_k} - f\big\|^2_{L^2(\mu)}.\end{aligned}$$

Since the last limit vanishes, our claim (2.10) follows.

From the identity (2.5), for $\mu$-a.e. $x \in \mathbb{R}^d$, taking a suitable sequence $\varepsilon_{k_j} \to 0$, by (2.6), (2.7), (2.8), (2.9), and (2.10), we infer that the identity (2.4) holds. □

## 3. The main Lemma

**Main Lemma 3.1.** *Let $\mu$ be a reflectionless measure for the $s$-Riesz transform. Let $B$ be a ball with thin boundary. Then, for $\mu$-a.e. $x \in B$,*

$$p_\mu(x, B, B) \lesssim \mathcal{P}(B)^2,$$

*with the implicit constant depending only on $s$ and $d$.*

*Proof.* Let $\nu$ be a measure supported on $B$ of type $\nu = g\mu_{|B}$ and set

$$F(\nu) = \int_B |R^s_\mu \chi_{B^c} + R^s_\nu \chi_B - m^\nu_B R^s_\mu \chi_{B^c}|^2 d\nu,$$

where $m^\nu_B f = \dfrac{1}{\nu(B)} \displaystyle\int_B f d\nu$. In what follows we write $m_B$ for $m^\mu_B$.

Notice that by antisymmetry and the reflectionless property of $\mu$,

$$F(\mu\chi_B) = \int_B |R^s\mu - m_B R^s_\mu \chi_{B^c}|^2 d\mu = \int_B |R^s\mu - m_B R^s\mu|^2 d\mu = 0.$$

For a small $t \in \mathbb{R}$ and a ball $\Delta$, centered at $x \in B$, consider the measure

$$\nu_t = \mu\chi_B(1 + t\chi_\Delta) = \mu\chi_B + t\mu\chi_{\Delta\cap B}$$

and set $g(t) = F(\nu_t)$. Then $g(t) \geq 0$ and $g'(0) = 0$.

Write $f_\nu = R^s_\mu\chi_{B^c} - m^\nu_B R^s_\mu\chi_{B^c}$ and $f_{\nu_t} = f_t$, so that

$$g(t) = \int_B |f_t + R^s_{\nu_t}\chi_B|^2 d\nu_t = \int_B |R^s_{\nu_t}\chi_B|^2 d\nu_t + 2\int_B f_t \cdot R^s_{\nu_t}\chi_B \, d\nu_t + \int_B |f_t|^2 d\nu_t.$$

Observe now that

$$f_t = R^s_\mu\chi_{B^c} - \frac{1}{\mu(B) + t\mu(\Delta\cap B)}\int_B R^s_\mu\chi_{B^c}\, d\mu - \frac{t}{\mu(B) + t\mu(\Delta\cap B)}\int_{\Delta\cap B} R^s_\mu\chi_{B^c}\, d\mu.$$

Notice that $\partial_t f_{t|t=0}$ is constant. Indeed, we have

$$\partial_t f_{t|t=0} = \frac{\mu(\Delta\cap B)}{\mu(B)^2}\int_B R^s_\mu\chi_{B^c}\, d\mu - \frac{1}{\mu(B)}\int_{\Delta\cap B} R^s_\mu\chi_{B^c}\, d\mu.$$

Hence, by using antisymmetry and the fact that $\partial_t f_{t|t=0}$ is constant,

$$\begin{aligned} 0 = g'(0) &= \int_{B\cap\Delta} |R^s_\mu\chi_B|^2 d\mu + 2\int_B R^s_\mu\chi_B \cdot R^s_\mu\chi_{B\cap\Delta} d\mu + 2\int_{B\cap\Delta} f_0 \cdot R^s_\mu\chi_B d\mu \\ &\quad + 2\int_B f_0 \cdot R^s_\mu\chi_{\Delta\cap B} d\mu + \int_{\Delta\cap B} |f_0|^2 d\mu + 2\int_B f_0 \cdot \partial_t f_{t|t=0} d\mu \\ &= \int_{B\cap\Delta} |R^s_\mu\chi_B|^2 d\mu + 2\int_{\Delta\cap B} R^{s\,*}_\mu[(R^s_\mu\chi_B)\chi_B] d\mu + 2\int_{B\cap\Delta} f_0 \cdot R^s_\mu\chi_B d\mu \\ &\quad + 2\int_{\Delta\cap B} R^{s\,*}_\mu(f_0\chi_B) d\mu + \int_{\Delta\cap B} |f_0|^2 d\mu + 2\int_B f_0 \cdot \partial_t f_{t|t=0} d\mu, \end{aligned}$$

where $R^{s\,*}_\mu$ is the adjoint of $R^s_\mu$. Then, assuming that $x$ is a Lebesgue point for the functions $|R^s_\mu\chi_B|^2$, $R^{s\,*}_\mu[(R^s_\mu\chi_B)\chi_B]$, $f_0 \cdot R^s_\mu\chi_B$, $R^{s\,*}_\mu(f_0\chi_B)$ and $|f_0|^2$, and that $x \in \mathring{B}$, we deduce

$$\begin{aligned} 0 = \lim_{r(\Delta)\to 0} \frac{g'(0)}{\mu(\Delta\cap B)} &= |R^s_\mu\chi_B(x)|^2 + 2R^{s\,*}_\mu((R^s_\mu\chi_B)\chi_B)(x) \\ &\quad + 2f_0(x)\cdot R^s_\mu\chi_B(x) + 2R^{s\,*}_\mu(f_0\chi_B)(x) + h(x), \end{aligned} \tag{3.1}$$

where

$$h(x) = |f_0(x)|^2 + \frac{2}{\mu(B)^2}\int_B R^s_\mu\chi_{B^c} d\mu \cdot \int_B f_0 d\mu - 2R^s_\mu\chi_{B^c}(x)\cdot\frac{1}{\mu(B)}\int_B f_0 d\mu.$$

Write

$$f_0 = \left(R^s_\mu\chi_{2B\setminus B} - m_B R^s_\mu\chi_{2B\setminus B}\right) + \left(R^s_\mu\chi_{(2B)^c} - m_B R^s_\mu\chi_{(2B)^c}\right) = h_1 + h_2,$$

the last identity being a definition for $h_1$ and $h_2$. Note that, for $w \in B$, by Lemma 2.2, $|h_1(w)| \lesssim \theta^s_\mu(2B)$. Moreover by standard estimates (which can be obtained by splitting the domain of integration into annuli), we get

$$|h_2(w)| \leq \frac{1}{\mu(B)}\int_B\int_{(2B)^c} |K(y-w) - K(y-z)| d\mu(y) d\mu(z) \lesssim \mathcal{P}(B).$$

Therefore, for all $w \in B$,

$$|f_0(w)| \lesssim \theta(2B) + \mathcal{P}(B) \lesssim \mathcal{P}(B). \tag{3.2}$$

Hence,

$$h(x) = |f_0(x)|^2 + 2m_B f_0 \cdot m_B(R^s_\mu \chi_{B^c} - R^s_\mu \chi_{B^c}(x)) \lesssim \mathcal{P}(B)^2. \tag{3.3}$$

Set

$$A_1 = |R^s_\mu \chi_B(x)|^2 + 2R^{s*}_\mu((R^s_\mu \chi_B)\chi_B)(x)$$

and

$$A_2 = 2f_0(x)\cdot R^s_\mu \chi_B(x) + 2R^{s*}_\mu(f_0\chi_B)(x).$$

In view of (3.1), (3.2), and (3.3),

$$|A_1 + A_2| \lesssim \mathcal{P}(B)^2. \tag{3.4}$$

Since $\theta^s_\mu(w) = 0$ $\mu-$almost everywhere (see [Pr, Theorem 4.4]), by Lemma 2.3, $A_1$ equals $p_\mu(x,B,B)$ $\mu$-a.e. on $B$. We deal now with $A_2$. Write

$$A_2 = 2\sum_{j=1}^{2}\left(h_j(x)\cdot R^s_\mu \chi_B(x) + R^{s*}_\mu(h_j\chi_B)(x)\right). \tag{3.5}$$

Notice that for any vectorial constant $k$, one has $kR^s_\mu\chi_B(x) + R^{s*}_\mu(k\chi_B)(x) = 0$. Therefore for $k = m_Q R^s_\mu \chi_{2B\setminus B}$ and $j=1$ in (3.5), for $\mu$-a.e. $x \in B$ we obtain

$$\begin{aligned} h_1(x)\cdot R^s_\mu\chi_B(x) + R^{s*}_\mu(h_1\chi_B)(x) &= R^s_\mu\chi_{2B\setminus B}(x)\cdot R^s_\mu\chi_B(x) + R^{s*}_\mu\left((R^s_\mu\chi_{2B\setminus B})\chi_B\right)(x) \\ &= p_\mu(x,B,2B\setminus B) - R^{s*}_\mu\left((R^s_\mu\chi_B)\chi_{2B\setminus B}\right)(x), \end{aligned}$$

the last step due to (2.4).

We deal now with $j=2$ in (3.5). Notice that for $y \in B$,

$$|h_2(x) - h_2(y)| = |R^s_\mu\chi_{(2B)^c}(x) - R^s_\mu\chi_{(2B)^c}(y)| \lesssim \int_{(2B)^c}\frac{|x-y|}{|z-y|^{s+1}}d\mu(z) \lesssim \frac{|x-y|}{r(B)}\mathcal{P}(B),$$

hence

$$\begin{aligned} h_2(x)\cdot R^s_\mu\chi_B(x) + R^{s*}_\mu(h_2\chi_B)(x) &= \left|\int_B \frac{x-y}{|x-y|^{s+1}}h_2(x)d\mu(y) - \int_B\frac{x-y}{|x-y|^{s+1}}h_2(y)d\mu(y)\right| \\ &\le \int_B \frac{|h_2(x)-h_2(y)|}{|x-y|^s}d\mu(y) \lesssim \frac{\mathcal{P}(B)}{r(B)}\int_B\frac{|x-y|}{|x-y|^s}d\mu(y) \\ &\lesssim \frac{\mathcal{P}(B)}{r(B)^s}\,\mu(B) \lesssim \mathcal{P}(B)^2. \end{aligned}$$

So we have shown that

$$|A_2 - 2\,p_\mu(x,B,2B\setminus B)| \le 2\,|R^{s*}_\mu\left((R^s_\mu\chi_B)\chi_{2B\setminus B}\right)(x)| + c\,\mathcal{P}(B)^2.$$

Therefore, from (3.4), taking into account that $A_1 = p_\mu(x,B,B)$ $\mu$-a.e. on $B$, we deduce

$$p_\mu(x,B,B) + 2\,p_\mu(x,B,2B\setminus B) \lesssim \mathcal{P}(B)^2 + |R^{s*}_\mu\left((R^s_\mu\chi_B)\chi_{2B\setminus B}\right)(x)| \quad \text{for } \mu\text{-a.e. } x\in B.$$

By using Lemma 2.2 with $a = R^s_\mu(\chi_B)\chi_{2B\setminus B}$, we have $\|a\|_\infty \lesssim \theta^s_\mu(2B)$, and hence we get that

$$|R^{s*}_\mu\left((R^s_\mu\chi_B)\chi_{2B\setminus B}\right)(x)| \lesssim \theta_\mu(2B)^2,$$

and then immediatly we obtain

$$p_\mu(x,B,B)+2p_\mu(x,B,2B\setminus B)\lesssim \mathcal{P}(B)^2 \quad \text{for } \mu\text{-a.e. } x\in B,$$

which proves our lemma. □

## 4. Proof of the theorem

**Lemma 4.1.** *Let $0<s<1$ and let $\mu$ be a reflectionless measure for the s-Riesz transform. There exists some constant $\delta>0$ small enough (depending only on $s$ and $d$) such that if $B_0\subset\mathbb{R}^d$ is a ball with*

$$\theta_\mu^s(B_0)\geq \sup_{x\in\mathbb{R}^d,r>0}\frac{\theta_\mu^s(B(x,r))}{2},$$

*then every ball $B\subset\mathbb{R}^d$ such that $B_0\subset\delta B$ satisfies $\theta_\mu^s(B)\approx\theta_\mu^s(B_0)$, with the implicit constant depending only on $s$ and $d$.*

Notice that in this lemma the ball $B$ can be arbitrarily big. We do *not* ask $r(B)\approx r(B_0)$.

*Proof.* We assume first that $B_0\subset\frac{1}{2}B$. Let $B'$ be a ball with thin boundary such that $\frac{1}{2}B\subset B'\subset B$. Then, by [Pr] and Lemma 3.1, for $\mu$-a.e. $x\in B$,

$$\theta_\mu^s(B_0)^2\lesssim\iint_{B'\times B'}\frac{d\mu(y)d\mu(z)}{\max\{|x-y|,|x-z|,|y-z|\}^{2s}}\lesssim p_\mu(x,B',B')\lesssim\mathcal{P}(B')^2.$$

Therefore

$$\theta_\mu^s(B_0)\lesssim\mathcal{P}(B')\lesssim\mathcal{P}(B).$$

We claim that this implies the existence of a ball $\widetilde{B}\supset B$, concentric with $B$, with comparable radius, that is $r(\widetilde{B})\approx r(B)$, and such that

$$\theta_\mu^s(\widetilde{B})\geq\theta_\mu^s(B).$$

To see this, notice that

$$\begin{aligned}\theta_\mu^s(B_0)\leq\mathcal{P}(B)=\sum_k\theta_\mu^s(2^kB)2^{-k}&\leq\max_{1\leq k\leq j}\theta_\mu^s(2^jB)+2\sum_{k\geq j+1}\theta_\mu^s(B_0)2^{-k}\\&\leq\max_{1\leq k\leq j}\theta_\mu^s(2^jB)+2^{-j+3}\theta_\mu^s(B_0),\end{aligned}$$

which implies

$$\theta_\mu^s(B_0)\lesssim\max_{1\leq k\leq j}\theta_\mu^s(2^jB)$$

for $j$ big enough and hence the claim. Thus, for every ball $B$ with $B_0\subset\delta B$ (for a sufficiently small $\delta$), $\theta_\mu^s(B)\gtrsim\theta_\mu^s(B_0)$.

The converse inequality $\theta_\mu^s(B)\lesssim\theta_\mu^s(B_0)$ holds by the choice of $B_0$. So the lemma is proved. □

We now prove the main result of this paper:

**Theorem.** *If $0<s<1$ there are no non-trivial reflectionless measures for the s-dimensional Riesz transform.*

*Proof.* Let $\mu$ be a non-trivial reflectionless measure for the $s$-dimensional Riesz transform. Let $B_0$ be a ball such that

$$\theta_\mu^s(B_0) \geq \sup_{x\in\mathbb{R}^d, r>0} \frac{\theta_\mu^s(B(x,r))}{2}.$$

Consider the sequence of balls $B_k = 2^k B_0$. By Lemma 4.1, there exists $k_0$ such that for $k \geq k_0$, $\theta_\mu^s(B_k) \approx \theta_\mu^s(B_0)$. Since for $m \geq 0$,

$$\mu(B_{k+m}) = r(B_{k+m})^s\, \theta_\mu^s(B_{k+m}) \approx 2^{ms} r(B_k)^s \theta_\mu^s(B_k) = 2^{ms}\mu(B_k),$$

there exists $m \geq 0$ such that $\mu(B_{k+m}) \geq \dfrac{\mu(B_k)}{2}$. Consider the balls $B_{k_0}, B_{k_0+m}, \cdots, B_{k_0+Nm}$, and write $\widetilde{B}_j = B_{k_0+jm}$ for $0 \leq j \leq N$. We can assume that $\widetilde{B}_N$ has thin boundary due to Lemma 2.1. Then Lemma 3.1 implies that for all $N$,

$$p_\mu(x, \widetilde{B}_N, \widetilde{B}_N) \lesssim \mathcal{P}(\widetilde{B}_N) \approx \theta_\mu^s(B_0). \tag{4.1}$$

But notice that

$$\begin{aligned}
p_\mu(x, \widetilde{B}_N, \widetilde{B}_N) &\approx \iint_{\widetilde{B}_N \times \widetilde{B}_N} \frac{d\mu(y)d\mu(z)}{\max\{|x-y|, |x-z|, |y-z|\}^{2s}} \\
&\gtrsim \sum_{j=1}^N \int_{\widetilde{B}_j} \int_{\widetilde{B}_j \setminus \widetilde{B}_{j-1}} \frac{d\mu(y)d\mu(z)}{r(\widetilde{B}_j)^{2s}} \\
&= \sum_{j=1}^N \theta_\mu^s(\widetilde{B}_j) \frac{\mu(\widetilde{B}_j \setminus \widetilde{B}_{j-1})}{r(\widetilde{B}_j)^s} \\
&\approx \sum_{j=1}^N \theta_\mu^s(\widetilde{B}_j)^2 \approx N\theta_\mu^s(\widetilde{B}_0)^2 \underset{N\to\infty}{\longrightarrow} \infty,
\end{aligned}$$

which contradicts (4.1). □

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