Classification of quadratic differential systems with two invariant conics: parabola and ellipse

Cristina Bujac, Nicolae Vulpe
State University of Moldova
Vladimir Andrunachievici Institute of Mathematics
and Computer Science (cristina.bujac@math.md; nvulpe@gmail.com)

**Abstract**

Consider the class **QS** of all non-degenerate planar quadratic differential systems, and its subclass **QSPE**, consisting of those systems that simultaneously possess an invariant parabola and an invariant ellipse. We investigate all possible configurations of these two invariant conics, along with any invariant straight lines that systems in **QSPE** may exhibit, and explore the geometric properties encoded in such configurations.

The classification presented here is given modulo the action of the group of real affine transformations and time rescaling, and is expressed in terms of affine invariant polynomials. The resulting classification yields an algorithm that allows one to determine whether a given real quadratic differential system belongs to the class **QSPE**; if it does, the corresponding configuration of this system can then be identified. This work aims to contribute to the study of the integrability of systems within the class **QSPE**.

**Key-words:** Quadratic vector fields, affine invariant polynomials, invariant algebraic curves, invariant parabolas and ellipses.

**2010 Mathematics Subject Classification:** 58K30, 34A26, 34C05, 34C40.



# 1 Introduction and statement of main results

We consider here differential systems of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where $P$, $Q \in \mathbb{R}[x,y]$, i.e. $P,\ Q$ are polynomials in $x,\ y$ over $\mathbb{R}$ and their associated vector fields

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

We call *degree* of a system (1) the integer $m = \max(\deg P, \deg Q)$. In particular we call *quadratic* a differential system (1) with $m = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

Quadratic systems appear in the modelling of many natural phenomena described in different branches of science, in biological and physical applications and applications of these systems became a subject of interest for the mathematicians. Many papers have been published about quadratic systems, see for example [14] for a bibliographical survey.

Here we always assume that the polynomials $P$ and $Q$ are coprime. Otherwise doing a rescaling of the time, systems (1) can be reduced to linear or constant systems. Quadratic systems under this assumption are called *non–degenerate quadratic systems*.

Let $V$ be an open and dense subset of $\mathbb{R}^2$, we say that a non-constant differentiable function $H : V \to \mathbb{R}$ is a first integral of a system (1) on $V$ if $H(x(t), y(t))$ is constant for all of the values of $t$ for which $(x(t), y(t))$ is a solution of this system contained in $V$. Obviously $H$ is a first integral of systems (1) if and only if

$$X(H) = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = 0,$$

for all $(x, y) \in V$. When a system (1) has a first integral we say that this system is integrable.

The knowledge of the first integrals is of particular interest in planar differential systems because they allow us to draw their phase portraits.

On the other hand given $f \in \mathbb{C}[x, y]$ we say that the curve $f(x, y) = 0$ is an *invariant algebraic curve* of systems (1) if there exists $K \in \mathbb{C}[x, y]$ such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

The polynomial $K$ is called the *cofactor* of the invariant algebraic curve $f = 0$. When $K = 0$, $f$ is a polynomial first integral.

The motivation for studying systems in the quadratic class stems not only from their wide range of applications but also from significant theoretical interests, as discussed by Schlomiuk and Vulpe in the introduction of [19]. The analysis of non-degenerate quadratic systems can be effectively approached through the use of normal forms and the application of invariant theory.

Systematic research on quadratic differential systems possessing an invariant conic began towards the end of the 20th century and the beginning of the 21st. Several important contributions have since been made in this area. Quadratic systems with an invariant ellipse acting as a limit cycle were studied by Qin Yuan-xum [8]. Druzhkova [5] established the necessary and sufficient conditions on the coefficients of both the quadratic system and the conic for the conic to be an invariant curve of the system. Cairo and Llibre [3] investigated the Darboux integrability of quadratic systems possessing invariant algebraic conics. Christopher [4] considered the class of quadratic systems having a parabola as an integral curve.

Oliveira, Rezende, and Vulpe [9] provided necessary and sufficient conditions, in terms of system coefficients, for a system in QS to admit at least one invariant hyperbola. In addition, Oliveira, Rezende, Schlomiuk, and Vulpe [10] derived the affine invariant conditions under which an ellipse is an invariant curve of the system. In [14], the authors presented a classification of quadratic systems with an invariant hyperbola based on configurations of hyperbolas and the presence or absence of invariant straight lines — an invariant classification independent of specific normal forms. A similar classification for systems possessing an invariant ellipse was given in [12]. Furthermore, Vulpe [21] provided the necessary and sufficient affine invariant conditions for a system in QS to possess a parabola as an invariant curve.

In this work, we study non-degenerate quadratic differential systems that simultaneously possess an invariant parabola and an invariant ellipse. We denote this family by **QSPE**. Our main objective is to characterize the systems in **QSPE** in terms of invariant polynomials. In our approach, the equalities and inequalities defining the bifurcation diagram, partitioning the parameter space into regions and subsets with distinct dynamics, are not expressed through the coefficients of a fixed normal form (or several such forms), whose coefficients often lack geometric meaning and are rigidly tied to those forms. Instead, they are expressed using invariant polynomials, which are flexible, geometrically meaningful objects. These polynomials can be readily computed by computer algebra systems for any chosen normal form and allow for an easy transition between different normal forms. On the other hand since number of the invariant curve increases we could obtain new classes of quadratic systems which are Darboux integrable and in particular rational integrable.

Our main results are stated in the following theorem.

**Main Theorem.** ***(A)*** *The condition $\chi_1 = \chi_2 = \widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ is necessary for a non-degenerate quadratic system to possess at least one invariant parabola and at least one invariant ellipse.*

***(B)*** *Assume that for a non-degenerate quadratic system $(S)$ the condition $\chi_1 = \chi_2 = \widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ holds. Then the system $(S)$ possesses simultaneously one invariant parabola and one invariant ellipse generating a configuration of invariant conics and invariant lines, if and only if the corresponding conditions indicated below are satisfied, respectively:*

- ***α)*** *For $\eta < 0$ the system $(S)$ could possess one of the configurations Config. $\mathcal{PE}$.1–Config. $\mathcal{PE}$.7 shown in Figure 1 if and only if the corresponding conditions provided by Diagram 1, in the branch defined by $\eta < 0$, are satisfied.*

- ***β)*** *For $C_2 = 0$ the system $(S)$ could possess one of the configurations Config. $\mathcal{PH}$.8–Config. $\mathcal{PH}$.10 shown in Figure 1 if and only if the corresponding conditions provided by Diagram 1, in the branch defined by $C_2 = 0$, are satisfied.*

***(C)*** *We construct the bifurcation diagram in the space $\mathbb{R}^{12}$ of the coefficients of the system in* ***QSPE****, as illustrated in Diagram 1, according to their configurations of invariant parabolas, ellipses, and invariant straight lines. Moreover, this diagram provides an algorithm for determining the configuration of any quadratic differential system possessing invariant parabola and invariant ellipses, expressed in any normal form.*

## 2 Some preliminary results

Assume that a conic

$$\Phi(x, y) \equiv p + qx + ry + sx^2 + 2vxy + uy^2 = 0 \tag{2}$$

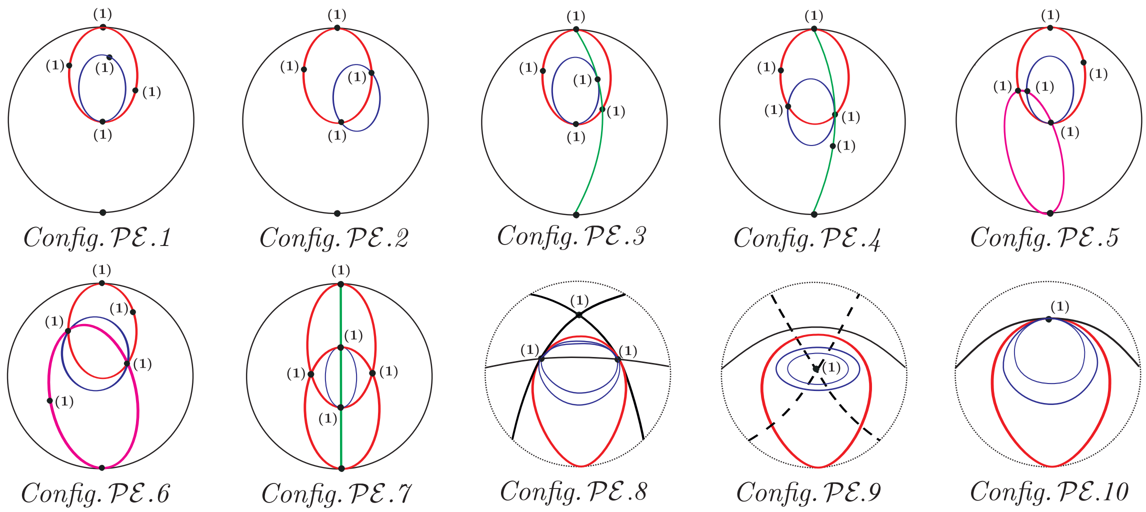


Figure 1: Configurations of systems in **QSPE**

is an affine algebraic invariant curve for quadratic systems, which we rewrite in the form:

$$\begin{aligned} \frac{dx}{dt} &= a + cx + dy + gx^2 + 2hxy + ky^2 \equiv P(x,y), \\ \frac{dy}{dt} &= b + ex + fy + lx^2 + 2mxy + ny^2 \equiv Q(x,y). \end{aligned} \tag{3}$$

We denote by $\tilde{a} = (a, b, c, d, g, h, c, l, m, n, n)$ the set of its coefficients.

**Remark 1.** *Following [7] we construct the determinant*

$$\Delta = \begin{vmatrix} s & v & q/2 \\ v & u & r/2 \\ q/2 & r/2 & p \end{vmatrix},$$

*associated to the conic* (2)*. By [7] this conic is irreducible (i.e. the polynomial* $\Phi$ *defining the conic is irreducible over* $\mathbb{C}$*) if and only if* $\Delta \neq 0$*. Moreover an irreducible conic is an ellipse (respectively, hyperbola; parabola) if* $\delta < 0$ *(respectively* $\delta > 0$*;* $\delta = 0$*) where* $\delta = v^2 - su$*. Furthermore if the conic* (2) *is an ellipse (i.e.* $\delta < 0$*) then it is real for* $(u+s)\Delta < 0$ *and it is imaginary for* $(u+s)\Delta > 0$.

## 2.1 Results involving the use of polynomial invariants

As previously established in [11], the necessary and sufficient conditions for a system in the class **QS** to possess at least one invariant ellipse are expressed in terms of its coefficients. Specifically, [11] defines the following 34 affine invariant polynomials that determine the existence of at least one invariant ellipse:

$$\eta,\ \widetilde{M}, C_2,\ \theta,\ \widetilde{N},\ N_7,\ H_2,\ H_9,\ H_{10},\ \mathcal{F},\ \mathcal{T}_3,\ \widehat{\beta}_1 - \widehat{\beta}_8,\ \widehat{\gamma}_1 - \widehat{\gamma}_9,\ \widehat{\mathcal{R}}_1 - \widehat{\mathcal{R}}_6. \tag{4}$$

The explicit forms of the invariant polynomials listed in (4) are provided in [11].

According to [11] we have the following proposition.

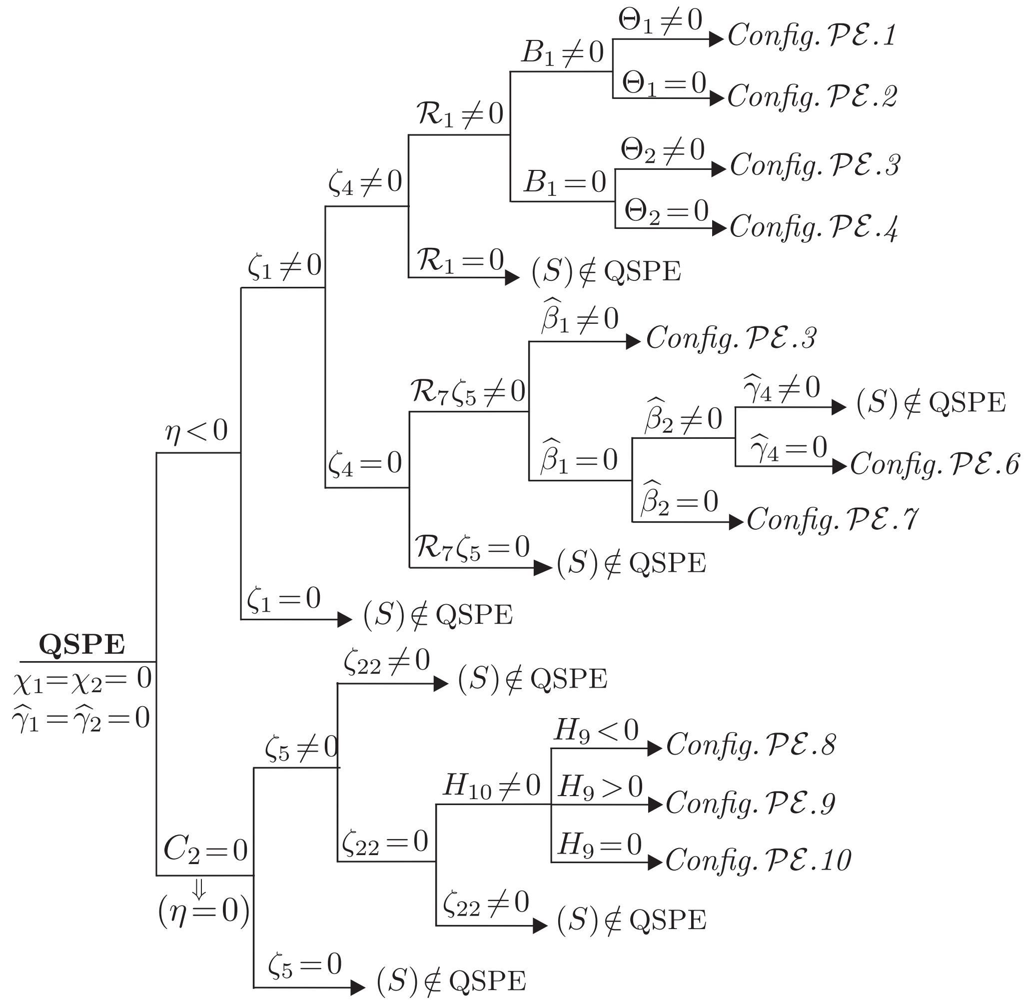


Diagram 1: Conditions for configurations of systems in **QSPE**

**Proposition 1.** *(A) The conditions* $\widehat{\gamma}_1=\widehat{\gamma}_2=0$ *and either* $\eta<0$ *or* $C_2=0$ *are necessary for a quadratic system in the class* $QS$ *to possess at least one invariant ellipse.*

*(B) Assume that for a system in the class* $QS$ *the condition* $\widehat{\gamma}_1=\widehat{\gamma}_2=0$ *is satisfied.*

1. *If* $\eta<0$*, the necessary and sufficient conditions for this system to possess at least one invariant ellipse are presented in Diagram 2. This diagram also specifies the conditions under which the ellipse is real, complex, or corresponds to a limit cycle.*

2. *If* $C_2=0$*, the corresponding necessary and sufficient conditions for the existence of at least one invariant ellipse are given in Diagram 3.*

3. *In the case* $C_2=0$ *the corresponding necessary and sufficient conditions for this system to possess at least one invariant ellipse are given in Diagram 3.*

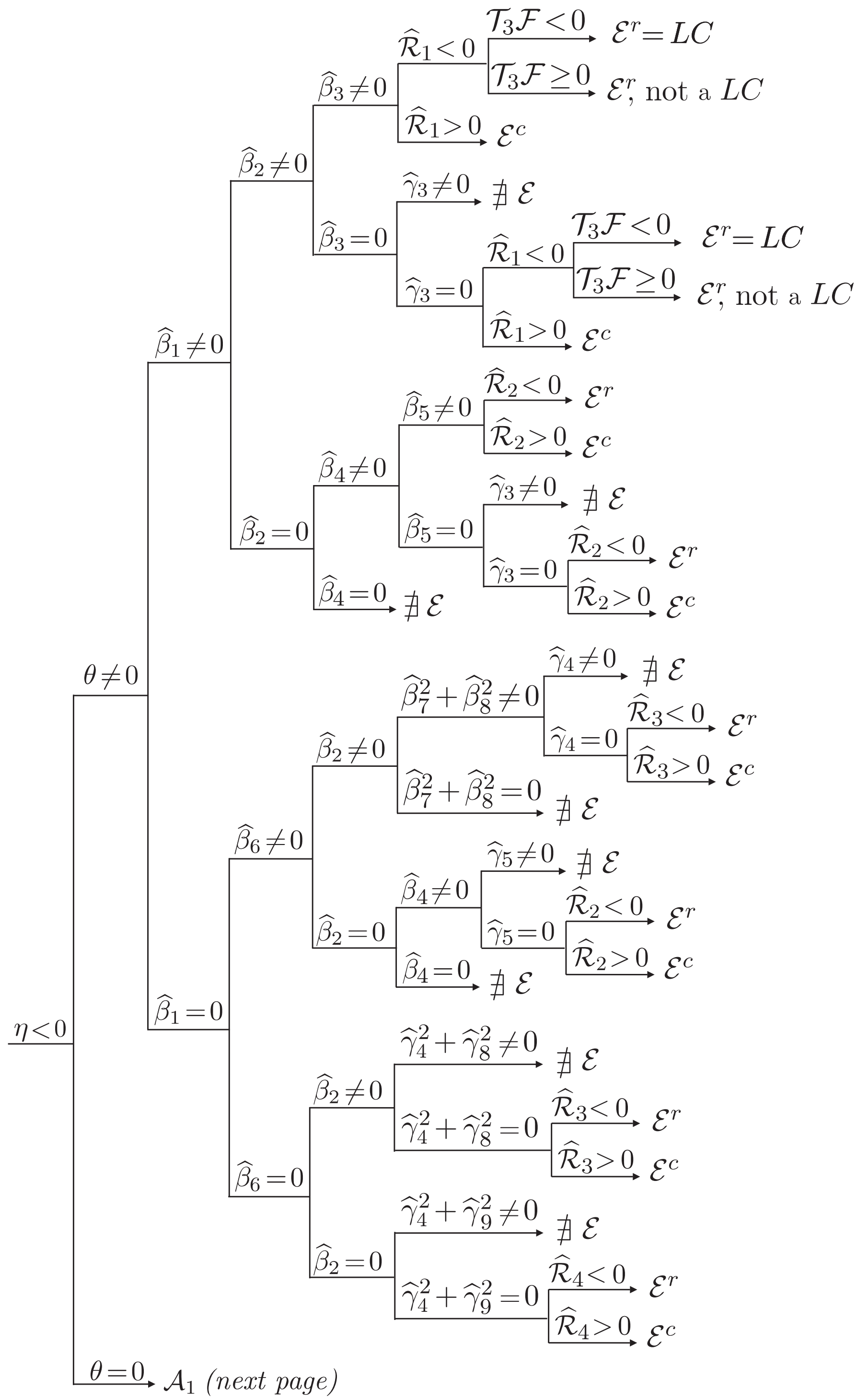


Diagram 2: Existence of invariant ellipses: the case $\eta < 0$

Next we mention that in [21] the necessary and sufficient affine invariant conditions for a quadratic systems to possess at least one invariant parabola are determined. To give these invariant criteria

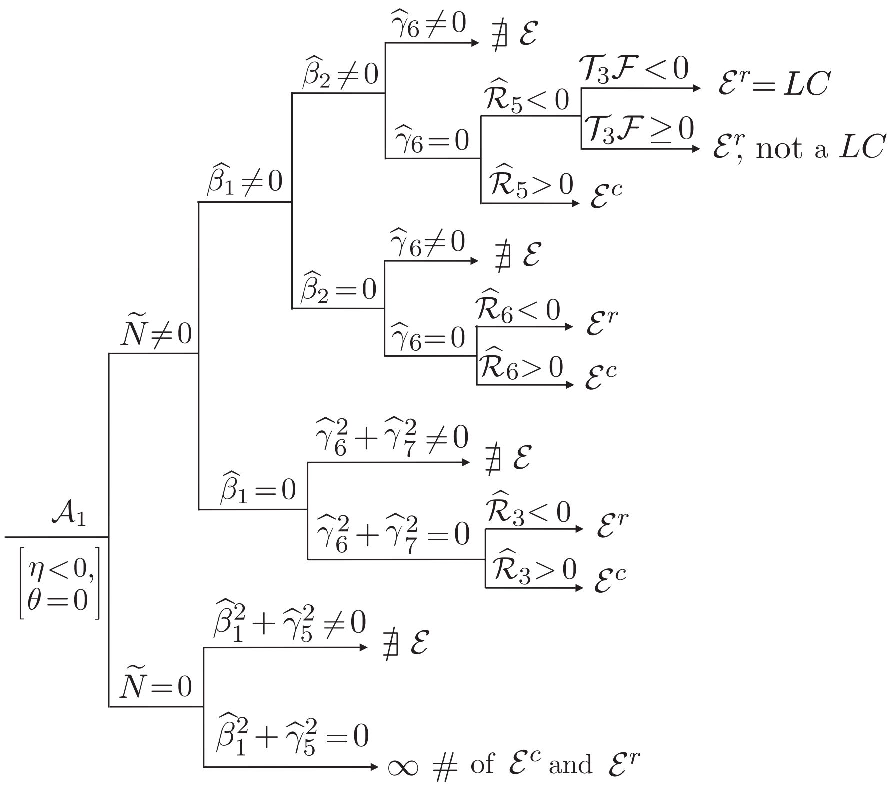


Diagram 2 *(cont.)* Existence of invariant ellipses: the case $\eta<0$

the following 43 affine invariant polynomials are provided:

$$\eta,\ M, C_2,\ \mu_0-\mu_4,\ \chi_1-\chi_4,\ \zeta_1-\zeta_{24},\ \mathcal{R}_1-\mathcal{R}_7. \tag{5}$$

The expressions for the above invariant polynomials can be found in [21].

Following [21] we shall use the next definition.

**Definition 1.** *We say that we have an invariant parabola in a simple (respectively double; triple) direction if the infinite singular point of this parabola is defined by a simple (respectively double; triple) factor in the factorization of the invariant polynomial $C_2(x,y)$ over $\mathbb{C}$. We denote this parabola by $\cup$ (respectively $\overset{2}{\cup}$; $\overset{3}{\cup}$). Moreover if the infinite invariant line $Z=0$ is filled up with singularities then we denote by $\overset{\infty}{\cup}$ the invariant parabola which is tangent to the line $Z=0$ at a non-isolated singular point.*

According to [21] in order to describe the invariant parabolas that a quadratic system could have we use the following notations:

- $\cup$ for a simple invariant parabola;
- $\Cup$ for two simple invariant parabolas in the same direction (they could intersect or be tangent to the Poincare circle at diametrically opposite points);

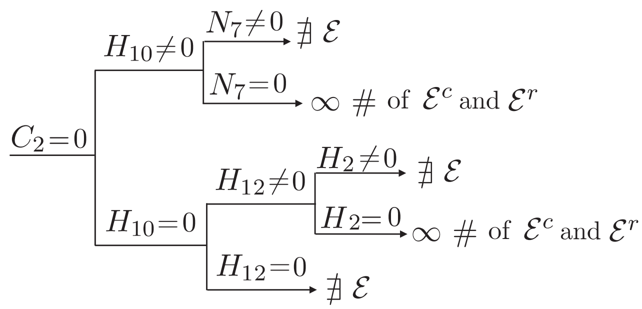


Diagram 3: Existence of invariant ellipses: the case $C_2 = 0$

- $\cup\subset$ for two simple invariant parabolas in different directions;
- $\bigcup^2$ for one double invariant parabola;
- $\overset{2}{\cup}$ for one simple invariant parabola in double direction;
- $\overset{3}{\cup}$ for one simple invariant parabola in triple direction;
- $\overset{\infty}{\cup}$ for one simple invariant parabola when the line at infinity is filled up with singularities;
- $\overset{2}{\cup}\subset$ for two simple invariant parabola: one in a double direction and one in a simple direction;
- $\mathbb{U}\subset$ for three simple invariant parabolas: two in one direction and one in another direction;
- $\overset{2}{\cup}\overset{2}{\mathbb{U}}$ for three real invariant parabolas in a double direction;
- $\overset{2}{\cup}\overset{2}{\mathbb{U}}{}^{c}$ for one real and two complex invariant parabolas in a double direction;
- $\overset{2}{\cup}\overset{2}{\bigcup}{}^{2}$ for one simple and one double real invariant parabolas in a double direction;
- $\overset{2}{\bigcup}{}^{3}$ for one triple real invariant parabola in a double direction;
- $\infty\overset{2}{\cup}$ for an 1-parameter family of invariant parabolas in a double direction;
- $\infty\overset{3}{\cup}$ for an 1-parameter family of invariant parabolas in a triple direction.

Our objective is to determine the conditions under which a quadratic system simultaneously possesses an invariant ellipse and an invariant parabola.

According to Proposition 1 the necessary conditions for the existence of an invariant ellipse of a quadratic system (3) are $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ and either $\eta < 0$ or $C_2(x, y) = 0$ (which implies $\eta = 0$). So we present here only two lemmas proved in [21] and corresponding to the cases $\eta < 0$ and $C_2(x, y) = 0$.

**Lemma 1.** *Assume that for a non-degenerate arbitrary quadratic system the condition $\eta < 0$ is satisfied. Then this system could possess only one of the following sets of invariant parabolas:* $\cup$*,*

$\Cup$ *and* $\bigcup^2$*. Moreover this system has one of the above sets of invariant parabolas if and only if* $\chi_1 = \chi_2 = 0$, $\zeta_1 \neq 0$ *and one of the following sets of conditions are satisfied, correspondingly:*

$(\boldsymbol{\mathcal{E}_1})$ $\zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0 \quad \Rightarrow \cup;$
$(\boldsymbol{\mathcal{E}_2})$ $\zeta_4 = 0,\ \mathcal{R}_7 \neq 0,\ \zeta_5 \neq 0 \quad \Rightarrow \Cup;$
$(\boldsymbol{\mathcal{E}_3})$ $\zeta_4 = 0,\ \mathcal{R}_7 \neq 0,\ \zeta_5 = 0 \quad \Rightarrow \bigcup^2;$
$(\boldsymbol{\mathcal{E}_4})$ $\zeta_4 = 0,\ \mathcal{R}_7 = 0,\ \zeta_5 \neq 0 \quad \Rightarrow \cup.$

*Furthermore a quadratic system* (3) *with* $\eta < 0$ *possessing an invariant parabola via an affine transformation and time rescaling could be brought to the following canonical form:*

$$\dot{x} = m + \frac{2n-1}{2}\,x - \frac{g}{2}\,y + gx^2 - xy, \quad \dot{y} = 2mx + 2ny - x^2 + gxy - 2y^2 \tag{6}$$

*possessing the invariant parabola* $\Phi(x,y) = x^2 - y$.

**Lemma 2.** *Assume that for a non-degenerate quadratic system the condition* $C_2 = 0$ *holds. Then this system possesses an invariant parabola (which is unique) if and only if the following set of conditions holds:*

$$(\boldsymbol{\mathcal{V}})\quad \zeta_5 \neq 0,\ \zeta_{22} = 0 \quad \Rightarrow \overset{\infty}{\cup}.$$

*Moreover via an affine transformation and time rescaling this system could be brought to the form*

$$\dot{x} = m + nx - y/2 + x^2, \quad \dot{y} = 2mx + 2ny + xy, \tag{7}$$

*which possess the invariant parabola* $\Phi(x,y) = x^2 - y$.

### 2.2 Invariant polynomials associated with configurations of invariant conics

We single out the following five polynomials, basic ingredients in constructing invariant polynomials for systems (3):

$$\begin{aligned} C_i(\tilde{a},x,y) &= yp_i(x,y) - xq_i(x,y),\ (i = 0,1,2) \\ D_i(\tilde{a},x,y) &= \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y},\ (i = 1,2). \end{aligned} \tag{8}$$

As it was shown in [19] these polynomials of degree one in the coefficients of systems (3) are $GL$–comitants of these systems. Let $f,\ g \in \mathbb{R}[\tilde{a},x,y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

The polynomial $(f,g)^{(k)} \in \mathbb{R}[\tilde{a},x,y]$ is called *the transvectant of index* $k$ *of* $(f,g)$ (cf. [6], [13])).

**Proposition 2** (see [20])**.** *Any* $GL$*–comitant of systems* (3) *can be constructed from the elements* (8) *by using the operations:* $+, -, \times$, *and by applying the differential operation* $(*,*)^{(k)}$.

**Remark 2.** *We point out that the elements* (8) *generate the whole set of* $GL$*–comitants and hence also the set of affine comitants as well as the set of* $T$*-comitants.*

We construct the following $GL$–comitants of the second degree with respect to the coefficients of the initial systems

$$\begin{array}{lll} T_1 = (C_0, C_1)^{(1)}, & T_2 = (C_0, C_2)^{(1)}, & T_3 = (C_0, D_2)^{(1)}, \\ T_4 = (C_1, C_1)^{(2)}, & T_5 = (C_1, C_2)^{(1)}, & T_6 = (C_1, C_2)^{(2)}, \\ T_7 = (C_1, D_2)^{(1)}, & T_8 = (C_2, C_2)^{(2)}, & T_9 = (C_2, D_2)^{(1)}. \end{array} \tag{9}$$

Using these $GL$–comitants as well as the polynomials (8) we construct the additional invariant polynomials. In order to be able to calculate the values of the needed invariant polynomials directly for every canonical system we shall define here a family of $T$–comitants expressed through $C_i$ $(i = 0, 1, 2)$ and $D_j$ $(j = 1, 2)$:

$$\begin{aligned}
\hat{A} &= \left(C_1, T_8 - 2T_9 + D_2^2\right)^{(2)}/144, \\
\widehat{D} &= \left[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6 - (C_1, T_5)^{(1)} + 6D_1(C_1D_2 - T_5) - 9D_1^2C_2\right]/36, \\
\widehat{E} &= \left[D_1(2T_9 - T_8) - 3\,(C_1, T_9)^{(1)} - D_2(3T_7 + D_1D_2)\right]/72, \\
\widehat{F} &= \left[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0\,(D_2, T_9)^{(1)} - 9D_2^2T_4 + 288D_1\widehat{E}\right. \\
&\quad \left. - 24\left(C_2, \widehat{D}\right)^{(2)} + 120\left(D_2, \widehat{D}\right)^{(1)} - 36C_1\,(D_2, T_7)^{(1)} + 8D_1\,(D_2, T_5)^{(1)}\right]/144,
\end{aligned}$$

$$\begin{aligned}
\widehat{B} = \Big\{ & 16D_1\,(D_2, T_8)^{(1)}\,(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0\,(D_2, T_9)^{(1)}\,(3D_1D_2 - 5T_6 + 9T_7) \\
& + 2\,(D_2, T_9)^{(1)}\left(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32\,(C_0, T_5)^{(1)}\right) \\
& + 6\,(D_2, T_7)^{(1)}\,[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5) + C_2(9T_4 + 96T_3)] \\
& + 6\,(D_2, T_6)^{(1)}\,[32C_0T_9 - C_1(12T_7 + 52D_1D_2) - 32C_2D_1^2] + 48D_2\,(D_2, T_1)^{(1)}\left(2D_2^2 - T_8\right) \\
& - 32D_1T_8\,(D_2, T_2)^{(1)} + 9D_2^2T_4\,(T_6 - 2T_7) - 16D_1\,(C_2, T_8)^{(1)}\left(D_1^2 + 4T_3\right) \\
& + 12D_1\,(C_1, T_8)^{(2)}\,(C_1D_2 - 2C_2D_1) + 6D_1D_2T_4\left(T_8 - 7D_2^2 - 42T_9\right) \\
& + 12D_1\,(C_1, T_8)^{(1)}\,(T_7 + 2D_1D_2) + 96D_2^2\left[D_1\,(C_1, T_6)^{(1)} + D_2\,(C_0, T_6)^{(1)}\right] - \\
& - 16D_1D_2T_3\left(2D_2^2 + 3T_8\right) - 4D_1^3D_2\left(D_2^2 + 3T_8 + 6T_9\right) + 6D_1^2D_2^2\,(7T_6 + 2T_7) \\
& - 252D_1D_2T_4T_9\Big\}/(2^8 3^3), \\
\widehat{K} = & (T_8 + 4T_9 + 4D_2^2)/72, \quad \widehat{H} = (8T_9 - T_8 + 2D_2^2)/72.
\end{aligned}$$

These polynomials in addition to (8) and (9) will serve as bricks in constructing affine invariant polynomials for systems (3).

The following 42 affine invariants $A_1, \ldots, A_{42}$ form the minimal polynomial basis of affine invariants up to degree 12. This fact was proved in [2] by constructing $A_1, \ldots, A_{42}$ using the above bricks.

$$
\begin{array}{llll}
A_1 & = \hat{A}, & A_{22} & = \frac{1}{1152}\llbracket C_2, \widehat{D})^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)} D_2\big)^{(1)}, \\
A_2 & = (C_2, \widehat{D})^{(3)}/12, & A_{23} & = \llbracket [\widehat{F}, \widehat{H})^{(1)}, \widehat{K}\big)^{(2)}/8, \\
A_3 & = \llbracket C_2, D_2)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/48, & A_{24} & = \llbracket C_2, \widehat{D})^{(2)}, \widehat{K}\big)^{(1)}, \widehat{H}\big)^{(2)}/32, \\
A_4 & = (\widehat{H}, \widehat{H})^{(2)}, & A_{25} & = \llbracket \widehat{D}, \widehat{D})^{(2)}, \widehat{E}\big)^{(2)}/16, \\
A_5 & = (\widehat{H}, \widehat{K})^{(2)}/2, & A_{26} & = (\widehat{B}, \widehat{D})^{(3)}/36, \\
A_6 & = (\widehat{E}, \widehat{H})^{(2)}/2, & A_{27} & = \llbracket \widehat{B}, D_2)^{(1)}, \widehat{H}\big)^{(2)}/24, \\
A_7 & = \llbracket [C_2, \widehat{E})^{(2)}, D_2\big)^{(1)}/8, & A_{28} & = \llbracket C_2, \widehat{K})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{E}\big)^{(2)}/16, \\
A_8 & = \llbracket \widehat{D}, \widehat{H})^{(2)}, D_2\big)^{(1)}/8, & A_{29} & = \llbracket \widehat{D}, \widehat{F})^{(1)}, \widehat{D}\big)^{(3)}/96, \\
A_9 & = \llbracket \widehat{D}, D_2)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/48, & A_{30} & = \llbracket C_2, \widehat{D})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{D}\big)^{(3)}/288, \\
A_{10} & = \llbracket \widehat{D}, \widehat{K})^{(2)}, D_2\big)^{(1)}/8, & A_{31} & = \llbracket \widehat{D}, \widehat{D})^{(2)}, \widehat{K}\big)^{(1)}, \widehat{H}\big)^{(2)}/64, \\
A_{11} & = (\widehat{F}, \widehat{K})^{(2)}/4, & A_{32} & = \llbracket \widehat{D}, \widehat{D})^{(2)}, D_2\big)^{(1)}, \widehat{H}\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{12} & = (\widehat{F}, \widehat{H})^{(2)}/4, & A_{33} & = \llbracket \widehat{D}, D_2)^{(1)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/128, \\
A_{13} & = \llbracket C_2, \widehat{H})^{(1)}, \widehat{H}\big)^{(2)}, D_2\big)^{(1)}/24, & A_{34} & = \llbracket \widehat{D}, \widehat{D})^{(2)}, D_2\big)^{(1)}, \widehat{K}\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{14} & = (\widehat{B}, C_2)^{(3)}/36, & A_{35} & = \llbracket \widehat{D}, \widehat{D})^{(2)}, \widehat{E}\big)^{(1)}, D_2\big)^{(1)}, D_2\big)^{(1)}/128, \\
A_{15} & = (\widehat{E}, \widehat{F})^{(2)}/4, & A_{36} & = \llbracket \widehat{D}, \widehat{E})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{H}\big)^{(2)}/16, \\
A_{16} & = \llbracket \widehat{E}, D_2)^{(1)}, C_2\big)^{(1)}, \widehat{K}\big)^{(2)}/16, & A_{37} & = \llbracket \widehat{D}, \widehat{D})^{(2)}, \widehat{D}\big)^{(1)}, \widehat{D}\big)^{(3)}/576, \\
A_{17} & = \llbracket \widehat{D}, \widehat{D})^{(2)}, D_2\big)^{(1)}, D_2\big)^{(1)}/64, & A_{38} & = \llbracket C_2, \widehat{D})^{(2)}, \widehat{D}\big)^{(2)}, \widehat{D}\big)^{(1)}, \widehat{H}\big)^{(2)}/64, \\
A_{18} & = \llbracket \widehat{D}, \widehat{F})^{(2)}, D_2\big)^{(1)}/16, & A_{39} & = \llbracket \widehat{D}, \widehat{D})^{(2)}, \widehat{F}\big)^{(1)}, \widehat{H}\big)^{(2)}/64, \\
A_{19} & = \llbracket \widehat{D}, \widehat{D})^{(2)}, \widehat{H}\big)^{(2)}/16, & A_{40} & = \llbracket \widehat{D}, \widehat{D})^{(2)}, \widehat{F}\big)^{(1)}, \widehat{K}\big)^{(2)}/64, \\
A_{20} & = \llbracket C_2, \widehat{D})^{(2)}, \widehat{F}\big)^{(2)}/16, & A_{41} & = \llbracket C_2, \widehat{D})^{(2)}, \widehat{D}\big)^{(2)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}/64, \\
A_{21} & = \llbracket \widehat{D}, \widehat{D})^{(2)}, \widehat{K}\big)^{(2)}/16, & A_{42} & = \llbracket \widehat{D}, \widehat{F})^{(2)}, \widehat{F}\big)^{(1)}, D_2\big)^{(1)}/16.
\end{array}
$$

In the above list, the bracket "$\llbracket$" is used in order to avoid placing the otherwise necessary up to five parentheses "(".

Using the elements of the minimal polynomial basis given above, in [10] (respectively [21]) are defined the invariant polynomials of the set (4) (respectively the set (5)) which are responsible for the existence of at least on invariant hyperbola (respectively parabola) for systems (3).

Here we present additional invariant polynomials responsible for the existence of at least one invariant hyperbola and one invariant parabola as well as for distinguishing the corresponding configurations of these conics together with invariant lines in case they exist.

First we mention the invariant polynomials $\mu_0(\tilde{a}), \mu_1(\tilde{a}, x, y), \ldots \mu_4(\tilde{a}, x, y)$ from the set (5). These polynomials are in fact comitants of systems (3) with respect to the group $GL(2, \mathbb{R})$ and their geometrical meaning is revealed in Lemma 5.2 of [1]. Using these invariant polynomials we construct the invariant polynomials $\mathbf{D}$ and $\mathbf{R}$ which are responsible for the existence of multiple finite singularities of a quadratic system:

$$\mathbf{D} = \Big[3\big((\mu_3, \mu_3)^{(2)},\ \mu_2\big)^{(2)} - \big(6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\ \mu_4\big)^{(4)}\Big]/48, \quad \mathbf{R} = 3\mu_1^2 - 8\mu_0\mu_2.$$

Next we construct the following $T$-comitants (for the definition of $T$-comitants see [16]) which are responsible for the existence of invariant straight lines of systems (3):

$$B_3(\tilde{a},x,y) = (C_2,\widehat{D})^{(1)} = \mathrm{Jacob}(C_2,\widehat{D}),$$
$$B_2(\tilde{a},x,y) = (B_3,B_3)^{(2)} - 6B_3(C_2,\widehat{D})^{(3)},$$
$$B_1(\tilde{a}) = \mathrm{Res}_x\left(C_2,\widehat{D}\right)/y^9 = -2^{-9}3^{-8}\,(B_2,B_3)^{(4)}\,.$$

**Lemma 3** (see [15])**.** *For the existence of invariant straight lines in one (respectively 2; 3 distinct) directions in the affine plane it is necessary that* $B_1 = 0$ *(respectively* $B_2 = 0$*;* $B_3 = 0$*).*

To detect the parallel invariant lines we need the following invariant polynomials:

$$\widetilde{N}(\tilde{a},x,y) = D_2^2 + T_8 - 2T_9 = 9\widehat{N},$$
$$\theta(\tilde{a}) = 2A_5 - A_4 \;(\equiv \mathrm{Discriminant}\big(\widetilde{N}(a,x,y)\big)/1296).$$

**Lemma 4** (see [15])**.** *A necessary condition for the existence of one couple (respectively two couples) of parallel invariant straight lines of a system* (3) *corresponding to* $\tilde{a} \in \mathbb{R}^{12}$ *is the condition* $\theta(\tilde{a}) = 0$ *(respectively* $N(\tilde{a},x,y) = 0$*).*

We point out some important $GL$-comitant in the study of the invariant conics. Considering $C_2(\tilde{a},x,y) = yp_2(\tilde{a},x,y) - xq_2(\tilde{a},x,y)$ as a cubic binary form of $x$ and $y$ we calculate

$$\eta(\tilde{a}) = \mathrm{Discrim}[C_2], \quad \widetilde{M}(\tilde{a},x,y) = \mathrm{Hessian}[C_2].$$

We point out (see [19]) that the invariant polynomials $C_2$, $\eta$ and $\widetilde{M}$ are responsible for the number of infinite singularities (real or complex).

**Remark 3.** To describe the various kinds of multiplicities of infinite singularities, we adopt the concepts and notations introduced in [15]. Accordingly, we denote by $(a,b)$ the maximum number $a$ (respectively, $b$) of infinite (respectively, finite) singularities which can arise from a perturbation of a multiple infinite singularity. In this case, we say that an infinite singular point *has multiplicity* $(a,b)$.

Using the elements $A_1,\ldots,A_{42}$ of the minimal polynomial basis, we additionally construct the affine invariant polynomials $\Theta_1$ and $\Theta_2$, which are required to complete the classification of the systems in QSPE according to the configurations of their invariant conics and lines.

$$\begin{aligned}
\Theta_1 = {}& 32(381978466A_8^2 - 64935936A_9^2 - 506684533A_{10}^2 - 710816967A_{10}A_{11} - 86444184A_{11}^2\\
&- 3A_8(143803448A_9 + 78196427A_{10} + 452848A_{11} - 20329512A_{12}) + 24A_9(13332575A_{10}\\
&+ 3471261A_{11} - 17668493A_{12}) + 254143185A_{10}A_{12} + 217943016A_{11}A_{12}) + 6144(3483132A_6A_{14}\\
&- 1657787A_7A_{14} + 6447090A_6A_{15} - 2462247A_7A_{15}) + A_4(-14474302849A_{19} + 1304419920A_{20}\\
&- 11495507137A_{21}) + 16A_5(1110517380A_{17} + 1793066174A_{18} + 1184797899A_{19} - 92273936A_{20}\\
&+ 1402960651A_{21}),\\
\Theta_2 = {}& 32A_3 - 65A_7.
\end{aligned}$$

# 3 The proof of Main Theorem

To prove the Main Theorem we consider each one of the statements $(\boldsymbol{A})$, $(\boldsymbol{B})$ and $(\boldsymbol{C})$.

## 3.1 The statements $(\boldsymbol{A})$ and $(\boldsymbol{C})$

According to [11] a quadratic system could have an invariant ellipse only if the condition $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ is satisfied. On the other hand by [21] the condition $\chi_1 = \chi_2 = 0$ is necessary for the existence of an invariant parabola for an arbitrary quadratic system. Therefore we conclude that the statement $(\boldsymbol{A})$ of the Main Theorem is valid.

The proof of the statement $(\boldsymbol{C})$ follow directly from the construction of the bifurcation diagram pictured here in Diagram 1.

## 3.2 The statement $(\boldsymbol{B})$

According to this statement we have to examine two possibilities: $\eta < 0$ and $C_2 = 0$ (which implies $\eta = \widetilde{M} = 0$).

### 3.2.1 The possibility $\eta < 0$

In this case systems (3) possess at infinity one real and two complex singularities. According to Lemma 1 if a quadratic system with $\eta < 0$ possesses an invariant parabola then via an affine transformation and time rescaling it can be brought to the canonical form (6). Thus in what follows we consider the following 3-parameter family of systems

$$\dot{x} = m + \frac{2n-1}{2}x - \frac{g}{2}y + gx^2 - xy, \quad \dot{y} = 2mx + 2ny - x^2 + gxy - 2y^2, \tag{10}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

It is clear that in order to detect the existence of an invariant ellipse for the above systems it is sufficient to impose the necessary condition $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ as well as the corresponding conditions given by Diagram 2 to be fulfilled. Therefore we follow step by step the statements $(\boldsymbol{\mathcal{E}_i})$ $(i = 1, 2, 3, 4)$ provided by Lemma 1.

**3.2.1.1 The statement $(\boldsymbol{\mathcal{E}_1})$.** For systems (10) calculations yield:

$$\begin{aligned} \zeta_4 &= (25 + g^2)(3g + 9g^3 - 4m - 6gn)/16 \neq 0, \\ \mathcal{R}_1 &= 15(1 + g^2)(25 + g^2)(3g + 9g^3 - 4m - 6gn)/2 \neq 0. \end{aligned} \tag{11}$$

On the other hand considering the conditions provided by Proposition 1 we evaluate the values of the invariant polynomials $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ for systems (10) and we calculate:

$$\begin{aligned} \operatorname{Res}_m(\widehat{\gamma}_1, \widehat{\gamma}_2) =& 2051166796875\, g(9 + g^2)^5(1 + 9g^2)(25 + 9g^2)^2 \times \\ & (4 + 7g^2 - 4n)^6(n - 1)(4g^2n - 7 - 4g^2)V_1V_2/131072, \end{aligned}$$

where

$$\begin{aligned} V_1 &= 121 + 56g^2 + 4g^4 + 88n - 16g^2n + 16n^2, \\ V_2 &= 676 + 2276g^2 + 169g^4 - 1872n - 936g^2n + 1296n^2. \end{aligned}$$

**Lemma 5.** *The conditions $\widehat{\gamma}_1=\widehat{\gamma}_2=0$ and $\zeta_4\neq 0$ imply $g(4+7g^2-4n)V_1V_2\neq 0$ and $(n-1)(4g^2n-7-4g^2)=0$.*

*Proof:* Suppose the contrary and assume first $g=0$. Then for systems (10) we calculate:

$$\widehat{\gamma}_1=3(n-1)\big[25m^2+4(n-1)^2\big]/4=0 \;\Leftrightarrow\; n=1$$

and for $n=1$ we have

$$\widehat{\gamma}_2=19845000m^3,\quad \zeta_4=-25m/4.$$

Thus the condition $\zeta_4\neq 0$ implies $\widehat{\gamma}_2\neq 0$.

Admit now $g\neq 0$. We observe that $V_1$ and $V_2$ are quadratic polynomials with respect to the parameter $n$ and we calculate:

$$\mathrm{Discrim}\,[V_1,n]=-6400g^2,\quad \mathrm{Discrim}\,[V_2,n]=-8294400g^2.$$

Due to $g\neq 0$ we have $V_1>0$ and $V_2>0$ and therefore we cannot have either $V_1=0$ or $V_2=0$ for real parameters $g$ and $n$.

Assume finally that the condition $4+7g^2-4n=0$ holds. Then $n=(4+7g^2)/4$ and calculations yield:

$$\widehat{\gamma}_1=175g(6g+3g^3+8m)^3/2048,\quad \zeta_4=-(25+g^2)(6g+3g^3+8m)/32.$$

Since $g\neq 0$ and $\zeta_4\neq 0$ we obtain $\widehat{\gamma}_1\neq 0$.

So considering (11) we conclude that the condition $(n-1)(4g^2n-7-4g^2)=0$ is necessary in order to have $\widehat{\gamma}_1=\widehat{\gamma}_2=0$ and $\zeta_4\neq 0$ and this completes the proof of Lemma 5. ∎

Thus, in what follows we consider the condition $(n-1)(4g^2n-7-4g^2)=0$.

Next we evaluate for systems (10) the following invariant polynomials:

$$B_1=-\big[(g-8m)^2+(1-4n)^2\big]U_1U_2/64,\quad \Theta_1=-64935936\zeta_4(4g^2n-7-4g^2)U_1, \tag{12}$$

where

$$U_1=(g+g^3+4m-2gn),\;\; U_2=16m^2+8gm(3+2n)+(4+g^2)\big[g^2+(1+2n)^2\big].$$

We observe that for $B_1\neq 0$ the condition $4g^2n-7-4g^2=0$ is equivalent to $\Theta_1=0$ (due to $\zeta_4\neq 0$). So we examine two cases:

***1:* The case $B_1\neq 0$.** Then $U_1U_2\neq 0$ and we consider two subcases: $\Theta_1\neq 0$ and $\Theta_1=0$

***1.1.:*** *The subcase* $\Theta_1\neq 0$. This implies $(4g^2n-7-4g^2)\neq 0$ and therefore we get $n=1$. Calculations yield:

$$\begin{aligned}\widehat{\gamma}_1=&7gm(225g^2+106g^4+9g^6+600gm-120g^3m+400m^2)/64,\\ \widehat{\gamma}_2=&1575m\big[g^2(9+g^2)(5670+7052g^2+2985g^4+279g^6)+4gm(35910+66954g^2+\\ &\quad 9743g^4+131g^6)+16(6300+26665g^2-1623g^4)m^2+157760gm^3\big]/8,\end{aligned} \tag{13}$$

As we can see the common root of the above polynomials is $m=0$ and this leads to the 1-parameter family of systems

$$\dot{x}=\frac{1}{2}x-\frac{g}{2}y+gx^2-xy,\quad \dot{y}=2y-x^2+gxy-2y^2, \tag{14}$$

for which $\eta = -4 < 0$. So considering the conditions provided by Diagram 2 we calculate:

$$\theta = -(9+g^2)/2, \quad \widehat{\beta}_1 = -g^2(9+g^2)(25+9g^2)/64, \quad \widehat{\beta}_2 = -g(25+g^2)/2,$$
$$\widehat{\beta}_3 = -7(1+9g^2)/2, \quad \widehat{\mathcal{R}}_1 = -3g^4(9+g^2)^4(25+9g^2)/2048, \quad \mathcal{T}_3\mathcal{F} = g^4(25+9g^2)^2/128.$$

Due to $g \neq 0$ we get the conditions

$$\theta\widehat{\beta}_1\widehat{\beta}_2\widehat{\beta}_3 \neq 0, \quad \widehat{\mathcal{R}}_1 < 0, \quad \mathcal{T}_3\mathcal{F} > 0. \tag{15}$$

Therefore according to Diagram 2 we conclude that systems (14) possess one real invariant ellipse, which is not a limit cycle. More exactly we determine that systems (14) possess the following invariant parabola and ellipse:

$$\Phi(x,y) = x^2 - y = 0, \quad \Psi_1(x,y) = x^2 + y^2 - y = 0. \tag{16}$$

For these systems we calculate

$$B_1 = -\frac{1}{64}g(g^2-1)(4+g^2)(9+g^2)^2$$

and since $B_1 \neq 0$, according to Lemma 3 systems (14) could not possess any invariant line. We determine that these systems have four finite real singularities $M_i(x_i, y_i)$ ($i = 1,2,3,4$) with the coordinates

$$\begin{gathered} x_1 = 0,\ y_1 = 0; \quad x_2 = -\frac{g}{1+g^2},\ y_2 = \frac{1}{1+g^2}; \\ x_{3,4} = \frac{1}{4}\big(g \pm \sqrt{g^2+8}\,\big), \quad y_{3,4} = \frac{1}{8}\big(4 + g^2 \pm g\sqrt{g^2+8}\,\big). \end{gathered} \tag{17}$$

For systems (14) we calculate: $\mathbf{D} = -12(8+g^2) < 0$ and according to [1, Proposition 5.1] we conclude that all four real singularities are distinct for any value of the parameter $g$.

We observe that the invariant conics (16) have the unique point of intersection $M_1(0,0)$ which is a point of tangency. The singular point $M_2$ is located on the invariant ellipse whereas $M_3$ and $M_4$ are located on the invariant parabola on different parts with respect to $M_1$ because $x_3x_4 = -1/2$.

It is clear that all the finite singularities are distinct and then we arrive at the unique configuration *Config.* $\mathcal{PE}$*.1.*

***1.2:*** *The subcase* $\Theta_1 = 0$. This implies $4g^2n - 7 - 4g^2 = 0$ and since $g \neq 0$ (due to Lemma 5) we get $n = \dfrac{7+4g^2}{4g^2}$ and we obtain that the polynomials $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ have the common factor $6 + 3g^2 + 8g^3m$. So we get the common root $m = -\dfrac{3(2+g^2)}{8g^3}$ and we arrive at the following 1-parameter family of systems:

$$\begin{aligned} \dot{x} &= -\frac{3(2+g^2)}{8g^3} + \frac{(7+2g^2)}{4g^2}\,x - \frac{g}{2}\,y + gx^2 - xy, \\ \dot{y} &= -\frac{3(2+g^2)}{4g^3}\,x + \frac{7+4g^2}{2g^2}\,y - x^2 + gxy - 2y^2. \end{aligned} \tag{18}$$

Considering Diagram 2 for these systems we calculate

$$\begin{gathered} \theta = -(9+g^2)/2, \quad \widehat{\beta}_1 = -(g-1)^2(1+g)^2(1+g^2)^2(9+g^2)(25+9g^2)/(64g^6), \\ \widehat{\beta}_2 = -g(25+g^2)/2, \quad \widehat{\beta}_3 = -7(1+9g^2)/2, \\ \widehat{\mathcal{R}}_1 = -3(g-1)^2(1+g)^2(1+g^2)^5(9+g^2)^4(25+9g^2)/(2048g^{10}), \\ \mathcal{T}_3\mathcal{F} = (g-1)^2(1+g)^2(1+g^2)^4(25+9g^2)^2/(32g^8). \end{gathered} \tag{19}$$

On the other hand considering the conditions provided by the statement $(\boldsymbol{\mathcal{E}_1})$ of Lemma 1 for systems (18) we calculate:

$$\zeta_4 = \frac{3}{16g^3}(g-1)(1+g)(1+g^2)(25+g^2)(3g^2-1) \neq 0.$$

Therefore taking into account (19) we obtain the conditions (15) and according to Diagram 2 we conclude that systems (18) possess one real invariant ellipse, which is not a limit cycle. More exactly we determine that systems (18) possess the following invariant parabola and ellipse:

$$\Phi(x,y) = x^2 - y = 0, \quad \Psi_2(x,y) = 16g^4(x^2+y^2) + 16gx - 8g^2(3+2g^2)y - 3 = 0. \tag{20}$$

Considering Lemma 3 for systems (18) we calculate

$$B_1 = -\frac{1}{64g^{15}}(g^2-3)(1+g^2)^8(4+g^2)(9+g^2)^2$$

and since $B_1 \neq 0$ according to Lemma 3 systems (18) could not possess any invariant line. We determine that these systems have four finite real singularities $M_i(x_i, y_i)$ $(i = 1,2,3,4)$ with the coordinates

$$\begin{gathered} x_1 = -\frac{3}{2g},\ y_1 = \frac{9}{4g^2};\quad x_2 = \frac{1}{2g},\ y_2 = \frac{1}{4g^2};\\ x_3 = \frac{2+g^2}{2g},\ y_3 = \frac{(2+g^2)^2}{4g^2},\quad x_4 = -\frac{3+2g^2}{2g^3},\ y_4 = \frac{1}{4g^2}. \end{gathered} \tag{21}$$

We observe that the invariant conics (20) have two points of intersection $M_1$ and $M_2$ and the singularity $M_3$ is located on the invariant parabola. On the other hand the singular point $M_4$ is not located on any of the invariant conics and it could not be located on any of them for some values of the parameter $g$ because calculations yield:

$$\Phi(x_4,y_4) = \frac{3}{4g^6}(1+g^2)(3+g^2) \neq 0, \quad \Psi_2(x_4,y_4) = \frac{12}{g^2}(1+g^2)^2 \neq 0.$$

In order to determine the position of the singular point $M_3$ with respect to $M_1$ and $M_2$ (all three located on the invariant parabola) we calculate

$$(x_3-x_1)(x_3-x_2) = \frac{1}{4g^2}(1+g^2)(5+g^2) > 0.$$

So we conclude that the singular point $M_3$ could not be located between the points of intersection of the invariant conics.

On the other hand for systems (18) we calculate:

$$\mathbf{D} = -\frac{27}{g^{18}}(3+g^2)^2(5+g^2)^2 < 0$$

and according to [1, Proposition 5.1] we conclude that all four real singularities are distinct for any value of the parameter $g$.

So taking into consideration these facts, in the case $B_1 \neq 0$ and $\Theta_1 = 0$ we arrive at the unique configuration *Config.* $\mathcal{PE}.2$.

***2:* The case $B_1 = 0$.** We prove the following lemma.

**Lemma 6.** *The conditions* $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0 = B_1$ *and* $\zeta_4 \neq 0$ *imply* $U_1 = 0$.

*Proof:* Assume that $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ and $\zeta_4 \neq 0$ hold. Then by Lemma 5 the condition $(n-1)(4g^2n - 7 - 4g^2) = 0$ has to be fulfilled.

Assume first $n = 1$. Then considering (13) we obtain that $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ implies $m = 0$ and taking into account (12) we calculate:

$$B_1 = -\frac{1}{64}(9+g^2)U_1U_2 = 0, \quad U_2 = (4+g^2)(9+g^2) \neq 0.$$

So the lemma is valid for $n = 1$.

Suppose now $4g^2n - 7 - 4g^2 = 0$, i.e. $n = \dfrac{7+4g^2}{4g^2}$ (due to $g \neq 0$). As it was shown earlier (see the subcase $\Theta_1 = 0$, on page 15) in this case the polynomials $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ have the common factor $6 + 3g^2 + 8g^3m$ and we get the common root $m = -\dfrac{3(2+g^2)}{8g^3}$. Calculation yield:

$$B_1 = \frac{1}{64g^6}(1+g^2)^2(4+g^2)(9+g^2)U_1U_2 = 0; \quad U_2 = \frac{1}{g^6}(1+g^2)^4(9+g^2) \neq 0$$

and this completes the proof of Lemma 6. ∎

Thus according to this lemma in the case $B_1 = 0$ we get $U_1 = 0$, i.e. $m = -g(1+g^2-2n)/4$. We calculate

$$\text{Res}_n(\widehat{\gamma}_1, \widehat{\gamma}_2) = 2^{-1}3^9 5^8 7^4\, g^7(1+g^2)^{13}(9+g^2)^5(1+9g^2)(25+9g^2)^2\,(g^2-1)(g^2-3) = 0$$

and due to $g \neq 0$ evidently this condition gives us $(g^2-1)(g^2-3) = 0$.

On the other hand in the case under consideration we have $\Theta_2 = 80(g^2-3)$ and we discuss two subcases: $\Theta_2 \neq 0$ and $\Theta_2 = 0$.

***2.1:*** *The subcase* $\Theta_2 \neq 0$. Then we get $g^2 - 1 = 0$ and we observe that $\widehat{\gamma}_1$ is a polynomial in $n$ and $g^2$. So setting $g^2 = 1$ we obtain

$$\widehat{\gamma}_1 = 5(n-1)(25-24n+16n^2)/2 = 0 \;\Rightarrow\; n = 1$$

due to the condition Discrim $[25 - 24n + 16n^2, n] = -1024 < 0$. Thus, we arrive at the systems (14) under the condition $g^2 = 1$, which possess the invariant curves (16) and the singular points (17).

We observe that the systems (14) with $g^2 = 1$ admit the invariant line $x = 1/2$ for $g = -1$ and $x = -1/2$ for $g = 1$. In each case, the corresponding invariant line is tangent to the invariant ellipse at the point $M_2$ and intersects the invariant parabola at the point $M_3$ or $M_4$. Consequently, we obtain the same configuration, denoted by *Config.* $\mathcal{PE}.3$.

***2.2:*** *The subcase* $\Theta_2 = 0$. This condition implies $g^2 - 3 = 0$ and arrive at the systems (18) under the condition $g = \pm\sqrt{3}$. These systems possess the invariant curves (20) and the finite singularities (21).

We observe that systems (18) possess the invariant line $x = -\sqrt{3}/2$ for $g = \sqrt{3}$ and $x = \sqrt{3}/2$ for $g = -\sqrt{3}$. In each one of these cases the corresponding invariant line intersects the invariants parabola at the point $M_1$ and passes through the singular point $M_4$. Moreover, this point lies on the invariant line below $M_1$. Consequently, in both cases we obtain the configuration *Config.* $\mathcal{PE}.4$.

**3.2.1.2 The statement ($\boldsymbol{\mathcal{E}_2}$).** In this case we have $\zeta_4 = 0$ and considering (11) this condition implies $3g + 9g^3 - 4m - 6gn = 0$. So we get $m = 3g(1+3g^2-2n)/4$ and we arrive at the 2-parameter family of systems

$$\begin{aligned}\dot{x} =& \frac{3}{4}g(1+3g^2-2n) + \frac{2n-1}{2}\,x - \frac{g}{2}\,y + gx^2 - xy,\\ \dot{y} =& \frac{3}{2}g(1+3g^2-2n)\,x + 2n\,y - x^2 + gxy - 2y^2\end{aligned} \tag{22}$$

which besides the parabola $\Phi(x, y) = x^2 - y = 0$ possess another one

$$\widetilde{\Phi}(x, y) = (3g^2 - 4n)(1 + 3g^2 - 2n) + 2g(3g^2 - 4n)x - 4(1 + g^2)x^2 + 2(2 + 5g^2 - 4n)y = 0.$$

These two parabolas are distinct due to the conditions provided by the statement $\boldsymbol{(\mathcal{E}_2)}$ of Lemma 1 because for systems (22) we have

$$\zeta_5 = 19(25 + g^2)(3g^2 - 4n)^2/4 \neq 0, \quad \mathcal{R}_7 = 16120(1 + 3g^2)(2 + 5g^2 - 4n) \neq 0.$$

Now we examine the conditions for the existence of an additional invariant ellipse for these systems. Calculations yield:

$$\widehat{\gamma}_1 = 3(3g^2 - 1)(1 + 9g^2)(4 + 7g^2 - 4n)^3/64, \quad \widehat{\beta}_1 = -(9g^2 - 1)(4 + 7g^2 - 4n)^2/8$$

and we discuss two possibilities: $\widehat{\beta}_1 \neq 0$ and $\widehat{\beta}_1 = 0$.

***1:*** *The possibility* $\widehat{\beta}_1 \neq 0$. In this case the condition $\widehat{\gamma}_1 = 0$ gives us $3g^2 - 1 = 0$, i.e. $g = \dfrac{\varepsilon}{\sqrt{3}}$, where $\varepsilon = \pm 1$. However we may assume $\varepsilon = 1$ because via the rescaling $x \to -x$ in systems (22) we may change the sign of the parameter $g$.

Then we calculate

$$\widehat{\gamma}_2 = \frac{5600}{\sqrt{3}}(n - 1)(4n - 25)(12n - 19)^2, \quad \widehat{\beta}_1 = -\frac{1}{36}(12n - 19)^2$$

and due to $\widehat{\beta}_1 \neq 0$ the condition $\widehat{\gamma}_2 = 0$ implies $(n - 1)(4n - 25) = 0$.

***1.1:*** *The case* $n - 1 = 0$. Then $n = 1$ and we get the system

$$\dot{x} = \frac{x}{2} - \frac{1}{2\sqrt{3}}\,y + \frac{1}{\sqrt{3}}\,x^2 - xy, \quad \dot{y} = 2y - x^2 + \frac{1}{\sqrt{3}}\,xy - 2y^2, \tag{23}$$

which possesses three invariant conics (two parabolas and an ellipse):

$$\Phi(x, y) = x^2 - y = 0, \quad \widetilde{\Phi}(x, y) = 3\sqrt{3}x + 8x^2 + y, \quad \Psi(x, y) = x^2 + y^2 - y.$$

We point out that for system (23) the conditions (15) provided by Diagram 2 are fulfilled. This means that the ellipse is not a limit cycle.

Next we examine the configurations of the invariant curves which could possess system (23). Since for this system we have $B_1 = \frac{637}{162\sqrt{3}} \neq 0$ by Lemma 3 we deduce that system (23) does not have any invariant line.

On the other hand we determine that this system has four finite real singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$x_1 = 0,\ y_1 = 0;\quad x_2 = -\frac{\sqrt{3}}{4},\ y_2 = \frac{3}{4};\quad x_3 = -\frac{1}{\sqrt{3}},\ y_3 = \frac{1}{3};\quad x_4 = \frac{\sqrt{3}}{2},\ y_4 = \frac{3}{4}.$$

To detect the position of these singularities with respect to the invariant conics we calculate:

$$\Phi(x_1, y_1) = \widetilde{\Phi}(x_1, y_1) = \Psi(x_1, y_1) = 0;\quad \widetilde{\Phi}(x_2, y_2) = \Psi(x_2, y_2) = 0;$$
$$\Phi(x_3, y_3) = \widetilde{\Phi}(x_3, y_3) = 0;\quad \Phi(x_4, y_4) = 0.$$

Therefore since we have not any parameter it is easy to detect for system (23) the unique configuration *Config.* $\mathcal{PE}.5$.

***1.2:*** *The case* $4n - 25 = 0$. This implies $n = 25/4$ and we obtain the system

$$\begin{aligned} \dot{x} &= -\frac{21\sqrt{3}}{8} + \frac{23}{4}\,x - \frac{1}{2\sqrt{3}}\,y + \frac{1}{\sqrt{3}}\,x^2 - xy,\\ \dot{y} &= -\frac{21\sqrt{3}}{4\,x} + \frac{25}{2}\,y - x^2 + \frac{1}{\sqrt{3}}\,xy - 2y^2, \end{aligned} \tag{24}$$

which possesses three invariant conics (two parabolas and an ellipse):

$$\Phi(x, y) = x^2 - y = 0,\quad \widetilde{\Phi}(x, y) = -189 + 12\sqrt{3}x + 4x^2 + 32y,$$
$$\Psi(x, y) = -27 + 48\sqrt{3}x + 16x^2 - 88y + 16y^2.$$

It is not too hard to determine that for systems (24) the conditions (15) are also satisfied.

For determining the configuration of invariant conics for system (24) we calculate its finite singularities. We obtain that this system has the singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$x_1 = \frac{7}{2\sqrt{3}},\ y_1 = \frac{49}{12};\quad x_2 = -\frac{3\sqrt{3}}{2},\ y_2 = \frac{27}{4};\quad x_3 = -\frac{11\sqrt{3}}{2},\ y_3 = \frac{3}{4};\quad x_4 = \frac{\sqrt{3}}{2},\ y_4 = \frac{3}{4}.$$

On the other hand calculations yield:

$$\Phi(x_1, y_1) = \widetilde{\Phi}(x_1, y_1) = 0;\quad \Phi(x_2, y_2) = \widetilde{\Phi}(x_2, y_2) = \Psi(x_2, y_2) = 0;$$
$$\widetilde{\Phi}(x_3, y_3) = 0;\quad \Phi(x_4, y_4) = \Psi(x_4, y_4) = 0.$$

It is not too difficult to detect that in this case we get a configuration with is equivalent to *Config.* $\mathcal{PE}.5$.

***2:*** *The possibility* $\widehat{\beta}_1 = 0$. Then due to the condition $\widehat{\gamma}_1 = 0$ we have $4 + 7g^2 - 4n = 0$ which gives us $n = (4 + 7g^2)/4$. This implies $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ and we arrive at the following 1-parameter family of systems:

$$\begin{aligned} \dot{x} &= -\frac{3g(2+g^2)}{8} + \frac{(2+7g^2)}{4}\,x - \frac{g}{2}\,y + gx^2 - xy,\\ \dot{y} &= -\frac{3g(2+g^2)}{4}\,x + \frac{4+7g^2}{2}\,y - x^2 + gxy - 2y^2 \end{aligned} \tag{25}$$

for which considering Diagram 2 we calculate:

$$\theta = -(9+g^2)/2, \quad \widehat{\beta}_2 = -g(25+g^2)/2, \quad \widehat{\beta}_6 = -5(25+9g^2)/8,$$
$$\widehat{\beta}_7 = -56(9+4g^2), \quad \widehat{\gamma}_4 = 1728(g-1)g(1+g)(1+g^2)^3(9+g^2)^2(1+25g^2). \tag{26}$$

We observe that the condition $\widehat{\beta}_2 = 0$ is equivalent to $g = 0$.

***2.1:*** *The case* $\widehat{\beta}_2 \neq 0$. Then we have $\theta \neq 0$, $\widehat{\beta}_1 = 0$, $\widehat{\beta}_6 \neq 0$, $\widehat{\beta}_2 \neq 0$, $\widehat{\beta}_7 \neq 0$. Therefore according to Diagram 2 systems (18) possess an invariant ellipse if and only if $\widehat{\gamma}_4 = 0$.

Thus considering (26) and the condition $\widehat{\beta}_2 \neq 0$ (i.e. $g \neq 0$) we obtain $g = \pm 1$. However we may assume $g = 1$ because system (25) with $g = -1$ via the transformation

$$x_1 = x - 1, \quad y_1 = -y + 5/2, \quad t_1 = -t$$

could be brought to the system (25) with $g = 1$.

So $g = 1$ and then we get the system

$$\dot{x} = -\frac{9}{8} + \frac{9}{4}x - \frac{1}{2}y + x^2 - xy, \quad \dot{y} = -\frac{9}{4}x + \frac{11}{2}y - x^2 + gxy - 2y^2, \tag{27}$$

which possesses the following three invariant conics:

$$\Phi(x,y) = x^2 - y = 0, \quad \widetilde{\Phi}(x,y) = -3 + 4x + 2x^2 + 2y,$$
$$\Psi = -3 + 16x + 16x^2 - 40y + 16y^2 = 0.$$

Since for the above system $B_1 = 4000 \neq 0$ by Lemma 3 we conclude that this system does not have any invariant line.

We determine that system (27) possesses four finite singularities $M_i(x_i, y_i)$ $(i = 1,2,3,4)$ with the coordinates

$$x_1 = -\frac{3}{2}, \; y_1 = \frac{9}{4}; \quad x_2 = \frac{1}{2}, \; y_2 = \frac{1}{4}; \quad x_3 = \frac{3}{2}, \; y_3 = \frac{9}{4}; \quad x_4 = -\frac{5}{2}, \; y_4 = \frac{1}{4}.$$

On the other hand calculations yield:

$$\Phi(x_1,y_1) = \widetilde{\Phi}(x_1,y_1) = \Psi(x_1,y_1) = 0; \quad \Phi(x_2,y_2) = \widetilde{\Phi}(x_2,y_2) = \Psi(x_2,y_2) = 0;$$
$$\widetilde{\Phi}(x_3,y_3) = 0; \quad \Phi(x_4,y_4) = 0.$$

So we observe that these three invariant conics have two common points: $M_1$ and $M_2$. It is not too difficult to detect that in this case we get the configuration *Config.*$\mathcal{PE}$*.6*.

It remains to point out that in this case we have $\widehat{\mathcal{R}}_3 = -41600 < 0$, i.e. the conditions provided by Diagram 2 are fulfilled.

***2.2:*** *The case* $\widehat{\beta}_2 = 0$. Then $g = 0$ and we get the system

$$\dot{x} = x(1-2y)/2, \quad \dot{y} = 2y - x^2 - 2y^2, \tag{28}$$

possessing two parabolas, one ellipse and one invariant line:

$$\Phi(x,y) = x^2 - y = 0, \quad \widetilde{\Phi}(x,y) = x^2 + y - 1 = 0, \quad \Psi(x,y) = x^2 - y + y^2, \quad x = 0.$$

On the other hand for this system we calculate

$$\theta = -9/2 \neq 0,\ \widehat{\beta}_1 = 0,\ \widehat{\beta}_6 = -125/8 \neq 0,\ \widehat{\beta}_2 = 0,\ \widehat{\beta}_4 = -4 \neq 0,\ \widehat{\gamma}_5 = 0,\ \widehat{\mathcal{R}}_2 = -5625/8 < 0,$$

and we conclude that the corresponding conditions provided by Diagram 2 are satisfied.

We determine that system (28) possesses four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$x_1 = -\frac{1}{\sqrt{2}},\ y_1 = \frac{1}{2};\quad x_2 = \frac{1}{\sqrt{2}},\ y_2 = \frac{1}{2};\quad x_3 = 0,\ y_3 = 0;\quad x_4 = 0,\ y_4 = 1.$$

On the other hand calculations yield:

$$\begin{aligned}
&\Phi(x_1, y_1) = \widetilde{\Phi}(x_1, y_1) = 0;\quad \Phi(x_2, y_2) = \widetilde{\Phi}(x_2, y_2) = 0;\\
&\Phi(x_3, y_3) = \Psi(x_3, y_3) = 0,\quad \widetilde{\Phi}(x_4, y_4) = \Psi(x_4, y_4) = 0.
\end{aligned}$$

We observe that the two invariant parabolas share two common points, denoted $M_1$ and $M_2$. In addition, each parabola intersects the invariant ellipse at one point. It is also worth noting that the invariant line $x = 0$ passes through the singular points $M_3$ and $M_4$; these points correspond to the points of tangency between the invariant ellipse and each of the hyperbolas.

It follows that, in this case, the resulting configuration is of type *Config.* $\mathcal{PE}.7$.

#### 3.2.1.3 The statement $(\boldsymbol{\mathcal{E}_3})$.

In this case by Lemma 1 we must have $\zeta_4 = \zeta_5 = 0$. So calculating for systems (10) the resultant of these two invariant polynomials we obtain:

$$\text{Res}_m(\zeta_4, \zeta_5) = 19(25 + g^2)^3(3g^2 - 4n)^2/64 = 0 \ \Leftrightarrow\ n = 3g^2/4.$$

Then we obtain that the invariant polynomials $\zeta_4$ and $\zeta_5$ have the common factor $6g + 9g^3 - 8m$ which we must force to be zero. So we get $m = 3g(2 + 3g^2)/8$ and we calculate:

$$\begin{aligned}
\widehat{\gamma}_1 =& 3(1 + g^2)^3(3g^2 - 1)(1 + 9g^2),\\
\widehat{\gamma}_2 =& 8100g(1 + g^2)^4(1 + 9g^2)(-109 + 1083g^2).
\end{aligned}$$

It is clear that these two polynomials in $g$ cannot have a common real root. Consequently, if the conditions stated in $(\boldsymbol{\mathcal{E}_3})$ of Lemma 1 are satisfied, then the systems (10) do not possess any invariant ellipse.

#### 3.2.1.4 The statement $(\boldsymbol{\mathcal{E}_4})$.

In this case we must have $\zeta_4 = \mathcal{R}_7 = 0$. So calculating for systems (10) the resultant of these two invariant polynomials we obtain:

$$\text{Res}_m(\zeta_4, \mathcal{R}_7) = 2015(25 + g^2)^2(1 + 3g^2)(2 + 5g^2 - 4n)/2 = 0 \ \Leftrightarrow\ n = (2 + 5g^2)/4.$$

Then we determine that the invariant polynomials $\zeta_4$ and $\zeta_5$ have the common factor $3g^3 - 8m$ which we must force to be zero. So we get $m = 3g^3/8$ and we calculate:

$$\begin{aligned}
\widehat{\gamma}_1 =& 3(1 + g^2)^3(3g^2 - 1)(1 + 9g^2)/8,\\
\widehat{\gamma}_2 =& 21600g(1 + g^2)^4(1 + 9g^2)(-6 + 25g^2).
\end{aligned}$$

Evidently, these two polynomials in $g$ cannot have a common real root. Consequently, if the conditions stated in $(\boldsymbol{\mathcal{E}_4})$ of Lemma 1 are satisfied, then the systems (10) do not possess any invariant ellipse.

So we have proved the following theorem.

**Theorem 1.** *Consider a quadratic system possessing a finite number of singularities (both finite and infinite). This system admits the following configurations if and only if* $\eta < 0$, $\zeta_1 \neq 0$, $\chi_1 = \chi_2 = \widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$, *and one of the corresponding sets of conditions is satisfied, respectively:*

$\zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0,\ B_1 \neq 0,\ \Theta_1 \neq 0 \Rightarrow$ *Config.* $\mathcal{PE}$*.1;*
$\zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0,\ B_1 \neq 0,\ \Theta_1 = 0 \Rightarrow$ *Config.* $\mathcal{PE}$*.2;*
$\zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0,\ B_1 = 0,\ \Theta_2 \neq 0 \Rightarrow$ *Config.* $\mathcal{PE}$*.3;*
$\zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0,\ B_1 = 0,\ \Theta_2 = 0 \Rightarrow$ *Config.* $\mathcal{PE}$*.4;*
$\zeta_4 = 0,\ \mathcal{R}_7\zeta_5 \neq 0,\ \widehat{\beta}_1 \neq 0 \Rightarrow$ *Config.* $\mathcal{PE}$*.5;*
$\zeta_4 = 0,\ \mathcal{R}_7\zeta_5 \neq 0,\ \widehat{\beta}_1 = 0,\ \widehat{\beta}_2 \neq 0,\ \widehat{\gamma}_4 = 0 \Rightarrow$ *Config.* $\mathcal{PE}$*.6;*
$\zeta_4 = 0,\ \mathcal{R}_7\zeta_5 \neq 0,\ \widehat{\beta}_1 = 0,\ \widehat{\beta}_2 = 0 \Rightarrow$ *Config.* $\mathcal{PE}$*.7.*

### 3.2.2 The possibility $C_2 = 0$

In this case, the line at infinity of system (3) is entirely filled with singularities. According to Lemma 2, if this system possesses an invariant parabola, it can be brought, via an affine transformation and time rescaling, to the following canonical form:

$$\dot{x} = m + nx - y/2 + x^2, \quad \dot{y} = 2mx + 2ny + xy \tag{29}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y$.

In what follows, we determine the conditions for the existence of at least one invariant ellipse in systems of the form (29). First, we note that for these systems, $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$; hence, the necessary conditions for the existence of an invariant ellipse are satisfied.

Next, following the approach in [18], we evaluate the following invariant polynomials:

$$H_{10} = 9, \quad H_9 = 576(2m - n^2)^3, \quad H_{12} = -2(2m - n^2)(8mx^2 - 16n^2x^2 - 12nxy - 3y^2). \tag{30}$$

Since $H_{10} \neq 0$, it follows from Diagram 3 that systems (29) can possess invariant ellipses if and only if$N_7 = 0$. For these systems we indeed obtain $N_7 = 0$; therefore, according to Diagram 3, systems (29) possess an infinite number of real invariant ellipses.

On the other hand, since $H_{10} \neq 0$, it follows from [18] (see Table 1) that a system from the family (29) (with $C_2 = 0$) possesses three real invariant lines if $H_9 < 0$, and one real and two complex invariant lines if $H_9 > 0$.

Considering (30), we observe that the condition $H_9 = 0$ implies $H_{12} = 0$. According to [18, Table 1], in this case a system (29) possesses a triple invariant affine line. Consequently, we distinguish three cases: $H_9 < 0$, $H_9 > 0$ and $H_9 = 0$.

**3.2.2.1 The case $H_9 < 0$.** Then $2m - n^2 < 0$ and we set $2m - n^2 = -z^2 < 0$, where $z$ is a new parameter. So we have $m = (n^2 - z^2)/2$ and this leads to the family of systems

$$\dot{x} = (n^2 - z^2)/2 + nx - y/2 + x^2, \quad \dot{y} = (n^2 - z^2)x + 2ny + xy \tag{31}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y$ and the following three invariant lines:

$$L_1 : n^2 - z^2 + 2nx + y = 0; \quad L_{2,3} : (n \pm z)^2 + 2(n \pm z)x + y = 0.$$

Moreover systems (31) possess the family of conics (depending on the parameter $w$)

$$F_w(x, y) = w(x^2 - y) + (n^2 - z^2 + 2nx + y)^2 = 0$$

for which considering Remark 1 we calculate

$$\Delta = -w^2(w + 4z^2)/4, \quad \delta = w.$$

According to this remark, a conic from the above family is irreducible if and only if $\Delta \neq 0$. Moreover, an irreducible conic is a hyperbola if $\delta > 0$, an ellipse if $\delta < 0$, and a parabola if $\delta = 0$. However, the condition $\delta = 0$ implies $\Delta = 0$; therefore, a parabola cannot belong to the family $F_w(x, y) = 0$.

Thus we have the following irreducible invariant conics defined by $F_w(x, y) = 0$:

- $w(w + 4z^2) \neq 0$ and $w < 0 \Rightarrow$ a family of ellipses;
- $w(w + 4z^2) \neq 0$ and $w > 0 \Rightarrow$ a family of hyperbolas.

We determine that systems (31) possess three real finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3$ with the coordinates

$$x_1 = -n, \ y_1 = (n^2 - z^2); \quad x_{2,3} = -(n \pm z), \ y_{2,3} = (n \pm z)^2$$

and we calculate

$$L_2(x_1, y_1) = L_3(x_1, y_1) = 0, \quad \Phi(x_2, y_2) = L_1(x_2, y_2) = L_2(x_2, y_2) = 0;$$
$$\Phi(x_3, y_3) = L_1(x_3, y_3) = L_3(x_3, y_3) = 0.$$

Moreover for any value of the parameter $w$ for the family of invariant conics $F_w(x, y) = 0$ we have

$$F_w(x_2, y_2) = F_w(x_3, y_3) = 0,$$

i.e. the points $M_2$ and $M_3$ are the intersections of all conics belonging to the family $F_w(x, y) = 0$. Furthermore, the singular point $M_2$ (respectively $M_3$) is the point of tangency of the invariant line $L_2(x, y) = 0$ (respectively $L_3(x, y) = 0$) and the invariant parabola $\Phi(x, y) = 0$. Moreover, every invariant conic in the family $F_w(x, y) = 0$ is tangent to the invariant parabola at these two singular points, $M_2$ and $M_3$.

Therefore, taking into account that the line at infinity is entirely filled with singularities, in the case $H_9 < 0$ we obtain the configuration *Config.* $\mathcal{PE}.8$.

#### 3.2.2.2 The case $H_9 > 0$.

Then $2m - n^2 > 0$ and we set $2m - n^2 = z^2 > 0$, where $z$ is a new parameter. So we have $m = (n^2 + z^2)/2$ and this leads to the family of systems

$$\dot{x} = (n^2 + z^2)/2 + nx - y/2 + x^2, \quad \dot{y} = (n^2 + z^2)x + 2ny + xy \tag{32}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y$ and the following three invariant lines:

$$L'_1 : n^2 + z^2 + 2nx + y = 0; \ \ L'_{2,3} : (n \pm iz)^2 + 2(n \pm iz)x + y = 0.$$

Moreover systems (32) possess the family of conics depending on the parameter $w$:

$$F'_w(x, y) = w(x^2 - y) + (n^2 - z^2 + 2nx + y)^2 = 0$$

for which considering Remark 1 we calculate

$$\Delta = -w^2(w - 4z^2)/4, \quad \delta = w.$$

Therefore we have the following invariant conics defined by $F'_w(x, y) = 0$:

- $w(w - 4z^2) \neq 0$ and $w < 0 \Rightarrow$ a family of ellipses;
- $w(w - 4z^2) \neq 0$ and $w > 0 \Rightarrow$ a family of hyperbolas.

We find that systems (32) possess only one real singular point, $M_1(-n, n^2 + z^2)$, which is the intersection of the complex invariant lines. Considering the invariant parabola $\Phi(x, y) = x^2 - y$, we observe that $\Phi(x_1, y_1) = -z^2 < 0$. Hence, $M_1$ lies within the domain bounded by this parabola and its point at infinity. Moreover, this singular point is surrounded by a family of real ellipses contained within this domain.

It follows that, in this case, the system possesses the configuration *Config.* $\mathcal{PE}$*.9*.

***3:*** *The possibility* $H_9 = 0$. This implies $m = n^2/2$ and we get the family of systems

$$\dot{x} = n^2/2 + nx - y/2 + x^2, \quad \dot{y} = n^2x + 2ny + xy \tag{33}$$

for which $H_{10} = 9 \neq 0$ and $H_9 = H_{12} = 0$. Therefore according to [18, Table 1] the above systems possess a triple invariant affine line $L''_1 : n^2 + 2nx + y = 0$ and a triple real singular point $M_1(-n, n^2)$.

On the other hand beside the invariant parabola $\Phi(x, y) = x^2 - y$ systems (33) possess the following family of the invariant conics

$$F''_w(x, y) = w(x^2 - y) + (n^2 + 2nx + y)^2 = 0$$

for which considering Remark 1 we calculate

$$\Delta = -w^3/4, \quad \delta = w.$$

Thus, according to Remark 1, for $w < 0$ we obtain a family of ellipses, whereas for $w > 0$ the conics $F''_w(x, y) = 0$ are hyperbolas.

Considering the coordinates $x_1 = -n$ and $y_1 = -n^2$ of the triple singular point $M_1$ we compute:

$$L''_1(x_1, y_1) = \Phi(x_1, y_1) = 0, \quad F''_w(x_1, y_1) = 0, \ \forall w \in \mathbb{R}.$$

So we conclude that $M_1$ is a common point of all the invariant conics and, moreover, a point of tangency between these invariant conics and the invariant line. Hence, in this case, we obtain the configuration *Config.* $\mathcal{PE}$*.10.*

Thus we we establish the following theorem.

**Theorem 2.** *Assume that for a non-degenerate quadratic system the conditions* $C_2 = 0$*,* $\zeta_5 \neq 0$ *and* $\zeta_{22} = 0$ *are satisfied. Then this system possess one invariant parabola and a family of non-parabolic invariant conics as well as invariant lines of total multiplicity three. More exactly beside the invariant parabola this system possesses a subfamily of invariant ellipses and a subfamily of invariant ellipses, and*

- *three real invariant lines if* $H_9 < 0 \;\Rightarrow\;$ *Config.* $\mathcal{PE}$*.8;*
- *one real and two complex invariant lines if* $H_9 > 0 \;\Rightarrow\;$ *Config.* $\mathcal{PE}$*.9;*
- *one triple real invariant line if* $H_9 = 0 \;\Rightarrow\;$ *Config.* $\mathcal{PE}$*.10.*

Thus, taking into account Theorems 1 and 2, together with Proposition 1 and Diagrams 2 and 3, we can construct Diagram 1, which summarizes the obtained results. According to this diagram, one can determine whether an arbitrary quadratic system belongs to the class **QSPE**; if it does, the corresponding configuration of this system can then be identified.

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