Family of quadratic differential systems with two invariant conics: parabola and hyperbola

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Abstract

Consider the class **QS** of all non-degenerate planar quadratic differential systems and its subclass **QSPH** of all its systems possessing simultaneously an invariant parabola and an invariant hyperbola. We investigate all possible configurations of these two invariant conics and invariant straight lines that systems in **QSPH** could possess and their geometric properties encoded in such configurations. The classification presented here is taken modulo the action of the group of real affine transformations and time rescaling and it is given in terms of affine invariant polynomials. It yields a total of 43 distinct configurations. The obtained classification is an algorithm which makes it possible for any given real quadratic differential system in **QSPH** to specify its configuration of invariant parabolas and hyperbolas and straight lines. This work will prove helpful in studying the integrability of the systems in **QSPH**.

Key-words: Quadratic vector fields, affine invariant polynomials, invariant algebraic curves, invariant parabolas and hyperbolas.

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1 Introduction and statement of main results

We consider here differential systems of the form

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where P, $Q \in \mathbb{R}[x,y]$, i.e. P, Q are polynomials in x, y over $\mathbb{R}$ and their associated vector fields

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

We call *degree* of a system (1) the integer $m = \max(\deg P, \deg Q)$. In particular we call *quadratic* a differential system (1) with $m = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

Quadratic systems appear in the modelling of many natural phenomena described in different branches of science, in biological and physical applications and applications of these systems became a subject of interest for the mathematicians. Many papers have been published about quadratic systems, see for example [14] for a bibliographical survey.

Here we always assume that the polynomials P and Q are coprime. Otherwise doing a rescaling of the time, systems (1) can be reduced to linear or constant systems. Quadratic systems under this assumption are called *non–degenerate quadratic systems.*

Let V be an open and dense subset of $\mathbb{R}^2$, we say that a non-constant differentiable function $H : V \to \mathbb{R}$ is a first integral of a system (1) on V if $H(x(t), y(t))$ is constant for all of the values of t for which $(x(t), y(t))$ is a solution of this system contained in V. Obviously H is a first integral of systems (1) if and only if

$$X(H) = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = 0,$$

for all $(x, y) \in V$. When a system (1) has a first integral we say that this system is integrable.

The knowledge of the first integrals is of particular interest in planar differential systems because they allow us to draw their phase portraits.

On the other hand given $f \in \mathbb{C}[x, y]$ we say that the curve $f(x, y) = 0$ is an *invariant algebraic curve* of systems (1) if there exists $K \in \mathbb{C}[x, y]$ such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf.$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. When $K = 0$, f is a polynomial first integral.

The motivation for studying the systems in the quadratic class is not only because of their usefulness in many applications but also for theoretical reasons, as discussed by Schlomiuk and Vulpe in the introduction of [15]. The study of non–degenerate quadratic systems could be done using normal forms and applying the invariant theory.

Systematic work on quadratic differential systems possessing an invariant conic began towards the end of the XX-th century and the beginning of this century. Quadratic systems having an invariant ellipse as a limit cycle were investigated by Qin Yuan-xum [8]; the necessary and sufficient conditions on the coefficients of a quadratic system and also on the coefficients of a conic so as to have the conic as an invariant curve of the system were presented by Druzhkova [5]; Cairó and Llibre in [3] have investigated the Darboux integrability of the quadratic systems having invariant algebraic conics; Christopher in [4] considered the class of quadratic systems having a parabola as an integral curve; Oliveira, Rezende and Vulpe [9] provided necessary and sufficient conditions for a system in **QS** to have at least one invariant hyperbola in terms of its coefficients and the necessary and sufficient affine invariant conditions for a system in **QS** so as to have the ellipse as an invariant curve of the system were presented by Oliveira, Rezende, Schlomiuk and Vulpe [11]. In [10] the authors classified the family of quadratic systems possessing an invariant hyperbola in terms of configurations of hyperbolas and presence or absence of invariant lines. This is an invariant classification, independent of specific normal forms. A similar classification in the case of an invariant ellipse is done in [12]. The necessary and sufficient affine invariant conditions for a system in **QS** so as to have the parabola as an invariant curve of the system were presented by Vulpe [20].

In this work we consider non-degenerate quadratic differential systems possessing simultaneously an invariant parabola and an invariant hyperbola. We denote this family by **QSPH**. Our goal in this paper is to obtain a characterization of systems in **QSPH** in terms of invariant polynomials. Thus our equalities and inequalities in the bifurcation diagram splitting the parameter space into regions and subsets with distinct dynamics, will not be expressed in terms of coefficients of a fixed normal form or several such forms, coefficients which do not have any geometrical meaning and are rigidly tied to these normal forms. They will be expressed in terms of invariant polynomials which are very supple objects that can be easily computed by a computer for any specific normal form and allowing us also to easily pass from one normal form to any other.

On the other hand since number of the invariant curve increases we could obtain new classes of quadratic systems which are Darboux integrable and in particular rational integrable.

Our main results are stated in the following theorem.

Main Theorem. ***(A)*** *The condition $\chi_1 = \chi_2 = \gamma_1 = \gamma_2 = 0$ is necessary for a non-degenerate quadratic system to possess at least one invariant parabola and at least one invariant hyperbola.*

(B) *Assume that for a non-degenerate quadratic system (S) the condition $\chi_1 = \chi_2 = \gamma_1 = \gamma_2 = 0$ holds. Then this system possesses simultaneously one invariant parabola and one invariant hyperbola generating a configuration of invariant conics and invariant lines if and only if the corresponding conditions indicated below are satisfied, respectively:*

- **$\boldsymbol{\alpha}$)** *For $\eta > 0$ the system (S) could possess one of the configurations Config. $\mathcal{PH}$.1–Config. $\mathcal{PH}$.38 given in Figure 1 if and only if the corresponding conditions provided by Diagram 1 are satisfied.*

- **$\boldsymbol{\beta}$)** *For $\eta = 0$ the system (S) could possess one of the configurations Config. $\mathcal{PH}$.39–Config. $\mathcal{PH}$.43 given in Figure 1 if and only if the corresponding conditions provided by Diagram 2 are satisfied.*

(C) *We construct the bifurcation diagram in the space $\mathbb{R}^{12}$ of the coefficients of the system in **QSPH** pictured in Diagrams 1 and 2, according to their configurations of invariant parabolas, hyperbolas and invariant straight lines. Moreover, these diagrams give an algorithm to compute the configuration of a system with an invariant parabola for any quadratic differential system, presented in any normal form.*

(D) *In Figure 1 are given all the configurations that could occur for systems in the class* **QSPH**. *We prove that all these configurations are realizable within* **QSPH** *and that these 43 configurations are distinct. This proof is done using geometric and numeric invariants and it is presented in the diagram Diagram 5.*

The article is organized as follows: In Section 2 we present some preliminary results involving the use of affine invariant polynomials. More exactly we present the invariant criteria for the existence of an invariant hyperbola determined in [10] as well as for the existence of an invariant parabola constructed in [20]. Moreover in the second case we have improved the results obtained in [20] (compare Proposition 1 with Proposition 1* on page 11). In the same Section 2 we present the necessary invariant polynomials associated with configurations of invariant conics.

In Section 3 we give the proof of the Main Theorem.

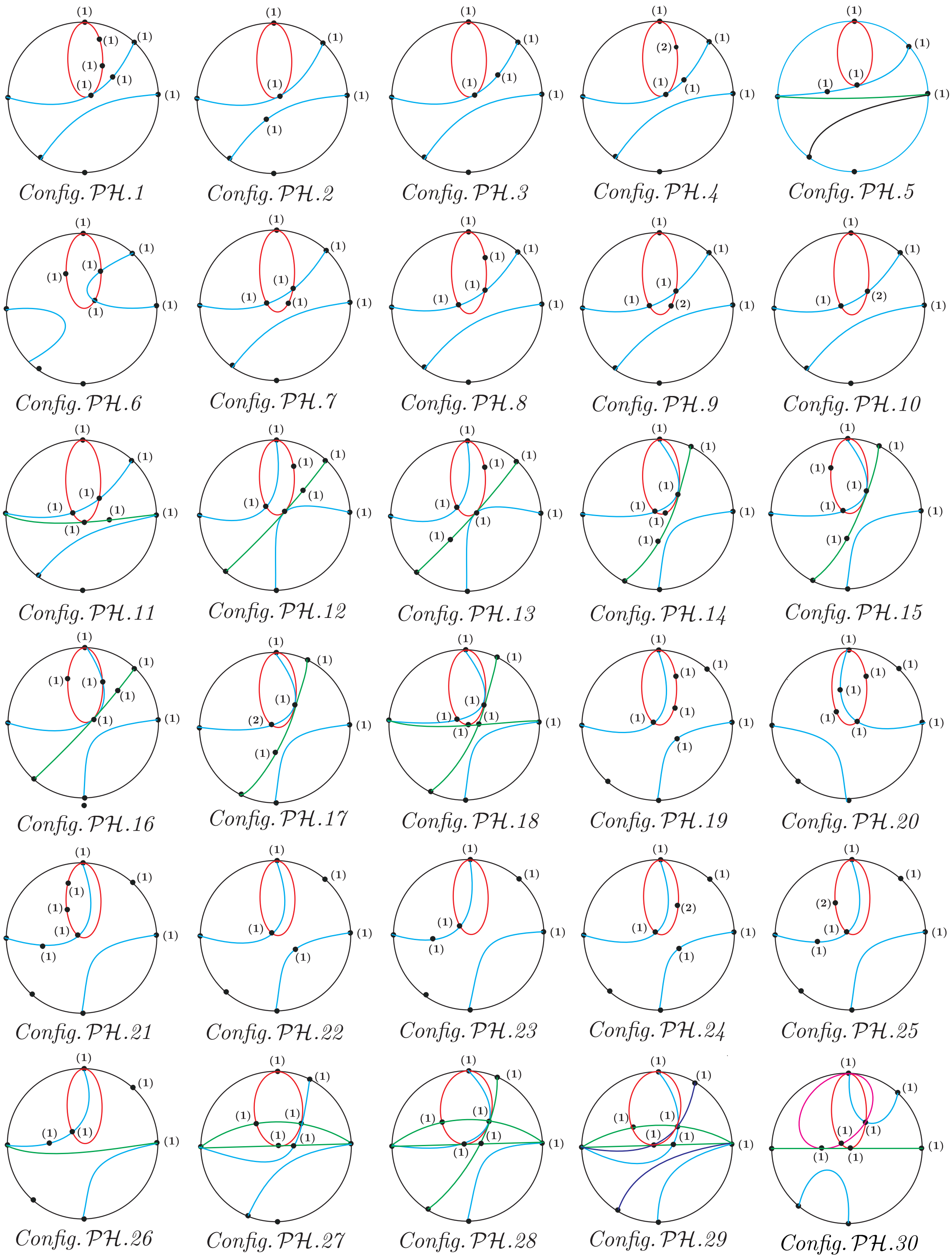

Figure 1: Configurations of quadratic systems in **QSPH**

2 Some preliminary results

Assume that a conic

$$\Phi(x, y) \equiv p + qx + ry + sx^2 + 2vxy + uy^2 = 0 \tag{2}$$

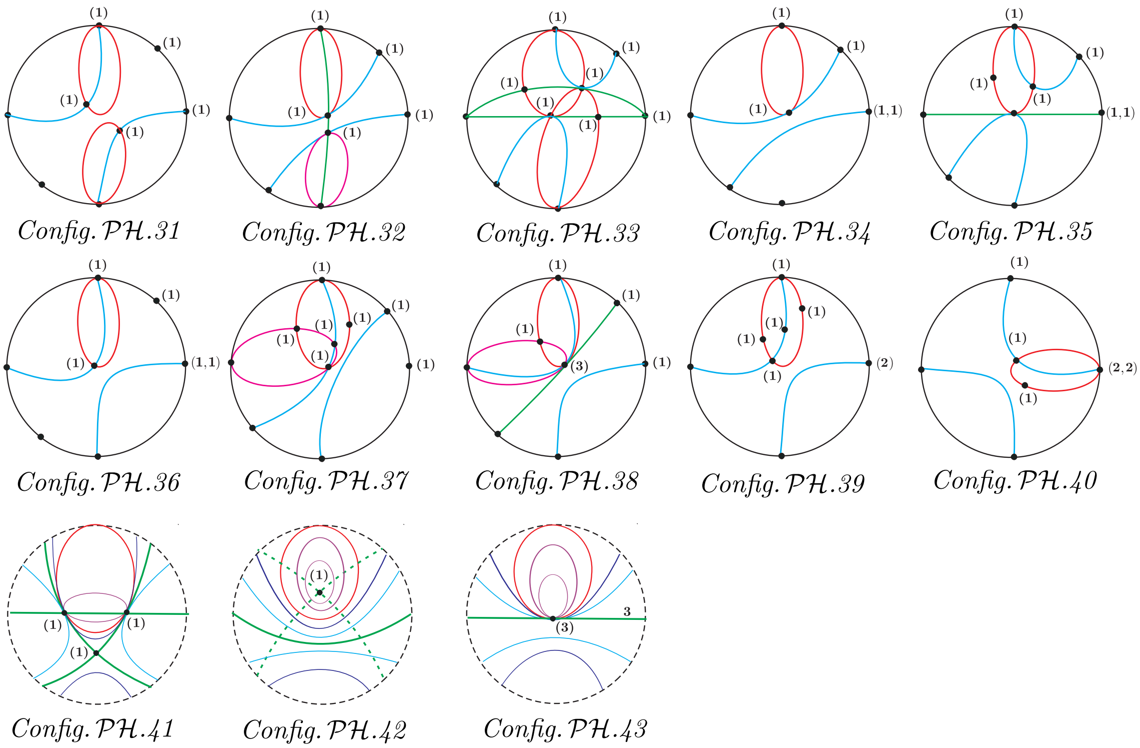

Figure 1 (*cont.*) Configurations of quadratic systems in **QSPH**

is an affine algebraic invariant curve for quadratic systems, which we rewrite in the form:

$$\begin{aligned}\frac{dx}{dt} &= a + cx + dy + gx^2 + 2hxy + ky^2 \equiv P(x,y),\\ \frac{dy}{dt} &= b + ex + fy + lx^2 + 2mxy + ny^2 \equiv Q(x,y)\end{aligned} \tag{3}$$

and we denote by $\tilde{a} = (a, b, c, d, g, h, c, l, m, n, n)$ the set of its coefficients.

Remark 1. *Following [7] we construct the determinant*

$$\Delta = \begin{vmatrix} s & v & q/2 \\ v & u & r/2 \\ q/2 & r/2 & p \end{vmatrix},$$

associated to the conic (2)*. By [7] this conic is irreducible (i.e. the polynomial* Φ *defining the conic is irreducible over* $\mathbb{C}$*) if and only if* $\Delta \neq 0$*. Moreover an irreducible conic is an ellipse (respectively, hyperbola; parabola) if* $\delta < 0$ *(respectively* $\delta > 0$*;* $\delta = 0$*) where* $\delta = v^2 - su$*. Furthermore if the conic* (2) *is an ellipse (i.e.* $\delta < 0$*) then it is real for* $(u+s)\Delta < 0$ *and it is imaginary for* $(u+s)\Delta > 0$*.*

2.1 Results involving the use of polynomial invariants

As it was mentioned earlier, in [9] the necessary and sufficient conditions for a system in **QS** to have at least one invariant hyperbola in terms of its coefficients are provided. More exactly in [9] the following 54 affine invariant polynomials that are responsible for the existence of at least one

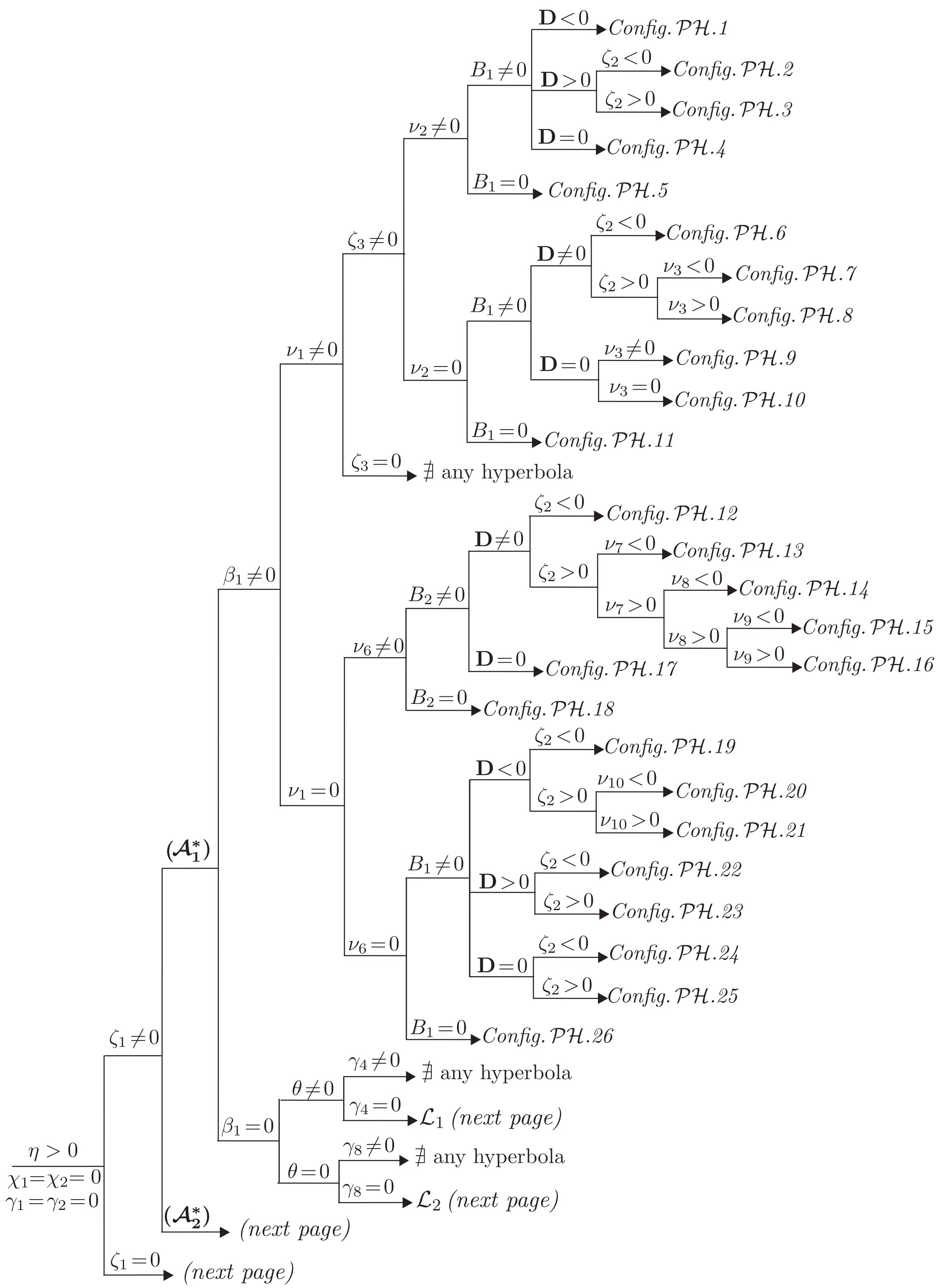

Diagram 1: Conditions for configurations for systems in **QSPH**: the case $\eta > 0$

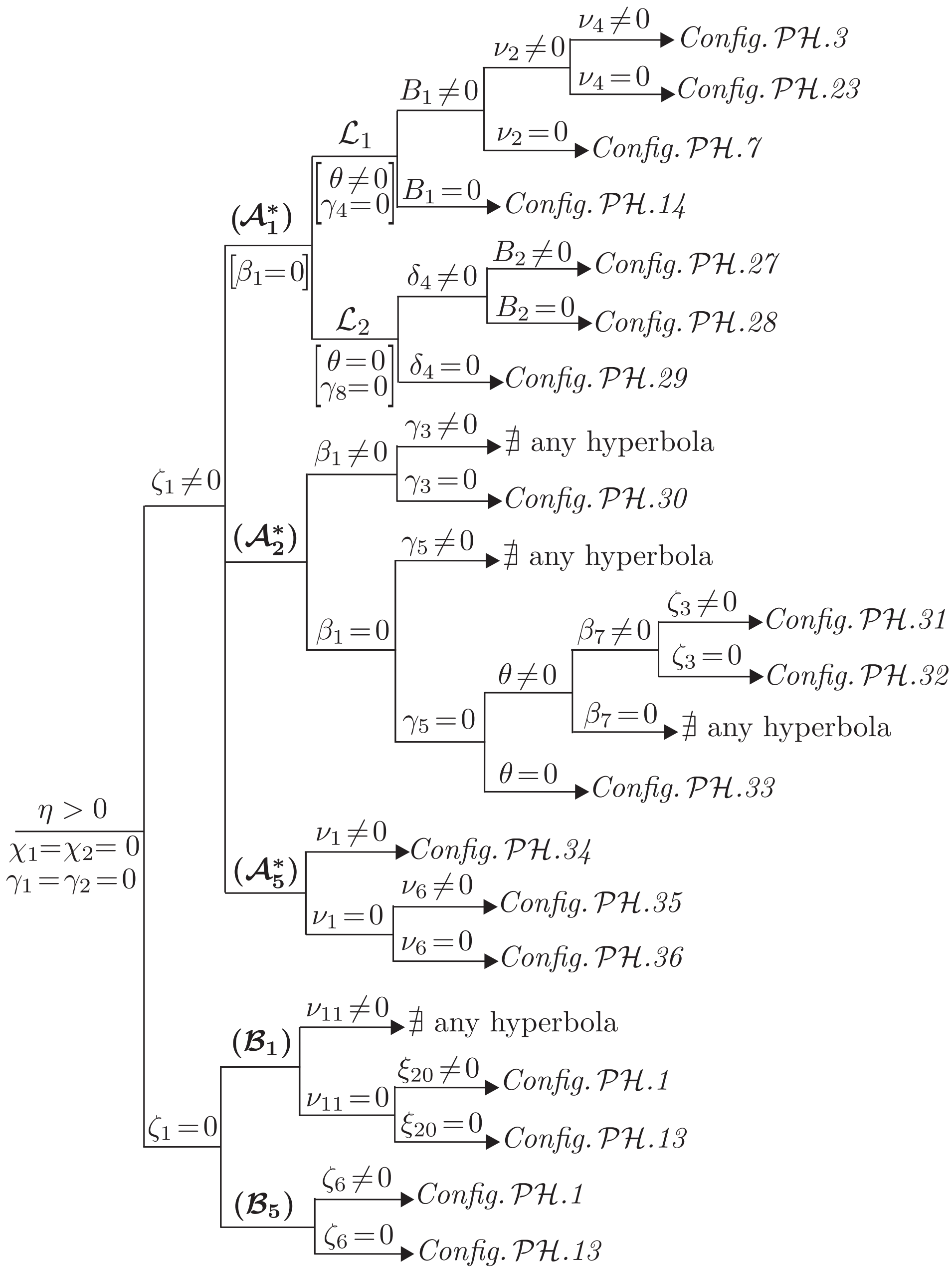

Diagram 1 (*cont.*) Conditions for configurations for systems in **QSPH**: the case $\eta > 0$

invariant hyperbola are defined:

$$\eta,\ M, C_2,\ \theta,\ N,\ N_7,\ \beta_1 - \beta_{13},\ \gamma_1 - \gamma_{19},\ \delta_1 - \delta_5,\ \widetilde{\mathcal{R}}_1 - \widetilde{\mathcal{R}}_{11}. \tag{4}$$

The expressions for the invariant polynomials (4) can be found in [9].

Remark 2. We point out that instead of the notations $\widetilde{\mathcal{R}}_i$ $(i = 1, \dots, 11)$ given in (4), in article [9] there are used the notations $\mathcal{R}_i$ $(i = 1, \dots, 11)$, correspondingly . However we add here tilde ' $\widetilde{\ }$' in order to distinguish from the the invariant polynomials $\mathcal{R}_i$ $(i = 1, \dots, 7)$ defined in [20]

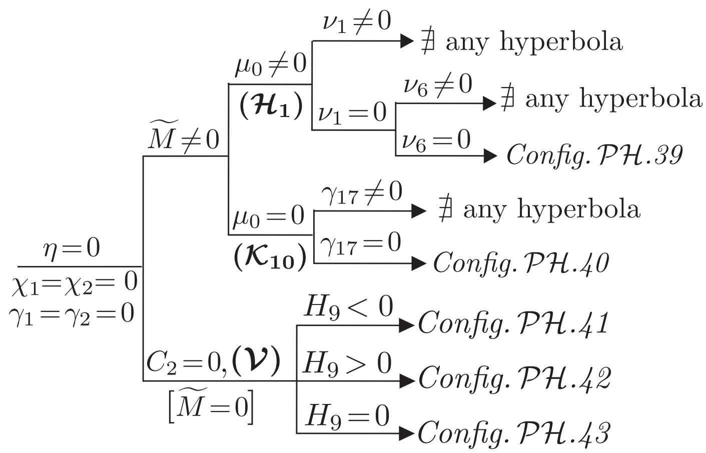

Diagram 2: Conditions for configurations for systems in **QSPH**: the case $\eta = 0$

(see the set (5) below). So in Diagrams 3 and 4 the names $\mathcal{R}_i$ $(i = 1, \ldots, 11)$ are also substituted by $\widetilde{\mathcal{R}}_i$.

According to [9] we have the following theorem.

Theorem 1. *(A) The conditions $\gamma_1 = \gamma_2 = 0$ and either $\eta \geq 0$, $M \neq 0$ or $C_2 = 0$ are necessary for a quadratic system in the class **QS** to possess at least one invariant hyperbola.*

*(B) Assume that for a system in the class **QS** the condition $\gamma_1 = \gamma_2 = 0$ is satisfied.*

1. *If $\eta > 0$ then the necessary and sufficient conditions for this system to possess at least one invariant hyperbola are given in Diagram 3, where we can also find the number and multiplicity of such hyperbolas.*

2. *In the case $\eta = 0$ and either $M \neq 0$ or $C_2 = 0$ the corresponding necessary and sufficient conditions for this system to possess at least one invariant hyperbola are given in Diagram 4, where we can also find the number and multiplicity of such hyperbolas.*

Remark 3. An invariant hyperbola is denoted by $\mathcal{H}$ if it is real and by $\overset{c}{\mathcal{H}}$ if it is complex. In the case we have two such hyperbolas then it is necessary to distinguish whether they have parallel or non-parallel asymptotes in which case we denote them by $\mathcal{H}^p$ $(\overset{c}{\mathcal{H}}{}^p)$ if their asymptotes are parallel and by $\mathcal{H}$ if there exists at least one pair of non-parallel asymptotes. We denote by $\mathcal{H}_k$ $(k = 2, 3)$ a hyperbola with multiplicity k; by $\mathcal{H}_2^p$ a double hyperbola, which after perturbation splits into two $\mathcal{H}^p$; and by $\mathcal{H}_3^p$ a triple hyperbola which splits into two $\mathcal{H}^p$ and one $\mathcal{H}$.

Next we mention that in [20] the necessary and sufficient affine invariant conditions for a quadratic systems to possess at least one invariant parabola are determined. To give these invariant criteria the following 43 affine invariant polynomials are provided:

$$\eta,\ M, C_2,\ \mu_0 - \mu_4,\ \chi_1 - \chi_4,\ \zeta_1 - \zeta_{24},\ \mathcal{R}_1 - \mathcal{R}_7. \tag{5}$$

The expressions for the above invariant polynomials can be found in [20].

Following [20] we shall use the next definition.

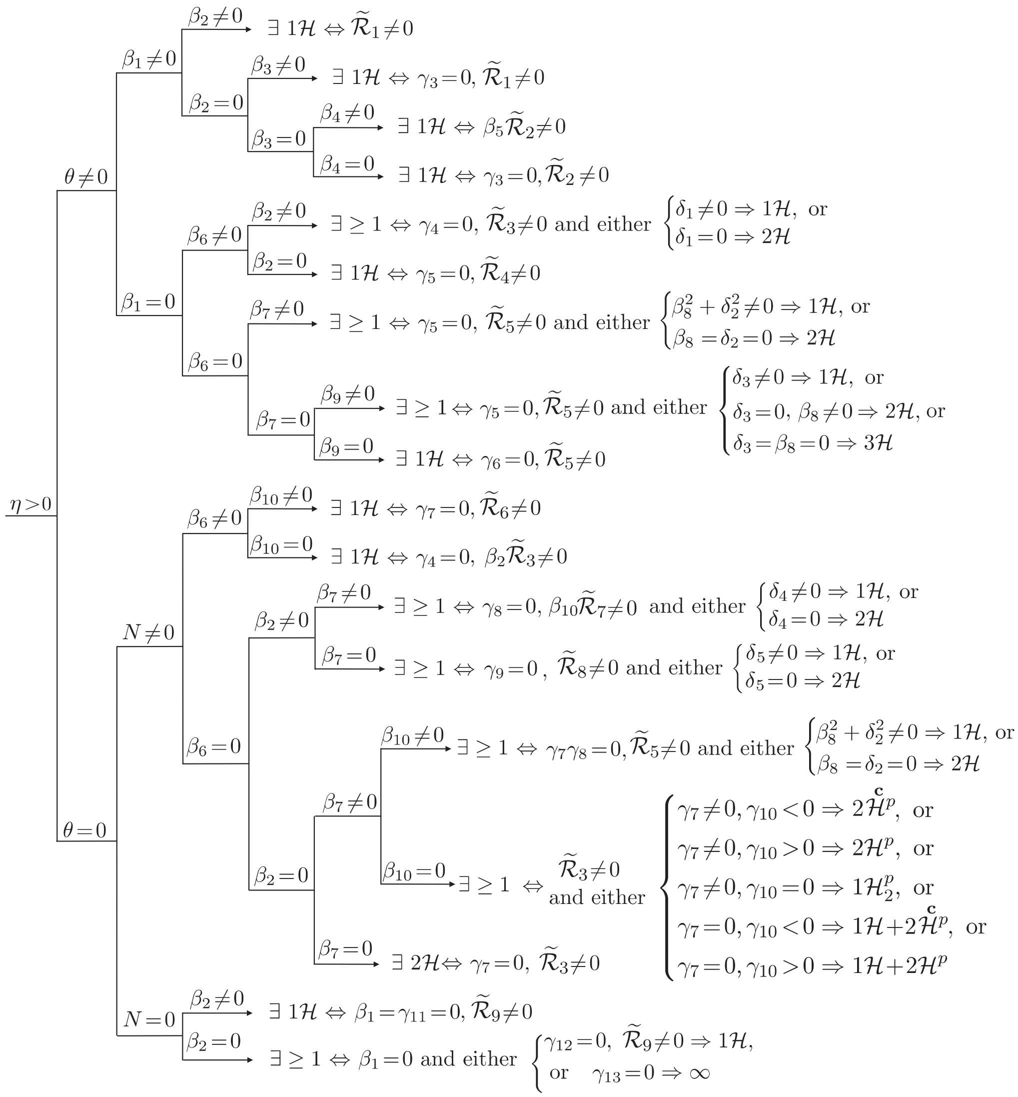

Diagram 3: Existence of invariant hyperbolas: the case $\eta>0$

Definition 1. *We say that we have an invariant parabola in a simple (respectively double; triple) direction if the infinite singular point of this parabola is defined by a simple (respectively double; triple) factor in the factorization of the invariant polynomial $C_2(x,y)$ over $\mathbb{C}$. We denote this parabola by $\bigcup$ (respectively $\overset{2}{\bigcup}$; $\overset{3}{\bigcup}$). Moreover if the infinite invariant line $Z=0$ is filled up with singularities then we denote by $\overset{\infty}{\bigcup}$ the invariant parabola which is tangent to the line $Z=0$ at a non-isolated singular point.*

According to [20] in order to describe the invariant parabolas that a quadratic system could have we use the following notations:

- $\bigcup$ for a simple invariant parabola;
- $\Cup$ for two simple invariant parabolas in the same direction (they could intersect or be tangent

- $\eta=0$
  - $M\neq 0$
    - $\theta\neq 0$
      - $\beta_2\neq 0$
        - $\beta_1\neq 0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \widetilde{\mathcal{R}}_1\neq 0$
        - $\beta_1=0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \gamma_4=0,\ \widetilde{\mathcal{R}}_3\neq 0:\ \begin{cases}\delta_1\neq 0 \Rightarrow 1\mathcal{H};\\ \delta_1=0 \Rightarrow 1\mathcal{H}_2\end{cases}$
      - $\beta_2=0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \beta_1=\gamma_{14}=0,\ \widetilde{\mathcal{R}}_{10}\neq 0:\ \begin{cases}\beta_7\beta_8\neq 0 \Rightarrow 1\mathcal{H};\\ \beta_7\beta_8=0 \Rightarrow 1\mathcal{H}_2\end{cases}$
    - $\theta=0$
      - $N\neq 0$
        - $\beta_4\neq 0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \beta_3=\gamma_8=0,\ \widetilde{\mathcal{R}}_7\neq 0:\ \begin{cases}\delta_4\neq 0 \Rightarrow 1\mathcal{H};\\ \delta_4=0 \Rightarrow 1\mathcal{H}_2\end{cases}$
        - $\beta_4=0$ → $\exists\geq 1 \Leftrightarrow \beta_6=0,\ \beta_{11}\widetilde{\mathcal{R}}_{11}\neq 0$ and either $\begin{cases}\beta_{12}\neq 0,\ \gamma_{15}=0:\ \begin{cases}\gamma_{16}^2+\delta_6^2\neq 0 \Rightarrow 1\mathcal{H};\\ \gamma_{16}=\delta_6=0 \Rightarrow 1\mathcal{H}_2\end{cases} \text{or}\\ \beta_{12}=\gamma_{16}=0,\ \gamma_{17}<0 \Rightarrow 2\mathcal{H}^p, \text{ or}\\ \beta_{12}=\gamma_{16}=0,\ \gamma_{17}>0 \Rightarrow 2\overset{\mathbf{c}}{\mathcal{H}}{}^p, \text{ or}\\ \beta_{12}=\gamma_{16}=\gamma_{17}=0 \Rightarrow 1\mathcal{H}_3^p\end{cases}$
      - $N=0$
        - $\beta_{13}\neq 0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \gamma_{10}=\gamma_{17}=0,\ \widetilde{\mathcal{R}}_{11}\neq 0$
        - $\beta_{13}=0$ → $\exists\ \infty \Leftrightarrow \gamma_9=\tilde{\gamma}_{18}=\tilde{\gamma}_{19}=0$
  - $C_2=0$ $(M=0)$ → $\exists\ \infty \Leftrightarrow N_7=0$

Diagram 4: Existence of invariant hyperbolas: the case $\eta=0$

to the Poincare circle at diametrically opposite points);

- $\cup\subset$ for two simple invariant parabolas in different directions;
- $\boldsymbol{\cup}^2$ for one double invariant parabola;
- $\overset{2}{\cup}$ for one simple invariant parabola in double direction;
- $\overset{3}{\cup}$ for one simple invariant parabola in triple direction;
- $\overset{\infty}{\cup}$ for one simple invariant parabola when the line at infinity is filled up with singularities;
- $\overset{2}{\cup}\subset$ for two simple invariant parabola: one in a double direction and one in a simple direction;
- $\mathbb{U}\subset$ for three simple invariant parabolas: two in one direction and one in another direction;
- $\overset{2}{\cup}\overset{2}{\mathbb{U}}$ for three real invariant parabolas in a double direction;
- $\overset{2}{\cup}\overset{2}{\mathbb{U}}{}^c$ for one real and two complex invariant parabolas in a double direction;
- $\overset{2}{\cup}\overset{2}{\boldsymbol{\cup}}{}^2$ for one simple and one double real invariant parabolas in a double direction;
- $\overset{2}{\boldsymbol{\cup}}{}^3$ for one triple real invariant parabola in a double direction;

- $\infty\overset{2}{\cup}$ for an 1-parameter family of invariant parabolas in a double direction;
- $\infty\overset{3}{\cup}$ for an 1-parameter family of invariant parabolas in a triple direction.

The proof of the next proposition could be found in [20].

Proposition 1. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta > 0$ and $\zeta_1 \neq 0$ are satisfied. Then this system could possess invariant parabolas only in one direction. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$ and $\boldsymbol{\cup}^2$. Moreover this system has one of the above sets of parabolas if and only if $\chi_1 = \chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$(\boldsymbol{\mathcal{A}_1})$	$\zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0$	$\Rightarrow \cup$;
$(\boldsymbol{\mathcal{A}_2})$	$\zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 \neq 0$	$\Rightarrow \Cup$;
$(\boldsymbol{\mathcal{A}_3})$	$\zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 = 0$	$\Rightarrow \boldsymbol{\cup}^2$;
$(\boldsymbol{\mathcal{A}_4})$	$\zeta_2 \neq 0,\ \zeta_3 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 = 0,\ \zeta_5 \neq 0$	$\Rightarrow \cup$;
$(\boldsymbol{\mathcal{A}_5})$	$\zeta_2 \neq 0,\ \zeta_3 = 0,\ \zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0$	$\Rightarrow \cup$;
$(\boldsymbol{\mathcal{A}_6})$	$\zeta_2 \neq 0,\ \zeta_3 = 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0\ \zeta_5 \neq 0$	$\Rightarrow \Cup$;
$(\boldsymbol{\mathcal{A}_7})$	$\zeta_2 \neq 0,\ \zeta_3 = 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 = 0$	$\Rightarrow \boldsymbol{\cup}^2$;
$(\boldsymbol{\mathcal{A}_8})$	$\zeta_2 \neq 0,\ \zeta_3 = 0,\ \zeta_4 = 0,\ \mathcal{R}_2 = 0,\ \zeta_5 \neq 0$	$\Rightarrow \cup$;
$(\boldsymbol{\mathcal{A}_9})$	$\zeta_2 = 0,\ \ \zeta_6 \neq 0,\ \mathcal{R}_1 = 0,\ \mathcal{R}_2 \neq 0$	$\Rightarrow \cup$.

Furthermore in the case of the existence of an invariant parabola a system with $\eta > 0$ and $\zeta_1 \neq 0$ could be brought via an affine transformation and time rescaling to the following canonical form:

$$\dot{x} = m + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + (g-1)xy + 2y^2 \tag{6}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

However examining the conditions $(\boldsymbol{\mathcal{A}_1})$–$(\boldsymbol{\mathcal{A}_9})$ we detect that some of the sets of these conditions could be joined. More exactly we observe that $(\boldsymbol{\mathcal{A}_1})$ and $(\boldsymbol{\mathcal{A}_5})$ contain the same conditions except for the condition involving ζ_3. Therefore by eliminating the conditions involving only ζ_3, we obtain a new set of conditions which we denote by $(\boldsymbol{\mathcal{A}_1^*})$: $\zeta_2 \neq 0$, $\zeta_4 \neq 0$, $\mathcal{R}_1 \neq 0$.

We perform the analogous operation on $(\boldsymbol{\mathcal{A}_2})$ and $(\boldsymbol{\mathcal{A}_6})$ (respectively $(\boldsymbol{\mathcal{A}_3})$ and $(\boldsymbol{\mathcal{A}_7})$; $(\boldsymbol{\mathcal{A}_4})$ and $(\boldsymbol{\mathcal{A}_8})$) resulting in the new conditions $(\boldsymbol{\mathcal{A}_2^*})$ (respectively $(\boldsymbol{\mathcal{A}_3^*})$; $(\boldsymbol{\mathcal{A}_4^*})$).

Thus we can replace the first part of Proposition 1 obtaining the following one:

Proposition 1^*. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta > 0$ and $\zeta_1 \neq 0$ are satisfied. Then this system could possess invariant parabolas only in one direction. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$ and $\boldsymbol{\cup}^2$. Moreover this system has one of the above sets of parabolas if and only if $\chi_1 = \chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$(\boldsymbol{\mathcal{A}_1^*})$	$\zeta_2 \neq 0,\ \zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0$	$\Rightarrow \cup$;
$(\boldsymbol{\mathcal{A}_2^*})$	$\zeta_2 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 \neq 0$	$\Rightarrow \Cup$;
$(\boldsymbol{\mathcal{A}_3^*})$	$\zeta_2 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 = 0$	$\Rightarrow \boldsymbol{\cup}^2$;
$(\boldsymbol{\mathcal{A}_4^*})$	$\zeta_2 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 = 0,\ \zeta_5 \neq 0$	$\Rightarrow \cup$;
$(\boldsymbol{\mathcal{A}_5^*})$	$\zeta_2 = 0,\ \ \zeta_6 \neq 0,\ \mathcal{R}_1 = 0,\ \mathcal{R}_2 \neq 0$	$\Rightarrow \cup$.

The proof of the next proposition could also be found in [20].

Proposition 2. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta > 0$ and $\zeta_1 = 0$ are satisfied. Then this system could possess invariant parabolas in one or two directions. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$, $\mathbf{U}^2$, $\cup\subset$ and $\Cup\subset$. Moreover this system has one of the above sets of invariant parabolas if and only if $\chi_1 = \chi_3 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$(\boldsymbol{\mathcal{B}_1})$ $\chi_4 \neq 0,\ \zeta_7 \neq 0,\ \mathcal{R}_3 \neq 0 \quad \Rightarrow \cup$;
$(\boldsymbol{\mathcal{B}_2})$ $\chi_4 \neq 0,\ \zeta_7 = 0,\ \mathcal{R}_4 \neq 0,\ \zeta_8 \neq 0 \quad \Rightarrow \Cup$;
$(\boldsymbol{\mathcal{B}_3})$ $\chi_4 \neq 0,\ \zeta_7 = 0,\ \mathcal{R}_4 \neq 0,\ \zeta_8 = 0 \quad \Rightarrow \mathbf{U}^2$;
$(\boldsymbol{\mathcal{B}_4})$ $\chi_4 \neq 0,\ \zeta_7 = 0,\ \mathcal{R}_4 = 0 \quad \Rightarrow \cup$;
$(\boldsymbol{\mathcal{B}_5})$ $\chi_4 = 0,\ \zeta_5 \neq 0,\ \zeta_9 \neq 0 \quad \Rightarrow \cup\subset$;
$(\boldsymbol{\mathcal{B}_6})$ $\chi_4 = 0,\ \zeta_5 \neq 0,\ \zeta_9 = 0,\ \zeta_{10} \neq 0 \quad \Rightarrow \cup$;
$(\boldsymbol{\mathcal{B}_7})$ $\chi_4 = 0,\ \zeta_5 = 0,\ \zeta_6 \neq 0 \quad \Rightarrow \Cup\subset$.

Furthermore in the case of the existence of an invariant parabola a system with $\eta > 0$ and $\zeta_1 = 0$ could be brought via an affine transformation and time rescaling to the systems (6) *with $g = 2$.*

We observe that the Propositions 1 and 2 are devoted to the case $\eta > 0$. On the other hand a quadratic system can possess an invariant hyperbola also in the case $\eta = 0$ and $M \neq 0$ as well as in the case $C_2 = 0$ (then $\eta = M = 0$). So we present here the corresponding statements which are proved in [20].

Proposition 3. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta = 0$, $\widetilde{M} \neq 0$ and $\mu_0 \neq 0$ are satisfied. Then this system could possess invariant parabolas only in one (simple) direction. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\Cup$ and $\mathbf{U}^2$. Moreover this system has one of these sets of invariant parabolas if and only if $\chi_1 = \chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$(\boldsymbol{\mathcal{H}_1})$ $\zeta_4 \neq 0,\ \mathcal{R}_1 \neq 0 \quad \Rightarrow \cup$;
$(\boldsymbol{\mathcal{H}_2})$ $\zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 \neq 0 \quad \Rightarrow \Cup$;
$(\boldsymbol{\mathcal{H}_3})$ $\zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 = 0 \quad \Rightarrow \mathbf{U}^2$;
$(\boldsymbol{\mathcal{H}_4})$ $\zeta_4 = 0,\ \mathcal{R}_2 = 0,\ \zeta_5 \neq 0 \quad \Rightarrow \cup$.

Furthermore if this system satisfies one of the statements $(\boldsymbol{\mathcal{H}_1})$ - $(\boldsymbol{\mathcal{H}_4})$ then it can be brought via an affine transformation and time rescaling to the following canonical form:

$$\dot{x} = m + nx - gy/2 + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + gxy + 2y^2, \tag{7}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

Proposition 4. *Assume that for a non-degenerate arbitrary quadratic system the conditions $\eta = 0$, $\widetilde{M} \neq 0$ and $\mu_0 = 0$ are satisfied. Then this system could possess invariant parabolas in two directions. More exactly it could only possess one of the following sets of invariant parabolas: $\cup$, $\overset{2}{\cup}$, $\overset{2}{\cup}\subset$, $\overset{2}{\cup}\overset{2}{\Cup}$, $\overset{2}{\cup}\overset{2}{\Cup}^c$, $\overset{2}{\cup}\overset{2}{\mathbf{U}}^2$, $\overset{2}{\mathbf{U}}^3$ and $\infty\overset{2}{\cup}$. Moreover this system has one of the above sets of invariant parabolas if and only if $\chi_1 = \chi_2 = 0$ and one of the following sets of conditions are satisfied, correspondingly:*

$$\begin{array}{lll}
(\boldsymbol{\mathcal{K}_1}) & \zeta_{11}\zeta_{12}\zeta_{13}\zeta_{14} \neq 0,\ \mathcal{R}_5 \neq 0,\ \begin{cases}\zeta_{15} \neq 0, \zeta_{16} = 0 \\ or\ \zeta_{15} = \zeta_{17} = 0,\end{cases} & \Rightarrow \overset{2}{\cup};\\
(\boldsymbol{\mathcal{K}_2}) & \zeta_{11} = 0,\ \mathcal{R}_3 \neq 0,\ \zeta_{16} = 0,\ \mathcal{R}_5 \neq 0 & \Rightarrow \overset{2}{\cup};\\
(\boldsymbol{\mathcal{K}_3}) & \zeta_{11} = 0,\ \mathcal{R}_3 = 0,\ \zeta_{16} \neq 0,\ \mathcal{R}_6 \neq 0 & \Rightarrow \cup\\
(\boldsymbol{\mathcal{K}_4}) & \zeta_{11} = 0,\ \mathcal{R}_3 = 0,\ \zeta_{16} = 0,\ \zeta_8 \neq 0 & \Rightarrow \overset{2}{\cup}\subset;\\
(\boldsymbol{\mathcal{K}_5}) & \zeta_{12} = 0,\ \zeta_{16} = 0,\ \zeta_8 \neq 0 & \Rightarrow \overset{2}{\cup};\\
(\boldsymbol{\mathcal{K}_6}) & \zeta_{13} = 0,\ \zeta_8 = 0,\ \zeta_{18} > 0,\ \mathcal{R}_5 \neq 0 & \Rightarrow \overset{2}{\cup}\,\overset{2}{\Cup};\\
(\boldsymbol{\mathcal{K}_7}) & \zeta_{13} = 0,\ \zeta_8 = 0,\ \zeta_{18} < 0,\ \mathcal{R}_5 \neq 0 & \Rightarrow \overset{2}{\cup}\,\overset{2}{\Cup}{}^{c};\\
(\boldsymbol{\mathcal{K}_8}) & \zeta_{13} = 0,\ \zeta_8 = 0,\ \zeta_{18} = 0,\ \chi_3 \neq 0,\ \mathcal{R}_5 \neq 0 & \Rightarrow \overset{2}{\cup}\,\overset{2}{\boldsymbol{\cup}}{}^{2}\\
(\boldsymbol{\mathcal{K}_9}) & \zeta_{13} = 0,\ \zeta_8 = 0,\ \zeta_{18} = 0,\ \chi_3 = 0,\ \mathcal{R}_5 \neq 0 & \Rightarrow \overset{2}{\boldsymbol{\cup}}{}^{3};\\
(\boldsymbol{\mathcal{K}_{10}}) & \zeta_{14} = 0,\ \zeta_{19} \neq 0,\ \zeta_{20} \neq 0,\ \zeta_{21} \neq 0,\ \mathcal{R}_5 = 0 & \Rightarrow \overset{2}{\cup};\\
(\boldsymbol{\mathcal{K}_{11}}) & \zeta_{14} = 0,\ \zeta_{19} = 0,\ \zeta_{20} \neq 0,\ \zeta_{21} = 0,\ \mathcal{R}_5 = 0 & \Rightarrow \infty\overset{2}{\cup}.
\end{array}$$

Furthermore if this system satisfies one of the statements $(\boldsymbol{\mathcal{K}_1})$–$(\boldsymbol{\mathcal{K}_{11}})$, *with except of the statement* $(\boldsymbol{\mathcal{K}_3})$, *then it can be brought via an affine transformation and time rescaling to the following canonical form:*

$$\dot{x} = 2mx + 2ny + (h-1)xy, \quad \dot{y} = n - (h+1)x/2 + my + hy^2$$

possessing the invariant parabola $\Phi(x,y) = y^2 - x$ *in the double direction.*

If the system satisfies the statement $(\boldsymbol{\mathcal{K}_3})$ *(i.e. we have a parabola in simple direction) then it can be brought via an affine transformation and time rescaling to the canonical form* (7).

Observation 1. We remark that there is an inaccuracy in the statement of Theorem 3.16 from [20]. More exactly, assume that for a quadratic system the conditions provided by the statement $(\boldsymbol{\mathcal{K}_3})$ of Theorem 3.16 from [20] are satisfied. Then this system could be brought via an affine transformation and time rescaling to the canonical form $(S^1_{\boldsymbol{\gamma}})$ with $g = 0$ (possessing the invariant parabola $\Phi(x,y) = x^2 - y$) and not to $(S^2_{\boldsymbol{\gamma}})$ (with the invariant parabola $\Phi(x,y) = x - y^2$). This follows directly from the proof of Lemma 3.15 (page 53) and Theorem 3.16 (page 59) from [20].

Finally, in the case $C_2 = 0$ according to [20] the following proposition is valid.

Proposition 5. *Assume that for a non-degenerate quadratic system the condition* $C_2 = 0$ *holds. Then this system possesses an invariant parabola (which is unique) if and only if the following set of conditions holds:*

$$(\boldsymbol{\mathcal{V}}) \quad \zeta_5 \neq 0, \zeta_{20} = 0 \quad \Rightarrow \overset{\infty}{\cup}.$$

Moreover via an affine transformation and time rescaling this system could be brought to the form

$$\dot{x} = m + nx - y/2 + x^2, \quad \dot{y} = 2mx + 2ny + xy,$$

which possess the invariant parabola $\Phi(x,y) = x^2 - y$.

2.2 Invariant polynomials associated with configurations of invariant conics

We single out the following five polynomials, basic ingredients in constructing invariant polynomials for systems (3):

$$\begin{aligned} C_i(\tilde{a},x,y) &= yp_i(x,y) - xq_i(x,y),\ (i=0,1,2)\\ D_i(\tilde{a},x,y) &= \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y},\ (i=1,2). \end{aligned} \tag{8}$$

As it was shown in [18] these polynomials of degree one in the coefficients of systems (3) are GL–comitants of these systems. Let $f,\, g \in \mathbb{R}[\tilde{a},x,y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

The polynomial $(f,g)^{(k)} \in \mathbb{R}[\tilde{a},x,y]$ is called *the transvectant of index k of (f,g)* (cf. [6], [13])).

Proposition 6 (see [19])**.** *Any GL–comitant of systems* (3) *can be constructed from the elements* (8) *by using the operations: $+$, $-$, $\times$, and by applying the differential operation $(*,*)^{(k)}$.*

Remark 4. *We point out that the elements* (8) *generate the whole set of GL–comitants and hence also the set of affine comitants as well as the set of T-comitants.*

We construct the following GL–comitants of the second degree with respect to the coefficients of the initial systems

$$\begin{array}{lll} T_1 = (C_0,C_1)^{(1)}, & T_2 = (C_0,C_2)^{(1)}, & T_3 = (C_0,D_2)^{(1)},\\ T_4 = (C_1,C_1)^{(2)}, & T_5 = (C_1,C_2)^{(1)}, & T_6 = (C_1,C_2)^{(2)},\\ T_7 = (C_1,D_2)^{(1)}, & T_8 = (C_2,C_2)^{(2)}, & T_9 = (C_2,D_2)^{(1)}. \end{array} \tag{9}$$

Using these GL–comitants as well as the polynomials (8) we construct the additional invariant polynomials. In order to be able to calculate the values of the needed invariant polynomials directly for every canonical system we shall define here a family of T–comitants expressed through C_i $(i = 0,1,2)$ and D_j $(j=1,2)$:

$$\begin{aligned} \hat{A} &= \left(C_1, T_8 - 2T_9 + D_2^2\right)^{(2)}/144,\\ \widehat{D} &= \Big[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6 - (C_1,T_5)^{(1)} + 6D_1(C_1D_2 - T_5) - 9D_1^2C_2\Big]/36,\\ \widehat{E} &= \Big[D_1(2T_9 - T_8) - 3\,(C_1,T_9)^{(1)} - D_2(3T_7 + D_1D_2)\Big]/72,\\ \widehat{F} &= \big[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0\,(D_2,T_9)^{(1)} - 9D_2^2T_4 + 288D_1\widehat{E}\\ &\quad - 24\left(C_2,\widehat{D}\right)^{(2)} + 120\left(D_2,\widehat{D}\right)^{(1)} - 36C_1\,(D_2,T_7)^{(1)} + 8D_1\,(D_2,T_5)^{(1)}\Big]/144, \end{aligned}$$

$$
\begin{aligned}
\widehat{B} = \Big\{ & 16D_1\,(D_2,T_8)^{(1)}\,(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0\,(D_2,T_9)^{(1)}\,(3D_1D_2 - 5T_6 + 9T_7)\\
& + 2\,(D_2,T_9)^{(1)}\,\big(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32\,(C_0,T_5)^{(1)}\,\big)\\
& + 6\,(D_2,T_7)^{(1)}\,[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5) + C_2(9T_4 + 96T_3)]\\
& + 6\,(D_2,T_6)^{(1)}\,[32C_0T_9 - C_1(12T_7 + 52D_1D_2) - 32C_2D_1^2] + 48D_2\,(D_2,T_1)^{(1)}\,\big(2D_2^2 - T_8\big)\\
& - 32D_1T_8\,(D_2,T_2)^{(1)} + 9D_2^2T_4\,(T_6 - 2T_7) - 16D_1\,(C_2,T_8)^{(1)}\,\big(D_1^2 + 4T_3\big)\\
& + 12D_1\,(C_1,T_8)^{(2)}\,(C_1D_2 - 2C_2D_1) + 6D_1D_2T_4\,\big(T_8 - 7D_2^2 - 42T_9\big)\\
& + 12D_1\,(C_1,T_8)^{(1)}\,(T_7 + 2D_1D_2) + 96D_2^2\Big[D_1\,(C_1,T_6)^{(1)} + D_2\,(C_0,T_6)^{(1)}\Big] -\\
& - 16D_1D_2T_3\,\big(2D_2^2 + 3T_8\big) - 4D_1^3D_2\,\big(D_2^2 + 3T_8 + 6T_9\big) + 6D_1^2D_2^2\,(7T_6 + 2T_7)\\
& - 252D_1D_2T_4T_9\Big\}/(2^83^3),
\end{aligned}
$$

$$
\widehat{K} = (T_8 + 4T_9 + 4D_2^2)/72, \qquad \widehat{H} = (8T_9 - T_8 + 2D_2^2)/72.
$$

These polynomials in addition to (8) and (9) will serve as bricks in constructing affine invariant polynomials for systems (3).

The following 42 affine invariants $A_1,\dots,A_{42}$ form the minimal polynomial basis of affine invariants up to degree 12. This fact was proved in [2] by constructing $A_1,\dots,A_{42}$ using the above bricks.

$$
\begin{array}{ll}
A_1 = \hat{A}, & A_{22} = \frac{1}{1152}[\![C_2,\widehat{D})^{(1)},D_2\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}D_2\big)^{(1)},\\
A_2 = (C_2,\widehat{D})^{(3)}/12, & A_{23} = [\![\widehat{F},\widehat{H})^{(1)},\widehat{K}\big)^{(2)}/8,\\
A_3 = [\![C_2,D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48, & A_{24} = [\![C_2,\widehat{D})^{(2)},\widehat{K}\big)^{(1)},\widehat{H}\big)^{(2)}/32,\\
A_4 = (\widehat{H},\widehat{H})^{(2)}, & A_{25} = [\![\widehat{D},\widehat{D})^{(2)},\widehat{E}\big)^{(2)}/16,\\
A_5 = (\widehat{H},\widehat{K})^{(2)}/2, & A_{26} = (\widehat{B},\widehat{D})^{(3)}/36,\\
A_6 = (\widehat{E},\widehat{H})^{(2)}/2, & A_{27} = [\![\widehat{B},D_2)^{(1)},\widehat{H}\big)^{(2)}/24,\\
A_7 = [\![C_2,\widehat{E})^{(2)},D_2\big)^{(1)}/8, & A_{28} = [\![C_2,\widehat{K})^{(2)},\widehat{D}\big)^{(1)},\widehat{E}\big)^{(2)}/16,\\
A_8 = [\![\widehat{D},\widehat{H})^{(2)},D_2\big)^{(1)}/8, & A_{29} = [\![\widehat{D},\widehat{F})^{(1)},\widehat{D}\big)^{(3)}/96,\\
A_9 = [\![\widehat{D},D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48, & A_{30} = [\![C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)}/288,\\
A_{10} = [\![\widehat{D},\widehat{K})^{(2)},D_2\big)^{(1)}/8, & A_{31} = [\![\widehat{D},\widehat{D})^{(2)},\widehat{K}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{11} = (\widehat{F},\widehat{K})^{(2)}/4, & A_{32} = [\![\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{H}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{12} = (\widehat{F},\widehat{H})^{(2)}/4, & A_{33} = [\![\widehat{D},D_2)^{(1)},\widehat{F}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{13} = [\![C_2,\widehat{H})^{(1)},\widehat{H}\big)^{(2)},D_2\big)^{(1)}/24, & A_{34} = [\![\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{K}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{14} = (\widehat{B},C_2)^{(3)}/36, & A_{35} = [\![\widehat{D},\widehat{D})^{(2)},\widehat{E}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{15} = (\widehat{E},\widehat{F})^{(2)}/4, & A_{36} = [\![\widehat{D},\widehat{E})^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/16,\\
A_{16} = [\![\widehat{E},D_2)^{(1)},C_2\big)^{(1)},\widehat{K}\big)^{(2)}/16, & A_{37} = [\![\widehat{D},\widehat{D})^{(2)},\widehat{D}\big)^{(1)},\widehat{D}\big)^{(3)}/576,\\
A_{17} = [\![\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},D_2\big)^{(1)}/64, & A_{38} = [\![C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{D}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{18} = [\![\widehat{D},\widehat{F})^{(2)},D_2\big)^{(1)}/16, & A_{39} = [\![\widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{H}\big)^{(2)}/64,\\
A_{19} = [\![\widehat{D},\widehat{D})^{(2)},\widehat{H}\big)^{(2)}/16, & A_{40} = [\![\widehat{D},\widehat{D})^{(2)},\widehat{F}\big)^{(1)},\widehat{K}\big)^{(2)}/64,\\
A_{20} = [\![C_2,\widehat{D})^{(2)},\widehat{F}\big)^{(2)}/16, & A_{41} = [\![C_2,\widehat{D})^{(2)},\widehat{D}\big)^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/64,\\
A_{21} = [\![\widehat{D},\widehat{D})^{(2)},\widehat{K}\big)^{(2)}/16, & A_{42} = [\![\widehat{D},\widehat{F})^{(2)},\widehat{F}\big)^{(1)},D_2\big)^{(1)}/16.
\end{array}
$$

In the above list, the bracket "$[\![$" is used in order to avoid placing the otherwise necessary up to five parentheses "(".

Using the elements of the minimal polynomial basis given above, in [10] (respectively [20]) are defined the invariant polynomials of the set (4) (respectively the set (5)) which are responsible for the existence of at least on invariant hyperbola (respectively parabola) for systems (3).

Here we present additional invariant polynomials responsible for the existence of at least one invariant hyperbola and one invariant parabola as well as for distinguishing the corresponding configurations of these conics together with invariant lines in case they exist.

First we mention the invariant polynomials $\mu_0(\tilde{a}), \mu_1(\tilde{a},x,y), \dots \mu_4(\tilde{a},x,y)$ from the set (5). These polynomials are in fact comitants of systems (3) with respect to the group $GL(2,\mathbb{R})$ and their geometrical meaning is revealed in Lemma 5.2 of [1]. Using these invariant polynomials we construct the invariant polynomials $\mathbf{D}$ and $\mathbf{R}$ which are responsible for the existence of multiple finite singularities of a quadratic system:

$$\mathbf{D} = \Big[3\big((\mu_3,\mu_3)^{(2)},\ \mu_2\big)^{(2)} - \big(6\mu_0\mu_4 - 3\mu_1\mu_3 + \mu_2^2,\ \mu_4\big)^{(4)}\Big]/48, \quad \mathbf{R} = 3\mu_1^2 - 8\mu_0\mu_2.$$

Next we construct the following T-comitants (for the definition of T-comitants see [16]) which are responsible for the existence of invariant straight lines of systems (3):

$$\begin{aligned} B_3(\tilde{a},x,y) &= (C_2,\widehat{D})^{(1)} = \mathrm{Jacob}(C_2,\widehat{D}),\\ B_2(\tilde{a},x,y) &= (B_3,B_3)^{(2)} - 6B_3(C_2,\widehat{D})^{(3)},\\ B_1(\tilde{a}) = \mathrm{Res}_x\left(C_2,\widehat{D}\right)/y^9 &= -2^{-9}3^{-8}\left(B_2,B_3\right)^{(4)}. \end{aligned}$$

Lemma 1 (see [15])**.** *For the existence of invariant straight lines in one (respectively 2; 3 distinct) directions in the affine plane it is necessary that* $B_1 = 0$ *(respectively* $B_2 = 0$*;* $B_3 = 0$*).*

To detect the parallel invariant lines we need the following invariant polynomials:

$$\begin{aligned} N(\tilde{a},x,y) &= D_2^2 + T_8 - 2T_9 = 9\widehat{N},\\ \theta(\tilde{a}) &= 2A_5 - A_4\ (\equiv \mathrm{Discriminant}\big(N(a,x,y)\big)/1296). \end{aligned}$$

Lemma 2 (see [15])**.** *A necessary condition for the existence of one couple (respectively two couples) of parallel invariant straight lines of a system* (3) *corresponding to* $\tilde{a} \in \mathbb{R}^{12}$ *is the condition* $\theta(\tilde{a}) = 0$ *(respectively* $N(\tilde{a},x,y) = 0$*).*

We point out some important GL-comitant in the study of the invariant conics. Considering $C_2(\tilde{a},x,y) = yp_2(\tilde{a},x,y) - xq_2(\tilde{a},x,y)$ as a cubic binary form of x and y we calculate

$$\eta(\tilde{a}) = \mathrm{Discrim}[C_2/x^3,\xi], \quad M(\tilde{a},x,y) = \mathrm{Hessian}[C_2],$$

where $\xi = y/x$ or $\xi = x/y$. We point out (see [18]) that the invariant polynomials C_2, η and M are responsible for the number of infinite singularities (real or complex).

Remark 5. In order to describe the various kinds of multiplicities for infinite singularities we use the concepts and notations introduced in [15]. Thus we denote by (a,b) the maximum number a (respectively, b) of infinite (respectively, finite) singularities which can be obtained by perturbation of a multiple infinite singularity. In this case we say that an infinite singular point *has multiplicity* (a,b).

Using the elements $A_1, \ldots, A_{42}$ of the minimal polynomial basis we construct the needed affine invariant polynomials $\nu_1 - \nu_{12}$ which are responsible for the configurations of the invariant conics and lines.

$$\begin{aligned}
\nu_1 = {} & 25(169A_1^2A_4 - 413A_6^2) - 1136(2A_7^2 + 2A_5A_9 - 9A_5A_{10} - 9A_5A_{11}) + 14110A_6A_7 - 6653A_5A_8 \\
& - 9351A_5A_{12}; \\
\nu_2 = {} & 32(567613A_7A_8 + 561656A_6A_9 + 1084090A_7A_{12} - 27183A_2A_{12}) - 80(498431A_6A_{10} \\
& - 1933A_6A_{11} + 458971A_7A_{11} + 330167A_6A_{12}) - 21060272A_7A_{10} - 679575A_4A_{14}; \\
\nu_3 = {} & 7A_4 - 16A_5; \\
\nu_4 = {} & 4621A_9 - 975A_{11} - 2850A_{12}; \\
\nu_5 = {} & 20227A_9 - 5700A_{11} - 11325A_{12}; \\
\nu_6 = {} & A_7(63632A_9 - 10324370A_8 - 2712970A_{10} - 3385206A_{11} - 9508962A_{12}) + A_6(40618285A_8 \\
& + 6108280A_9 - 4455879A_{10} + 14451927A_{11} + 15048477A_{12}) + 13241360A_5A_{14} \\
& + 8068554A_4A_{15} - 2858192A_5A_{15}; \\
\nu_7 = {} & 3155A_1^2 + 499A_9 - 267A_{10} + 1838A_{11} - 1777A_{12}; \\
\nu_8 = {} & -391487A_8 + 167912A_9 + 557021A_{10} + 1736321A_{11} - 1495779A12; \\
\nu_9 = {} & -53197A_1^2 - 2622A_9 + 3934A_{10} - 7221A_{11} + 9752A_{12}; \\
\nu_{10} = {} & 2124A_9 - 1443346A_1^2 - 169493A_{10} - 369693A_{11} + 145346A_{12}; \\
\nu_{11} = {} & 3797A_1^2 + 384A_{11} - 464A_{12}; \\
\nu_{12} = {} & 20A_2^2 + 56A_{18} - 344A_{19} - 20A_{20} + 351A_{21}.
\end{aligned}$$

3 The proof of Main Theorem

To prove the Main Theorem we consider each one of the statements from $(\boldsymbol{A})$ to $(\boldsymbol{D})$.

3.1 The statements $(\boldsymbol{A})$ and $(\boldsymbol{C})$

According to [9] a quadratic system could have an invariant hyperbola only if the condition $\gamma_1 = \gamma_2 = 0$ is satisfied. On the other hand by [20] the condition $\chi_1 = \chi_2 = 0$ is necessary for the existence of an invariant parabola for an arbitrary quadratic system. Therefore we conclude that the statement $(\boldsymbol{A})$ of the Main Theorem is valid.

The proof of the statement $(\boldsymbol{C})$ follow directly from the construction of the bifurcation diagrams pictured here in Diagrams 1 and 2.

3.2 The statement $(\boldsymbol{B})$

According to this statement we have to examine two possibilities: $\eta > 0$ and $\eta = 0$.

3.2.1 The possibility $\eta > 0$

In this case systems (3) possess three distinct real singularities at infinity and according to [20] if a quadratic system with $\eta > 0$ possesses an invariant parabola then via an affine transformation and time rescaling it can be brought to the canonical form (6), i.e.

$$\dot{x} = m + nx - \frac{1}{2}(1+g)y + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + (g-1)xy + 2y^2 \tag{10}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$. Following [20] we distinguish two cases: $\zeta_1 \neq 0$ and $\zeta_1 = 0$.

3.2.1.1 The case $\zeta_1 \neq 0$. According to Proposition 1^* for the existence of an invariant parabola for a quadratic system with $\eta > 0$ and $\zeta_1 \neq 0$ one of the sets of conditions from $(\boldsymbol{\mathcal{A}_1^*})$ to $(\boldsymbol{\mathcal{A}_5^*})$ it is necessary and sufficient.

Next we determine the conditions for a system (10) to possess at least one invariant hyperbola, i.e. to be in the class **QSPH**. Moreover we are interested to construct all the configurations which could have systems in the class **QSPH** taking into consideration the possible existence of invariant lines.

According to Theorem 1 for the existence of at least one invariant hyperbola for systems (10) the condition $\gamma_1 = \gamma_2 = 0$ is necessary. So for systems (10) we calculate:

$$\begin{aligned}
\gamma_1 =& \frac{1}{512}\mathcal{V}_1\mathcal{V}_2\mathcal{V}_3, \quad \nu_1 = \frac{1607}{128}(g-2)(3+g)\mathcal{V}_1\mathcal{V}_2,\\
\mathcal{V}_1 =& 1 - 3g - g^2 + 3g^3 - 20m - 6n + 6gn,\\
\mathcal{V}_2 =& 3 + 13g + 13g^2 + 3g^3 - 20m + 2n + 6gn,\\
\mathcal{V}_3 =& 1 - 3g - g^2 + 3g^3 - 28m - 56gm - 10n - 2gn + 12g^2n.
\end{aligned} \tag{11}$$

In what follows we examine step by step each one of the statements $(\boldsymbol{\mathcal{A}_1^*})$–$(\boldsymbol{\mathcal{A}_5^*})$ provided by Proposition 1^*.

3.2.1.1.1 The statement $(\boldsymbol{\mathcal{A}_1^*})$. Considering the conditions provided by this statement we evaluate for systems (10) the following invariant polynomials:

$$\begin{aligned}
\zeta_1 = 2(g-2)(3+g), \quad \zeta_2 = 4g(1+g),\\
\zeta_4 = \frac{1}{16}(g-2)(3+g)(1 + 7g + 15g^2 + 9g^3 - 4m + 2n + 6gn),\\
\mathcal{R}_1 = -\frac{15}{2}g(1+g)(g-2)(3+g)(1 + 7g + 15g^2 + 9g^3 - 4m + 2n + 6gn).
\end{aligned}$$

On the other hand considering Diagram 3 we observe that one of the most generic condition is given by the invariant polynomial β_1 and we examine here two possibilities: $\beta_1 \neq 0$ and $\beta_1 = 0$.

1: *The possibility* $\beta_1 \neq 0$. As it was mentioned above we have to force the condition $\gamma_1 = \gamma_2 = 0$ to be fulfilled. From (11) we observe that the condition $\gamma_1 = 0$ implies $\mathcal{V}_1\mathcal{V}_2\mathcal{V}_3 = 0$. We discuss two cases: $\nu_1 \neq 0$ and $\nu_1 = 0$.

1.1: *The case $\nu_1 \neq 0$.* Then $\mathcal{V}_1\mathcal{V}_2 \neq 0$ and the condition $\gamma_1 = 0$ gives us $\mathcal{V}_3 = 0$. Since

$$\mathcal{V}_3 = 8(1+2g)m + 2(g-1)(5+6g)n + (g-1)(1+g)(3g-1)$$

and this polynomial is linear in m if $1+2g \neq 0$. Since for systems (10) we have $\zeta_3 = 8(1+2g)^2$ we consider two subcases: $\zeta_3 \neq 0$ and $\zeta_3 = 0$.

1.1.1: *The subcase $\zeta_3 \neq 0$.* Then $1+2g \neq 0$ and the condition $\mathcal{V}_3 = 0$ yields

$$m = \frac{1}{28(1+2g)}(g-1)(2g-1+3g^2+10n+12gn). \tag{12}$$

In this case for systems (10) calculations yield

$$\begin{aligned}\gamma_2 = \frac{675}{784(1+2g)^3}&(g-1)^2(2+g)^2(3g-1)(4+3g)\mathcal{W}_1\mathcal{W}_2^2\mathcal{W}_3,\\ \mathcal{W}_1 &= 4g-3+16n, \quad \mathcal{W}_2 = 1+8g+7g^2+4n,\\ \mathcal{W}_3 &= 1-2g+g^2+4g^3+4n+16gn+16g^2n.\end{aligned}$$

Moreover for these systems we

$$\beta_1 = -\frac{1}{784(1+2g)^2}(g-1)(2+g)(3g-1)(4+3g)\mathcal{W}_2^2 \tag{13}$$

and the condition $\beta_1 \neq 0$ implies $(g-1)(2+g)(3g-1)(4+3g)\mathcal{W}_2 \neq 0$. Therefore the condition $\gamma_2 = 0$ is equivalent to $\mathcal{W}_1\mathcal{W}_3 = 0$.

On the other hand for systems (10) with the value of the parameter m given in (12) we calculate

$$\nu_2 = -\frac{553500}{49}(g-1)(2+g)(1+3g+3g^2)\mathcal{W}_2\mathcal{W}_3$$

and since $\text{Discrim}\,[1+3g+3g^2, g] = -3 < 0$ and $\beta_1 \neq 0$ we conclude that the condition $\nu_2 = 0$ is equivalent to $\mathcal{W}_3 = 0$. So we discuss two possibilities: $\nu_2 \neq 0$ and $\nu_2 = 0$.

1.1.1.1: *The possibility $\nu_2 \neq 0$.* Then the condition $\gamma_2 = 0$ implies $\mathcal{W}_1 = 0$, i.e. we have $n = (3-4g)/16$ and we arrive at the 1-parameter family of systems

$$\begin{aligned}\dot{x} =& \frac{1}{32}(g-1) + \frac{1}{16}(3-4g)x - \frac{1}{2}(1+g)y + gx^2 + xy,\\ \dot{y} =& \frac{1}{16}(g-1)x + \frac{1}{8}(3-4g)y + (g-1)xy + 2y^2\end{aligned} \tag{14}$$

which possess the following invariant parabola and hyperbola:

$$\Phi(x,y) = x^2 - y = 0, \quad \Psi(x,y) = -1 + 16x - 96y + 256xy - 256y^2 = 0. \tag{15}$$

For these systems the condition $\gamma_1 = \gamma_2 = 0$ is satisfied and taking into consideration Diagram 3 we calculate

$$\begin{aligned}&\theta = -(g-1)(2+g)/2, \quad \beta_1 = -\frac{1}{256}(g-1)(2+g)(3g-1)(4+3g)(1+2g)^2,\\ &\beta_2 = \frac{7}{16}(1+2g)^2, \quad \widetilde{\mathcal{R}}_1 = -\frac{3}{2048}(g-1)^2(2+g)^2(3g-1)(4+3g)(1+2g)^2.\end{aligned} \tag{16}$$

We observe that the condition $\beta_1 \neq 0$ implies $\theta\beta_2\widetilde{\mathcal{R}}_1 \neq 0$ and therefore we conclude that the corresponding conditions provided by Diagram 3 are fulfilled.

Next we examine systems (14). Considering Lemma 1 for these systems we calculate

$$B_1 = -\frac{1}{65536}(g-1)^2(2+g)^2(2g-1)(1+2g)(3+2g)(1+2g+2g^2) \tag{17}$$

and we examine two cases: $B_1 \neq 0$ and $B_1 = 0$.

1.1.1.1.1: *The case* $B_1 \neq 0$. According to Lemma 1 systems (14) could not possess any invariant line. These systems possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$x_1 = \frac{1}{4},\ y_1 = \frac{1}{16};\quad x_2 = \frac{2g^2-1}{8g(1+g)},\ y_2 = \frac{g-1}{16g(1+g)};\quad x_{3,4} = \frac{1}{8}\big(1-2g \pm \sqrt{U_1}\,\big),$$
$$y_{3,4} = \frac{1}{32}\big(4g^2 - 3 \pm (1-2g)\sqrt{U_1}\,\big),\quad U_1 = 4g(1+g) - 7.$$

We observe that the invariant conics (15) have the unique point of intersection $M_1(1/4, 1/16)$ which is a point of tangency. The singular point M_2 is located on the invariant hyperbola whereas M_3 and M_4 are located on the invariant parabolas. Moreover these last points are either real (for $U_1 > 0$) or complex (for $U_1 < 0$) or coinciding (for $U_1 = 0$).

Remark 6. *The singular points M_2 and M_1 are located on the same branch of the invariant hyperbola if and only if the condition $g(g+1) > 0$ holds.*

Indeed calculations yield:

$$\Psi(x,y)\big|_{x=x_1} = -(16y-1)(3+16y) = 0 \ \Rightarrow\ y_1 = 1/16,\ y_1' = -3/16,\ y_1 - y_1' = 1/4 > 0,$$
$$\Psi(x,y)\big|_{x=x_2} = -\frac{1}{g(1+g)}(2+3g+16gy)(1-g+16y+16gy) = 0 \ \Rightarrow$$
$$y_2 = \frac{g-1}{16g(1+g)},\ y_2' = -\frac{2+3g}{16g},\quad y_2 - y_2' = \frac{1+2g+2g^2}{8g(1+g)},$$

where y_1' (respectively y_2') is the second point of intersection of the invariant hyperbola with the line $x = x_1$ (respectively $x = x_2$.) Since $\text{Discrim}\,[1+2g+2g^2, g] = -4 < 0$ we conclude that $\text{sign}\,(y_2 - y_2') = \text{sign}\,(g(1+g))$. So considering that $y_1 - y_1' > 0$ we conclude that the remark is true.

To determine the position of the real singularities $M_{3,4}$ with respect to M_1 all located on the invariant parabola we calculate

$$(x_3 - x_1)(x_4 - x_1) = \frac{1}{8} > 0,\quad (x_3 - x_1) + (x_4 - x_1) = -\frac{1+2g}{4},$$
$$\text{sign}\,\big((x_1 - x_3) + (x_1 - x_4)\big) = -\text{sign}\,(1+2g).$$

Therefore we conclude that the singular point $M_1(1/4, 1/16)$ could not be located between M_3 and M_4. Moreover the singularities M_3 and M_4 are located to the right (respectively left) of point M_1 if $2g+1 < 0$ (respectively $2g+1 > 0$).

We have to determine also the position of the singular point M_2 with respect to M_1 both located on the invariant hyperbola. Calculations yield

$$x_2 - x_1 = -\frac{1+2g}{8g(1+g)},\quad \text{sign}\,(x_2 - x_1) = -\text{sign}\,\big(g(1+g)(1+2g)\big).$$

On the other hand for systems (14) we calculate

$$\mathbf{D} = -\frac{3}{65536}(4g^2+4g-7) = -\frac{3}{65536}U_1 \quad \Rightarrow \ \text{sign}\,(\mathbf{D}) = -\text{sign}\,(U_1)$$

and therefore we consider three subcases: $\mathbf{D}<0$, $\mathbf{D}>0$ and $\mathbf{D}=0$.

1.1.1.1.1.1: *The subcase* $\mathbf{D}<0$. Then $U_1>0$, i.e. $4g(1+g)-7>0$ and this implies $g(g+1)>0$ (as $\zeta_2\neq 0$). So we conclude that

$$\text{sign}\,(x_2-x_1) = -\text{sign}\,(1+2g) = \text{sign}\,\big((x_1-x_3)+(x_1-x_4)\big).$$

and hence all three real singularities M_2, M_3 and M_4 are located on the same semi-plane $x>1/4$ (respectively $x<1/4$) if $2g-1<0$ (respectively $2g-1>0$). Moreover according to Remark 6 the singularities M_2 and M_1 are located on the same branch of invariant hyperbola.

Thus in the case $\mathbf{D}<0$ we obtain the configuration *Config.* $\mathcal{PH}$*.1* if $2g-1<0$. In the case $2g-1>0$ we arrive at a configuration which is equivalent to *Config.* $\mathcal{PH}$*.1*.

1.1.1.1.1.2: *The subcase* $\mathbf{D}>0$. This implies $U_1<0$ and then the singularities M_3 and M_4 are complex. So it remains two real finite singularities M_1 (the point of tangency of the invariant hyperbola and parabola) and M_2 located on the invariant hyperbola. Moreover according to Remark 6 these two singularities are located on the same branch (respectively different branches) of the hyperbola if $g(g+1)>0$ (respectively $g(g+1)<0$).

Therefore in the case $g(g+1)<0$ we get the unique configuration *Config.* $\mathcal{PH}$*.2*.

Assume now $g(g+1)>0$. Then M_1 and M_2 are located on the same branch of hyperbola and since

$$\text{sign}\,(x_2-x_1) = -\text{sign}\,\big(g(1+g)(1+2g)\big) = -\text{sign}\,\big(1+2g\big)$$

we arrive at the configuration *Config.* $\mathcal{PH}$*.3* if $1+2g<0$. In the case $1+2g>0$ we obtain a configuration which is equivalent to *Config.* $\mathcal{PH}$*.3*.

On the other hand we observe that $\text{sign}\,(g(1+g)) = \text{sign}\,(\zeta_2)$ and therefore we obtain *Config.* $\mathcal{PH}$*.2* for $\zeta_2<0$ and *Config.* $\mathcal{PH}$*.3* for $\zeta_2>0$.

1.1.1.1.1.3: *The subcase* $\mathbf{D}=0$. This implies $U_1=0$ and then the singularities M_3 and M_4 coalesced. Since in this case we have $4g(1+g)=7>0$ by Remark 6 the simple singularities M_1 and M_2 are located on the same branch of hyperbola.

So in the case $1+2g<0$ we arrive at the configuration *Config.* $\mathcal{PH}$*.4*. If $1+2g>0$ we obtain a configuration which is equivalent to *Config.* $\mathcal{PH}$*.4*.

1.1.1.1.2: *The case* $B_1=0$. Considering (17) and (16) we obtain that due to $\beta_1\neq 0$ the condition $B_1=0$ is equivalent to $(2g-1)(3+2g)=0$.

If $2g-1=0$ then we have $g=1/2$ and we calculate $\mathbf{D}=3\cdot 2^{-14}>0$. Therefore system (14) with $g=1/2$ has two real and two complex singularities. Moreover in addition to the invariant parabola $\Phi(x,y)=0$ and hyperbola $\Psi(x,y)=0$ the system (14) with $g=1/2$ possesses the invariant line $y=-1/16$. We detect that this invariant line is an asymptote of the invariant hyperbola and intersect invariant parabola at the complex singularities M_3 and M_4. So we obtain that this specific system possesses the configuration *Config.* $\mathcal{PH}$*.5*.

Assume now $3+2g=0$. This implies $g=-3/2$ and we get again $\mathbf{D}=3\cdot 2^{-14}>0$. In this case system (14) with $g=-3/2$ possesses the invariant line $y=x-5/16$ which is the second asymptote of the invariant hyperbola and also intersect invariant parabola at the complex singularities M_3 and M_4. As a result we obtain a configuration which is equivalent to *Config.* $\mathcal{PH}$*.5*.

1.1.1.2: *The possibility* $\nu_2=0$. Then considering (13) and the condition $\beta_1\neq 0$ we obtain

$$\mathcal{W}_3=(1+g)(1-3g+4g^2)+4(1+2g)^2n=0.$$

Therefore due to $\zeta_3\neq 0$ (i.e. $1+2g\neq 0$) we obtain:

$$n=\frac{(1+g)(1-3g+4g^2)}{4(1+2g)^2}$$

and we arrive at the systems

$$\begin{aligned}\dot{x}=&\frac{(g-1)(1+g)^2(2g-1)}{8(1+2g)^3}-\frac{(1+g)(1-3g+4g^2)}{4(1+2g)^2}x-\frac{1}{2}(1+g)y+gx^2+xy,\\ \dot{y}=&\frac{(g-1)(1+g)^2(2g-1)}{4(1+2g)^3}x-\frac{(1+g)(1-3g+4g^2)}{2(1+2g)^2}y+(g-1)xy+2y^2.\end{aligned}\tag{18}$$

These systems possess the invariant parabola $\Phi(x,y)=x^2-y=0$ and the invariant hyperbola

$$\begin{aligned}\Psi(x,y)=&(1-g)(1+g)^3+4(1+g)^2(4g^2-1)x-24g(1+g)(1+2g)^2y+\\ &16(1+2g)^4(x-y)y=0.\end{aligned}$$

For the above systems the condition $\gamma_1=\gamma_2=0$ is satisfied and taking into consideration Diagram 3 we calculate

$$\begin{aligned}\beta_1=&-\frac{g^2}{4(1+2g)^6}(g-1)(1+g)^2(2+g)(3g-1)(4+3g)(1+2g+2g^2)^2,\\ \beta_2=&\frac{7g}{2(1+2g)^2}(1+g)(1+2g+2g^2),\quad \theta=-(g-1)(2+g)/2,\\ \widetilde{\mathcal{R}}_1=&-\frac{3g^4}{4(1+2g)^8}(g-1)^2(1+g)^4(2+g)^2(3g-1)(4+3g)(1+2g+2g^2)\end{aligned}$$

We observe that the condition $\beta_1\neq 0$ implies $\theta\beta_2\widetilde{\mathcal{R}}_1\neq 0$, i.e. the conditions provided by Diagram 3 in this case are fulfilled. Considering Lemma 1 for systems (18) we calculate

$$B_1=-\frac{2}{(1+2g)^{15}}(g-1)^2g^8(1+g)^8(2+g)^2(2g-1)(3+2g)(1+g+g^2)\tag{19}$$

and we examine two cases: $B_1\neq 0$ and $B_1=0$.

1.1.1.2.1: *The case* $B_1\neq 0$. According to Lemma 1 systems (18) could not possess any invariant line. These systems possess four finite singularities $M_i(x_i,y_i)$ $(i=1,2,3,4)$ with the coordinates

$$\begin{gathered}x_1=\frac{g-1}{2(1+2g)},\ y_1=\frac{(g-1)^2}{4(1+2g)^2};\quad x_2=\frac{g+1}{2(1+2g)},\ y_2=\frac{(g+1)^2}{4(1+2g)^2};\\ x_3=-\frac{(g+1)(2g-1)}{2(1+2g)},\quad y_3=\frac{(g+1)^2(2g-1)^2}{4(1+2g)^2};\\ x_4=\frac{(g+1)(1-2g+4g^2)}{2(1+2g)^3},\ y_4=\frac{(g^2-1)(2g-1)}{4(1+2g)^3}.\end{gathered}$$

Calculations yield

$$\Phi(x_1,y_1)=\Psi(x_1,y_1)=\Phi(x_2,y_2)=\Psi(x_2,y_2)=\Phi(x_3,y_3)=0,$$
$$\Psi(x_3,y_3)=-16g^3(1+g)^3(g^2+g-1)=-16g^3(1+g)^3\rho_1,$$
$$\Phi(x_4,y_4)=\frac{3g(1+g)}{2(1+2g)^6}(2g^2+2g-1)=\frac{3g(1+g)}{2(1+2g)^6}\varkappa_1,\quad \Psi(x_4,y_4)=\frac{12g^2(1+g)^2}{(1+2g)^2}.$$

Therefore the singular points M_1 and M_2 are the points of intersection of the invariant parabola with invariant hyperbola. The singularity M_3 also is located on the invariant parabola, whereas the singular point M_4 in generic case is outside of the invariant curves. However due to the condition $\zeta_2\neq 0$ (i.e. $g(g+1)\neq 0$) we have $\Psi(x_3,y_3)=0$ if and only if $\rho_1=0$ and $\Phi(x_4,y_4)=0$ if and only if $\varkappa_1=0$.

On the other hand for systems (18) we have

$$\mathbf{D}=-\frac{432}{(1+2g)^18}g^8(1+g)^8(g^2+g-1)^2(2g^2+2g-1)^2=-\frac{432}{(1+2g)^18}g^8(1+g)^8\rho_1^2\varkappa_1^2$$

and due to $\zeta_2\neq 0$ the condition $\mathbf{D}=0$ is equivalent to $\rho_1\varkappa_1=0$. So we consider two subcases: $\mathbf{D}\neq 0$ and $\mathbf{D}=0$.

1.1.1.2.1.1: *The subcase* $\mathbf{D}\neq 0$. Then $\rho_1\varkappa_1\neq 0$ and on the invariant conics are located only three finite singularities: M_1, M_2 and M_3. To detect the position of the point M_3 (located on the invariant parabola) with respect to M_1 and M_2, which are the points of intersection of the invariant conics we calculate:

$$x_3-x_1=-\frac{\rho_1}{1+2g},\quad x_3-x_2=-\frac{g(1+g)}{1+2g},\quad x_2-x_1=\frac{1}{1+2g},$$
$$\operatorname{sign}(x_2-x_1)=\operatorname{sign}(1+2g),\ \operatorname{sign}(x_3-x_1)=-\operatorname{sign}\big(\rho_1(1+2g)\big),$$
$$\operatorname{sign}(x_3-x_2)=-\operatorname{sign}\big(g(1+g)(1+2g)\big)$$

and since $\operatorname{sign}(g(g+1))=\operatorname{sign}(\zeta_2)$ we consider two possibilities: $\zeta_2<0$ and $\zeta_2>0$.

1.1.1.2.1.1.1: *The possibility* $\zeta_2<0$. Then $g(1+g)<0$ and this implies $\rho_1=g(g+1)-1<0$. So we have $-1<g<0$ and considering the above relations among the coordinates of the singularities M_1, M_2 and M_3 we arrive at the configuration *Config.* $\mathcal{PH}.6$ in the case $-1<g<-1/2$. For $-1/2<g<0$ we obtain a configuration which is equivalent to *Config.* $\mathcal{PH}.6$.

1.1.1.2.1.1.2: *The possibility* $\zeta_2>0$. Then we have $g(1+g)>0$ and in this case the sign of ρ_1 is important. Indeed in the case $\delta_1<0$ we obtain the configuration *Config.* $\mathcal{PH}.7$ if $2g+1<0$ and a configuration equivalent to *Config.* $\mathcal{PH}.7$ if $2g+1>0$.

Assuming $\delta_1>0$ we arrive at the configuration *Config.* $\mathcal{PH}.8$ if $2g+1<0$ and at a configuration equivalent to *Config.* $\mathcal{PH}.8$ if $2g+1>0$.

It remains to point out that for systems (18) we have $\nu_3=8\rho_1$ and hence for $\zeta_2>0$ we obtain the configuration *Config.* $\mathcal{PH}.7$ if $\nu_3<0$ and *Config.* $\mathcal{PH}.8$ if $\nu_3>0$.

1.1.1.2.1.2: *The subcase* $\mathbf{D}=0$. Then $\rho_1\varkappa_1=0$ and since $\nu_3=8\rho_1$ we discuss two possibilities: $\nu_3\neq 0$ and $\nu_3=0$.

1.1.1.2.1.2.1: *The possibility* $\nu_3 \neq 0$. Then the condition $\mathbf{D} = 0$ yields

$$\varkappa_1 = 2g^2 + 2g - 1 = 0 \;\Rightarrow\; g = (-1 \pm \sqrt{3})/2$$

and then we obtain:

$$x_3 = \frac{1}{12}(3 \mp \sqrt{3}) = x_4, \quad y_3 = \frac{1}{24}(2 \pm \sqrt{3}) = y_4.$$

So we conclude that for $\varkappa_1 = 0$ the singular point M_4 coalesces with M_3 producing a double singularity on the invariant parabola. Moreover this double singularity is located between M_1 and M_2. As a result we arrive at the configuration *Config.* $\mathcal{PH}$*.9*

1.1.1.2.1.2.2: *The possibility* $\nu_3 = 0$. In this case we obtain s

$$\rho_1 = g^2 + g - 1 = 0 \;\Rightarrow\; g = (-1 \pm \sqrt{5})/2$$

and then we calculate:

$$x_1 = \frac{1}{20}(5 \mp \sqrt{5}) = x_3, \quad y_1 = \frac{1}{40}(7 \mp \sqrt{5}) = y_3.$$

So for $\rho_1 = 0$ we obtain that the singular point M_3 coalesces with M_1 producing a double singularity that is an intersection point of invariant parabola with hyperbola. As a result we get the configuration *Config.* $\mathcal{PH}$*.10*.

1.1.1.2.2: *The case* $B_1 = 0$. Considering (19) and the condition $\beta_1 \neq 0$ we conclude that the condition $B_1 = 0$ implies $(2g - 1)(3 + 2g) = 0$.

If $2g - 1 = 0$ then we have $g = 1/2$ and system (18) possesses the invariant line $y = 0$ which is one of the asymptote of the invariant hyperbola and is tangent at the point $M_3(0, 0)$ to the invariant parabola. Moreover the singular point $M_4(3/32, 0)$ is located on this invariant line. So we arrive at the configuration *Config.* $\mathcal{PH}$*.11*.

Assume now $3 + 2g = 0$. This implies $g = -3/2$ and in this case system (14) with $g = -3/2$ possesses the invariant line $y = x - 1/4$ which is the second asymptote of the invariant hyperbola and also is tangent at the point $M_3(1/2, 1/4)$ to the invariant parabola. Moreover the singular point $M_4(13/32, 5/32)$ is located at this invariant line. Therefore in this case we obtain a configuration which is equivalent to *Config.* $\mathcal{PH}$*.11*.

1.1.2: *The subcase* $\zeta_3 = 0$. Then we have $1 + 2g = 0$, i.e. $g = -1/2$ and for systems (10) we calculate

$$\begin{aligned}
\zeta_1 &= -25/2 \neq 0, \quad \zeta_2 = -1 \neq 0, \quad \zeta_4 = 25(32m - 1 + 8n)/512 \neq 0,\\
\gamma_1 &= -\frac{3}{262144}(5 + 160m + 8n)(16n - 5)(160m + 72n - 15),\\
\nu_1 &= -\frac{40175}{32768}(5 + 160m + 8n)(160m + 72n - 15).
\end{aligned} \tag{20}$$

and since $\nu_1 \neq 0$ we obtain that the condition $\gamma_1 = 0$ implies $16n - 5 = 0$, i.e. $n = 5/16$. Then we obtain:

$$\gamma_2 = -\frac{2480625}{524288}(3 + 64m)^3, \quad \zeta_4 = \frac{25}{1024}(3 + 64m).$$

Clearly the condition $\zeta_4 \neq 0$ implies $\gamma_2 \neq 0$ and we conclude that in the case $\zeta_3 = 0$ we cannot have any invariant hyperbola.

1.2: *The case* $\nu_1 = 0$. Then considering (11) and the condition $\zeta_1 \neq 0$ we obtain $\mathcal{V}_1\mathcal{V}_2 = 0$. We have the following lemma.

Lemma 3. *The condition* $\mathcal{V}_2 = 0$ *for systems* (10) *could be transferred to the condition* $\mathcal{V}_1 = 0$ *via an affine transformation.*

Proof: Applying to systems (10) the transformation

$$x_1 = -x + 1/2, \quad y_1 = -x + y + 1/4,$$

we obtain the systems

$$\begin{aligned}\dot{x}_1 =& -\frac{1}{8}(g + 8m + 4n) + \frac{1}{4}(1 + 2g + 4n)x_1 + \frac{g}{2}y_1 - (1 + g)x_1^2 + x_1y_1,\\ \dot{y}_1 =& -\frac{1}{4}(g + 8m + 4n)x_1 + \frac{1}{2}(1 + 2g + 4n)y_1 - (g + 2)x_1y_1 + 2y_1^2.\end{aligned}$$

So setting the new parameters

$$\begin{aligned}m_1 = -\frac{1}{8}(g + 8m + 4n), \quad n_1 = \frac{1}{4}(1 + 2g + 4n), \quad g_1 = -(1 + g) \;\Rightarrow\\ m = -\frac{1}{8}(g_1 + 8m_1 + 4n_1), \quad n = \frac{1}{4}(1 + 2g_1 + 4n_1) \quad g = -(1 + g_1),\end{aligned} \tag{21}$$

we obtain the family of systems

$$\dot{x}_1 = m_1 + n_1x_1 - \frac{1 + g_1}{2}y_1 + g_1x_1^2 + x_1y_1, \quad \dot{y} = 2m_1x_1 + 2n_1y_1 + (g_1 - 1)x_1y_1 + 2y_1^2$$

which has the same form as (10). Then considering (21) calculations yield:

$$\begin{aligned}\mathcal{V}_2(g, m, n) = 3 + 13g + 13g^2 + 3g^3 - 20m + 2n + 6gn =\\ -(1 - 3g_1 - g_1^2 + 3g_1^3 - 20m_1 - 6n_1 + 6g_1n_1) = -\mathcal{V}_1(g_1, m_1, n_1).\end{aligned}$$

and this completes the proof of the lemma. ∎

Thus we may assume

$$V_1 = 1 - 3g - g^2 + 3g^3 - 20m - 6n + 6gn = 0$$

and we obtain

$$m = \frac{1}{20}(g - 1)(2g - 1 + 3g^2 + 6n). \tag{22}$$

In this case for systems (10) calculations yield

$$\begin{aligned}\gamma_2 = -\frac{63}{80}(g - 1)^2(2 + g)(3g - 1)(2 + 3g)\mathcal{W}_1'(\mathcal{W}_2')^2\mathcal{W}_3',\\ \mathcal{W}_1' = 1 - 2g + 2g^2 + 4n, \quad \mathcal{W}_2' = 1 + 8g + 7g^2 + 4n,\\ \mathcal{W}_3' = -11 + 2g + 13g^2 + 36n.\end{aligned}$$

On the other for systems (10) with the value of the parameter m from (22) we have

$$\beta_1 = -\frac{1}{80}(g - 1)(3g - 1)(\mathcal{W}_2')^2$$

and hence the condition $\beta_1 \neq 0$ implies $(g-1)(3g-1)\mathcal{W}_2' \neq 0$. Therefore the condition $\gamma_2 = 0$ is equivalent to $\mathcal{W}_1'\mathcal{W}_3' = 0$.

On the other hand for systems (10) with the value of the parameter m given in (22) we calculate

$$\nu_6 = -\frac{3977}{40} g^3(g-2)(3+g)(2+3g)\mathcal{W}_2'\mathcal{W}_3', \quad \zeta_4 = \frac{1}{40}(g-2)(3+g)(2+3g)\mathcal{W}_2' \tag{23}$$

and since and $\zeta_2\zeta_4 \neq 0$ we conclude that the condition $\nu_6 = 0$ is equivalent to $\mathcal{W}_3' = 0$. So we discuss two subcases: $\nu_6 \neq 0$ and $\nu_6 = 0$.

1.2.1: *The subcase* $\nu_6 \neq 0$. Then the condition $\gamma_2 = 0$ implies $\mathcal{W}_1' = 0$, i.e. we have $n = (2g-1-2g^2)/4$ and we arrive at the 1-parameter family of systems

$$\begin{aligned} \dot{x} =& \frac{1}{8}(g-1)(2g-1) + \frac{1}{4}(2g-1-2g^2)x - \frac{1}{2}(1+g)y + gx^2 + xy, \\ \dot{y} =& \frac{1}{4}(g-1)(2g-1)x + \frac{1}{2}(2g-1-2g^2)y + (g-1)xy + 2y^2 \end{aligned} \tag{24}$$

which possess the invariant line $y = x - 1/4$ and the following invariant parabola and hyperbola:

$$\Phi(x,y) = x^2 - y = 0, \quad \Psi(x,y) = 1 - g + 2(2g-1)x - 4(1+g)y + 8xy = 0.$$

For these systems the condition $\gamma_1 = \gamma_2 = 0$ is satisfied and taking into account Diagram 3 we calculate

$$\begin{aligned} \theta = -(g-1)(2+g)/2,\ \beta_1 = -\frac{5g^2}{16}(g-1)(2+g)^2(3g-1),\ \beta_2 = -\frac{g}{4}(g-2)(2+g)(2+3g), \\ \widetilde{\mathcal{R}}_1 = \frac{15g^3}{32}(g-1)^2(2+g)^2(3g-1)(g-2), \quad \zeta_4 = \frac{g}{8}(g-2)(2+g)(3+g)(2+3g). \end{aligned} \tag{25}$$

We observe that the condition $\zeta_4\beta_1 \neq 0$ implies $\theta\beta_2\widetilde{\mathcal{R}}_1 \neq 0$ and therefore we conclude that the corresponding conditions provided by Diagram 3 are fulfilled and systems (24) possess a unique invariant hyperbola.

We examine systems (24). Considering Lemma 1 for these systems we calculate

$$B_1 = 0,\ B_2 = -\frac{243}{2}(g-1)^2 g^4(2g-1)(x-y)^4 \tag{26}$$

and we examine two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1.2.1.1: *The possibility* $B_2 \neq 0$. According to Lemma 1 systems (24) could not possess an invariant line different from $y = x - 1/4$. These systems possess four finite singularities $M_i(x_i, y_i)$ $(i = 1,2,3,4)$ with the coordinates

$$\begin{aligned} x_1 = \frac{1}{2},\ y_1 = \frac{1}{4};\ \ x_2 = \frac{1-2g}{2},\ y_2 = \frac{(1-2g)^2}{4};\ \ x_3 = \frac{g-1}{2},\ y_3 = \frac{(g-1)^2}{4}; \\ x_4 = \frac{1-g+g^2}{2(1+g)},\ y_4 = \frac{(g-1)(2g-1)}{4(1+g)}. \end{aligned} \tag{27}$$

Since $g + 1 \neq 0$ (due to the condition $\zeta_2 \neq 0$) we deduce that all singularities are located on the finite part of the phase plan.

We observe that the singular point M_1 is a point of tangency of the invariant line $y = x-1/4$ with invariant parabola as well as with a branch of the hyperbola. The point M_3 is a point of intersection of invariant parabola with hyperbola and the singularity M_2 (respectively, M_4) is located on the invariant parabola (respectively, line).

On the other hand for systems (24) we calculate

$$\mathbf{D} = -\frac{3}{4}(g-2)^6 g^{10}(3g-2)^2 \leq 0$$

and according to [1, Table 5.1] the condition $\mathbf{D} = 0$ is necessary for a quadratic system to possess multiple finite singularity. So we consider two cases: $\mathbf{D} \neq 0$ and $\mathbf{D} = 0$.

1.2.1.1.1: *The case* $\mathbf{D} \neq 0$. So all the finite singularities are distinct and we have to determine the positions of the singular points M_2 and M_4 with respect to the points of intersection of invariant curves. So we calculate:

$$x_2 - x_1 = -g,\ x_4 - x_1 = -\frac{g(g-2)}{2(1+g)},\ x_3 - x_2 = \frac{2-3g}{2};\quad \operatorname{sign}(x_2 - x_1) = -\operatorname{sign}(g),$$
$$\operatorname{sign}(x_4 - x_1) = -\operatorname{sign}\big(g(g+1)(g-2)\big),\ \operatorname{sign}(x_3 - x_2) = \operatorname{sign}(2-3g)$$

and since $\operatorname{sign}(g(g+1)) = \operatorname{sign}(\zeta_2)$ we consider two subcases: $\zeta_2 < 0$ and $\zeta_2 > 0$.

1.2.1.1.1.1: *The subcase* $\zeta_2 < 0$. Then $g(1+g) < 0$ (i.e. $-1 < g < 0$) and this implies

$$x_2 - x_1 > 0,\quad x_4 - x_1 > 0,\quad x_3 - x_2 < 0.$$

Therefore we arrive at the configuration *Config.* $\mathcal{PH}$*.12*.

1.2.1.1.1.2: *The subcase* $\zeta_2 > 0$. Then we have $g(g+1) > 0$ and we observe that we have the following possible bifurcation values for the parameter g: $g \in \{-1, 0, 2/3, 2\}$.

On the other hand for systems (24) we calculate

$$\nu_7 = \frac{1227}{4} g^2(2g+1),\quad \nu_8 = \frac{1353975}{4} g^2(3g-2),\quad \nu_9 = \frac{47035}{8} g^2(g-2).$$

We observe that $\nu_7\nu_8\nu_9 \neq 0$ due to $g(g+1) > 0$ and $\mathbf{D} \neq 0$ (i.e. $(3g-2)(g-2) \neq 0$). Therefore we have

$$\operatorname{sign}(\nu_7) = \operatorname{sign}(2g+1),\quad \operatorname{sign}(\nu_8) = \operatorname{sign}(3g-2),\quad \operatorname{sign}(\nu_9) = \operatorname{sign}(g-2).$$

So for $g(g+1) > 0$ and $\mathbf{D} \neq 0$ we detect the following configurations:

$\nu_7 < 0$ (i.e. $g < -1$) $\Rightarrow$ $x_2 < x_1$, $x_4 < x_1$, $x_2 > x_3$ $\Rightarrow$ *Config.* $\mathcal{P}$*.13*;
$\nu_7 > 0$, $\nu_8 < 0$ (i.e. $0 < g < 2/3$) $\Rightarrow$ $x_2 > x_1$, $x_4 < x_1$, $x_2 > x_3$ $\Rightarrow$ *Config.* $\mathcal{P}$*.14*;
$\nu_7 > 0$, $\nu_8 > 0$, $\nu_9 < 0$ (i.e. $2/3 < g < 2$) $\Rightarrow$ $x_2 < x_1$, $x_4 < x_1$, $x_2 < x_3$ $\Rightarrow$ *Config.* $\mathcal{P}$*.15*;
$\nu_7 > 0$, $\nu_8 > 0$, $\nu_9 > 0$ (i.e. $g > 2$) $\Rightarrow$ $x_2 < x_1$, $x_4 > x_1$, $x_2 < x_3$ $\Rightarrow$ *Config.* $\mathcal{P}$*.16*.

1.2.1.1.2: *The case* $\mathbf{D} = 0$. Then we have $(g-2)(3g-2) = 0$ and since $\zeta_4 \neq 0$ we have $g-2 \neq 0$ (see (25)). So the condition $\mathbf{D} = 0$ implies $g = 2/3$ and considering (27) we determine that the finite singularities M_2 and M_3 coalesce producing a double point which is a point of intersection of the invariant parabola and hyperbola. As a result we arrive at the configuration *Config.* $\mathcal{PH}$*.17*.

1.2.1.2: *The possibility* $B_2 = 0$. Considering (26) and (25) we conclude that due to $\beta_1 \neq 0$ the condition $B_2 = 0$ gives us $g = 1/2$. So we arrive at the system

$$\dot{x} = -\frac{1}{8}x - \frac{3}{4}y + \frac{1}{2}x^2 + xy, \quad \dot{y} = -\frac{1}{4}y(1 + 2x - 8y)$$

which possesses two invariant lines: $y = 0$ and $y = x - 1/4$. Moreover this system beside the invariant parabola possesses the invariant hyperbola $\Psi(x, y) = 1 - 12y + 16xy = 0$.

Considering (27) we determine that the above systems possess four real singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$ with the coordinates

$$x_1 = \frac{1}{2},\ y_1 = \frac{1}{4};\quad x_2 = 0,\ y_2 = 0;\quad x_3 = -\frac{1}{4},\ y_3 = \frac{1}{16};\quad x_4 = \frac{1}{4},\ y_4 = 0.$$

Since we have $0 < g = 1/2 < 2/3$ it was detected earlier that systems (24) in this case possess the configuration *Config.* $\mathcal{PH}$*.14*. Therefore adding the invariant line $y = 0$ which is tangent to the parabola at the singular point $M_2(0, 0)$ we get the new configuration *Config.* $\mathcal{PH}$*.18*.

1.2.2: *The subcase* $\nu_6 = 0$. Considering (23) and the condition $\zeta_2\zeta_4 \neq 0$ we conclude that the condition $\nu_6 = 0$ is equivalent to

$$\mathcal{W}_3' = -11 + 2g + 13g^2 + 36n = 0 \ \Rightarrow\ n = (1 + g)(11 - 13g)/36.$$

So we arrive at the 1-parameter family of systems

$$\begin{aligned} \dot{x} =& \frac{1}{24}(g-1)(1+g)^2 - \frac{1}{36}(1+g)(13g-11)x - \frac{1}{2}(1+g)y + gx^2 + xy, \\ \dot{y} =& \frac{1}{12}(g-1)(1+g)^2x - \frac{1}{18}(1+g)(13g-11)y + (g-1)xy + 2y^2 \end{aligned} \tag{28}$$

which beside the invariant parabola $\Phi(x, y) = x^2 - y = 0$ possess the invariant hyperbola:

$$\Psi(x, y) = -(1 + g)^3 + 18(1 + g)^2x - 108(1 + g)y + 216xy = 0. \tag{29}$$

Considering Diagram 3 for systems (28) we calculate

$$\begin{gathered} \theta = -(g-1)(2+g)/2,\ \beta_1 = -\frac{5}{324}(g-1)(1+g)^2(3g-1)(2+5g)^2, \\ \beta_2 = -\frac{1}{18}(g-2)(1+g)(2+3g)(2+5g), \\ \widetilde{\mathcal{R}}_1 = \frac{5}{324}(g-2)(g-1)^2(g+1)^4(2+g)(3g-1)(2+5g), \\ \zeta_4 = \frac{1}{36}(g-2)(1+g)(3+g)(2+3g)(2+5g) \end{gathered} \tag{30}$$

and following Diagram 3 we have to consider two possibilities: $\theta \neq 0$ and $\theta = 0$.

1.2.2.1: *The possibility* $\theta \neq 0$. Then $(g - 1)(2 + g) \neq 0$ and therefore the condition $\zeta_4\beta_1 \neq 0$ implies $\beta_2\widetilde{\mathcal{R}}_1 \neq 0$. So the corresponding conditions provided by Diagram 3 are fulfilled and systems (28) possess a unique invariant hyperbola.

Considering Lemma 1 for systems (28) we calculate

$$B_1 = \frac{2}{19683}(g-2)^3(g-1)^2(2g-1)(1+g)^8(2+3g) \tag{31}$$

and we examine two cases: $B_1 \neq 0$ and $B_1 = 0$.

1.2.2.1.1: *The case* $B_1 \neq 0$. According to Lemma 1 systems (28) could not possess any invariant line. These systems possess four finite singularities $M_i(x_i, y_i)$ $(i = 1, 2, 3, 4)$ with the coordinates

$$\begin{gathered} x_1 = \frac{1+g}{6},\ y_1 = \frac{(1+g)^2}{36};\quad x_2 = \frac{5g^2+g-4}{18g},\ y_2 = \frac{g^2-1}{12};\quad x_{3,4} = -\frac{1}{8}\big(2g-1 \pm \sqrt{U_2}\,\big), \\ y_{3,4} = \frac{1}{36}\big(17g^2-8g-7 \pm 2(2g-1)\sqrt{U_2}\,\big),\quad U_2 = 13g^2-4g-8. \end{gathered} \tag{32}$$

We observe that the invariant conics of systems (28) have the unique point of intersection $M_1(0, 0)$. The singular point M_2 is located on the invariant hyperbola whereas M_3 and M_4 are located on the invariant parabola. Moreover these last singularities are either real (for $U_2 > 0$), or complex (for $U_2 < 0$), or coinciding (for $U_2 = 0$).

We prove the next lemma.

Lemma 4. *If $\zeta_2 < 0$ that the singular points M_2 and M_1 are located on different branches of the invariant hyperbola.*

Proof: We determine that the invariant parabola $\Psi(x, y) = 0$ given in (29) has the asymptotes $x = (g + 1)/2$ and $y = -(g + 1)^2/12$ and it is clear the only the branch located over the asymptote $y = -(g+1)^2/12$ intersects the invariant parabola at the point $M_1(x_1, y_1)$. Moreover since $g(g+1) < 0$, i.e. $-1 < g < 0$ (due to $\zeta_2 < 0$) we deduce that the upper branch is located on the semi-plane $x < (g + 1)/2$ whereas the second branch is on the semi-plane $x > (g + 1)/2$.

On the other hand we have

$$\begin{gathered} x_2 - x_1 = \frac{(1+g)(g-2)}{9g},\quad y_2 - y_1 = \frac{(1+g)(g-2)}{8} \\ \operatorname{sign}(x_2 - x_1) = \operatorname{sign}\big(g(1+g)(g-2)\big),\quad \operatorname{sign}(y_2 - y_1) = \operatorname{sign}\big((1+g)(g-2)\big). \end{gathered} \tag{33}$$

Since $-1 < g < 0$ we obtain $x_2 > x_1$ and $y_2 < y_1$ and it is clear that the point M_2 could not be located on the upper branch of hyperbola. Therefore it is located on the second branch and this completes the proof of the lemma. ∎

For systems (28) we calculate

$$\mathbf{D} = -\frac{16}{3^{13}}(g-2)^8(1+g)^8(-8-4g+13g^2) = -\frac{16}{3^{13}}(g-2)^8(1+g)^8U_2 \Rightarrow \operatorname{sign}(\mathbf{D}) = -\operatorname{sign}(U_2)$$

and we discuss three subcases: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

1.2.2.1.1.1: *The subcase* $\mathbf{D} < 0$. Then $U_2 > 0$ and hence the singularities $M_{3,4}$ are real. So it is clear that we have to know the position of these singularities with respect to M_1 all located on the invariant parabola. We calculate:

$$\begin{gathered} (x_3 - x_1)(x_4 - x_1) = \frac{1}{9}(2+g-g^2) = \frac{1}{9}\rho_2,\quad (x_3 - x_1) + (x_4 - x_1) = -g, \\ \operatorname{sign}\big((x_3 - x_1)(x_4 - x_1)\big) = \operatorname{sign}(\rho_2),\ \operatorname{sign}\big((x_3 - x_1) + (x_4 - x_1)\big) = -\operatorname{sign}(g). \end{gathered}$$

We point out that the position of the singular point M_2 with respect to M_1 is determined by the relations (33) and since $\operatorname{sign}(g(g + 1)) = \operatorname{sign}(\zeta_2)$ we consider two possibilities: $\zeta_2 < 0$ and $\zeta_2 > 0$.

If $\zeta_2 < 0$ then $g(1+g) < 0$ (i.e $-1 < g < 0$) and this implies $\rho_2 = (2-g)(g+1) > 0$ and $g-2 < 0$. Thus considering the above relations among the coordinates of the singularities M_1, M_2, M_3 and M_4 and Lemma 4 we obtain $x_3 > x_1$, $x_4 > x_1$ and we arrive at the unique configuration *Config.* $\mathcal{PH}$*.19*.

Assume now that $\zeta_2 > 0$. This implies $g(g+1) > 0$ and we observe that we have the following possible bifurcation values for the parameter g: $g \in \{-1, 0, 2\}$.

So in the case $\mathbf{D} < 0$ (i.e. $U_2 > 0$) and $\zeta_2 > 0$ (i.e. $g(g+1) > 0$) we obtain the following configurations:

$$\begin{array}{lll} g < -1 & \Rightarrow\ x_2 < x_1,\, x_3 < x_1,\, x_4 > x_1 & \Rightarrow\ \textit{Config.}\,\mathcal{P}.20; \\ 0 < g < 2 & \Rightarrow\ x_2 < x_1,\, x_3 < x_1,\, x_4 < x_1 & \Rightarrow\ \textit{Config.}\,\mathcal{P}.21; \\ g > 2 & \Rightarrow\ x_2 > x_1,\, x_3 < x_1,\, x_4 > x_1 & \Rightarrow\ \simeq \textit{Config.}\,\mathcal{P}.20; \end{array}$$

We observe that due to $g(g+1) > 0$ the condition $\rho_2 = (2-g)(g+1) > 0$ is equivalent to $0 < g < 2$ whereas for $\rho_2 < 0$ we have either $g < -1$ or $g > 2$.

On the other hand for systems (28) we calculate

$$\nu_{10} = \frac{338845}{18}(2-g)g(1+g)^2 = \frac{338845}{18}g(g+1)\rho_2 \ \Rightarrow\ \operatorname{sign}(\nu_{10}) = \operatorname{sign}(\rho_2)$$

and therefore in the case $\mathbf{D} < 0$ and $\zeta_2 > 0$ systems (28) possess the configuration *Config.* $\mathcal{PH}$*.20* if $\nu_{10} < 0$ and *Config.* $\mathcal{PH}$*.21* if $\nu_{10} > 0$.

1.2.2.1.1.2: *The subcase* $\mathbf{D} > 0$. Then $U_2 < 0$ and hence the singularities $M_{3,4}$ are complex. As it was shown earlier (see (33)) the position of the singular point M_2 with respect to M_1 is determined by the sign of $g(1+g)(g-2)$.

If $\zeta_2 < 0$ (i.e. $g(g+1) < 0$) then considering Lemma 4 we arrive at the configuration *Config.* $\mathcal{PH}$*.22*.

Assume now $\zeta_2 > 0$, i.e. $g(g+1) < 0$. Then we have either $g < -1$ or $g > 0$. In this case the condition $U_2 < 0$ implies $g_1 < g < g_2$, where $U_2(g_1) = U_2(g_2) = 0$ and

$$g_1 = \frac{2}{13}(1-3\sqrt{3}) \approx -0.6456, \quad g_2 = \frac{2}{13}(1+3\sqrt{3}) \approx 0.9532 \ \Rightarrow\ g-2 < 0.$$

Therefore we get the unique configuration *Config.* $\mathcal{PH}$*.23*.

1.2.2.1.1.3: *The subcase* $\mathbf{D} = 0$. This implies $U_2 = 0$ and clearly the singular points M_3 and M_4 coalesce. Then since $\zeta_2 = 4g(1+g)$ we conclude the condition $U_2 = 0$ gives us $g = g_1$ if $\zeta_2 < 0$ and $g = g_2$ if $\zeta_2 > 0$. Therefore we arrive at the configuration *Config.* $\mathcal{PH}$*.24* if $\zeta_2 < 0$ and *Config.* $\mathcal{PH}$*.25* if $\zeta_2 > 0$.

1.2.2.1.2: *The case* $B_1 = 0$. Considering (31) and (30) we conclude that due to $\zeta_4\beta_1 \neq 0$ the condition $B_1 = 0$ is equivalent to $2g-1 = 0$. Then $g = 1/2$ and systems (28) possess additionally the invariant line $y = -3/16$. This line is the horizontal asymptote of the invariant hyperbola and intersects the invariant parabola at the complex points M_3 and M_4. As a result we get the configuration *Config.* $\mathcal{PH}$*.26*.

1.2.2.2: *The possibility* $\theta = 0$. This implies $(g-1)(g+2) = 0$ and due to $\beta_1 \neq 0$ (i.e. $g-1 \neq 0$) we obtain $g = -2$.

Considering Diagram 3 for systems (28) we calculate:

$$\theta = 0,\ \widetilde{N} = 27(x-y)^2 \neq 0,\ \beta_6 = \frac{4}{3} \neq 0,\ \beta_{10} = -70 \neq 0,\ \gamma_7 = 0,\ \widetilde{\mathcal{R}}_6 = \frac{5056}{81} \neq 0$$

and by Diagram 3 in the case $g=-2$ systems (28) could possess only one invariant hyperbola.

On the other hand in the case under consideration we have $B_1=-2560/2187\neq 0$ and by Lemma 1 we could not have any invariant line. So considering (32) we obtain a configuration which is equivalent to *Config.* $\mathcal{PH}$*.20*. It is important to point out the following remark.

Remark 7. *As it was proved earlier in the case $\theta\neq 0$ systems* (28) *possess the configuration Config.* $\mathcal{PH}$*.20 if and only if the conditions* $\mathbf{D}<0$, $\zeta_2>0$ *and* $\nu_{10}<0$ *are satisfied. In the case* $\theta=0$ *(i.e.* $g=-2$*) for systems* (28) *we calculate:*

$$\mathbf{D}=-13\cdot 2^{22}3^{-13}<0,\quad \zeta_2=8>0,\quad \nu_{10}=-1355380/9<0$$

and we conclude that the condition given by $\theta=0$ is not a bifurcation one and we could omit it in the bifurcation diagram presented in the parameter space $\mathbb{R}^{12}$ for general quadratic systems.

2: *The possibility* $\beta_1=0$. Since for systems (10) the condition $\gamma_1=\gamma_2=0$ is necessary we calculate

$$Res_m(\beta_1,\gamma_1)=25\cdot 2^{-18}(g-1)(2+g)(3g-1)(4+3g)(1+8g+7g^2+4n)^4.$$

We claim that due to $\zeta_4\neq 0$ the condition $1+8g+7g^2+4n\neq 0$ must hold. Indeed supposing the contrary we get $n=-(1+g)(1+7g)/4$ and then calculations yield:

$$\beta_1=-\frac{25}{256}(8m-1-3g+g^2+3g^3)^2,\quad \zeta_4=-\frac{1}{32}(g-2)(3+g)(8m-1-3g+g^2+3g^3)$$

and clearly the condition $\zeta_4\neq 0$ implies $\beta_1\neq 0$. This proves our claim.

Thus we conclude that the condition $Res_m(\beta_1,\gamma_1)=0$ yields

$$(g-1)(2+g)(3g-1)(4+3g)=0$$

and since $\theta=-(g-1)(2+g)/2$ we examine two cases: $\theta\neq 0$ and $\theta=0$.

2.1: *The case* $\theta\neq 0$. Then we obtain $(3g-1)(4+3g)=0$. We claim that via an affine transformation the condition $4+3g=0$ for systems (10) could be transferred to the condition $3g-1=0$. Indeed as it is shown in the proof of Lemma 3 systems (10) with the parameters (m,n,g) could be brought to the same systems but with the new parameters (m_1,n_1,g_1) determined in (21). In particular we have $g_1=-(1+g)$ (then $g=-(1+g_1)$) and we obtain

$$4+3g=4-3(1+g_1)=-(3g_1-1)$$

and this completes the proof of our claim.

Thus we may assume $g=1/3$ and then for systems (10) we calculate:

$$\beta_1=-\frac{1}{108}(5m+n)(80+135m+99n),\quad \gamma_1=\frac{7}{216}(5m+n)^2(20-45m+9n)$$

and evidently the condition $\beta_1=\gamma_1=0$ yields $m=-n/5$ and we arrive at the family of systems

$$\dot{x}=-\frac{n}{5}+nx-\frac{2}{3}y+\frac{1}{3}x^2+xy,\quad \dot{y}=-\frac{2n}{5}x+2ny-\frac{2}{3}xy+2y^2. \tag{34}$$

For these systems considering Diagram 3 we calculate:

$$\beta_1 = 0, \quad \beta_6 = \frac{2}{135}(10+9n), \quad \beta_2 = \frac{1}{2}(10+9n), \quad \widetilde{\mathcal{R}}_3 = -\frac{10}{3}n^2,$$
$$\gamma_4 = \frac{16}{98415}(5+36n)(48n-5)(4+75n)(81n-20n), \quad \delta_1 = -\frac{2}{45}n(1257n-50),$$
$$\zeta_4 = -\frac{5}{27}(10+9n), \quad B_1 = 16\cdot 3^{-7}5^{-5}n^2(5+36n)(20+39n)(20+63n)$$

Therefore due to $\zeta_4 \neq 0$ we obtain $\beta_6\beta_2 \neq 0$ and according to Diagram 3 for the existence of at least one invariant hyperbola the conditions $\gamma_4 = 0$ and $\widetilde{\mathcal{R}}_3 \neq 0$ are necessary and sufficient. We observe that for $\gamma_4 = 0$ we get $\widetilde{\mathcal{R}}_3\delta_1 \neq 0$ and by Diagram 3 systems (34) could possess a single invariant parabola. We discuss two subcases: $B_1 \neq 0$ and $B_1 = 0$

2.1.1: *The subcase* $B_1 \neq 0$. Then we have $5+36n \neq 0$ and therefore the condition $\gamma_4 = 0$ gives us $(4+75n)(48n-5)(81n-20n) = 0$.

On the other hand for system (34) we have

$$\nu_2 = \frac{656000}{243}(10+9n)(4+75n), \quad \nu_4 = \frac{250}{81}(81n-20) \tag{35}$$

and we examine two possibilities: $\nu_2 \neq 0$ and $\nu_2 = 0$.

2.1.1.1: *The possibility* $\nu_2 \neq 0$. Then $4+75n \neq 0$ and hence the condition $\gamma_4 = 0$ implies $(48n-5)(81n-20) = 0$. Considering the value of the invariant polynomial ν_4 we discuss two cases: $\nu_4 \neq 0$ and $\nu_4 = 0$.

2.1.1.1.1: *The case* $\nu_4 \neq 0$. Then we have $81n-20 \neq 0$ and we obtain $48n-5 = 0$. This implies $n = 5/48$ and we arrive at the system

$$\dot{x} = -\frac{1}{48} + \frac{5}{48}x - \frac{2}{3}y + \frac{1}{3}x^2 + xy, \quad \dot{y} = -\frac{1}{24}x - \frac{5}{24}y - \frac{2}{3}xy + 2y^2, \tag{36}$$

which possess the following invariant parabola and hyperbola:

$$\Phi(x,y) = x^2 - y = 0, \quad \Psi(x,y) = -1 + 16x - 96y + 256xy - 256y^2 = 0.$$

The system (36) possesses two real and two complex singularities $M_i(x_i, y_i)$, $i = 1,2,3,4$ with the coordinates

$$x_1 = -\frac{7}{32},\ y_1 = -\frac{1}{32};\quad x_2 = \frac{1}{4},\ y_2 = \frac{1}{16};\quad x_{3,4} = \frac{1}{24}\big(1 \pm i\sqrt{47}\big),\ y_{3,4} = \frac{1}{288}\big(-23 \pm i\sqrt{47}\big).$$

We determine that the singular point M_2 is a point of tangency of the invariant parabola and hyperbola and the real point M_1 lies on the invariant hyperbola. So we get a configuration which is equivalent to *Config.* $\mathcal{PH}$*.3*.

2.1.1.1.2: *The case* $\nu_4 = 0$. This implies $n = 20/81$ and we arrive at the system

$$\dot{x} = -\frac{4}{81} + \frac{20}{81}x - \frac{2}{3}y + \frac{1}{3}x^2 + xy, \quad \dot{y} = -\frac{8}{81}x + \frac{40}{81}y - \frac{2}{3}xy + 2y^2, \tag{37}$$

which possess the following invariant parabola and hyperbola:

$$\Phi(x,y) = x^2 - y = 0, \quad \Psi(x,y) = -8 + 108x - 486y + 729xy = 0.$$

The system (37) possesses two real and two complex singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$ with the coordinates

$$x_1 = -\frac{14}{27},\ y_1 = -\frac{2}{27};\quad x_2 = \frac{2}{9},\ y_2 = \frac{4}{81};\quad x_{3,4} = \frac{1}{18}\big(1 \pm i\sqrt{71}\big),\ y_{3,4} = \frac{1}{162}\big(-35 \pm i\sqrt{71}\big).$$

We observe that M_2 is a point of intersection of the invariant conics whereas M_1 is located on the invariant hyperbola. Taking into consideration that the asymptotes of the hyperbola are $x = 0$ and $y = 0$ we obtain that the parabola intersect the hyperbola at the infinite singular point $N[0 : 1 : 0]$. As a result we arrive at a configuration which is equivalent to *Config.* $\mathcal{PH}$*.23*.

2.1.1.2: *The possibility* $\nu_2 = 0$. Considering (35) due to $\zeta_4 \neq 0$ (i.e. $10+9n \neq 0$) we get $n = -4/75$ and we arrive at the system

$$\dot{x} =\frac{4}{375} - \frac{4}{75}x - \frac{2}{3}y + \frac{1}{3}x^2 + xy,\quad \dot{y} = \frac{8}{375}x - \frac{8}{75}y - \frac{2}{3}xy + 2y^2, \tag{38}$$

which possess the following invariant parabola and hyperbola:

$$\Phi(x, y) = x^2 - y = 0,\quad \Psi(x, y) = 8 - 20x - 150y + 625xy - 625y^2 = 0.$$

The system (38) possess four real singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$ with the coordinates

$$x_1 = -\frac{1}{5},\ y_1 = \frac{1}{25};\quad x_2 = \frac{14}{125},\ y_2 = \frac{2}{125};\quad x_3 = \frac{2}{15},\ y_3 = \frac{4}{225},\ x_4 = \frac{2}{5},\ y_3 = \frac{4}{25}.$$

In this case we arrive at a configuration which is equivalent to *Config.* $\mathcal{PH}$*.7*.

2.1.2: *The subcase* $B_1 = 0$. Since the condition $\gamma_4 = 0$ is necessary we conclude that we must have $5 + 36n = 0$. Then $n = -5/36$ and we arrive at the systems

$$\dot{x} =\frac{1}{36} - \frac{5}{36}x - \frac{2}{3}y + \frac{1}{3}x^2 + xy,\quad \dot{y} = \frac{1}{18}x - \frac{5}{18}y - \frac{2}{3}xy + 2y^2,$$

which belongs to the family of systems (24) for $g = 1/3$. This family was investigated earlier and since $0 < g = 1/3 < 2/3$ we get the configuration *Config.* $\mathcal{PH}$*.14*.

2.2: *The case* $\theta = 0$. This implies $(g - 1)(g + 2) = 0$ and again considering the proof of Lemma 3 (see (21)) we obtain

$$g + 2 = -(1 + g_1) + 2 = -(g_1 - 1)$$

and therefore due to an affine transformation we may assume $g = 1$. In this case for systems (10) we calculate:

$$\beta_1 = -\frac{1}{4}m(128 + 25m + 32n),\quad \gamma_1 = \frac{105}{8}m^2(8 - 5m + 2n)$$

and evidently the condition $\beta_1 = \gamma_1 = 0$ yields $m = 0$ and we arrive at the family of systems

$$\dot{x} =nx - y + x^2 + xy,\quad \dot{y} = 2y(n + y) \tag{39}$$

for which $\gamma_1 = \gamma_2 = 0$. We observe that systems (39) beside the invariant parabola $\Phi(x, y) = x^2 - y = 0$ possess two invariant lines: $y = 0$ and $y = -n$. In order to determine the conditions for the existence of an invariant hyperbolas considering Diagram 3 we calculate:

$$\theta = 0,\ \widetilde{N} = 27y^2,\ \beta_6 = 0,\quad \beta_2 = n + 4,\ \beta_7 = -30,\ \gamma_8 = 126(1 + 4n)(1 + 9n)^2(1 + 16n),$$

$$\beta_{10} = -70,\ \widetilde{\mathcal{R}}_7 = -\frac{3}{2}(1 + 7n + 18n^2),\quad \delta_4 = 6(1 + n)(1 + 9n),$$

$$\zeta_4 = -2(4 + n),\ B_1 = 0,\ B_2 = -648(1 + n)^2(1 + 4n)y^4.$$

We observe that the condition $\zeta_4 \neq 0$ implies $\beta_2 \neq 0$ and therefore according to Diagram 3 the systems (39) possess at least one invariant hyperbola if and only if $\gamma_8 = 0$ and $\beta_{10}\widetilde{\mathcal{R}}_7 \neq 0$. On the other hand it is clear that the condition $\gamma_8 = 0$ implies $\widetilde{\mathcal{R}}_7 \neq 0$.

Thus by Diagram 3 for the existence of at least one invariant hyperbola we must impose the condition $\gamma_8 = 0$ and we discuss two subcases: $\delta_4 \neq 0$ and $\delta_4 = 0$.

2.2.1: *The subcase* $\delta_4 \neq 0$. Then the condition $\gamma_8 = 0$ implies $(1+4n)(1+16n) = 0$ and we examine two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

2.2.1.1: *The possibility* $B_2 \neq 0$. Then $1+4n \neq 0$ and we obtain $1+16n = 0$, i.e. $n = -1/16$. This leads to the system

$$\dot{x} = -\frac{1}{16}\,x - y + x^2 + xy, \quad \dot{y} = \frac{1}{8}\,y(16y-1) \tag{40}$$

which beside the invariant lines $y = 0$ and $y = 1/16$ possesses the following invariant parabola and hyperbola:

$$\Phi(x,y) = x^2 - y = 0, \quad \Psi(x,y) = 1 - 16x + 96y - 256xy + 256y^2 = 0.$$

The system (40) possesses four real singularities $M_i(x_i, y_i)$, $i = 1,2,3,4$ with the coordinates

$$x_1 = 0,\ y_1 = 0;\quad x_2 = -\frac{1}{4},\ y_2 = \frac{1}{16};\quad x_3 = \frac{1}{16},\ y_3 = 0;\quad x_4 = \frac{1}{4},\ y_4 = \frac{1}{16}.$$

It is easy to determine that the singular point M_1 is a point of the tangency of the invariant line $y = 0$ with the invariant parabola.

On the other hand M_4 is a point of intersection of the invariant line $y = 1/16$ with both invariant conics and moreover is a point of tangency of the invariant hyperbola with parabola. As a result we get a new configuration *Config.* $\mathcal{PH}$*.27*.

2.2.1.2: *The possibility* $B_2 = 0$. In this case considering the condition $\gamma_8 = 0$ we get $1+4n = 0$, i.e. $n = -1/4$ and we obtain the system

$$\dot{x} = -\frac{1}{4}\,x - y + x^2 + xy, \quad \dot{y} = \frac{1}{2}\,y(4y-1). \tag{41}$$

This system possesses three invariant lines: $y = 0$, $y = 1/4$ and $y = x - 1/4$. Moreover this system beside the invariant parabola possesses the invariant hyperbola $\Psi(x,y) = x - 4y + 4xy = 0$.

The system (41) possesses four real singularities $M_i(x_i, y_i)$, $i = 1,2,3,4$ with the coordinates

$$x_1 = 0,\ y_1 = 0;\quad x_2 = -\frac{1}{2},\ y_2 = \frac{1}{4};\quad x_3 = \frac{1}{4},\ y_3 = 0;\quad x_4 = \frac{1}{2},\ y_4 = \frac{1}{4}.$$

We obtain that M_1 is a point of the tangency of the invariant line $y = 0$ with the invariant parabola and moreover through this point passes one of the branch of hyperbola. On the other hand the singular point M_4 is a point of intersection of four invariant curves: two invariant lines ($y = 1/4$ and $y = x - 1/4$) and two invariant conics. Moreover M_4 is point of tangency of the invariant line $y = x - 1/4$ with both invariant conics.

Therefore we determine that system (41) has the configuration *Config.* $\mathcal{PH}$*.28*.

2.2.2: The subcase $\delta_4 = 0$. By Diagram 3 in this case we have two invariant hyperbolas. Considering the condition $\gamma_8 = 0$ we get $1 + 9n = 0$, i.e. $n = -1/9$ and we obtain the system

$$\dot{x} = -\frac{1}{9}x - y + x^2 + xy, \quad \dot{y} = \frac{2}{9}y(9y - 1). \tag{42}$$

This system possesses two invariant lines $y = 0$ and $y = 1/9$, the invariant parabola $\Phi(x, y) = x^2 - y = 0$ and the following two invariant hyperbolas:

$$\Psi_1(x, y) = -1 + 9x - 27y + 27xy = 0, \quad \Psi_2(x, y) = -x + 9y - 27xy + 27y^2 = 0.$$

The system (42) possess four real singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$ with the coordinates

$$x_1 = 0,\ y_1 = 0; \quad x_2 = -\frac{1}{3},\ y_2 = \frac{1}{9}; \quad x_3 = \frac{1}{9},\ y_3 = 0; \quad x_4 = \frac{1}{3},\ y_4 = \frac{1}{9}.$$

We detect that the singular point M_4 is a point of intersection of the invariant line $y = 1/9$ with all three invariant conics. Moreover one of the branches of the invariant hyperbola intersects the invariant parabola in two singular points: M_1 and M_4. So we arrive at the configuration *Config.*$\mathcal{PH}$*.29*
.

3.2.1.1.2 The statement ($\mathcal{A}_2^*$). According to this statement the following conditions must be satisfied:

$$\zeta_1 \neq 0,\ \zeta_2 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 \neq 0.$$

For systems (10) we calculate

$$\zeta_1 = 2(g - 2)(3 + g), \quad \zeta_4 = \frac{1}{16}(g - 2)(3 + g)\mathcal{S}(g, m, n)$$

where

$$\mathcal{S}(g, m, n) = 1 + 7g + 15g^2 + 9g^3 - 4m + 2n + 6gn.$$

Therefore due to $\zeta_1 \neq 0$ the condition $\zeta_4 = 0$ implies $\mathcal{S}(g, m, n) = 0$ and this gives us

$$m = \frac{1}{4}(1 + 3g)(1 + 4g + 3g^2 + 2n). \tag{43}$$

Then for systems (10) with this value of the parameter m we obtain

$$\begin{gathered}
\zeta_1 = 2(g - 2)(3 + g),\ \zeta_2 = 4g(1 + g),\ \zeta_4 = 0, \\
\zeta_5 = \frac{19}{4}(g - 2)(3 + g)(1 + 4g + 3g^2 + 4n)^2, \\
\mathcal{R}_2 = -\frac{1}{16}(g - 2)(3 + g)(8 + 27g + 27g^2)(1 + 6g + 5g^2 + 4n).
\end{gathered} \tag{44}$$

On the other hand for the existence of an invariant hyperbola the condition $\gamma_1 = \gamma_2 = 0$ is necessary. So for systems (10) with the parameter m given in (43) we calculate:

$$\begin{aligned}
\gamma_1 &= -\frac{3}{64}(1 + 3g)(2 + 3g)(1 + 3g + 3g^2)(1 + 8g + 7g^2 + 4n)^3, \\
\gamma_2 &= -\frac{675}{16}(1 + 2g)(1 + 3g)(2 + 3g)(1 + 8g + 7g^2 + 4n)^2\Upsilon(g, n), \\
\beta_1 &= -\frac{1}{16}(5 + 18g + 18g^2)(1 + 8g + 7g^2 + 4n)^2,
\end{aligned} \tag{45}$$

where $\Upsilon(g,n)$ is a pronominal in the parameter g and n. Since Discrim$[1+3g+3g^2, g] = -3 < 0$ we conclude that the condition $\gamma_1 = 0$ implies $\gamma_2 = 0$. So we consider two possibilities: $\beta_1 \neq 0$ and $\beta_1 = 0$.

1: *The possibility* $\beta_1 \neq 0$. In this case the condition $\gamma_1 = 0$ is equivalent to $(1+3g)(2+3g) = 0$. We claim that we may assume $1 + 3g = 0$ due to an affine transformation the condition. Indeed as it is shown in the proof of Lemma 3 systems (10) with the parameters (m, n, g) could be brought to the same systems but with the new parameters (m_1, n_1, g_1) determined in (21). Using these formulae we obtain

$$2+3g = -(1+3g_1), \quad \mathcal{S}(g,m,n) = -(1+7g_1+15g_1^2+9g_1^3-4m_1+2n_1+6g_1n_1 = -\mathcal{S}(g_1,m_1,n_1)$$

and clearly the conditions $\mathcal{S}(g,m,n) = 0$ and $2+3g = 0$ are transferred to the conditions $\mathcal{S}(g_1,m_1,n_1) = 0$ and $1 + 3g_1 = 0$, respectively. This completes the proof of our claim.

Thus we assume $g = -1/3$ and considering Diagram 3 for systems (10) with the parameter m given in (43) and we calculate:

$$\theta = \frac{10}{9}, \ \beta_1 = -\frac{1}{81}(9n-2)^2, \quad \beta_2 = 0, \ \beta_3 = -\frac{4}{81}(9n-2), \quad \widetilde{\mathcal{R}}_1 = \frac{4}{729}(9n-2)^3,$$
$$\gamma_3 = \frac{61256}{243}(9n-2)(8+9n)(81n-8).$$

We observe that the condition $\beta_1 \neq 0$ implies $\beta_3\widetilde{\mathcal{R}}_1 \neq 0$ and by Diagram 3 we could have an invariant hyperbola if and only if $\gamma_3 = 0$. Therefore due to $\beta_1 \neq 0$ we obtain $(8+9n)(81n-8) = 0$.

Assume first $8 + 9n = 0$. Then $n = -8/9$ and we arrive at the system

$$\dot{x} = -\frac{8}{9}x - \frac{1}{3}y - \frac{1}{3}x^2 + xy, \quad \dot{y} = \frac{2}{9}y(-8-6x+9y) \tag{46}$$

which beside the invariant line $y = 0$ possesses the following two invariant parabolas and a hyperbola:

$$\Phi_1(x,y) = x^2 - y = 0, \quad \Phi_2(x,y) = 64+48x+9x^2-81y = 0, \quad \Psi(x,y) = 3x^2+y-3xy = 0.$$

The system (46) possesses four real singularities $M_i(x_i, y_i)$, $i = 1,2,3,4$ with the coordinates

$$x_1 = 0, \ y_1 = 0; \quad x_2 = -\frac{8}{3}, \ y_2 = 0; \quad x_3 = -\frac{2}{3}, \ y_3 = \frac{4}{9}; \quad x_4 = \frac{4}{3}, \ y_4 = \frac{16}{9}.$$

We observe that the invariant line $y = 0$ is tangent to the invariant parabola $\Phi_1(x,y) = 0$ (respectively $\Phi_2(x,y) = 0$) at the singular point M_1 (respectively M_2). Moreover M_1 is a point of tangency of one of the branches of the hyperbola passing through the origin.

On the other hand M_4 is a point of intersection of the invariant hyperbola with both invariant parabolas. As a result we get the configuration *Config.* $\mathcal{PH}$*.30*.

Suppose now $81n - 8 = 0$. This yields $n = 8/81$ and we arrive at the system

$$\dot{x} = \frac{8}{81}x - \frac{1}{3}y - \frac{1}{3}x^2 + xy, \quad \dot{y} = \frac{2}{81}y(8-54x+81y).$$

We observe that the above system can be brought to system (46) via the change $(x,y,t) \mapsto \big((3x+8)/27, y/9, 9t\big)$ and hence we get *Config.* $\mathcal{PH}$*.30*.

***2:** The possibility* $\beta_1=0$. then considering (45) we obtain

$$1+8g+7g^2+4n=0 \;\Rightarrow\; n=-(1+8g+7g^2)/4$$

and taking into account (43) we get $m=(1+3g-g^2-3g^3)/8$. Then considering Diagram 3 we calculate:

$$\begin{gathered}\beta_1=\beta_6=0,\ \beta_7=-6(2g-1)(3+2g),\ \beta_9=-2(9+4g+4g^2),\\ \gamma_5=108(g-1)^2g^3(1+g)^3(2+g)^2(1+2g)(2+5g)(3+5g)(1+2g+2g^2),\\ \widetilde{\mathcal{R}}_5=-18g(1+g)\big[(1+4g+3g^2)x-(1+2g)y\big]\big[(g-1)^2x^2+2(1+3g)xy+y^2\big],\\ \theta=-(g-1)(2+g)/2,\ \zeta_5=76(g-2)g^2(1+g)^2(3+g).\end{gathered}$$

We observe that $\beta_9\neq 0$ because $\operatorname{Discrim}[9+4g+4g^2,g]=-128<0$. Therefore according to Diagram 3 in the case $\theta\neq 0$ and $\beta_1=\beta_6=0$ we could have an invariant hyperbola if and only if $\gamma_5=0$ and $\widetilde{\mathcal{R}}_5\neq 0$. We observe that due to $\zeta_5\neq 0$ (i.e. $g(g+1)\neq 0$) the condition $\gamma_5=0$ implies $\widetilde{\mathcal{R}}_5\neq 0$. So we conclude that for the existence of an invariant hyperbola in the case under consideration the condition $\gamma_5=0$ is necessary and sufficient.

So we discuss two cases: $\theta\neq 0$ and $\theta=0$.

***2.1:** The case* $\theta\neq 0$. Then by Diagram 3 we have to split our examination into two subcases: $\beta_7\neq 0$ and $\beta_7=0$. However we claim that for $\beta_7=0$ we could not have an invariant hyperbola. Indeed, the condition $\beta_7=0$ implies $(2g-1)(3+2g)=0$. Then calculations yield

$$\theta=5/8\neq 0,\ \beta_1=\beta_6=\beta_7=0,\ \beta_9=-24\neq 0,\ \gamma_5=9021375/1024\neq 0$$

if $g=1/2$ and

$$\theta=5/8\neq 0,\ \beta_1=\beta_6=\beta_7=0,\ \beta_9=-24\neq 0,\ \gamma_5=-9021375/1024\neq 0$$

if $g=-3/2$. In both cases by Diagram 3 the systems under consideration could not have any invariant hyperbola.

Thus our claim is proved and we assume that the condition $\beta_7\neq 0$ is satisfied. Then due to the conditions $\theta\zeta_5\neq 0$ and $\operatorname{Discrim}[1+2g+2g^2,g]=-4<0$ we obtain that the condition $\gamma_5=0$ gives us $(1+2g)(2+5g)(3+5g)=0$. Since $\zeta_3=8(1+2g)^2$ we discuss two subcases: $\zeta_3\neq 0$ and $\zeta_3=0$.

***2.1.1:** The subcase* $\zeta_3\neq 0$. Then the condition $\gamma_5=0$ implies $(2+5g)(3+5g)=0$.

Assume first $2+5g=0$, i.e. $g=-2/5$ and we arrive at the system

$$\dot{x}=-\frac{21}{1000}+\frac{27}{100}x-\frac{3}{10}y-\frac{2}{5}x^2+xy,\quad \dot{y}=-\frac{21}{500}x+\frac{27}{50}y-\frac{7}{5}xy+2y^2 \tag{47}$$

which possesses the following two invariant parabolas and a hyperbola:

$$\begin{gathered}\Phi_1(x,y)=x^2-y=0,\ \Phi_2(x,y)=21-60x+50x^2+50y=0,\\ \Psi(x,y)=-1+30x-300y+1000xy=0.\end{gathered}$$

The system (47) possesses two real and two complex finite singularities $M_i(x_i,y_i)$, $i=1,2,3,4$ with the coordinates

$$x_1=\frac{1}{10},\ y_1=\frac{1}{100};\quad x_2=\frac{1}{2},\ y_2=-\frac{7}{100};\quad x_{3,4}=\frac{1}{10}\big(3\pm 2i\sqrt{3}\big),\ y_{3,4}=\frac{3}{100}\big(-1\pm 4i\sqrt{3}\big).$$

We observe that M_1 (respectively M_2) is the point of intersection of the first (respectively, second) invariant parabola with upper (respectively, lower) branch of the invariant hyperbola. Therefore we arrive at the configuration *Config.* $\mathcal{PH}$*.31*.

Suppose now $3+5g=0$. This yields $g=-3/5$ and we arrive at the system

$$\dot{x}=-\frac{8}{125}+\frac{8}{25}\,x-\frac{1}{5}\,y-\frac{3}{5}\,x^2+xy,\quad \dot{y}=-\frac{16}{125}\,x+\frac{16}{25}\,y-\frac{8}{5}\,xy+2y^2,$$

that via the following affine transformation and time rescaling

$$x_1=x+1/10,\quad y_1=x-y-31/100,\quad t_1=-t$$

is brought to systems (47). Hence we get *Config.* $\mathcal{PH}$*.31*.

2.1.2: *The subcase* $\zeta_3=0$. Then $g=-1/2$ and considering (20) the condition $\zeta_4=0$ implies

$$32m+8n-1=0\ \Rightarrow\ m=(1-8n)/32.$$

Then calculations yield:

$$\gamma_1=\frac{3}{65536}(16n-5)^3,\ \gamma_2=0,\ \ \zeta_5=-\frac{475}{256}(16n-1)^2,\ \ \mathcal{R}_2=\frac{125}{1024}(16n-3).$$

Therefore the condition $\gamma_1=0$ gives us $n=5/16$ and this leads to the system

$$\dot{x}=\frac{1}{64}(4x-1)(3-8x+16y),\quad \dot{y}=-\frac{3}{32}\,x+\frac{5}{8}\,y-\frac{3}{2}xy+2y^2, \tag{48}$$

which beside the invariant line $x=1/4$ possesses the following two invariant parabolas and a hyperbola:

$$\Phi_1(x,y)=x^2-y=0,\ \ \Phi_2(x,y)=3-8x+8x^2+8y=0,\ \ \Psi(x,y)=-1+16x-96y+256xy-256y^2=0.$$

The system (48) possesses two real and two complex finite singularities $M_i(x_i,y_i)$, $i=1,2,3,4$ with the coordinates

$$x_1=\frac{1}{4},\ y_1=\frac{1}{16};\ \ x_2=\frac{1}{4},\ y_2=-\frac{3}{16};\ \ x_{3,4}=\frac{1}{4}\big(1\pm i\sqrt{2}\big),\ y_{3,4}=\frac{1}{16}\big(-1\pm 2i\sqrt{2}\big).$$

We observe that M_1 (respectively, M_2) is the point of intersection of the invariant line with the first (respectively the second) parabola and with upper (respectively, lower) branch of the invariant hyperbola. Moreover each one of the singular points M_1 and M_2 is a point of tangency of the invariant hyperbola with invariant parabolas, correspondingly. As a result we get a new configuration which we denote by *Config.* $\mathcal{PH}$*.32*.

2.2: *The case* $\theta=0$. This implies $(g-1)(g+2)=0$ and we examine each one of the cases.

Assume first $g=1$. This leads to the system

$$\dot{x}=-4x-y+x^2+xy,\quad \dot{y}=2y(y-4), \tag{49}$$

which beside the invariant lines $y=0$ and $y=4$ possesses the following two invariant parabolas and a hyperbola:

$$\Phi_1(x,y)=x^2-y=0,\quad \Phi_2(x,y)=-4x+x^2+y=0,\quad \Psi(x,y)=x^2-y+xy=0.$$

The system (49) possesses four real singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$ with the coordinates

$$x_1 = 0,\ y_1 = 0;\quad x_2 = -2,\ y_2 = 4;\quad x_3 = 2,\ y_3 = 4;\quad x_4 = 4,\ y_4 = 0.$$

We observe that the invariant line $y = 0$ is tangent at the singular point M_1 to the first invariant parabola as well as to the lower branch of the invariant hyperbola.

On the other hand the invariant line $y = 4$ is tangent at the singular point M_3 to the second invariant parabola as well as to the upper branch of the invariant hyperbola. Considering other points of intersections of these two parallel invariant lines we get the configuration *Config.* $\mathcal{PH}.33$.

Suppose now $g = -2$. In this case and we arrive at the system

$$\dot{x} = \frac{15}{8} - \frac{13}{4}\,x + \frac{1}{2}\,y - 2\,x^2 + xy, \quad \dot{y} = \frac{15}{4}\,x - \frac{13}{2}\,y - 3xy + 2y^2.$$

It is easy to determine that via the following affine transformation and time rescaling

$$x_1 = x + 3/2, \quad y_1 = x - y + 15/4, \quad t_1 = -t$$

the above systems can be brought to systems (49) and hence we get *Config.* $\mathcal{PH}.33$.

3.2.1.1.3 The statement ($\boldsymbol{\mathcal{A}_3^*}$). According to this statement the following conditions must be satisfied:

$$\zeta_1 \neq 0,\ \zeta_2 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 \neq 0,\ \zeta_5 = 0.$$

As it was shown earlier, for systems (10) the condition $\zeta_4 = 0$ implies (43) and the we obtain the values of the invariant polynomials ζ_5 and $\mathcal{R}_2$ given in (44). So since $\mathcal{R}_2 \neq 0$ considering (44) we obtain that the condition $\zeta_5 = 0$ implies

$$1 + 4g + 3g^2 + 4n = 0 \ \Rightarrow\ n = -(1+g)(1+3g)/4$$

and taking into account (43) we get $m = (1+g)(1+3g)^2/8$.

On the other hand for the existence of an invariant hyperbola the condition $\gamma_1 = \gamma_2 = 0$ is necessary. So for systems (10) with the parameters m and n given above we calculate:

$$\begin{aligned}\gamma_1 &= -3g^3(1+g)^3(1+3g)(2+3g)(1+3g+3g^2),\\ \gamma_2 &= -4050g^4(1+g)^4(1+2g)(1+3g)(2+3g)(298+1083g+1083g^2).\end{aligned}$$

Since $\text{Discrim}[1+3g+3g^2, g] = -3 < 0$ we conclude that the condition $\gamma_1 = 0$ implies $\gamma_2 = 0$. However we claim that in this case we cannot have any invariant hyperbola. Indeed, due to the condition $\zeta_2 \neq 0$ (i.e. $g(g+1) \neq 0$) we conclude that the condition $\gamma_1 = 0$ gives us $(1+3g)(2+3g) = 0$. Then it is not difficult to convince ourselves that for $g = -1/3$ as well as for $g = -2/3$ we obtain

$$\theta = 10/9 \neq 0,\ \beta_1 = -4/81 \neq 0,\ \beta_2 = 0,\ \beta_3 = 8/81 \neq 0,\ \gamma_3 = 7840768/243 \neq 0$$

Therefore in both cases by Diagram 3 the systems under consideration could not have any invariant hyperbola and this completes the proof of our claim.

Thus we have proved that if the conditions provided by the statement ($\boldsymbol{\mathcal{A}_3^*}$) of Proposition 1 are satisfied then systems (10) cannot have any invariant hyperbola.

3.2.1.1.4 The statement ($\mathcal{A}_4^*$). By Proposition 1^* in this case for systems (10) the following conditions are satisfied
$$\zeta_1 \neq 0,\ \zeta_2 \neq 0,\ \zeta_4 = 0,\ \mathcal{R}_2 = 0,\ \zeta_5 \neq 0.$$
Since $\zeta_5 \neq 0$, considering (44) the condition $\mathcal{R}_2 = 0$ implies
$$1 + 6g + 5g^2 + 4n = 0 \ \Rightarrow\ n = -(1+g)(1+5g)/4$$
and taking into account (43) we get $m = (1+g)^2(1+3g)/8$. Then calculations yield
$$\gamma_1 = -\frac{3}{8}g^3(1+g)^3(1+3g)(2+3g)(1+3g+3g^2),$$
$$\gamma_2 = -2700g^4(1+g)^4(1+2g)(1+3g)(2+3g)(31+100g+100g^2)$$
and we observe that the condition $\gamma_1 = 0$ implies $\gamma_2 = 0$. Since $\zeta_2 \neq 0$ (i.e. $g(g+1) \neq 0$) we conclude that the condition $\gamma_1 = 0$ gives us $(1+3g)(2+3g) = 0$. Then for $g = -1/3$ as well as for $g = -2/3$ we obtain
$$\theta = 10/9 \neq 0,\ \beta_1 = -1/81 \neq 0,\ \beta_2 = 0,\ \beta_3 = 4/81 \neq 0,\ \gamma_3 = -61256/27 \neq 0.$$
Therefore by Diagram 3 the systems under consideration could not have any invariant hyperbola.

3.2.1.1.5 The statement ($\mathcal{A}_5^*$). By Proposition 1^* in this case for systems (10) the following conditions are satisfied
$$\zeta_1 \neq 0,\ \zeta_2 = 0,\ \zeta_6 \neq 0,\ \mathcal{R}_1 = 0,\ \mathcal{R}_2 \neq 0.$$
Since for systems (10) we have $\zeta_2 = 4g(1+g)$ we obtain $g(g+1) = 0$. We claim that via an affine transformation the condition $g+1 = 0$ for systems (10) could be transferred to the condition $g = 0$. Indeed as it is shown in the proof of Lemma 3, systems (10) with the parameters (m, n, g) could be brought to the same systems but with the new parameters (m_1, n_1, g_1) determined in (21). In particular we have $g_1 = -(1+g)$ (then $g = -(1+g_1)$) and we obtain
$$1 + g = 1 - (1+g_1) = -g_1$$
and this completes the proof of our claim.

Therefore we may assume $g = 0$ and we arrive at the 2-parameter family of systems
$$\dot{x} = m + nx - \frac{y}{2} + xy, \quad \dot{y} = 2mx + 2ny - xy + 2y^2. \tag{50}$$
For these systems we calculate
$$\mu_0 = 0, \quad \mu_1 = -2(2m+n)y, \quad \zeta_6 = (2m+n)/2$$
and due to $\zeta_6 \neq 0$ we have $\mu_1 \neq 0$. Therefore according to [1, Lemma 5.2, statement (i)] we have the following remark.

Remark 8. *One of the singular points of systems* (50) *has gone to infinity and coalesced with the infinite singularity* $N[1:0:0]$ *producing a double infinite singularity of multiplicity* $(1,1)$.

For systems (50) and we calculate

$$\gamma_1 = -\frac{1}{512}(20m-2n-3)(20m+6n-1)(28m+10n-1),$$
$$\nu_1 = -\frac{4821}{64}(20m-2n-3)(20m+6n-1)$$

and we discuss two possibilities: $\nu_1 \neq 0$ and $\nu_1 = 0$.

1: *The possibility* $\nu_1 \neq 0$. In this case the condition $\gamma_1 = 0$ implies $28m + 10n - 1 = 0$, i.e. $m = (1 - 10n)/28$ and calculations yield

$$\gamma_2 = \frac{675}{49}(1+4n)^3(16n-3) = 0, \quad \nu_1 = -\frac{4821}{98}(1+4n)^2 \neq 0.$$

So we get $16n - 3 = 0$, i.e. $n = 3/16$ and we arrive at the system

$$\dot{x} = -\frac{1}{32} + \frac{3}{16}x - \frac{1}{2}y + xy, \quad \dot{y} = -\frac{1}{16}x + \frac{3}{8}y - xy + 2y^2, \tag{51}$$

which possesses the following invariant parabola and hyperbola:

$$\Phi(x,y) = x^2 - y = 0, \quad \Psi(x,y) = -1 + 16x - 96y + 256xy - 256y^2 = 0.$$

The system (51) possesses one real and two complex finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3$ with the coordinates

$$x_1 = \frac{1}{4}, \ y_1 = \frac{1}{16}; \quad x_{2,3} = \frac{1}{8}\big(1 \pm i\sqrt{7}\big), \ y_{2,3} = \frac{1}{32}\big(-3 \pm i\sqrt{7}\big).$$

We observe that M_1 is a point of tangency of the invariant parabola with the upper branch of the invariant hyperbola. Therefore considering the existence of a double infinite singularity $N[1:0:0]$ (see Remark 8) we arrive at the configuration *Config.* $\mathcal{PH}$*.34*.

2: *The possibility* $\nu_1 = 0$. Then we obtain $(20m - 2n - 3)(20m + 6n - 1) = 0$ (this implies $\gamma_1 = 0$) and we discuss two cases generated by this condition.

2.1: *The case* $20m - 2n - 3 = 0$. This implies $m = (3 + 2n)/20$ and calculations yield

$$\gamma_2 = 1134(3+2n)(1+4n)^3/5, \quad \zeta_6 = 3(1+4n)/20$$

and since by statement $\boldsymbol{(\mathcal{A}_5^*)}$ we have $\zeta_6 \neq 0$ we deduce that the condition $\gamma_2 = 0$ gives us $n = -3/2$ and then $m = 0$. So we get the system

$$\dot{x} = -\frac{3}{2}x - \frac{1}{2}y + xy, \quad \dot{y} = y(-3 - x + 2y), \tag{52}$$

which possesses the invariant line $y = 0$ and the following invariant parabola and hyperbola:

$$\Phi(x,y) = x^2 - y = 0, \quad \Psi(x,y) = 2x^2 + y - 2xy = 0.$$

The system (52) possesses three real finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3$ with the coordinates

$$x_1 = 0, \ y_1 = 0; \quad x_2 = -1, \ y_2 = 1; \quad x_3 = \frac{3}{2}, \ y_3 = \frac{9}{4}.$$

We observe that M_1 is a point of tangency of the invariant line $y = 0$ with the invariant parabola as well as with the lower branch of the invariant hyperbola. Therefore considering the existence of a double infinite singularity $N[1:0:0]$ (see Remark 8) we arrive at the configuration *Config.* $\mathcal{PH}$*.35*.

***2.2:** The case* $20m + 6n - 1 = 0$. Then $m = (1 - 6n)/20$ and calculations yield

$$\gamma_2 = 63(1 + 4n)^3(36n - 11)/20, \quad \zeta_6 = (1 + 4n)/20$$

and due to $\zeta_6 \neq 0$ the condition $\gamma_2 = 0$ gives us $n = 11/36$ and then $m = -1/24$. So we arrive at the system

$$\dot{x} = -\frac{1}{24} + \frac{11}{36}x - \frac{1}{2}y + xy, \quad \dot{y} = -\frac{1}{12}x + \frac{11}{18}y - xy + 2y^2. \tag{53}$$

This system possesses the following invariant parabola and hyperbola:

$$\Phi(x, y) = x^2 - y = 0, \quad \Psi(x, y) = -1 + 18x - 108y + 216xy = 0.$$

The system (53) possesses one real and two complex finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3$ with the coordinates

$$x_1 = \frac{1}{6}, \ y_1 = \frac{1}{36}; \quad x_{2,3} = \frac{1}{6}\big(1 \pm 2i\sqrt{2}\big), \ y_{2,3} = \frac{1}{36}\big(-7 \pm 4i\sqrt{2}\big).$$

It is easy to determine that M_1 is the point of intersection of the invariant parabola with upper branch of the invariant hyperbola. Therefore considering the existence of a double infinite singularity $N[1:0:0]$ (see Remark 8) we arrive at the configuration *Config.* $\mathcal{PH}$*.36*.

Next we have to determine the conditions for distinguishing the *Config.* $\mathcal{PH}$*.35* from *Config.* $\mathcal{PH}$*.36* in the case $\nu_1 = 0$. It was proved above that the system of equations $\nu_1 = 0$ and $\gamma_2 = 0$ could have only the following two solutions $\mathcal{S}_i = (m_i, n_i)$:

$$\mathcal{S}_1 = \Big(0, -\frac{3}{2}\Big), \quad \mathcal{S}_2 = \Big(-\frac{1}{24}, \frac{11}{36}\Big).$$

Moreover we proved that at the point $\mathcal{S}_1$ of the parameter space (m, n) of systems (50) the configuration *Config.* $\mathcal{PH}$*.35*, whereas at the point $\mathcal{S}_2$ we have *Config.* $\mathcal{PH}$*.36*.

On the other hand evaluating the invariant polynomial ν_6 at these points we obtain:

$$\nu_6(\mathcal{S}_1) = 536895/4 \neq 0, \quad \nu_6(\mathcal{S}_2) = 0.$$

Therefore we conclude that in the case $\nu_1 = 0$ we obtain the configuration *Config.* $\mathcal{PH}$*.35* for $\nu_6 \neq 0$ and *Config.* $\mathcal{PH}$*.36* for $\nu_6 = 0$.

Thus all the statements of Proposition 1 are examined and we conclude that the following theorem is proved.

Theorem 2. *Assume that for a non-degenerate quadratic system the conditions* $\eta > 0$ *and* $\zeta_1 \neq 0$ *are satisfied and in addition one of the sets of conditions provided by Proposition 1** *holds. Then this system possesses at least one invariant parabola and at least one invariant hyperbola and has one of the configurations presented in Figure 1 if and only if* $\gamma_1 = \gamma_2 = 0$ *and the following conditions are fulfilled, correspondingly.*

(i) If the statement $(\boldsymbol{\mathcal{A}_1^*})$ *holds then we obtain:*

- $\beta_1 \neq 0,\ \nu_1 \neq 0,\ \zeta_3 \neq 0,\ \nu_2 \neq 0,\ B_1 \neq 0,\ \mathbf{D} < 0\ \Rightarrow$ *Config.* $\mathcal{PH}$*.1;*
- $\beta_1 \neq 0,\ \nu_1 \neq 0,\ \zeta_3 \neq 0,\ \nu_2 \neq 0,\ B_1 \neq 0,\ \mathbf{D} > 0,\ \zeta_2 < 0\ \Rightarrow$ *Config.* $\mathcal{PH}$*.2;*
- $\beta_1 \neq 0,\ \nu_1 \neq 0,\ \zeta_3 \neq 0,\ \nu_2 \neq 0,\ B_1 \neq 0\ \mathbf{D} > 0,\ \zeta_2 > 0\ \Rightarrow$ *Config.* $\mathcal{PH}$*.3;*
- $\beta_1 \neq 0,\ \nu_1 \neq 0,\ \zeta_3 \neq 0,\ \nu_2 \neq 0,\ B_1 \neq 0,\ \mathbf{D} = 0\ \Rightarrow$ *Config.* $\mathcal{PH}$*.4;*
- $\beta_1 \neq 0,\ \nu_1 \neq 0,\ \zeta_3 \neq 0,\ \nu_2 \neq 0,\ B_1 = 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.5;*
- $\beta_1 \neq 0,\ \nu_1 \neq 0,\ \zeta_3 \neq 0,\ \nu_2 = 0,\ B_1 \neq 0,\ \mathbf{D} \neq 0,\ \zeta_2 < 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.6;*
- $\beta_1 \neq 0,\ \nu_1 \neq 0,\ \zeta_3 \neq 0,\ \nu_2 = 0,\ B_1 \neq 0,\ \mathbf{D} \neq 0,\ \zeta_2 > 0,\ \nu_3 < 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.7;*
- $\beta_1 \neq 0,\ \nu_1 \neq 0,\ \zeta_3 \neq 0,\ \nu_2 = 0,\ B_1 \neq 0,\ \mathbf{D} \neq 0,\ \zeta_2 > 0,\ \nu_3 > 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.8;*
- $\beta_1 \neq 0,\ \nu_1 \neq 0,\ \zeta_3 \neq 0,\ \nu_2 = 0,\ B_1 \neq 0,\ \mathbf{D} = 0,\ \nu_3 \neq 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.9;*
- $\beta_1 \neq 0,\ \nu_1 \neq 0,\ \zeta_3 \neq 0,\ \nu_2 = 0,\ B_1 \neq 0,\ \mathbf{D} = 0,\ \nu_3 = 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.10;*
- $\beta_1 \neq 0,\ \nu_1 \neq 0,\ \zeta_3 \neq 0,\ \nu_2 = 0,\ B_1 = 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.11;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 \neq 0,\ B_2 \neq 0,\ \mathbf{D} \neq 0,\ \zeta_2 < 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.12;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 \neq 0,\ B_2 \neq 0,\ \mathbf{D} \neq 0,\ \zeta_2 > 0,\ \nu_7 < 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.13;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 \neq 0,\ B_2 \neq 0,\ \mathbf{D} \neq 0,\ \zeta_2 > 0,\ \nu_7 > 0,\ \nu_8 < 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.14;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 \neq 0,\ B_2 \neq 0,\ \mathbf{D} \neq 0,\ \zeta_2 > 0,\ \nu_7 > 0,\ \nu_8 > 0,\ \nu_9 < 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.15;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 \neq 0,\ B_2 \neq 0,\ \mathbf{D} \neq 0,\ \zeta_2 > 0,\ \nu_7 > 0,\ \nu_8 > 0,\ \nu_9 > 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.16;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 \neq 0,\ B_2 \neq 0,\ \mathbf{D} = 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.17;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 \neq 0,\ B_2 = 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.18;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 = 0,\ B_1 \neq 0,\ \mathbf{D} < 0,\ \zeta_2 < 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.19;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 = 0,\ B_1 \neq 0,\ \mathbf{D} < 0,\ \zeta_2 > 0,\ \nu_{10} < 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.20;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 = 0,\ B_1 \neq 0,\ \mathbf{D} < 0,\ \zeta_2 > 0,\ \nu_{10} > 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.21;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 = 0,\ B_1 \neq 0,\ \mathbf{D} > 0,\ \zeta_2 < 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.22;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 = 0,\ B_1 \neq 0,\ \mathbf{D} > 0,\ \zeta_2 > 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.23;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 = 0,\ B_1 \neq 0,\ \mathbf{D} = 0,\ \zeta_2 < 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.24;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 = 0,\ B_1 \neq 0,\ \mathbf{D} = 0,\ \zeta_2 > 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.25;*
- $\beta_1 \neq 0,\ \nu_1 = 0,\ \nu_6 = 0,\ B_1 = 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.26;*
- $\beta_1 = 0,\ \theta \neq 0,\ \gamma_4 = 0,\ B_1 \neq 0,\ \nu_2 \neq 0,\ \nu_4 \neq 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.3;*
- $\beta_1 = 0,\ \theta \neq 0,\ \gamma_4 = 0,\ B_1 \neq 0,\ \nu_2 \neq 0,\ \nu_4 = 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.23;*

- $\beta_1 = 0,\ \theta \neq 0,\ \gamma_4 = 0,\ B_1 \neq 0,\ \nu_2 = 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.7;*
- $\beta_1 = 0,\ \theta \neq 0,\ \gamma_4 = 0,\ B_1 = 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.14;*
- $\beta_1 = 0,\ \theta = 0,\ \gamma_8 = 0,\ \delta_4 \neq 0,\ B_2 \neq 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.27;*
- $\beta_1 = 0,\ \theta = 0,\ \gamma_8 = 0,\ \delta_4 \neq 0,\ B_2 = 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.28;*
- $\beta_1 = 0,\ \theta = 0,\ \gamma_8 = 0,\ \delta_4 = 0 \Rightarrow$ *Config.* $\mathcal{PH}$*.29.*

(ii) If the statement $(\boldsymbol{\mathcal{A}_2^*})$ *holds then we obtain:*

- $\beta_1 \neq 0,\ \gamma_3 = 0 \ \Rightarrow$ *Config.* $\mathcal{PH}$*.30;*
- $\beta_1 = 0,\ \gamma_5 = 0,\ \theta \neq 0,\ \beta_7 \neq 0,\ \zeta_3 \neq 0 \ \Rightarrow$ *Config.* $\mathcal{PH}$*.31;*
- $\beta_1 = 0,\ \gamma_5 = 0,\ \theta \neq 0,\ \beta_7 \neq 0,\ \zeta_3 = 0 \ \Rightarrow$ *Config.* $\mathcal{PH}$*.32;*
- $\beta_1 = 0,\ \gamma_5 = 0,\ \theta = 0 \ \Rightarrow$ *Config.* $\mathcal{PH}$*.33.*

(iii) If the statement $(\boldsymbol{\mathcal{A}_5^*})$ *holds then we obtain:*

- $\nu_1 \neq 0 \ \Rightarrow$ *Config.* $\mathcal{PH}$*.34;*
- $\nu_1 = 0,\ \nu_6 \neq 0 \ \Rightarrow$ *Config.* $\mathcal{PH}$*.35;*
- $\nu_1 = 0,\ \nu_6 = 0 \ \Rightarrow$ *Config.* $\mathcal{PH}$*.36.*

3.2.1.2 The case $\zeta_1 = 0$. According to Proposition 2 for the existence of an invariant parabola for a quadratic system with $\eta > 0$ and $\zeta_1 = 0$ one of the sets of conditions from $(\boldsymbol{\mathcal{B}_1})$ to $(\boldsymbol{\mathcal{B}_7})$ it is necessary and sufficient. Moreover in this case a quadratic system can be brought via an affine transformation and time rescaling to the canonical form (10) with $g = 2$. So we consider the 2-parameter family of systems

$$\dot{x} = m + nx - \frac{3}{2}y + 2x^2 + xy, \quad \dot{y} = 2mx + 2ny + xy + 2y^2, \tag{54}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

By Theorem 1 for the existence of at least one invariant hyperbola for systems (54) the condition $\gamma_1 = \gamma_2 = 0$ is necessary. So for systems (54) we calculate:

$$\begin{gathered}\gamma_1 = -\frac{1}{512}\mathcal{U}_1\mathcal{U}_2\mathcal{U}_3, \quad \nu_{11} = -\frac{13}{64}\mathcal{U}_1\mathcal{U}_2; \ \ \mathcal{U}_1 = (140m - 34n - 15),\\ \mathcal{U}_2 = (20m - 14n - 105), \quad \mathcal{U}_3 = (20m - 6n - 15).\end{gathered} \tag{55}$$

In what follows we examine step by step each one of the statements $(\boldsymbol{\mathcal{B}_1})$ to $(\boldsymbol{\mathcal{B}_7})$ provided by Proposition 2.

3.2.1.2.1 The statement ($\boldsymbol{\mathcal{B}_1}$). We observe that the condition $\nu_{11} = 0$ implies $\gamma_1 = 0$ and therefore we discuss two possibilities: $\nu_{11} \neq 0$ and $\nu_{11} = 0$.

***1:** The possibility* $\nu_{11} \neq 0$. Then $\mathcal{U}_1\mathcal{U}_2 \neq 0$ and hence the condition $\gamma_1 = 0$ implies $\mathcal{U}_3 = 0$. This gives us $m = 3(5 + 2n)/20$ and we calculate

$$\gamma_2 = -1134(5 + 4n)^2(45 + 4n)^2, \quad \zeta_7 = \frac{30456}{5}(5 + 4n)^2(45 + 4n)(1745 - 152n + 400n^2).$$

We observe that the condition $\zeta_7 \neq 0$ provided by the statement $\boldsymbol{(\mathcal{B}_1)}$ of Proposition 2 implies $\gamma_2 \neq 0$. So we deduce that in the case $\nu_{11} \neq 0$ systems (54) could not possess any invariant hyperbola.

***2:** The possibility* $\nu_{11} = 0$. Considering (55) we obtain $\mathcal{U}_1\mathcal{U}_2 = 0$ and we examine each one of the possible cases.

***2.1:** The case* $\mathcal{U}_1 = 0$. Then we get $m = (15 + 34n)/140$ and calculations yield:

$$\gamma_2 = -\frac{270}{49}(45 + 4n)^2(5 + 16n)(33 + 100n),$$
$$\zeta_7 = \frac{24111}{3430}(45 + 4n)(25 + 52n)(33 + 100n)(375 + 280n + 1008n^2).$$

So due to $\zeta_7 \neq 0$ the condition $\gamma_2 = 0$ implies $n = -5/16$ and then $m = 1/32$. In this case we arrive at the system

$$\dot{x} = \frac{1}{32} - \frac{5}{16}x - \frac{3}{2}y + 2x^2 + xy, \quad \dot{y} = \frac{1}{16}x - \frac{5}{8}y + xy + 2y^2. \tag{56}$$

This system possesses the following invariant parabola and hyperbola:

$$\Phi(x, y) = x^2 - y = 0, \quad \Psi(x, y) = -1 + 16x - 96y + 256xy - 256y^2 = 0.$$

The system (56) possesses four real finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$ with the coordinates

$$x_1 = \frac{7}{48},\ y_1 = \frac{1}{48};\quad x_2 = \frac{1}{4},\ y_2 = \frac{1}{16};\quad x_{3,4} = \frac{1}{8}\big(-3 \pm \sqrt{17}\big),\ y_{3,4} = \frac{1}{32}\big(13 \mp 3\sqrt{17}\big).$$

We calculate

$$\Phi(x_2, y_2) = \Phi(x_3, y_3) = \Phi(x_4, y_4) = \Psi(x_2, y_2) = \Psi(x_1, y_1) = 0$$

and we conclude that M_2 is a common point of the invariant parabola and hyperbola. Moreover M_2 is a point of tangency of these two invariant conics. On the other hand M_1 is located on the upper branch of the hyperbola whereas M_3 an M_3 are located on the invariant parabola. As a result we obtain a configuration which is equivalent to *Config.* $\mathcal{PH}$*.1*.

***2.2:** The case* $\mathcal{U}_2 = 0$. Then we obtain $m = 7(15 + 2n)/20$ and we calculate

$$\gamma_2 = -1764(15 + 2n)(45 + 4n)^2(145 + 36n),$$
$$\zeta_7 = \frac{2961}{10}(45 + 4n)(55 + 12n)(145 + 36n)(26145 + 168n + 400n^2).$$

So due to $\zeta_7 \neq 0$ the condition $\gamma_2 = 0$ implies $n = -15/2$ and then $m = 0$. In this case we arrive at the system

$$\dot{x} = -\frac{15}{2}x - \frac{3}{2}y + 2x^2 + xy, \quad \dot{y} = y(-15 + x + 2y) \tag{57}$$

which possesses the invariant line $y = 0$ and the following invariant parabola and hyperbola:

$$\Phi(x, y) = x^2 - y = 0, \quad \Psi(x, y) = 2x^2 + y - 2xy = 0.$$

The system (57) possesses four real finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$ with the coordinates

$$x_1 = 0,\ y_1 = 0; \quad x_2 = -3,\ y_2 = 9; \quad x_3 = \frac{5}{2},\ y_3 = \frac{25}{4}; \quad x_4 = \frac{15}{4},\ y_4 = 0.$$

Calculations yield:

$$\Phi(x_1, y_1) = \Phi(x_2, y_2) = \Phi(x_3, y_3) = \Psi(x_1, y_1) = \Psi(x_3, y_3) = 0$$

and hence M_1 is a common point of invariant line $y = 0$ with the invariant parabola and lower branch of the hyperbola. Moreover the singular point M_1 is a point of tangency of these three invariant curves. On the other hand M_3 is point of intersection of the invariant parabola with the upper branch of the hyperbola. It remains to observe that M_2 is located on the invariant parabola whereas M_4 lies on the invariant line. As a result we obtain a configuration which is equivalent to *Config.* $\mathcal{PH}$*.13*.

Next we have to determine the conditions for distinguishing the *Config.* $\mathcal{PH}$*.1* from *Config.* $\mathcal{PH}$*.13* in the case $\nu_{11} = 0$. It was proved above that the system of equations $\nu_{11} = 0$ and $\gamma_2 = 0$ for systems (54) could have only the following two solutions $\mathcal{S}_i = (m_i, n_i)$:

$$\mathcal{S}_1 = \left(\frac{1}{32}, -\frac{5}{16}\right), \quad \mathcal{S}_2 = \left(0, -\frac{15}{2}\right).$$

Moreover we proved that at the point $\mathcal{S}_1$ of the parameter space (m, n) of systems (54) we have the configuration *Config.* $\mathcal{PH}$*.1*, whereas at the point $\mathcal{S}_2$ we have *Config.* $\mathcal{PH}$*.13*.

On the other hand evaluating the invariant polynomial ξ_{20} at these points we obtain:

$$\xi_{20}(\mathcal{S}_1) = -100035/32768 \neq 0, \quad \xi_{20}(\mathcal{S}_2) = 0.$$

Therefore we conclude that in the case $\nu_{11} = 0$ we obtain the configuration *Config.* $\mathcal{PH}$*.1* for $\xi_{20} \neq 0$ and *Config.* $\mathcal{PH}$*.13* for $\xi_{20} = 0$.

3.2.1.2.2 The statements from ($\mathcal{B}_2$) to ($\mathcal{B}_4$). We observe that these three statements have two common conditions: $\chi_4 \neq 0$ and $\zeta_7 = 0$. We claim that if for systems (54) these two conditions are satisfied then these systems could not have any invariant hyperbola.

Indeed, for systems (54) we calculate

$$\begin{gathered}\zeta_7 = -\frac{52875}{2}\mathcal{V}_1\mathcal{V}_2\mathcal{V}_3, \quad \chi_4 = 61875\mathcal{V}_2\mathcal{V}_3; \quad \mathcal{V}_1 = (4m - 14n - 147),\\ \mathcal{V}_2 = (1 + 4m + 2n), \quad \mathcal{V}_3 = (18m + 1372m^2 - 84mn + 27n^2 + 144n^3).\end{gathered} \tag{58}$$

So due to $\chi_4 \neq 0$ (i.e. $\mathcal{V}_2\mathcal{V}_3 \neq 0$) we obtain that the condition $\zeta_7 = 0$ implies $\mathcal{V}_1 = 0$. This gives us $m = 7(21 + 2n)/4$ and calculations yield

$$\gamma_1 = -399(45 + 4n)^3/8, \quad \gamma_2 = -47250(45 + 4n)^2(1287477 + 228336n + 10096n^2)$$

and clearly the condition $\gamma_1 = \gamma_2 = 0$ implies $45+4n = 0$. Then $n = -45/4$ (this yields $m = -21/8$) and considering Diagram 3 for systems (54) in this case we calculate:

$$\theta = -2 \neq 0, \quad \beta_1 = \beta_6 = 0, \quad \beta_7 = -126 \neq 0.$$

So by Diagram 3 for the existence of an invariant hyperbola in this case the condition $\gamma_5 = 0$ is necessary. However for systems (54) we have $\gamma_5 = 3784734720 \neq 0$ and this completes the proof of our claim.

3.2.1.2.3 The statement ($\boldsymbol{\mathcal{B}_5}$). In this case the condition $\chi_4 = 0$ must be fulfilled for systems (54). Considering (58) we obtain $\mathcal{V}_2\mathcal{V}_3 = 0$ and we examine each one of the possibilities provided by this condition.

1: *The possibility* $\mathcal{V}_2 \neq 0$. In this case we have

$$\mathcal{V}_3 = 18m + 1372m^2 - 84mn + 27n^2 + 144n^3 = 0$$

and we calculate:

$$\begin{aligned}\operatorname{Res}_m(\mathcal{V}_3, \gamma_1) =&63(5+4n)(145+36n)(33+100n)(1745-152n+400n^2)\times\\ &(26145+168n+400n^2)(375+280n+1008n^2)/4096.\end{aligned}$$

Since we have

$$\begin{aligned}&\text{Discrim}\,[1745-152n+400n^2] = -2768896 < 0,\\ &\text{Discrim}\,[26145+168n+400n^2] = -41803776 < 0,\\ &\text{Discrim}\,[375+280n+1008n^2] = -1433600 < 0\end{aligned}$$

we deduce that the polynomials $\mathcal{V}_3$ and γ_1 have a common solution in m if and only if

$$(5+4n)(145+36n)(33+100n) = 0.$$

So we discuss each one of the three cases provided by the above condition.

1.1: *The case 5 + 4 n=0.* Then $n = -5/4$ and we calculate

$$\mathcal{V}_3 = \frac{1}{16}(8m-3)(1275+2744m), \quad \mathcal{V}_2 = \frac{1}{2}(8m-3)$$
$$\gamma_1 = -\frac{125}{4096}(8m-3)(8m-35)(11+56m)$$

and therefore due to the condition $\mathcal{V}_2 \neq 0$ we obtain that the system of equation $\mathcal{V}_3 = 0$ and $\gamma_1 = 0$ cannot have any common solution in this case.

1.2.: *The case 145 + 36 n=0.* This implies $n = -145/36$ and we calculate

$$\mathcal{V}_3 = \frac{1}{1296}(72m-175)(66439+24696m), \quad \mathcal{V}_2 = \frac{1}{18}(72m-127)$$
$$\gamma_1 = -\frac{125}{995328}(11+24m)(72m-175)(439+504m)$$

and we obtain that the condition $\mathcal{V}_3 = \gamma_1 = 0$ yields $m = 175/72$. This leads to the system

$$\dot{x} = \frac{175}{72} - \frac{145}{36}x - \frac{3}{2}y + 2x^2 + xy, \quad \dot{y} = \frac{175}{36}x - \frac{145}{18}y + xy + 2y^2. \tag{59}$$

possessing the following two invariant parabolas and hyperbola:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = -625 + 2000x - 1800y + 432y^2 = 0,$$
$$\Psi(x, y) = 125 - 450x + 324y + 216x^2 - 216xy = 0.$$

We determine that system (59) possesses four real finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$ with the coordinates

$$x_1 = -\frac{5}{2},\ y_1 = \frac{25}{4};\quad x_2 = \frac{5}{6},\ y_2 = \frac{25}{36};\quad x_3 = \frac{7}{6},\ y_3 = \frac{49}{36};\quad x_4 = \frac{65}{54},\ y_4 = \frac{175}{108}.$$

Next we calculate

$$\Phi_1(x_1, y_1) = \Phi_2(x_1, y_1) = 0, \quad \Phi_1(x_3, y_3) = 0,\ \Phi_2(x_4, y_4) = \Psi(x_4, y_4) = 0,$$
$$\Phi_1(x_2, y_2) = \Phi_2(x_2, y_2) = \Psi(x_2, y_2) = 0$$

and we conclude that M_2 is a common point of all three invariant conics. Moreover M_1 is a point of intersection of two invariant parabolas, whereas M_4 is a point of intersection of the hyperbola with the second parabola. It remains to observe that M_3 is located on the parabola $\Phi_1(x, y) = 0$. As a result we obtain the configuration *Config.* $\mathcal{PH}$*.37*.

1.3: *The case 33 + 100 n=0.* This implies $n = -33/100$ and calculations yield

$$\mathcal{V}_3 = \frac{1}{250000}(1000m - 27)(20691 + 343000m), \quad \mathcal{V}_2 = \frac{1}{50}(17 + 200m),$$
$$\gamma_1 = -\frac{7}{64000000}(1000m - 27)(1000m - 5019)(1000m - 651).$$

We observe that the condition $\mathcal{V}_3 = \gamma_1 = 0$ gives us $m = 27/1000$ and this leads to the system

$$\dot{x} = \frac{27}{1000} - \frac{33}{100}x - \frac{3}{2}y + 2x^2 + xy, \quad \dot{y} = \frac{27}{500}x - \frac{33}{50}y + xy + 2y^2$$

possessing the following two invariant parabolas and hyperbola:

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = -243 + 2160x - 5400y + 10000y^2 = 0,$$
$$\Psi(x, y) = -27 + 540x - 3600y + 10000xy - 10000y^2 = 0.$$

The above system possesses four real finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$ with the coordinates

$$x_1 = \frac{3}{10},\ y_1 = \frac{9}{100};\quad x_2 = -\frac{9}{10},\ y_2 = \frac{81}{100};\quad x_3 = \frac{1}{10},\ y_3 = \frac{1}{100};\quad x_4 = \frac{39}{250},\ y_3 = \frac{9}{500}.$$

We calculate

$$\Phi_1(x_1, y_1) = \Phi_2(x_1, y_1) = \Psi_2(x_1, y_1) = 0, \quad \Phi_1(x_2, y_2) = \Phi_2(x_2, y_2) = 0,$$
$$\Phi_1(x_3, y_3) = \Psi(x_3, y_3) = 0, \quad \Phi_2(x_4, y_4) = 0$$

and we conclude that M_1 is a common point of all three invariant conics. On the other hand M_2 is a point of intersection of two invariant parabolas whereas M_3 is a common point of the first parabola with the upper branch of the hyperbola. It remains to observe that M_4 is located on the parabola $\Phi_2(x, y) = 0$ and this leads to a configuration which is equivalent to *Config. $\mathcal{PH}$.37*.

2: *The possibility* $\mathcal{V}_2 = 0$. In this case we have

$$\mathcal{V}_2 = 1 + 4m + 2n = 0 \;\Rightarrow\; m = -(1 + 2n)/4$$

and calculations yield:

$$\gamma_1 = 1/32(5 + 4n)(55 + 12n)(25 + 52n),$$
$$\gamma_2 = -225(-3 + 4n)(5 + 4n)(22555 + 52060n + 10176n^2)$$

Clearly the condition $\gamma_1 = \gamma_2 = 0$ implies $5 + 4n = 0$. Then we get $n = -5/4$ and this leads to the system

$$\dot{x} = \frac{3}{8} - \frac{5}{4}x - \frac{3}{2}y + 2x^2 + xy, \quad \dot{y} = \frac{3}{4}x - \frac{5}{2}y + xy + 2y^2. \tag{60}$$

This system possesses the invariant line $y = x - 1/4$ and three invariant conics: two parabolas and one hyperbola

$$\Phi_1(x, y) = x^2 - y = 0, \quad \Phi_2(x, y) = -3 + 16x - 24y + 16y^2 = 0,$$
$$\Psi(x, y) = -1 + 6x - 12y + 8xy = 0.$$

We determine that the above system possesses two real finite singularities: $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$ with the coordinates

$$M_1\Big(\frac{1}{2}, \frac{1}{4}\Big), \quad M_2\Big(-\frac{3}{2}, \frac{9}{4}\Big).$$

We claim that the singular point M_1 is a triple singularity of systems (60). Indeed, applying the corresponding translation, we could place M_1 at the origin of coordinates and we arrive at the systems

$$\dot{x} = x - y + 2x^2 + xy, \quad \dot{y} = x - y + xy + 2y^2,$$

where $M_0(0, 0)$ is a singularity of the above systems corresponding to the singularity M_1.

Considering [1], we calculate the following invariant polynomials

$$\mu_4 = \mu_3 = \mu_2 = 0, \quad \mu_1 = -24(x + y) \neq 0$$

and by [1, Lemma 5.2, statement (ii)] the point M_0 is of multiplicity 3. This completes the proof of our claim.

Next we calculate

$$\Phi_1(x_1, y_1) = \Phi_2(x_1, y_1) = \Psi(x_1, y_1) = 0, \; \Phi_1(x_2, y_2) = \Phi_2(x_2, y_2) = 0$$

and hence M_1 is a common point of all three invariant conics. Moreover the invariant line $y = x - 1/4$ passes through this point and it is tangent to each one of the conics. On the other hand M_2 is a point of intersection of the invariant parabolas. As a result we obtain the configuration *Config. $\mathcal{PH}$.38*.

Next we have to determine the conditions for distinguishing the configurations *Config.* $\mathcal{PH}$*.37* and *Config.* $\mathcal{PH}$*.38*. It was proved earlier that the system of equations $\chi_4 = 0$ and $\gamma_1 = 0$ for systems (54) could have only the following three solutions $\mathcal{S}_i = (m_i, n_i)$:

$$\mathcal{S}_1 = \Big(\frac{3}{8}, -\frac{5}{4}\Big), \quad \mathcal{S}_2 = \Big(\frac{175}{72}, -\frac{145}{36}\Big), \quad \mathcal{S}_3 = \Big(\frac{27}{1000}, -\frac{33}{100}\Big). \tag{61}$$

Moreover we shown that in the point $\mathcal{S}_1$ (respectively, $\mathcal{S}_2$; $\mathcal{S}_3$) of the parameter space (m, n) of systems (54) we have the configuration *Config.* $\mathcal{PH}$*.38* (respectively, *Config.* $\mathcal{PH}$*.37*; $\cong$ *Config.* $\mathcal{PH}$*.37*).

On the other hand evaluating the invariant polynomial ζ_6 at these points we obtain:

$$\zeta_6(\mathcal{S}_1) = 0, \quad \zeta_6(\mathcal{S}_2) = 130/9 \neq 0, \quad \zeta_6(\mathcal{S}_3) = -78/25 \neq 0.$$

Thus if for a system (54) the conditions provided by the statement $(\boldsymbol{\mathcal{B}_5})$ of Proposition 2 are satisfied and in addition we have $\gamma_1 = 0$ (this implies $\gamma_2 = 0$) then this system possesses the configuration *Config.* $\mathcal{PH}$*.37* for $\zeta_6 \neq 0$ and *Config.* $\mathcal{PH}$*.38* for $\zeta_6 = 0$.

3.2.1.2.4 The statements $(\boldsymbol{\mathcal{B}_6})$ and $(\boldsymbol{\mathcal{B}_7})$. According to Proposition 2 we have $\chi_4 = 0$ and either $\zeta_9 = 0$ (in the case $(\boldsymbol{\mathcal{B}_6})$) or $\zeta_5 = 0$ (in the case $(\boldsymbol{\mathcal{B}_7})$).

On the other hand we have shown earlier (see Subsection 3.2.1.2.3) that the system of equations $\chi_4 = 0$ and $\gamma_1 = 0$ for systems (54) could have only three solutions $\mathcal{S}_i = (m_i, n_i)$ presented in (61). Via a straightforward calculation it could be checked the following relations:

$$\zeta_5(\mathcal{S}_1)\zeta_5(\mathcal{S}_2)\zeta_5(\mathcal{S}_3) \neq 0, \quad \zeta_9(\mathcal{S}_1)\zeta_9(\mathcal{S}_2)\zeta_9(\mathcal{S}_3) \neq 0$$

Therefore we conclude that the conditions provided by the statements $(\boldsymbol{\mathcal{B}_6})$ or $(\boldsymbol{\mathcal{B}_7})$ are incompatible with the condition $\gamma_1 = 0$, i.e. in these cases we cannot have any invariant hyperbola.

Thus all the statements of Proposition 2 are examined and hence the following theorem is proved.

Theorem 3. *Assume that for a non-degenerate quadratic system the conditions $\eta > 0$, $\zeta_1 = 0$ are satisfied and in addition one of the sets of conditions provided by Proposition 2 holds. Then this system possesses at least one invariant parabola and one invariant hyperbola and has one of the configurations presented in Figure 1 if and only if $\gamma_1 = \gamma_2 = 0$ and the following conditions are fulfilled, correspondingly.*

(i) If the statement $(\boldsymbol{\mathcal{B}_1})$ holds then we obtain:

1. $\nu_{11} = 0$, $\xi_{20} \neq 0 \ \Rightarrow\ \cong$ *Config.* $\mathcal{PH}$*.1;*

2. $\nu_{11} = 0$, $\xi_{20} = 0 \ \Rightarrow\ \cong$ *Config.* $\mathcal{PH}$*.13.*

(ii) If the statement $(\boldsymbol{\mathcal{B}_5})$ holds then we obtain:

1. $\zeta_6 \neq 0 \ \Rightarrow$ *Config.* $\mathcal{PH}$*.37;*

2. $\zeta_6 = 0 \ \Rightarrow$ *Config.* $\mathcal{PH}$*.38.*

3.2.2 The possibility $\eta = 0$, $\widetilde{M} \neq 0$

According to [20, Main Theorem] we have the following lemma.

Lemma 5. *If a quadratic system possess an invariant parabola and in addition the conditions $\eta = 0$ and $\widetilde{M} \neq 0$ are satisfied, then this system could be brought via an affine transformation and time rescaling to one of the following two normal forms:*

$$\dot{x} = m + nx - gy/2 + gx^2 + xy, \quad \dot{y} = 2mx + 2ny + gxy + 2y^2, \tag{$S^1_{\boldsymbol{\gamma}}$}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y$, or

$$\dot{x} = 2mx + 2ny + (h - 1)xy, \quad \dot{y} = n - (h + 1)x/2 + my + hy^2 \tag{$S^2_{\boldsymbol{\gamma}}$}$$

possessing the invariant parabola $\Phi(x, y) = y^2 - x$.

For systems ($S^1_{\boldsymbol{\gamma}}$) calculations yield $\mu_0 = 2g^2$, whereas for ($S^2_{\boldsymbol{\gamma}}$) we have $\mu_0 = 0$. So we discuss two cases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

3.2.2.1 The case $\mu_0 \neq 0$.

According to Proposition 3 for the existence of an invariant parabola for a quadratic system with $\eta = 0$, $\widetilde{M} \neq 0$ and $\mu_0 \neq 0$ one of the sets of conditions from **($\boldsymbol{\mathcal{H}_1}$)** to **($\boldsymbol{\mathcal{H}_4}$)** it is necessary and sufficient.

Since for systems ($S^1_{\boldsymbol{\gamma}}$) we have $\mu_0 = 2g^2 \neq 0$, we may assume $g = 1$ due to the rescaling $(x, y, t) \mapsto (gx, g^2y, t/g^2)$. So in what follows we consider the following canonical form:

$$\dot{x} = m + nx - y/2 + x^2 + xy, \quad \dot{y} = 2mx + 2ny + xy + 2y^2 \tag{62}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y = 0$.

According to Proposition 1 for the existence of at least one invariant hyperbola for systems (62) the condition $\gamma_1 = \gamma_2 = 0$ is necessary. So for systems (62) we calculate:

$$\gamma_1 = -\frac{1}{128}(20m - 6n - 3)^2(14m - 3n), \quad \nu_1 = \frac{1607}{128}(20m - 6n - 3)^2. \tag{63}$$

On the other hand considering Diagram 4 for systems (62) we calculate:

$$\eta = 0, \quad \widetilde{M} = -8y^2 \neq 0, \quad \theta = -1/2 \neq 0, \quad \beta_2 = (14m - 3n)/2. \tag{64}$$

In what follows we examine step by step each one of the statements **($\boldsymbol{\mathcal{H}_1}$)**–**($\boldsymbol{\mathcal{H}_4}$)** provided by Proposition 3.

3.2.2.1.1 The statement ($\boldsymbol{\mathcal{H}_1}$).

Considering (63) we observe that the condition $\nu_1 = 0$ implies $\gamma_1 = 0$ and so we discuss two possibilities: $\nu_1 \neq 0$ and $\nu_1 = 0$.

1: it The possibility $\nu_1 \neq 0$. In this case the condition $\gamma_1 = 0$ is equivalent to $14m - 3n = 0$, i.e. $m = 3n/14$ and calculations yield

$$\gamma_2 = -\frac{12150}{49}n^2(7 + 4n)^2, \quad \nu_1 = \frac{14463}{6272}(7 + 4n)^2.$$

Therefore due to $\nu_1 \neq 0$ the condition $\gamma_2 = 0$ yields $n = 0$ (this implies $m = 0$) and considering (64) we obtain $\beta_2 = 0$.

So if the conditions (64) and $\beta_2 = 0$ are fulfilled, by Diagram 4 a system (62) could possess an invariant parabola only if $\beta_1 = 0$. However for a system (62) with $m = n = 0$ we have $\beta_1 = -1/2 \neq 0$. So we conclude that for $\nu_1 \neq 0$ systems (62) cannot have any invariant hyperbola.

2: it The possibility $\nu_1 = 0$. This implies $20m - 6n - 3 = 0$, i.e. $m = 3(1 + 2n)/20$ and we calculate

$$\gamma_2 = -\frac{567}{40}(1+2n)(7+4n)^2(13+36n), \quad \zeta_4 = \frac{3}{40}(7+4n).$$

So due to the condition $\zeta_4 \neq 0$ provided by the statement $(\boldsymbol{\mathcal{H}_1})$ we obtain that the condition $\gamma_2 = 0$ implies $(1 + 2n)(13 + 36n) = 0$. On the other hand we have

$$\nu_6 = -11931(7+4n)(13+36n)/40$$

and we examine two cases: $\nu_6 \neq 0$ and $\nu_6 = 0$.

2.1: *The case* $\nu_6 \neq 0$. Then $13 + 36n \neq 0$ and we get $n = -1/2$ which implies $m = 0$. Taking into account (64) we calculate

$$\beta_2 = -3/4 \neq 0, \quad \beta_1 = -15/16 \neq 0$$

and by Diagram 4 a system (62) possesses an invariant hyperbola if and only if $\widetilde{\mathcal{R}}_1 \neq 0$. However in the case $m = 0$ and $n = -1/2$ for systems (62) we get $\widetilde{\mathcal{R}}_1 = 0$. Therefore we conclude that we cannot have any invariant hyperbola in the case under consideration.

2.2: *The case* $\nu_6 = 0$. Due to $\zeta_4 \neq 0$ we have $7+4n \neq 0$ and then we get $13+36n = 0$. This implies $n = -13/36$ (then $m = 1/24$) and we arrive at the system

$$\dot{x} = \frac{1}{24} - \frac{13}{36}x - \frac{1}{2}y + x^2 + xy, \quad \dot{y} = \frac{1}{12}x - \frac{13}{18}y + xy + 2y^2.$$

This system possesses the following invariant parabola and hyperbola:

$$\Phi(x, y) = x^2 - y = 0, \quad \Psi(x, y) = -1 + 18x - 108y + 216xy = 0.$$

We determine that the above system possesses four real finite singularities $M_i(x_i, y_i)$, $i = 1, 2, 3, 4$ with the coordinates

$$x_1 = \frac{1}{6},\ y_1 = \frac{1}{36};\quad x_2 = \frac{5}{18},\ y_2 = \frac{1}{12};\quad x_{3,4} = -\frac{1}{8}\big(2 \pm \sqrt{13}\,\big),\ y_{3,4} = \frac{1}{36}\big(17 \pm 4\sqrt{13}\big).$$

We calculate

$$\Phi(x_1, y_1) = \Phi(x_3, y_3) = \Phi(x_4, y_4) = \Psi(x_1, y_1) = \Psi(x_2, y_2) = 0$$

and therefore M_1 is a common point of the invariant parabola and the upper branch of the hyperbola. On the other hand M_2 is located on the upper branch of the hyperbola whereas M_3 an M_4 are located on the invariant parabola.

Since $\eta = 0$, $\widetilde{M} \neq 0$ and $C_2 = -xy^2$ we conclude that at infinity we have a double point which is located at $N[1 : 0 : 0]$. As a result we obtain the configuration *Config.* $\mathcal{PH}$*.39*.

3.2.2.1.2 The statements from $(\mathcal{H}_2)$ to $(\mathcal{H}_4)$. We observe that these three statements have the common condition $\zeta_4 = 0$. We claim that if for systems (62) this condition is satisfied then they could not have any invariant hyperbola.

Indeed for systems (62) we have

$$\zeta_4 = (9 - 4m + 6n)/16 = 0 \ \Rightarrow \ m = 3(3 + 2n)/4.$$

Then we calculate

$$\gamma_1 = -81(7 + 4n)^3/64 = 0 \ \Rightarrow \ n = -7/4 \ \ (\text{then } m = -3/8)$$

and for a system (62) with $m = -3/8$ and $n = -7/4$ we obtain

$$\beta_2 = \beta_1 = 0, \quad \gamma_{14} = 200 \neq 0.$$

So considering (64) and Diagram 4 we conclude that the system (62) defined by the conditions $m = -3/8$ and $n = -7/4$ cannot have any invariant hyperbola. This completes the proof of our claim.

Thus we proved the following theorem.

Theorem 4. *Assume that for a non-degenerate quadratic system the conditions $\eta = 0$, $\widetilde{M} \neq 0$, $\chi_1 = 0$ and $\mu_0 \neq 0$ are satisfied. Then this system could possess one invariant parabola and one invariant hyperbola and this occurs if and only if $\nu_1 = \nu_6 = 0$ and the statement $(\mathcal{H}_1)$ of Proposition 3 holds. Moreover this system has the configuration Config. $\mathcal{PH}.39$.*

3.2.2.2 The case $\mu_0 = 0$ According to Lemma 5 we have to consider the families of systems $(S^1_{\boldsymbol{\gamma}})$ with $g = 0$ (due to $\mu_0 = 2g^2 = 0$) and $(S^2_{\boldsymbol{\gamma}})$ for which we have $\mu_0 = 0$. We examine each one of these families.

3.2.2.2.1 The family $(S^1_{\boldsymbol{\gamma}})$. Taking into account the condition $g = 0$ we examine the family of systems

$$\dot{x} = m + nx + xy, \quad \dot{y} = 2mx + 2ny + 2y^2, \tag{65}$$

which possess the invariant parabola $\Phi(x, y) = x^2 - y$. For these systems we have $\eta = 0$, $\widetilde{M} = -8y^2 \neq 0$ and $\gamma_1 = \gamma_2 = 0$. So considering the conditions provided by Diagram 4, for systems (65) we calculate:

$$\theta = 0, \ N = 27y^2, \ \beta_4 = 0, \ \beta_6 = -3m/2, \ \beta_{11} = 12y^2, \ \widetilde{\mathcal{R}}_{11} = 3my(16mx^3 - 4nx^2y + y^3).$$

According to Diagram 4 for the existence of at least one invariant hyperbola the conditions $\beta_6 = 0$ and $\widetilde{\mathcal{R}}_{11} \neq 0$ are necessary. However the condition $\beta_6 = 0$ gives us $m = 0$ and this implies $\widetilde{\mathcal{R}}_{11} = 0$. Therefore we conclude that systems (65) could not have any hyperbola beside the invariant parabola $\Phi(x, y) = x^2 - y$.

On the other hand we observe that for these systems we have

$$\zeta_{11} = \mathcal{R}_3 = 0, \quad \zeta_{16} = 180m^3(343m^2 + 36n^3), \quad \mathcal{R}_6 = 35700m^5.$$

We observe that in the case $\zeta_{16}\mathcal{R}_6 \neq 0$ the conditions $(\mathcal{K}_3)$ of Proposition 4 are satisfied. Therefore we have the next remark.

Remark 9. *Assume that for a quadratic system the conditions provided by the statement* $(\mathcal{K}_3)$ *of Proposition 4 are fulfilled. Then this system could not have any hyperbola beside the invariant parabola* $\Phi(x,y)=x^2-y$.

3.2.2.2.2 The family (S^2_γ). According to Proposition 4 for the existence of an invariant parabola for a quadratic system with $\eta=0$, $\widetilde{M}\neq 0$ and $\mu_0=0$ one of the sets of conditions from $(\mathcal{K}_1)$ to $(\mathcal{K}_{11})$ it is necessary and sufficient. Moreover in this case if for a quadratic system are satisfied the conditions of one of the sets from $(\mathcal{K}_1)$ to $(\mathcal{K}_{11})$ (except $(\mathcal{K}_3)$) then this system could be brought via an affine transformation and time rescaling to the canonical form (S^2_γ).

Thus in what follows we consider the 3-parameter family of systems

$$\dot{x}=2mx+2ny+(h-1)xy,\quad \dot{y}=n-(h+1)x/2+my+hy^2 \tag{66}$$

possessing the invariant parabola $\Phi(x,y)=y^2-x$ and determine the conditions for the existence at least one invariant hyperbola of systems (66).

We point out that for these systems we have $\gamma_1=\gamma_2=0$, i.e. the necessary condition for the existence of an invariant hyperbola are satisfied. So we examine step by step each one of the statements from $(\mathcal{K}_1)$ to $(\mathcal{K}_{11})$ provided by Proposition 4 except the statement $(\mathcal{K}_3)$ because according to Remark 9 in this case we could not have any invariant hyperbola.

1: *The statement* $(\mathcal{K}_1)$. Considering the conditions provided by this statement (see Proposition 4) for systems (66) we calculate

$$\begin{gathered}\zeta_{14}=(1+h)^2y^2,\ \zeta_{15}=(h-1)^2y^2/4,\quad \zeta_{16}=0,\\ \zeta_{17}=-144m(h-1)((1+h)^2)(3h-1)\mathcal{U}(h,m,n),\end{gathered}\tag{67}$$

where $\mathcal{U}(h,m,n)$ is a polynomial in the indicated parameters. We observe that the condition $\zeta_{15}=0$ implies $\zeta_{17}=0$ and we discuss two possibilities: $\zeta_{15}\neq 0$ and $\zeta_{15}=0$.

1.1: *The possibility* $\zeta_{15}\neq 0$. Considering Diagram 4 for systems (66) we calculate:

$$\eta=0,\quad M=-8y^2\neq 0,\quad \theta=0,\quad N=(h-1)(h+1)y^2,\quad \beta_4=0$$

and we observe that due to the condition $\zeta_{14}\zeta_{15}\neq 0$ we obtain $N\neq 0$. In this case according to Diagram 4 systems (66) could possess at least one invariant hyperbola only if $\beta_6=0$. However for systems (66) we have $\beta_6=(h-1)(1+h)^2/8\neq 0$ due $\zeta_{14}\zeta_{15}\neq 0$. Therefore we conclude that in the case $\zeta_{15}\neq 0$ we could not have any invariant hyperbola.

1.2: *The possibility* $\zeta_{15}=0$. Considering (67) we obtain $h=1$ and then calculations yield

$$\widetilde{N}=\beta_{13}=0,\quad \gamma_9=-6\neq 0.$$

Therefore by Diagram 4 systems (66) cannot have any invariant parabola.

2: *The statements* $(\mathcal{K}_2)$, $(\mathcal{K}_4)$. We observe that these two statements have the common condition $\zeta_{11}=0$. We claim that if for systems (66) this condition is satisfied then these systems could not have any invariant hyperbola.

Indeed for systems (66) we calculate

$$\zeta_{11} = -4(h-2)y^2 = 0 \;\Rightarrow\; h = 2$$

and considering Diagram 4 for systems (66) with $h = 2$ we calculate:

$$\eta = 0,\quad M = -8y^2 \neq 0,\quad \theta = 0,\quad N = 3y^2 \neq 0,\quad \beta_4 = 0,\quad \beta_6 = 9/8 \neq 0.$$

Therefore by Diagram 4 systems (66) in this case cannot have any invariant hyperbola and this completes the proof of our claim.

3: *The statement* $(\mathcal{K}_5)$. According to Proposition 4 in this case the condition $\zeta_{12} = 4hy^2 = 0$ holds. This implies $h = 0$ and calculations yield:

$$N = -y^2 \neq 0,\quad \beta_4 = 0,\quad \beta_6 = -1/8 \neq 0.$$

So according to Diagram 4 we obtain that systems (66) in this case cannot have any invariant hyperbola.

4: *The statements from* $(\mathcal{K}_6)$ *to* $(\mathcal{K}_9)$. We observe that these four statements have the common condition $\zeta_{13} = 0$. We claim that if for systems (62) this condition is satisfied then these systems could not have any invariant hyperbola.

Indeed for systems (66) we calculate

$$\zeta_{13} = (1+3h)y^2 = 0 \;\Rightarrow\; h = -1/3$$

and considering Diagram 4 for systems (66) with $h = -1/3$ we calculate:

$$\eta = 0,\quad M = -8y^2 \neq 0,\quad \theta = 0,\quad N = -8y^2/9 \neq 0,\quad \beta_4 = 0,\quad \beta_6 = -2/27 \neq 0.$$

Therefore by Diagram 4 systems (66) in this case cannot have any invariant hyperbola and this completes the proof of our claim.

5: *The statement* $(\mathcal{K}_{10})$. According to Proposition 4 in this case the condition $\zeta_{14} = (1+h)^2y^2 = 0$ holds. This implies $h = -1$ and calculations yield:

$$N = 0,\quad \beta_{13} = y^2 \neq 0$$

and by Diagram 4 systems (66) could possess one invariant hyperbola if and only if $\gamma_{10} = \gamma_{17} = 0$ and $\widetilde{R}_{11} \neq 0$. For systems (66) with $h = -1$ we calculate

$$\gamma_{10} = 0,\quad \gamma_{17} = -8(3m^2+16n)y^2,\quad \widetilde{R}_{11} = 24mny^4$$

and clearly the condition $\gamma_{17} = 0$ gives us $n = -3m^2/16 \neq 0$ due to $\widetilde{R}_{11} \neq 0$. This leads to the one-parameter family of systems

$$\dot{x} = 2mx - \frac{3m^2}{8}y - 2xy,\quad \dot{y} = -\frac{1}{16}(m-4y)(3m-4y) \tag{68}$$

possessing two parallel invariant lines $y = m/4$ and $y = 3m/4$ ($m \neq 0$) and the following invariant parabola and hyperbola:

$$\Phi(x, y) = y^2 - x = 0, \quad \Psi(x, y) = m^3 + 48mx - 12m^2y - 64xy = 0.$$

We determine that the above systems possess two real finite singularities $M_i(x_i, y_i)$, $i = 1, 2$ with the coordinates

$$x_1 = \frac{m^2}{16}, \ y_1 = \frac{m}{4}; \quad x_2 = \frac{9m^2}{16}, \ y_2 = \frac{3m}{4}.$$

For systems (68) we have $\mu_0 = \mu_1 = 0$ and $\mu_2 = 3m^2y^2/4$ and according to [1, Lemma 5.2, statement (i)] we conclude that two of the singular points of systems (68) have gone to infinity and coalesced with the infinite singularity $N[0 : 1 : 0]$ producing a singularity of the multiplicity $(2, 2)$.

We point out that the coordinates of the finite singularities M_1 and M_2 are distinct (due to $m \neq 0$) and we have

$$x_2 - x_1 = m^2/2 > 0, \quad y_2 - y_1 = m/2 \ \Rightarrow \ \text{sign}\,(y_2 - y_1) = \text{sign}\,(m)$$

On the other hand we calculate

$$\Phi(x_1, y_1) = \Psi(x_1, y_1) = 0, \quad \Phi(x_2, y_2) = 0$$

and therefore M_1 is a point of intersection of the invariant parabola with the lower branch of the hyperbola. Moreover M_2 is located on the invariant parabola and its position with respect to the point of intersection depends on the sign of the parameter m. It is not too difficult to determine that in the case $m < 0$ we arrive at the configuration *Config.* $\mathcal{PH}$*.40*. On the other hand for $m > 0$ we get a configuration which is equivalent to *Config.* $\mathcal{PH}$*.40*.

6: *The statement* $(\boldsymbol{\mathcal{K}_{11}})$. In this case by Proposition 4 the conditions $\zeta_{14} = \zeta_{19} = 0$ and $\zeta_{20} \neq 0$ hold. As it was shown in the previous case above the condition $\zeta_{14} = 0$ implies for systems (66) $h = -1$ and then we have

$$\zeta_{19} = 24mny^4, \quad \zeta_{20} = 32ny^2.$$

So due to $\zeta_{20} \neq 0$ (i.e. $n \neq 0$) the condition $\zeta_{19} = 0$ implies $m = 0$. In this case calculations yield

$$\eta = 0, \quad M = -8y^2 \neq 0, \quad \theta = 0, \quad N = 0, \quad \beta_{13} = y^2 \neq 0, \quad \gamma_{10} = 0, \quad \gamma_{17} = -128ny^2.$$

According to Diagram 4 in the case under consideration systems (66) could possess one invariant hyperbola only if $\gamma_{17} = 0$, i.e. $n = 0$. However due to the condition $\zeta_{20} \neq 0$ we have $n \neq 0$ and therefore in the case $(\boldsymbol{\mathcal{K}_{11}})$ systems (66) could not possess any invariant hyperbola.

Thus we proved the following theorem.

Theorem 5. *Assume that for a non-degenerate quadratic system the conditions $\eta = 0$, $\widetilde{M} \neq 0$ and $\chi_1 = \mu_0 = 0$ are satisfied. Then this system could possess one invariant parabola and one invariant hyperbola and this occurs if and only if $\gamma_{17} = 0$ and the statement $(\boldsymbol{\mathcal{K}_{10}})$ of Proposition 4 holds. Moreover this system has the configuration Config.* $\mathcal{PH}$*.40.*

3.2.3 The possibility $\eta = \widetilde{M} = C_2 = 0$

In this case the line at infinity of a system (3) is filled up with singularities. According to Proposition 5 if this system possess an invariant parabola it could be brought via an affine transformation and time rescaling to the following canonical form

$$\dot{x} = m + nx - y/2 + x^2, \quad \dot{y} = 2mx + 2ny + xy \tag{69}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y$.

In what follows we determine the conditions for the existence of at least one invariant hyperbola of systems (69). First of all we point out that for these systems we have $\gamma_1 = \gamma_2 = 0$, i.e. the necessary conditions for the existence of an invariant hyperbola are satisfied. According to [9, Theorem 2.18] a quadratic system with $C_2 = 0$ possesses an invariant hyperbola if and only if the condition $N_7 = 0$ holds. For systems (69) we calculate $N_7 = 0$ and we conclude that these systems possess at least one invariant hyperbola. Moreover according to Diagram 4 in this case systems (69) possess an infinite number of invariant hyperbolas.

On the other hand considering [17] we evaluate the following invariant polynomials:

$$H_{10} = 9, \quad H_9 = 576(2m - n^2)^3, \quad H_{12} = -2(2m - n^2)(8mx^2 - 16n^2x^2 - 12nxy - 3y^2).$$

Therefore since $H_{10} \neq 0$, according to [17] (see Table 1) a system from the family (69) (with $C_2 = 0$) possesses three real invariant lines if $H_9 < 0$ and one real and two complex invariant lines if $H_9 > 0$.

We observe that the condition $H_9 = 0$ implies $H_{12} = 0$ and by [17, Table 1] a system (69) in this case possesses a triple invariant affine line. So we consider three cases: $H_9 < 0$, $H_9 > 0$ and $H_9 = 0$.

3.2.3.1 The case $H_9 < 0$. Then $2m - n^2 < 0$ and we set $2m - n^2 = -w^2 < 0$, where w is a new parameter. So we have $m = (n^2 - w^2)/2$ and this leads to the family of systems

$$\dot{x} = (n^2 - w^2)/2 + nx - y/2 + x^2, \quad \dot{y} = (n^2 - w^2)x + 2ny + xy \tag{70}$$

possessing the invariant parabola $\Phi(x, y) = x^2 - y$ and the following three invariant lines:

$$L_1 : \, n^2 - w^2 + 2nx + y = 0; \quad L_{2,3} : \, (n \pm w)^2 + 2(n \pm w)x + y = 0.$$

Moreover systems (70) possess the family of conics (depending on the parameter s)

$$F_s(x, y) = s(x^2 - y) + (n^2 - w^2 + 2nx + y)^2 = 0$$

for which considering Remark 1 we calculate

$$\Delta = -s^2(s + 4w^2)/4, \quad \delta = s.$$

According to this remark a conic belonging to the above family is irreducible if and only if $\Delta \neq 0$. Moreover an irreducible conic is a hyperbola if $\delta > 0$; an ellipse if $\delta < 0$ and it is a parabola if $\delta = 0$. However the condition $\delta = 0$ implies $\Delta = 0$ and hence we could not have a parabola belonging to the family $F_s(x, y) = 0$.

Therefore we have the following irreducible invariant conics defined by $F_s(x, y) = 0$:

- $s(s+4w^2)\neq 0$ and $s>0 \Rightarrow$ a family of hyperbolas;
- $s(s+4w^2)\neq 0$ and $s<0 \Rightarrow$ a family of ellipses.

We determine that systems (70) possess three real finite singularities $M_i(x_i, y_i)$, $i=1,2,3$ with the coordinates

$$x_1=-n,\ y_1=(n^2-w^2);\quad x_{2,3}=-(n\pm w),\ y_{2,3}=(n\pm w)^2$$

and we calculate

$$L_2(x_1,y_1)=L_3(x_1,y_1)=0,\quad \Phi(x_2,y_2)=L_1(x_2,y_2)=L_2(x_2,y_2)=0;$$
$$\Phi(x_3,y_3)=L_1(x_3,y_3)=L_3(x_3,y_3)=0.$$

Moreover for any value of the parameter s for the family of invariant conics $F_s(x,y)=0$ we have

$$F_s(x_2,y_2)=F_s(x_3,y_3)=0,$$

i.e. M_2 and M_3 are the points of intersections of any two conics belonging to the family $F_s(x,y)=0$. We also observe that the singular point M_2 (respectively M_3) is a point of tangency of the invariant line $L_2(x,y)=0$ (respectively $L_3(x,y)=0$) with the invariant parabola $\Phi(x,y)=0$. Furthermore any invariant conic the family $F_s(x,y)=0$ is tangent to the invariant parabola at these two singular points M_2 and M_3.

Therefore taking into account that the infinite invariant line is fulfilled up with the singularities, in the case $H_9<0$ we arrive at the configuration *Config.*$\mathcal{PH}$*.41*.

3.2.3.2 The case $H_9>0$. Then $2m-n^2>0$ and we set $2m-n^2=z^2>0$, where z is a new parameter. So we have $m=(n^2+z^2)/2$ and this leads to the family of systems

$$\dot{x}=(n^2+z^2)/2+nx-y/2+x^2,\quad \dot{y}=(n^2+z^2)x+2ny+xy \tag{71}$$

possessing the invariant parabola $\Phi(x,y)=x^2-y$ and the following three invariant lines:

$$L_1':\ n^2+z^2+2nx+y=0;\quad L_{2,3}':\ (n\pm iz)^2+2(n\pm iz)x+y=0.$$

Moreover systems (71) possess the family of conics depending on the parameter s:

$$F_s'(x,y)=s(x^2-y)+(n^2-z^2+2nx+y)^2=0$$

for which considering Remark 1 we calculate

$$\Delta=-s^2(s-4z^2)/4,\quad \delta=s.$$

Therefore we have the following invariant conics defined by $F_s'(x,y)=0$:

- $s(s-4z^2)\neq 0$ and $s>0 \Rightarrow$ a family of hyperbolas;
- $s(s-4z^2)\neq 0$ and $s<0 \Rightarrow$ a family of ellipses.

We determine that systems (71) possess only one real singular point $M_1(-n, n^2+z^2)$ which is a point of intersection of the complex invariant lines L'_2 and L'_3. Considering the invariant parabola $\Phi(x,y) = x^2 - y$ we observe that $\Phi(x_1,y_1) = -z^2 < 0$ and therefore we conclude that M_1 is located the domain bordered by this parabola and its infinite point. Moreover this singular point is surrounded by the family of real ellipses placed in this domain.

Therefore it is not too difficult to determine that in this case we arrive at the configuration *Config.* $\mathcal{PH}.42$.

3: *The possibility* $H_9 = 0$. This implies $m = n^2/2$ and we get the family of systems

$$\dot{x} = n^2/2 + nx - y/2 + x^2, \quad \dot{y} = n^2x + 2ny + xy \tag{72}$$

for which $H_{10} = 9 \neq 0$ and $H_9 = H_{12} = 0$. Therefore according to [17, Table 1] the above systems possess a triple invariant affine line $L''_1 : n^2+2nx+y = 0$ and a triple real singular point $M_1(-n, n^2)$.

On the other hand beside the invariant parabola $\Phi(x,y) = x^2-y$ systems (72) possess the following family of the invariant conics

$$F''_s(x,y) = F_s(x,y) = s(x^2-y) + (n^2+2nx+y)^2 = 0 = 0$$

for which considering Remark 1 we calculate

$$\Delta = -s^3/4, \quad \delta = s.$$

Thus by Remark 1 for $s > 0$ we have a family of hyperbolas, whereas for $s < 0$ the conics $F''_s(x,y) = 0$ are ellipses.

Considering the coordinates $x_1 = -n$ and $y_1 = -n^2$ of the singular point M_1 we calculate:

$$L''_1(x_1,y_1) = \Phi(x_1,y_1) = 0, \quad F''_s(x_1,y_1) = 0, \ \forall s \in \mathbb{R}.$$

So we conclude that M_1 a common point of all the invariant curves and moreover it is a point of tangency of the invariant conics with the invariant line. So in this case we obtain the the configuration *Config.* $\mathcal{PH}.43$.

Thus we proved the following theorem.

Theorem 6. *Assume that for a non-degenerate quadratic system the conditions $C_2 = 0$, $\zeta_5 \neq 0$ and $\zeta_{22} = 0$ are satisfied. Then this system possess one invariant parabola and a family of non-parabolic invariant conics as well as invariant lines of total multiplicity three. More exactly beside the invariant parabola this system possesses:*

- *three real invariant lines, a subfamily of invariant hyperbolas and a subfamily of real invariant ellipses if $H_9 < 0 \;\Rightarrow\;$ Config. $\mathcal{PH}.41$;*
- *one real and two complex invariant lines, a subfamily of invariant hyperbolas and a subfamily of invariant ellipses (real and complex) if $H_9 > 0 \;\Rightarrow\;$ Config. $\mathcal{PH}.42$;*
- *one triple real invariant line, a subfamily of invariant hyperbolas and a subfamily of real invariant ellipses if $H_9 = 0 \;\Rightarrow\;$ Config. $\mathcal{PH}.43$.*

3.3 Geometric invariants and the proof of the statement $(\boldsymbol{D})$

In this subsection we complete the proof of the Main Theorem by showing that all 43 configurations of invariant parabolas and hyperbolas and invariant lines we constructed are non-equivalent according to Definitions 1.8 and 1.9 given in [10]. For this we define the invariants that split the configurations of the family **QSPH** into the 43 *distinct* ones (see Diagram 5). We hope these invariants are among those best suited for describing the geometric phenomena that are specific to this class.

The basic algebraic-geometric definitions of use here are the notion of an integer valued r-cycle and its type:

Definition 2. *Let V be an irreducible algebraic variety of dimension n over a field K. A cycle of dimension r or r-cycle on V is a formal sum $\sum_W m(W)W$ where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $m(W) \in \mathbb{Z}$, and only a finite number of $m(W)$'s are non-zero. We call degree of an r-cycle the sum $\sum_W$. An $(n-1)$-cycle is called a divisor.*

Definition 3. *We call type of an r-cycle the set of all ordered couples (n_1, n_2) where n_1 is a coefficient, $n_1 = m(W)$ appearing in the $r-$cyle and n_2 is the number of W's in the cycle whose coefficient is $m(W)$.*

We define a list of numeric and geometric invariants that allow us to distinguish the obtained configurations.

- $(N_{\mathcal{P}}, N_{\mathcal{H}})$ = (number of invariant parabolas, number of invariant hyperbolas);
- $n_{\mathcal{L}}^{\mathbb{R}}$ = number of real distinct invariant affine lines;
- $N_{\mathcal{P}\cap\mathcal{H}}^{\infty}$ = number of the the intersection points of invariant parabolas and invariant hyperbolas at infinity.
- $M_{\mathcal{P}\cap\mathcal{H}}^{n}$ = the maximum number of points of intersections of an parabola with an hyperbola at real finite points.
- $\tau_{\mathcal{P}}^{f}$ = the type of the divisor of the multiplicities of real finite singularities of systems on the parabolas. If on the parabola is not located any real singularity we set $\tau_{\mathcal{P}}^{f} = \{\emptyset\}$.
- $M_{1b\mathcal{H}}^{n}$ = the maximum number of real finite singularities of systems located on one branch of a hyperbola.
- M_s^{∞} = the maximum multiplicity of infinite singular points of a system
- Suppose an affine line has a point of tangency with the parabola $x^2 - y = 0$. Let Δ^c be the curvilinear triangle on the Poincaré disk with vertex at the infinite point of the parabola formed by the a closed branch of the parabola together with the adjacent segment of the invariant line and the open arc a_∞ which does not contain a singular infinite point in its interior. We denote by $n(\Delta^c)$ the number of the singularities located on Δ^b.

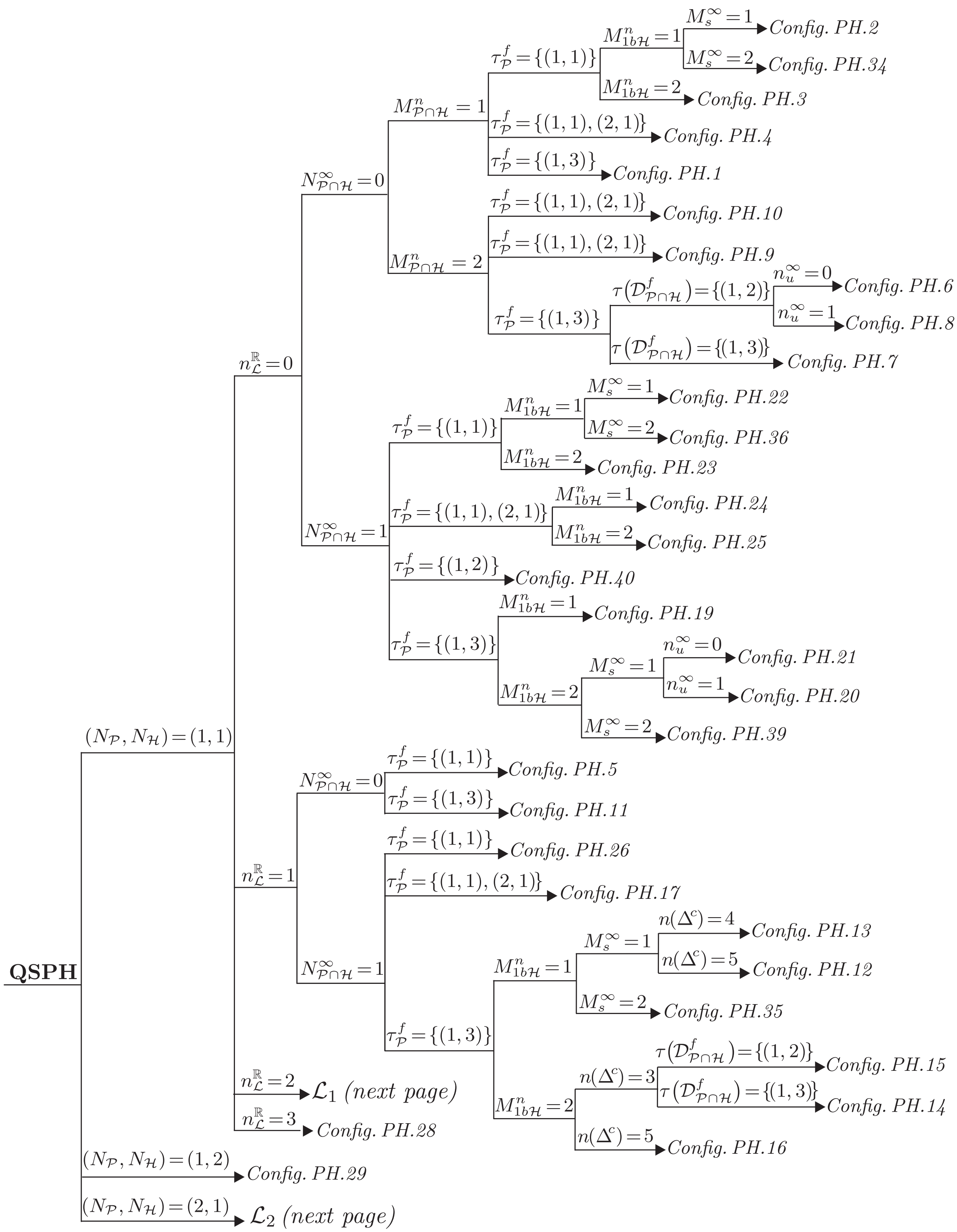

Diagram 5: Non-equivalent configurations of systems in **QSPH**

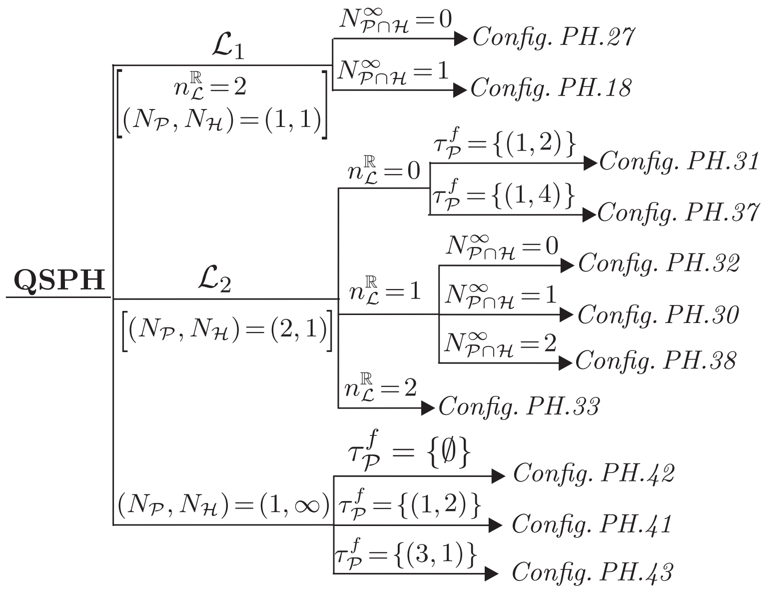

Diagram 5 (*contin.*) Non-equivalent configurations of systems in **QSPH**

- Assume that one branch of an invariant hyperbola intersect an parabola at two distinct finite points. Then we obtain a finite domain $\mathcal{D}^{f}_{\mathcal{P}\cap\mathcal{H}}$ delimited by the closed branches of the parabola and hyperbola. We denote by $\tau\big(\mathcal{D}^{f}_{\mathcal{P}\cap\mathcal{H}}\big)$ the type of the divisor of the multiplicities of singularities of the systems located on the border of $\mathcal{D}^{f}_{\mathcal{P}\cap\mathcal{H}}$.
- Suppose one branch of an invariant hyperbola intersect an parabola at two distinct points (finite or infinite). Let u_1 and u_2 be the intersection points of this branch with the infinite line $Z=0$. We denote by n_u^{∞} the number of infinite singularities located inside the open arc a_{∞} delimited by u_1 and u_2.

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