CUNTZ SEMIGROUPS OF ULTRAPRODUCT C^*-ALGEBRAS

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Abstract. We prove that the category of abstract Cuntz semigroups is bicomplete. As a consequence, the category admits products and ultraproducts. We further show that the scaled Cuntz semigroup of the (ultra)product of a family of C^*-algebras agrees with the (ultra)product of the scaled Cuntz semigroups of the involved C^*-algebras.

As applications of our results, we compute the non-stable K-Theory of general (ultra)products of C^*-algebras and we characterize when ultraproducts are simple. We also give criteria that determine order properties of these objects, such as almost unperforation.

1. Introduction

Product and ultraproduct constructions are important tools in mathematics. For C^*-algebras, they are technical devices in which one can suitably encompass notions that have come to the forefront of the theory. Most notably, ultrapoducts of C^*-algebras appear in aspects of the Toms-Winter conjecture; see, for example, [BBS$^+$19]. They also play an important role in detecting tensorial absorption by the Jiang-Su algebra for C^*-algebras with finite nuclear dimension; see [RT17].

The Cuntz semigroup is an invariant for C^*-algebras, first introduced in [Cun78] with the aim of proving that simple, stably finite, C^*-algebras admit quasitraces. Due to the fact that it contains a great deal of information about the C^*-algebra, its uses have been proved to be far-reaching in subsequent works. For example, the Cuntz semigroup has become key to distinguish C^*-algebras that fail to satisfy regularity conditions and thus are not amenable to classification using the Elliott invariant (see, for example, [Rør03] and [Tom08]). Further, for the class of simple, separable, nuclear C^*-algebras that absorb the Jiang-Su algebra tensorially, it was proved in [ADPS14] that the Cuntz semigroup after tensoring with the circle is functorially equivalent to the Elliott invariant.

The categorical framework needed to analyse the Cuntz semigroup as an invariant was developed in the pioneering work by Coward, Elliott and Ivanescu in [CEI08]. They defined the category Cu of abstract Cuntz semigroups (also called Cu-semigroups) and showed that the process of assigning to each C^*-algebra A the (concrete) Cuntz semigroup of its stabilisation is a sequentially continuous functor. The properties of the category Cu and its relation with the category of C^*-algebras was further developed in [APT18b], [APT18], and [APT18a]. It is therefore interesting to seek for efficient methods to compute the Cuntz semigroup of products and ultraproducts. This is one of the main objectives of this paper.

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The main structural features of the category Cu may be summarized as follows:

Theorem. *The category* Cu *of abstract Cuntz semigroups is a closed, symmetric, monoidal, bicomplete category.*

That Cu is symmetric and monoidal was established in [APT18b]; on the other hand, it was shown in [APT18] that Cu is closed. We prove in this paper that Cu is bicomplete, that is, complete and cocomplete (see Theorems 3.8 and 3.16 below). This means that Cu admits arbitrary small limits and colimits, and thus arbitrary products and inductive limits. (The latter was already proved in [APT18b].)

In order to show that Cu is complete, we take advantage of the fact that it sits coreflectively in a bigger category, termed $\mathcal{Q}$ in [APT18]. It follows that the product of a family of Cu-semigroups is not just their set-theoretic product, but rather a completion of the latter by means of the τ-construction (see Paragraph 2.12 for more details). Roughly speaking, the τ-construction consists of considering certain equivalence classes of continuous paths indexed on the interval $(-\infty, 0]$.

We further apply our results to C^*-algebras. To this end, we first study to what extent the Cuntz semigroup functor preserves these constructions. The fact that the Cuntz semigroup functor is stable by definition introduces a difficulty when dealing with products and ultraproducts of arbitrary infinite families of C^*-algebras, since these are not stable even if the algebras are. In order to circumvent this difficulty, we need a convenient substitute for stability. This is found in property (S), as introduced by Hjelmborg and Rørdam in [HR98].

Another important ingredient in our discussion is the notion of scale. For a C^*-algebra A, the natural scale of $\mathrm{Cu}(A)$ captures the information of equivalence classes coming from positive elements in A. We define in Section 4 scales in general abstract Cu-semigroups and introduce the category $\mathrm{Cu}_{\mathrm{sc}}$, in which Cu sits reflexively. This is the natural categorical framework in which Cuntz semigroups of products and ultraproducts of arbitrary families of C^*-algebras can be computed.

Ultraproducts over an ultrafilter $\mathcal{U}$ are introduced categorically in Paragraph 7.1 as inductive limits of products over sets in $\mathcal{U}$ following, for example, [Ekl77]. This is essential once we show that the scaled Cuntz semigroup functor preserves arbitrary inductive limits and products. We prove:

Theorem (Theorems 4.6, 5.13 and 7.5)**.** *The category* $\mathrm{Cu}_{\mathrm{sc}}$ *is bicomplete, and the scaled Cuntz semigroup functor preserves arbitrary products and ultraproducts of* C^**-algebras.*

More precisely, given a family $(A_j)_{j\in J}$ of C^*-algebras and an ultrafilter $\mathcal{U}$ over J, there is a natural commutative diagram

$$\begin{tikzcd} \mathrm{Cu}(\prod_{j\in J} A_j) \arrow[r, "\Phi"] \arrow[d, two heads] & \prod_{j\in J} \mathrm{Cu}(A_j) \arrow[d, two heads] \\ \mathrm{Cu}(\prod_{\mathcal{U}} A_j) \arrow[r, "\Phi_{\mathcal{U}}"] & \prod_{\mathcal{U}} \mathrm{Cu}(A_j) \end{tikzcd} .$$

where the horizontal maps are order-embeddings whose images are ideals. These maps induce order-isomorphisms between scaled Cuntz semigroups. In the unital setting, the said isomorphisms can be viewed as

$$\mathrm{Cu}(\prod_j A_j) \cong \{x \in \prod_j \mathrm{Cu}(A_j) \mid x \leq \infty u\},$$

where u is the image of the class of the unit via the map Φ and $\infty u = \sup_n nu$. A similar description is obtained for the ultraproduct.

Since the categorical picture of an ultraproduct is not the most commonly used in the category of C^*-algebras, we make the connection explicit in Paragraph 7.6 to also view an ultraproduct of C^*-algebras as the quotient of their product modulo the ideal of strings that converge to zero along the ultrafilter. This is also a convenient way to compute its Cuntz semigroup, as a quotient of the Cuntz semigroup of the product of the algebras modulo a suitable ideal, and it becomes a useful way to deal with ultraproducts in concrete examples (see Theorem 7.11 and Example 7.12).

As applications of our results, we compute the monoid of Murray-von Neumann classes of projections of a product and ultraproduct of a family of stably finite C^*-algebras, recovering and completing previous work of Li ([Li05]). We also characterize in Theorem 8.14 when an ultraproduct of C^*-algebras over a countably incomplete ultrafilter is simple; this unifies previous results due to Kirchberg (see [Rør02]) and also results from [FHL$^+$16].

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2. Preliminaries: Categorical Framework

2.1. **The category** Cu **of abstract Cuntz semigroups.** In this subsection, we recall the definition of the category Cu as introduced in [CEI08].

2.1. A *positively ordered monoid* is a commutative monoid M, written additively with neutral element 0, equipped with a translation invariant partial order $\leq$ for which 0 is the smallest element. The category of positively ordered monoids will be denoted by PoM, where the morphisms are maps that preserve addition, order, and the zero element. See [APT18b, Appendix B.2] for details.

2.2. Let $(X, \leq)$ be a partially ordered set. A binary relation $\prec$ on X is called an *auxiliary relation* (see [GHK$^+$03, Definition I-1.11, p.57]) if:

(i) If $x \prec y$ then $x \leq y$, for all $x, y \in X$.
(ii) If $w \leq x \prec y \leq z$ then $w \prec z$, for all $w, x, y, z \in X$.

If, further, X is a monoid, then an auxiliary relation $\prec$ is said to be *additive* if it is compatible with addition and $0 \prec x$ for every $x \in X$.

Of particular importance for us is the (sequential version of the) *compact containment relation*, also called *way-below relation*. Given $x, y \in X$, we say that x is *compactly contained* in y, or that x is *way-below* y, in symbols $x \ll y$, provided that for every increasing sequence $(y_n)_n$ such that the supremum $\sup_n y_n$ exists and satisfies $y \leq \sup_n y_n$, there exists $k \in \mathbb{N}$ such that $x \leq y_k$.

Definition 2.3 ([CEI08])**.** A Cu-*semigroup*, also called an *abstract Cuntz semigroup*, is a positively ordered monoid S that satisfies the following axioms:

(O1) Every increasing sequence $(x_n)_n$ in S has a supremum $\sup_n x_n \in S$.
(O2) Every element of S is the supremum of a $\ll$-increasing sequence.
(O3) If $x' \ll x$ and $y' \ll y$ then $x' + y' \ll x + y$, for all $x', y', x, y \in S$.
(O4) If $(x_n)_n$ and $(y_n)_n$ are increasing sequences in S, then $\sup_n(x_n + y_n) = \sup_n x_n + \sup_n y_n$.

A Cu-*morphism* is a PoM-morphism that preserves the way-below relation and suprema of increasing sequences. Given Cu-semigroups S and T, the set of all Cu-morphisms from S to T is denoted by $\mathrm{Cu}(S,T)$.

An *ideal* in a Cu-semigroup S is a submonoid $J \subseteq S$ that is downward hereditary (if $x, y \in S$ satisfy $x \leq y \in J$, then $x \in J$) and closed under passing to suprema of increasing sequences.

2.4. Let A be a C^*-algebra, and let a, b be positive elements in the stabilization $A \otimes \mathcal{K}$. Recall that a is *Cuntz subequivalent* to b, in symbols $a \precsim b$, provided that there is a sequence $(x_n)_n$ in $A \otimes \mathcal{K}$ such that $x_n b x_n^*$ converges to a in norm. In case that $a \precsim b$ and $b \precsim a$, then a and b are said to be *Cuntz equivalent*, and we write $a \sim b$. Then $\sim$ is an equivalence relation on $(A \otimes \mathcal{K})_+$. The *Cuntz semigroup* of A is the quotient set

$$\mathrm{Cu}(A) := (A \otimes \mathcal{K})_+/\!\sim .$$

The equivalence class of an element $a \in (A \otimes \mathcal{K})_+$ in $\mathrm{Cu}(A)$ is denoted by $[a]$. Addition on $\mathrm{Cu}(A)$ is defined by $[a] + [b] = [\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)]$, for $a, b \in (A \otimes \mathcal{K})_+$ (using an isomorphism $M_2(\mathcal{K}) \cong \mathcal{K}$). The neutral element for this operation is the class of the zero element in $A \otimes \mathcal{K}$. Finally, we define a partial order on $\mathrm{Cu}(A)$ by setting $[a] \leq [b]$ provided $a \precsim b$. In this way $\mathrm{Cu}(A)$ becomes a positively ordered monoid.

It was proved in [CEI08] that $\mathrm{Cu}(A)$ is a Cu-semigroup. Moreover, a *-homomorphism $\varphi\colon A \to B$ induces a Cu-morphism $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ by $\mathrm{Cu}(\varphi)([a]) = [(\varphi \otimes \mathrm{id}_{\mathcal{K}})(a)]$, for $a \in (A \otimes \mathcal{K})_+$. This defines a functor $\mathrm{Cu}(\_)$, from the category C^* of C^*-algebras with *-homomorphisms to the category Cu.

It was also shown in [CEI08] that the category Cu admits sequential inductive limits and that the functor $\mathrm{Cu}(\_)$ is sequentially continuous, that is, it preserves sequential inductive limits. This was extended to inductive limits over arbitrary directed sets in [APT18b, Corollary 3.2.9]. In Theorem 3.16 we show that Cu is even a cocomplete category, that is, it admits arbitrary (small) colimits.

We will often use the following result, which is called 'Rørdam's lemma' (see [Rør92, Proposition 2.4] or [Thi17, Theorem 2.30]): Let $a, b \in A_+$. Then the following are equivalent:

(1) $a \precsim b$.
(2) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $(a-\varepsilon)_+ \precsim (b-\delta)_+$.
(3) For every $\varepsilon > 0$ there exist $\delta > 0$ and $s \in A$ such that $(a-\varepsilon)_+ = ss^*$ and $s^*s \in \mathrm{Her}((b-\delta)_+)$.

2.2. **The category** W.

In this subsection, we recall the definition of the category W from [APT18b], with a slight deviation; see Remark 2.6.

Definition 2.5. A W-*semigroup* is a commutative monoid S together with a transitive binary relation $\prec$ satisfying $0 \prec x$ for every $x \in S$, and such that the following conditions are satisfied:

(W1) For each $x \in S$, there is a $\prec$-increasing sequence $(x_n)_n$ with $x_n \prec x$ for each n, and such that for any $y \prec x$ there is $n \in \mathbb{N}$ satisfying $y \prec x_n$.
(W3) If $x' \prec x$ and $y' \prec y$, then $x' + y' \prec x + y$, for all $x', y', x, y \in S$.
(W4) If $x, y, z \in S$ satisfy $x \prec y + z$, then there are $y', z' \in S$ such that $y' \prec y$, $z' \prec z$, and $x \prec y' + z'$.

Given W-semigroups S and T, a W-*morphism* is a map $f \colon S \to T$ that preserves addition, the relation $\prec$, the zero element, and that is continuous in the sense that for every $x \in S$ and $y \in T$ satisfying $y \prec f(x)$ there exists $x' \in S$ such that $x' \prec x$ and $y \prec f(x')$. The set of all W-morphisms from S to T is denoted by $\mathrm{W}(S, T)$.

Remark 2.6. In [APT18b, Defintition 2.1.1], we introduced a W-semigroup as a positively ordered monoid S together with an auxiliary relation $\prec$ satisfying $0 \prec x$ for every $x \in S$, satisfying (W1), (W3) and (W4) as in Definition 2.5, and satisfying (W2) which requires that each x in S is the supremum of $x^{\prec} := \{x' : x' \prec x\}$.

Let us clarify the difference between the definition above and the one given in [APT18b]. If S is a W-semigroup in the sense of [APT18b], then by forgetting the partial order on S we obtain a W-semigroup in the sense of Definition 2.5.

Conversely, let S be a W-semigroup in the sense of Definition 2.5. We define a binary relation on S by setting $x \leq y$ if and only if $x^{\prec} \subseteq y^{\prec}$. Then $\leq$ is a pre-order. Given $w, x, y, z \in S$, we have that $x \prec y$ implies $x \leq y$, and we have that $w \leq x \prec y \leq z$ implies $w \prec z$. Given $x \in S$, let us verify $x = \sup x^{\prec}$. Clearly, x is an upper bound for $x^{\prec}$. To show that x is the smallest upper bound, let $y \in S$ be another upper bound for $x^{\prec}$. Given $x' \in x^{\prec}$, we can apply (W1) to choose x'' with $x' \prec x'' \prec x$. Then $x'' \leq y$, and therefore $x' \in y^{\prec}$. It follows that $x^{\prec} \subseteq y^{\prec}$, that is, $x \leq y$. Thus, $(S, \leq, \prec)$ is a W-semigroup in the sense of [APT18b] if and only if the pre-order $\leq$ is a *partial order*.

Hence, the only difference between Definition 2.5 and [APT18b, Defintition 2.1.1] is that in the latter case the auxiliary relation is required to induce a partial order instead of a pre-order. However, in the proofs it is not necessary to require that $\prec$ induces a partial order. Moreover, by dropping this requirement, we may forget about the (partial) order altogether, which further simplifies some arguments.

2.7. Given a Cu-semigroup S, the monoid S together with the relation $\ll$ is a W-semigroup. Moreover, given two Cu-semigroups S and T, a map $\varphi \colon S \to T$ is a Cu-morphism if and only if φ, considered as a map from the W-semigroup $(S, \ll)$ to the W-semigroup $(T, \ll)$, is a W-morphism. Thus, we obtain a functor $\iota \colon \mathrm{Cu} \to \mathrm{W}$ that embeds Cu as a full subcategory of W.

2.8. Given a W-semigroup S, let $\bar{S}$ denote the set of all $\prec$-increasing sequences in S. We equip $\bar{S}$ with componentwise addition, giving it the structure of a commutative monoid. For sequences $\mathbf{x} := (x_n)_n$ and $\mathbf{y} := (y_n)_n$ in $\bar{S}$, we write $\mathbf{x} \subseteq \mathbf{y}$ if for every $k \in \mathbb{N}$, there is $l \in \mathbb{N}$ such that $x_k \prec y_l$. Then $\subseteq$ is a preorder on $\bar{S}$. Set $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{x} \subseteq \mathbf{y}$ and $\mathbf{y} \subseteq \mathbf{x}$. Then $\sim$ is an equivalence relation on $\bar{S}$. Define

$$\gamma(S) := \bar{S}/\sim .$$

Given $\mathbf{x} \in \bar{S}$, we use use $[\mathbf{x}]$ to denote its class in $\gamma(S)$. The relation $\subseteq$ induces a partial order on $\gamma(S)$, by setting $[\mathbf{x}] \leq [\mathbf{y}]$ if and only if $\mathbf{x} \subseteq \mathbf{y}$.

The arguments in [APT18b, Proposition 3.1.6] show that $\gamma(S)$ is a Cu-semigroup, which we call the Cu-*completion* of S. Further, given a W-morphism $\varphi \colon S \to T$, we define $\gamma(\varphi) \colon \gamma(S) \to \gamma(T)$ by $\gamma(\varphi)([(x_n)_n]) := [(\varphi(x_n))_n]$, for $(x_n)_n \in \bar{S}$. One can show that $\gamma(\varphi)$ is a Cu-morphism. This defines a covariant functor $\gamma \colon \mathrm{W} \to \mathrm{Cu}$.

Let S be a W-semigroup. We define $\alpha_S\colon S \to \gamma(S)$ by $\alpha_S(x) := [(x_n)_n]$, for $x \in S$, where $(x_n)_n$ is a $\prec$-increasing sequence obtained by applying (W1) to x. One verifies that α_S is a well-defined W-morphism. Adapting the proof of [APT18b, Theorem 3.1.10], we obtain the following result.

Theorem 2.9. *The category* Cu *is a full, reflective subcategory of* W*. The functor* $\gamma\colon \mathrm{W} \to \mathrm{Cu}$ *is a left adjoint to the inclusion functor* $\iota\colon \mathrm{Cu} \to \mathrm{W}$*. Moreover, given a* W*-semigroup* S*, the map* $\alpha_S\colon S \to \gamma(S)$ *is a universal* W*-morphism: For every* Cu*-semigroup* T*, there is a natural bijection*

$$\mathrm{Cu}\big(\gamma(S), T\big) \cong \mathrm{W}\big(S, \iota(T)\big),$$

implemented by sending a Cu*-morphism* $\varphi\colon \gamma(S) \to T$ *to* $\varphi \circ \alpha_S$*.*

2.3. **The category $\mathcal{Q}$.** In this subsection, we recall the definition of the category $\mathcal{Q}$ and of the τ-construction, as introduced in [APT18]. Since we will apply the τ-construction only to $\mathcal{Q}$-semigroups, we use a simplified definition of path that is different from that used in [APT18]. We remark that the τ-construction can be applied to a much more general class of semigroups equipped with an auxiliary relation by using a more general concept of paths; see [APT18, Definition 3.3].

Definition 2.10. A *$\mathcal{Q}$-semigroup* is a positively ordered monoid satisfying axioms (O1) and (O4) from Definition 2.3, together with an additive auxiliary relation $\prec$. A *$\mathcal{Q}$-morphism* is a PoM-morphism that preserves the auxiliary relation and suprema of increasing sequences. Given $\mathcal{Q}$-semigroups S and T, the set of all $\mathcal{Q}$-morphisms from S to T is denoted by $\mathcal{Q}(S, T)$.

2.11. Given a Cu-semigroup S, then S together with the relation $\ll$ is a $\mathcal{Q}$-semigroup. Moreover, given two Cu-semigroups S and T, a map $\varphi\colon S \to T$ is a Cu-morphism if and only if it is a $\mathcal{Q}$-morphism when considered as a map from $(S, \ll)$ to $(T, \ll)$. We obtain a functor $\iota\colon \mathrm{Cu} \to \mathcal{Q}$ that sends a Cu-semigroup S to the $\mathcal{Q}$-semigroup $(S, \ll)$, and embeds Cu as a full subcategory of $\mathcal{Q}$.

2.12. Let $S = (S, \prec)$ be a $\mathcal{Q}$-semigroup. A *path* in S is an order-preserving map $f\colon (-\infty, 0] \to S$ such that $f(t) = \sup_{t'<t} f(t')$ for all $t \in (-\infty, 0]$, and such that $f(t') \prec f(t)$ whenever $t' < t$. We denote the set of paths in S by $\mathcal{P}(S)$.

For $f, g \in \mathcal{P}(S)$ we define $f + g$ by $(f + g)(t) = f(t) + g(t)$. Together with the constant zero path, this gives $\mathcal{P}(S)$ the structure of a commutative monoid.

Given $f, g \in \mathcal{P}(S)$, we write $f \precsim g$ if, for every $t < 0$, there is $t' < 0$ such that $f(t) \prec g(t')$. Set $f \sim g$ if $f \precsim g$ and $g \precsim f$. It follows from [APT18, Lemma 3.4] that the relation $\precsim$ is reflexive, transitive, and compatible with the addition of paths. We set

$$\tau(S) := \mathcal{P}(S)/\!\sim .$$

Given $f \in \mathcal{P}(S)$, its equivalence class in $\tau(S)$ is denoted by $[f]$. Equip $\tau(S)$ with an addition and order by setting $[f] + [g] := [f + g]$ and $[f] \leq [g]$ provided $f \precsim g$. It is proved in [APT18, Theorem 3.15] that $\tau(S)$ is a Cu-semigroup.

We define $\varphi_S\colon \tau(S) \to S$ by $\varphi_S([f]) := f(0)$, for $f \in \mathcal{P}(S)$. One verifies that φ_S is a well-defined $\mathcal{Q}$-morphism. Given a path $f \in \mathcal{P}(S)$, we think of $f(0)$ as the endpoint of f. Accordingly, we call φ_S the *endpoint map*.

The construction just outlined will be referred to as the *τ-construction*. Given a $\mathcal{Q}$-morphism $\varphi\colon S \to T$, we define $\tau(\varphi)\colon \tau(S) \to \tau(T)$ by $\tau(\varphi)([f]) := [\varphi \circ f]$, for $f \in \mathcal{P}(S)$. One can show that $\tau(\varphi)$ is a Cu-morphism. This defines a covariant functor $\tau\colon \mathcal{Q} \to \mathrm{Cu}$.

Theorem 2.13 ([APT18, Theorem 4.12]). *The category* Cu *is a full, coreflective subcategory of $\mathcal{Q}$. The functor $\tau\colon \mathcal{Q}\to \mathrm{Cu}$ is a right adjoint to the inclusion functor $\iota\colon \mathrm{Cu}\to\mathcal{Q}$. Moreover, given a $\mathcal{Q}$-semigroup S, the endpoint map $\varphi_S\colon \tau(S)\to S$ is a universal $\mathcal{Q}$-morphism: For every* Cu*-semigroup T, there is a natural bijection*

$$\mathrm{Cu}\big(T,\tau(S)\big)\cong \mathcal{Q}\big(\iota(T),S\big),$$

implemented by sending a Cu*-morphism $\psi\colon T\to\tau(S)$ to $\varphi_S\circ\psi$.*

3. Bicompleteness of Cu

In this section we show that the category Cu is bicomplete, that is, complete and cocomplete. To achieve this, we use that Cu is a full, coreflective subcategory of $\mathcal{Q}$ and also a full, reflective subcategory of W; see Theorems 2.13 and 2.9.

We begin by briefly recalling the notions of small (co)limits and (co)completeness in arbitrary categories; see, for example, [Bor94, Section 2.6].

3.1 (Small limits and colimits). Let I be a small category, and let $F\colon I\to\mathcal{C}$ be a functor to a category $\mathcal{C}$. Recall that a *cone* to F is a pair (L,φ), where L is an object in $\mathcal{C}$ and $\varphi=(\varphi_i)_{i\in I}$ is a collection of $\mathcal{C}$-morphisms $\varphi_i\colon L\to F(i)$ with the property that, whenever $f\colon i\to j$ is a morphism in I, we have $F(f)\circ\varphi_i=\varphi_j$. This means that the left diagram below commutes.

By definition, a *limit* of F is a cone (L,φ) that is universal in the sense that every cone (L',φ') factors through (L,φ), that is, there exists a unique $\mathcal{C}$-morphism $\alpha\colon L'\to L$ such that $\varphi_i'=\varphi_i\circ\alpha$ for every i. This means that the right diagram below commutes.

$$\begin{tikzcd} & F(i) \arrow[d, "F(f)"] \\ L \arrow[ur, "\varphi_i"] \arrow[dr, "\varphi_j"'] & F(j) \end{tikzcd} \qquad \begin{tikzcd} & & F(i) \arrow[d, "F(f)"] \\ L' \arrow[r, "\alpha"] \arrow[urr, bend left, "\varphi_i'"] \arrow[drr, bend right, "\varphi_j'"'] & L \arrow[ur, "\varphi_i"] \arrow[dr, "\varphi_j"'] & F(j) \end{tikzcd}$$

If a limit of F exists then it is unique up to natural isomorphism and we shall denote it by $(\mathcal{C}\text{-}\varprojlim F,\pi)$ or just by $\mathcal{C}\text{-}\varprojlim F$. Recall that the category $\mathcal{C}$ is termed *complete* if it has small limits.

The notions of *cocone* to a functor F and *colimit* of F are defined dually. If a colimit of F exists then it is unique up to natural isomorphism and we shall denote it by $(\mathcal{C}\text{-}\varinjlim F,\sigma)$, or simply by $\mathcal{C}\text{-}\varinjlim F$. The category $\mathcal{C}$ is termed *cocomplete* if it has small colimits.

3.2. Let $F\colon I\to\mathcal{C}$ be a functor from a small category I to a category $\mathcal{C}$. Recall that, if I has no morphisms (except the identities at each object), then F corresponds to a selection of a family $(X_i)_{i\in I}$ of objects in $\mathcal{C}$. In this case, a limit of F is called a *product* of the family $(X_i)_{i\in I}$ and is denoted by $\prod_{i\in I}X_i$. Dually we define a *coproduct* which we denote by $\coprod_{\in I}X_i$.

In case I is an upward directed, partially ordered set, then the functor F corresponds to a family $(X_i)_{i\in I}$ of objects in $\mathcal{C}$ together with compatible $\mathcal{C}$-morphisms $\varphi_{j,i}\colon X_i\to X_j$ whenever $i\leq j$. A colimit of F is called an *inductive limit* or *direct limit*. Inverse limits are defined similarly.

Finally, if $I=\{0,1\}$ is the category with two objects and two morphisms $0\to 1$, then $F\colon I\to\mathcal{C}$ corresponds to two $\mathcal{C}$-morphisms $\varphi,\psi\colon X\to Y$. A colimit of F is then called a *coequalizer*. Thus, a coequalizer for (φ, ψ) is a universal pair (C, η), where C is an object of $\mathcal{C}$ and $\eta\colon Y \to C$ is a $\mathcal{C}$-morphism such that $\eta \circ \varphi = \eta \circ \psi$.

3.1. **Completeness.** In this subsection we prove that Cu is a complete category; see Theorem 3.8. In particular, Cu has products over infinite index sets. We will prove in the sequel that the scaled Cuntz semigroup functor preserves (ultra)products of C^*-algebras; see Theorem 5.13 and Theorem 7.5.

3.3. Let I be a set. Let $(S_i)_{i\in I}$ be a family of positively ordered monoids. Define

$$\text{PoM-}\prod_{i\in I} S_i := \left\{ (s_i)_{i\in I} : s_i \in S_i \text{ for all } i \in I \right\}.$$

Equipped with componentwise order and addition, PoM-$\prod_{i\in I} S_i$ is a positively ordered monoid. Given $i \in I$, we let $\pi_i\colon \text{PoM-}\prod_i S_i \to S_i$ be the projection map given by $\pi_i\big((s_j)_{j\in I}\big) := s_i$. It is easy to verify that $\big(\text{PoM-}\prod_{i\in I} S_i, (\pi_i)_i\big)$ is the product of the family $(S_i)_i$ in PoM.

Next, let I be a small category and let $F\colon I \to \text{PoM}$ be a functor. Set

$$S := \left\{ (s_i)_{i\in I} \in \text{PoM-}\prod_{i\in I} F(i) : F(f)(s_i) = s_j \text{ for all } f\colon i \to j \text{ in } I \right\}. \tag{3.1}$$

It is straightforward to verify that $0 \in S$ and that S is closed under addition in PoM-$\prod_i F(i)$. Thus, S inherits the structure of a positively ordered monoid.

For each $i \in I$, the projection map $\pi_i\colon \text{PoM-}\prod_j F(j) \to F(i)$ restricts to a PoM-morphism $\pi_i\colon S \to F(i)$. It is straightforward to verify that (S, π), where $\pi = (\pi_i)_{i\in I}$, is the limit of F in PoM. (Observe that PoM-$\varprojlim F$ agrees, as a set, with the limit of the functor F taking as target category the category of sets.)

Lemma 3.4. *Let I be a small category, let $F\colon I \to \text{PoM}$ be a functor, and let (S, π) be the limit of F in* PoM*. Assume that S_i satisfies (O1) and (O4), for each $i \in I$, and assume that $F(f)\colon F(i) \to F(j)$ preserves suprema of increasing sequences for each morphism $f\colon i \to j$ in I. Then S satisfies (O1) and (O4), and for each $i \in I$ the projection map $\pi_i\colon S \to S_i$ preserves suprema of increasing sequences.*

Proof. We may assume that S is defined as in (3.1). To verify (O1) for S, let $(s^{(n)})_{n\in\mathbb{N}}$ be an increasing sequence in S. For each n, we have $s^{(n)} = (s_i^{(n)})_{i\in I}$. Since the order in S is defined componentwise and S_i satisfies (O1) for each i, we may set $s_i := \sup_n s_i^{(n)}$. To verify that $s := (s_i)_i$ belongs to S, let $f\colon i \to j$ be a morphism in I. Since $F(f)\colon F(i) \to F(j)$ preserves suprema of increasing sequences, we deduce that

$$F(f)(s_i) = F(f)(\sup_n s_i^{(n)}) = \sup_n F(f)(s_i^{(n)}) = \sup_n s_j^{(n)} = s_j.$$

Thus, s belongs to S, and it is easy to verify that s is the supremum of $(s^{(n)})_n$ in S. This shows that S satisfies (O1).

The first part of the proof shows immediately that the projection maps π_i preserve suprema of increasing sequences. Moreover, since addition in S is also taken componentwise, we obtain that S satisfies (O4). $\square$

Definition 3.5. Let I be a small category, and let $F\colon I \to \mathcal{Q}$ be a functor. Let (S, π) be the limit of F in PoM with S defined as in (3.1). Define a binary relation $\prec_{\text{pw}}$ on S by setting $s \prec_{\text{pw}} t$ for $s = (s_i)_{i\in I}$ and $t = (t_i)_{i\in I}$ in S if and only if $s_i \prec t_i$ in S_i for each $i \in I$.

Proposition 3.6. *The category $\mathcal{Q}$ is complete. More precisely, given a small category I, and a functor $F\colon I \to \mathcal{Q}$, let $(S,(\pi_i)_i)$ be the limit of F in* PoM*, with S defined as in* (3.1)*. Then $(S,\prec_{\mathrm{pw}})$ is a $\mathcal{Q}$-semigroup and $\pi_i\colon S\to S_i$ is a $\mathcal{Q}$-morphism for each $i\in I$. Moreover, $(S,\prec_{\mathrm{pw}})$ with $(\pi_i)_i$ is a limit of F in $\mathcal{Q}$.*

Proof. By Lemma 3.4, S satisfies (O1) and (O4). It is straightforward to check that $\prec_{\mathrm{pw}}$ defines an additive auxiliary relation on S. Thus, $(S,\prec_{\mathrm{pw}})$ is a $\mathcal{Q}$-semigroup. Given $i\in I$, the projection map $\pi_i\colon S\to S_i$ clearly preserves the auxiliary relation. By Lemma 3.4, π_i is also a PoM-morphism that preserves suprema of increasing sequences, and consequently π_i is a $\mathcal{Q}$-morphism. To simplify, we denote the $\mathcal{Q}$-semigroup $(S,\prec_{\mathrm{pw}})$ by S. It follows that (S,π) is a cone to F.

To show that (S,π) is a limit of F in $\mathcal{Q}$, let (T,σ) be a cone to F. By the universal property of the limit in PoM, there is a (unique) PoM-morphism $\sigma\colon T\to S$ such that $\sigma_i=\pi_i\circ\sigma$ for all $i\in I$. Note that, $\sigma(t)=(\sigma_i(t))_{i\in I}$, for $t\in T$. We have to show that σ is a $\mathcal{Q}$-morphism.

To verify that σ preserves the auxiliary relation, let $t',t\in T$ satisfy $t'\prec t$. Since σ_i preserves the auxiliary relation, we obtain that $\sigma_i(t')\prec\sigma_i(t)$, for each $i\in I$. By definition, we then have $\sigma(t')\prec_{\mathrm{pw}}\sigma(t)$ in S, as desired.

To show that σ preserves suprema of increasing sequences, let $(t_n)_n$ be an increasing sequence in T. Set $t=\sup_n t_n$. Since σ_i preserves suprema of increasing sequences, we have $\sigma_i(t)=\sup_n\sigma_i(t_n)$ for each $i\in I$. We deduce that

$$\sigma(t)=(\sigma_i(t))_i=(\sup_n\sigma_i(t_n))_i=\sup_n(\sigma_i(t_n))_i=\sup_n\sigma(t_n). \qquad \square$$

Remark 3.7. It follows from Proposition 3.6 that $\mathcal{Q}$ has products. The product in $\mathcal{Q}$ of a family $(S_i)_{i\in I}$ of $\mathcal{Q}$-semigroups is precisely $\mathcal{Q}\text{-}\prod_i S_i=\left(\mathrm{PoM}\text{-}\prod_i S_i,\prec_{\mathrm{pw}}\right)$

Theorem 3.8. *The category* Cu *is complete. More precisely, given a small category I and a functor $F\colon I\to\mathrm{Cu}$, let $(S,(\pi_i)_i)$ be the limit of F in $\mathcal{Q}$. Let $\tau(S)$ be the τ-completion of S. Identifying $\tau(S_i)$ with S_i, we set $\psi_i:=\tau(\pi_i)\colon\tau(S)\to\tau(S_i)\cong S_i$. Then $\tau(S)$ together with $(\psi_i)_i$ is the limit of F in* Cu*, that is:*

$$\mathrm{Cu}\text{-}\varprojlim F=\tau\left(\mathcal{Q}\text{-}\varprojlim F\right)=\tau\left(\mathrm{PoM}\text{-}\varprojlim F,\ll_{\mathrm{pw}}\right)$$

Proof. By Theorem 2.13, Cu is a coreflective subcategory of $\mathcal{Q}$, which in turn is complete by Proposition 3.6. It follows that Cu is complete, since completeness passes to coreflective subcategories; see, for example, [Bor94, Proposition 3.5.3]. $\square$

Corollary 3.9. *The category* Cu *has arbitrary products, arbitrary inverse limits, and finite pullbacks. In particular:*

(1) Let $(S_i)_{i\in I}$ be a family of Cu*-semigroups. Then*

$$\prod_{i\in I}S_i=\tau\left(\mathcal{Q}\text{-}\prod_{i\in I}S_i\right)=\tau\left(\mathrm{PoM}\text{-}\prod_{i\in I}S_i,\ll_{\mathrm{pw}}\right).$$

(2) Let $((S_i)_{i\in I},(\varphi_{ij})_{i,j\in I,i\geq j})$ be an inverse system of Cu*-semigroups. Then*

$$\mathrm{Cu}\text{-}\varprojlim_{i\in I}S_i=\tau\left(\mathcal{Q}\text{-}\varprojlim_{i\in I}S_i\right).$$

The proof of the following result is straightforward, using that the τ-construction applied to Cu-semigroups is vacuous; see [APT18, Proposition 4.10].

Proposition 3.10. *Let $(S_j)_{j\in J}$ be a* finite *family of* Cu*-semigroups. Then $\prod_{j\in J} S_j$ is naturally isomorphic to* PoM-$\prod_j S_j$*, the (set-theoretic) product of the S_j equipped with componentwise order and addition.*

3.2. **Cocompleteness.** In this subsection we prove that Cu is a cocomplete category; see Theorem 3.16. In particular, it has coproducts and pushouts.

3.11. Let $\mathcal{C}$ be a category that has finite coproducts and inductive limits. Let $(X_j)_{j\in J}$ be a family of objects in $\mathcal{C}$. Denote $F \Subset J$ to mean that F is a finite subset of J. The collection of such subsets is upward directed with respect to inclusion, and we have

$$\coprod_{j\in J} X_j \cong \varinjlim_{F\Subset J} \coprod_{j\in F} X_j.$$

Thus, $\mathcal{C}$ has coproducts; see, for example, [Mac71, Theorem IX.1.1, p.212].

Now assume that $\mathcal{C}$ also has finite products and a zero object. Using universality we obtain, for $F \Subset J$, a natural morphism $\coprod_{j\in F} X_j \to \prod_{j\in F} X_j$. Given $F \subseteq G \Subset J$, the existence of a zero object allows us to construct a canonical morphism $\prod_{j\in F} X_j \to \prod_{j\in G} X_j$. In this way, one obtains an inductive system. Its inductive limit is termed the *direct sum* of the family $(X_j)_{j\in J}$ and is denoted by $\bigoplus_{j\in J} X_j$. It follows from our discussion that there is a natural map

$$\coprod_{j\in J} X_j \cong \varinjlim_{F\Subset J} \coprod_{j\in F} X_j \to \varinjlim_{F\Subset J} \prod_{j\in F} X_j = \bigoplus_{j\in J} X_j.$$

In the particular case that finite products and finite coproducts are isomorphic, it follows that this morphism is in fact an isomorphism and hence we may identify the coproduct of a family of objects in the category with their direct sum. Therefore, it should not be confusing that we use the symbol '$\bigoplus$' to denote both. As we shall see below, this is the case for the category Cu.

Next, suppose that $\mathcal{C}$ has arbitrary products. Since there are maps $\bigoplus_{j\in J} X_j \to X_i$ for each $i \in J$, we obtain by the universal property of the product a natural map $\bigoplus_{j\in J} X_j \to \prod_{j\in J} X_j$ and a commutative diagram

$$\begin{tikzcd}
\coprod_{j\in J} X_j \arrow[r] \arrow[dr] & \prod_{j\in J} X_j \\
& \bigoplus_{j\in J} X_j. \arrow[u]
\end{tikzcd}$$

Now, let $(S_j)_{j\in J}$ be a finite family of W-semigroups. Let $\bigoplus_j S_j$, together with maps $\psi_i\colon S_i \to \bigoplus_{j\in J} S_j$ for $i \in J$, be the coproduct of the underlying abelian monoids. Note that $\bigoplus_j S_j$ is a W-semigroup when endowed with the componentwise auxiliary relation. Moreover, this turns each ψ_i into a W-morphism. It follows easily that this is a coproduct of $(S_j)_{j\in J}$ in the category W. It is straightforward to verify that it also agrees with the finite product of $(S_j)_{j\in J}$ in W. Thus, according to the above discussion, we shall denote this coproduct by W-$\bigoplus_j S_j$.

Lemma 3.12. *The category* W *has coproducts.*

Proof. By Paragraph 3.11, W has finite coproducts. An argument analogous to [APT18b, Theorem 2.1.10, p.15] shows that W has inductive limits. This implies that W has small coproducts, as discussed in Paragraph 3.11. $\square$

Lemma 3.13. *Let S, T be* W*-semigroups, and let $\varphi, \psi\colon S \to T$ be* W*-morphisms. For $x, y \in T$, set $x \sim_0 y$ provided there are $z \in T$, and $a, b \in S$ such that*

$$x = z + \varphi(a) + \psi(b) \quad \textit{and} \quad y = z + \varphi(b) + \psi(a).$$

Let $\sim$ be the transitive closure of the relation $\sim_0$. Then $\sim$ is a congruence on T.

For $x, y \in T$, set $x \precsim_0 y$ provided there are $a, b \in T$ such that

$$x \sim a \prec b \sim y.$$

Let $\precsim$ be the transitive closure of $\precsim_0$. Then $\precsim$ is a transitive, additive relation.

Given $u, x, y \in T$ satisfying $u \precsim x \sim y$ there exists $y' \in T$ such that $u \precsim y' \prec y$.

Proof. Is is straightforward to check that $\sim_0$ is additive. It follows that $\sim$ is a congruence, and that $\precsim$ is additive and transitive.

Claim 1: Let $u, x, y \in T$ satisfy $u \prec x \sim_0 y$. Then there exists $y' \in T$ such that $u \precsim y' \prec y$. To prove the claim, choose $z \in T$ and $a, b \in S$ such that

$$x = z + \varphi(a) + \psi(b), \quad \text{and} \quad y = z + \varphi(b) + \psi(a).$$

Apply (W4) to choose $z', z'', v, w \in T$ such that

$$z'' \prec z' \prec z, \quad \text{and} \quad v \prec \varphi(a), \quad \text{and} \quad w \prec \psi(b), \quad \text{and} \quad u \prec z'' + v + w.$$

Now, as φ and ψ are continuous, there are $a', b' \in S$ such that $a' \prec a$, $b' \prec b$, $v \prec \varphi(a')$, and $w \prec \psi(b')$. Set $x' := z' + \varphi(b') + \psi(c')$, and $y' := z' + \varphi(c') + \psi(b')$. Then

$$u \prec z'' + v + w \prec z' + \varphi(a') + \psi(b') = x'.$$

We have $x' \sim_0 y'$, and therefore $u \precsim y'$. Further, we have $y' \prec y$ since both φ and ψ preserve the auxiliary relation.

Claim 2: Let $u, x, y \in T$ satisfy $u \prec x \sim y$. Then there exists $y' \in T$ such that $u \precsim y' \prec y$. To prove the claim, we use that $\sim$ is the transitive closure of $\sim_0$ to choose $n \in \mathbb{N}$ and elements $x_0, \ldots, x_n \in T$ such that

$$x = x_0 \sim_0 x_1 \sim_0 \ldots \sim_0 x_n = y.$$

Applying claim 1 to $u \prec x_0 \sim_0 x_1$, we obtain $x_1' \in T$ with $x_0 \precsim x_1' \prec x_1$. Applying claim 1 repeatedly, we obtain x_k' for $k = 1, \ldots, n$ with $x_{k-1}' \precsim x_k' \prec x_k$. Then $u \precsim x_n' \prec x_n$, which shows that $y' := x_n'$ has the desired properties.

Claim 3: Let $u, x, y \in T$ satisfy $u \precsim_0 x \sim y$. Then there exists $y' \in T$ such that $u \precsim y' \prec y$. To prove the claim, choose $a, b \in T$ with $u \sim a \prec b \sim x$. Then $a \prec b \sim y$. Applying claim 2, we obtain y' such that $a \precsim y' \prec y$. It follows that $u \precsim y' \prec y$, as desired.

Finally, to prove the last statement of the lemma, let $u, x, y \in T$ satisfy $u \precsim x \sim y$. Since $\precsim$ is the transitive closure of $\precsim_0$, we can choose $x' \in T$ such that

$$u \precsim x' \precsim_0 x \sim y.$$

By claim 3, we obtain $y' \in T$ with $x' \precsim y' \prec y$. Then $u \precsim y' \prec y$, as desired. □

Lemma 3.14. *The category* W *has coequalizers.*

Proof. Let S and T be W-semigroups, and let $\varphi, \psi\colon S \to T$ be W-morphisms. For ease of notation, we shall denote the auxiliary relations of both S and T by $\prec$. We have to construct a W-semigroup L and a universal W-morphism $\eta\colon T \to L$ such that $\eta \circ \varphi = \eta \circ \psi$.

Let $\sim$ be the congruence on T defined as in Lemma 3.13. Notice that, with this definition, we have $\varphi(d) \sim \psi(d)$ for any $d \in S$. Set $L := T/\sim$, and let $\eta\colon T \to L$

denote the quotient map. Given $x \in T$, we let $[x]$ denote the class of x in L. Thus, we have $\eta(x) = [x]$, for $x \in T$.

Let $\precsim$ be the transitive relation on T defined as in Lemma 3.13. This induces a binary relation $\prec$ on L by setting $[x] \prec [y]$ if and only if $x \precsim y$, for $x, y \in T$. It follows from Lemma 3.13 that $\prec$ is a well-defined, transitive relation on L.

We claim that $(L, \prec)$ is a W-semigroup. In order to establish this claim, we only need to verify axioms (W1) and (W4), as (W3) is trivially satisfied.

To verify (W1), let $x \in T$. Using that T satisfies (W1), we can choose a $\prec$-increasing sequence $(x_n)_n$ that is cofinal in $x^{\prec} := \{x' \in T : x' \prec x\}$. It is clear that $x_n \prec x_{n+1}$ implies $[x_n] \prec [x_{n+1}]$. Thus, $([x_n])_n$ is a $\prec$-increasing sequence in $[x]^{\prec}$. To show that this sequence is cofinal in $[x]^{\prec}$, let $a \in T$ satisfy $[a] \prec [x]$. By definition, we have $a \precsim x$. By Lemma 3.13, there exists $x' \in T$ with $a \precsim x' \prec x$. Choose n such that $x' \prec x_n$. Then $[a] \prec [x_n]$, as desired.

To verify (W4), let $a, x, y \in T$ satisfy $[a] \prec [x] + [y] = [x + y]$. By definition, we have $a \precsim x + y$. By Lemma 3.13, there exists $u \in T$ with $a \precsim u \prec x + y$. Using that T satisfies (W4), we can choose $x', y' \in T$ such that $u \prec x' + y'$, $x' \prec x$, and $y' \prec y$.

Then the element $[u]$ in L satisfies $[a] \prec [u]$, $[u] \prec [x'] + [y']$, $[x'] \prec [x]$, and $[y'] \prec [y]$, as desired.

We have verified that L is a W-semigroup. Moreover, the map $\eta\colon T \to L$ is a monoid morphism and preserves the auxiliary relation. To verify that η is continuous, let $a, b \in T$ satisfy $[a] \prec \eta(b)$. Then $a \precsim b$. By Lemma 3.13, we can choose $b' \in T$ such that $a \precsim b' \prec b$. Then b' has the desired properties. It is clear by the construction that $\eta \circ \varphi = \eta \circ \psi$.

We now claim that (L, η) is a coequalizer of the pair (φ, ψ). Thus, suppose that R is a W-semigroup and $h\colon T \to R$ is a W-morphism such that $h \circ \varphi = h \circ \psi$.

Define $\tilde{h}\colon L \to R$ by $\tilde{h}([x]) := h(x)$, for $x \in T$. If we prove that $\tilde{h}$ is W-morphism, then clearly it will be be the unique W-morphism satisfying $\tilde{h} \circ \eta = h$.

In order to see that $\tilde{h}$ is well defined, let $x, y \in T$ satisfy $[x] = [y]$, that is, $x \sim y$. We need to show that $h(x) = h(y)$. Since $\sim$ is the transitive closure of $\sim_0$, we may assume that $x \sim_0 y$. By definition, we can choose $a, b \in S$ and $z \in T$ such that

$$x = z + \varphi(a) + \psi(b) \quad \text{and} \quad y = z + \varphi(b) + \psi(a)\,.$$

Using at the second step that $h \circ \varphi = h \circ \psi$, we obtain

$$h(x) = h(z) + h(\varphi(a)) + h(\psi(b)) = h(z) + h(\psi(a)) + h(\varphi(b)) = h(y).$$

Using that h is additive, it follows that $\tilde{h}$ is additive. To show that $\tilde{h}$ preserves the auxiliary relation, let $x, y \in T$ satisfy $[x] \prec [y]$. Then, $x \precsim y$. Since $\precsim$ is the transitive closure of $\precsim_0$, we may assume that $x \precsim_0 y$. By definition, we can choose $a, b \in T$ such that

$$x \sim a \prec b \sim y.$$

Using at the third step that h preserves the auxiliary relation, we deduce

$$\tilde{h}([x]) = h(x) = h(a) \prec h(b) = h(y) = \tilde{h}([y]).$$

Finally, to show that $\tilde{h}$ is continuous, let $x \in T$ and $z \in R$ satisfy $z \prec \tilde{h}([x])$. This means that $z \prec h(x)$, and since h is continuous, there is $x' \in T$ with $x' \prec y$ and $z \leq h(x')$. We have $[x'] \prec [x]$ in L, and clearly $z \leq \tilde{h}([x'])$ in R, which shows that $[x']$ has the desired properties. $\square$

Proposition 3.15. *The category* W *is cocomplete.*

Proof. By Lemmas 3.12 and 3.14, W has small coproducts and coequalizers. This implies that W is cocomplete; see for example [Mac71, Corollary V.2.2, p.113] or (the dual statement of) [Bor94, Theorem 2.8.1]. ☐

Theorem 3.16. *The category* Cu *is cocomplete. More precisely, if I is a small category and $F\colon I \to \mathrm{Cu}$ is a functor, then* $\mathrm{Cu}\text{-}\varinjlim F = \gamma(\mathrm{W}\text{-}\varinjlim F)$.

Proof. By Proposition 3.15, W is cocomplete. By Theorem 2.9, Cu is a full, reflective subcategory of W, with reflector given by $\gamma\colon \mathrm{W} \to \mathrm{Cu}$. It follows that Cu is cocomplete, with the colimit of a functor F given by the reflection of the colimit of F in W; see [Bor94, Proposition 3.5.3]. ☐

Corollary 3.17. *The category* Cu *has coproducts and pushouts. In particular, the coproduct of a family $(S_j)_{j\in J}$ of* Cu*-semigroups is* $\bigoplus_j S_j = \gamma\big(\mathrm{W}\text{-}\bigoplus_j S_j\big)$.

The following result will be used in Section 6. Its proof, as the one of Proposition 3.10, is also straightforward, using that the γ-completion applied to Cu-semigroups is vacuous.

Proposition 3.18. *Let $(S_j)_{j\in J}$ be a* finite *family of* Cu*-semigroups. Then $\bigoplus_j S_j$ is naturally isomorphic to $\prod_{j\in J} S_j$.*

3.19. It was shown in [APT18] that Cu is a closed, symmetric, monoidal category. The tensor product construction in the category Cu was introduced and developed in [APT18b, Chapter 6]. It was proved in [APT18b, Proposition 6.4.1] that the bifunctor $\_\otimes\_ \colon \mathrm{Cu}\times\mathrm{Cu} \to \mathrm{Cu}$ is continuous in both variables. This result is recovered below. On the other hand, [APT18, Theorem 5.11] shows that, for any Cu-semigroup T, the functor $\_\otimes_{\mathrm{Cu}} T$ has a right adjoint, denoted by $[\![T,\_]\!]$. The bifunctor $[\![\_,\_]\!]\colon \mathrm{Cu}\times\mathrm{Cu} \to \mathrm{Cu}$ is referred to as the *internal-hom bifunctor*.

Theorem 3.20. *Let T be a* Cu*-semigroup, let $((S_i)_{i\in I}, (\varphi_{j,i})_{i\leq j})$ be an inductive system in* Cu*, and let $((R_i)_{i\in I}, (\psi_{j,i})_{i\leq j})$ be an inverse system in* Cu*. Then:*

(1) We have an inductive system $((S_i\otimes T)_{i\in I}, (\varphi_{j,i}\otimes \mathrm{id}_T)_{i\leq j})$ in Cu*, and*
$$(\mathrm{Cu}\text{-}\varinjlim_{i\in I} S_i)\otimes T \cong \mathrm{Cu}\text{-}\varinjlim_{i\in I}(S_i\otimes T).$$
(2) We have an inverse system $(([\![T,R_i]\!])_{i\in I}, ((\psi_{j,i})_*)_{i\leq j})$ in Cu*, and*
$$[\![T, \mathrm{Cu}\text{-}\varprojlim_{i\in I} R_i]\!] \cong \mathrm{Cu}\text{-}\varprojlim_{i\in I}[\![T,R_i]\!].$$
(3) We have an inverse system $(([\![S_i,T]\!])_{i\in I}, ((\varphi^*_{j,i})_{i\leq j})$ in Cu*, and*
$$[\![\mathrm{Cu}\text{-}\varinjlim_{i\in I} S_i, T]\!] \cong \mathrm{Cu}\text{-}\varprojlim_{i\in I}[\![S_i,T]\!].$$

Proof. The functors $F_T := \_\otimes T$ and $G_T := [\![T,\_]\!]$ form an adjoint pair by [APT18, Theorem 5.10]. Then (1) and (2) follow from general category theory.

Let us prove statement (3). Using the notation in [APT18, Paragraph 5.5], we have that $(([\![S_i,T]\!])_{i\in I}, (\varphi^*_{i,j})_{i,j\in I})$ is an inverse system of Cu-semigroups. Let R be any other Cu-semigroup. Since $\tau\colon \mathcal{Q}\to\mathrm{Cu}$ is a coreflection (Theorem 2.13), we obtain natural bijections:
$$\mathrm{Cu}\big(R, \mathrm{Cu}\text{-}\varprojlim[\![S_i,T]\!]\big) \cong \mathrm{Cu}\big(R, \tau(\mathcal{Q}\text{-}\varprojlim[\![S_i,T]\!])\big) \cong \mathcal{Q}\big(R, \mathcal{Q}\text{-}\varprojlim[\![S_i,T]\!]\big).$$

Given an inverse system of $\mathcal{Q}$-semigroups $(Y_i)_i$, the set $\mathcal{Q}(R, \mathcal{Q}\text{-}\varprojlim_i Y_i)$ is naturally bijective to the (set-theoretic) inverse limit $\varprojlim_i \mathcal{Q}(R, Y_i)$. Similarly, given an inductive system of W-semigroups $(X_i)_i$, the set $\mathrm{W}(\mathrm{W}\text{-}\varinjlim X_i, T)$ is naturally bijective to $\varprojlim \mathrm{W}(X_i, T)$. Using these observations, and the adjunction between tensor product and internal-hom, the result follows. ☐

4. Scaled Cuntz semigroups

In this section, we introduce the category $\mathrm{Cu}_{\mathrm{sc}}$ of scaled Cu-semigroups, which we show to be complete and cocomplete.

Definition 4.1. Let S be a Cu-semigroup. A *scale* for S is a downward hereditary subset $\Sigma \subseteq S$ that is closed under passing to suprema of increasing sequences in S, and that generates S as an ideal. We call (S, Σ) a *scaled* Cu-*semigroup*.

Given scaled Cu-semigroups (S, Σ) and (T, Θ), a *scaled* Cu-*morphism* is a Cu-morphism $\alpha\colon S \to T$ satisfying $\alpha(\Sigma) \subseteq \Theta$. We let $\mathrm{Cu}_{\mathrm{sc}}$ denote the category of scaled Cu-semigroups and scaled Cu-morphisms.

4.2. Let A be a C^*-algebra. Then

$$\Sigma_A := \big\{ x \in \mathrm{Cu}(A) : \text{for every } x' \ll x \text{ there exists } a \in A_+ \text{ such that } x' \ll [a] \big\}$$

is a scale for $\mathrm{Cu}(A)$. We call $\mathrm{Cu}_{\mathrm{sc}}(A) := (\mathrm{Cu}(A), \Sigma_A)$ the *scaled Cuntz semigroup* of A. Given a *-homomorphism $\varphi\colon A \to B$ between C^*-algebras, the induced Cu-morphism $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ (see Paragraph 2.4) maps Σ_A into Σ_B. Hence, we obtain a functor $C^* \to \mathrm{Cu}_{\mathrm{sc}}$. In Theorems 4.7, 5.13 and 7.5, we show that this functor preserves inductive limits and (ultra)products.

The following result is straightforward to prove. We omit the details.

Proposition 4.3. *Let A be a C^*-algebra, let $J \subseteq A$ be an ideal, and let $\pi\colon A \to A/J$ denote the quotient map. Then*

$$\Sigma_J = \mathrm{Cu}(J) \cap \Sigma_A, \quad \textit{and} \quad \Sigma_{A/J} = \mathrm{Cu}(\pi)(\Sigma_A).$$

4.4. Given a Cu-semigroup S, the whole of S is a scale for S, and we say that (S, S) is *trivially scaled*. A map between Cu-semigroups is a Cu-morphism if and only if it is a scaled Cu-morphism when both Cu-semigroups are equipped with trivial scales. Hence, Cu is isomorphic to the full subcategory of trivially scaled Cu-semigroups in $\mathrm{Cu}_{\mathrm{sc}}$. In the other direction, we have the forgetful functor $\mathcal{F}\colon \mathrm{Cu}_{\mathrm{sc}} \to \mathrm{Cu}$, that maps (S, Σ) to S. For every scaled Cu-semigroup (S, Σ) and every Cu-semigroup T, there is a natural bijection

$$\mathrm{Cu}\big(S, T\big) \cong \mathrm{Cu}_{\mathrm{sc}}\big((S, \Sigma), (T, T)\big),$$

showing that $\mathcal{F}$ is a left adjoint to the inclusion $\mathrm{Cu} \to \mathrm{Cu}_{\mathrm{sc}}$, and thus Cu is a full, reflective subcategory of $\mathrm{Cu}_{\mathrm{sc}}$.

4.5. Let I be a small category, and let $F\colon I \to \mathrm{Cu}_{\mathrm{sc}}$, $i \mapsto F(i) = (S_i, \Sigma_i)$ be a functor. Consider the underlying functor $I \to \mathrm{PoM}$, $i \mapsto S_i$, and let $(S, (\pi_i)_i)$ be the limit of F in PoM (see Paragraph 3.3) with S defined as in (3.1). Set $\Sigma_0 := S \cap \prod_i \Sigma_i$. Note that

$$\Sigma_0 = \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \Sigma_i : F(f)(s_i) = s_j \text{ for all } f\colon i \to j \text{ in } I \right\}.$$

Then Σ_0 is a downward hereditary subset of S satisfying $\pi_i(\Sigma_0) \subseteq \Sigma_i$ for each i.

The limit of the underlying functor $\mathcal{F}\circ F\colon I\to \mathrm{Cu}$, $i\mapsto S_i$, is given as $\tau(S,\ll_{\mathrm{pw}})$ together with maps $\psi_i:=\tau(\pi_i)\colon \tau(S,\ll_{\mathrm{pw}})\to\tau(S_i,\ll)\cong S_i$; see Theorem 3.8. Set

$$\Sigma := \big\{[(\mathbf{x}_t)_{t\leq 0}]\in\tau(S,\ll_{\mathrm{pw}}) : \mathbf{x}_t\in\Sigma_0 \text{ for all } t<0\big\}.$$

Then Σ is a downward hereditary subset of $\tau(S,\ll_{\mathrm{pw}})$ that is closed under passing to suprema of increasing sequences. Let $\langle\Sigma\rangle$ denote the ideal of $\tau(S,\ll_{\mathrm{pw}})$ generated by Σ. Then $(\langle\Sigma\rangle,\Sigma)$ is a scaled Cu-semigroup. Moreover, for each $i\in I$ we have $\psi_i(\Sigma)\subseteq\Sigma_i$, which shows that $\psi_i\colon(\langle\Sigma\rangle,\Sigma)\to(S_i,\Sigma_i)$ is a scaled Cu-morphism. One can show that this defines a limit for F in $\mathrm{Cu}_{\mathrm{sc}}$. We omit the details.

Theorem 4.6. *The category* $\mathrm{Cu}_{\mathrm{sc}}$ *is bicomplete.*

Proof. It remains to show that $\mathrm{Cu}_{\mathrm{sc}}$ is cocomplete. Let I be a small category, and let $F\colon I\to\mathrm{Cu}_{\mathrm{sc}}$, $i\mapsto F(i)=(S_i,\Sigma_i)$ be a functor. By Theorem 3.16, Cu is cocomplete, and so there is a Cu-semigroup S and Cu-morphisms $\sigma_i\colon S_i\to S$ that are compatible with $F(f)\colon S_i\to S_j$ for each $f\colon i\to j$ in I, and such that $(S,(\sigma_i)_i)$ is universal with these properties. Let Σ be the smallest subset of S that contains $\bigcup_i\sigma_i(\Sigma_i)$, and that is downward hereditary and closed under suprema of increasing sequences in S, that is,

$$\Sigma=\Big\{x\in S:\text{ for every } x'\ll x \text{ there exists } y\in\bigcup_{i\in I}\sigma_i(\Sigma_i) \text{ with } x'\leq y\Big\}.$$

Then (S,Σ) is a scaled Cu-semigroup, and each σ_i is a scaled Cu-morphism. To show that this gives a limit for F in $\mathrm{Cu}_{\mathrm{sc}}$, let (T,Θ) be a scaled Cu-semigroup together with compatible scaled Cu-morphisms $\tau_i\colon(S_i,\Sigma_i)\to(T,\Theta)$. Using the universal property of $(S,(\sigma_i)_i)$, we obtain a (unique) Cu-morphism $\beta\colon S\to T$ such that $\tau_i=\beta\circ\sigma_i\colon S_i\to T$. Since each τ_i is a scaled Cu-morphism, we have

$$\beta\Big(\bigcup_{i\in I}\sigma_i(\Sigma_i)\Big)=\bigcup_{i\in I}(\beta\circ\sigma_i)(\Sigma_i)=\bigcup_{i\in I}\tau_i(\Sigma_i)\subseteq\Theta,$$

which implies that $\beta(\Sigma)\subseteq\Theta$, and thus β is a scaled Cu-morphism. □

Theorem 4.7. *The scaled Cuntz semigroup functor preserves inductive limits: Given an inductive system* $((A_j)_{j\in J},(\varphi_{k,j})_{j\leq k})$ *of* C^**-algebras, we have*

$$\mathrm{Cu}_{\mathrm{sc}}\big(\varinjlim_j A_j\big)\cong\varinjlim_j\mathrm{Cu}_{\mathrm{sc}}(A_j).$$

Proof. Set $A:=\varinjlim_j A_j$ and let $\varphi_{\infty,j}\colon A_j\to A$ denote the maps to the inductive limit. They induce scaled Cu-morphisms $\psi_{\infty,j}\colon\mathrm{Cu}_{\mathrm{sc}}(A_j)\to\mathrm{Cu}_{\mathrm{sc}}(A)$, which in turn induce a scaled Cu-morphism

$$\psi\colon\varinjlim_j\mathrm{Cu}_{\mathrm{sc}}(A_j)\to\mathrm{Cu}_{\mathrm{sc}}(A).$$

It follows from [APT18b, Corollary 3.2.9, p.29] that ψ is an order-isomorphism. Let Σ denote the scale of $\varinjlim_j\mathrm{Cu}_{\mathrm{sc}}(A_j)$. We have $\psi(\Sigma)\subseteq\Sigma_A$, and it remains to show that ψ maps Σ onto Σ_A. Let $x\in\Sigma_A$, and let $x'\ll x$. It is enough to verify $x'\in\psi(\Sigma)$. By definition of Σ_A, there exists $a\in A_+$ with $x'\ll[a]$. Thus, there is $\varepsilon>0$ with $x'\ll[(a-\varepsilon)_+]$. Choose j and $b\in(A_j)_+$ such that $\|a-\varphi_{\infty,j}(b)\|<\varepsilon$. Set $y:=[b]\in\Sigma_{A_j}$, and let $\tilde{y}$ be the image of y in $\varinjlim_j\mathrm{Cu}(A_j)$. Then $\tilde{y}\in\Sigma$, and

$$x'\ll[(a-\varepsilon)_+]\leq[\varphi_{\infty,j}(b)]=\psi_{\infty,j}(y)=\psi(\tilde{y}).$$

It follows that $\psi^{-1}(x')\in\Sigma$, and thus $x'\in\psi(\Sigma)$. □

5. Cuntz semigroups of products

In this section, we show that the scaled Cuntz semigroup functor preserves products; see Theorem 5.13.

5.1. Let $(A_j)_{j\in J}$ be a family of C^*-algebras. For each $i \in J$, the natural projection $\pi_i \colon \prod_j A_j \to A_i$ induces a Cu-morphism $\tilde{\pi}_i \colon \mathrm{Cu}(\prod_j A_j) \to \mathrm{Cu}(A_i)$. Then, by the universal property of the product, there is a unique Cu-morphism

$$\Phi \colon \mathrm{Cu}(\prod_{j\in J} A_j) \to \prod_{j\in J} \mathrm{Cu}(A_j)$$

such that $\tilde{\pi}_i = \sigma_i \circ \Phi$ for all $i \in I$, where $\sigma_i \colon \prod_j \mathrm{Cu}(A_j) \to \mathrm{Cu}(A_j)$ denotes the natural Cu-morphism associated to the product in the category Cu.

Let us describe the map Φ more concretely. Given a positive element $\underline{a} = (a_j)_{j\in J}$ in $\prod_j A_j$, and $t \in (-\infty, 0]$, set

$$\varphi_t(\underline{a}) := ([(a_j + t)_+])_{j\in J}.$$

Then $t \mapsto \varphi_t(\underline{a})$ is a path in $(\text{PoM-}\prod_j \mathrm{Cu}(A_j), \ll_{\mathrm{pw}})$, and we have

$$\Phi([\underline{a}]) = [(\varphi_t(\underline{a}))_{t\in(-\infty,0]}].$$

Next, we need to deal with the stabilization that occurs in the construction of $\mathrm{Cu}(\prod_j A_j)$. We note that even if each A_j is stable, the product $\prod_j A_j$ is *not* stable (unless J is finite); see Remark 8.6. Nevertheless, we will see that $\prod_j A_j$ satisfies property (S) of Hjelmborg-Rørdam, which is enough for our purposes.

Definition 5.2 (Hjelmborg, Rørdam, [HR98])**.** A C^*-algebra A is said to have property (S) if for every $a \in A_+$ and every $\varepsilon > 0$ there exist $b \in A_+$ and $x \in A$ such that $a = x^*x$, $b = xx^*$ and $\|ab\| \leq \varepsilon$.

Theorem 5.3 ([HR98])**.** *Let A be a C^*-algebra. If A is stable, then it has property (S). The converse holds if A is σ-unital (in particular, if A is separable).*

Lemma 5.4. *Property (S) passes to products.*

Proof. Let $(A_j)_{j\in J}$ be a family of C^*-algebras with property (S). To verify that $A := \prod_{j\in J} A_j$ has property (S), let $a = (a_j)_j \in A_+$ and $\varepsilon > 0$. Given $j \in J$, use that A_j satisfies (S), to choose $b_j \in (A_j)_+$ and $x_j \in A_j$ such that

$$a_j = x_j^* x_j, \quad \text{and} \quad b_j = x_j x_j^*, \quad \text{and} \quad \|a_j b_j\| \leq \varepsilon.$$

Set $b := (b_j)_j$ and $x := (x_j)_j$. For each j, we have $\|x_j\| = \|a_j\|^{1/2} \leq \|a\|^{1/2}$ and therefore $\|x\| = \sup_i \|x_i\| < \infty$. Similarly, we have $\|b\| < \infty$. Hence, b and x belong to A, and we have $a = x^*x$, $b = xx^*$ and $\|ab\| \leq \varepsilon$, as desired. □

We remark that it is straightforward to verify that property (S) passes to quotients, and hence to ultraproducts. The following result is well-known (although we couldn't locate it in the literature). It can be proved using Blackadar's technique of constructing separable sub-C^*-algebras with prescribed properties ([Bla06, Section II.8.5, p.176ff]), which is the C^*-algebraic version of the downward Löwenheim-Skolem theorem from model theory.

Lemma 5.5. *Let A be a C^*-algebra with property (S), and let $A_0 \subseteq A$ be a separable sub-C^*-algebra. Then there exists a separable sub-C^*-algebra $B \subseteq A$ with property (S) (and hence B is stable by Theorem 5.3), and such that $A_0 \subseteq B$.*

Lemma 5.6. *Let A be a C^*-algebra with property (S), and let $p \in \mathcal{K}$ be a rank one projection. Then the map $\iota\colon A \to A \otimes \mathcal{K}$, given by $\iota(a) = a \otimes p$, induces a natural bijection between Cuntz equivalence classes in A_+ and $(A\otimes\mathcal{K})_+$. Moreover, $a, b \in A_+$ satisfy $\iota(a) \precsim \iota(b)$ in $A \otimes \mathcal{K}$ if and only if $a \precsim b$ in A.*

Proof. In general, if $D \subseteq B$ is a hereditary sub-C^*-algebra, and given $x, y \in D_+$, then x is Cuntz (sub)equivalent to y in D if and only if x is Cuntz (sub)equivalent to y in B; see e.g. [Thi17, Proposition 2.18]. Since ι identifies A with a hereditary sub-C^*-algebra of $A \otimes \mathcal{K}$, it follows that the map $\tilde{\iota}\colon (A_+)_{/\sim} \to \mathrm{Cu}(A)$, given by $\tilde{\iota}([a]) = [a \otimes p]$, is a well-defined order-embedding.

To show that $\tilde{\iota}$ is surjective, let $a \in (A\otimes\mathcal{K})_+$. Choose a separable sub-C^*-algebra $A_0 \subseteq A$ such that a belongs to $A_0 \otimes \mathcal{K} \subseteq A \otimes \mathcal{K}$. (For instance, let $(e_{kl})_{k,l\geq 1}$ denote matrix units for $\mathcal{K}$ such that $p = e_{11}$, and let A_0 be generated by $\iota^{-1}(e_{1k}ae_{l1})$ for $k, l \geq 1$.) Apply Lemma 5.5 to obtain a separable, stable sub-C^*-algebra $B \subseteq A$ with $A_0 \subseteq B$. Note that a belongs to $B \otimes \mathcal{K}$. Since B is stable, there exists a unitary in $M(B \otimes \mathcal{K})$ such that $uxu^* \in B \otimes p$ for each $x \in B \otimes \mathcal{K}$. Then, uau^* belongs to $B \otimes p$, and hence to the image of ι. Further, uau^* is Cuntz equivalent to a in $M(B \otimes \mathcal{K})$, and consequently in the hereditary sub-C^*-algebra $B \otimes \mathcal{K}$, and therefore also in $A \otimes \mathcal{K}$. □

The following technical lemmas will be used to compute the Cuntz semigroups of product C^*-algebras.

Lemma 5.7. *Let $(A_j)_{j\in J}$ be a family of C^*-algebras, and let $\underline{a} = (a_j)_{j\in J}$ and $\underline{b} = (b_j)_{j\in J}$ be positive elements in $\prod_j A_j$. Then the following are equivalent:*

1. *We have $\underline{a} \precsim \underline{b}$ in $\prod_j A_j$.*
2. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that $(a_j - \varepsilon)_+ \precsim (b_j - \delta)_+$ in A_j for every $j \in J$.*
3. *We have $\Phi([\underline{a}]) \leq \Phi([\underline{b}])$ in $\prod_j \mathrm{Cu}(A_j)$.*

Proof. As in Paragraph 5.1, set

$$\varphi_t(\underline{a}) = ([(a_j + t)_+])_j, \quad \text{and} \quad \varphi_t(\underline{b}) = ([(b_j + t)_+])_j,$$

for each $t \in (-\infty, 0]$. Then

$$\Phi([\underline{a}]) = [\varphi_t(\underline{a})_{t\in(-\infty,0]}], \quad \text{and} \quad \Phi([\underline{b}]) = [\varphi_t(\underline{b})_{t\in(-\infty,0]}].$$

By definition of the order in $\prod_j \mathrm{Cu}(A_j) = \tau(\text{PoM-}\prod_j \mathrm{Cu}(A_j), \ll_{\mathrm{pw}})$, we have $\Phi([\underline{a}]) \leq \Phi([\underline{b}])$ if and only if for every $t < 0$ there exists $t' < 0$ such that $\varphi_t(\underline{a}) \ll_{\mathrm{pw}} \varphi_{t'}(\underline{b})$. Equivalently, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\varphi_{-\varepsilon}(\underline{a}) \ll_{\mathrm{pw}} \varphi_{-\delta}(\underline{b})$. We deduce that (2) and (3) are equivalent.

It follows directly from Rørdam's Lemma (Paragraph 2.4) that (1) implies (2). Conversely, assume (2), and let us verify (1) with Rørdam's Lemma. Let $\varepsilon > 0$. We need to find $\underline{s}$ in $\prod_j A_j$ such that $\underline{s}\underline{s}^* = (\underline{a} - \varepsilon)_+$ and $\underline{s}^*\underline{s} \in \mathrm{Her}(\underline{b})$. By assumption, for each $j \in J$ there is $\delta > 0$ such that $(a_j - \frac{\varepsilon}{2})_+ \precsim (b_j - \delta)_+$ in A_j. Apply Rørdam's Lemma for $(a_j - \frac{\varepsilon}{2})_+ \precsim (b_j - \delta)_+$ and $\frac{\varepsilon}{2}$ to obtain $s_j \in A_j$ such that

$$(a_j - \varepsilon)_+ = \big((a_j - \tfrac{\varepsilon}{2})_+ - \tfrac{\varepsilon}{2}\big)_+ = s_j s_j^*, \quad \text{and} \quad s_j^* s_j \in \mathrm{Her}((b_j - \delta)_+).$$

Observe that $\|s_j\| \leq \|a_j\|^{1/2}$ for all $j \in J$, and hence $(\|s_j\|)_j$ is bounded. Set $\underline{s} := (s_j)_{j\in J}$. Then $\underline{s}$ is an element in $\prod_j A_j$ satisfying $\underline{s}\underline{s}^* = (\underline{a} - \epsilon)_+$. Let $f_\delta\colon \mathbb{R} \to [0, 1]$ be a continuous function satisfying $f(t) = 0$ for $t \leq 0$ and $f(t) = 1$ for

$t \geq \delta$. Then $f_\delta(b_j)s_j^*s_jf_\delta(b_j) = s_j^*s_j$ for each j, and therefore $f_\delta(\underline{b})\underline{s}^*\underline{s}f_\delta(\underline{b}) = \underline{s}^*\underline{s}$. This implies that $\underline{s}^*\underline{s} \in \mathrm{Her}(\underline{b})$, as desired. □

Lemma 5.8. *Let A be a C^*-algebra, let $a \in A_+$, and let $x \in \mathrm{Cu}(A)$ with $x \ll [a]$. Then there exists a positive, contractive $b \in A$ such that $x \ll [(b-\frac{1}{2})_+]$ and $[b] = [a]$.*

Proof. Choose $\delta > 0$ such that $x \ll [(a-\delta)_+]$. We may assume that $\delta < 1/2$. Let $h\colon \mathbb{R} \to [0,1]$ be defined by

$$h(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{1}{2\delta}t, & \text{if } 0 \leq t \leq 2\delta \\ 1, & \text{if } 2\delta \leq t \end{cases}.$$

Set $b := h(a)$. Then $[a] = [b]$. Consider the functions $\mathbb{R} \to \mathbb{R}$, given by $t \mapsto (h(t) - \frac{1}{2})_+$ and $t \mapsto (t-\delta)_+$. They have the same support (namely (δ, ∞)), which implies that $(h(a) - 1/2)_+$ and $(a-\delta)_+$ are Cuntz equivalent (first in the commutative C^*-algebra $C^*(a) \cong C(\sigma(a))$, and thus also in A). Hence,

$$x \ll [(a-\delta)_+] = [(h(a) - \frac{1}{2})_+] = [(b - \frac{1}{2})_+].$$ □

Proposition 5.9. *Let $(A_j)_{j\in J}$ be a family of stable C^*-algebras. Then, the map $\Phi\colon \mathrm{Cu}(\prod_j A_j) \to \prod_j \mathrm{Cu}(A_j)$ from Paragraph 5.1 is an isomorphism.*

Proof. Set $A := \prod_j A_j$. By Lemma 5.4, A has property (S). Hence, by Lemma 5.6 there is a natural identification of $\mathrm{Cu}(A)$ with the Cuntz equivalence classes in A_+. Thus, it remains to show that Φ is an order-isomorphism when restricted to Cuntz classes of elements in A_+.

It follows from Lemma 5.7 that Φ is an order-embedding. It remains to show that Φ is surjective. Since Φ is an order-embedding, and since $\mathrm{Cu}(A)$ has suprema of increasing sequences that moreover are preserved by Φ, it is enough to verify that the image of Φ is order-dense.

Let $x, y \in \prod_j \mathrm{Cu}(A_j)$ satisfy $x \ll y$. We will find $\underline{b} \in A_+$ such that $x \ll \Phi([\underline{b}]) \ll y$. Choose $\ll_{\mathrm{pw}}$-increasing paths $(\mathbf{x}_t)_{t\in(-\infty,0]}$ and $(\mathbf{y}_t)_{t\in(-\infty,0]}$ in PoM-$\prod_j \mathrm{Cu}(A_j)$ representing x and y, respectively. It follows from [APT18, Lemma 3.16] that there exists $t < 0$ such that $\mathbf{x}_0 \ll_{\mathrm{pw}} \mathbf{y}_t$.

Choose elements $x_{0,j} \in \mathrm{Cu}(A_j)$ such that $\mathbf{x}_0 = (x_{0,j})_j$, and choose positive elements $a_j \in A_j$ such that $\mathbf{y}_t = ([a_j])_j$. Then, for each $j \in J$, we have $x_{0,j} \ll [a_j]$. Applying Lemma 5.8, we obtain a positive, contractive $b_j \in A_j$ such that

$$x_{0,j} \ll [(b_j - \frac{1}{2})_+], \quad \text{and} \quad [b_j] = [a_j].$$

Set $\underline{b} := (b_j)_j$, which is a positive, contractive element in $\prod_j A_j$. We have

$$\mathbf{x}_0 \ll_{\mathrm{pw}} ([(b_j - \frac{1}{2})_+])_j = \varphi_{-1/2}(\underline{b}), \quad \text{and} \quad \varphi_0(\underline{b}) = ([b_j])_j = ([a_j])_j = \mathbf{y}_t \ll_{\mathrm{pw}} \mathbf{y}_{t/2},$$

and therefore

$$x = [(\mathbf{x}_t)_{t\leq 0}] \ll [(\varphi_t(\underline{b}))_{t\leq 0}] = \Phi([\underline{b}]) \ll [(\mathbf{y}_t)_{t\leq 0}] = y.$$

This shows that $\underline{b}$ has the claimed properties. □

Remark 5.10. More generally, Proposition 5.9 (and consequently Proposition 7.4) holds for every family of C^*-algebras that satisfy property (S). However, Example 5.15 shows that it does not hold for every family of C^*-algebras.

The proof of the following result is standard. We therefore omit it.

Lemma 5.11. *Let $A \subseteq B$ be a hereditary sub-C^*-algebra. Then the inclusion $\iota\colon A \to B$ induces an order-embedding $\mathrm{Cu}(\iota)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ that identifies $\mathrm{Cu}(A)$ with an ideal of $\mathrm{Cu}(B)$.*

5.12. Let $(S_j, \Sigma_j)_{j\in J}$ be a family of scaled Cu-semigroups. Let (S, Σ) be their product in $\mathrm{Cu}_{\mathrm{sc}}$, which we also call their *scaled product*.

Let us describe (S, Σ) more concretely. The set-theoretic product $\prod_j \Sigma_j$ is a downward hereditary subset of PoM-$\prod_j S_j$. Then

$$\Sigma = \big\{[(\mathbf{x}_t)_{t\leq 0}] \in \prod_{j\in J} S_j : \mathbf{x}_t \in \prod_{j\in J} \Sigma_j \text{ for every } t < 0\big\},$$

and S is the ideal of $\prod_j S_j$ generated by Σ. Given $[(\mathbf{x}_t)_{t\leq 0}] \in \prod_j S_j$, we have $[(\mathbf{x}_t)_{t\leq 0}] \in S$ if and only if for every $t < 0$ there exist $\sigma^{(1)}, \ldots, \sigma^{(N)} \in \prod_j \Sigma_j$ such that $\mathbf{x}_t \ll_{\mathrm{pw}} \sigma^{(1)} + \ldots + \sigma^{(N)}$.

Theorem 5.13. *The scaled Cuntz semigroup functor preserves products.*

More concretely, let $(A_j)_{j\in J}$ be a family of C^*-algebras, and let (S, Σ) be the scaled product of $(\mathrm{Cu}_{\mathrm{sc}}(A_j))_j$ with $\Sigma \subseteq S \subseteq \prod_j \mathrm{Cu}(A_j)$ as in Paragraph 5.12. Then, the map $\Phi\colon \mathrm{Cu}(\prod_j A_j) \to \prod_j \mathrm{Cu}(A_j)$ from Paragraph 5.1 is an order-embedding with image S that moreover identifies the scale of $\mathrm{Cu}(\prod_{j\in J} A_j)$ with Σ:

$$\mathrm{Cu}_{\mathrm{sc}}(\prod_{j\in J} A_J) = (\mathrm{Cu}(\prod_{j\in J} A_j), \Sigma_{\prod_j A_j}) \cong \prod_{j\in J} (\mathrm{Cu}(A_j), \Sigma_{A_j}) = (S, \Sigma).$$

Proof. Let $p \in \mathcal{K}$ be a rank-one projection. For each j, let $\iota_j\colon A_j \to A_j \otimes \mathcal{K}$ be given by $\iota_j(a) := a \otimes p$. This induces a natural map $\iota\colon \prod_j A_j \to \prod_j (A_j \otimes \mathcal{K})$, which identifies $\prod_j A_j$ with a hereditary sub-C^*-algebra of $\prod_j (A_j \otimes \mathcal{K})$. By Lemma 5.11, $\mathrm{Cu}(\iota)$ is an order-embedding whose image is an ideal.

Let Φ' be the natural map from Paragraph 5.1 for the product $\prod_j (A_j \otimes \mathcal{K})$. For each j, we have a Cu-morphism $\mathrm{Cu}(\iota_j)\colon \mathrm{Cu}(A_j) \to \mathrm{Cu}(A_j \otimes \mathcal{K})$, which induces a Cu-morphism $\alpha\colon \prod_j \mathrm{Cu}(A_j) \to \prod_j \mathrm{Cu}(A_j \otimes \mathcal{K})$. Then

$$\alpha \circ \Phi = \Phi' \circ \mathrm{Cu}(\iota),$$

which means that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Cu}(\prod_{j\in J} A_j) \arrow[r, "\Phi"] \arrow[d, "\mathrm{Cu}(\iota)"] & \prod_{j\in J} \mathrm{Cu}(A_j) \arrow[d, "\alpha", "\cong"'] \\
\mathrm{Cu}(\prod_{j\in J} A_j \otimes \mathcal{K}) \arrow[r, "\cong", "\Phi'"'] & \prod_{j\in J} \mathrm{Cu}(A_j \otimes \mathcal{K})\ .
\end{tikzcd}$$

By Proposition 5.9, Φ' is an order-isomorphism. Further, each $\mathrm{Cu}(\iota_j)$ is an isomorphism, and consequently so is α. It follows that Φ is an order-embedding whose image is an ideal. It remains to show that Φ maps $\Sigma_{\prod_j A_j}$ onto Σ.

First, let $x \in \Sigma_{\prod_j A_j}$. By definition, for every $x' \ll x$, there exists a positive $\underline{a} \in \prod_j A_j$ with $x' \leq [\underline{a}]$. Since Φ preserves suprema of increasing sequences, and since Σ is downward hereditary, it is enough to verify that $\Phi([\underline{a}]) \in \Sigma$ for every positive $\underline{a} \in \prod_j A_j$. But this follows directly from the description of Φ in Paragraph 5.1.

Conversely, using that Φ is an order-embedding, and that $\Sigma_{\prod_j A_j}$ is closed under suprema of increasing sequences that moreover are preserved by Φ, it is enough to verify that the image of $\Phi(\Sigma_{\prod_j A_j})$ is order-dense in Σ. Thus, let $y', y \in \Sigma$ satisfy $y' \ll y$. Choose an $\ll_{\mathrm{pw}}$-increasing path $(\mathbf{y}_t)_{t\in(-\infty,0]}$ such that $y = [(\mathbf{y}_t)_{t\leq 0}]$, and choose $y_{t,j} \in \mathrm{Cu}(A_j)$ such that $\mathbf{y}_t = (y_{t,j})_j$. Then choose $\varepsilon > 0$ such that $y' \leq [(\mathbf{y}_{t-3\varepsilon})_{t\leq 0}]$. By definition of Σ, we have $\mathbf{y}_t \in \text{PoM-}\prod_j \Sigma_{A_j}$ for each $t < 0$. In particular, $\mathbf{y}_{-\varepsilon} \in \text{PoM-}\prod_j \Sigma_{A_j}$. For each $j \in J$, we have $y_{-2\varepsilon,j} \ll y_{-\varepsilon,j} \in \Sigma_{A_j}$. By definition of Σ_{A_j}, we obtain $a_j \in (A_j)_+$ such that $y_{-2\varepsilon,j} \leq [a_j]$.

Applying Lemma 5.8, we obtain a positive, contractive $b_j \in A_j$ such that

$$x_{-3\varepsilon,j} \ll [(b_j - \frac{1}{2})_+], \quad \text{and} \quad [b_j] = [a_j].$$

Then $\underline{b} := (b_j)_j$ is a positive element in $\prod_j A_j$. Hence, the Cuntz class of $\underline{b}$ belongs to $\Sigma_{\prod_j A_j}$. As in the proof of Proposition 5.9, we obtain $y' \leq \Phi([\underline{b}]) \leq y$. □

In the case of finitely many C^*-algebras, the map Φ from Theorem 5.13 is easily seen to be surjective, whence we obtain the following (well-known) result. We note that the product of finitely many Cu-semigroups is simply their set-theoretic product, equipped with componentwise order and addition; see Proposition 3.10.

Corollary 5.14. *Let $(A_j)_{j\in J}$ be a* finite *family of C^*-algebras. Then the map from Paragraph 5.1 induces a natural isomorphism $\mathrm{Cu}(\prod_{j\in J} A_j) \cong \prod_{j\in J} \mathrm{Cu}(A_j)$.*

Example 5.15. Set $A_j := \mathbb{C}$ for each $j \in \mathbb{N}$. Set $A := \prod_j A_j \cong \ell^\infty(\mathbb{N})$. We have $\mathrm{Cu}(A_j) \cong \overline{\mathbb{N}}$ for each j, and the product $\prod_j \overline{\mathbb{N}}$ is defined as the equivalence classes of $\ll_{\mathrm{pw}}$-increasing paths $(-\infty, 0] \to \text{PoM-}\prod_j \overline{\mathbb{N}}$. It follows that $\prod_j \overline{\mathbb{N}}$ may be identified with equivalence classes of componentwise increasing paths $(-\infty, 0) \to \text{PoM-}\prod_j \mathbb{N}$. In particular, compact elements in $\prod_j \overline{\mathbb{N}}$ naturally corresponds to functions $\mathbb{N} \to \mathbb{N}$. (In Corollary 8.2 we will generalize this to give a description of compact elements in arbitrary products.)

Let us see that the natural order-embedding $\Phi\colon \mathrm{Cu}(A) \to \prod_j \overline{\mathbb{N}}$ is not surjective. Recall that an element x in a Cu-semigroup S is said to be compact if $x \ll x$. Note that Φ maps the Cuntz class of the unit of A to the compact element in $\prod_j \overline{\mathbb{N}}$ that correspond to the function $f\colon \mathbb{N} \to \mathbb{N}$ with $f(j) = 1$ for all j. Consider the compact element x in $\prod_j \overline{\mathbb{N}}$ that corresponds to the function $g\colon \mathbb{N} \to \mathbb{N}$ with $g(j) = j$ for all j. Since $g \not\leq nf$ for every $n \in \mathbb{N}$, it follows that $x \not\leq \infty\Phi([1_A])$. In particular, $\Phi([1_A])$ is not full, and $\mathrm{Cu}(\prod_j A_j)$ is isomorphic to a proper ideal of $\prod_j \mathrm{Cu}(A_j)$.

It is natural to ask to what extent the Cuntz semigroup functor preserves arbitrary limits. We are thankful to Eusebio Gardella for pointing out this example.

Example 5.16. Let Γ be a countable, discrete, exact group satisfying the approximation property. (For instance, any countable, discrete, amenable group.) In [Suz17, Theorem A], Suzuki constructs a decreasing sequence of C^*-algebras $(A_n)_{n\in\mathbb{N}}$, each of which is isomorphic to $\mathcal{O}_2$, whose intersection is isomorphic to the reduced group C^*-algebra $C^*_r(\Gamma)$.

It follows that $C^*_r(\Gamma) \cong \varprojlim A_n$. For each $n \in \mathbb{N}$, we have $\mathrm{Cu}(A_n) \cong \{0, \infty\}$. Since the maps in the inverse system given by the algebras A_n are defined by the inclusions $A_n \subseteq A_{n-1}$, at the level of the Cuntz semigroup they induce the identity maps. It follows that $\text{Cu-}\varprojlim \mathrm{Cu}(A_n) \cong \{0, \infty\}$. However, $\mathrm{Cu}(C^*_r(\Gamma)) \not\cong \{0, \infty\}$ since, for example, $C^*_r(\Gamma)$ has a normalized trace.

Remark 5.17. One can consider a product over an infinite index set J as the inverse limit of the subproducts of finite subsets of J. By Proposition 5.9, the Cuntz semigroup functor preserves products of stable C^*-algebras, and therefore it *does* preserve this particular inverse limit. One difference between Example 5.16 and a product of C^*-algebras is that, in the latter, all connecting maps are surjective. Therefore, the following question is pertinent:

Question 5.18. Does the Cuntz semigroup functor preserve inverse limits with surjective connecting morphisms? That is, do we have

$$\mathrm{Cu}(\varprojlim A_n) \cong \mathrm{Cu}\text{-}\varprojlim \mathrm{Cu}(A_n)$$

for every (sequential) inverse system $(A_n)_{n\in\mathbb{N}}$ of (stable) C^*-algebras where each map $A_{n+1} \to A_n$ is surjective?

6. Cuntz semigroups of direct sums and sequence algebras

In this section we show that the Cuntz semigroup functor preserves direct sums. It follows that the scaled Cuntz semigroup functor preserves sequence algebras; see Corollary 6.6.

In Cu, product and coproduct of finite families are naturally isomorphic; see Propositions 3.10 and 3.18. It follows that for any family $(S_j)_{j\in J}$ of Cu-semigroups, the direct sum $\bigoplus_{j\in J} S_j$ is naturally isomorphic to the coproduct $\coprod_{j\in J} S_j$; see Paragraph 3.11. The analogous statement holds in $\mathrm{Cu}_{\mathrm{sc}}$. In the category of C^*-algebras, it is not true that finite products and coproducts coincide (see Example 6.2). Nevertheless, there is a zero object and the categorical construction of the direct sum yields the usual definition of direct sums for C^*-algebras.

Theorem 6.1. *Let $(A_j)_{j\in J}$ be a family of C^*-algebras. Then there is are natural isomorphisms*

$$\mathrm{Cu}\big(\bigoplus_{j\in J} A_j\big) \cong \bigoplus_{j\in J} \mathrm{Cu}(A_j), \quad \text{and} \quad \mathrm{Cu}_{\mathrm{sc}}\big(\bigoplus_{j\in J} A_j\big) \cong \bigoplus_{j\in J} \mathrm{Cu}_{\mathrm{sc}}(A_j).$$

Proof. We use that direct sums are defined as inductive limits of finite products. Using that the Cuntz semigroup functor preserves inductive limits ([APT18b, Corollary 3.2.9, p.29]) and finite products (Corollary 5.14), we obtain the first isomorphism. The second isomorphism follows analogously, using Theorem 4.7. □

Example 6.2. The Cuntz semigroup functor does not preserve coproducts. Given two C^*-algebras A and B, their coproduct in the category of C^*-algebras is the (full) free product $A * B$. Consider for example $A = B = \mathbb{C}$. It is known that

$$\mathbb{C} * \mathbb{C} \cong \big\{ f \in C([0,1], M_2(\mathbb{C})) : f(0) \text{ diagonal}, f(1) = \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right), \text{ some } x \in \mathbb{C} \big\},$$

see [Bla06, Example IV.1.4.2, p.330]. We have $\mathrm{Cu}(\mathbb{C}) \cong \overline{\mathbb{N}}$, and the coproduct of $\overline{\mathbb{N}}$ and $\overline{\mathbb{N}}$ in the category Cu is simply $\overline{\mathbb{N}} \times \overline{\mathbb{N}}$ with coordinatewise order and addition. On the other hand, it is easy to see that $\mathrm{Cu}(\mathbb{C} * \mathbb{C}) \not\cong \overline{\mathbb{N}} \times \overline{\mathbb{N}}$. (Using [APS11, Corollary 3.5], one can in fact explicitly compute $\mathrm{Cu}(\mathbb{C} * \mathbb{C})$.)

Notation 6.3. Let $(S_j)_{j\in J}$ be a family of Cu-semigroups. Given $\mathbf{x} = (x_j)_{j\in J}$ in $\mathrm{PoM}\text{-}\prod_{j\in J} S_j$, we define its *support* as $\mathrm{supp}(\mathbf{x}) := \{ j \in J : x_j \neq 0 \}$. Set

$$\mathrm{c}_0\big((S_j)_j\big) := \big\{ [(\mathbf{x}_t)_{t\leq 0}] \in \prod_j S_j : \mathrm{supp}(\mathbf{x}_t) \text{ is finite, for every } t < 0 \big\}.$$

Given a Cu-semigroup S, we write $\mathrm{c}_0(S)$ for $\mathrm{c}_0\big((S)_{n\in\mathbb{N}}\big)$.

Proposition 6.4. *Let $(S_j)_{j\in J}$ be a family of* Cu*-semigroups. Then $\mathrm{c}_0\big((S_j)_j\big)$ is an ideal in $\prod_j S_j$, and the canonical map $\bigoplus_j S_j \to \prod_j S_j$ is an order-embedding whose image is $\mathrm{c}_0\big((S_j)_j\big)$.*

Proof. For each $F \Subset J$, we let $\varphi_{J,F}\colon \prod_{j\in F} S_j \to \prod_{j\in J} S_j$ be the natural morphism. Since these maps are compatible with the inductive system defining $\bigoplus_j S_j$, they induce a natural morphism $\varphi\colon \bigoplus_j S_j = \varinjlim_{F\Subset J} \prod_{j\in F} S_j \to \prod_{j\in J} S_j$. Using that each $\varphi_{J,F}$ is an order-embedding, it follows that φ is an order-embedding as well. To simplify, we write c_0 for $\mathrm{c}_0\big((S_j)_j\big)$. It is easy to verify that c_0 is an ideal in $\prod_j S_j$. Since the image of each $\varphi_{J,F}$ is contained in c_0, so is the image of φ.

To show that the image of φ is c_0, let $x \in \mathrm{c}_0$. Choose $\mathbf{x}_t = (x_{t,j})_j \in \mathrm{PoM}\text{-}\prod_j S_j$ such that $x = [(\mathbf{x}_t)_{t\leq 0}]$. For $n \geq 1$, set $F_n := \mathrm{supp}(\mathbf{x}_{-1/n})$. Since $x \in \mathrm{c}_0$, we obtain that F_n is finite. For $t \leq 0$ set

$$\mathbf{x}_t^{(n)} := (x_{t-\frac{1}{n},j})_{j\in F_n} \in \mathrm{PoM}\text{-}\prod_{j\in F_n} S_j.$$

Then $(\mathbf{x}_t^{(n)})_{t\leq 0}$ is a path, and we let $x^{(n)}$ denote its class in $\prod_{j\in F_n} S_j$. By construction, we have $\varphi_{J,F}(x^{(n)}) = [(\mathbf{x}_{t-1/n})_{t\leq 0}]$. The images of $x^{(n)}$ in $\bigoplus_j S_j$ form an increasing sequence, and we let y denote their supremum. Then

$$\varphi(y) = \sup_n \varphi_{J,F}(x^{(n)}) = \sup_n [(\mathbf{x}_{t-1/n})_{t\leq 0}] = [(\mathbf{x}_t)_{t\leq 0}] = x. \qquad \square$$

The following result is straightforward to prove. We omit the details.

Proposition 6.5. *Let $(S_j, \Sigma_j)_{j\in J}$ be a family of scaled* Cu*-semigroups, and let (S, Σ) be their scaled product with $\Sigma \subseteq S \subseteq \prod_j S_j$ as in Paragraph 5.12. Then $\mathrm{c}_0\big((S_j)_j\big) \subseteq S$, and we have a canonical isomorphism:*

$$\bigoplus_{j\in J} (S_j, \Sigma_j) \cong \big(\mathrm{c}_0\big((S_j)_j\big), \mathrm{c}_0\big((S_j)_j\big) \cap \Sigma\big).$$

Recall that the *sequence algebra* of a C^*-algebra A is defined as

$$A_\infty := \ell^\infty(A)/c_0(A) \cong \frac{\prod_{n\in\mathbb{N}} A}{\bigoplus_{n\in\mathbb{N}} A}.$$

By combining Theorems 6.1 and 5.13, Proposition 6.4, [CRS10, Theorem 5] we can now describe the Cuntz semigroup of the sequence algebra:

Corollary 6.6. *Let A be a C^*-algebra. Let (S, Σ) be the scaled product of the family $(\mathrm{Cu}(A))_{n\in\mathbb{N}}$, with $\Sigma \subseteq S \subseteq \prod_n \mathrm{Cu}(A)$ as in Paragraph 5.12. Then*

$$\mathrm{Cu}_{sc}(A_\infty) \cong S/\mathrm{c}_0(\mathrm{Cu}(A)).$$

7. Cuntz semigroups of ultraproducts

In this section, we show that the scaled Cuntz semigroup functor preserves ultraproducts; see Theorem 7.5. In particular, the Cuntz semigroup functor preserves ultraproducts of *stable* C^*-algebras; see Proposition 7.4. We take the opportunity to offer a more categorical view on ultraproducts, which suits both C^*-algebras and Cu-semigroups.

7.1. **Categorical ultraproducts.** We first recall the basic theory of categorical ultraproducts.

7.1. Let $\mathcal{C}$ be a category that has (small) products and inductive limits, let J be a set, let $\mathcal{U}$ be an ultrafilter on J, and let $(X_j)_{j\in J}$ be a family of objects in $\mathcal{C}$. Let $G \subseteq F \subseteq J$. By the universal property of the product we obtain a morphism

$$\varphi_{G,F} \colon \prod_{j\in F} X_j \to \prod_{j\in G} X_j,$$

such that $\pi_{j,F} = \pi_{j,G} \circ \varphi_{G,F}$ for each $j \in G$.

We order the elements of $\mathcal{U}$ by reversed inclusion. Then $\mathcal{U}$ is upward directed, and we obtain an inductive system indexed over $\mathcal{U}$, with objects $\prod_{j\in F} X_j$ for $F \in \mathcal{U}$, and with morphism $\varphi_{G,F}$ for $F, G \in \mathcal{U}$ with $F \supseteq G$. The inductive limit of this system is called the (categorical) *ultraproduct* of $(X_j)_{j\in J}$ along $\mathcal{U}$:

$$\prod_{\mathcal{U}} X_j := \varinjlim_{F\in\mathcal{U}} \prod_{j\in F} X_j.$$

We let $\pi_{\mathcal{U}} \colon \prod_{j\in J} X_j \to \prod_{\mathcal{U}} X_j$ denote the natural morphism to the inductive limit.

The ultraproduct of a constant family of objects is called *ultrapower*. Given an object X, we denote its ultrapower by $X_{\mathcal{U}}$, that is, $X_{\mathcal{U}} := \prod_{\mathcal{U}} X$.

The ultraproduct construction is functorial in the following sense: Given morphisms $\alpha_j \colon X_j \to Y_j$, we obtain a natural morphism $\alpha_{\mathcal{U}} \colon \prod_{\mathcal{U}} X_j \to \prod_{\mathcal{U}} Y_j$ as follows: For every $F \in \mathcal{U}$, the universal property of products implies that the morphisms $(\alpha_j)_{j\in F}$ induce a natural morphism $\alpha_F \colon \prod_{j\in F} X_j \to \prod_{j\in F} Y_j$. These morphisms then induce a morphism between the ultraproducts, defined as respective inductive limits. Similarly, a morphism $X \to Y$ induces a morphism $X_{\mathcal{U}} \to Y_{\mathcal{U}}$.

7.2. Let $\mathcal{C}$ and $\mathcal{D}$ be categories that have products and inductive limits (and hence categorical ultraproducts), and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor that preserves inductive limits. Let $\mathcal{U}$ be an ultrafilter on a set J, and let $(X_j)_{j\in J}$ be a family of objects in $\mathcal{C}$. Then there is a natural morphism

$$\Phi_{\mathcal{U}} \colon F(\prod_{\mathcal{U}} X_j) \to \prod_{\mathcal{U}} F(X_j).$$

If the functor F also preserves products, then $\Phi_{\mathcal{U}}$ is a natural isomorphism.

7.3. Let $\mathcal{U}$ be an ultrafilter on a set J, and let $(A_j)_{j\in J}$ be a family of C^*-algebras. The category of C^*-algebras, and the categories Cu and $\mathrm{Cu}_{\mathrm{sc}}$ of (scaled) Cu-semigroups are complete and cocomplete, and thus each of the categories admit categorical ultraproducts in the sense of Paragraph 7.1. By [APT18b, Corollary 3.2.9, p.29] and Theorem 4.7, the (scaled) Cuntz semigroup functor preserves arbitrary inductive limits. (It was shown in [CEI08] that the Cuntz semigroup functor preserves *sequential* inductive limits; for the application to ultraproducts it is however crucial that the functor also preserves inductive limits over non-sequential directed sets.) As explained in Paragraph 7.2, we obtain natural (scaled) Cu-morphisms

$$\Phi_{\mathcal{U}} \colon \mathrm{Cu}(\prod_{\mathcal{U}} A_j) \to \prod_{\mathcal{U}} \mathrm{Cu}(A_j), \quad \text{and} \quad \Phi_{\mathcal{U},\mathrm{sc}} \colon \mathrm{Cu}_{\mathrm{sc}}(\prod_{\mathcal{U}} A_j) \to \prod_{\mathcal{U}} \mathrm{Cu}_{\mathrm{sc}}(A_j).$$

Applying Proposition 5.9 and Theorem 5.13, we obtain the following results:

Proposition 7.4. *Let $\mathcal{U}$ be an ultrafilter on a set J, and let $(A_j)_{j\in J}$ be a family of stable C^*-algebras. Then, the map $\Phi_{\mathcal{U}}\colon \mathrm{Cu}(\prod_{\mathcal{U}} A_j) \to \prod_{\mathcal{U}} \mathrm{Cu}(A_j)$ from Paragraph 7.3 is an isomorphism.*

Theorem 7.5. *The scaled Cuntz semigroup functor preserves ultraproducts: Given an ultrafilter $\mathcal{U}$ on a set J, and a family $(A_j)_{j\in J}$ of C^*-algebras, the map $\Phi_{\mathcal{U},\mathrm{sc}}$ from Paragraph 7.3 is an isomorphism:*

$$\mathrm{Cu}_{\mathrm{sc}}(\prod_{\mathcal{U}} A_j) \cong \prod_{\mathcal{U}} \mathrm{Cu}_{\mathrm{sc}}(A_j).$$

7.2. **A different picture of ultraproducts.** In many 'everyday' categories, the (categorical) ultraproduct admits a more concrete description, which is the one used in practice. We will first recall this for the category of C^*-algebras, and then obtain a similar result for the category Cu.

7.6. Let $\mathcal{U}$ be an ultrafilter on a set J, and let $(A_j)_{j\in J}$ be a family of C^*-algebras. Set $A := \prod_{j\in J} A_j$. We define the support of an element $\underline{a} = (a_j)_j \in A$ as $\mathrm{supp}(\underline{a}) := \{j\in J : a_j \neq 0\}$. Given $F\in\mathcal{U}$, we set

$$I_F := \big\{ a\in A : \mathrm{supp}(a)\cap F = \emptyset \big\}.$$

Then I_F is a (closed, two-sided) ideal in A that is naturally isomorphic to $\prod_{j\in F^c} A_j$. (We use the convention that the product over the empty set is the zero C^*-algebra.) Moreover, the canonical map $\varphi_{F,J}\colon A\to\prod_{j\in F} A_j$ is a surjective *-homomorphism with kernel I_F. Thus, we obtain a natural isomorphism $\prod_{j\in F} A_j \cong A/I_F$.

Given $F\supseteq G$, we have $I_F\subseteq I_G$. Under the isomorphisms $\prod_{j\in F} A_j\cong A/I_F$ and $\prod_{j\in G} A_j \cong A/I_G$, the connecting morphism $\varphi_{G,F}$ (defined as in Paragraph 7.1) corresponds to the natural quotient map $A/I_F\to A/I_G$. Let I be the ideal of A generated by the I_F, that is,

$$I := \overline{\bigcup_{F\in\mathcal{U}} I_F}^{\|\cdot\|} \subseteq A.$$

The inductive limit of the system A/I_F is isomorphic to A/I. It follows that the ultraproduct $\prod_{\mathcal{U}} A_j$ is naturally isomorphic to A/I, as shown in the following commutative diagram:

$$\begin{array}{ccccccccc}
A/I_F & \longrightarrow & A/I_G & \longrightarrow & \dots & \longrightarrow & \varinjlim_{F\in\mathcal{U}} A/I_F & = & A/I \\
\downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & & \downarrow\cong \\
\prod_{j\in F} A_j & \xrightarrow{\varphi_{G,F}} & \prod_{j\in G} A_j & \longrightarrow & \dots & \longrightarrow & \varinjlim_{F\in\mathcal{U}} \prod_{j\in F} X_j & = & \prod_{\mathcal{U}} A_j.
\end{array}$$

It remains to describe the ideal I. Given $(a_j)_j\in A=\prod_j A_j$, we have a bounded function $J\to\mathbb{R}$, $j\mapsto\|a_j\|$, which allows us to consider $\lim_{j\to\mathcal{U}}\|a_j\|$. For $a=(a_j)_j\in A$ we have $\lim_{j\to\mathcal{U}}\|a_j\|=0$ if and only if $\{j\in J : \|a_j\|<\varepsilon\}\in\mathcal{U}$ for every $\varepsilon>0$. It follows that

$$I = \big\{ (a_j)_j \in A : \lim_{j\to\mathcal{U}} \|a_j\| = 0 \big\}.$$

This shows that the 'category theoretic' ultraproduct of C^*-algebras agrees with the notion of ultraproduct of C^*-algebras considered in (continuous) model theory; see for example [GH01]. The ideal I is also denoted by $c_{\mathcal{U}}((A_j)_j)$.

For every positive element $a_j \in A_j$ and $\varepsilon > 0$ we have $\|a_j\| \leq \varepsilon$ if and only if $(a_j - \varepsilon)_+ = 0$. Thus,

$$I_+ = \big\{(a_j)_j \in A_+ : \mathrm{supp}((a_j-\varepsilon)_+)_{j\in J}) \notin \mathcal{U} \text{ for every } \varepsilon > 0\big\}.$$

7.7. Let $\mathcal{U}$ be an ultrafilter on a set J, and let $(S_j)_{j\in J}$ be a family of Cu-semigroups. Set $P := \text{PoM-}\prod_j S_j$, the set-theoretic product of the S_j. The product $S := \prod_{j\in J} S_j$ in Cu is given by equivalence classes of $\ll_{\mathrm{pw}}$-increasing paths $(-\infty, 0] \to P$. Given $F \in \mathcal{U}$, we set

$$I_F := \big\{[(\mathbf{x}_t)_{t\leq 0}] \in S \text{ such that } \mathrm{supp}(\mathbf{x}_0) \cap F = \emptyset\big\}.$$

Then I_F is an ideal of S that is canonically isomorphic to $\prod_{j\in F^c} S_j$. As for C^*-algebras, the map $\varphi_{F,J}\colon S \to \prod_{j\in F} S_j$ induces an isomorphism $\prod_{j\in F} S_j \cong S/I_F$. Let I be the ideal of S generated by the I_F, that is,

$$I := \overline{\bigcup_{F\in\mathcal{U}} I_F}^{\sup} = \big\{ \sup_n x_n : (x_n)_n \text{ increasing sequence in } \bigcup_{F\in\mathcal{U}} I_F \big\} \subseteq S.$$

We obtain natural isomorphisms

$$\prod_{\mathcal{U}} S_j := \varinjlim_{F\in\mathcal{U}} \prod_{j\in F} S_j \cong \varinjlim_{F\in\mathcal{U}} S/I_F \cong S/I.$$

Let $\pi_{\mathcal{U}}\colon \prod_{j\in J} S_j \to \prod_{\mathcal{U}} S_j$ denote the quotient map with kernel I. The next result describes the ideal I.

Lemma 7.8. *We retain the notation from Paragraph 7.7. Let $(\boldsymbol{x}_t)_{t\leq 0}$ be a path in $(\text{PoM-}\prod_j S_j, \ll_{\mathrm{pw}})$. Then the following are equivalent:*

(1) We have $[(\boldsymbol{x}_t)_{t\leq 0}] \in I$.
(2) For every $\varepsilon > 0$, there exists $F \in \mathcal{U}$ such that the ε-cut-down $[(\boldsymbol{x}_{t-\varepsilon})_{t\leq 0}]$ belongs to I_F.
(3) For every $t < 0$, we have $\mathrm{supp}(\boldsymbol{x}_t) \notin \mathcal{U}$.

Proof. (1)⇒(2): Suppose $[(\mathbf{x}_t)_{t\leq 0}] \in I$. Then there is an increasing sequence $(x_n)_n$ in $\bigcup_{F\in\mathcal{U}} I_F$ such that $[(\mathbf{x}_t)_{t\leq 0}] = \sup x_n$. Let $\varepsilon > 0$. Using that $[(\mathbf{x}_{t-\varepsilon})_{t\leq 0}] \ll [(\mathbf{x}_t)_{t\leq 0}]$, we obtain n such that $[(\mathbf{x}_{t-\varepsilon})_{t\leq 0}] \leq x_n$. Since $x_n \in \bigcup_{F\in\mathcal{U}} I_F$ and since each I_F is an ideal of S, there exists $F \in \mathcal{U}$ such that $[(\mathbf{x}_{t-\varepsilon})_{t\leq 0}]$ belongs to I_F.

(2)⇒(3): Let $t < 0$. Set $\varepsilon := -t$. By assumption, there exists $F \in \mathcal{U}$ such that $[(\mathbf{x}_{s-\varepsilon})_{s\leq 0}]$ belongs to I_F. By definition, this means $\mathrm{supp}(\mathbf{x}_{0-\varepsilon}) \cap F = \emptyset$, and thus $\mathrm{supp}(\mathbf{x}_t) \notin \mathcal{U}$.

(3)⇒(1): Suppose that for every $t < 0$, we have $\mathrm{supp}(\mathbf{x}_t) \notin \mathcal{U}$. Now, write

$$[(\mathbf{x}_t)_{t\leq 0}] = \sup_{n\geq 1} [(\mathbf{x}_{t-\frac{1}{n}})_{t\leq 0}],$$

and put $F_n = \mathrm{supp}(\mathbf{x}_{-\frac{1}{n}})^c$. It is clear now that $F_n \in \mathcal{U}$ and that $[(\mathbf{x}_{t-\frac{1}{n}})_{t\leq 0}] \in I_{F_n}$. Therefore $[(\mathbf{x}_t)_{t\leq 0}] \in I$, as desired. □

Notation 7.9. Let $\mathcal{U}$ be an ultrafilter on a set J, and let $(S_j)_{j\in J}$ be a family of Cu-semigroups. Set

$$\mathrm{c}_{\mathcal{U}}\big((S_j)_j\big) := \big\{[(\mathbf{x}_t)_{t\leq 0}] \in \prod_j S_j : \mathrm{supp}(\mathbf{x}_t) \notin \mathcal{U} \text{ for each } t < 0\big\}.$$

Proposition 7.10. *We retain the notation from Paragraph 7.7. Then $\mathrm{c}_{\mathcal{U}}\big((S_j)_j\big)$ is an ideal in $\prod_j S_j$, and we have a natural isomorphism*

$$\prod_{\mathcal{U}} S_j \cong \big(\prod_{j\in J} S_j\big)/c_{\mathcal{U}}((S_j)_j).$$

Theorem 7.11. *Let $\mathcal{U}$ be an ultrafilter on a set J, and let $(A_j)_{j\in J}$ be a family of C^*-algebras. Let (S,Σ) be the scaled product of $(\mathrm{Cu}_{\mathrm{sc}}(A_j))_j$ with $\Sigma\subseteq S\subseteq\prod_j \mathrm{Cu}(A_j)$ as in Paragraph 5.12. Then we have a commutative diagram, where each row is a short exact sequence, where the upper vertical maps are isomorphisms, and where the lower vertical maps are inclusions as ideals:*

$$\begin{tikzcd}
\mathrm{Cu}(\mathrm{c}_{\mathcal{U}}((A_j)_j)) \arrow[r, hook] \arrow[d, "\cong"'] & \mathrm{Cu}(\prod_j A_j) \arrow[r, two heads] \arrow[d, "\cong"', "\Phi"] & \mathrm{Cu}(\prod_{\mathcal{U}} A_j) \arrow[d, "\cong"', "\Phi_{\mathcal{U}}"] \\
S\cap \mathrm{c}_{\mathcal{U}}\big((\mathrm{Cu}(A_j))_{j\in J}\big) \arrow[r, hook] \arrow[d] & S \arrow[r, two heads] \arrow[d] & S/\big(S\cap \mathrm{c}_{\mathcal{U}}\big((\mathrm{Cu}(A_j))_{j\in J}\big)\big) \arrow[d] \\
\mathrm{c}_{\mathcal{U}}\big((\mathrm{Cu}(A_j))_{j\in J}\big) \arrow[r, hook] & \prod_j \mathrm{Cu}(A_j) \arrow[r, two heads] & \prod_{\mathcal{U}} \mathrm{Cu}(A_j).
\end{tikzcd}$$

In particular, the map $\Phi_{\mathcal{U}}$ from Paragraph 7.3 is an order-embedding that identifies $\mathrm{Cu}(\prod_{\mathcal{U}} A_j)$ with the image of S under the map $\pi_{\mathcal{U}}$ from Paragraph 7.7. Further, $\Phi_{\mathcal{U}}$ identifies the scale of $\mathrm{Cu}(\prod_{\mathcal{U}} A_j)$ with $\pi_{\mathcal{U}}(\Sigma)$.

Proof. In general, if J is an ideal in a C^*-algebra A, then the inclusion $J\to A$ induces a map that identifies $\mathrm{Cu}(J)$ with an ideal of $\mathrm{Cu}(A)$, and such that $\mathrm{Cu}(A)/\mathrm{Cu}(J)$ is naturally isomorphism to $\mathrm{Cu}(A/J)$; see [CRS10, Theorem 4.1], and note that the assumption that the ideal be σ-unital can be removed; see also [APT18b, Section 5.1, p.37ff]. This shows that the top row in the diagram is exact.

By Theorem 5.13, Φ is an order-embedding with image S. To simplify, we denote $\mathrm{c}_{\mathcal{U}}((A_j)_j)$ by $\mathrm{c}^A_{\mathcal{U}}$, and we denote $\mathrm{c}_{\mathcal{U}}\big((\mathrm{Cu}(A_j))_{j\in J}\big)$ by $\mathrm{c}^S_{\mathcal{U}}$.

Claim: Let $\underline{a}\in(\prod_j A_j)_+$. Then $\Phi([\underline{a}])\in \mathrm{c}^S_{\mathcal{U}}$ if and only if $\underline{a}\in \mathrm{c}^A_{\mathcal{U}}$.

To prove the claim, let a_j such that $\underline{a}=(a_j)_j$, and set $\mathbf{x}_t:=([(a_j+t)_+])_j$ for $t\leq 0$. Then $\Phi([\underline{a}])=[(\mathbf{x}_t)_{t\leq 0}]$; see Paragraph 5.1. As noted at the end of Paragraph 7.6, we have $\underline{a}\in c^A_{\mathcal{U}}$ if and only if $\mathrm{supp}(((a_j-\varepsilon)_+)_{j\in J})\notin\mathcal{U}$ for every $\varepsilon>0$, which in turn is equivalent to $\mathrm{supp}(\mathbf{x}_t)\notin\mathcal{U}$ for every $t<0$. Now, the claim follows from Lemma 7.8.

It follows that for $x\in\Sigma_{\prod_j A_j}$ we have $x\in\mathrm{Cu}(\mathrm{c}_{\mathcal{U}}((A_j)_j))$ if and only if $\Phi(x)\in \mathrm{c}^S_{\mathcal{U}}$. Since $\Phi(\Sigma_{\prod_j A_j})=\Sigma$, it follows that $\Phi(\mathrm{Cu}(\mathrm{c}^A_{\mathcal{U}})\cap\Sigma_{\prod_j A_j})=\mathrm{c}^S_{\mathcal{U}}\cap\Sigma$. Using Proposition 4.3 at the third step, we obtain

$$\Phi(\mathrm{Cu}(\mathrm{c}^A_{\mathcal{U}}))=\Phi(\langle\Sigma_{\mathrm{c}^A_{\mathcal{U}}}\rangle)=\langle\Phi(\Sigma_{\mathrm{c}^A_{\mathcal{U}}})\rangle=\langle\Phi(\mathrm{Cu}(\mathrm{c}^A_{\mathcal{U}})\cap\Sigma_{\prod_j A_j})\rangle=\langle \mathrm{c}^S_{\mathcal{U}}\cap\Sigma\rangle=\mathrm{c}^S_{\mathcal{U}}\cap S,$$

where $\langle\_\rangle$ denotes the generated ideal. It follows that the upper vertical maps in the diagram are isomorphisms.

Lastly, to show the statement about the scales, let $\psi\colon \mathrm{Cu}(\prod_j A_j)\to\mathrm{Cu}(\prod_{\mathcal{U}} A_j)$ be the Cu-morphism induced by the quotient *-homomorphism. Using Proposition 4.3 at the last step, we get

$$\pi_{\mathcal{U}}(\Sigma)=\pi_{\mathcal{U}}(\Phi(\Sigma_{\prod_j A_j}))=\Phi_{\mathcal{U}}(\psi(\Sigma_{\prod_j A_j}))=\Phi_{\mathcal{U}}(\Sigma_{\prod_{\mathcal{U}} A_j}).\qquad\square$$

Example 7.12. Let J be an infinite set, and let $\mathcal{U}$ be an ultrafilter on J. We identify the Stone Čech compactification $\beta(J)$ of J with the set of ultrafilters on J, and we equip it with the *Stone topology*, which has a basis of clopen sets given by

$$\mathcal{U}_A := \big\{\mathcal{F} \in \beta(J) : A \in \mathcal{F}\big\},$$

for each $A \subseteq J$. Let $\mathrm{cpct}_\sigma(\beta(J))$ denote the family of σ-compact, open subsets of $\beta(J)$. Note that $\mathrm{cpct}_\sigma(\beta(I))$ is a positively ordered monoid, where addition of sets is given by their union, and order is defined by inclusion. Then we have isomorphisms of Cu-semigroups:

$$\prod_{j\in J}\{0,\infty\} \cong \mathrm{cpct}_\sigma(\beta(J)), \quad \text{and} \quad \prod_{\mathcal{U}}\{0,\infty\} \cong \{0,\infty\}.$$

Proof. Given $A, B \subseteq J$, we have $\mathcal{U}_{A\cup B} = \mathcal{U}_A \cup \mathcal{U}_B$, and $\mathcal{U}_A \subseteq \mathcal{U}_B$ if and only if $A \subseteq B$. We first compute $\prod_J\{0,\infty\}$. Let $\mathcal{P}(J)$ denote the powerset of J considered as a positively ordered monoid with addition given by union and order given by inclusion. It is clear that PoM-$\prod_j\{0,\infty\} \cong \mathcal{P}(J)$. Under this identification, the auxiliary relation $\ll_{\mathrm{pw}}$ becomes inclusion and it is easy to see that in this case the τ-construction agrees with the (sequential) round ideal completion γ. Hence,

$$\prod_{j\in J}\{0,\infty\} = \tau\big(\text{PoM-}\prod_{j\in J}\{0,\infty\}, \ll_{\mathrm{pw}}\big) \cong \tau\big(\mathcal{P}(J), \subseteq\big) \cong \gamma\big(\mathcal{P}(J), \subseteq\big).$$

The elements in $\gamma(\mathcal{P}(J),\subseteq)$ are equivalence classes of increasing sequences $\mathbf{A} = (A_n)_{n\in\mathbb{N}}$ of subsets of J, under the antisymmetrization of the relation $(A_n)_n \precsim (B_n)_n$ if for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $A_n \subseteq B_m$. We use $[\mathbf{A}]$ to denote the class of $\mathbf{A}$, and recall that addition is given by $[(A_n)_n]+[(B_n)_n] = [(A_n\cup B_n)_n]$. Define $\Phi\colon \gamma(\mathcal{P}(J),\subseteq) \to \mathrm{cpct}_\sigma(\beta(J))$ by

$$\Phi([\mathbf{A}]) := \bigcup_{n\in\mathbb{N}} \mathcal{U}_{A_n},$$

for every increasing sequence $\mathbf{A} = (A_n)_{n\in\mathbb{N}}$. Using the properties of the clopen sets $\mathcal{U}_A$ mentioned above, it follows that Φ is well-defined, order-preserving and additive.

To see that Φ is surjective, let $U \subseteq \beta(J)$ be a σ-compact, open subset. Choose an increasing sequence $(K_n)_n$ of compact subsets such that $U = \bigcup_n K_n$. Since the $\mathcal{U}_A$ form a basis of the topology, and the K_n are compact, we can find sets D_n such that $K_n \subseteq \mathcal{U}_{D_n} \subseteq U$. Therefore, setting $A_n = \bigcup_{i=1}^n D_i$ to make them increasing, we have $U = \bigcup_n \mathcal{U}_{A_n}$ and $U = \Phi([\mathbf{A}])$ where $\mathbf{A} = (A_n)_n$.

Now suppose that $\Phi([\mathbf{A}]) \leq \Phi([\mathbf{B}])$. This implies that for each n, $\mathcal{U}_{A_n} \subseteq \bigcup_{m\in\mathbb{N}}\mathcal{U}_{B_m}$. Since $\mathcal{U}_{A_n}$ is compact and the sequence $(B_m)_m$ is increasing, there is $m \in \mathbb{N}$ such that $\mathcal{U}_{A_n} \subseteq \mathcal{U}_{B_m}$ which implies $A_n \subseteq B_m$. Hence $\mathbf{A} \precsim \mathbf{B}$ proving that Φ is an order-embedding, and thus $\prod_j\{0,\infty\} \cong \mathrm{cpct}_\sigma(\beta(J))$.

We now compute the ultraproduct. Let us write $c_\mathcal{U}$ for $c_\mathcal{U}((\{0,\infty\})_j)$. It follows from Proposition 7.10 that $c_\mathcal{U}$ corresponds to equivalence classes of $\mathbf{A} = (A_n)_n$ with all $A_n \notin \mathcal{U}$. Let $[\mathbf{A}], [\mathbf{B}] \in \gamma(\mathcal{P}(J),\subseteq)$ with $[\mathbf{B}] \notin c_\mathcal{U}$. Then there is n_0 such that $B_{n_0} \in \mathcal{U}$. For each n, we have

$$A_n = (A_n \cap B_{n_0}) \cup (A_n \cap B_{n_0}^c) \subseteq B_{n_0} \cup (B_n \cap A_{n_0}^c).$$

Set $\mathbf{C} := (A_n \cap B_{n_0}^c)_n$. Then $\mathbf{A} \precsim \mathbf{B} + \mathbf{C}$. Since $(A_n \cap B_{n_0}^c) \notin \mathcal{U}$, we have $\mathbf{C} \in c_\mathcal{U}$.

By Proposition 7.10, we have $\prod_\mathcal{U}\{0,\infty\} \cong \prod_j\{0,\infty\}/c_\mathcal{U}$. It follows from the above argument that $\prod_\mathcal{U}\{0,\infty\}$ has only two elements, $\{0,\infty\}$, as desired. ☐

8. Applications

8.1. **Compact elements and K-Theory.** Recall that, if S is a Cu-semigroup, then an element $a \in S$ is *compact* provided $a \ll a$. If S is only a $\mathcal{Q}$-semigroup, we shall also use this terminology for an element $a \in S$ such that $a \prec a$. The subsemigroup of compact elements of S will be denoted by S_c. This semigroup is a positively ordered monoid, regardless of whether S is a Cu-semigroup or a $\mathcal{Q}$-semigroup. We next describe the compact elements of an ultraproduct of Cu-semigroups, for which we use the picture of categorical ultraproducts; see Paragraph 7.1. This will be used later to describe the Murray-von Neumann semigroup of an ultraproduct of unital C^*-algebras, thus recovering [Li05, Theorem 4.1] (see also [GH01, Proposition 2.1]).

Lemma 8.1. *Let S be a $\mathcal{Q}$-semigroup. Then, the endpoint map (Paragraph 2.11) induces a* PoM-*isomorphism $\tau(S)_c \cong S_c$.*

Proof. Since the endpoint map $\varphi_S\colon \tau(S) \to S$ is a $\mathcal{Q}$-morphism, we see that its restriction to $\tau(S)_c$ is a PoM-morphism that maps compact elements to compact elements. Given $a \in S_c$, the map $f_a\colon (-\infty, 0] \to S$ given by $f_a(t) = a$ for every t is clearly a path with $[f_a] \in \tau(S)_c$ and $\varphi_S([f_a]) = a$. Thus, φ_S induces an additive, order-preserving bijection $\tau(S)_c \to S_c$.

Now, let $[f], [g] \in \tau(S)_c$ satisfy $\varphi_S([f]) \leq \varphi_S([g])$. By [APT18, Lemma 3.16] applied to $[g] \ll [g]$, there is $t_0 < 0$ such that $g(t) \prec g(t_0)$ for all $t < 0$. Therefore $\varphi_S([g]) = \sup_{t<0} g(t) \leq g(t_0)$ and thus, for every $t < 0$, we have $f(t) \prec g(t_0)$. This implies $[f] \leq [g]$ and therefore φ_S induces an order-embedding. □

Corollary 8.2. *Let $(S_j)_{j\in J}$ be a family of* Cu-*semigroups. Then*
$$(\prod_{j\in J} S_j)_c \cong \text{PoM-}\prod_{j\in J} (S_j)_c\,.$$

Proof. We have $\prod_j S_j = \tau(\mathcal{Q}\text{-}\prod_j S_j)$, by Corollary 3.9. Therefore, Lemma 8.1 implies that $(\prod_j S_j)_c \cong (\mathcal{Q}\text{-}\prod_j S_j)_c$. Since the auxiliary relation in $\mathcal{Q}\text{-}\prod_J S_j$ is $\ll_{\mathrm{pw}}$, it follows that $(\mathcal{Q}\text{-}\prod_j S_j)_c$ is isomorphic to PoM-$\prod_j (S_j)_c$. □

Proposition 8.3. *Let $\mathcal{U}$ be an ultrafilter on a set J, and let $(S_j)_{j\in J}$ be a family of* Cu-*semigroups. Then there is a natural* PoM-*isomorphism*
$$(\prod_{\mathcal{U}} S_j)_c \cong \prod_{\mathcal{U}} (S_j)_c.$$
In particular, for every compact $x \in \prod_{\mathcal{U}} S_j$ there exists a compact $y \in \prod_j S_j$ with $x = \pi_{\mathcal{U}}(y)$, where $\pi_{\mathcal{U}}\colon \prod_{j\in J} S_j \to \prod_{\mathcal{U}} S_j$ is the natural quotient map.

Proof. We have $\prod_{\mathcal{U}} S_j = \varinjlim_{F\in\mathcal{U}} \prod_{j\in F} S_j$, and thus $(\prod_{\mathcal{U}} S_j)_c \cong \varinjlim_{F\in\mathcal{U}} (\prod_{j\in F} S_j)_c$ with the restriction of the same connecting maps. Now, by Corollary 8.2, for each $F \in \mathcal{U}$ the end point map induces an isomorphism $(\prod_{j\in F} S_j)_c \cong \prod_{j\in F} (S_j)_c$, which is easily seen to be compatible with the connecting maps. Therefore
$$(\prod_{\mathcal{U}} S_j)_c \cong \varinjlim_{F\in\mathcal{U}} \big(\prod_{j\in F} S_j\big)_c \cong \varinjlim_{F\in\mathcal{U}} \prod_{j\in F} (S_j)_c = \prod_{\mathcal{U}} (S_j)_c. \qquad \square$$

For a C^*-algebra A, we denote as customary by $V(A)$ the semigroup of Murray-von Neumann equivalence classes of projections in $A \otimes \mathcal{K}$. By [BC09], if A is stably finite, then $V(A)$ can be identified with the semigroup of compact elements in $\mathrm{Cu}(A)$. The following result for ultraproducts is implicit in [Li05, Theorem 4.1].

Corollary 8.4. *Let $\mathcal{U}$ be an ultrafilter on a set J, and let $(A_j)_{j\in J}$ be a family of stably finite, unital C^*-algebras. Put $u_j = [1_{A_j}] \in V(A_j)$ and $u = (u_j)_{j\in J}$. Then:*

(1) $V(\prod_{j\in J} A_j) \cong \{x \in \prod_{j\in J} V(A_j) \colon x \leq nu$ for some $n \in \mathbb{N}\}$.
(2) $V(\prod_{\mathcal{U}} A_j) \cong \{x \in \prod_{\mathcal{U}} V(A_j) \colon x \leq n\pi_{\mathcal{U}}(u)$ for some $n \in \mathbb{N}\}$.

Proof. First observe that both $\prod_i A_i$ and $\prod_{\mathcal{U}} A_i$ are stably finite C^*-algebras. Therefore $V(\prod_j A_j)$ is identified with $\mathrm{Cu}(\prod_j A_j)_c$. By Theorem 5.13, we have

$$\mathrm{Cu}(\prod_j A_j) \cong \{x \in \prod_j \mathrm{Cu}(A_j) : x \leq \infty\, u\},$$

and thus the desired conclusions follow from Corollary 8.2. Statement (2) follows analogously from Theorem 7.5 and Proposition 8.3. □

8.2. **Simplicity.**

8.5. Recall that an ultrafilter $\mathcal{U}$ is called *countably complete* if it is closed under taking intersections of countably many elements. If $\mathcal{U}$ it not countably complete, then it is called *countably incomplete.* Every free ultrafilter on a countable set is countably incomplete.

It is well-known that the ultrapower of a separable C^*-algebra A over a countably complete (free) ultrafilter is canonically isomorphic to A itself; see for example [GH01, Proposition 6.1]. Thus, with respect to a separable C^*-algebra, a countably complete, free ultrafilter behaves like a principal ultrafilter. Let now A be a simple, stable, separable, infinite-dimensional C^*-algebra, and let $\mathcal{U}$ be a free ultrafilter. If $\mathcal{U}$ is countably incomplete, then $\prod_{\mathcal{U}} A$ is neither stable (see Remark 8.6), nor separable, and usually not simple (see Theorem 8.14 for a characterization of simplicity of ultrapowers). It follows that the behaviour of ultraproducts and ultrapowers is immensely different for countably complete and countably incomplete ultrafilters.

We remark that it is not obvious that countably complete, free utrafilters exist. In fact, the existence of a countably complete, free ultrafilter is equivalent to the existence of a measurable cardinal, which puts this question outside of our usual axiomatic setting of ZFC and into the realm of set theory.

Remark 8.6. In general, stability does not pass to ultraproducts or ultrapowers. More precisely, let A be a (stable) C^*-algebra, let I be a set and let $\mathcal{U}$ be a free ultrafilter on I. Farah and Hart, [FH13], showed that $\prod_{\mathcal{U}} A$ is a SAW*-algebra if I is countable. On the other hand, it was shown by Ghasemi, [Gha15, Theorem 2.6], that a SAW*-algebra is not isomorphic to the tensor product of two infinite-dimensional C^*-algebras. In particular, infinite-dimensional SAW*-algebras are not stable. If A is infinite-dimensional then so is $\prod_{\mathcal{U}} A$. Thus, if I is countable and A is infinite-dimensional, then $\prod_{\mathcal{U}} A$ is not stable.

This can also be deduced from a result of Kania, [Kan15, Theorem 1.1], who showed that a C^*-algebra that is a Grothendieck space is not isomorphic to the tensor product of two infinite-dimensional C^*-algebras. Moreover, he showed that $\prod_{\mathcal{U}} A$ is a Grothendieck space if $\mathcal{U}$ is countably incomplete. Thus, if $\mathcal{U}$ is countably incomplete and A is infinite-dimensional, then $\prod_{\mathcal{U}} A$ is not stable.

Definition 8.7. We say that a scaled Cu-semigroup (S, Σ) has property (C_n) if for all $x, y \in \Sigma$ with $y \neq 0$, we have $x \leq ny$.

Lemma 8.8. *Let $\mathcal{U}$ be a free ultrafilter on a set J, and let $(S_j, \Sigma_j)_{j\in J}$ be a family of scaled* Cu*-semigroups. Let $n \in \mathbb{N}$ and $F \in \mathcal{U}$ such that (S_j, Σ_j) has (C_n) for all $j \in F$. Then $\prod_{\mathcal{U}}(S_j, \Sigma_j)$ has (C_n).*

Proof. Let $x, y \in \prod_{\mathcal{U}} \Sigma_j$ with $y \neq 0$. We need to show that $x \leq ny$. Choose $\tilde{x}, \tilde{y} \in \prod_j (S_j, \Sigma_j)$ with $\pi_{\mathcal{U}}(\tilde{x}) = x$ and $\pi_{\mathcal{U}}(\tilde{y}) = y$. Then, choose $x_{t,j}, y_{t,j} \in S_j$ such that $\tilde{x} = [((x_{t,j})_j)_t]$ and $\tilde{y}_t = [((y_{t,j})_j)_t]$. Since $y \neq 0$, we have $\tilde{y} \notin c_{\mathcal{U}}$ (see Proposition 7.10). Hence, we can fix $s < 0$ such that $G := \mathrm{supp}((y_{s,j})_j)$ belongs to $\mathcal{U}$.

For $t \leq 0$ and $j \in J$ set

$$z_{t,j} := \begin{cases} x_{t,j}, & \text{if } j \notin F \cap G \\ 0, & \text{if } j \in F \cap G \end{cases}.$$

Then $z := [((z_{t,j})_j)_t]$ belongs to $c_{\mathcal{U}} \cap \prod_j (S_j, \Sigma_j)$. We claim that $\tilde{x} \leq n\tilde{y} + z$. To prove this, let $t < 0$. Choose $\tilde{s}$ with $t, s < \tilde{s} < 0$ (with s as chosen above).

Case 1: Let $j \in F \cap G$. Then S_j has (C_n), and we have $x_{\tilde{s},j}, y_{s,j} \in \Sigma_j$ with $y_{s,j} \neq 0$. We obtain that $x_{\tilde{s},j} \leq n y_{s,j}$, and thus

$$x_{t,j} \ll x_{\tilde{s},j} \leq n y_{s,j} = n y_{\tilde{s},j} + u_{\tilde{s},j}.$$

Case 2: Let $j \notin F \cap G$. Then

$$x_{t,j} \ll x_{\tilde{s},j} = z_{\tilde{s},j} \leq n y_{\tilde{s},j} + z_{\tilde{s},j}.$$

Thus, $(x_{t,j})_j \ll_{\mathrm{pw}} n(y_{\tilde{s},j})_j + (z_{\tilde{s},j})_j$. Hence, $\tilde{x} \leq n\tilde{y} + z$ in $\prod_j S_j$, and thus $x \leq ny$ in $\prod_{\mathcal{U}}(S_j, \Sigma_j)$. $\square$

Recall that a Cu-semigroup S satisfies (O6) if whenever $a', a, b, c \in S$ satisfy $a' \ll a \leq b + c$, there are elements $e, f \in S$ with $e \leq a, b$ and $f \leq a, c$, and such that $a' \leq e + f$. It is well known that this is satisfied by the Cuntz semigroup of any C^*-algebra.

Lemma 8.9. *Let (S, Σ) be a scaled* Cu*-semigroup satisfying (O6). Assume that (S, Σ) has (C_n) for some $n \geq 1$. Then S is simple.*

Proof. Pick $n \geq 1$ such that (S, Σ) has (C_n). Let $x, y \in S$ with $y \neq 0$. We need to show that $x \leq \infty y$.

Claim: There exists $z \in \Sigma$ with $0 \neq z \leq y$. To show this, choose $y'', y' \in S$ with $0 \neq y'' \ll y' \ll y$. Using that Σ generates S as an ideal, we can then choose $y_1, \ldots, y_K \in \Sigma$ such that

$$y' \leq y_1 + \ldots + y_K.$$

Applying (O6) for $y'' \ll y' \leq y_1 + \ldots + y_K$, we obtain $e_1, \ldots, e_K \in S$ such that

$$y'' \leq e_1 + \ldots + e_K, \quad \text{and} \quad e_k \leq y', y_k$$

for each $k = 1, \ldots, K$. Since $y'' \neq 0$, there exists $k \in \{1, \ldots, K\}$ with $e_k \neq 0$. Then $z := e_k$ has the desired properties to verify the claim.

It is enough to verify that $x' \leq \infty y$ for every $x' \ll x$. Thus, let $x' \ll x$. Using again that Σ generates S as an ideal, we obtain $x_1, \ldots, x_L \in \Sigma$ such that

$$x' \leq x_1 + \ldots + x_L.$$

Since (S, Σ) has (C_n), we have $x_l \leq nz$ for every $l = 1, \ldots, L$. Hence,

$$x' \leq x_1 + \ldots + x_L \leq Lnz \leq Lny \leq \infty y. \quad \square$$

Lemma 8.10. *Let $\mathcal{U}$ be a countably incomplete, free ultrafilter on a set J, and let $(S_j,\Sigma_j)_{j\in J}$ be a family of scaled* Cu*-semigroups such that $\prod_{\mathcal{U}}(S_j,\Sigma_j)$ is simple. Then there exist $n\in\mathbb{N}$ and $F\in\mathcal{U}$ such that S_j has (C_n) for all $j\in F$.*

Proof. We argue by contradiction. Thus, assume that for all $n\in\mathbb{N}$, and for all $F\in\mathcal{U}$, there is $j\in F$ such that S_j does not have (C_n). Let us write $c_{\mathcal{U}}$ for $c_{\mathcal{U}}((S_j)_j)$. By Proposition 7.10, we have $\prod_{\mathcal{U}} S_j\cong(\prod_j S_j)/c_{\mathcal{U}}$, and we let $\pi_{\mathcal{U}}\colon\prod_j S_j\to\prod_{\mathcal{U}} S_j$ denote the quotient map.

Since $\mathcal{U}$ is countably incomplete, we may find a decreasing sequence $(F_n)_n$ of elements in $\mathcal{U}$ such that $\bigcap_n F_n=\emptyset$. By our assumption the sets

$$G_n:=\big\{j\in F_n : S_j \text{ does not have } (C_n)\big\}$$

also belong to $\mathcal{U}$, and they also form a decreasing sequence with empty intersection. Passing to this subsequence we may assume that for every $j\in F_n$, the semigroup S_j does not have (C_n).

For each $n\geq 1$, and each $j\in F_n\setminus F_{n+1}$, choose elements $x_j,\,y_j\in\Sigma_j$ with

$$y_j\neq 0,\quad\text{and}\quad x_j\not\leq ny_j.$$

Further, choose paths $(x_{t,j})_{t\leq 0}$ in S_j with $x_j=x_{0,j}$ such that $x_{t,j}\not\leq ny_j$ for all $t\leq 0$. Likewise, choose paths $(y_{t,j})_{t\leq 0}$ in S_j with $y_j=y_{0,j}$ and such that $y_{t,j}\neq 0$ for all $t\leq 0$. For $t\leq 0$ set

$$\mathbf{x}_t:=\begin{cases} x_{t,j}, & \text{if } j\in F_n\setminus F_{n+1}\\ 0, & \text{otherwise;}\end{cases}\quad\text{and}\quad \mathbf{y}_t:=\begin{cases} y_{t,j}, & \text{if } j\in F_n\setminus F_{n+1}\\ 0, & \text{otherwise.}\end{cases}$$

Set $x=[(\mathbf{x}_t)_{t\leq 0}]$ and $y=[(\mathbf{y}_t)_{t\leq 0}]$ in $\prod_j S_j$. By construction, we have $x,y\in\prod_j(S_j,\Sigma_j)$. Since $\mathrm{supp}(\mathbf{y}_{-1})=F_1\in\mathcal{U}$, we have $y\notin c_{\mathcal{U}}$, and therefore $\pi_{\mathcal{U}}(y)\neq 0$. Using that $\prod_{\mathcal{U}}(S_j,\Sigma_j)$ is simple, it follows that $\pi_{\mathcal{U}}(x)\leq\infty\pi_{\mathcal{U}}(y)$.

Let $\varepsilon>0$, and let $x_\varepsilon:=[(\mathbf{x}_{t-\varepsilon})_{t\leq 0}]$ be the ε-cut-down of x. We have $x_\varepsilon\ll x$ in $\prod_j(S_j,\Sigma_j)$, and therefore

$$\pi_{\mathcal{U}}(x_\varepsilon)\ll\pi_{\mathcal{U}}(x)\leq\infty\pi_{\mathcal{U}}(y)=\sup_m m\pi_{\mathcal{U}}(y),$$

in $\prod_{\mathcal{U}}(S_j,\Sigma_j)$. We obtain $m\in\mathbb{N}$ such that $\pi_{\mathcal{U}}(x_\varepsilon)\leq m\pi_{\mathcal{U}}(y)$, and thus there is $z\in c_{\mathcal{U}}$ such that $x_\varepsilon\leq my+z$. Choose $z_{t,j}\in S_j$ such that $z=[((z_{t,j})_{j\in J})_{t\leq 0}]$.

Fix any $t<0$. Then there is $s<0$ such that $\mathbf{x}_{t-\varepsilon}\ll_{\mathrm{pw}} m\mathbf{y}_s+\mathbf{z}_s$. Set $T:=(\mathrm{supp}(\mathbf{z}_s))^c$. Since $z\in c_{\mathcal{U}}$, we have $T\in\mathcal{U}$, and thus there is $N\geq m$ with $T\cap(F_N\setminus F_{N+1})\neq\emptyset$. Choose j in $T\cap(F_N\setminus F_{N+1})$. Then $z_{s,j}=0$, and thus

$$(\mathbf{x}_{t-\varepsilon})_j=x_{t-\varepsilon,j}\leq my_{s,j}\leq my_j\leq Ny_j,$$

in contradiction with our choice. $\square$

Theorem 8.11. *Let $\mathcal{U}$ be a countably incomplete, free ultrafilter on a set J, and let $(S_j,\Sigma_j)_{j\in J}$ be a family of scaled* Cu*-semigroups that satisfy (O6). Then the following are equivalent:*

1. *$\prod_{\mathcal{U}}(S_j,\Sigma_j)$ is simple.*
2. *$\prod_{\mathcal{U}}(S_j,\Sigma_j)$ has (C_n) for some $n\geq 1$.*
3. *There exist $n\in\mathbb{N}$ and $F\in\mathcal{U}$ such that S_j has (C_n) for all $j\in F$.*

In particular, if (S,Σ) is a scaled Cu*-semigroup satisfying (O6), then the ultrapower $\prod_{\mathcal{U}}(S,\Sigma)$ is simple if and only if (S,Σ) has (C_n) for some n.*

Proof. By Lemma 8.10, (1) implies (3), and by Lemma 8.8 (3) implies (2). It is straightforward to verify that (O6) passes to products and quotients in Cu. Hence, $\prod_{\mathcal{U}}(S_j, \Sigma_j)$ satisfies (O6), whence we can apply Lemma 8.9 to deduce that (2) implies (1). □

Let $n \in \mathbb{N}$. Adapting [KR02, Definition 4.3] from C^*-algebras to Cu-semigroups, let us say that a Cu-semigroup S is *pi-n* if $2nx = nx$ for every $x \in S$. Further, we say that S is *weakly purely infinite* if it is pi-n for some n.

Note that a Cu-semigroup S is simple and pi-n if and only if for all $x, y \in S$ with $y \neq 0$ we have $x \leq ny$. Thus, S is simple and pi-n if and only if the trivially scaled Cu-semigroup (S, S) has (C_n).

Corollary 8.12. *Let $\mathcal{U}$ be a countably incomplete, free ultrafilter on a set J, and let $(S_j)_{j\in J}$ be a family of* Cu*-semigroups. Then the following are equivalent:*

(1) $\prod_{\mathcal{U}} S_j$ is simple.
(2) $\prod_{\mathcal{U}} S_j$ is simple and weakly purely infinite.
(3) There exist $n \in \mathbb{N}$ and $F \in \mathcal{U}$ such that S_j is simple and pi-n for all $j \in F$.

In particular, the ultrapower $\prod_{\mathcal{U}} S$ of a Cu*-semigroup S is simple if and only if S is simple and weakly purely infinite.*

Proof. It is clear that (2) implies (1). As in the proof of Theorem 8.11, we obtain that (1) implies (3), and that (3) implies (2). (Note that Lemmas 8.10 and 8.8 do not require (O6).) □

Lemma 8.13. *Let A be a C^*-algebra, and let $n \geq 1$. Then $(\mathrm{Cu}(A), \Sigma_A)$ satisfies (C_n) if and only if $A \cong M_k(\mathbb{C})$ for $k \leq n$, or A is simple and purely infinite.*

Proof. The backward implication is easy to show. Thus, assume that $(\mathrm{Cu}(A), \Sigma_A)$ satisfies (C_n). Since Cuntz semigroups of C^*-algebras satisfy (O6), it follows from Lemma 8.9 that A is simple. Then A is either elementary (that is, A contains a minimal nonzero projection), or not. In the first case, we obtain $\mathrm{Cu}(A) \cong \overline{\mathbb{N}}$. It is easy to verify that $(\overline{\mathbb{N}}, \Sigma)$ has (C_n) if and only if $\Sigma \subseteq \{0, \ldots, n\}$. It follows that A contains only projections of rank at most n, and thus $A \cong M_k(\mathbb{C})$ for some $k \leq n$.

In the other case, it follows from the Glimm Halving Lemma that for every nonzero $x \in \mathrm{Cu}(A)$ there exists a nonzero y with $2y \leq x$. Let $x \in \mathrm{Cu}(A)$ be nonzero. From the Claim in the proof of Lemma 8.9 we obtain a nonzero $y \in \Sigma_A$ with $y \leq x$. Applying the Glimm Halving Lemma several times, we obtain a nonzero z with $2nz \leq y$. By (C_n), we also have $y \leq nz$, and thus $2y \leq 2nz \leq y$. Therefore $y = \infty$, and then $x = \infty$. It follows that $\mathrm{Cu}(A) \cong \{0, \infty\}$, which implies that A is simple and purely infinite. □

Using Theorem 8.11, we obtain a characterization for when an ultraproduct of C^*-algebras is simple. In the case that we have a purely infinite simple C^*-algebra A, and a free ultrafilter $\mathcal{U}$ over $\mathbb{N}$, it was proved by Kirchberg (see [Rør02, Proposition 6.2.6]) that the ultrapower $A_{\mathcal{U}}$ is again purely infinite simple. The result that an ultraproduct of purely infinite, simple C^*-algebras is again purely infinite simple can be obtained as a consequence of [FHL$^+$16, Theorem 2.5.1].

Theorem 8.14. *Let $\mathcal{U}$ be a countably incomplete, free ultrafilter on a set J, and let $(A_j)_{j\in J}$ be a family of C^*-algebras. Then $\prod_{\mathcal{U}} A_j$ is simple if and only if either A_j is simple and purely infinite for almost all j in J, or else there is $n \in \mathbb{N}$ such that $A_j \cong M_n(\mathbb{C})$ for almost all j in J.*

In particular, the ultrapower $\prod_{\mathcal{U}} A$ of a C^*-algebra A is simple if and only if A is either simple and purely infinite or $A \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$.

Proof. If $A_j \cong M_n(\mathbb{C})$ for almost all $j \in J$, then $\prod_{\mathcal{U}} A_j \cong M_n(\mathbb{C})$, which is simple. Note that a C^*-algebra A is simple and purely infinite if and only if $\mathrm{Cu}(A) \cong \{0, \infty\}$, with the trivial scale. Thus, if A_j is simple and purely infinite for almost all $j \in J$, then using Theorem 5.13 and Example 7.12 we obtain

$$\mathrm{Cu}(\prod_{\mathcal{U}} A_j) \cong \prod_{\mathcal{U}} \mathrm{Cu}(A_j) \cong \prod_{\mathcal{U}} \{0, \infty\} \cong \{0, \infty\},$$

which implies that $\prod_{\mathcal{U}} A_j$ is simple (and purely infinite).

To show the converse implication, assume that $\prod_{\mathcal{U}} A_j$ is simple. Since Cuntz semigroups of C^*-algebras satisfy (O6), and since the Cuntz semigroup of a C^*-algebra is simple if and only if the C^*-algebra is simple, we can apply Theorem 8.11 to deduce that there exist $n \geq 1$ and $F \in \mathcal{U}$ such that the scaled Cu-semigroups $\mathrm{Cu}(A_j)$ have (C_n) for all $j \in F$. Let F_0 be set of $j \in F$ such that A_j is simple and purely infinite, and for $k = 1, \ldots, n$ let F_k be the set of $j \in F$ such that $A_j \cong M_k(\mathbb{C})$. By Lemma 8.13, we have $F = F_0 \cup \ldots \cup F_n$, and since $\mathcal{U}$ is an ultrafilter, we have $F_k \in \mathcal{U}$ for precisely one k, which proves the claim. □

8.3. **Comparability.** We show that several comparability properties pass to (ultra)products of Cu-semigroups. In particular, in Corollary 8.18 we recover an unpublished result of A. Tikuisis, that was brought to our attention by L. Robert (private communication). We refer to [APT18b, Section 5.6, p.72ff], for definitions of the unperforation properties considered in the next results.

Proposition 8.15. *The following properties pass to products, direct sums, and ultraproducts of* Cu*-semigroups: unperforation, almost unperforation, near unperforation.*

Proof. It is straightforward to verify that each of the considered properties passes to ideals and quotients. Thus, it is enough to consider products. We only prove the statement for unperforation, as the other situations are similar.

Let $(S_j)_{j \in J}$ be a family of unperforated Cu-semigroups, and let $x, y \in \prod_j S_j$ and $n \geq 1$ satisfy $nx \leq ny$. Choose $\mathbf{x}_t = (x_{t,j})_j$ and $\mathbf{y}_t = (y_{t,j})_j$ such that $x = [(\mathbf{x}_t)_{t \leq 0}]$ and $y = [(\mathbf{y}_t)_{t \leq 0}]$. Given $t < 0$, there is $s < 0$ such that $n\mathbf{x}_t \ll_{\mathrm{pw}} n\mathbf{y}_s$, that is, $nx_{t,j} \ll ny_{s,j}$ for all $j \in J$. Therefore $x_{t,j} \ll y_{s,j}$ for all j, which implies $\mathbf{x}_t \ll_{\mathrm{pw}} \mathbf{y}_s$. Hence, $x \leq y$, as desired. □

Corollary 8.16. *Given a family of C^*-algebras whose Cuntz semigroups are unperforated (almost unperforated, nearely unperforated), the Cuntz semigroups of their product, direct sum, and ultraproduct over some ultrafilter are also unperforated (almost unperforated, nearly unperforated).*

Proof. This is a direct consequence of Proposition 8.15, using Theorems 5.13, 6.1 or 7.5, and using that each of the properties passes to ideals, and hence from (ultra)products to scaled (ultra)products. □

Proposition 8.17. *Let $\mathcal{U}$ be an ultrafilter on a set J, and let $(S_j)_{j \in J}$ be a family of totally ordered* Cu*-semigroups. Then $\prod_{\mathcal{U}} S_j$ is totally ordered.*

Proof. Let $x, y \in \prod_j S_j$. We need to show that $\pi_{\mathcal{U}}(x) \leq \pi_{\mathcal{U}}(y)$ or $\pi_{\mathcal{U}}(y) \leq \pi_{\mathcal{U}}(x)$. Choose $\mathbf{x}_t = (x_{t,j})_j$ and $\mathbf{y}_t = (y_{t,j})_j$ such that $x = [(\mathbf{x}_t)_{t\leq 0}]$ and $y = [(\mathbf{y}_t)_{t\leq 0}]$. For every $\varepsilon > 0$ and $t < 0$ let us define the following set:

$$F_{\varepsilon,t} := \big\{ j \in J : x_{-\varepsilon,j} \leq y_{t,j} \big\}.$$

If $F_{\varepsilon,t} \in \mathcal{U}$ for some $\varepsilon > 0$ and $t < 0$, then $\pi_{\mathcal{U}}([(\mathbf{x}_{t-\varepsilon})_{t\leq 0}]) \leq \pi_{\mathcal{U}}([(\mathbf{y}_t)_{t\leq 0})$. Hence, if for all $\varepsilon > 0$ there is $t < 0$ with $F_{\varepsilon,t} \in \mathcal{U}$, then $\pi_{\mathcal{U}}(x) \leq \pi_{\mathcal{U}}(y)$. Otherwise, there is $\varepsilon > 0$ with $F_{\varepsilon,t} \notin \mathcal{U}$ for all $t < 0$. Since the S_j are totally ordered, we get

$$F_{\varepsilon,t}^c = \big\{ j \in J : y_{t,j} \leq x_{-\varepsilon,j} \big\} \in \mathcal{U},$$

for all $t < 0$, which then implies $\pi_{\mathcal{U}}(y) \leq \pi_{\mathcal{U}}(x)$. □

Corollary 8.18. *Let $\mathcal{U}$ be an ultrafilter over some set, and let A be a strongly self-absorbing, stably finite C^*-algebra satisfying the UCT. Then the Cuntz semigroup of the ultrapower $A_{\mathcal{U}}$ is totally ordered.*

Proof. By [APT18b, Paragraph 7.6.1, p.136], the Cuntz semigroup $\mathrm{Cu}(A)$ is totally ordered. By Proposition 8.17, $\prod_{\mathcal{U}} \mathrm{Cu}(A)$ is totally ordered. Now, the statement follows using that $\mathrm{Cu}(\prod_{\mathcal{U}} A)$ is an ideal in $\prod_{\mathcal{U}} \mathrm{Cu}(A)$; see Theorem 7.5. □

Remark 8.19. Recall that a functional on a Cu-semigroup S is a PoM-morphism $S \to [0,\infty]$ that preserves suprema of increasing sequences. Given a family $(S_j)_{j\in J}$ of Cu-semigroups and an ultrafilter $\mathcal{U}$ on J, it is natural to study functionals on the ultraproduct $\prod_{\mathcal{U}} S_j$. Given a functional $\varphi_j \colon S_j \to [0,\infty]$, for each $j \in J$, there is a naturally induced functional on $\prod_{\mathcal{U}} S_j$. The construction of this functional and the question whether the set of such functionals is dense in the set of all functionals on $\prod_{\mathcal{U}} S_j$ will be analysed in detail in forthcoming work, [APRT19].

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