

MINIMAL SETS OF LEFSCHETZ PERIODS OF THE MORSE–SMALE DIFFEOMORPHISMS ON $\mathbb{S}^n$ AND $\mathbb{S}^m \times \mathbb{S}^n$

CLARA CUFÍ-CABRÉ AND JAUME LLIBRE

ABSTRACT. We study the set of periods of the Morse–Smale diffeomorphisms on the n -dimensional sphere $\mathbb{S}^n$, and on the products of two spheres of arbitrary dimension $\mathbb{S}^m \times \mathbb{S}^n$. We classify the minimal sets of Lefschetz periods for such Morse–Smale diffeomorphisms. This characterization is done using the induced maps on the homology and the parity of the dimension of the manifolds. The main tool used is the Lefschetz zeta function.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In this paper we deal with discrete dynamical systems defined by a diffeomorphism on a compact manifold.

Let M be a compact manifold, and let $f : M \rightarrow M$ be a continuous map, and denote by f^m the m -th iterate of f . A point $x \in M$ such that $f(x) = x$ is called a *fixed point*, or a *periodic point of period 1* of f . A point $x \in M$ is called *periodic of period $k > 1$* if $f^k(x) = x$ and $f^m(x) \neq x$ for all $m = 1, \dots, k - 1$, and the set formed by the iterates of x , i.e. $\{x, f(x), \dots, f^{k-1}(x)\}$, is called the *periodic orbit* of the periodic point x .

As usual $\mathbb{N}$ denotes the set of all positive integers. Then $\text{Per}(f)$ is the set $\{k \in \mathbb{N} : f \text{ has a periodic orbit of period } k\}$.

A fixed point x of a C^1 map f is called *hyperbolic* if all the eigenvalues of $Df(x)$ have modulus different than one.

A periodic point x of f of period k is called a *hyperbolic periodic point* if it is a hyperbolic fixed point of f^k .

We denote by $\text{Diff}(M)$ the space of all C^1 diffeomorphisms on a compact Riemannian manifold M . As it is well known the set $\text{Diff}(M)$

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is a topological space endowed with the topology of the norm of the supremum with respect to f and its differential Df . Along this paper all the diffeomorphisms will be C^1 .

We say that two maps $f, g \in \text{Diff}(M)$ are *topologically equivalent* if there exists a homeomorphism $h : M \rightarrow M$ such that $h \circ f = g \circ h$. A map $f \in \text{Diff}(M)$ is called *structurally stable* if there exists a neighbourhood $U \subset \text{Diff}(M)$ of f such that every $g \in U$ is topologically equivalent to f .

An important class of diffeomorphisms from the dynamical point of view is the one formed by the *Morse–Smale diffeomorphisms*, which we will introduce next, because this family is structurally stable inside the class of all diffeomorphisms, see for details [7, 18, 19, 22].

We say that $x \in M$ is a *nonwandering point* of f if for any neighborhood U of x there is a positive integer m such that $f^m(U) \cap U \neq \emptyset$. We denote by $\Omega(f)$ the set of nonwandering points of f . Clearly if γ is a periodic orbit of f , then $\gamma \subseteq \Omega(f)$.

Let d be the metric on M induced by the norm of the supremum, and suppose that $x \in M$ is a hyperbolic fixed point of f . We define the *stable manifold* of x as the set

$$W^s(x) = \{y \in M : d(x, f^m(y)) \rightarrow 0 \text{ as } m \rightarrow \infty\},$$

and the *unstable manifold* of x as the set

$$W^u(x) = \{y \in M : d(x, f^{-m}(y)) \rightarrow 0 \text{ as } m \rightarrow \infty\}.$$

In the same way we define the stable and unstable manifolds of a hyperbolic periodic point $x \in M$ of period k as the stable and unstable manifolds of the hyperbolic fixed point x under f^k , respectively. We say that the submanifolds $W^s(x)$ and $W^u(x)$ have a *transversal intersection* if at every point of intersection, their separate tangent spaces at that point together generate the tangent space of the ambient manifold M at that point.

Definition 1. A diffeomorphism $f : M \rightarrow M$ is *Morse–Smale* if

- (1) $\Omega(f)$ is finite,
- (2) all the periodic points of f are hyperbolic, and
- (3) for every $x, y \in \Omega(f)$, $W^s(x)$ and $W^u(x)$ have a transversal intersection.

It is known that condition (1) implies that $\Omega(f)$ is the set of all periodic points of f , see again [7, 19, 22].

An important result about the class of Morse–Smale diffeomorphisms is the fact that they are *quasi unipotent* on homology (see [7, 20]). This means that if a homotopy class admits a Morse–Smale diffeomorphism, then all the eigenvalues of the nontrivial linear maps induced on the homology are roots of unity.

In the last quarter of the 20th century appeared some papers dedicated to understand the connections between the dynamics of the Morse–Smale diffeomorphisms and the topology of the manifold where they are defined, without trying to be exhaustive, see for instance [5, 17, 20, 21, 22]. This interest continues during this first part of the 21st century, see for example [2, 3, 8, 14, 15, 16].

The objective of this paper is to study the set of periods of the Morse–Smale diffeomorphisms on the n –dimensional sphere $\mathbb{S}^n$ and on products of two spheres of arbitrary dimension $\mathbb{S}^m \times \mathbb{S}^n$. More precisely, our goal is to describe the set $MPer_L(f)$ (see Definition 9) that those diffeomorphisms can exhibit for arbitrary values of n and m .

Our main results are Theorems 14, 15 and 20, where we characterize the possible sets for $MPer_L(f)$ in function of the action of f on the homology and on the parity of the dimensions n and m . For the sake of simplicity we state here the following weak versions of these theorems.

Theorem 2. *Let $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ be a Morse–Smale diffeomorphism. Then $MPer_L(f)$ is either $\emptyset$, or $\{1\}$.*

The set of periods for the Morse–Smale diffeomorphisms on the two dimensional sphere have been studied with more details in [1, 9].

Theorem 3. *Let $M = \mathbb{S}^m \times \mathbb{S}^n$, with $m \neq n$, and let $f : M \rightarrow M$ be a Morse–Smale diffeomorphism. Then $MPer_L(f)$ is either $\emptyset$, or $\{1\}$.*

Theorem 4. *Let $f : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{S}^n \times \mathbb{S}^n$ be a Morse–Smale diffeomorphism. Then $MPer_L(f)$ is either $\emptyset$, or $\{1\}$, or $\{1, 3\}$.*

The main tools for proving our results are the Lefschetz zeta function and the properties of the cyclotomic polynomials.

Related with these results the reader can look at the set of minimal periods for homeomorphisms (respectively continuous maps) on $\mathbb{S}^n$ and on $\mathbb{S}^m \times \mathbb{S}^n$ which have been studied in [11] (respectively [12]).

Finally we remark that the results obtained along this paper hold for Morse–Smale diffeomorphisms in any compact manifold with the same homology as the manifolds considered here. More precisely, they

hold for any manifold homotopic to $\mathbb{S}^n$ and $\mathbb{S}^m \times \mathbb{S}^n$, respectively. Furthermore the techniques here used for proving our results also could be used for determining the set of minimal Lefschetz periods on products of more spheres, namely, $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_r}$.

The paper is organized as follows. In Section 2 we recall the notions and some results on the Lefschetz numbers and the Lefschetz zeta function. Some notions and properties related with the minimal set of Lefschetz periods for Morse–Smale diffeomorphisms are considered in Section 3. In Section 6 we recall some properties of the cyclotomic polynomials. Finally, in Sections 4, 5 and 7 we prove the strong versions of Theorems 2, 3 and 4.

2. LEFSCHETZ NUMBERS AND THE LEFSCHETZ ZETA FUNCTION

Let $f : M \rightarrow M$ be a continuous map on a compact manifold of dimension n . We denote by $H_0(M, \mathbb{Q}), \dots, H_n(M, \mathbb{Q})$ the homology groups of M with rational coefficients. A continuous map $f : M \rightarrow M$ induces $n + 1$ morphisms on the homology groups of M ,

$$f_{*i} : H_i(M, \mathbb{Q}) \rightarrow H_i(M, \mathbb{Q}), \quad i \in \{0, \dots, n\}.$$

If $[\sigma]$ is an element of $H_i(M)$, then the i -th induced map on the homology f_{*i} is given by $f_{*i}([\sigma]) = [f \circ \sigma]$. The homology groups $H_0(M, \mathbb{Q}), \dots, H_n(M, \mathbb{Q})$ are finite dimensional vector spaces on $\mathbb{Q}$, and hence, once we choose a basis, each morphism f_{*i} can be written as a matrix of integer coefficients with respect to this basis. For more details, see [24].

Definition 5. Let M be a compact manifold and let $f : M \rightarrow M$ be a continuous map. The *Lefschetz number* of f is defined as

$$L(f) = \sum_{i=0}^n (-1)^i \operatorname{tr}(f_{*i}),$$

where $\operatorname{tr}(f_{*i})$ denotes the trace of f_{*i} .

A very important result which relates the Lefschetz number of a map f and the existence of fixed points of f is the Lefschetz Fixed Point Theorem, for a proof see [4],

Theorem 6 (Lefschetz fixed point theorem). *Let $f : M \rightarrow M$ be a continuous map on a compact manifold. If $L(f) \neq 0$, then f has a fixed point.*

In order to obtain information about the set of periods of a map $f : M \rightarrow M$, it is useful to have information of the sequence $\{L(f^m)\}_{m=0}^{\infty}$ of the Lefschetz numbers of all the iterates of f . For this purpose it has been introduced the *Lefschetz zeta function*,

Definition 7. Let $f : M \rightarrow M$ be a continuous map on a compact manifold. The Lefschetz zeta function of f is defined as

$$\mathcal{Z}_f(t) = \exp \left(\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m \right).$$

This function generates the whole sequence of Lefschetz numbers, and it may be independently computed through

$$(1) \quad \mathcal{Z}_f(t) = \prod_{k=0}^n \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}},$$

(see for more details [6]), where n is the dimension of M , n_k is the dimension of $H_k(M, \mathbb{Q})$, I_{n_k} is the $n_k \times n_k$ identity matrix, and we take $\det(I_{n_k} - t f_{*k}) = 1$ if $n_k = 0$. Note that the function given in (1) is a rational function of t . Hence the information about the infinite sequence of Lefschetz numbers of f is contained in two polynomials with integer coefficients.

For a map f having a finite number of periodic points, all of them being hyperbolic, we give another characterization of the Lefschetz zeta function introduced by Franks in [6].

Let $f : M \rightarrow M$ be a C^1 map on a compact manifold without boundary, and let γ be a hyperbolic periodic orbit of f of period p . For each $x \in \gamma$, let E_x^u denote the linear subspace of the tangent space $T_x M$ of M at x , generated by the eigenvectors of Df^p corresponding to the eigenvalues whose moduli are greater than 1, and let E_x^s denote the linear subspace of $T_x M$ generated by the remaining eigenvectors. We denote by u and s the dimensions of the spaces E_x^u and E_x^s , respectively.

We define the *orientation type* Δ of γ to be $+1$ if $Df^p(x) : E_x^u \rightarrow E_x^u$ preserves orientation, that is, if $\det Df^p(x) > 0$ with $x \in \gamma$, and to be -1 if it reverses orientation, that is, if $\det Df^p(x) < 0$. Note that the definitions of Δ and u do not depend on the periodic point x , but only on the periodic orbit γ .

For a C^1 map $f : M \rightarrow M$ having only finitely many periodic orbits, all of them hyperbolic, we define the *periodic data* as the collection composed by all triples (p, u, Δ) corresponding to all the hyperbolic periodic orbits of f , where a same triple can occur more than once

provided that it corresponds to different periodic orbits. The following result was proved by Franks in [6].

Theorem 8. *Let $f : M \rightarrow M$ be a C^1 map on a compact manifold without boundary, having finitely many periodic points, all hyperbolic, and let Σ be the periodic data of f . Then the Lefschetz zeta function of f satisfies*

$$\mathcal{Z}_f(t) = \prod_{(p, u, \Delta) \in \Sigma} (1 - \Delta t^p)^{(-1)^{u+1}}.$$

This result will be useful to obtain information on the set of periods of a given map f from the comparison of the expressions of the Lefschetz zeta function obtained from the definition and from Theorem 8.

3. MINIMAL SET OF LEFSCHETZ PERIODS OF MORSE–SMALE DIFFEOMORPHISMS

Definition 9. Let f be a map satisfying the hypotheses of Theorem 8. The *minimal set of Lefschetz periods of f* , $\text{MPer}_L(f)$, is the set given by the intersection of all sets of periods forced by the different representations of $\mathcal{Z}_f(t)$ as products of the form $(1 \pm t^p)^{\pm 1}$.

As an example we consider the following Lefschetz zeta function of a Morse–Smale diffeomorphism f on the four-dimensional torus $\mathbb{T}^4$,

$$\mathcal{Z}_f^{(1)}(t) = \frac{(1 - t^3)^2(1 + t^3)}{(1 - t)^6(1 + t)^3},$$

see for instance [10]. Note that $\mathcal{Z}_f^{(1)}(t)$ can also be expressed as products of the form $(1 \pm t^p)^{\pm 1}$ as a quotient of two polynomials of degree 9 in the following different ways,

$$\begin{aligned} \mathcal{Z}_f^{(2)}(t) &= \frac{(1 - t^3)(1 - t^6)}{(1 - t)^6(1 + t)^3}, \\ \mathcal{Z}_f^{(3)}(t) &= \frac{(1 - t^3)(1 - t^6)}{(1 - t)^3(1 - t^2)^3}, \\ \mathcal{Z}_f^{(4)}(t) &= \frac{(1 - t^3)^2(1 + t^3)}{(1 - t)^3(1 - t^2)^3}, \end{aligned}$$

where clearly $\mathcal{Z}_f^{(1)}(t) = \mathcal{Z}_f^{(2)}(t) = \mathcal{Z}_f^{(3)}(t) = \mathcal{Z}_f^{(4)}(t)$, and one can also express $\mathcal{Z}_f^{(k)}(t) = g(t)\mathcal{Z}_f^{(k)}(t)/g(t)$, for any $g(t)$ given by a product of terms of the form $(1 \pm t^p)^{\pm 1}$.

Since f is a Morse–Smale diffeomorphism, it satisfies the hypotheses of Theorem 8. Then the expression of the Lefschetz zeta function given by $\mathcal{Z}_f^{(1)}(t)$ or by $g(t)\mathcal{Z}_f^{(1)}(t)/g(t)$ will ensure the existence of periodic orbits of periods 1 and 3 for f . In the same way, the expression of $\mathcal{Z}_f^{(2)}(t)$ will provide the periods $\{1, 3, 6\}$ for f , the expression of $\mathcal{Z}_f^{(3)}(t)$ will provide the periods $\{1, 2, 3, 6\}$, and finally $\mathcal{Z}_f^{(4)}(t)$ will provide the periods $\{1, 2, 3\}$.

If a certain period p appears in all the possible expressions of the Lefschetz zeta function of f written as a product of elements of the form $(1 \pm t^p)^{\pm 1}$, then f will have a periodic orbit of that period. More precisely, in this case we have that the minimal set of Lefschetz periods of f is

$$\text{MPer}_L(f) = \{1, 3\} \cap \{1, 3, 6\} \cap \{1, 2, 3, 6\} \cap \{1, 2, 3\} = \{1, 3\}.$$

Note that if $\mathcal{Z}_f(t)$ is constant equal to 1, then $\text{MPer}_L(f) = \emptyset$.

For a Morse–Smale diffeomorphism $f : M \rightarrow M$ on a compact manifold we define the *minimal set of periods of f* , $\text{MPer}_{ms}(f)$, as

$$\text{MPer}_{ms} = \bigcap_{h \sim f} \text{Per}(h),$$

where h runs over all the Morse–Smale diffeomorphisms of M which are homotopic to f .

It is clear that any Morse–Smale diffeomorphism f on a compact manifold without boundary satisfies the hypotheses of Theorem 8. Then, from the definition of the minimal set of Lefschetz periods, it always follows that

$$\text{MPer}_L(f) \subseteq \text{MPer}_{ms}(f),$$

because two homotopic maps on a manifold M induce the same maps on the homology groups of M , and consequently they induce the same Lefschetz zeta functions.

Remark 10. By Theorem 8 an even period n , can never be contained in the set $\text{MPer}_L(f)$. Indeed, the following expressions

$$1 - t^n = (1 + t^{n/2})(1 - t^{n/2}), \quad 1 + t^n = \frac{1 - t^{2n}}{1 - t^n} = \frac{1 - t^{2n}}{(1 + t^{n/2})(1 - t^{n/2})},$$

show that if the term $1 - t^n$ or $1 + t^n$, with n even, appears in one of the expressions of $\mathcal{Z}_f(t)$, one can always obtain a new expression of $\mathcal{Z}_f(t)$ that does not contain the period n . This implies that the period n can

not be in the intersection of the set of periods forced by the different expressions of $\mathcal{Z}_f(t)$, and thus, n can never be an element of $\text{MPer}_L(f)$.

Remark 11. Along the paper, for every possible Lefschetz zeta function of a given map f , in general we will write only one of the possible equivalent expressions of $\mathcal{Z}_f(t)$. We will provide the expression of $\mathcal{Z}_f(t)$ that, by Theorem 8, forces a smaller set of periods, and thus, that is sufficient to describe the set $\text{MPer}_L(f)$. For example, consider the Lefschetz zeta function given by

$$(2) \quad \frac{1}{(1-t^2)(1+t^2)} = \frac{1}{1-t^4} = \frac{1}{(1-t)(1+t)(1+t^2)},$$

where the first expression provides the period 2, the second one provides the period 4 and the third one provides the periods 1 and 2. We have then $\text{MPer}_L(f) = \{2\} \cap \{4\} \cap \{1, 2\} = \emptyset$.

Suppose that an expression for a Lefschetz zeta function $\mathcal{Z}_f(t)$ forces only one odd period, p . Then, it is clear that we will have either $\text{MPer}_L(f) = \{p\}$ or $\text{MPer}_L(f) = \emptyset$, since we do not need to consider the even periods. To discard the first case, it is sufficient to find another expression for $\mathcal{Z}_f(t)$ that does not force the period p . In (2) we could have $\text{MPer}_L(f) = \{1\}$ from the last expression, but the other two expressions force to discard this option. Thus, in this case, providing only the expression $\frac{1}{(1-t^2)(1+t^2)}$ or $\frac{1}{1-t^4}$ is already sufficient to ensure that $\text{MPer}_L(f) = \emptyset$.

Remark 12. Note that even if the minimal set of Lefschetz periods of a Morse–Smale diffeomorphism is empty, one can still obtain some information about the set of periods from Theorem 8. For example, suppose that the Lefschetz zeta function of a map f satisfying the hypotheses of Theorem 8 is $\mathcal{Z}_f(t) = 1 + t^2$. This function can be expressed as products of terms of the form $(1 \pm t^p)^{\pm 1}$ in infinitely many ways,

$$1 + t^2 = \frac{1 - t^4}{1 - t^2} = \frac{1 - t^4}{(1+t)(1-t)} = \frac{1 - t^8}{(1+t)(1-t)(1+t^4)},$$

and so on. In this case it is clear that $\text{MPer}_L(f) = \emptyset$, but by Remark 10 one can see that each of the infinitely many expressions of $\mathcal{Z}_f(t)$ forces either the period 2 or the periods $\{2, 4\}$, or the periods $\{1, 4\}$. Hence, by Theorem 8 we have that f has either a periodic orbit of period 2, or periodic orbits of periods 2 and 4, or periodic orbits of periods 1 and 4.

Remark 13. We note that if f preserves the orientation, then we have $\Delta = 1$ for all the periodic orbits of f . If f reverses the orientation,

then we have $\Delta = -1$ for periodic orbits of odd period and $\Delta = 1$ for periodic orbits of even period.

4. PERIODS OF MORSE-SMALE DIFFEOMORPHISMS ON $\mathbb{S}^n$

Let $n \in \mathbb{N}$ and let $\mathbb{S}^n$ be the n -dimensional sphere. Our aim is to describe all the possible minimal sets of Lefschetz periods for the Morse-Smale diffeomorphisms on $\mathbb{S}^n$. The main tool that we will use is the Lefschetz zeta function and Theorem 8.

The homology groups of $\mathbb{S}^n$ over $\mathbb{Q}$ are

$$H_k(\mathbb{S}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k \in \{0, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

For a continuous map $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ the nontrivial induced maps on the homology can be written as the integer matrices $f_{*0} = (1)$ and $f_{*n} = (d)$, where d is called the *degree* of f .

Let $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ be a Morse-Smale diffeomorphism. As we already mentioned before, if a homotopy class of maps admits a Morse-Smale diffeomorphism, then the linear maps induced on the homology are quasi unipotent, which means that all their eigenvalues are roots of unity. Then we must have either $d = 1$, or $d = -1$.

Since f is a diffeomorphism, the function $\det Df(x)$ has constant sign on $\mathbb{S}^n$, and so the orientation of f is constant on $\mathbb{S}^n$ and it is determined by the sign of the degree d of f . More precisely, if $d = 1$ then f is an orientation-preserving diffeomorphism, and if $d = -1$ then f is an orientation-reversing diffeomorphism.

The following theorem is the main result of this section.

Theorem 14. *Let $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ be a Morse-Smale diffeomorphism. Then,*

- (a) *If n is even and f preserves the orientation, then $MPer_L(f) = 1$.*
- (b) *If n is even and f reverses the orientation, then $MPer_L(f) = \emptyset$ but $\{1, 2\} \cap Per(f) \neq \emptyset$.*
- (c) *If n is odd and f preserves the orientation, then $MPer_L(f) = \emptyset$.*
- (d) *If n is odd and f reverses the orientation, then $MPer_L(f) = 1$.*

Proof. Computing the Lefschetz zeta function for f using equation (1), we get

$$\mathcal{Z}_f(t) = \prod_{k=0}^n \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}} = (1-t)^{-1} (1-td)^{(-1)^{n+1}},$$

and so for n even we have

$$\begin{aligned} \mathcal{Z}_f(t) &= \frac{1}{(1-t)^2} \text{ if } d = 1, \\ \mathcal{Z}_f(t) &= \frac{1}{1-t^2} = \frac{1}{(1-t)(1+t)} \text{ if } d = -1, \end{aligned}$$

and for n odd we have

$$\begin{aligned} \mathcal{Z}_f(t) &\equiv 1 \text{ if } d = 1, \\ \mathcal{Z}_f(t) &= \frac{1+t}{1-t} \text{ if } d = -1. \end{aligned}$$

It is clear that f satisfies the hypotheses of Theorem 8. Then the results of the statements of the theorem follow directly applying Theorem 8 to each of the expressions obtained for the Lefschetz zeta function $\mathcal{Z}_f(t)$.

For $1/(1-t)^2$ and $(1+t)/(1-t)$ we have $\text{MPer}_L(f) = 1$, because any other expression of the same Lefschetz zeta function as a product of terms the form $(1 \pm t^p)^{\pm 1}$ would provide at least the period 1.

For the function $1/(1-t^2) = 1/((1-t)(1+t))$ it is clear that $\text{MPer}_L(f) = \emptyset$, but taking into account the considerations of Remarks 10 and 12, we can ensure that $\{1, 2\} \cap \text{Per}(f) \neq \emptyset$. Finally we have $\text{MPer}_L(f) = \emptyset$ when $\mathcal{Z}_f(t) \equiv 1$. $\square$

5. PERIODS OF MORSE–SMALE DIFFEOMORPHISMS ON $\mathbb{S}^m \times \mathbb{S}^n$, $m \neq n$

In this section we describe the minimal sets of Lefschetz periods for Morse–Smale diffeomorphisms on a product of two spheres of arbitrary dimension, $\mathbb{S}^m \times \mathbb{S}^n$ with $m \neq n$.

For every $m, n \in \mathbb{N}$, $m \neq n$, the homology groups over $\mathbb{Q}$ of $\mathbb{S}^m \times \mathbb{S}^n$ are given by

$$H_k(\mathbb{S}^m \times \mathbb{S}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k \in \{0, m, n, m+n\}, \\ 0 & \text{otherwise.} \end{cases}$$

For a continuous map $f : \mathbb{S}^m \times \mathbb{S}^n \rightarrow \mathbb{S}^m \times \mathbb{S}^n$, we can write the expressions for the linear maps induced on the homology as the integer matrices $f_{*0} = (1)$, $f_{*m} = (a)$, $f_{*n} = (b)$, $f_{*m+n} = (d)$, where d is the degree of f . Clearly the rest of the induced maps are the zero map.

Let $f : \mathbb{S}^m \times \mathbb{S}^n \rightarrow \mathbb{S}^m \times \mathbb{S}^n$ be a Morse–Smale diffeomorphism. Since the linear maps induced by f on the homology are quasi unipotent, in this case we have $a, b, d \in \{-1, 1\}$.

The following theorem is the main result of this section.

Theorem 15. *Let $M = \mathbb{S}^m \times \mathbb{S}^n$ be with $m \neq n$, and let $f : M \rightarrow M$ be a Morse–Smale diffeomorphism.*

- (i) *If m and n are even, then*
 - (a) *If $\{a, b, d\} = \{1, -1, -1\}$, then $MPer_L(f) = \emptyset$ but $\{1, 2\} \cap Per(f) \neq \emptyset$.*
 - (b) *Otherwise, $MPer_L(f) = 1$.*
- (ii) *If m and n are odd, then*
 - (a) *If $a = b = d = 1$ or $\{a, b\} = \{-1, 1\}$ and $d = -1$, then $MPer_L(f) = \emptyset$.*
 - (b) *Otherwise, $MPer_L(f) = 1$.*
- (iii) *If m is even and n is odd, then*
 - (a) *If $a = b = d = 1$ or $\{b, d\} = \{-1, 1\}$ and $a = -1$, then $MPer_L(f) = \emptyset$.*
 - (b) *Otherwise, $MPer_L(f) = 1$.*
- (iv) *If m is odd and n is even, then*
 - (a) *If $a = b = d = 1$ or $\{a, d\} = \{-1, 1\}$ and $b = -1$, then $MPer_L(f) = \emptyset$.*
 - (b) *Otherwise, $MPer_L(f) = 1$.*

Proof. Computing the Lefschetz zeta function for f using equation (1) we obtain

$$\begin{aligned} \mathcal{Z}_f(t) &= \prod_{k=0}^{m+n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}} \\ &= (1-t)^{-1} (1-at)^{(-1)^{m+1}} (1-bt)^{(-1)^{n+1}} (1-dt)^{(-1)^{m+n+1}}. \end{aligned}$$

Depending on whether m and n are even or odd, and for each value of $a, b, d \in \{-1, 1\}$, the expressions obtained for the Lefschetz zeta functions in each case are displayed in Tables 1, 2, 3 and 4.

The proof follows directly from the information obtained from the tables and applying Theorem 8, taking into account the considerations of Remarks 10, 11 and 12. $\square$

Values for a, b, d	$\mathcal{Z}_f(t)$
$a = b = d = 1$	$\frac{1}{(1-t)^4}$
$a = b = d = -1$	$\frac{1}{1-t} \frac{1}{(1+t)^3}$
$\{a, b, d\} = \{1, 1, -1\}$	$\frac{1}{(1-t)^3} \frac{1}{1+t}$
$\{a, b, d\} = \{1, -1, -1\}$	$\frac{1}{(1-t)^2} \frac{1}{(1+t)^2} = \frac{1}{(1-t^2)^2}$

Table 1. $\mathcal{Z}_f(t)$ for a Morse–Smale diffeomorphism f on $\mathbb{S}^m \times \mathbb{S}^n$, with $m \neq n$ and n, m even.

Values for a, b, d	$\mathcal{Z}_f(t)$
$a = b = d = 1$	1
$a = b = d = -1$	$\frac{1+t}{1-t}$
$a = b = 1, d = -1$	$\frac{1-t}{1+t}$
$a = b = -1, d = 1$	$\frac{(1+t)^2}{(1-t)^2}$
$\{a, b\} = \{-1, 1\}, d = -1$	1
$\{a, b\} = \{-1, 1\}, d = 1$	$\frac{1+t}{1-t}$

Table 2. $\mathcal{Z}_f(t)$ for a Morse–Smale diffeomorphism f on $\mathbb{S}^m \times \mathbb{S}^n$, with $m \neq n$ and m, n odd.

Values for a, b, d	$\mathcal{Z}_f(t)$
$a = b = d = 1$	1
$a = b = d = -1$	$\frac{1+t}{1-t}$
$b = d = 1, a = -1$	$\frac{1-t}{1+t}$
$b = d = -1, a = 1$	$\frac{(1+t)^2}{(1-t)^2}$
$\{b, d\} = \{-1, 1\}, a = 1$	$\frac{1+t}{1-t}$
$\{b, d\} = \{-1, 1\}, a = -1$	1

Table 3. $\mathcal{Z}_f(t)$ for a Morse–Smale diffeomorphism f on $\mathbb{S}^m \times \mathbb{S}^n$, with $m \neq n$ and m even, n odd.

6. CYCLOTOMIC POLYNOMIALS

We shall need the class of the cyclotomic polynomials and some of their properties to study the Lefschetz zeta function of the Morse–Smale diffeomorphisms on $\mathbb{S}^n \times \mathbb{S}^n$.

Values for a, b, d	$\mathcal{Z}_f(t)$
$a = b = d = 1$	1
$a = b = d = -1$	$\frac{1+t}{1-t}$
$a = d = 1, b = -1$	$\frac{1-t}{1+t}$
$a = d = -1, b = 1$	$\frac{(1+t)^2}{(1-t)^2}$
$\{a, d\} = \{-1, 1\}, b = 1$	$\frac{1+t}{1-t}$
$\{a, d\} = \{-1, 1\}, b = -1$	1

Table 4. $L(f)$, $l(f^2)$ and $\mathcal{Z}_f(t)$ for a Morse–Smale diffeomorphism on $\mathbb{S}^m \times \mathbb{S}^n$, with $m \neq n$ and m odd, n even.

The n –th *cyclotomic polynomial* is defined as the monic polynomial with integer coefficients that is the minimal polynomial of any primitive n –th root of unity. That is, the monic polynomial of lowest degree with integer coefficients that has any primitive n –th root of unity as a root.

The n –th cyclotomic polynomial can be written as

$$c_n(t) = \prod_{1 \leq k \leq n : \gcd(k, n) = 1} (e^{\frac{2\pi i k}{n}} - t),$$

where the set $\{e^{\frac{2\pi i k}{n}} : 1 \leq k \leq n \text{ and } \gcd(k, n) = 1\}$ is formed by the primitive n –th roots of the unity.

An alternative way to express $c_n(t)$ is

$$c_n(t) = \frac{1 - t^n}{\prod_{d|n, d < n} c_d(t)},$$

from which is clear that c_n is the unique irreducible polynomial that is a divisor of $x^n - 1$ but is not a divisor of $x^k - 1$ for any $k < n$.

The degree of the cyclotomic polynomial $c_n(t)$ is given by the Euler’s totient function, $\varphi(n) = n \prod_{p|n} (1 - 1/p)$, where the product is over the distinct prime numbers p dividing n . It is known that $\varphi(n) \geq 4$ for $n > 6$.

For more information about the cyclotomic polynomials see for instance [13, 23].

The following propositions, which will be useful for our purposes, provide information relating cyclotomic polynomials and the characteristic polynomials of quasi unipotent matrices. Since the proofs are short, we provide them for completeness.

Proposition 16. *For $0 < k < n$, w_k is a root of $c_n(t)$ if and only if w_k^{-1} is also a root of $c_n(t)$.*

Proof. The roots of $c_n(t)$ are given by $w_k = e^{\frac{2\pi i k}{n}}$ for every k coprime with respect to n . Since $w_k^{-1} = e^{-\frac{2\pi i k}{n}}$, one has then $w_k^{-1} = e^{\frac{2\pi i (n-k)}{n}}$. Also, it is clear that k and n are coprime if and only if $n - k$ and n are coprime. Therefore the proposition follows. $\square$

Proposition 17. *Let A be a $n \times n$ integer matrix such that all its eigenvalues are roots of unity. Then the characteristic polynomial of A is product of cyclotomic polynomials.*

Proof. Let $p(t)$ be the characteristic polynomial of A , and denote by $\{\alpha_1, \dots, \alpha_n\}$ the roots of $p(t)$, where each α_i is a k_i -th primitive root of unity for some k_i depending on α_i . In the set $\{\alpha_1, \dots, \alpha_n\}$ a root appears repeated as many times as its multiplicity in the polynomial $p(t)$.

Let $m_1(t)$ be the polynomial with integer coefficients of minimal degree having α_1 as a root. Then $m_1(t)$ is a cyclotomic polynomial. Moreover $m_1(t)$ must divide $p(t)$. Indeed, writing

$$p(\alpha_1) = m_1(\alpha_1)q_1(\alpha_1) + r_1(\alpha_1),$$

where $q_1(t)$ denotes the quotient of the division of $p(t)$ by $m_1(t)$ in the polynomial ring $\mathbb{Z}[t]$, and $r_1(t)$ denotes the remainder, we get $r_1(\alpha_1) = 0$. Hence we have $r_1(t) \equiv 0$, because the degree of the polynomial $r_1(t)$ has to be lower than the degree of $m_1(t)$.

Thus we have $p(t) = m_1(t)q_1(t)$, where $q_1(t)$ is a polynomial having the roots of the set $R = \{\alpha_2, \dots, \alpha_n\}$ which are not roots of $m_1(t)$. More precisely, all the roots of $m_1(t)$ are simple, α_1 is a root of $m_1(t)$ and all the other roots of $m_1(t)$ are also roots of $p(t)$ and consequently we must remove them from the set R . Denote by S the roots which have not been removed from R .

Now repeating the same process that we have done with the root α_1 and the polynomial $p(t)$, but now for the first root α_i of the set S and the polynomial $q_1(t)$, there is a cyclotomic polynomial $m_2(t)$ such that $q_1(t) = m_2(t)q_2(t)$. Applying a finite number of times these arguments we obtain that $p(t)$ is a product of cyclotomic polynomials. $\square$

Proposition 18. *Let A be a $n \times n$ integer matrix such that all its eigenvalues are roots of unity. Then*

$$\begin{aligned} \det(I_n - tA) &= \det(A - tI_n) & \text{if } \det A = 1, \\ \det(I_n - tA) &= -\det(A - tI_n) & \text{if } \det A = -1. \end{aligned}$$

Proof. Let $p(t) = \det(A - tI_n)$ be the characteristic polynomial of A , and $q(t) = \det(I_n - tA)$. The polynomials $p(t)$ and $q(t)$ are related by

$$(3) \quad (-1)^n q(t) = t^n p(t^{-1}).$$

Since A is quasi unipotent, by Proposition 17 we have that $p(t)$ is product of cyclotomic polynomials. So, by Proposition 16 and (3) it follows that the polynomials $p(t)$ and $q(t)$ have the same roots with the same multiplicity.

Since $p(t)$ is monic, we have $q(t) = \lambda p(t)$, for some $\lambda \in \mathbb{Z}$. To determine λ , note that $1 = q(0) = \lambda p(0) = \lambda \det(A)$, and so $\lambda = (\det(A))^{-1}$. Since all the eigenvalues of A are roots of unity and A is an integer matrix, then either $\det A = 1$ or $\det A = -1$. $\square$

We display in Table 5 the list of the first 15 cyclotomic polynomials.

$c_1(t) = 1 - t$	$c_2(t) = 1 + t$	$c_3(t) = \frac{1 - t^3}{1 - t}$
$c_4(t) = 1 + t^2$	$c_5(t) = \frac{1 - t^5}{1 - t}$	$c_6(t) = \frac{1 + t^3}{1 + t}$
$c_7(t) = \frac{1 - t^7}{1 - t}$	$c_8(t) = 1 + t^4$	$c_9(t) = \frac{1 - t^9}{1 - t^3}$
$c_{10}(t) = \frac{1 + t^5}{1 + t}$	$c_{11}(t) = \frac{1 - t^{11}}{1 - t}$	$c_{12}(t) = \frac{1 + t^6}{1 + t^2}$
$c_{13}(t) = \frac{1 - t^{13}}{1 - t}$	$c_{14}(t) = \frac{1 + t^7}{1 + t}$	$c_{15}(t) = \frac{(1 - t^{15})(1 - t)}{(1 - t^3)(1 - t^5)}$

Table 5. The first 15 cyclotomic polynomials.

7. PERIODS OF MORSE-SMALE DIFFEOMORPHISMS ON $\mathbb{S}^n \times \mathbb{S}^n$

In this section we consider Morse–Smale diffeomorphisms on a product of two spheres of the same dimension, $\mathbb{S}^n \times \mathbb{S}^n$, for any $n \in \mathbb{N}$.

The homology groups over $\mathbb{Q}$ of $\mathbb{S}^n \times \mathbb{S}^n$ are given by

$$H_k(\mathbb{S}^n \times \mathbb{S}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k \in \{0, 2n\}, \\ \mathbb{Q} \oplus \mathbb{Q} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{S}^n \times \mathbb{S}^n$ be a Morse–Smale diffeomorphism. Then, the induced map on the homology $f_{0*} : \mathbb{Q} \rightarrow \mathbb{Q}$ is the identity map and so it is given by the integer matrix (1). The other nontrivial maps induced on the homology are $f_{*n} : \mathbb{Q} \oplus \mathbb{Q} \rightarrow \mathbb{Q} \oplus \mathbb{Q}$, which can be represented by a 2×2 matrix with integer coefficients, and $f_{*2n} : \mathbb{Q} \rightarrow \mathbb{Q}$, which can be represented by the integer matrix (d), where d is the degree of f .

Since the linear maps induced on the homology by a Morse–Smale diffeomorphism are quasi unipotent, it holds that $d \in \{-1, 1\}$, and that the eigenvalues of f_{*n} are roots of unity. However now there are infinitely many possible values for the coefficients of f_{*n} with respect to a chosen basis. Thus, to characterize the possible Lefschetz zeta functions of a Morse–Smale diffeomorphism on $\mathbb{S}^n \times \mathbb{S}^n$ we will use Propositions 17 and 18.

In the next proposition we give all the possible expressions, up to their sign, for the Lefschetz zeta function of a Morse–Smale diffeomorphism, $f : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{S}^n \times \mathbb{S}^n$, depending on the parity of n and on the degree of f , which determines when f preserves or not the orientation.

Proposition 19. *Let $f : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{S}^n \times \mathbb{S}^n$ be a Morse–Smale diffeomorphism, and let d be its degree. The following Lefschetz zeta functions $\mathcal{Z}_f(t)$ are given up to their sign.*

(a) *If n is even and $d = 1$, then the possible $\mathcal{Z}_f(t)$ are*

$$\begin{aligned} & \frac{1}{(1-t)^4}; \quad \frac{1}{(1-t^2)^2}; \quad \frac{1}{(1-t)^3(1+t)}; \\ & \frac{1}{(1-t)(1-t^3)}; \quad \frac{1}{(1-t)^2(1+t^2)}; \quad \frac{1+t}{(1-t)^2(1+t^3)}. \end{aligned}$$

(b) *If n is even and $d = -1$, then the possible $\mathcal{Z}_f(t)$ are*

$$\begin{aligned} & \frac{1}{(1-t)^3(1+t)}; \quad \frac{1}{(1+t)^3(1-t)}; \quad \frac{1}{(1-t^2)^2}; \\ & \frac{1}{(1-t^3)(1+t)}; \quad \frac{1}{1-t^4}; \quad \frac{1}{(1-t)(1+t^3)}. \end{aligned}$$

(c) If n is odd and $d = 1$, then the possible $\mathcal{Z}_f(t)$ are

$$1; \quad \frac{(1+t)^2}{(1-t)^2}; \quad \frac{1+t}{1-t}; \quad \frac{1-t^3}{(1-t)^3}; \quad \frac{1+t^2}{(1-t)^2}; \quad \frac{1+t^3}{(1+t)(1-t)^2}.$$

(d) If n is odd and $d = -1$, then the possible $\mathcal{Z}_f(t)$ are

$$\frac{1-t}{1+t}; \quad \frac{1+t}{1-t}; \quad 1; \quad \frac{1-t^3}{(1-t)^2(1+t)}; \quad \frac{1+t^2}{1-t^2}; \quad \frac{1+t^3}{(1+t)^2(1-t)}.$$

Proof. From (1) the Lefschetz zeta function of f is

$$\mathcal{Z}_f(t) = \prod_{k=0}^{2n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}} = \frac{\det(I_2 - t f_{*n})^{(-1)^{n+1}}}{(1-t)(1-dt)}.$$

Since f is a Morse–Smale diffeomorphism, the linear maps induced on the homology are quasi unipotent, and so we have $d \in \{-1, 1\}$ and that the eigenvalues of f_{*n} are roots of unity. Thus, by Proposition 17, the characteristic polynomial $\det(f_{*n} - t I_2)$ of f_{*n} is product of cyclotomic polynomials, and from Proposition 18 we have that $\det(I_2 - t f_{*n}) = \det(f_{*n}) \det(f_{*n} - t I_2)$, where $\det(f_{*n}) \in \{-1, 1\}$. Therefore

$$\mathcal{Z}_f(t) = \det(f_{*n}) \frac{\det(f_{*n} - t I_2)^{(-1)^{n+1}}}{(1-t)(1-dt)},$$

where $d \in \{-1, 1\}$ and where $\det(f_{*k} - t I_2)$ is a product of cyclotomic polynomials.

Since $\det(f_{*k} - t I_2)$ is a polynomial of degree 2, to compute all its possible expressions it is sufficient to consider the product of all the cyclotomic polynomials such that the degree of the product be equal to 2. Then $\mathcal{Z}_f(t)$ is given, up to the sign, by

$$\mathcal{Z}_f(t) = \pm \frac{(\prod_k c_k(t))^{(-1)^{n+1}}}{(1-t)(1-dt)},$$

where k runs over the positive integers that satisfy $\sum_k \deg(c_k) = 2$, being $\deg(c_k)$ the degree of the cyclotomic polynomial c_k .

Since the degree of the cyclotomic polynomial $c_n(t)$ satisfies $\deg(c_n) = \varphi(n) \geq 4$ for $n > 6$, from the list of cyclotomic polynomials displayed in Table 5 it follows that such a product $\prod_k c_k$ can only be given by the products c_1^2 , c_2^2 , $c_1 c_2$, c_3 , c_4 , or c_6 . Then taking into account when n is even or odd and the sign of d we obtain the six different expressions for $\mathcal{Z}_f(t)$ in each of the statements of the proposition. $\square$

Note that in order to study the set of periods of f using Theorem 8, we do not need to take into account the sign of the Lefschetz zeta function of f .

In the following theorem, which is the main result of this section, we characterize the set $MPer_L(f)$ of the Morse–Smale diffeomorphisms on $\mathbb{S}^n \times \mathbb{S}^n$ using Theorem 8 and the information about $\mathcal{Z}_f(t)$ obtained in Proposition 19.

Theorem 20. *Let $f : \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{S}^n \times \mathbb{S}^n$ be a Morse–Smale diffeomorphism. Then the following statements hold.*

(a) *If n is even and $d = 1$, then $MPer_L(f)$ is*

$$\begin{aligned} \emptyset & \text{ if } \mathcal{Z}_f(t) = \pm \frac{1}{(1-t^2)^2}; \\ \{1\} & \text{ if } \mathcal{Z}_f(t) \in \left\{ \pm \frac{1}{(1-t)^4}, \pm \frac{1}{(1-t)^3(1+t)}, \pm \frac{1}{(1-t)^2(1+t^2)} \right\}; \\ \{1, 3\} & \text{ if } \mathcal{Z}_f(t) \in \left\{ \pm \frac{1}{(1-t)(1-t^3)}, \pm \frac{1+t}{(1-t)^2(1+t^3)} \right\}, \end{aligned}$$

but if $\mathcal{Z}_f(t) = \pm 1/(1-t^2)^2$ then $\{1, 2\} \cap Per(f) \neq \emptyset$.

(b) *If n is even and $d = -1$, then $MPer_L(f)$ is*

$$\begin{aligned} \emptyset & \text{ if } \mathcal{Z}_f(t) \in \left\{ \pm \frac{1}{1-t^4}, \pm \frac{1}{(1-t^2)^2} \right\}; \\ \{1\} & \text{ if } \mathcal{Z}_f(t) \in \left\{ \pm \frac{1}{(1-t)^3(1+t)}, \pm \frac{1}{(1+t)^3(1-t)} \right\}; \\ \{1, 3\} & \text{ if } \mathcal{Z}_f(t) \in \left\{ \pm \frac{1}{(1-t^3)(1+t)}, \pm \frac{1}{(1-t)(1+t^3)} \right\}, \end{aligned}$$

but if $\mathcal{Z}_f(t) = \pm 1/(1-t^2)^2$ then $\{1, 2\} \cap Per(f) \neq \emptyset$, and if $\mathcal{Z}_f(t) = \pm 1/(1-t^4)$ then $\{1, 2, 4\} \cap Per(f) \neq \emptyset$.

(c) *If n is odd and $d = 1$, then $MPer_L(f)$ is*

$$\begin{aligned} \emptyset & \text{ if } \mathcal{Z}_f(t) \equiv \pm 1; \\ \{1\} & \text{ if } \mathcal{Z}_f(t) \in \left\{ \pm \frac{(1+t)^2}{(1-t)^2}, \pm \frac{1+t}{1-t}, \pm \frac{1+t^2}{(1-t)^2} \right\}; \\ \{1, 3\} & \text{ if } \mathcal{Z}_f(t) \in \left\{ \pm \frac{1-t^3}{(1-t)^3}, \pm \frac{1+t^3}{(1+t)(1-t)^2} \right\}. \end{aligned}$$

(d) If n is odd and $d = -1$, then $MPer_L(f)$ is

$$\begin{aligned} & \emptyset \text{ if } \mathcal{Z}_f(t) \in \left\{ \pm 1, \pm \frac{1+t^2}{1-t^2} \right\}; \\ & \{1\} \text{ if } \mathcal{Z}_f(t) \in \left\{ \pm \frac{1-t}{1+t}, \pm \frac{1+t}{1-t} \right\}; \\ & \{1, 3\} \text{ if } \mathcal{Z}_f(t) \in \left\{ \pm \frac{1-t^3}{(1-t)^2(1+t)}, \pm \frac{1+t^3}{(1+t)^2(1-t)} \right\}. \\ & \text{but if } \mathcal{Z}_f(t) = \pm(1+t^2)/(1-t^2) \text{ then } \{1, 2, 4\} \cap Per(f) \neq \emptyset. \end{aligned}$$

Proof. The result is a consequence of Proposition 19 and the application of Theorem 8.

By Remark 10 an even positive integer m cannot be contained in $MPer_L(f)$, and therefore we do not consider the even periods. For each of the cases (a), (b), (c) and (d), we consider the six possible Lefschetz zeta functions for f given in Proposition 19.

In each case $MPer_L(f)$ is given by the intersection of the sets of periods forced by each representation of $\mathcal{Z}(t)$ in products of the form $1 \pm t^p$ that give rise to different sets of periods.

In the cases where we have $MPer_L(f) = \emptyset$ but the Lefschetz zeta function is not constantly equal to 1, we obtain the results from Remarks 10 and 12.

For $\mathcal{Z}_f(t) = \pm 1/(1-t^2)^2$ we have the equality

$$\frac{1}{(1-t^2)^2} = \frac{1}{(1+t)^2(1-t)^2},$$

and by Remark 10 all the possible expressions for $\mathcal{Z}_f(t)$ force either the period 2, or the period 1.

For $\mathcal{Z}_f(t) = \pm 1/(1-t^4)$ we have the equalities

$$\frac{1}{1-t^4} = \frac{1}{(1+t^2)(1-t^2)} = \frac{1}{(1+t^2)(1-t)(1+t)},$$

and by Remark 10 all the possible expressions for $\mathcal{Z}_f(t)$ force either the period 1, or 2, or 4.

For $\mathcal{Z}_f(t) = \pm(1+t^2)/(1-t^2)$, we have the equalities

$$\frac{1+t^2}{1-t^2} = \frac{1-t^4}{(1-t^2)^2} = \frac{1-t^4}{(1+t)^2(1-t)^2} = \frac{(1+t^2)(1-t^2)}{(1+t)^2(1-t)^2},$$

and by Remark 10 all the possible expressions for $\mathcal{Z}_f(t)$ also force either the period 1, or 2, or 4. $\square$

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REFERENCES

- [1] S. BATTERSON, M. HANDEL AND C. NARASIMHAN, *Orientation reversing Morse–Smale diffeomorphisms of $\mathbb{S}^2$* , Invent. Math. **64** (1980/81), 345–356.
- [2] P. BERRIZBEITIA, M.J. GONZÁLEZ AND V.F. SIRVENT, *On the Lefschetz zeta function and the minimal sets of Lefschetz periods for Morse–Smale diffeomorphisms on products of ℓ -spheres*, Topology Appl. **235** (2018), 428–444.
- [3] C. BONATTI, V. GRINES AND O. POCHINKA, *Topological classification of Morse–Smale diffeomorphisms on 3-manifolds*, Duke Math. J. **168** (2019), 2507–2558.
- [4] R.F. BROWN, *The Lefschetz fixed point theorem*, Scott, Foresman and Company, Glenview, IL, 1971.
- [5] J. FRANKS AND C. NARASIMHAN, *The periodic behavior of Morse–Smale diffeomorphisms*, Invent. Math. **48** (1978), 279–292.
- [6] J. FRANKS, *Homology and dynamical systems*, vol. **49**, CBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, 1982.
- [7] J. FRANKS AND M. SHUB, *The Existence of Morse–Smale Diffeomorphisms*, Topology **20**(1981), 273–290.
- [8] G. GRAFF, M. LEBIEDŹ AND A. MYSZKOWSKI, *Periodic expansion in determining minimal sets of Lefschetz periods for Morse–Smale diffeomorphisms*, J. Fixed Point Theory Appl. **21** (2019), no. 2, Art. 47, 21 pp.
- [9] J.L.G. GUIRAO AND J. LLIBRE, *Periods for the Morse–Smale diffeomorphisms on $\mathbb{S}^2$* , Colloquium Math. **200** (2008), 477–483.
- [10] J.L.G. GUIRAO AND J. LLIBRE, *Minimal Lefschetz sets of periods for Morse–Smale diffeomorphisms on the n -dimensional torus*, J. Difference Equ. Appl. **16** (2010), 689–703.
- [11] J.L.G. GUIRAO AND J. LLIBRE, *Periods of homeomorphisms of some compact spaces*, Houston J. Math. **42** (2016), 1047–1058.
- [12] J.L.G. GUIRAO AND J. LLIBRE, *Periods of continuous maps of some compact spaces*, J. Difference Equ. Appl. **23** (2017), 1–7.
- [13] S. LANG, *Algebra*, Addison-Wesley, New York, 1971.
- [14] J. LLIBRE AND V. SIRVENT, *Minimal sets of periods for Morse–Smale diffeomorphisms on orientable compact surfaces*, Houston J. of Math. **35** (2009), 835–855; **36** (2010), 335–336.

- [15] J. LLIBRE AND V. SIRVENT, *Minimal sets of periods for Morse–Smale diffeomorphisms on non-orientable compact surfaces without boundary*, J. Difference Equ. Appl. **19** (2013), no. 3, 402–417.
- [16] J. LLIBRE AND V. SIRVENT, *A survey on the minimal sets of Lefschetz periods for Morse–Smale diffeomorphisms on some closed manifolds*, Publ. Mat. Urug. **14** (2013), 155–169.
- [17] C. NARASIMHAN, *The periodic behavior of Morse–Smale diffeomorphisms on compact surfaces*, Trans. Amer. Math. Soc. **248** (1979), 145–169.
- [18] J. PALIS AND S. SMALE, *Structural stability theorems*, Proc. Sympos. Pure Math. **14**, Amer Math. Soc., Providence, R.I., 1970, 223–231.
- [19] C. ROBINSON, *Structural stability for C^1 diffeomorphisms*, J. Differential Equations **22** (1976), 28–73.
- [20] M. SHUB, *Morse–Smale diffeomorphisms are unipotent on homology*, Dynamical Systems (Proceedings of a symposium held at the University of Bahia, Salvador, 1971), Academic Press, New York, 1973.
- [21] M. SHUB AND D. SULLIVAN, *Homology theory and dynamical systems*, Topology **14** (1975), 109–132.
- [22] S. SMALE, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 797–817.
- [23] I. STEWART, *Galois theory*, Chapman & Hall, 1973.
- [24] J.W. VICK, *Homology theory. An introduction to algebraic topology*, Second edition, Graduate Texts in Mathematics **145**, Springer-Verlag, New York, 1994.

DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA,
08193 BELLATERRA, BARCELONA, CATALONIA, SPAIN

Email address: clara@mat.uab.cat, jllibre@mat.uab.cat