MAPPINGS WITH FINITE DISTORTION IN $\mathsf{L}^p_{\mathrm{loc}}$: MODULUS OF CONTINUITY AND COMPRESSION OF HAUSDORFF MEASURE

ALBERT CLOP AND DAVID A HERRON

Abstract. We study mappings of finite distortion whose distortion functions belong to the Lebesgue space $\mathsf{L}^p_{\mathrm{loc}}$. We establish a local modulus of continuity estimate for the inverse of such a mapping. As an application, we describe the possible compression of Hausdorff measure under such mappings. We also exhibit examples that describe the extent to which our results are sharp.

1. Introduction

We call $\mathsf{R}^n \supset \Omega \xrightarrow{f} \mathsf{R}^n$ a *mapping of finite distortion* provided

- f belongs to the Sobolev space $\mathsf{W}^{1,1}_{\mathrm{loc}}(\Omega;\mathsf{R}^n)$,
- $J(\cdot,f)$ belongs to the Lebesgue space $\mathsf{L}^1_{\mathrm{loc}}(\Omega;\mathsf{R})$,
- there exists a measurable function $K:\Omega\to[1,\infty]$ that is finite almost everywhere and is such that for almost every $x\in\Omega$,

$$|Df(x)|^n \le K(x)\,J(x,f)\,. \tag{1.1}$$

Here Ω is a domain (open and connected) in Euclidean space R^n with $n\ge 2$, $|Df(x)|$ denotes the operator norm of the differential matrix of f at the point x, and $J(\cdot,f)$ is the Jacobian determinant of f. We call (1.1) the *distortion inequality*; any measurable function K satisfying (1.1) is a *distortion function* for f. When $K\in\mathsf{L}^\infty_{\mathrm{loc}}$ we recover the well studied class of mappings of bounded distortion, also known as the quasiregular mappings; see for instance [Res89]. Nonconstant mappings of finite distortion are continuous, discrete, and open, provided their distortion function satisfies certain conditions. For example, the finite distortion mappings of *exponentially integrable distortion*, that is, those for which $e^{pK}\in\mathsf{L}^1_{\mathrm{loc}}$, have been extensively studied. See [IKO01, KKM+03] and also the monograph [IM01] and the references therein.

In this work we study finite distortion homeomorphisms whose distortion functions K belong to the Lebesgue space $\mathsf{L}^p_{\mathrm{loc}}$ for some $p>n-1$. We establish a sharp local modulus of continuity inequality for the inverse map. Then, using this inequality, we prove an estimate for the possible compression of Hausdorff measures induced by such maps. We exhibit examples that illustrate the sharpness of these estimates. We include a result concerning the expansion of Hausdorff measure for planar finite distortion homeomorphisms with exponentially integrable distortion.

Date: April 27, 2012.

2010 *Mathematics Subject Classification.* Primary: 30C65; Secondary: 28A78, 26B10.

Key words and phrases. mappings of finite distortion, modulus of continuity, compression, Hausdorff measure, Hausdorff dimension.

This work was done while the first author was visiting the University of Cincinnati; he is grateful for their hospitality and financial support. The second author was partially supported by the Charles Phelps Taft Research Center.

Here is the modulus of continuity result.

Theorem A. *Let $\Omega, \Omega' \subset \mathsf{R}^n$ be domains. Suppose $\Omega \xrightarrow{f} \Omega'$ is a homeomorphism with finite distortion $K := K_f \in \mathsf{L}^p_{\text{loc}}(\Omega)$ for some $p \in (n-1, \infty)$. Then for each ball $B := \mathsf{B}(f(z), R)$ with $\bar{B} \subset \Omega'$ and all points $x \in G := f^{-1}(B) \subset \Omega$,*

$$|f(x) - f(z)| \geq R \exp\left(- \frac{C(p,n) \|K\|^{1/(n-1)}_{\mathsf{L}^p(G)}}{|x-z|^{n/[p(n-1)]}} \right). \tag{1.2}$$

Moreover, the exponent $n/[p(n-1)]$ in (1.2) *is optimal.*

For future reference we note that (1.2) is equivalent to the inequality

$$|g(y) - g(a)| \leq C(p,n) \|K\|^{\frac{p}{n}}_{\mathsf{L}^p(G)} \left(\log \frac{R}{|y-a|} \right)^{-p(n-1)/n} \tag{1.3}$$

where $g := f^{-1}$, $y := f(x)$, and $a := f(z)$. We recognize (1.3) as the n-dimensional analog of [KT10, Theorem 3]; our proof is similar to theirs.

As an application of Theorem A, we establish the following result that describes the possible compression of Hausdorff measure under certain finite distortion homeomorphisms.

Theorem B. *Let $p \in (n-1, \infty)$ and $s \in (0, n]$. Define the Hausdorff gauge function*

$$h(t) = h_{s,p,n}(t) := \left(\log \frac{1}{t} \right)^{-p\,s\,(n-1)/n}.$$

Suppose $\mathsf{R}^n \supset \Omega \xrightarrow{f} \Omega' \subset \mathsf{R}^n$ is a homeomorphism with finite distortion in $\mathsf{L}^p_{\text{loc}}(\Omega)$. Then for each $E \subset \Omega$ with $\mathcal{H}^s(E) > 0$, $\mathcal{H}^h(f(E)) > 0$.

We present our proofs of Theorems A and B in §3.A and §3.B respectively. In §3.C we observe a simple corollary of these results that describes the possible expansion of certain Hausdorff measures under finite distortion homeomorphisms that have exponentially integrable distortion.

Examples 4.2 and 4.4 reveal that the exponent $p\,s\,(n-1)/n$ in Theorem B has the optimal behavior as $s \to 0$. Moreover, these examples also suggest that Theorem B can be improved, by replacing the exponent $p\,s\,(n-1)/n$ with $p\,s\,(n-1)/(n-s)$. However, since finite distortion homeomorphisms with distortion in $\mathsf{L}^p_{\text{loc}}$ need not satisfy Lusin's condition N (see [KKM01]), it is not clear whether or not we need the factor $1/(n-s)$.

2. Preliminaries

Our notation is relatively standard. We write $C = C(a, \dots)$ to indicate a constant C that depends only on the parameters $a, \dots$; the notation $A \lesssim B$ means there exists a finite constant c with $A \leq cB$, and $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$ hold. Typically $a, b, c, C, K, \dots$ will be constants that depend on various parameters, and we try to make this as clear as possible often giving explicit values, however, at times C will denote a generic constant whose value depends only on the data present but may differ even on the same line of inequalities.

The Euclidean distance between points x, y in Euclidean space R^n is denoted $|x-y|$. Then $\mathsf{B}(x;r) := \{y : |x-y| < r\}$ and $\mathsf{S}(x;r) := \{y : |x-y| = r\}$ are the open ball and the sphere of radius r centered at the point x. We let $\mathsf{B}^n := \mathsf{B}(0;1)$ denote the unit ball.

Given a ball B and $\sigma > 0$, we let $\sigma\, B$ denote the dilated ball with the same center; that is, $\sigma\, \mathsf{B}(x, r) := \mathsf{B}(x, \sigma\, r)$.

2.A. **Hausdorff Measure.** A non-decreasing function $(0,\infty) \xrightarrow{h} (0,\infty)$ is called a *Hausdorff dimension gauge* provided $\lim_{t\to 0^+} h(t) = 0$. We use a dimension gauge h to define the *Hausdorff measure* $\mathcal{H}^h$ via

$$\mathcal{H}^h(E) := \lim_{r\to 0^+} \left[\inf \left\{ \sum_i h(\operatorname{diam} A_i) : E \subset \bigcup_i A_i, \operatorname{diam}(A_i) \le r \right\} \right] .$$

Typically we are only interested in knowing whether this quantity is zero or positive and finite or infinite. For this we can assume that the covering sets A_i are balls $\mathsf{B}(a_i, r_i)$ with $r_i \le r$, and then $h(\operatorname{diam} A_i)$ is replaced with $h(2r_i)$; doing this does not change the positivity or the finiteness of $\mathcal{H}^h(E)$.

When $h(t) = t^s$ for some $s > 0$, we use the standard notation $\mathcal{H}^s$ instead of $\mathcal{H}^h$, and then $\mathcal{H}^s(E)$ is called the *s-dimensional Hausdorff measure* of a set E. The *Hausdorff dimension* of E is defined by

$$\dim_{\mathcal{H}}(E) := \inf \{ s > 0 : \mathcal{H}^s(E) = 0 \} .$$

However, in this paper we require a finer notion of "size"; e.g., we will need to distinguish the "sizes" of certain 0-dimensional sets.

It is easy to check that for any two Hausdorff gauge functions g and h,

$$\mathcal{H}^h(E) \le \limsup_{t\to 0^+} \frac{h(t)}{g(t)}\, \mathcal{H}^g(E) .$$

With this in mind, we impose an ordering on Hausdorff gauge functions as follows: given two gauges g and h we write

$$g \preceq h \iff \limsup_{t\to 0^+} \frac{h(t)}{g(t)} < \infty \quad \text{and} \quad g \prec h \iff \lim_{t\to 0^+} \frac{h(t)}{g(t)} = 0 .$$

Here are some simple examples.

(1) When $r > 0$ and $s > 0$: $\quad r < s \iff t^r \prec t^s$.
(2) When $p > 0$ and $q > 0$: $\quad t^p[\log(1/t)]^q \prec t^p \prec t^p/[\log(1/t)]^q$.
(3) For any $\alpha, \beta \in \mathsf{R}$: $\alpha < \beta \iff [\log(1/t)]^{-\alpha} \prec [\log(1/t)]^{-\beta}$.
(4) When $p > 0$: $\quad \alpha < \beta \iff \exp(-\alpha[\log(1/t)]^p) \prec \exp(-\beta[\log(1/t)]^p)$.

Notice that when $g \preceq h$, $\mathcal{H}^h << \mathcal{H}^g$. Also, if $g \prec h$, then $\mathcal{H}^g(E) < \infty \implies \mathcal{H}^h(E) = 0$; i.e., sets that are "small" with respect to $\mathcal{H}^g$ are $\mathcal{H}^h$-null sets.

We can use this order to see what are the "best" gauges. For example, in Theorem B, we verify that a certain set has positive measure. In this setting, the "best" gauge is the biggest: if $g \prec h$, then h is a better gauge in the sense that $\mathcal{H}^h(E) > 0$ is a stronger statement than $\mathcal{H}^g(E) > 0$. On the other hand, in our examples we construct a certain set with zero or finite measure, and in this setting the "best" gauge is the smallest: if $g \prec h$, then $\mathcal{H}^g(E) < \infty \implies \mathcal{H}^h(E) = 0$.

We remark that for any $\alpha \in \mathsf{R}$ and any $p < 1$, both of the gauges $[\log(1/t)]^{-\alpha}$ and $\exp(-\alpha[\log(1/t)]^p)$ are zero dimensional. That is, if the measure of a set E (with respect to either of these gauges) is finite, then $\dim_{\mathcal{H}}(E) = 0$.

2.B. **Capacity Estimate.** Our proof of Theorem A depends on a certain capacity estimate. We provide this estimate here.

The *(variational) p-capacity* of a compact set $E \subset \Omega$, relative to Ω, is

$$\operatorname{cap}_p(E;\Omega) := \inf_{u\in\mathcal{W}} \int_\Omega |\nabla u|^p$$

where $\mathcal{W} := \mathcal{C}(\Omega)\cap \mathsf{W}_0^{1,1}(\Omega)$ is the family of all functions u that are continuous in Ω, possess weak derivatives that are locally integrable, have 'zero boundary values', and satisfy $u \geq 1$ on E. Standard arguments permit us to assume that $u \in \mathcal{C}_0^\infty(\Omega)$ with $0 \leq u \leq 1$, and we call these latter functions *admissible* for $\operatorname{cap}_p(E;\Omega)$; see [HKM93, pp.27-28].

We recall the following standard estimates for the capacity of a spherical ring one; see [HKM93, Example 2.12, p.35] or [Res89, p.106].

2.1. *Fact.* Let $0 < r < R < \infty$ and $1 < p < \infty$. Then

$$\operatorname{cap}_p(\bar{\mathsf{B}}(r);\mathsf{B}(R)) = \begin{cases} \omega_{n-1}[\log{(R/r)}]^{1-n} & \text{when } p = n \\ \omega_{n-1}\,|\alpha|^{p-1}\,|R^\alpha - r^\alpha|^{1-p} & \text{when } p \neq n\,. \end{cases}$$

where $\alpha := \dfrac{p-n}{p-1}$.

The following estimate was established in [KT10, Lemma 1] for dimension $n = 2$. As they show, when $n = 2$ we can allow $p = 1$; however, their argument does not apply when $n > 2$.

2.2. **Lemma.** *Let $E \subsetneq \Omega \subsetneq \mathsf{R}^n$ with E a continuum and Ω a bounded domain. Suppose that $u \in \mathcal{C}(\Omega)\cap \mathsf{W}_0^{1,1}(\Omega)$ with $u \geq 1$ on E. Then for each $n-1 < p < n$, there is a constant $C(p,n) > 0$ such that*

$$\int_\Omega |\nabla u|^p \geq C(p,n)\ (\operatorname{diam}(E))^{n-p}\ .$$

Proof. We assume that the origin lies in E and that for all $r \in [0,d]$, $E \cap \mathsf{S}(r) \neq \emptyset$; here $d := \operatorname{diam}(E)$. Pick $R \geq 2d$ with $\Omega \subset \mathsf{B}(R)$.

Suppose there exists $r_0 \in [d/2,d]$ with $u \geq 1/2$ on $\mathsf{S}(r_0)$. Define

$$v := \begin{cases} 2u & \text{for } |x| \geq r_0\,, \\ \max\{1,2u\} & \text{for } |x| < r_0\,. \end{cases}$$

Then $v \geq 1$ in $\bar{\mathsf{B}}(r_0)$, so v is admissible for $\operatorname{cap}_p(\bar{\mathsf{B}}(r_0);\mathsf{B}(R))$. Since $|\nabla v| \leq 2\,|\nabla u|$, we obtain

$$\int_\Omega |\nabla u|^p = \int_{\mathsf{B}(R)} |\nabla u|^p \geq 2^{-p} \int_{\mathsf{B}(R)} |\nabla v|^p \geq 2^{-p}\operatorname{cap}_p(\bar{\mathsf{B}}(r_0);\mathsf{B}(R))\,.$$

Using the inequality $2r_0 \leq R$ in conjunction with Fact 2.1, we deduce that

$$\int_\Omega |\nabla u|^p \geq C(p,n)\,d^{n-p}$$

where $C(p,n) := \omega^{n-1}4^{p-n}|\alpha|^{p-1}(1-2^\alpha)$.

Suppose there is no such r_0; thus, for each $r \in [d/2,d]$ there exists a point $x \in \mathsf{S}(r)$ with $u(x) < 1/2$.

We may assume that $u \in \mathsf{W}^{1,p}(\mathsf{B}(R))$. Then by Fubini's theorem, for a.e. $r \in (0,R)$ $u \in \mathsf{W}^{1,p}(\mathsf{S}(r))$. The Sobolev embedding theorem now permits us to assert that for a.e. $r \in (0,R)$ and all $x, y \in S(r)$,

$$|u(x) - u(y)| \leq C_1(p,n)\, r \left(\fint_{\mathsf{S}(r)} |\nabla u|^p \right)^{1/p} .$$

Here

$$\fint_{\mathsf{S}(r)} |\nabla u|^p = \frac{1}{\omega_{n-1} r^{n-1}} \int_{\mathsf{S}(r)} |\nabla u(r\,\omega)|^p r^{n-1}\, d\omega \,.$$

Thus we obtain

$$\int_{\mathsf{S}(r)} |\nabla u(r\,\omega)|^p r^{n-1}\, d\omega \geq \frac{\omega_{n-1}\, r^{n-1}}{(2\,C_1\, r)^p}$$

which yields

$$\int_{\Omega} |\nabla u|^p \geq C(p,n)\, d^{n-p}$$

where $C(p,n) := \dfrac{\omega^{n-1}}{(2\,C_1)^p (n-p)} \dfrac{2^{n-p}-1}{4^{p-n}}$. □

2.C. **Quasiconformal Compression.** We recall that for $\lambda \geq 1$, the map $x \mapsto |x|^{\lambda-1}\, x$ defines a K-quasiconformal self-homeomorphism of R^n, with $K = \lambda^{n-1}$. Given $\lambda \geq 1$ and $\sigma \in (0,1)$, we define

$$\Psi(x) := \begin{cases} x & \text{for } x \in \mathsf{R}^n \setminus \mathsf{B}^n \,, \\ |x|^{\lambda-1}\, x & \text{for } x \in \mathsf{B}^n \setminus \sigma \mathsf{B}^n \,, \\ \sigma^{\lambda-1}\, x & \text{for } x \in \sigma \mathsf{B}^n \,. \end{cases}$$

We note that

$$\Psi(x) = x \;\text{ on } |x| = 1 \quad \text{and} \;\; \Psi(x) = \sigma^{\lambda-1}\, x \;\text{ on } |x| = \sigma \,.$$

In particular, Ψ is a λ^{n-1}-quasiconformal self-homeomorphism of R^n that is the identity in $\mathsf{R}^n \setminus \mathsf{B}^n$, conformal in $\sigma\mathsf{B}^n$, and with

$$\Psi(\mathsf{B}^n) = \mathsf{B}^n \quad \text{and} \quad \Psi(\sigma \mathsf{B}^n) = \sigma^\lambda \mathsf{B}^n \,, \quad \text{so} \quad \Psi(\mathsf{B}^n \setminus \sigma \mathsf{B}^n) = \mathsf{B}^n \setminus \sigma^\lambda \mathsf{B}^n \,.$$

Moreover, the distortion of Ψ "lives" in $\mathsf{B}^n \setminus \sigma\mathsf{B}^n$; i.e., Ψ is conformal in $\sigma\mathsf{B}^n \cup (\mathsf{R}^n \setminus \bar{\mathsf{B}}^n)$.

By employing auxiliary similarity maps, we can transport the action of Ψ to any ball $\mathsf{B} := \mathsf{B}(a,r)$; see Figure 1. We define $\Psi^{\sigma,\lambda}_{a,r}$ via

$$\Psi^{\sigma,\lambda}_{a,r}(x) := \begin{cases} x & \text{for } x \in \mathsf{R}^n \setminus \mathsf{B} \,, \\ a + \left| \dfrac{x-a}{r} \right|^{\lambda-1} (x-a) & \text{for } x \in \mathsf{B} \setminus \sigma\mathsf{B} \,, \\ a + \sigma^{K-1}\, (x-a) & \text{for } x \in \sigma\mathsf{B} \,. \end{cases}$$

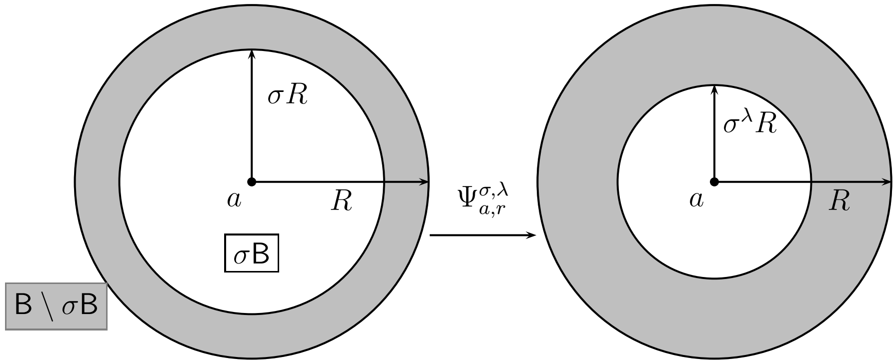

FIGURE 1. The radial squeeze-stretch map $\Psi^{\sigma,\lambda}_{a,r}$

Then $\Psi^{\sigma,\lambda}_{a,r}$ is a λ^{n-1}-quasiconformal self-homeomorphism of R^n that is the identity in $\mathsf{R}^n\setminus\mathsf{B}$, conformal in $\sigma\mathsf{B}$, and with

$$\Psi^{\sigma,\lambda}_{a,r}(\mathsf{B})=\mathsf{B} \quad\text{and}\quad \Psi^{\sigma,\lambda}_{a,r}(\sigma\mathsf{B})=\sigma^\lambda\mathsf{B}\,, \quad\text{so}\quad \Psi^{\sigma,\lambda}_{a,r}(\mathsf{B}\setminus\sigma\mathsf{B})=\mathsf{B}\setminus\sigma^\lambda\mathsf{B}\,,$$
$$\text{and also for all points }\; x\in\mathsf{R}^n\,,\quad |\Psi^{\sigma,\lambda}_{a,r}(x)-x|\le r\,. \tag{2.3}$$

We call $\Psi^{\sigma,\lambda}_{a,r}$ a *radial squeeze-stretch mapping*: it "squeezes" the ball $\sigma\mathsf{B}$ to $\sigma^\lambda\mathsf{B}$ via scaling by $\sigma^{\lambda-1}$ and "stretches" the spherical ring $\mathsf{B}\setminus\sigma\mathsf{B}$ to the ring $\mathsf{B}\setminus\sigma^\lambda\mathsf{B}$ via the radial map $x\mapsto|x|^{\lambda-1}x$. In addition, the distortion of $\Psi^{\sigma,\lambda}_{a,r}$ "lives" in the ring $\mathsf{B}\setminus\sigma\bar{\mathsf{B}}$; that is, $\Psi^{\sigma,\lambda}_{a,r}$ is conformal in $\sigma\mathsf{B}\cup(\mathsf{R}^n\setminus\bar{\mathsf{B}})$. Finally, we note that the radial squeeze-stretch map $\Psi^{\sigma,\lambda}_{a,r}$ is uniquely determined by the concentric ball triple

$$(\mathsf{B},\sigma\mathsf{B},\sigma^\lambda\mathsf{B}):=(\mathsf{B}(a,r),\mathsf{B}(a,\sigma r),\mathsf{B}(a,\sigma^\lambda r))\,.$$

3. PROOFS OF THEOREMS

Here corroborate Theorems A and B. Then we present a simple application that describes the possible expansion of certain Hausdorff measures under finite distortion homeomorphisms that have exponentially integrable distortion.

3.A. **Proof of Theorem A.** We assume $\Omega\xrightarrow{f}\Omega'$ is a homeomorphism with a distortion function $K_f\in\mathsf{L}^p_{\mathrm{loc}}(\Omega)$ for some $p\in(n-1,\infty)$. Set $q:=p\,n/(p+1)$, so $n-1<q<n$. Let $z\in\Omega$, $a:=f(z)$, $B:=\mathsf{B}(a,R)$, and suppose that $\bar B\subset\Omega'$. Put $g:=f^{-1}$ and $G:=g(B)$. Fix a point $x\in G$. Let $y:=f(x)$, $A:=[a,y]$, and $E:=g(A)$. We demonstrate that inequality (1.3) holds.

An appeal to the capacity estimate in Lemma 2.2 provides the inequality

$$\mathrm{cap}_q(E;G)\ge C(p,n)\,(\mathrm{diam}(E))^{n-q}\ge C\,|g(y)-g(a)|^{n-q}\,.$$

Let u be an admissible test function for the conformal capacity $\mathrm{cap}_n(A;B)$ and put $v:=u\circ f$. The Chain Rule in conjunction with the distortion inequality (1.1) imply that

$$|\nabla v|^q\le|\nabla(u\circ f)|^qK^{q/n}J^{q/n}\,,$$

where $K:=K_f$, $J:=J(\cdot,f)$, and $q/n=p/(p+1)$.

To estimate $\int_G |\nabla v|^q$, we use Hölder's inequality and a change of variables to get

$$\int_G |\nabla v|^q \leq \int_G (|\nabla(u\circ f)|^n J)^{q/n} K^{q/n} \leq \left(\int_G |\nabla(u\circ f)|^n J\right)^{q/n} \left(\int_G (K^{q/n})^{n/(n-q)}\right)^{(n-q)/n}$$
$$= \left(\int_B |\nabla u|^n\right)^{p/(p+1)} \left(\int_G K^p\right)^{1/(p+1)} = \|K\|_{\mathsf{L}^p(G)}^{p/(p+1)} \left(\int_B |\nabla u|^n\right)^{p/(p+1)} .$$

Taking an infimum over all such admissible test functions u, and appealing to Fact 2.1, we obtain

$$\operatorname{cap}_q^{p+1}(E;G) \leq \|K\|_{\mathsf{L}^p(G)}^p \operatorname{cap}_n^p(A;B) \leq \|K\|_{\mathsf{L}^p(G)}^p \omega_{n-1}^p \left(\log \frac{R}{|y-a|}\right)^{-p(n-1)} .$$

Since $n-q = n/(p+1)$, by combining the above with our initial capacity inequality we arrive at (1.3).

Next we demonstrate that the exponent $n/[p(n-1)]$ in (1.2) is best possible in the sense that this inequality fails to hold for any smaller exponent. (Similarly, in (1.3) the exponent $p(n-1)/n$ is best possible in that the inequality fails to hold for any larger exponent.) More precisely, we show that for each $\alpha \in (0, n/[p(n-1)])$ there is a finite distortion homeomorphism $f : \mathsf{R}^n \to \mathsf{R}^n$ with $K_f \in \mathsf{L}^p_{\mathrm{loc}}$ and so that for all $C > 0$ and any $\varepsilon > 0$, there exists a point $x \in f^{-1}(\mathsf{B}(f(0), \varepsilon))$ such that

$$|f(x) - f(0)| < \varepsilon \exp(-C/|x|^\alpha) .$$

To this end, let $\alpha \in (0, n/[p(n-1)])$. Choose $\beta \in (\alpha, n/[p(n-1)])$. Define a radial homeomorphism $f : \mathsf{R}^n \to \mathsf{R}^n$ by $f(0) = 0$ and for $x \neq 0$,

$$f(x) := \rho(|x|)\, \frac{x}{|x|} \quad \text{where} \quad \rho(t) := \exp(-t^{-\beta}) .$$

Since f is radial, there are only two directional derivatives to check, and we find that

$$|Df(x)| = \max\{\rho'(r), \rho(r)/r\} \qquad \text{and} \qquad J_f(x) = \rho'(r)\,(\rho(r)/r)^{n-1} ,$$

where $r = |x|$. We compute

$$\rho'(t) = \frac{\beta}{t}\,\frac{\rho(t)}{t}$$

and conclude that for $|x| \leq \beta$

$$K_f(x) = \frac{|Df(x)|^n}{J(x,f)} = \left(\frac{\beta}{t}\right)^{n-1} .$$

Since $\beta\, p\,(n-1) < n$, $K_f \in \mathsf{L}^p_{\mathrm{loc}}$.

Now let $C > 0$ and $0 < \varepsilon < 1$ be given. Set $\tau := (2C)^{-1/(\beta-\alpha)}$ and $\lambda := (\log \varepsilon^{-2})^{-1/\beta}$. Then for $0 < t < \lambda \wedge \tau$,

$$\log \frac{1}{\varepsilon} < \frac{1}{2\,t^\beta} \quad \text{and} \quad C\, t^{\beta-\alpha} \leq \frac{1}{2}$$

so

$$\exp\left(\frac{C}{t^\alpha}-\frac{1}{t^\beta}\right)\le \exp\left(-\frac{1}{2\,t^\beta}\right)<\varepsilon\,.$$

Thus any x with $t=|x|<\lambda\wedge\tau$ lies in $f^{-1}(\mathsf{B}(f(0),\varepsilon))$ and has

$$|f(x)-f(0)|=\rho(|x|)=\exp\left(-\frac{1}{t^\beta}\right)<\varepsilon\,\exp\left(-\frac{C}{t^\alpha}\right)=\varepsilon\,\exp\left(-\frac{C}{|x|^\alpha}\right)\,.$$

□

3.B. **Proof of Theorem B.** We assume $\Omega\xrightarrow{f}\Omega'$ is a homeomorphism with a distortion function $K_f\in\mathsf{L}^p_{\mathrm{loc}}(\Omega)$ for some $p\in(n-1,\infty)$. Let $s>0$ and put

$$h(t):=h_{s,p,n}(t):=\left(\log\frac{1}{t}\right)^{-p\,s\,\frac{n-1}{n}}\,.$$

We demonstrate that for each $E\subset\Omega$, $\mathcal{H}^h(f(E))=0$ implies that $\mathcal{H}^s(E)=0$.

Set $g:=f^{-1}$. For each point $a\in\Omega'$, let $R(a):=\frac{1}{2}\,\mathrm{dist}(a,\partial\Omega')$. According to (1.3), for each $r\in(0,R(a))$,

$$\mathrm{diam}(g[\mathsf{B}(a,r)])\le D(a)\left(\log\frac{R(a)}{r}\right)^{-p\,\frac{n-1}{n}}$$

where $D(a):=2\,C(p,n)\|K_f\|^{p/n}_{\mathsf{L}^p(g[\mathsf{B}(a,R(a))])}$. For integers j, k with $j\ge 2$, $k\ge 1$ we define

$$F_{jk}:=\{a\in\Omega'\mid R(a)\ge 1/j\ ,D(a)\le k\}\,.$$

Then $\Omega'=\bigcup_{j,k}F_{jk}$. Also, for each $a\in F_{jk}$ and all $r\in(0,1/j)$,

$$\mathrm{diam}(g[\mathsf{B}(a,r)])\le k\,\left(\log\frac{1}{j\,r}\right)^{-p\,\frac{n-1}{n}}\,.$$

Suppose $E\subset\Omega$ with $\mathcal{H}^h(f(E))=0$. Fix integers j, k with $j\ge 2$, $k\ge 1$. We show that $\mathcal{H}^s(E\cap g(F_{jk}))=0$. Let $\varepsilon>0$ be given. Select $\rho\in(0,2/j^2)$. Note that

$$0<r<\frac{2}{j^2}\implies\frac{1}{2r}<\frac{1}{(j\,r)^2}\quad\text{and so}\quad\log\frac{1}{2r}<2\,\log\frac{1}{j\,r}\,.$$

Since $\mathcal{H}^h(f(E)\cap F_{jk})=0$, there are balls $B_i:=\mathsf{B}(a_i,r_i)$ with $a_i\in f(E)\cap F_{jk}$, $r_i\in(0,\rho)$, $f(E)\cap F_{jk}\subset\bigcup_i B_i$ and such that $\sum_i h(2r_i)<\varepsilon$. As $a_i\in F_{jk}$, $r_i<\rho<2/j^2\le 1/j\le R(a_i)$ and thus

$$\mathrm{diam}(g(B_i))\le k\left(\log\frac{1}{j\,r_i}\right)^{-p\,\frac{n-1}{n}}\le 2^{p\frac{n-1}{n}}k\left(\log\frac{1}{2\,r_i}\right)^{-p\,\frac{n-1}{n}}\,.$$

Therefore

$$\sum_i(\mathrm{diam}(g(B_i)))^s\le 2^{p\,s\,\frac{n-1}{n}}k^s\sum_i\left(\log\frac{1}{2\,r_i}\right)^{-p\,s\,\frac{n-1}{n}}=C\sum_i h(2\,r_i)\le C\,\varepsilon\,.$$

where $C=C(k,s,p,n)$. Since $E\cap g(F_{jk})\subset\bigcup_i g(B_i)$, by letting $\varepsilon\searrow 0$ in the above we conclude that $\mathcal{H}^s(E\cap g(F_{jk}))=0$. □

3.C. **Expansion of Hausdorff Measure.** Thanks to work of Gill [Gil10], we know that finite distortion homeomorphisms with exponentially integrable distortion have inverses that are finite distortion homeomorphisms with distortion functions in certain $\mathsf{L}^p_{\text{loc}}$ classes. Thus we get the following consequence of Theorems A and B that describes the possible expansion of certain Hausdorff measures under such maps.

3.1. **Corollary.** *Suppose $\mathsf{R}^2 \supset \Omega \xrightarrow{g} \Omega' \subset \mathsf{R}^2$ is a homeomorphism with finite distortion $K := K_g$ that satisfies $\exp(\lambda K) \in \mathsf{L}^1_{\text{loc}}(\Omega)$ for some $\lambda > 1$. Let $p \in [1, \lambda)$, $s \in (0, 2]$ and define*

$$h(t) = h_{s,p}(t) := \left(\log \frac{1}{t}\right)^{-ps/2}.$$

Then for each $A \subset \Omega$ with $\mathcal{H}^h(A) = 0$, $\mathcal{H}^s(g(A)) = 0$.

Further information concerning the expansion problem in the plane can be found in the works [KZZ09, KZZ10, Raj11] where the authors demonstrate that sets with dimension strictly less than two cannot be arbitrarily expanded.

4. Compression Examples

Here we present two examples that illustrate to what extent the gauge function $h_{s,p,n}$ in Theorem B is optimal. Note that $h_{s,p,n} = h_{\beta_0}$ where $\beta_0 := p\,s\,(n-1)/n$ and

$$h_\beta(t) := \left(\log \frac{1}{t}\right)^{-\beta}.$$

We provide information concerning the following questions: Can a similar result be established with a Hausdorff gauge h that is larger than $h_{s,p,n}$? Can a similar result be established with some Hausdorff gauge h_β that is larger than $h_{s,p,n}$ (so with some $\beta > \beta_0$)?

In Example 4.2 we show that any such gauge h (with the conclusion of Theorem B holding) cannot have $h_\alpha \preceq h$ for any $\alpha > \alpha_0 := p\,s\,(n-1)/(n-s)$; in particular, if $h = h_\beta$, then $\beta \le \alpha_0$. The more complicated Example 4.4 provides finer information as explained in our very last paragraph.

Our examples are based on Cantor sets. A *generalized Cantor dust* is a compact set $E = \bigcap_1^\infty E_N$ where $E_1 \supset E_2 \supset \cdots \supset E_N \supset \dots$ is a decreasing sequence of compact sets and each E_N is a finite union of disjoint closed balls. In our examples, E_N will be a union of certain closed subballs that are chosen from each of the balls that comprise E_{N-1}. We first give an overview, then describe our general construction, and then give two specific examples.

4.A. **General Construction.** We start with the closed unit ball $E_0 := \bar{\mathsf{B}} := \bar{\mathsf{B}}^n \subset \mathsf{R}^n$. We pick m_1 closed balls $E^1_i \subset E_0$ ($1 \le i \le m_1$) and put $E_1 := \bigcup_1^{m_1} E^1_i$. Next, for each $1 \le i \le m_1$, we pick m_2 closed balls $E^2_{ij} \subset E^1_i$ ($1 \le j \le m_2$) and put $E_2 := \bigcup_{i=1}^{m_1} \bigcup_{j=1}^{m_2} E^2_{ij}$. (In fact, we do this so that the sets $E^1_i \setminus \bigcup_{j=1}^{m_2} E^2_{ij}$ are "isomorphic"). Continuing in this manner we get

$$E_N := \bigcup_{\ell(J)=N} E^N_J \quad \text{where } J = (j_1, \dots, j_N) \in \{1, \dots, m_1\} \times \cdots \times \{1, \dots, m_N\};$$

here $\ell(J)$ denotes the length of the tuple $J = (j_1, \dots, j_N)$. Thus E_N is a union of $m_1 \cdots m_N$ disjoint closed balls E^N_J. By appropriately specifying the radii of these balls, we obtain a finite upper bound for the Hausdorff measure of $E = \bigcap_1^\infty E_N$, and by choosing the balls "fairly uniformly distributed" we also get a positive lower bound for this measure; see [Mat95, pp.63,64].

We follow this method for our general construction. We require the fact that for each positive integer $m \in \mathsf{N}$, there are m disjoint closed balls in B^n each with the same radius r and such that $m\, r^n = \kappa^n$ where $\kappa = \kappa(n)$ is a dimensional constant. By working with dyadic cubes, it is straightforward to confirm this with $\kappa(n) := 1/\sqrt{8n}$. We start with a given $s \in (0,n)$ (and later a given $p > 0$ and a given Hausdorff gauge h). We construct generalized Cantor dusts $E, F \subset \mathsf{B}^n$ and a self-homeomorphism f of R^n such that

$$f(E) = F\,,\ \mathcal{H}^s(E) \simeq 1\,,\ \text{ either }\ \mathcal{H}^h(F) = 0\ \text{ or }\ \mathcal{H}^h(F) < \infty\,,$$

and so that f has finite distortion $K_f \in \mathsf{L}^p_{\mathrm{loc}}$. The precise details for these latter conditions will be provided in each example.

In each specific example, we will select integers $m_N \geq 2$ and distortion constants $\lambda_N \geq 1$. At each step $1, 2, \dots, N, \dots$ we choose m_N disjoint closed balls $\bar{\mathsf{B}}(a_i^N, R_N) \subset \mathsf{B}^n$ (so here $1 \leq i \leq m_N$) each of radius R_N where R_N is chosen so that $m_N R_N^n = \kappa_N^n$; we choose these balls "fairly uniformly distributed" in B^n. Here $0 < \kappa_N \leq \kappa(n) = 1/\sqrt{8n}$. We also select $\sigma_N \in (0,1)$ so that $m_N(\sigma_N R_N)^s = 1$. (Such a σ_N exists provided $m_N\, R_N^s > 1$, so provided we take $m_N > (1/\kappa_N)^{ns/(n-s)}$.)

Thus, starting with $0 < \kappa_N \leq \kappa(n)$ and $m_N > (1/\kappa_N)^{ns/(n-s)}$, we take

$$R_N := \kappa_N\, m_N^{-1/n} \quad \text{and} \quad \sigma_N := \kappa_N^{-1} m_N^{(1/n)-(1/s)} = \kappa_N^{-1} m_N^{(s-n)/ns}\,.$$

Let φ_i^N and ϑ_i^N be the similarities of R^n given by

$$\varphi_i^N(x) := a_i^N + \sigma_N\, R_N\, x \quad \text{and} \quad \vartheta_i^N(x) := a_i^N + \sigma_N^{\lambda_N} R_N\, x\,,$$

so that

$$\varphi_i^N(\mathsf{B}^n) = \mathsf{B}(a_i^N, \sigma_N R_N) \quad \text{and} \quad \vartheta_i^N(\mathsf{B}^n) = \mathsf{B}(a_i^N, \sigma_N^{\lambda_N} R_N)\,;$$

here $\lambda_N \geq 1$ are auxiliary parameters that will be chosen later to determine the distortion of f. Notice that $a_i^N = \varphi_i^N(0)$. Next—see Figure 2—we define

$$\begin{aligned} E_i^1 &:= \varphi_i^1(\bar{\mathsf{B}}) && \text{for } 1 \leq i \leq m_1\,, \\ E_{ij}^2 &:= \varphi_i^1 \circ \varphi_j^2(\bar{\mathsf{B}}) && \text{for } 1 \leq i \leq m_1 \text{ and } 1 \leq j \leq m_2\,, \end{aligned}$$

and, in general, for $J = (j_1, \dots, j_N) \in \{1, \dots, m_1\} \times \cdots \times \{1, \dots, m_N\}$,

$$E_J^N := \Phi_J^N(\bar{\mathsf{B}}) \qquad \text{where} \quad \Phi_J^N := \varphi_{j_1}^1 \circ \varphi_{j_2}^2 \dots \varphi_{j_N}^N\,.$$

Similarly, define

$$F_J^N := \Theta_J^N(\bar{\mathsf{B}}) \qquad \text{where} \quad \Theta_J^N := \vartheta_{j_1}^1 \circ \vartheta_{j_2}^2 \dots \vartheta_{j_N}^N\,.$$

We obtain generalized Cantor dusts

$$E := \bigcap_1^\infty E_N \quad \text{and} \quad F := \bigcap_1^\infty F_N$$

where

$$E_N := \bigcup_{\text{all } J} E_J^N \quad \text{and} \quad F_N := \bigcup_{\text{all } J} E_J^N\,.$$

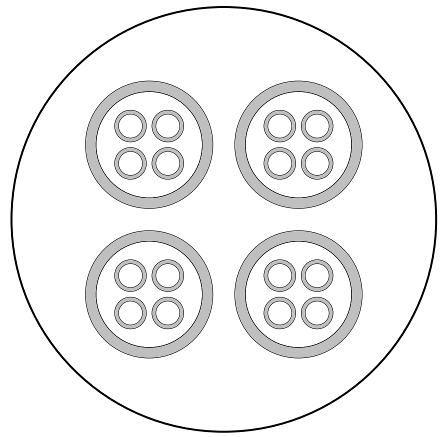

FIGURE 2. The 2^{nd} generation set E_2 with $m_1 = m_2 = 4$

It is straightforward to calculate the centers and radii of the balls E_J^N, F_J^N. For example, the latter ball has center

$$\Theta_J^N(0) = a_{j_1}^1 + \sigma_1^{\lambda_1} R_1 \left(a_{j_2}^2 + \sigma_2^{\lambda_2} R_2 \left[\cdots + \sigma_{N-2}^{\lambda_{N-2}} R_{N-2} \left(a_{j_{N-1}}^{N-1} + \sigma_{N-1}^{\lambda_{N-1}} R_{N-1} a_{j_N}^N \right) \right] \right) .$$

Also, the balls E_J^N each have radius $\sigma_1 R_1 \cdots \sigma_N R_N$. Since these balls form a cover of E with

$$\sum_{\text{all } J} (\sigma_1 R_1 \cdots \sigma_N R_N)^s = m_1 \cdots m_N (\sigma_1 R_1 \cdots \sigma_N R_N)^s = 1 \,,$$

it is clear that $\mathcal{H}^s(E) \lesssim 1$. In fact, since these balls are chosen "fairly uniformly uniformly distributed" in their parent, it follows that $\mathcal{H}^s(E) \simeq 1$; see [Mat95, pp.63,64]. In the examples that follow, we also determine the size of F. For this it is useful to know that each F_J^N has radius $t_N := \sigma_1^{\lambda_1} R_1 \cdots \sigma_N^{\lambda_N} R_N$.

Finally, we construct a finite distortion homeomorphism $\mathsf{R}^n \xrightarrow{f} \mathsf{R}^n$ with the property that $f(E) = F$. This map is given as the limit of a sequence $(f_N)_{N=1}^\infty$ of quasiconformal self-homeomorphisms of R^n; the maps f_N are defined via a recursive relation. In order to accomplish this task, we introduce triples (B_J, C_J, D_J) of concentric balls defined by

$$\begin{aligned}
B_J &:= \mathsf{B}(c_J, r_N) \,, \\
C_J &:= \sigma_N \, B_J = \mathsf{B}(c_J, \sigma_N r_N) \,, \\
D_J &:= \sigma_N^{\lambda_N} \, B_J = \mathsf{B}(c_J, \sigma_N^{\lambda_N} r_N) \,, \\
\text{where } \; r_N &:= \sigma_{N-1}^{\lambda_{N-1}} r_{N-1} R_N \text{ (with } r_0 = \sigma_0 = \lambda_0 := 1)
\end{aligned}$$

and it remains to specify the centers c_J. In fact, $c_J := \Theta_J^N(0)$, but this is more easily understood by starting at the beginning. Write f_0 to denote the identity map: $f_0(x) = x$.

Step 1. For $1 \le i \le m_1$, put

$$\begin{aligned}
B_i &:= f_0 \circ \varphi_i^1(\sigma_1^{-1} \mathsf{B}) \,, \\
C_i &:= \sigma_1 B_i = f_0 \circ \varphi_i^1(\mathsf{B}) \,, \\
D_i &:= \sigma_1^{\lambda_1} B_i \,.
\end{aligned}$$

One can readily check that

$$B_i = \mathsf{B}(c_i, r_1) \quad \text{and} \quad \bar{C}_i = f_0(E_i^1)$$

where $c_i := \vartheta_i^1(0) = a_i^1$ and $r_1 := R_1$. For each triple (B_i, C_i, D_i) we have a radial squeeze-stretch map $\Psi_i^1 := \Psi_{c_i,r_1}^{\sigma_1,\lambda_1}$ (see §2.C) and we define

$$g_1(x) := \begin{cases} \Psi_i^1(x) & \text{for } x \in B_i\,,\ 1 \le i \le m_1\,, \\ x & \text{for } x \in \mathsf{R}^n \setminus \bigcup_{i=1}^{m_1} B_i\,. \end{cases}$$

Thus $\mathsf{R}^n \xrightarrow{g_1} \mathsf{R}^n$ is K_1-quasiconformal, $K_1 := \lambda_1^{n-1}$, and conformal in $\mathsf{R}^n \setminus \bigcup_1^{m_1} (\bar{B}_i \setminus C_i)$ with

$$g_1(B_i) = B_i \quad \text{and} \quad g_1(C_i) = D_i \;\text{ via a scaling by } \sigma_1^{\lambda_1 - 1} \quad \text{and}$$
$$g_1(B_i \setminus C_i) = B_i \setminus D_i \;\text{ via the radial stretch } x \mapsto |x|^{\lambda_1 - 1} x\,.$$

We set $\boxed{f_1 := g_1 \circ f_0}$. Note that $B_i \setminus C_i = \varphi_i^1(\sigma_1^{-1}\mathsf{B} \setminus \mathsf{B})$, so the distortion of f_1 is given via

$$K_{f_1} = \begin{cases} K_1 & \text{in } \bigcup_{i=1}^{m_1} \varphi_i^1(\sigma_1^{-1}\mathsf{B} \setminus \bar{\mathsf{B}})\,, \\ 1 & \text{in } \mathsf{R}^n \setminus \bigcup_{i=1}^{m_1} \varphi_i^1(\sigma_1^{-1}\bar{\mathsf{B}} \setminus \mathsf{B})\,. \end{cases}$$

Also, by comparing centers and radii, we see that

$$f_1(E_i^1) = f_1 \circ \varphi_i^1(\bar{\mathsf{B}}) = g_1(\bar{C}_i) = \bar{D}_i = F_i^1 \quad \text{and so} \quad f_1(E_1) = F_1\,.$$

Step 2. For $1 \le i \le m_1$ and $1 \le j \le m_2$, put

$$\begin{aligned} B_{ij} &:= f_1 \circ \Phi_{ij}^2(\sigma_2^{-1}\mathsf{B})\,, \\ C_{ij} &:= \sigma_2 B_{ij} = f_1 \circ \Phi_{ij}^2(\mathsf{B})\,, \\ D_{ij} &:= \sigma_2^{\lambda_2} B_{ij}^2\,. \end{aligned}$$

One can readily check that

$$B_{ij} = f_1 \circ \varphi_i^1[\mathsf{B}(a_j^2, R_2)] = \mathsf{B}(c_{ij}, r_2) \quad \text{and} \quad \bar{C}_{ij} = f_1(E_{ij}^2)$$

where $c_{ij} := \Theta_{ij}^2(0)$ and $r_2 := \sigma_1^{\lambda_1} r_1 R_1$. For each triple (B_{ij}, C_{ij}, D_{ij}) we have radial a squeeze-stretch map $\Psi_{ij}^2 := \Psi_{c_{ij},r_2}^{\sigma_2,\lambda_2}$ (see §2.C) and we define

$$g_2(x) := \begin{cases} \Psi_{ij}^2(x) & \text{for } x \in B_{ij}\,,\ 1 \le i \le m_1\,,\ 1 \le j \le m_2\,, \\ x & \text{for } x \in \mathsf{R}^n \setminus \bigcup_{i=1}^{m_1} \bigcup_{j=1}^{m_2} B_{ij}\,. \end{cases}$$

Thus $\mathsf{R}^n \xrightarrow{g_2} \mathsf{R}^n$ is K_2-quasiconformal, $K_2 := \lambda_2^{n-1}$, and conformal in $\mathsf{R}^n \setminus \bigcup_{i,j} (\bar{B}_{ij} \setminus C_{ij})$ with

$$g_2(B_{ij}) = B_{ij} \quad \text{and} \quad g_2(C_{ij}) = D_{ij} \;\text{ via a scaling by } \sigma_2^{\lambda_2 - 1} \quad \text{and}$$
$$g_2(B_{ij} \setminus C_{ij}) = B_{ij} \setminus D_{ij} \;\text{ via the radial squeeze-stretch } x \mapsto |x|^{\lambda_2 - 1} x\,.$$

We set $\boxed{f_2 := g_2 \circ f_1}$. Then, since

$$f_1^{-1}(B_{ij}) = \Phi_{ij}^2(\sigma_2^{-1}\mathsf{B})\,,$$

we see that

$$f_2(x) = \begin{cases} \Psi_{ij}^2 \circ f_1(x) & \text{for } x \in \Phi_{ij}^2(\sigma_2^{-1}\mathsf{B})\,, \\ f_1(x) & \text{otherwise}\,. \end{cases}$$

Note that $\Phi^2_{ij}(\sigma_2^{-1}\mathsf{B}) = \varphi^1_i[\mathsf{B}(a^2_j, R_2)] \subset \varphi^1_i(\mathsf{B}) = C_i$. In C_i, $f_1 = g_1$ is conformal (being a linear scaling/squeeze by $\sigma_1^{\lambda_1 - 1}$). Therefore, the distortion of f_2 in $\Phi^2_{ij}(\sigma_2^{-1}\mathsf{B})$ comes only from Ψ^2_{ij} (which has distortion K_2 in $B^2_{ij} \setminus \bar{C}_{ij}$ and is conformal elsewhere). In particular, we deduce that the distortion of f_2 is given via

$$K_{f_2} = \begin{cases} K_2 & \text{in } \bigcup_{i,j} \Phi^2_{ij}(\sigma_2^{-1}\mathsf{B} \setminus \bar{\mathsf{B}})\,, \\ K_1 & \text{in } \bigcup_{i=1}^{m_1} \varphi^1_i(\sigma_1^{-1}\mathsf{B} \setminus \bar{\mathsf{B}})\,, \\ 1 & \text{everywhere else}\,. \end{cases}$$

Also, by comparing centers and radii, we confirm that

$$f_2(E^2_{ij}) = g_2(\bar{C}_{ij}) = \bar{D}_{ij} = F^2_{ij} \quad \text{and so} \quad f_2(E_2) = F_2\,.$$

Step N. For each $J = I \times \{j\} = (j_1, \dots, j_N) \in \{1, \dots, m_1\} \times \cdots \times \{1, \dots, m_N\}$, put

$$\begin{aligned} B_J &:= f_{N-1} \circ \Phi^N_J(\sigma_N^{-1}\mathsf{B})\,, \\ C_J &:= \sigma_N B^N_J = f_{N-1} \circ \Phi^N_J(\mathsf{B})\,, \\ D_J &:= \sigma_N^{\lambda_N} B^N_J\,. \end{aligned}$$

One can check that

$$B_J = f_{N-1} \circ \Phi^{N-1}_I[\mathsf{B}(a^N_j, R_N)] = \mathsf{B}(c_J, r_N) \quad \text{and} \quad \bar{C}_J = f_{N-1}(E^N_J)$$

where $c_J := \Theta^N_J(0)$ and $r_N := \sigma_{N-1}^{\lambda_{N-1}} r_{N-1} R_N$. For each triple (B_J, C_J, D_J) we have radial a squeeze-stretch map $\Psi^N_J := \Psi^{\sigma_N, \lambda_N}_{c_J, r_N}$ (see §2.C) and we define

$$g_N(x) := \begin{cases} \Psi^N_J(x) & \text{for } x \in B_J\,, \\ x & \text{for } x \in \mathsf{R}^n \setminus \bigcup_J B_J\,. \end{cases}$$

Thus $\mathsf{R}^n \xrightarrow{g_N} \mathsf{R}^n$ is K_N-quasiconformal, $K_N := \lambda_N^{n-1}$, and conformal in $\mathsf{R}^n \setminus \bigcup_J (\bar{B}_J \setminus C_J)$ with

$$g_N(B_J) = B^N_J \quad \text{and} \quad g_N(C_J) = D_J \ \text{ via a scaling by } \sigma_N^{\lambda_N - 1} \quad \text{and}$$
$$g_N(B_J \setminus C_J) = B_J \setminus D_J \ \text{ via the radial squeeze-stretch } x \mapsto |x|^{\lambda_N - 1} x\,.$$

We set $\boxed{f_N := g_N \circ f_{N-1}}$. Then, since

$$f_{N-1}^{-1}(B_J) = \Phi^N_J(\sigma_N^{-1}\mathsf{B})\,,$$

we see that

$$f_N(x) = \begin{cases} \Psi^N_J \circ f_{N-1}(x) & \text{for } x \in \Phi^N_J(\sigma_N^{-1}\mathsf{B})\,, \\ f_{N-1}(x) & \text{otherwise}\,. \end{cases}$$

Note that for $J = I \times \{j\}$, $\Phi^N_J(\sigma_N^{-1}\mathsf{B}) = \Phi^{N-1}_I[\mathsf{B}(a^N_j, R_N)] \subset \Phi^{N-1}_I(\mathsf{B})$.

In $\Phi^{N-1}_I(\mathsf{B})$, f_{N-2} is conformal (in fact, a linear scaling/squeeze) with $f_{N-2}[\Phi^{N-1}_I(\mathsf{B})] = C_I$. In C_I, g_{N-1} is conformal (being a dilation by $\sigma_{N-1}^{\lambda_{N-1} - 1}$). Therefore, in each ball $\Phi^{N-1}_I(\mathsf{B})$, $f_{N-1} = g_{N-1} \circ f_{N-2}$ is conformal.

It now follows that the distortion of f_N in $\Phi_J^N(\sigma_N^{-1}\mathsf{B})$ comes only from Ψ_J^N (which has distortion K_N in $B_J^N \setminus \bar{C}_J$ and is conformal elsewhere). In particular, we deduce that the distortion of f_N is given via

$$K_{f_N} = \begin{cases} K_N & \text{in } \bigcup_J \Phi_J^N(\sigma_N^{-1}\mathsf{B} \setminus \bar{\mathsf{B}})\,, \\ \vdots & \vdots \\ K_2 & \text{in } \bigcup_{i,j} \Phi_{ij}^2(\sigma_2^{-1}\mathsf{B} \setminus \bar{\mathsf{B}})\,, \\ K_1 & \text{in } \bigcup_{i=1}^{m_1} \varphi_i^1(\sigma_1^{-1}\mathsf{B} \setminus \bar{\mathsf{B}})\,, \\ 1 & \text{everywhere else}\,. \end{cases}$$

Also, by comparing centers and radii, we corroborate that

$$f_N(E_J^N) = g_N(\bar{C}_J) = \bar{D}_J = F_J^N \quad \text{and so} \quad f_N(E_N) = F_N\,.$$

Final Step. We thus have a sequence $(f_N)_1^\infty$ of quasiconformal self-homeomorphisms of R^n. In fact, using (2.3) we see that this sequence is uniformly Cauchy, so there is a limit map $f := \lim_{N\to\infty} f_N$ that is evidently a homeomorphism. Since $f_N(E_N) = F_N$ for each N, $f(E) = F$. Also, since

$$f_N = f_{N-1} \quad \text{in } \mathsf{R}^n \setminus \bigcup_{\text{all } J} \Phi_J^N(\sigma_N^{-1}\mathsf{B})\,,$$

and, for $J = I \times \{j\}$,

$$\Phi_J^N(\sigma_N^{-1}\mathsf{B}) = \Phi_I^{N-1}[\mathsf{B}(a_j^N, R_N)] \subset E_I^{N-1}\,,$$

we deduce that

$$\{f_N \neq f_{N-1}\} \subset \bigcup_{\text{all } J} \Phi_J^N(\sigma_N^{-1}\mathsf{B}) \subset E_{N-1}\,.$$

Recalling that $E = \bigcap E_N$ is a Lebesgue null set, we see that for almost every x in R^n, the tail of the sequence $(f_N(x))_1^\infty$ is constant. It follows that f is absolutely continuous on lines and differentiable almost everywhere with a Jacobian that is positive almost everywhere. The absolute continuity guarantees that f belongs to $\mathsf{W}^{1,1}_{\text{loc}}(\mathsf{R}^n, \mathsf{R}^n)$. Being a Sobolev homeomorphism, we know that the Jacobian of f belongs to $\mathsf{L}^1_{\text{loc}}(\mathsf{R}^n)$; see for example [IM01, Cor.6.3.1, p.108].

Thus f is a finite distortion homeomorphism with distortion function

$$K_f = \begin{cases} K_N & \text{in } \bigcup_J \Phi_J^N(\sigma_N^{-1}\mathsf{B} \setminus \bar{\mathsf{B}})\,, \\ 1 & \text{everywhere else}\,. \end{cases}$$

We note that $K_f = K_N$ occurs in the union of $M_N := m_1\, m_2 \cdots m_N$ spherical rings each with outer radius $\sigma_1 R_2 \cdots \sigma_{N-1} R_{N-1} R_N$ and inner radius $\sigma_1 R_2 \cdots \sigma_N R_N$; here we set $\sigma_0 = R_0 := 1$.

To determine the $\mathsf{L}^p_{\text{loc}}$ integrability of the distortion function $x \mapsto K_f(x)$, it suffices to examine the convergence of the series

$$\sum_{N=1}^{\infty} M_N\, [(\sigma_1 R_2 \cdots \sigma_{N-1} R_{N-1} R_N)^n - (\sigma_1 R_2 \cdots \sigma_N R_N)^n]\, K_N^p\,.$$

Recalling that $\sigma_N R_N^n = m_N^{-n/s}$ and $\sigma_N = \kappa_N^{-1} m_N^{(s-n)/ns}$, we find that the above series equals

$$\sum_{N=1}^{\infty} M_N(\sigma_N^{-n}-1)(\sigma_1 R_2 \cdots \sigma_N R_N)^n K_N^p = \sum_{N=1}^{\infty}(\sigma_N^{-n}-1) M_N^{1-n/s} K_N^p$$

$$= \sum_{N=1}^{\infty}(\kappa_N^n\, m_N^{(n-s)/s}-1) M_N^{1-n/s} K_N^p\,. \tag{4.0}$$

In certain of our specific examples we consider *regular Cantor dusts* by which we mean that $\kappa_N = \kappa$ and $m_N = m$ are some fixed constants. In this setting, the above convergence question simplifies to looking at convergence of the series

$$\sum_{N=1}^{\infty} m^{N(1-n/s)} K_N^p\,. \tag{4.1}$$

We also need to estimate $\mathcal{H}^h(F)$, at least to show that this is zero or finite. For this it suffices to examine the behavior of $M_N\, h(\mathrm{diam}(F_N^J))$ as $N \to \infty$. Recall that $F = \bigcap F_N$ where F_N is the union of M_N disjoint closed balls each of radius $t_N := \sigma_1^{\lambda_1} R_1 \cdots \sigma_N^{\lambda_N} R_N$; this simplifies to $t_N = \sigma^{\lambda_1+\cdots\lambda_N} R^N$ when F is a regular Cantor dust.

In summary, the above construction produces a finite distortion homeomorphism $\mathsf{R}^n \xrightarrow{f} \mathsf{R}^n$ and Cantor dusts $E, F \subset \mathsf{B}^n$ with $f(E) = F$ and $\mathcal{H}^s(E) \simeq 1$. The integrability of the distortion of f can be determined by checking the convergence of the appropriate series in (4.0) or (4.1). Finally, we can provide upper estimates for the Hausdorff measure $\mathcal{H}^h(F)$ by controlling the radii t_N of the balls used to construct F.

4.B. **Via Regular Cantor Dusts.** In Example 4.2 we show that any Hausdorff dimension gauge h with the conclusion of Theorem B holding cannot have $h_\alpha \preceq h$ for any $\alpha > \alpha_0$; in particular, if $h = h_\beta$, then $\beta \le \alpha_0$.

4.2. *Example.* Let $p \in [1,\infty)$, $s \in (0,n)$, and $\alpha_0 = \alpha_0(s,p,n) := p\,s\,(n-1)/(n-s)$. Fix $\alpha > \alpha_0$ and put

$$h(t) = h_\alpha(t) := \left(\log\frac{1}{t}\right)^{-\alpha}.$$

There exists a finite distortion homeomorphism $\mathsf{R}^n \xrightarrow{f} \mathsf{R}^n$ and a regular Cantor dust $E \subset \mathsf{B}^n$ such that f has distortion $K_f \in \mathsf{L}^p_{\mathrm{loc}}(\mathsf{R}^n)$, $\mathcal{H}^s(E) \simeq 1$, and $\mathcal{H}^h(f(E)) = 0$.

Proof. Fix an integer $m \ge 2 \vee \kappa(n)^{-sn/(n-s)}$. Choose K with

$$m^{(n-1)/\alpha} < K < m^{(n-s)/ps}\,.$$

Then $\lambda := K^{1/(n-1)} > 1$. Let f, E, and F be constructed as in §4.A using the parameters s, $m_N := m$, and $\lambda_N := \lambda^N$. According to (4.1), f has distortion $K_f \in \mathsf{L}^p_{\mathrm{loc}}(\mathsf{R}^n)$ precisely when the series

$$\sum_{N=1}^{\infty}(m^{1-n/s})^N K_N^p$$

converges. Since $K_N = \lambda_N^{n-1} = K^N$, the above is a geometric series; the upper bound on K gives us $m^{1-n/s}K^p < 1$, so the series converges and $K_f \in \mathsf{L}^p_{\mathrm{loc}}(\mathsf{R}^n)$.

Note that as E is a regular Cantor dust, we have

$$R_N = R := \kappa\, m^{-1/n} \quad \text{and} \quad \sigma_N = \sigma := \kappa^{-1} m^{(s-n)/sn}$$

for all N, where $\kappa = \kappa(n)$.

It remains to verify that $\mathcal{H}^h(F) = 0$. Recall that $F = \bigcap_1^\infty F_N$ where F_N is a disjoint union of m^N closed balls each of radius

$$t_N := \sigma_1^{\lambda_1} R_1 \cdots \sigma_N^{\lambda_N} R_N = \sigma^{S_N}\, R^N$$

where

$$S_N := \lambda_1 + \cdots + \lambda_N = \lambda + \lambda^2 + \cdots + \lambda^N = \frac{\lambda}{\lambda - 1}\left(\lambda^N - 1\right) .$$

We examine

$$m^N h(t_N) = \frac{m^N}{S_N^\alpha}\left(\log\frac{1}{\sigma} + \frac{N}{S_N}\,\log\frac{1}{R}\right)^{-\alpha} .$$

Evidently,

$$\frac{N}{S_N} = \frac{\lambda - 1}{\lambda}\,\frac{N}{\lambda^N - 1} \longrightarrow 0 \;\text{ as } N \to \infty .$$

The lower bound on K gives $m^{N/\alpha} < K^{N/(n-1)} = \lambda^N$, and thus

$$\frac{m^N}{S_N^\alpha} = \left(\frac{\lambda - 1}{\lambda}\right)^\alpha \left(\frac{m^{N/\alpha}}{\lambda^N - 1}\right)^\alpha \longrightarrow 0 \;\text{ as } N \to \infty .$$

From this we deduce that $m^N h(t_N) \longrightarrow 0$ as $N \to \infty$ and hence that $\mathcal{H}^h(F) = 0$. $\square$

The above example reveals several things regarding Theorem B. Assume h is a Hausdorff gauge with the conclusion of Theorem B holding, and with $h_{\beta_0} \preceq h$. Then it cannot be that $h_\alpha \preceq h$ for any $\alpha > \alpha_0$. This means that

$$\forall\, \alpha > \alpha_0 , \quad \limsup_{t \to 0^+} h(t)\left(\log\frac{1}{t}\right)^\alpha = \infty .$$

For gauges of the form $h = h_\beta$ this implies that $\beta_0 \le \beta \le \alpha_0$. In particular, we see that the exponent β_0 is asymptotically sharp as $s \to 0$ in the following sense: any such β will satisfy

$$1 \le \frac{\beta}{\beta_0} \le \frac{\alpha_0}{\beta_0} = \frac{n}{n - s} \longrightarrow 1 \quad \text{as } s \to 0^+ .$$

The above discussion leads to the following.

4.3. *Questions.*

(a) Does Theorem B hold for some Hausdorff gauge h with $h_{\beta_0} \prec h$?
(b) Does Theorem B hold for some Hausdorff gauge h_β with $\beta > \beta_0$?
(c) Does Theorem B hold for the Hausdorff gauge h_{α_0}?
(d) Does Example 4.2 hold with $\alpha = \alpha_0$?
(e) Does Example 4.2 hold for some Hausdorff gauge h with $h \preceq h_{\alpha_0}$?

4.C. **Via Generalized Cantor Dusts.** We close this paper with an example that provides some partial information regarding Questions 4.3. We consider generalized Cantor dusts E, F and allow F to possibly have positive, but finite, Hausdorff measure.

4.4. *Example.* Let $p \in [1, \infty)$, $s \in (0, n)$, and $\alpha = \alpha_0(s, p, n) := p\, s\,(n-1)/(n-s)$. Put

$$h(t) = h_\alpha(t) := \left(\log \frac{1}{t}\right)^{-\alpha} .$$

There exists a finite distortion homeomorphism $\mathsf{R}^n \xrightarrow{f} \mathsf{R}^n$ and a Cantor dust $E \subset \mathsf{B}^n$ such that f has distortion $K_f \in \mathsf{L}^p_{\mathrm{loc}}(\mathsf{R}^n)$, $\mathcal{H}^s(E) \simeq 1$, and $\mathcal{H}^h(f(E)) < \infty$.

Proof. As in the general construction, for each N we select m_N disjoint closed balls each of radius R_N so that $m_N\, R_N^n = \kappa_N^n$ and $m_N(\sigma_N\, R_N)^s = 1$; here $0 < \kappa_N \le \kappa(n) = 1/\sqrt{8n}$. Thus, given κ_N and $m_N > 1/\kappa_N^{ns/(n-s)}$, we set

$$R_N := \kappa_N\, m_N^{-1/n} \quad \text{and} \quad \sigma_N := \kappa_N^{-1} m_N^{(1/n)-(1/s)} ;$$

the lower bound on m_N ensures that $0 < \sigma_N < 1$. These choices of R_N and σ_N, together with appropriate placement of the balls, guarantee that $\mathcal{H}^s(E) \simeq 1$ (and produce a finite distortion homeomorphism $f : \mathsf{R}^n \to \mathsf{R}^n$ with $f(E) = F$.) In fact, below we take

$$m_N := \lceil (2/\kappa_N)^{ns/(n-s)} \rceil \quad \text{with} \quad \kappa_N \to 0$$

where κ_N are chosen to ensure that $K_f \in \mathsf{L}^p_{\mathrm{loc}}(\mathsf{R}^n)$.

Now we explain how to define the distortion numbers λ_N so that $\mathcal{H}^h(F) < \infty$. Recall that the target dust $F = \bigcap F_N$ is constructed with F_N a union of $M_N := m_1 \ldots m_N$ disjoint closed balls each of radius

$$t_N = \sigma_1^{\lambda_1} R_1 \cdots \sigma_N^{\lambda_N} R_N .$$

It is not difficult to choose λ_N so that $M_N\, h(t_N) = 1$; this guarantees that $\mathcal{H}^h(F) < \infty$. To this end, notice that (with $t_0 = 1$)

$$t_N = h^{-1}\left(M_N^{-1}\right) = \exp\left(-M_N^{1/\alpha}\right)$$

holds if and only if

$$\sigma_N^{\lambda_N} R_N = \frac{t_N}{t_{N-1}} = \exp\left(M_{N-1}^{1/\alpha} - M_N^{1/\alpha}\right) = \exp\left(M_N^{1/\alpha}[m_N^{-1/\alpha} - 1]\right)$$

which is valid precisely when

$$\lambda_N = \frac{-M_N^{1/\alpha}[m_N^{-1/\alpha} - 1] + \log R_N}{-\log \sigma_N} .$$

That is, by taking

$$\lambda_N := \frac{M_N^{1/\alpha}(1 - m_N^{-1/\alpha}) - (1/n)\log m_N + \log \kappa_N}{[(1/s) - (1/n)]\log m_N + \log \kappa_N} , \tag{4.5}$$

we obtain $M_N\, h(t_N) = 1$ and hence $\mathcal{H}^h(F) < \infty$. (The alert reader recognizes that we require $\lambda_N \ge 1$ in order for (4.5) to be a valid choice of the distortion constants. Evidently, there is an integer $m_0 = m_0(p, s, n)$ with the property that

$$\forall\, m \ge m_0 , \quad m^{1/\alpha} - 1 \ge \frac{1}{s} \log m .$$

It is straightforward to check that when $m_N \geq m_0$, equation (4.5) gives $\lambda_N \geq 1$.)

Now we examine the requirement that $K_f \in \mathsf{L}^p_{\text{loc}}(\mathsf{R}^n)$. According to (4.0), this holds provided

$$\sum_{N=1}^{\infty} \left(\kappa_N^n m_N^{(n/s)-1} - 1 \right) M_N^{1-n/s} K_N^p < \infty \,,$$

where $K_N = \lambda_N^{n-1}$ and $M_N = m_1 \cdots m_N$. Again, λ_N is given by (4.5), m_N will depend on κ_N, and we get to choose κ_N. Since

$$\left(\kappa_N^n m_N^{(n/s)-1} - 1 \right) M_N^{1-n/s} = \left(\kappa_N^n - m_N^{1-(n/s)} \right) M_{N-1}^{1-n/s} < \kappa_N^n \, M_{N-1}^{1-n/s} \,,$$

we see that the above series is dominated by

$$\sum_{N=1}^{\infty} \kappa_N^n \, M_{N-1}^{1-n/s} \, \lambda_N^{p(n-1)}$$

which in turn is easily seen—by using (4.5)—to be comparable to

$$\sum_{N=1}^{\infty} \kappa_N^n M_{N-1}^{1-n/s} \left(\frac{M_N^{1/\alpha}}{\log m_N} \right)^{p(n-1)} .$$

This latter sum is just

$$\sum_{N=1}^{\infty} \kappa_N^n m_N^{p(n-1)/\alpha} M_{N-1}^{1-(n/s)+p(n-1)/\alpha} \left(\log m_N\right)^{-p(n-1)} = \sum_{N=1}^{\infty} \kappa_N^n m_N^{(n/s)-1} \left(\log m_N\right)^{-p(n-1)} \,;$$

here equality holds because $\alpha = p\, s\,(n-1)/(n-s)$, and so $p(n-1)/\alpha = (n/s) - 1$.

It remains to verify that we can choose κ_N and m_N so that the above series converges. Fix $q \in (0, p\,(n-1))$. Define

$$\kappa_N := \kappa(n)\, 2^{-N^{1/q}} \quad \text{and then set} \quad m_N := \lceil (2/\kappa_N)^{ns/(n-s)} \rceil \,.$$

Evidently, $\kappa_N^n m_N^{(n/s)-1} \leq 2^{n-1+n/s} = C(s,n)$. Also,

$$\log m_N \geq \frac{ns}{n-s} \, \log \frac{2}{\kappa_N} \geq \log 2 \, \frac{ns}{n-s} \, N^{1/q}$$

and therefore

$$\sum_{N=1}^{\infty} \kappa_N^n m_N^{(n/s)-1} \left(\log m_N\right)^{-p(n-1)} \lesssim \sum_{N=1}^{\infty} \frac{1}{N^{p(n-1)/q}} < \infty \,.$$

We note that as $\kappa_N \to 0$, there is an N_0 so that for all $N \geq N_0$, $m_N \geq m_0$. We take $\lambda_1 = \lambda_2 = \cdots = \lambda_{N_0} = 1$ and use (4.5) to define λ_N for $N > N_0$. □

We close by pointing out that Example 4.4 provides the following information regarding the "best possible" gauge for Theorem B. If this theorem is true for some Hausdorff gauge h, then $h_{\alpha_0} \prec h$ cannot hold, and thus

$$\limsup_{t \to 0^+} h(t) \left(\log \frac{1}{t} \right)^{\alpha_0} > 0 \,.$$

References

[Gil10] James T. Gill, *Planar maps of sub-exponential distortion*, Ann. Acad. Sci. Fenn. Math. **35** (2010), no. 1, 197–207.

[HKM93] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Univ. Press, Oxford, 1993.

[IKO01] T. Iwaniec, P. Koskela, and J. Onninen, *Mappings of finite distortion: monotonicity and continuity*, Invent. Math. **144** (2001), no. 3, 507–531.

[IM01] T. Iwaniec and G.J. Martin, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs, Oxford University Press, New York, 2001.

[KKM01] Janne Kauhanen, Pekka Koskela, and Jan Malý, *Mappings of finite distortion: condition N*, Michigan Math. J. **49** (2001), no. 1, 169–181.

[KKM+03] Janne Kauhanen, Pekka Koskela, Jan Malý, Jani Onninen, and Xiao Zhong, *Mappings of finite distortion: sharp Orlicz-conditions*, Rev. Mat. Iberoamericana **19** (2003), no. 3, 857–872.

[KT10] Pekka Koskela and Juhani Takkinen, *Mappings of finite distortion: formation of cusps. III*, Acta Math. Sin. (Engl. Ser.) **26** (2010), no. 5, 817–824.

[KZZ09] Pekka Koskela, Aleksandra Zapadinskaya, and Thomas Zürcher, *Generalized dimension distortion under planar Sobolev homeomorphisms*, Proc. Amer. Math. Soc. **137** (2009), no. 11, 3815–3821.

[KZZ10] ———, *Mappings of finite distortion: generalized Hausdorff dimension distortion*, J. Geom. Anal. **20** (2010), no. 3, 690–704.

[Mat95] P. Mattila, *Geometry of sets and measures in euclidean in spaces: Fractals and rectifiability*, Cambridge studies in advanced math., no. 44, Cambridge Univ. Press, Cambridge, 1995.

[Raj11] Tapio Rajala, *Planar Sobolev homeomorphisms and Hausdorff dimension distortion*, Proc. Amer. Math. Soc. **139** (2011), no. 5, 1825–1829.

[Res89] Yu. G. Reshetnyak, *Space mappings with bounded distortion*, Translations of Mathematical Monographs, vol. 73, American Mathematical Society, Providence, RI, 1989, Translated from the Russian by H. H. McFaden.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, Facultat de Ciències, Campus de la U.A.B., 08193 Bellaterra (Barcelona), Catalonia

E-mail address: albertcp@mat.uab.cat

Department of Mathematics, University of Cincinnati, OH 45221

E-mail address: David.Herron@UC.edu