POISSON TYPE GENERATORS FOR $L^1(\mathbb{R})$

GERARD ASCENSI AND JOAQUIM BRUNA

Abstract. We characterize the discrete sets $\Lambda \subseteq \mathbb{R}$ such that $\{\varphi(t-\lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$, φ being an $L^1(\mathbb{R})$-function whose Fourier transform behaves like $e^{-2\pi|\xi|}$.

1. Introduction

The study of the generators by translations for $L^p(\mathbb{R})$ has been a classical topic of study in harmonic analysis. Results in [Bru06] and [BOU06] characterize the discrete sets $\Lambda \subseteq \mathbb{R}$ for which there exists a function $\varphi \in L^1(\mathbb{R})$ with the property that $\{\varphi(t-\lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$ as those having infinite Beurling-Malliavin density. In [Ole97] and [OlU04] we can find results proving that in $L^2(\mathbb{R})$ there are more sets whit this property, and that a characterization in terms of densities is not posible.

Given a function φ, a natural problem is to characterize the discrete sets Λ such that $\{\varphi(t-\lambda), \lambda \in \Lambda\}$ span $L^p(\mathbb{R})$. There are very few complete results of this kind. In [BrM07] Bruna and Melnikov give a complete characterization for the Poisson function:

$$P(t) = \frac{1}{\pi}\frac{1}{1+t^2}.$$

Theorem 1.1 (Bruna, Melnikov)**.** *The translates $\{P(t-\lambda), \lambda \in \Lambda\}$ span $L^p(\mathbb{R})$, $1 \leqslant p < \infty$ if and only if*

$$\sum_{\lambda \in \Lambda} e^{-\frac{\pi}{2}|\lambda|} = \infty. \tag{1}$$

In the $L^2(\mathbb{R})$ case, by Fourier transform, $\{\varphi(t-\lambda), \lambda \in \Lambda\}$ span $L^2(\mathbb{R})$ if and only if the set of exponentials $\{e^{2\pi i\lambda\xi}, \lambda \in \Lambda\}$ span the weighted space $L^2(\mathbb{R}, |\widehat{\varphi}|^2)$, and hence the above characterization holds

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for any function φ such that $|\widehat{\varphi}| \simeq e^{-2\pi|\xi|}$. The aim of this note is to give a generalization of this type for the $L^1(\mathbb{R})$ case.

Theorem 1.2. *Assume $\varphi \in L^1(\mathbb{R})$ has non-vanishing Fourier transform satisfying*

$$Ae^{-2\pi|\xi|} \leq |\widehat{\varphi}(\xi)| \leq Be^{-2\pi|\xi|}$$

for some constants A, B. Assume also that $|(\widehat{\varphi})'|(\xi) = O(e^{-2\pi|\xi|})$. Then $\{\varphi(t-\lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$ if and only if condition (1) *holds.*

In fact the proof will show that condition (1) is necessary if $Ae^{-2\pi|\xi|} \leqslant |\widehat{\varphi}(\xi)|$ and it is sufficient if both $\widehat{\varphi}(\xi)$ and $(\widehat{\varphi})'(\xi)$ are $O(e^{-2\pi|\xi|})$.

2. Proof of the theorem

Notice first that if certain translates of $\varphi \in L^1(\mathbb{R})$ span $L^1(\mathbb{R})$, then obviously $\widehat{\varphi}(\xi) \neq 0$ for all ξ. In fact, non-vanishing of $\widehat{\varphi}$ characterizes (as a consequence of Wiener's Tauberian theorem) those φ such that all its translates span $L^1(\mathbb{R})$. Analogously, for $p = 2$, a necessary condition in order than some translates of φ span $L^2(\mathbb{R})$ is that $\widehat{\varphi}(\xi) \neq 0$ for almost all ξ, this being equivalent to the fact that all translates of φ span $L^2(\mathbb{R})$.

Lemma 1. *Assume $h \in L^1(\mathbb{R})$ and that $\widehat{h}(\xi) \neq 0$ for all ξ. Then, if $f \in L^p(\mathbb{R}), 1 \leq p \leq \infty$, and the convolution $f * h$ is zero, then $f = 0$. The same holds if $h \in L^2(\mathbb{R})$ and $\widehat{h}(\xi) \neq 0$ almost everywhere.*

Proof. For $1 \leq p \leq 2$, the Fourier transform of f is a function in $L^q, \frac{1}{p} + \frac{1}{q} = 1$, and the Fourier transform of $f * h$ is $\widehat{f}\,\widehat{h}$, so the lemma follows. In the general case, we consider the closed subspace E of $L^1(\mathbb{R})$ consisting of functions g such that $f * g = 0$; since E is translation invariant and contains h, Wiener's tauberian theorem implies that E is the whole $L^1(\mathbb{R})$, and this implies $f = 0$. When $h \in L^2(\mathbb{R})$ we use Beurling's theorem describing all closed translation-invariant subspaces of $L^2(\mathbb{R})$ to reach the same result. $\square$

We note in passing that the version of Wiener or Beurling theorem for $L^p(\mathbb{R}), 1 < p < 2$, is still unknown.

In the following, we assume that $\varphi \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ satisfies $\widehat{\varphi}(\xi) \neq 0$ for all ξ or else $\varphi \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ with $\widehat{\varphi}(\xi) \neq 0$ for almost all ξ. By duality, $\{\varphi(t-\lambda), \lambda \in \Lambda\}$ span $L^p(\mathbb{R})$ if and only if $f \in L^q(\mathbb{R})$ and

$$\widetilde{\varphi} * f(\lambda) = \int_{\mathbb{R}} f(t)\varphi(t-\lambda)\, dt = 0 \quad \forall \lambda \in \Lambda$$

implies $f = 0$. Here $\widetilde{\varphi}(t) = \varphi(-t)$. By the lemma, $f = 0$ is equivalent to $\widetilde{\varphi} * f = 0$, whence $\{\varphi(t-\lambda), \lambda \in \Lambda\}$ span $L^p(\mathbb{R})$ if and only if Λ is a uniqueness set for the space

$$E^q_\varphi = \{F = f * \widetilde{\varphi}; f \in L^q(\mathbb{R})\}$$

meaning that $F \in E^q_\varphi, F(\lambda) = 0, \lambda \in \Lambda$, implies $F = 0$.

Lemma 2. *Assume $h \in L^1(\mathbb{R})$ and $\widehat{h}(\xi) \neq 0$ for every ξ (respectively, $h \in L^2(\mathbb{R})$ with $\widehat{h}(\xi) \neq 0$ almost everywhere). Then if $\{\varphi(t-\lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$ then $\{(\varphi * h)(t-\lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$ (respectively $L^2(\mathbb{R})$).*

Proof. For $f \in L^q(\mathbb{R})$,

$$\int_{\mathbb{R}} f(t)\,(\varphi * h)(t-\lambda)\,dt = \int_{\mathbb{R}} (\widetilde{h} * f)(x)\varphi(x-\lambda)\,dx,$$

whence the result follows from lemma 1. □

Lemma 3. *If $\phi(t) = P * \widehat{P}(t)$ then $\{\phi(t-\lambda), \lambda \in \Lambda\}$ span $L^p(\mathbb{R}), 1 \leq p < \infty$ if and only if* (1) *holds*

Proof. It is clear that this condition is sufficient, since ϕ is a convolution of P with a function of $L^1(\mathbb{R})$ and we can apply lemma 2. For the necessity we will revise the proof of theorem 1.1. By duality, we must see that if $\sum_{\lambda\in\Lambda} e^{-\frac{\pi}{2}|\lambda|} < \infty$ then we can find $g \in L^q(\mathbb{R}), \neq 0$ such that:

$$\int_{\mathbb{R}} g(t)\phi(t-\lambda)\,dt = 0 \qquad \forall \lambda \in \Lambda,$$

where we can think that g is real. The above integral equals

$$\int_{\mathbb{R}} g(t)\big(P * \widehat{P}\big)(t-\lambda)\,dt = \frac{1}{\pi}\int_{\mathbb{R}} \big(g * \widehat{P}\big)(t)\frac{1}{1+(t-\lambda)^2}\,dt$$

Now we complexify this expression:

$$F(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{(t-z)^2+1}\,dt \tag{2}$$

with $f = g * \widehat{P}$. When f ranges in $L^q(\mathbb{R})$, F ranges in the space $E^q(B)$ which in [BrM07] is shown to be the space of holomorphic functions in $B = \Im z < 1$ such that:

$$\|F\| = \sup_{|y|<1}\int_{\mathbb{R}} |\Re F(x+iy)|^q\,dx = \|F\|_q^q < \infty$$
$$F(\overline{z}) = \overline{F(z)}, \quad z \in B.$$

For $E^\infty(B)$ the first condition is replaced by $\Re F$ bounded. Hence we have to find $F \in E^q(B)$ such that $F(\lambda) = 0$ for every $\lambda \in \Lambda$ and that it can be written as (2) with $f = g * \widehat{P}$ for some $g \in L^q(\mathbb{R})$. We use the Fourier transform to see that:

$$F(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{-2\pi|\xi|} e^{2\pi i x\xi}\, d\xi.$$

By analytical continuation we obtain:

$$F(z) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{-2\pi|\xi|} e^{2\pi i z\xi}\, d\xi,$$

where we want that $\widehat{f}(\xi) = \frac{1}{\pi}\frac{\widehat{g}(\xi)}{1+\xi^2}$ with $g \in L^q(\mathbb{R})$. That is, we search for $F \in E^q(B)$ that can be written as:

$$F(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\widehat{g}(\xi)}{1+\xi^2} e^{-2\pi|\xi|} e^{2\pi i z\xi}\, d\xi$$

with $g \in L^q(\mathbb{R})$. Since

$$F''(z) = \int_{\mathbb{R}} \widehat{f}(\xi)(2\pi i\xi)^2 e^{-2\pi|\xi|} e^{2\pi i z\xi}\, d\xi,$$

this amounts to $F'' \in E^q(B)$. So we have reduced the problem to find $F \in E^q(B)$ such that $F(\lambda) = 0$ for $\lambda \in \Lambda$ and such that $F'' \in E^q(B)$.

Now, as in [BrM07], we translate the problem to the disk. The comformal map from B to the disk is given by

$$w = \Phi(z) = \frac{e^{\frac{\pi}{2}z} - 1}{e^{\frac{\pi}{2}z} + 1}.$$

Let $\Gamma = \Phi(\Lambda) \subset \mathbb{R} \cap \mathbb{D}$, as shown in [BrM07] finiteness of $\sum_{\lambda\in\Lambda} e^{-\frac{\pi}{2}|\lambda|}$ is equivalent to

$$\sum_{\gamma\in\Gamma} \log\frac{1}{|\gamma|} < \infty,$$

that is the Blaschke condition. This guarantees that the product

$$\beta(w) = \prod_{\gamma\in\Gamma} \frac{w-\gamma}{1-\gamma w}$$

is convergent (it is necessary to multiply by w if $0 \in \Gamma$).

We suppose that H is a holomorphic function of the disk. If $F(z) = H(\Phi(z))$ then $F \in E^q(B)$ exactly when $g(s) = F(s \pm i) = H\left(\frac{ie^{\frac{\pi}{2}s}-1}{ie^{\frac{\pi}{2}s}+1}\right)$ is in $L^q(\mathbb{R})$, that is if

$$\int_{\mathbb{R}} |g(s)|^q\, ds = \frac{1}{\pi}\int_{|z|=1} |H(z)|^q \frac{|dz|}{|1-z^2|} < \infty. \tag{3}$$

We need $F'' \in E^q(B)$ as well. Computing

$$F''(z) = H''\big(\Phi(z)\big)\left(\frac{\pi e^{\frac{\pi}{2}z}}{(e^{\frac{\pi}{2}z}+1)^2}\right)^2 + H'\big(\Phi(z)\big)\frac{\pi^2}{2}\frac{e^{\frac{\pi}{2}z}(1-e^{\frac{\pi}{2}z})}{(e^{\frac{\pi}{2}z}+1)^3}$$

and changing variables again we need

$$\int_{|z|=1}\left|H''(z)\frac{\pi i(z+1)(z-1)}{\big(i(z+1)+(z-1)\big)^2}\right|^q\frac{|dz|}{|1-z^2|}<\infty \tag{4}$$

$$\int_{|z|=1}\left|H'(z)\frac{\pi^2}{2}\frac{i(z+1)(z-1)\big((z-1)-i(z+1)\big)}{\big(i(z+1)+(z+1)\big)^3}\right|^q\frac{|dz|}{|1-z^2|}<\infty. \tag{5}$$

Therefore we have to find a holomorphic function H in the disk with $H(\gamma)=0$ for $\gamma\in\Gamma$, $H(\overline{z})=\overline{H(z)}$ and so that (3), (4) and (5) are fulfilled. We choose $H(z)=(1-z^2)^n\beta(z)$ with n big enough. We first bound the derivatives of β:

$$\beta'(z)=\sum_{\gamma\in\Gamma}\frac{1-\gamma^2}{(1-\gamma z)^2}\prod_{\lambda\in\Gamma,\lambda\neq\gamma}\frac{z-\lambda}{1-\lambda z}.$$

The product is bounded by 1 independently of γ for almost all z. Moreover, for $|z|=1$ we have $|1-\gamma z|=|z-\lambda|\geqslant\frac{1}{2}|1-z^2|$ and $|1-\gamma^2|\leqslant 2(1-|\gamma|)$. Therefore

$$|\beta'(z)|\leqslant\frac{2}{|1-z^2|^2}\sum_{\gamma\in\Gamma}|1-\gamma^2|\leqslant\frac{4}{|1-z^2|^2}\sum_{\gamma\in\Gamma}1-|\gamma|\leqslant\frac{2K}{|1-z^2|^2},$$

where we have used the Blaschke condition. For the second derivative we have

$$\beta''(z)=2\sum_{\gamma_1\neq\gamma_2\in\Gamma}\frac{1-\gamma_1^2}{(1-\gamma_1 z)^2}\frac{1-\gamma_2^2}{(1-\gamma_2 z)^2}\prod_{\lambda\in\Gamma,\lambda\neq\gamma_1,\gamma_2}\frac{z-\lambda}{1-\lambda z}+$$
$$+2\sum_{\gamma\in\Gamma}\frac{1-\gamma^2}{(1-\gamma z)^3}\prod_{\lambda\in\Gamma,\lambda\neq\gamma}\frac{z-\lambda}{1-\lambda z},$$

which similarly can be bound by

$$|\beta''(z)|\leqslant\frac{12K^2}{|1-z^2|^4}.$$

Then, choosing $H(z)=(1-z^2)^4\beta(z)$ all required conditions are fulfilled and the proof is finished. $\square$

Lemma 4. *Let $\psi(t)=P(t)-P''(t)$. Then $\{\psi(t-\lambda),\lambda\in\Lambda\}$ span $L^p(\mathbb{R})$ if and only if condition* (1) *holds*

Proof. Notice that $\widehat{\psi}(\xi) = (1 + 4\pi^2\xi^2)e^{-2\pi|\xi|}$. The space E^q_ψ consists of the functions

$$F(z) = \langle f(t), \psi(t-z)\rangle = \int_{\mathbb{R}} \widehat{f}(\xi)(1+4\pi^2\xi^2)e^{-2\pi|\xi|}e^{-2\pi i\xi z} = G(z)-G''(z)$$

with $G \in E^q(B)$, that is with $f \in L^q(\mathbb{R})$. Clearly this space contains $E^q(B)$, and so the condition (1) is necessary, for if the series converges we already know that there is $H \in E^q(B)$ vanishing on Λ.

For the sufficiency we find a growth condition fulfilled by the second derivative of a function of $E^q(B)$. There is a constant c_q such that whenever G is holomorphic in a disk $D(a, R)$ of center a and radius R one has

$$|G''(a)|^q \leq c_q \frac{1}{R^{2q+2}} \int_{D(a,R)} |G(z)|^q dA(z).$$

Let $G \in E^q(B)$. For $z \in B$ we apply the above to the ball of center z and radius $\frac{1-|y|}{2}$ ($z = x + iy$) to get

$$\begin{aligned}\int_B (1 - |y|)^{2q}|G''(z)|^q \, dm(z) \\ \leqslant c_q \int_B \frac{1}{(1 - |y|)^2} \int_{B\left(z, \frac{1-|y|}{2}\right)} |G(w)|^q \, dm(w) \, dm(z).\end{aligned}$$

Applying Fubini and noticing that $\left\{z : w \in B\left(z, \frac{1-|y|}{2}\right)\right\} \subset B(w, 1 - |\Im w|)$ and that for z in this set $(1 - |\Im w|)^2 \leq c(1 - |y|)^2$ we obtain

$$\int_B (1 - |y|)^{2q}|G''(z)|^q \, dm(z) \leqslant c_q \int_B |G(w)|^q \, dm(w).$$

This last integral is bounded since

$$\int_B |G(z)|^q \, dm(z) = \int_{-1}^{1} \int_{\mathbb{R}} |G(x + iy)|^q \, dx \, dy \leqslant \int_{-1}^{1} \|G\|^q \, dy = 2\|G\|^q.$$

This says that G'' is in a Bergman type space. Obviously G satisfies this condition too, and so the above holds with G replaced by $F \in E^q_\psi$. We next translate this integral to the disk.

We can check that if $w = \Phi(z) = \frac{e^{\frac{\pi}{2}z}-1}{e^{\frac{\pi}{2}z}+1}$ then $1 - |y| \geqslant 1 - |w|$. If $H(w) = F(\Phi^{-1}(w))$ then

$$\begin{aligned}\int_{\mathbb{D}} |H(w)|^q(1 - |w|)^{2q-1} \, dm(w) &\leqslant \int_D (1 - |w|)^{2q}|H(w)|^q \frac{dm(w)}{|1 - w^2|} \\ &\leqslant \int_B (1 - |y|)^{2q}|F''(z)|^q \, dm(z).\end{aligned}$$

Therefore H is in the Bergman space in the disk with weight $(1-|w|)^{2q-1}$. The set of zeros contained in a diameter of a function of this space satisfies the Blaschke condition [Kor75]. Therefore the zeros of a function $F \in E^q_\psi$ satisfy $\sum_{\lambda\in\Lambda} e^{-\frac{\pi}{2}|\lambda|} < \infty$ and so (1) is sufficient. $\square$

Now, theorem 1.2 can be deduced using the previous lemmas. Assume that $\{\varphi(t-\lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$ and that $Ae^{-2\pi|\xi|} \leqslant |\widehat{\varphi}(\xi)|$. Writing

$$\phi = P * \widehat{P} = h * \varphi$$

with

$$\widehat{h}(\xi) = \frac{e^{-2\pi|\xi|}}{\widehat{\varphi}(\xi)(1+\xi^2)}.$$

By lemma 2 (in the L^2 case), the functions $\{\phi(t-\lambda)\}_{\lambda\in\Lambda}$ span $L^2(\mathbb{R})$ and therefore by lemma 3 condition (1) must hold. Assume next that $\widehat{\varphi}(\xi)$ and $(\widehat{\varphi})'(\xi)$ are $O(e^{-2\pi|\xi|})$ and that (1) holds; we write

$$\varphi = h * (P - P'')$$

with

$$\widehat{h}(\xi) = \frac{\widehat{\varphi}(\xi)}{e^{-2\pi|\xi|}(1+4\pi^2\xi^2)}.$$

The hypothesis on φ implies that both $\widehat{h}$ and $(\widehat{h})'$ are in $L^2(\mathbb{R})$, whence both $h(x)$ and $xh(x)$ are in $L^2(\mathbb{R})$, so $h \in L^1(\mathbb{R})$. Since (1) holds, by lemma 4 the functions $\{\psi(t-\lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$ for $\psi = P - P''$ and then lemma 2 implies that the functions $\{\varphi(t-\lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$.

3. Generalizations and comments.

Using the same ideas we can also prove

Theorem 3.1. *Fix $n \in \mathbb{N}$. Let φ be a function for which there exists constants $A, B > 0$ such that:*

$$A\frac{e^{-2\pi|\xi|}}{1+\xi^{2n}} \leqslant |\widehat{\varphi}(\xi)| \leqslant B(1+\xi^{2n})e^{-2\pi|\xi|}. \tag{6}$$

We also suppose that:

$$|\widehat{\varphi}'(\xi)| \leqslant C(1+\xi^{2n}e^{-2\pi|\xi|}).$$

Then the set $\{\varphi(t-\lambda), \lambda \in \Lambda\}$ spans $L^1(\mathbb{R})$ if and only if condition (1) holds.

Only slight modifications are needed. For instance, one must use that the $2n$-th derivative of a function of $E^q(B)$ is in a Bergman type space (whit a different weight); the Korenblum's result quoted before applies to all the Bergman spaces as it is in fact true for functions in the class $A^{-\infty}$. Another fact which is needed is that the $2n$-th derivative of the Blaschke product appearing above can be bounded by $\frac{K}{|1-z^2|^{4n}}$.

As used in the proof, for the function P the space E^q_P is exactly $E^q(B)$. For the functions φ considered here we have not exactly described this space, yet we can describe its uniqueness sets.

In [Zal78] the Gaussian function G is considered. A complete characterization is not achieved. In fact, one can show that in this case the space E^2_φ can be identified with the Fock space, for which the description of the uniqueness sets is an open question. It is known that a sufficient condition in order that the translates $\{G(t-\lambda), \lambda \in \Lambda\}$ span $L^2(\mathbb{R})$ is that the series $\sum_n \frac{1}{|\lambda_n|^{2+\varepsilon}}$ diverges for some ε, while it is necessary that it diverges for $\varepsilon = 0$.

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Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, 1090 Vienna, Austria

E-mail address: gerard.ascensi@univie.ac.at

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Cerdanyola del Valles, SPAIN

E-mail address: bruna@mat.uab.cat