COHOMOLOGY OF KAC-MOODY GROUPS OVER A FINITE FIELD

JAUME AGUADÉ AND ALBERT RUIZ

Abstract. We compute the mod p cohomology algebra of a family of infinite discrete Kac-Moody groups of rank two defined over finite fields of characteristic different from p.

1. Introduction

A functorial definition of discrete Kac-Moody groups over any commutative ring was established by Tits in 1987 ([Tit87]) and there is now a great deal of interest in these objects as the monograph by Rémy ([Rém02]) and the references that it contains may illustrate. The work of Kac-Peterson ([KP85]) and Kitchloo ([Kit98]) on the classifying spaces $BK(\mathbb{C})$ of the topological Kac-Moody groups over the complex field were the starting point for the study of these objects from the point of view of the so-called *homotopical group theory* (cf. the address by Grodal at the 2010 ICM, [Gro10]). Since then, some significant progress has been made in this area. For instance, we know that the mod p cohomology of $BK(\mathbb{C})$ is noetherian ([BK02]). In the case of rank two, the mod p cohomology of $BK(\mathbb{C})$ has been explicitely computed ([ABKS05]), and the self maps of $BK(\mathbb{C})$ have been described and classified ([AR03]).

The main results in this paper (Proposition 8.2 and Theorem 8.3) provide an explicit description of the cohomology algebra $H^*(BG_{\mathcal{D}}(k);\mathbb{F}_p)$ for $G_{\mathcal{D}}(k)$ a discrete infinite Kac-Moody group of rank two over a finite field k of characteristic different from p (under some restrictions, as stated in 8.3). To arrive to this result we need detailed descriptions of the root system (section 3), the parabolic subgroups and the Lévi subgroups (section 4) of $G_{\mathcal{D}}(k)$. The modular invariant theory of the dihedral group plays also a main role in our computation (sections 5 and 7). Some of these partial steps in the main result may also have interest in their own.

In the classic case of a finite Weyl group, Friedlander ([Fri76]) discovered a very interesting relationship between the p-completion of the classifying space of the Chevalley group over the field $\mathbb{F}_q$ and the homotopy fixed points of an unstable Adams map ψ^q defined on the classifying space of the corresponding compact connected Lie group. In a final section of this paper we see that this result does not generalize to the Kac-Moody group case (infinite Weyl group). This suggests a discrepancy between the algebraic and the homotopical definitions of Kac-Moody groups over finite fields, in the case of an infinite Weyl group.

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Remark 1.1. Once completed this work we became aware of results in J. Foley's PhD Thesis ([Fol12]), having a significant overlap with the results of this work and, in some points, reaching beyond (cf. Remark 4.4).

2. Kac-Moody groups over a field

In this preliminary section we recall the basic notions and notations of Kac-Moody groups over a field. Our main references are the original paper by Tits ([Tit87]) and the book by Rémy ([Rém02]).

A Kac-Moody group functor depends on a set of data $\mathcal{D}$ consisting of a $n \times n$ generalized Cartan matrix A, a free abelian group Λ, and elements $\alpha_1, \dots, \alpha_n \in \Lambda$ and $h_1, \dots, h_n \in \Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z})$ such that $\langle h_i, \alpha_j \rangle = A_{ij}$. From these data $\mathcal{D}$ one can construct a functor $G_{\mathcal{D}}(-)$ from commutative rings to groups which coincides with the classic Chevalley-Demazure functor if A is a Cartan matrix. When restricted to fields, the functor $G_{\mathcal{D}}(-)$ can be characterized by a small set of axioms.

Let k be a field and let k^+ and $k^\times$ be the additive and multiplicative groups of k, respectively. The abelian group $T = \mathrm{Hom}(\Lambda, k^\times)$ plays the role of a maximal torus in $G_{\mathcal{D}}(k)$ through a monomorphism $\eta\colon T \to G_{\mathcal{D}}(k)$. For each $1 \le i \le n$ there is a homomorphism $\phi_i\colon SL_2(k) \to G_{\mathcal{D}}(k)$ as well as two monomorphisms $x_i^+, x_i^-\colon k^+ \to G_{\mathcal{D}}(k)$ such that $x_i^+(r) = \phi_i \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$ and $x_i^-(r) = \phi_i \left(\begin{smallmatrix} 1 & 0 \\ -r & 1 \end{smallmatrix}\right)$. The kernel of each ϕ_i is central in $SL_2(k)$. The group $G_{\mathcal{D}}(k)$ is generated by T and the images of all ϕ_i. Each ϕ_i sends diagonal matrices into the maximal torus T as follows: for any $r \in k^\times$ we have

$$\phi_i \left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix}\right)(\lambda) = r^{h_i(\lambda)}. \qquad (*)$$

T normalizes $U_i^+ = x_i^+(k^+)$ and $U_i^- = x_i^-(k^+)$ according to the formula

$$t\, x_i^\pm(r)\, t^{-1} = x_i^\pm(t(\alpha_i)^{\pm 1} r) \qquad (**)$$

for any $r \in k$.

The equations $\omega_i(a_j) = a_j - A_{ij} a_i$, $i, j = 1, \dots, n$ define an action of a group W (the Weyl group) on $\mathbb{Q}a_1 \oplus \cdots \oplus \mathbb{Q}a_n$. The orbit $\Phi = W\{a_1, \dots, a_n\}$ is the root system of $G_{\mathcal{D}}(k)$. Every root is an integral linear combination of $\{a_1, \dots, a_n\}$ with coefficients all positive or all negative. We talk of positive roots Φ^+ and negative roots Φ^-. W is a finite group if and only if the matrix A is a Cartan matrix (i. e. A is the product of a diagonal matrix and a positive definite symmetric matrix). The root system depends only on the matrix A and not on the full data $\mathcal{D}$. The Weyl group W acts also on Λ (and on T) by by $\omega_i(\lambda) = \lambda - \langle h_i, \lambda \rangle\, \alpha_i$, for $\lambda \in \Lambda$ and $i = 1, \dots, n$.

$G_{\mathcal{D}}(k)$ is a group with a double BN-pair. There are Borel subgroups B^+, B^-, standard parabolic subgroups P_I^+, P_I^- for any $I \subset \{1, \dots, n\}$, root groups U_a for any $a \in \Phi$, and all the rich theory of double BN-pairs applies.

From a topological point of view, one of the most remarkable properties of Kac-Moody groups with infinite Weyl group, which does not hold in the classic case of a finite Weyl group, is the following.

Theorem 2.1. *([Mit88]) Let G be a Kac-Moody group with an infinite Weyl group and let $\mathcal{C}$ denote the poset of proper standard parabolic subgroups P_I of G. Then there is a homotopy equivalence $BG \simeq \operatorname{hocolim}_{\mathcal{C}} BP_I$.* □

The purpose of this paper is to compute $H^*(BG_{\mathcal{D}}(k);\mathbb{F}_p)$ when the free abelian group Λ is of rank two, k is a finite field, the Weyl group of $G_{\mathcal{D}}(k)$ is infinite and p is an odd prime different from the characteristic of the field k.

Throughout this paper we fix the following notations:

(1) p, ℓ are different primes, p is odd and k is a finite field of order q and characteristic ℓ.
(2) a, b are positive integers such that $ab \geq 4$ and A is the generalized Cartan matrix
$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}.$$
Without loss of generality, we assume $a \leq b$.
(3) n_i, m_i, s_i, t_i for $i = 1, 2$ are positive integers such that
$$\begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix} \begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix} = A.$$
We denote $\Delta = s_1t_2 - s_2t_1$, $\nabla = n_1m_2 - n_2m_1$, so that $\Delta\nabla = 4 - ab$.
(4) Λ is a free abelian group of rank two $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $\Lambda^\vee = \mathbb{Z}e_1^\vee \oplus \mathbb{Z}e_2^\vee$ is its $\mathbb{Z}$-dual.
(5) $\alpha_1, \alpha_2 \in \Lambda$ and $h_1, h_2 \in \Lambda^\vee$ are the elements $\alpha_i = n_ie_1 + m_ie_2$, $h_i = s_ie_1^\vee + t_ie_2^\vee$, $i = 1, 2$.
(6) G is the Kac-Moody group $G_{\mathcal{D}}(k)$ where $\mathcal{D}$ consists of the matrix A, the lattice Λ and the elements $\{\alpha_1, \alpha_2\}$ and $\{h_1, h_2\}$.

In particular, we want to emphasize that through all this paper the prime p is always assumed to be odd and different from the characteristic of the field k.

3. The root system in rank two

In this section we will study the root system Φ associated to the generalized Cartan matrix A. Recall that Φ is the orbit of the basis vectors of $\mathbb{Q}^2$ under the action of the infinite dihedral group $W = \langle \omega_1, \omega_2 \rangle$ acting as
$$\omega_1 = \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} 1 & 0 \\ b & -1 \end{pmatrix}.$$
This root system Φ has been studied in [Tit86], [KP85], [Kit98], [Rui01]. The lattice Λ and the elements $\alpha_1, \alpha_2 \in \Lambda$, $h_1, h_2 \in \Lambda^\vee$ do not play any role in this section.

If we write $\tau = \omega_1\omega_2$, then the matrices in W can be described as follows (cf. [Rui01], [Kit98])
$$\tau^n = \begin{pmatrix} d_{2n+1} & -d_{2n} \\ c_{2n} & -c_{2n-1} \end{pmatrix}, \quad \tau^n\omega_1 = \begin{pmatrix} -d_{2n+1} & d_{2n+2} \\ -c_{2n} & c_{2n+1} \end{pmatrix}$$

where the integers c_n, d_n are defined inductively for any integer n as follows:

$$c_0 = d_0 = 0, \quad c_1 = d_1 = 1,$$
$$c_{n+1} = bd_n - c_{n-1}, \quad d_{n+1} = ac_n - d_{n-1}.$$

The following proposition lists many properties of these integers that we will use in this section.

Proposition 3.1.

(1) $c_{2n+1} = d_{2n+1}$, $bd_{2n} = ac_{2n}$.
(2) $c_{2n} \equiv 0\,(b)$, $c_{2n+1} \equiv (-1)^n\,(b)$, $d_{2n} \equiv 0\,(a)$, $d_{2n+1} \equiv (-1)^n\,(a)$.
(3) $c_{-n} = -c_n$, $d_{-n} = -d_n$.
(4) *If* $a = 1$ *and* $b = 4$, *then* $d_{2n} = n$, $c_{2n} = 4n$, $c_{2n+1} = d_{2n+1} = 2n + 1$.
(5) *If* $ab > 4$, *let* $\zeta > 1$ *be a real root of* $X^4 - (ab-2)X^2 + 1$. *Then*

$$c_n = \begin{cases} \dfrac{\zeta^{2n} - 1}{\zeta^{n-1}(\zeta^2 - 1)} & n \text{ odd} \\ \\ \dfrac{b(\zeta^{2n} - 1)}{\zeta^{n-2}(\zeta^4 - 1)} & n \text{ even} \end{cases}$$

(6) $c_n, d_n > 0$ *for* $n > 0$.
(7) *For* $ab > 4$ *the function* $f(n) = c_{2n+1} - c_{2n-1}$, $n > 0$, *is strictly increasing.*
(8) *If* $a, b > 1$ *then the sequences* $\{c_n\}$, $\{d_n\}$ *are strictly increasing.*
(9) *The sequences* $\{c_{2n}\}$, $\{d_{2n}\}$, $\{c_{2n+1}\}$, $\{d_{2n+1}\}$ *are strictly increasing.*
(10) *If* $(a, b) \neq (1, 4)$ *then* $d_{2n} < d_{2n+1} < d_{2n+4}$ *for* $n \geq 0$.
(11) *If* $a = 1$ *then for* $n > 1$ *we have* $d_{2n+1} < \min\{(b-1)d_{2n}, (b-2)d_{2n-1}\}$.

Proof. Most of these properties are either evident or can be easily proven by induction. To prove (7), use (5) to obtain a formula for $f(n)$ and then check that $(d/dx)f(x) > 0$ (cf. [Rui01]). To prove (8) by induction notice that if n is odd then

$$c_{n+1} = bd_n - c_{n-1} = bc_n - c_{n-1} \geq 2c_n - c_{n-1} > c_n$$

and if n is even then

$$c_{n+1} = bd_n - c_{n-1} = ac_n - c_{n-1} \geq 2c_n - c_{n-1} > c_n$$

and similarly with $\{d_n\}$.

To prove (9) it is enough to consider the case $a = 1$. If $b = 4$, the result follows from (4). If $b \geq 5$ we know from (7) that $\{c_{2n+1}\}$ is strictly increasing. Then, $d_{2n} = (c_{2n+1} + c_{2n-1})/b$ and $c_{2n} = bd_{2n}$ are also strictly increasing.

In (10) we can also assume $a = 1$ and $b \geq 5$. If we write $d_{2n+1} - d_{2n}$ a a function of ζ using (5) then we see that $d_{2n} < d_{2n+1}$. We have

$$\zeta^4 = (b-2)\,\zeta^2 - 1 \geq 3\,\zeta^2 - 1 > \zeta^2 + 1.$$

Then if $k = 2n + 1$ we see that $\zeta^{2k+2} > \zeta^{2k} + \zeta^{2k-2} - 1$ and so

$$\zeta^{2k+6} - 1 > \zeta^2(\zeta^{2k} - 1)(\zeta^2 + 1)$$

which implies $d_{k+3} > d_k$.

If $a = 1$ we have

$$d_{2n+1} = bd_{2n} - d_{2n-1} = (b-1)d_{2n} - d_{2n-1} + d_{2n}$$

and

$$d_{2n} = d_{2n-1} - d_{2n-2} < d_{2n-1}$$

for $n > 1$. This proves the first inequality in (11). To get the second one, observe

$$\begin{aligned} d_{2n+1} &= bd_{2n} - d_{2n-1} = (b-1)d_{2n-1} - bd_{2n-2} \\ &= (b-2)d_{2n-1} + bd_{2n-2} - d_{2n-3} - bd_{2n-2} < (b-2)d_{2n-1} \end{aligned}$$

if $n > 1$. □

In this section, it is convenient to denote $\{u_0, v_0\}$ the standard basis of $\mathbb{Q}^2$. We will use the following notation ($i \geq 0$):

$$u_i = \tau^i u_0 = (d_{2i+1}, c_{2i}), \quad v_i = \tau^{-i} v_0 = (d_{2i}, c_{2i+1}),$$
$$\overline{u}_i = \tau^i \omega_1 v_0 = (d_{2i+2}, c_{2i+1}), \quad \overline{v}_i = \tau^{-i} \omega_2 u_0 = (d_{2i+1}, c_{2i+2}).$$

It is clear that these are the positive roots Φ^+ of the infinite root system Φ. We write

$$\mathcal{A} = \big\{u_i, \overline{u}_i \,|\, i \geq 0\big\}, \quad \mathcal{B} = \big\{v_i, \overline{v}_i \,|\, i \geq 0\big\}.$$

Recall ([Tit87]) that a set of roots Ψ is called *pre-nilpotent* if there are elements $\omega, \omega' \in W$ such that $\omega\Psi \subset \Phi^+$ and $\omega'\Psi \subset \Phi^-$.

Proposition 3.2. *If $e, w \in \mathcal{A}$ (resp. $\mathcal{B}$), then the pair $\{e, w\}$ is pre-nilpotent.*

Proof. If $e, w \in \mathcal{A}$ and $N > 0$ is large enough then we have $\tau^{-N}e, \tau^{-N}w \in \Phi^-$. If $e, w \in \mathcal{B}$ then we may use τ^N for $N > 0$ large enough. □

Proposition 3.3. *If $e, w \in \mathcal{A}$ (resp. $\mathcal{B}$) and $a > 1$, then $e + w \notin \Phi$.*

Proof. If $e, w \in \mathcal{A}$ and $e + w \in \mathcal{B}$ then for $N >$ large enough $\tau^N(e + w) \in \Phi^-$ while $\tau^N(e), \tau^N(w) \in \Phi^+$. Hence, if $e + w \in \Phi$ then $e + w \in \mathcal{A}$. Moreover, W leaves invariant the quadratic form

$$Q(X, Y) = bX^2 + aY^2 - abXY$$

and $Q(e) = Q(w) = Q(e + w)$ is impossible. Hence, any relation $e + w \in \Phi$ should be one of these ($i, j, k \geq 0$): 1) $u_i + u_j = \overline{u}_k$, 2) $u_i + \overline{u}_j = u_k$, 3) $u_i + \overline{u}_j = \overline{u}_k$, 4) $\overline{u}_i + \overline{u}_j = u_k$.

Each of these equalities can be translated into an equality between coefficients c_n, d_n as follows:

$$\begin{aligned} &1) \quad (d_{2i+1}, c_{2i}) + (d_{2j+1}, c_{2j}) = (d_{2k+2}, c_{2k+1}) \\ &2) \quad (d_{2i+1}, c_{2i}) + (d_{2j+2}, c_{2j+1}) = (d_{2k+1}, c_{2k}) \\ &3) \quad (d_{2i+1}, c_{2i}) + (d_{2j+2}, c_{2j+1}) = (d_{2k+2}, c_{2k+1}) \\ &4) \quad (d_{2i+2}, c_{2i+1}) + (d_{2j+2}, c_{2j+1}) = (d_{2k+1}, c_{2k}) \end{aligned}$$

In each case, reducing modulo a or b and applying 3.1(2) we get a contradiction. If we start with $e, w \in \mathcal{B}$ and assume $e + w \in \Phi$, then applying $-\tau^N$ for N large enough we obtain a relation $e' + w' \in \Phi$ with $e', w' \in \mathcal{A}$ which we have seen is not possible. □

The above result fails for $a = 1$ since in this case one sees immediately that

$$\overline{u}_0 + \overline{u}_1 = u_1, \quad \overline{v}_0 + \overline{v}_1 = v_1$$

and so, applying $\tau^{\pm i}$ we obtain relations

$$\overline{u}_i + \overline{u}_{i+1} = u_{i+1}, \quad \overline{v}_i + \overline{v}_{i+1} = v_{i+1}. \quad (*)$$

Proposition 3.4. *If $a = 1$, $b > 4$, $e, w \in \mathcal{A}$ (resp. $\mathcal{B}$) and $e + w = f \in \Phi$, then the equality $e + w = f$ is as in $(*)$ above.*

Proof. Assume $e + w = f \in \Phi$ with $e, w \in \mathcal{A}$, the case of $e, w \in B$ being equivalent. The same arguments as in 3.3 imply that $e + w = f$ has to be one of these equalities $(i, j, k \geq 0)$: 3) $u_i + \overline{u}_j = \overline{u}_k$, 4) $\overline{u}_i + \overline{u}_j = u_k$.

Notice that if $ijk > 0$ then τ^{-1} gives a relation of the same type with smaller subscripts. Hence, we can assume that at least one of the subscripts i, j, k is equal to zero.

Equation 3) implies

$$d_{2i+1} + d_{2j+2} = d_{2k+2}, \quad c_{2i} + c_{2j+1} = c_{2k+1}.$$

Hence $d_{2k+2} > d_{2j+2}$ and since the sequence $\{d_{2n}\}$ is strictly increasing, we get $k > j$. Also, since the sequence $\{c_{2n+1}\}$ is strictly increasing, we get $i > 0$. Hence, we can assume $j = 0$ which amounts to

$$d_{2i+1} + 1 = d_{2k+2}, \quad c_{2i} + 1 = c_{2k+1}.$$

Then,

$$bd_{2i} - d_{2i-1} = d_{2i+1} = d_{2k+2} - 1 = d_{2k+1} - d_{2k} - 1 = bd_{2i} - d_{2k}.$$

Using 3.1(10) we have $d_{2i+2} > d_{2i-1} = d_{2k} < d_{2k+1}$ and so $i = k$ which is impossible since this would imply $1 + d_{2k+1} = d_{2k+2} = d_{2k+1} - d_{2k}$.

On the other side, equation 4) implies

$$d_{2i+2} + d_{2j+2} = d_{2k+1}, \quad c_{2i+1} + c_{2j+1} = c_{2k1}.$$

In particular, $k > 0$ and without loss of generality we can assume $i = 0$. We have

$$1 + d_{2j+2} = d_{2k+1}, \quad 1 + c_{2j+1} = c_{2k}$$

and then

$$\begin{aligned} 1 + d_{2j+2} &= d_{2k+1} = c_{2k} - d_{2k-1} = 1 + d_{2j+1} - d_{2k-1} \\ d_{2j+2} &= d_{2j+1} - d_{2j} \end{aligned}$$

and so $d_{2j} = d_{2k-1} < d_{2k+2}$ (using 3.1(10)) and $j < k + 1$ because $\{d_{2n}\}$ is strictly increasing. On the other hand, also by 3.1(10), we have $d_{2k-2} < d_{2k-1} = d_{2j}$ and since $\{d_{2n}\}$ is strictly increasing we have $k - 1 < j$. Hence, $j = k$. Then, $c_{2k} = 1 + c_{2k+1} = 1 + c_{2k} - c_{2k-1}$ which yields $k = 1$. □

The only remaining case is $a = 1$, $b = 4$. In this case, the roots are explicitly computed in 3.1(4) and the following result can be easily obtained.

Proposition 3.5. *If $a = 1$, $b = 4$, $e, w \in \mathcal{A}$ (resp. $\mathcal{B}$) and $e + w = f \in \Phi$, then the equality $e + w = f$ is*

$$\overline{u}_i + \overline{u}_{i+2k-1} = u_{i+k}$$

for some $i \geq 0, k > 0$ (resp. $\overline{v}_i + \overline{v}_{i+2k-1} = v_{i+k}$). □

To summarize, we have seen that in the root system Φ the sum of two positive roots *of the same type* ($\mathcal{A}$ or $\mathcal{B}$) is almost never a root. The only exceptions are

$$\begin{aligned} \overline{u}_i + \overline{u}_{i+1} &= u_{i+1},\ i \geq 0 \text{ which occur for } a = 1; \\ \overline{u}_i + \overline{u}_{i+2k-1} &= u_{i+k},\ i \geq 0,\ k > 0, \text{ which occur for } (a, b) = (1, 4) \end{aligned}$$

for type $\mathcal{A}$ and similarly for type $\mathcal{B}$. Our final step in this section is to compute, in each of the cases $e + w \in \Phi$ above, the sets

$$(\mathbb{N}e + \mathbb{N}w) \cap \Phi.$$

Proposition 3.6. *If $e, w \in \mathcal{A}$ (resp. $\mathcal{B}$) and $e+w \in \Phi$, then $(\mathbb{N}e+\mathbb{N}w)\cap\Phi = \{e+w\}$.*

Proof. We know that we can assume $a = 1$. Consider first the case $n\overline{u}_0+m\overline{u}_1 = f \in \Phi$, $n, m > 0$. Then, there are only two possibilities: either $f = \overline{u}_i$ or $f = u_i$. In the first case, we can solve $n\overline{u}_0 + m\overline{u}_1 = \overline{u}_i$ for n, m and get

$$\begin{aligned} n &= (b-1)\, d_{2i+2} - (b-2)\, d_{2i+1} \\ m &= d_{2i+1} - d_{2i+2}. \end{aligned}$$

Then, $n = d_{2i+1} - (b-1)\, d_{2i}$ and then 3.1(11) yields $n = 0$. If $n\overline{u}_0 + m\overline{u}_1 = u_i$ then we can also solve for n, m:

$$\begin{aligned} n &= (b-1)\, d_{2i+1} - (b-2)\, c_{2i} \\ m &= c_{2i} - d_{2i+1}. \end{aligned}$$

Hence, $n = d_{2i+1} - (b-2)\, d_{2i-1}$ and 3.1(11) yields $n = m = 1$.

In the general case of $n\overline{u}_i + m\overline{u}_{i+1} \in \Phi$ we can apply τ^{-1} enough times to get $n\overline{u}_0 + m\overline{u}_1 \in \Phi$ which we have already ruled out.

It remains only the case $(a, b) = (1, 4)$ where we have to consider

$$n\overline{u}_i + m\overline{u}_{i+2k-1} \in \Phi, \quad i \geq 0,\ k > 0.$$

Using τ^{-i} we can assume that $i = 0$. Recall that the explicit values of c_n and d_n are given by 3.1(4). Then, the solutions for $n\overline{u}_0 + m\overline{u}_{2k-1} = \overline{u}_j$ are

$$n = 1 - \frac{j}{2k-1}, \quad m = \frac{j}{2k-1}$$

which are not possible. Also, the solutions for $n\overline{u}_0 + m\overline{u}_{2k-1} = u_j$ are

$$n = 2 - \frac{2j-1}{2k-1}, \quad m = \frac{2j-1}{2k-1}$$

which are only possible for $n = m = 1$. The proposition is proven. □

4. The parabolic subgroups and the Lévi decomposition

In the case of the Kac-Moody group G that we are considering, the poset of the proper (positive) parabolic subgroups consists of only three groups $P_\varnothing = B^+$, P_1, P_2 and 2.1 reduces to the fact that BG is the homotopy colimit of the diagram $BP_1 \leftarrow BP_\varnothing \to BP_2$. This is equivalent to saying that G is the amalgamated product of P_1 and P_2 over B^+ ([Bro82], Th. 7.3). In this section we want to investigate the group theoretical structure of these groups.

Like in the case of a finite Weyl group, there is a Lévi decomposition for parabolic subgroups of Kac-Moody groups ([Rém02], 6.2) $P_I = H_I \ltimes V_I$. In particular, we have that $P_\varnothing = B = T \ltimes U^+$ where U^+ is the subgroup generated by all positive root groups. Using the notations of section 3, we have:

Proposition 4.1. *(*[KP85]*, proposition 4.3 and remark 2) Let $U_{\mathcal{A}}$ (resp. $U_{\mathcal{B}}$) be the subgroup generated by all root groups U_e for $e \in \mathcal{A}$ (resp. $e \in \mathcal{B}$). Denote $E = \oplus_0^\infty k^+$. Then,*

1. *$U^+ = U_{\mathcal{A}} * U_{\mathcal{B}}$.*
2. *If $a > 1$, then both $U_{\mathcal{A}}$ and $U_{\mathcal{B}}$ are isomorphic to E.*
3. *If $a = 1$, then both $U_{\mathcal{A}}$ and $U_{\mathcal{B}}$ are extensions of E by E.*

Proof. (1) is in [KP85] 4.3. (2) and (3) follow from the analysis in section 3. If $a > 1$ we have seen that the sum of two roots in $\mathcal{A}$ (resp. $\mathcal{B}$) is not a root. Hence, $U_{\mathcal{A}}$ and $U_{\mathcal{B}}$ are abelian groups generated by countably many root groups, each one isomorphic to k^+. If $a = 1$ we have seen that in some cases, the sum of two roots in $\mathcal{A}$ can be a root (cf. 3.4, 3.5). However, the subgroup of $U_{\mathcal{A}}$ generated by the root groups U_{u_i} for $i \geq 0$ is abelian, isomorphic to E and normal in $U_{\mathcal{A}}$. Also, it follows from 3.4, 3.5 and 3.6 that the quotient of $U_{\mathcal{A}}$ by this normal subgroup is also abelian and isomorphic to E. The same holds for $U_{\mathcal{B}}$. □

To determine the structure of the subgroups V_I we need the following lemma.

Lemma 4.2. *Let K and M be groups and assume N is a normal subgroup of index r in K. Consider the inclusion $N * M < K * M$. Then the normal closure of $N * M$ in $K * M$ is isomorphic to the free product $N * (*_{i=1}^r M)$.*

Proof. There is an easy topological proof for this lemma. The inclusion $N \lhd K$ can be realized topologically by a pointed map of classifying spaces $\pi\colon BN \to BK$ which is a regular r-fold covering. Let $\{x_1, \dots, x_r\}$ be the fiber of π over the base point of BK and consider the map $\pi'\colon \widetilde{X} \to X$ defined as follows. $X = BK \vee BM$, $\widetilde{X} = BN \vee (BM)_1 \vee \cdots \vee (BM)_r$ where each $(BM)_i$ is glued to BN at the point x_i and the map π' is defined so that $\pi'|_{BN} = \pi$ and $\pi'|_{(BM)_i} = \mathrm{id}$.

Then, it is clear that π' is also a regular covering and π'^* identifies $\pi_1(\widetilde{X}) \cong N * (*_{i=1}^r M)$ to a normal subgroup of $K * M$. Hence, the normal closure of $N * M$ in $K * M$ is contained in $\pi_1(\widetilde{X})$.

On the other side, if γ_i is a path in BN from the base point to the point x_i, then the copy of M in $*_{i=1}^r M$ corresponding to the summand $(BM)_i$ is sent by π'^* to the conjugate $[\pi\gamma_i]^{-1} M [\pi\gamma_i]$ in $K * M$. Hence, $\pi_1(\widetilde{X})$ is contained in the normal closure of $N * M$ in $K * M$. □

We are now ready to prove that for our cohomology computations we can replace the parabolic subgroups P_I by the Lévi subgroups H_I.

Proposition 4.3. *For each $I \subsetneq \{1,2\}$, the homomorphism $P_I \to H_I$ induces an isomorphism in cohomology with coefficients in $\mathbb{F}_p$.*

Proof. P_I is an extension of H_I by V_I, hence it is enough to prove that the groups V_I are p-acyclic. Recall that we are assuming that p is prime to the order of k and so the group k^+ is p-acyclic. For $I = \varnothing$ we have $V_\varnothing = U^+$ and Proposition 4.1 shows that U^+ is indeed p-acyclic. Consider V_1. According to [Rém02] 6.2, V_1 is the normal closure in U^+ of the subgroup generated by the root groups U_e for $e \in \Phi^+ - \{u_0\}$. 4.1 and 4.2 allows us to compute this normal closure and we see that it is also mod p acyclic. □

Remark 4.4. The mod p triviality of the unipotent subgroups of Kac-Moody groups over finite fields of characteristic different from p has been proved independently in [Fol12].

This last result implies that if we take coefficients in $\mathbb{F}_p$, then the cohomology of G is isomorphic to the cohomology of the group $H_1 *_T H_2$. Notice that the hypothesis $p \neq \ell$ is crucial.

The remainder of this section is devoted to study the structure of the groups $H_i = \langle T, U_i^+, U_i^- \rangle = \langle T, \phi_i(SL_2(k)) \rangle$ as well as the homomorphism $\eta\colon T \to H_i$. Recall from section 2 the meaning of the integers n_i, m_i, s_i, t_i for $i = 1, 2$. Recall also the action of the Weyl group W on Λ given by $\omega_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$, for $\lambda \in \Lambda$. Since the analysis for H_1 is the same as for H_2, we omit all subscripts $i = 1, 2$ and we write H, n, m, s, t, ω to simplify the typography.

The homomorphism $\phi\colon SL_2(k) \to H$ can have a non trivial central kernel. We say that H is *monic* if ϕ is injective. The property $(*)$ in section 2 shows that H is monic if and only if $\ell = 2$ or s, t are relatively prime.

The proof of the following proposition is straightforward.

Proposition 4.5. *Assume H is monic and consider $\psi\colon k^\times \to k^\times \times k^\times$ given by $\psi(\zeta) = (\zeta^s, \zeta^t)$. Then, there is a split exact sequence of abelian groups*

$$k^\times \xrightarrowtail{\psi} k^\times \times k^\times \xrightarrow{\pi} k^\times$$

with $\pi(\zeta, \tau) = \zeta^{-t}\tau^s$. If s, t are relatively prime, a section is given by $\sigma(\zeta) = (\zeta^{-\mu}, \zeta^\lambda)$ where λ, μ are integers such that $\lambda s + \mu t = 1$. If s, t are both even, then $\ell = 2$ and a section is given by $\sigma(\zeta) = ((\zeta^{1/2})^{-m}, (\zeta^{1/2})^n)$. □

Proposition 4.6. *Let $E = SL_2(k) \rtimes k^\times$ with action given by $\zeta \cdot \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & \zeta^r y \\ \zeta^{-r} z & t \end{smallmatrix}\right)$ for some integer r. Then, $E \cong GL_2(k)$ if r is odd and $E \cong SL_2(k) \times k^\times$ if r is even.*

Proof. If $r = 2r' - 1$, consider the homomorphism

$$(M, \zeta) \mapsto M \left(\begin{smallmatrix} \zeta^{r'} & 0 \\ 0 & \zeta^{1-r'} \end{smallmatrix}\right)$$

from $SL_2(k) \rtimes k^\times$ to $GL_2(k)$. If $r = 2r'$, consider the homomorphism

$$(M, \zeta) \mapsto \left[M \left(\begin{smallmatrix} \zeta^{r'} & 0 \\ 0 & \zeta^{-r'} \end{smallmatrix}\right), \zeta \right]$$

from $SL_2(k) \rtimes k^\times$ to $SL_2(k) \times k^\times$. □

Let $\overline{H}$ denote any of the groups $GL_2(k)$, $SL_2(k) \times k^\times$ and let $\overline{\eta}$ be the homomorphism $\overline{\eta}\colon T \to \overline{H}$ given by $\overline{\eta}(\zeta,\tau) = \left(\begin{smallmatrix}\zeta & 0\\ 0 & \tau\end{smallmatrix}\right) \in GL_2(k)$ and $\overline{\eta}(\zeta,\tau) = \left[\left(\begin{smallmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{smallmatrix}\right), \tau\right] \in SL_2(k) \times k^\times$. Let $\overline{\omega}$ be the automorphism of Λ given by $\overline{\omega}(e,v) = (v,e)$ if $\overline{H} = GL_2(k)$ and $\overline{\omega}(e,v) = (-e,v)$ if $\overline{H} = SL_2(k) \times k^\times$.

We say that H is *split* if n, m are both even. With these notations, we can state the following structure theorem.

Proposition 4.7. *Assume H is monic. Let $\overline{H} = SL_2(k) \times k^\times$ if H is split and $\overline{H} = GL_2(k)$ if it is not. Then,*

(1) *There is an isomorphism $\gamma\colon H \cong \overline{H}$ such that the following diagram is commutative*

$$\begin{array}{ccc} T & \xrightarrow{\eta} & H \\ {\scriptstyle \gamma|_T}\downarrow & & \cong\downarrow{\scriptstyle \gamma} \\ T & \xrightarrow{\overline{\eta}} & \overline{H} \end{array}$$

(2) *$\gamma|_T$ is induced by $M\colon \Lambda \to \Lambda$ given by*

$$M = \begin{cases} \begin{pmatrix} n/2 & -t \\ m/2 & s \end{pmatrix} & \text{if } H \text{ is split} \\ \frac{1}{2}\begin{pmatrix} n-t & -n-t \\ m+s & -m+s \end{pmatrix} & \text{if } H \text{ is not split.} \end{cases}$$

(3) *$M^{-1}\omega M = \overline{\omega}$.*

Proof. Recall that $H = \langle \phi(SL_2(k)), T\rangle$ and T normalizes $SL_2(k)$ according to the formula $(**)$ in section 2. Since H is monic we have that s, t are relatively prime and we can apply 4.5. We have a commutative diagram were each row is an exact sequence and the bottom row is the split exact sequence in 4.5.

$$\begin{array}{ccccc} SL_2(k) & \rightarrowtail & SL_2(k) \rtimes k^\times & \twoheadrightarrow & k^\times \\ \| & & \delta\uparrow & & \| \\ SL_2(k) & \xrightarrow{\phi} & H & \xrightarrow{\pi} & k^\times \\ \uparrow & & \eta\uparrow & & \| \\ k^\times & \xrightarrow{\phi} & T & \xrightarrow{\pi} & k^\times \\ \| & & \cong\uparrow{\scriptstyle \alpha} & & \| \\ k^\times & \xrightarrow{\psi} & k^\times \times k^\times & \underset{\sigma}{\overset{\pi}{\rightleftarrows}} & k^\times \end{array}$$

This proves that $H \cong SL_2(k) \rtimes k^\times$. To compute the action, choose λ, μ such that $\lambda s + \mu t = 1$ and let $r = \lambda m - \mu n$. It is easy to see that r is even if and only if n, m

are both even, i. e. if H is split. Then we conclude that the action of $k^\times$ on $SL_2(k)$ is given by

$$\zeta \cdot \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} x & \zeta^r y \\ \zeta^{-r} z & t \end{pmatrix}$$

and then 4.6 proves the first part of this proposition.

To compute the matrix M, notice that δ is given by $\delta(g) = (\phi^{-1}(g\,\sigma\pi(g)^{-1}), \pi(g))$. Then, some diagram chasing yields

$$M = \begin{cases} \begin{pmatrix} \lambda - r't & -\lambda + r't - t \\ \mu + sr' & -\mu - sr' + s \end{pmatrix}, \ r = 2r' - 1, \ H \text{ not split}; \\ \begin{pmatrix} \lambda - r't & -t \\ \mu + r's & s \end{pmatrix}, \ r = 2r', \ H \text{ split}. \end{cases}$$

It is easy to see that these matrices coincide with the ones in (2). Knowing these explicit values for M, the equality in (3) is immediate. □

Consider now the case in which H is not monic. This means that $\ell \neq 2$ and $s = 2s'$, $t = 2t'$.

Proposition 4.8. *Assume H is not monic and consider $\psi\colon k^\times \to k^\times \times k^\times$ given by $\psi(\zeta) = (\zeta^s, \zeta^t)$. Let $d = (q-1)/2$. There is an exact sequence of abelian groups*

$$k^\times/\{\pm 1\} \xrightarrow{\psi'} k^\times \times k^\times \xrightarrow{\pi'} k^\times \times \{\pm 1\}$$

where ψ' is induced by ψ and π' is given by $\pi'(\zeta, \tau) = (\zeta^{-t'}\tau^{s'}, (\zeta^n \tau^m)^d)$.

Proof. Since H is not monic, we have that q is odd and $(s,t) = 2$. We know that $ns' + mt' = 1$. The injectivity of ψ' as well as the identity $\pi'\psi' = 1$ are clear. Notice that $(\zeta, 1) = \pi'(\zeta^{-m}, \zeta^n)$. Also, if θ is a generator of $k^\times$, we have $(1, -1) = \pi'(\theta^{s'}, \theta^{t'})$. This proves the surjectivity of π'. Assume $\pi'(\zeta, \tau) = (1,1)$. If n is even, then m is odd and this implies that τ is a square $\tau = \gamma^2$. Then, $(\zeta, \tau) = \psi'(\zeta^{n/2}\gamma^m)$. The case m even is similar. Finally, if n, m are both odd, we have $(\zeta, \tau)^d = 1$ and so ζ, τ are both squares or both non squares. If $\zeta = \delta^2$, $\tau = \gamma^2$, then $(\zeta, \tau) = \psi'(\delta^n \gamma^m)$. If ζ, τ were both non squares, the equality $\zeta^{t'} = \tau^{s'}$ is in contradiction to $ns' + mt' = 1$ with n, m odd. □

Notice that this exact sequence is always split over $k^\times \times \{1\}$, a section being given by $\sigma(\zeta, 1) = (\zeta^{-m}, \zeta^n)$. But one sees easily that this exact sequence is not always split over $\{1\} \times \{\pm 1\}$.

Proposition 4.9. *Assume H is not monic. Let $\eta\colon T \to PGL_2(k) \times k^\times$ given by $\overline{\eta}(\zeta, \tau) = \left(\left[\left(\begin{smallmatrix} \zeta & 0 \\ 0 & 1 \end{smallmatrix}\right)\right], \tau\right)$ and let $\overline{\omega}\colon \Lambda \to \Lambda$ be given by $\overline{\omega}(e, v) = (-e, v)$. Then*

(1) *There is an isomorphism* $\gamma\colon H \cong PGL_2(k)\times k^\times$ *such that the following diagram is commutative.*

$$\begin{array}{ccc} T & \xrightarrow{\eta} & H \\ {\scriptstyle \gamma|_T}\downarrow & & \cong\downarrow{\scriptstyle \gamma} \\ T & \xrightarrow{\overline{\eta}} & PGL_2(k)\times k^\times \end{array}$$

(2) $\gamma|_T$ *is induced by* $M\colon \Lambda\to\Lambda$ *given by*

$$M=\begin{pmatrix} n & -t/2 \\ m & s/2 \end{pmatrix}.$$

(3) $M^{-1}\omega M=\overline{\omega}$.

Proof. Since H is not monic, $\phi\colon SL_2(k)\to H$ factors through a monomorphism $\phi'\colon PSL_2(k)\to H$ which fits into a commutative diagram

$$\begin{array}{ccccc} PSL_2(k) & \overset{\phi'}{\rightarrowtail} & H & \overset{\pi'}{\twoheadrightarrow} & k^\times\times\{\pm1\} \\ \uparrow & & {\scriptstyle\eta}\uparrow & & \| \\ k^\times/\{\pm1\} & \overset{\phi'}{\rightarrowtail} & T & \overset{\pi'}{\twoheadrightarrow} & k^\times\times\{\pm1\} \\ \| & & \cong\uparrow{\scriptstyle\alpha} & & \| \\ k^\times/\{\pm1\} & \overset{\psi'}{\rightarrowtail} & k^\times\times k^\times & \overset{\pi'}{\twoheadrightarrow} & k^\times\times\{\pm1\} \end{array}$$

where the bottom row coincides with the exact sequence in 4.8. This bottom exact sequence is split over $k^\times\times\{1\}$ and a section is $\sigma(\zeta,1)=(\zeta^{-m},\zeta^n)$. In general, it is not split over $\{1\}\times\{\pm1\}$. However, the top exact sequence is split. To see this, consider the element

$$\beta=\phi\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\eta\alpha(\theta^{s'},\theta^{t'})\in H$$

where $\theta\in k^\times$ is a generator. Then $\pi'(\beta)=(1,-1)$ and an easy computation using formulas $(*)$ and $(**)$ in section 2 shows that $\beta^2=1$. Hence, H is a semidirect product $H\cong PSL_2(k)\rtimes(k^\times\times\{\pm1\})$.

Then, if we compute the induced action of $k^\times\times\{\pm1\}$ on $PSL_2(k)$, it turns out that $k^\times$ acts trivially, while $\{\pm1\}$ acts through

$$\left(\begin{smallmatrix}x&y\\z&t\end{smallmatrix}\right)\mapsto\begin{pmatrix}t&-\zeta^{-1}z\\-\zeta y&x\end{pmatrix}.$$

It is easy to identify this extension. Consider the homomorphism $\epsilon\colon PGL_2(k)\to\{\pm1\}$ which sends to -1 all matrices whose determinant is not a square. The kernel of ϵ is $PSL_2(k)$ and we have an extension

$$PSL_2(k)\overset{\phi'}{\rightarrowtail}PGL_2(k)\overset{\epsilon}{\twoheadrightarrow}\{\pm1\}\ .$$

This extension has a section given by $-1\mapsto\left(\begin{smallmatrix}0&-\zeta^{-1}\\1&0\end{smallmatrix}\right)$ and the action of $\{\pm1\}$ on $PSL_2(k)$ is exactly the same as above. This proves that $H\cong PGL_2\times k^\times$.

We want to compute now the homomorphism

$$k^\times \times k^\times \xrightarrow{\alpha} T \xrightarrow{\eta} H \xrightarrow{\delta} PSL_2(k) \rtimes (k^\times \times \{\pm 1\}) \longrightarrow PGL_2(k) \times k^\times.$$

We omit the details of this computation which is straightforward once we have an explicit section of the extension and we recall that δ is given by

$$\delta(g) = (\phi'^{-1}(g\sigma(\pi'(g)^{-1})), \pi'(g)).$$

Once we have the matrix M, the equality in (3) follows immediately. □

5. The Weyl group and its invariants

The Weyl group W of the Kac-Moody group G is an infinite dihedral group. There are two relevant integral representation of W. First, there is the action of W on $\mathbb{Q}^2$ given by the root system, as studied in section 3. On the other side, W acts on the lattice Λ by $\omega_i(\lambda) = \lambda - \langle h_i, \lambda\rangle \alpha_i$ for $i = 1, 2$. In this section we are interested in the representation given by the action of W on $\Lambda \otimes \mathbb{Z}_p$.

The integral p-adic representations of an infinite dihedral group were classified in [ABS07] (see also [Agu09]) using some explicit invariants which can be easily computed. Let us denote by ν_p the p-adic valuation.

Theorem 5.1. *(*[ABS07],[Agu09]*) Assume p is an odd prime. There are invariants $\Gamma(\rho) \in \mathbb{Z}_p$, $\delta_1(\rho), \delta_2(\rho) \in \{0, 1, 2, \dots, \infty\}$ which classify all representations $\rho\colon W \to GL_2(\mathbb{Z}_p)$. These invariants can take any value in their range, subject to the relation $\delta_1 + \delta_2 = \nu_p(\Gamma) + \nu_p(\Gamma - 1)$. If a representation ρ is given by $\omega_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\omega_2 = M^{-1}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) M$ with $M = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$ (any representation is like that), then $\Gamma(\rho) = xt/\det(M)$, $\delta_1(\rho) = \nu_p(xz)$ and $\delta_2(\rho) = \nu_p(yt)$.* □

There is also a similar result for $p = 2$ which is more complex and will not be needed in this work (cf. [ABS07],[Agu09]).

Let us apply this result to classify the action of W on $\Lambda \otimes \mathbb{Z}_p$. Recall from section 2 the meaning of the integers a, b, Δ, ∇.

Proposition 5.2. *The p-adic representation of the Weyl group of G given by the action of W on $\Lambda \otimes \mathbb{Z}_p$ is classified by $\Gamma = ab/4$, $\delta_1 = \nu_p(a) + \nu_p(\Delta)$, $\delta_2 = \nu_p(b) + \nu_p(\nabla)$.*

Proof. The representation is given by the matrices

$$\omega_i = \begin{pmatrix} 1 - s_i n_i & -t_i n_i \\ -s_i n_i & 1 - t_i n_i \end{pmatrix}, \quad i = 1, 2.$$

The matrices $P_i = \left(\begin{smallmatrix} t_i & n_i \\ -s_i & m_i \end{smallmatrix}\right)$ for $i = 1, 2$ have determinant 2 and satisfy $P_i^{-1}\omega_i P_i = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Hence, the matrix M in 5.1 is $M = P_2^{-1}P_1 = \frac{1}{2}\left(\begin{smallmatrix} -a & \nabla \\ -\Delta & -b \end{smallmatrix}\right)$. □

The action of W on Λ gives an action of W on $T = \mathrm{Hom}(\Lambda, k^\times)$ and on $H^1(T; \mathbb{F}_p)$. Of course, $H^1(T; \mathbb{F}_p)$ vanishes unless $q \equiv 1\,(p)$ and in this case $H^1(T; \mathbb{F}_p) = {\mathbb{F}_p}^2$ and we have a representation of W in $GL_2(\mathbb{F}_p)$ which is the reduction modulo p of the p-adic representation of W studied in 5.2 and factors through some finite dihedral group W_p. We are interested in these representations and in their rings of invariants. The mod p reductions of the representations of an infinite dihedral group in $GL_2(\mathbb{Z}_p)$ were studied in [ABKS05] (see Table 1 in [ABKS05]). Let us summarize the results

of [ABKS05] that we need here. We denote by P a polynomial algebra $P = \mathbb{F}_p[x, y]$ with two generators of degree 2, with the given action action of W_p. There are six types of representations:

- Type **I**. $\Gamma \equiv 0\ (p)$, $\delta_1, \delta_2 > 0$. W_p is elementary abelian of order 4 and its invariants are $P^W = \mathbb{F}_p[x^2, y^2]$.
- Type **II**. A representation in this type is given, up to the outer automorphism of W, by $\Gamma \equiv 0\ (p)$, $\delta_1 = 0, \delta_2 > 0$. W_p is a dihedral group of order $4p$ and its invariants P^{W_p} form a polynomial ring on generators y^2, $x^2(x^{p-1} - y^{p-1})^2$.
- Type **III**. W_p is of order two. $\Gamma \not\equiv 0\ (p)$, $\delta_1, \delta_2 > 0$, $P^W = \mathbb{F}_p[x, y^2]$.
- Type **IV**. This type occurs when $\Gamma \not\equiv 0\ (p)$, $\delta_1 = 0$, $\delta_2 > 0$. W_p turns out to be a dihedral group of order $2p$ in $GL_2(\mathbb{F}_p)$ which is studied in pages 128–129 of [Smi95]. The invariants are polynomial in degrees 2 and $4p$ namely $P^W = \mathbb{F}_p[x, (yx^{p-1} - y^p)^2]$.
- Type **V**. This type occurs when $\Gamma \not\equiv 0\ (p)$, $\delta_1 > 0$, $\delta_2 = 0$. W_p turns out to be a dihedral group of order $2p$ in $GL_2(\mathbb{F}_p)$ which is also studied in pages 128–129 of [Smi95]. The invariants are polynomial in degrees 4 and $2p$ namely $P^W = \mathbb{F}_p[y^2, x(x^{p-1} - y^{p-1})]$.
- Type **VI**. This type occurs when $\Gamma \not\equiv 0\ (p)$, $\delta_1 = \delta_2 = 0$. The order $2m$ of the dihedral group W_p is twice the multiplicative order of the roots (in $\mathbb{F}_{p^2}$) of the polynomial $X^2 - 2(2\Gamma - 1)X + 1$. Thus, $p \equiv \pm 1\ (m)$. In particular, the order of W_p is prime to p and then the invariants P^W form a polynomial ring on generators z_4, t_{2m} of degrees 4 and $2m$, respectively. (cf. [ABKS05].)

Let us study in more detail the invariant theory of the representations of type VI. What follows is probably known, but we have not been able to find references for everything that we need in this paper. Hence, this may have some independent interest.

Proposition 5.3. *Let ρ be the representation of type VI of the dihedral group of order $2m$ in $GL_2(\mathbb{Z}_p)$ classified by $\Gamma \in \mathbb{Z}_p$. Let $\theta \in \mathbb{F}_{p^2}$ be a primitive m-th root of unity. Then, there is a basis x, y such that the ring of invariants $\mathbb{F}_p[x, y]^{D_{2m}} = \mathbb{F}_p[f, g]$ where the polynomials f, g can be chosen as follows:*

1. *If $p \equiv 1\ (m)$, then $f = xy$, $g = x^m + y^m$.*
2. *If $p \equiv -1\ (m)$, then $f = x^2 + (2 - 4\Gamma)xy + y^2$.*
3. *If $p \equiv -1\ (m)$, let $u = \theta x - y$, $v = \theta y - x$. Then, $g = u^m + v^m$ if m is even, $m \neq 2$, and $g = (u^m + v^m)/(\theta - \theta^{-1})$ if m is odd.*

Proof. If $p \equiv 1\ (m)$, then $\theta \in \mathbb{F}_p$ and the matrices

$$\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}$$

generate the representation ρ. The computation of the invariants of this representation is trivial:

$$\mathbb{F}_p[x, y]^{D_{2m}} = \mathbb{F}_p[f, g], \quad f = xy,\ g = x^m + y^m.$$

If $m = p + 1$, then $\theta \notin \mathbb{F}_p$, but $\theta^i + \theta^{-i}$ and $(\theta^i - \theta^{-i})/(\theta - \theta^{-1})$ are in $\mathbb{F}_p$ for all values of i (they are invariant under the Frobenius). Let $\gamma = \theta + \theta^{-1}$. Actually,

θ, θ^{-1} are the roots of the irreducible polynomial $X^2 - \gamma X + 1$ and so $\theta \in \mathbb{F}_{p^2}$. Also, $\gamma = 4\Gamma - 2$. Let us consider the matrices

$$\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} \gamma & 1 \\ -1 & 0 \end{pmatrix}, \quad \omega, \tau \in GL_2(\mathbb{F}_p).$$

If we extend scalars to $\mathbb{F}_{p^2}$, then the change of basis given by $P = \left(\begin{smallmatrix} \theta & -1 \\ -1 & \theta \end{smallmatrix} \right)$ transforms ω, τ into

$$P^{-1}\omega P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P^{-1}\tau P = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}$$

and we see that ω, τ generate a dihedral group $D_{2m} \subset GL_2(\mathbb{F}_p)$ which gives the representation ρ. Like before, the computation of the invariants over $\mathbb{F}_{p^2}$ is trivial:

$$\mathbb{F}_{p^2}[u, v]^{D_{2m}} = \mathbb{F}_{p^2}[f, g], \quad f = uv,\ g = u^{p+1} + v^{p+1},$$

but we are interested in having explicit descriptions of the invariants over $\mathbb{F}_p$.

We have $f = uv = -\theta(x^2 - \gamma xy + y^2)$ and thus $\theta^{-1} f$ is a degree two invariant in $\mathbb{F}_p[x, y]^{D_{2m}}$. (Notice that $\gamma^2 - 4 = (\theta - \theta^{-1})^2$ is not a square in $\mathbb{F}_p$ and so no change of basis can produce a degree two invariant of the form xy, in contrast to what is written in page 137 of [Smi95].)

Let us consider now the invariant $g(u, v)$. If we write it in the basis x, y we have

$$g = u^{p+1} + v^{p+1} = \sum_{i=0}^{p+1} (-1)^i \binom{p+1}{i} (\theta^i + \theta^{-i})\, x^i y^{p-i+1}.$$

Hence, $g \in \mathbb{F}_p[x, y]$ and we have seen that $\mathbb{F}_p[x, y]^{D_{2m}} = \mathbb{F}_p[\theta^{-1}uv, u^{p+1} + v^{p+1}]$.

Consider now the general case $p \equiv -1\,(m)$, $m \neq 2$. If $mr = p + 1$, then $\omega, \tau^r \in GL_2(\mathbb{F}_p)$ produce the representation ρ of D_{2m} in $GL_2(\mathbb{F}_p)$ and its invariants over $\mathbb{F}_{p^2}$ are generated by $f = uv$ and $g_m = u^m + v^m$. Like before, $x^2 - \gamma xy + y^2$ is an invariant in degree two. The situation with respect to g_m is slightly more complicated. If m is even, then

$$g_m = u^m + v^m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} (\theta^i + \theta^{-i})\, x^i y^{m-i} \in \mathbb{F}_p[x, y]$$

and $\mathbb{F}_p[x, y]^{D_{2m}} = \mathbb{F}_p[x^2 - \gamma xy + y^2, u^m + v^m]$ like before. On the other side, if m is odd we have

$$g_m = u^m + v^m = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} (\theta^i - \theta^{-i})\, x^i y^{m-i} \notin \mathbb{F}_p[x, y]$$

and so $g_m/(\theta - \theta^{-1}) \in \mathbb{F}_p[x, y]$. Thus, we have

$$\mathbb{F}_p[x, y]^{D_{2m}} = \mathbb{F}_p\left[x^2 - \gamma xy + y^2, \frac{u^m + v^m}{\theta - \theta^{-1}}\right].$$

□

6. Cohomology of the parabolic subgroups

We have seen in 4.3 that the proper parabolic subgroups P_I have the same mod p cohomology as their Lévi factors which are T, H_1, H_2. We have also seen in 4.7 and 4.9 that the groups H_1, H_2 are isomorphic to $GL_2(k)$, $SL_2(k)\times k^\times$ or $PGL_2(k)\times k^\times$. In this section we recollect some know facts about the cohomology of these finite groups. Most of this can be found in [AM04] and [FP78]. Recall that we are always assuming that p is an odd prime different from ℓ.

If $q\not\equiv\pm1\ (p)$, then p does not divide the order of these groups and cohomology is trivial in all cases.

Let $D=k^\times\times k^\times\cong(\mathbb{Z}/(q-1)\mathbb{Z})^2$ be the subgroup of diagonal matrices in $GL_2(k)$. If $q\equiv1\ (p)$, then $H^*(BGL_2(k);\mathbb{F}_p)$ is a ring of invariants, namely

$$H^*(BGL_2(k);\mathbb{F}_p)\cong H^*(BD;\mathbb{F}_p)^{C_2}$$

where C_2 is a cyclic group of order two acting on D by permuting the two factors. Also, if r is such that $q-1=sp^r$ with s prime to p, then $H^*(BD;\mathbb{F}_p)\cong H^*(B\mathbb{Z}/p^r\mathbb{Z}\times B\mathbb{Z}/p^r\mathbb{Z};\mathbb{F}_p)$ and

$$H^*(BGL_2(k);\mathbb{F}_p)\cong\mathbb{F}_p[x_2,x_4]\otimes E(y_1,y_3)$$

(the subscripts denote the degrees of the generators) with secondary Bocksteins of height r relating the exterior generators to the polynomial generators.

If $q\equiv-1\ (p)$, then $H^*(BD;\mathbb{F}_p)$ is trivial but $H^*(BGL_2(k);\mathbb{F}_p)$ is also a ring of invariants. If $q+1=sp^r$ with s prime to p, then

$$H^*(BGL_2(k);\mathbb{F}_p)\cong H^*(B\mathbb{Z}/p^r\mathbb{Z};\mathbb{F}_p)^{C_2}\cong\mathbb{F}_p[x_4]\otimes E(y_3)$$

where C_2 acts on $\mathbb{Z}/p^r\mathbb{Z}$ by multiplication by -1.

The case of $SL_2(k)=Sp_2(k)$ is similar. If $q\equiv-1\ (p)$ then

$$H^*(B(SL_2(k)\times k^\times);\mathbb{F}_p)\cong H^*(BSL_2(k);\mathbb{F}_p)\cong H^*(BGL_2(k);\mathbb{F}_p).$$

If $q\equiv1\ (p)$ then

$$H^*(BSL_2(k);\mathbb{F}_p)\cong H^*(B(D\cap SL_2(k));\mathbb{F}_p)^{C_2}\cong\mathbb{F}_p[x_4]\otimes E(y_3)$$

where C_2 acts on $D\cap SL_2(k)\cong\mathbb{Z}/(q-1)\mathbb{Z}$ as multiplication by -1.

Let us consider now the case of $PGL_2(k)$. If $q\equiv-1\ (p)$ then $H^*(Bk^\times;\mathbb{F}_p)$ is trivial and the extension $k^\times\rightarrowtail GL_2(k)\twoheadrightarrow PGL_2(k)$ gives an isomorphism

$$H^*(BPGL_2(k);\mathbb{F}_p)\cong H^*(BGL_2(k);\mathbb{F}_p).$$

On the other side, if $q\equiv1\ (p)$ and r is such that $q-1=sp^r$ with s prime to p, then the Sylow p-subgroup of $PGL_2(k)$ is cyclic

$$Syl=\left\{[\left(\begin{smallmatrix}x&0\\0&1\end{smallmatrix}\right)]\ \middle|\ x\in k^\times,\ x^{p^r}=1\right\}\cong\mathbb{Z}/p^r\mathbb{Z}$$

and it is easy to compute the stable elements in $H^*(BSyl;\mathbb{F}_p)$. It turns out that the only automorphism of Syl that appears is the involution $x\mapsto x^{-1}$ and so

$$H^*(BPGL_2(k);\mathbb{F}_p)\cong H^*(BSyl;\mathbb{F}_p)^{C_2}\cong\mathbb{F}_p[x_4]\otimes E(y_3).$$

If we apply all this to the parabolic subgroups H_1, H_2 in G, we obtain the following.

Proposition 6.1. *Let H be one of the Lévi subgroups of G and let $\omega \in W$ be the corresponding generating reflection. Then, $H^*(BH;\mathbb{F}_p)$ is trivial unless $q \equiv \pm 1$ (p). If $q \equiv 1$ (p) then $H^*(BH;\mathbb{F}_p) \cong H^*(BT;\mathbb{F}_p)^\omega$. If $q \equiv -1$ (p) then $H^*(BT;\mathbb{F}_p)$ is trivial and $H^*(BH;\mathbb{F}_p) \cong \mathbb{F}_p[x_4] \otimes E(y_3)$ with a Bockstein relating y_3 to x_4.*

Proof. This follows from 4.7, 4.9 and the discussion above. □

In this paper we are assuming that p is an odd prime. It happens that the situation at the prime two is far more complex, even for the Lévi factors. For example, if $p = 2$ and $q \equiv 3$ (4), we have $H^*(BD;\mathbb{F}_p) \cong H^*(B\mathbb{Z}/2\mathbb{Z} \times B\mathbb{Z}/2\mathbb{Z};\mathbb{F}_2) \cong \mathbb{F}_2[u_1, v_1]$ and $H^*(BGL_2(k);\mathbb{F}_2)$ is the subalgebra of $H^*(BD;\mathbb{F}_2)$ generated by the elements

$$x_1 = u_1 + v_1$$
$$x_3 = u_1v_1^2 + u_1^2v_1$$
$$x_4 = u_1^2v_1^2$$

(see [FP78], page 342). Then, $H^*(BGL_2;\mathbb{F}_2)$ is neither a polynomial ring nor a ring of invariants. Actually,

$$H^*(BGL_2(k);\mathbb{F}_2) \cong \mathbb{F}_2[x_4, x_1, x_3]/(x_3^2 + x_4x_1^2).$$

7. The structure of Coker(ϕ)

In this section we will continue the study of the invariant theory of a dihedral group of type VI that we introduced in section 5. So, p is an odd prime and D is a dihedral group with a representation in $GL_2(\mathbb{F}_p)$ of type VI, classified by $\Gamma \in \mathbb{F}_p$, $\Gamma \neq 0$. The order of D is $2m$ where m is the order of the roots of $X^2 - 2(2\Gamma - 1)X + 1$. Depending on the values of a, b, Δ, ∇ (see Proposition 5.2), the Weyl group of the Kac-Moody group G acting on $H^1(T;\mathbb{F}_p)$ provides an example of such a representation, but this section is written independently of the theory of Kac-Moody groups.

We use the following notation. $P = \mathbb{F}_p[v, v']$ is a polynomial algebra on two generators of degree two. $S = \mathbb{F}_p[v, v'] \otimes E(u, u')$ is the tensor product of P and an exterior algebra on two generators of degree one. We consider P as a subalgebra of S. From a topological point of view, we can think of P and S as $H^*(BS^1 \times BS^1;\mathbb{F}_p)$ and $H^*(B\mathbb{Z}/p \times B\mathbb{Z}/p;\mathbb{F}_p)$, respectively.

Let us consider the following (graded) derivations acting on S. d has degree -1 and is given by

$$d(u) = d(u') = 0,\ d(v) = u,\ d(v') = u'.$$

δ has degree $+1$ and is given by

$$\delta(v) = \delta(v') = 0,\ \delta(u) = v,\ \delta(u') = v'.$$

Under the identification $S \cong H^*(B\mathbb{Z}/p \times B\mathbb{Z}/p;\mathbb{F}_p)$, δ corresponds to the Bockstein homomorphism. It is easy to see that if $x \in P$ is a polynomial of degree $2n$, then $\delta d(x) = nx$.

D acts on the algebras P and S and it is clear that the derivations d and δ commute with this action. From 5.3 we know that the ring of invariants of P under D is a polynomial algebra $P^D = \mathbb{F}_p[x_4, x_{2m}]$ and we have explicit descriptions of the

generators x_4, x_{2m}. If we consider the action of D on S, theorem 9.3.2 in [Smi95] says that the ring of invariants is

$$S^D = \mathbb{F}_p[x_4, x_{2m}] \otimes E(dx_4, dx_{2m}).$$

D is generated by two elements ω_1, ω_2 of order two and we can also consider the rings of invariants of each generating reflection S^{ω_1}, S^{ω_2}.

Let ϕ be the P^D linear map

$$S^{\omega_1} \oplus S^{\omega_2} \to S$$

defined by $\phi(t,h) = t + h$. We want to investigate the structure of $\mathcal{S} = \mathrm{Coker}(\phi)$ as a P^D-module.

Let us introduce now two *relative invariants* α, J of D in S.

$$\alpha = uu', \quad J = \begin{vmatrix} \frac{\partial x_4}{\partial v} & \frac{\partial x_4}{\partial v'} \\ \frac{\partial x_{2m}}{\partial v} & \frac{\partial x_{2m}}{\partial v'} \end{vmatrix}.$$

Of course, J is the Jacobian of the algebraically independent polynomials x_4 and x_{2m} and so it does not vanish. It is a polynomial of degree $2m$ in P. One sees immediately that both α and J are relative invariants of D with respect to the determinant: for any $g \in D$ we have $g\alpha = \det(g)\,\alpha$ and $gJ = \det(g)\,J$.

Since the derivations d and δ commute with the action of D on S, we can define two new relative invariants as follows.

$$\alpha' = \delta(\alpha) = vu' - v'u, \quad J' = \frac{1}{m}\, d(J).$$

Notice that $\delta(J') = J$.

Proposition 7.1. *$[\alpha]$, $[\alpha']$, $[J]$, $[J']$ are P^D-linearly independent elements in $\mathcal{S}$.*

Proof. Let $f[\alpha'] + g[J] = 0$ in $\mathcal{S}$ with $f, g \in P^D$. This means that there are $p_1 \in S^{\omega_1}$, $p_2 \in S^{\omega_2}$ such that

$$f\alpha' + gJ = p_1 + p_2.$$

applying ω_1 and subtracting we get

$$2f\alpha' + 2gJ = p_2 - \omega_1 p_2.$$

Applying ω_2 we get

$$-2f\alpha' - 2gJ = p_2 - \omega_2\omega_1 p_2.$$

Notice that α' and J are left invariant by $\omega_2\omega_1$. Hence, applying $\omega_2\omega_1$ and adding, we get

$$-4f\alpha' - 4gJ = p_2 - (\omega_2\omega_1)^2 p_2.$$

Inductively, we get

$$-2mf\alpha' - 2mgJ = p_2 - (\omega_2\omega_1)^m p_2 = 0.$$

Since m is prime to p, this yields $f\alpha' + gJ = 0$ in S. But $gJ \in P$ and so $f\alpha' \in P$ and this can only happen if $f, g = 0$.

We have proven that $[\alpha']$ and $[J]$ are P^D-linearly independent in $\mathcal{S}$. Consider now a vanishing linear combination $f[\alpha] + g[\alpha'] + h[J] + l[J'] = 0$. This means that there are $p_1 \in S^{\omega_1}$, $p_2 \in S^{\omega_2}$ such that

$$f\alpha + g\alpha' + hJ + lJ' = p_1 + p_2.$$

If we apply the derivation δ we get

$$f\alpha' + hJ = \delta(p_1) + \delta(p_2) = 0$$

and the argument above yields $f = h = 0$. Then, $g\alpha' + lJ = p_1 + p_2$ and again this implies $g = l = 0$. □

The following consequence of the proposition above is a key step in our computation of the cohomology of the Kac-Moody group G.

Theorem 7.2. *$\mathcal{S}$ is a free P^D-module with basis $\{[\alpha], [\alpha'], [J], [J']\}$.*

Proof. We know that the free P^D-module with basis $\{[\alpha], [\alpha'], [J], [J']\}$ is a submodule of $\mathcal{S}$. It is enough to prove that both graded vector spaces have the same Poincaré series. The kernel of ϕ is S^D. It is very easy to write down the Poincaré series of S, $S^{\omega_1} \oplus S^{\omega_2}$ and S^D. Then, a straightforward computation with power series proves that the Poincaré series of $\mathcal{S}$ is as expected. □

The computation of the multiplicative structure of $H^*(BG; \mathbb{F}_p)$ in the next section requires that we work out some relations that hold in S. Let $y_3 = dx_4$, $y_{2m-1} = dx_{2m}$. Let θ, γ be as in the proof of 5.3.

Theorem 7.3. *The following identities hold in S:*

(1) $y_3 J' = m\lambda x_{2m}\alpha$, $y_{2m-1}J' = 2m^2\mu x_4^{m-1}\alpha$.
(2) $y_3 J = 2x_4 J' - m\lambda x_{2m}\alpha'$, $y_{2m-1}J = mx_{2m}J' - 2m^2\mu x_4^{m-1}\alpha'$.

where the coefficients λ and μ are as follows:

(a) $\lambda = 1$ *if* $p \equiv 1\,(m)$; $\lambda = \gamma^2 - 4$ *if* $p \equiv -1\,(m)$.
(b) $\mu = 1$ *if* $p \equiv 1\,(m)$; $\mu = \gamma^2 - 4$ *if* $p \equiv -1\,(m)$ *and* $m \neq 2$ *is even;* $\mu = -1$ *if* $p \equiv -1\,(m)$ *and* m *is odd.*

Proof. The identities in (2) follow from (1) applying the derivation δ. To prove (1) consider

$$y_3 J' = \frac{1}{m}\, dx_4\, dJ = \frac{1}{m}\mathcal{J}(x_4, \mathcal{J}(x_4, x_{2m}))\alpha$$
$$y_{2m-1}J' = \frac{1}{m}\, dx_{2m}\, dJ = \frac{1}{m}\mathcal{J}(x_{2m}, \mathcal{J}(x_4, x_{2m}))\alpha$$

where $\mathcal{J}(f, g)$ denotes the determinant of the Jacobian matrix of the polynomials f, g. To compute these double Jacobians, notice that the Jacobian is a relative invariant of the determinant and so if we change bases then the Jacobian is multiplied by the determinant of the change of basis matrix.

We will use the descriptions of x_4 and x_{2m} as given in 5.3.

If $p \equiv 1\,(m)$, then $x_4 = vv'$, $x_{2m} = v^m + (v')^m$. Then, a trivial direct calculation gives

$$\mathcal{J}(x_4, \mathcal{J}(x_4, x_{2m})) = m^2 x_{2m}$$
$$\mathcal{J}(x_{2m}, \mathcal{J}(x_4, x_{2m})) = 2m^3 x_4^{m-1}.$$

If $p \equiv -1\,(m)$, we extend scalars as to have the m-th root of unity θ in our field of coefficients. Then x_4 and x_{2m} have simple descriptions in a new base w, w' which is related to the basis v, v' through a matrix P of determinant $\theta^2 - 1$ (see the proof of 5.3).

If $p \equiv -1\,(m)$ and m is even, $m \neq 2$, then $x_4 = -\theta^{-1}ww'$ and $x_{2m} = w^m + (w')^m$. In this new basis, the computation of the double Jacobians is as easy as in the case $p \equiv 1\,(m)$. We get

$$\mathcal{J}(x_4, \mathcal{J}(x_4, x_{2m})) = \theta^{-2}(\theta^2-1)^2 m^2 x_{2m} = m^2(\gamma^2-4)x_{2m}$$
$$\mathcal{J}(x_{2m}, \mathcal{J}(x_4, x_{2m})) = -\theta^{-1}(\theta^2-1)^2 2m^3 (ww')^{m-1} = 2m^3(\gamma^2-4)x_4^{m-1}.$$

If $p \equiv -1\,(m)$ and m is odd, then $x_4 = -\theta^{-1}ww'$ and $x_{2m} = (w^m + (w')^m)/(\theta - \theta^{-1})$. Then

$$\mathcal{J}(x_4, \mathcal{J}(x_4, x_{2m})) = \theta^{-2}(\theta-\theta^{-1})^{-1}(\theta^2-1)^2 m^2 (w^m + (w')^m) = m^2(\gamma^2-4)x_{2m}$$
$$\mathcal{J}(x_{2m}, \mathcal{J}(x_4, x_{2m})) = -\theta^{-1}(\theta-\theta^{-1})^{-2}(\theta^2-1)^2 2m^3 (ww')^{m-1} = -2m^3 x_4^{m-1}.$$

□

8. The cohomology of BG

The preceding sections have provided all ingredients that we need to compute the cohomology algebra of the Kac-Moody group G with coefficients in the field $\mathbb{F}_p$. By 2.1 we know that BG is homotopy equivalent to the push out of $BP_1 \leftarrow BP_\varnothing \rightarrow BP_2$. By 4.3 we know that $H^*(BG;\mathbb{F}_p) \cong H^*(B(H_1 *_T H_2);\mathbb{F}_p)$. We have computed H_1 and H_2 in section 4 and we have computed $H^*(BH_i;\mathbb{F}_p)$ in 6.1. Then, the obvious tool to use to compute $H^*(B(H_1 *_T H_2);\mathbb{F}_p)$ is the Mayer-Vietoris exact sequence. The following lemma on the behavior of the connecting homomorphism of the Mayer-Vietoris exact sequence will be useful.

Lemma 8.1. *Let*

$$\begin{array}{ccc} A & \xrightarrow{i_1} & B_1 \\ \downarrow{\scriptstyle i_2} & \searrow^{l} & \downarrow{\scriptstyle j_1} \\ B_2 & \xrightarrow{j_2} & X \end{array}$$

be a push out diagram and let $\Delta\colon H^*(A) \to H^*(X)$ be the connecting homomorphism of the corresponding Mayer-Vietoris exact sequence (coefficient ring omitted). Then, for any $x \in H^*(X)$, $t \in H^*(A)$, we have $\Delta(l^*(x)\,t) = x\,\Delta(t)$.

Proof. Let us assume that all maps in the diagram are cofibrations. Then, Δ is defined through this commutative diagram (cf. [Dol80], p. 49):

$$\begin{array}{ccc} H^*(A) & \xrightarrow{\Delta} & H^*(X) \\ \downarrow{\scriptstyle \delta} & & \uparrow{\scriptstyle r^*} \\ H^*(B_1, A) & \xleftarrow[\cong]{k^*} & H^*(X, B_2) \end{array}$$

where r and k are the obvious inclusions, δ is the connecting homomorphism for the couple (B_1, A) and k is an isomorphism because of excision. The way that δ behaves with respect to cup products (cf. [Dol80], p. 220) gives

$$\delta(i_1^*(b_1)\,t) = b_1\,\delta(t), \quad \text{for } b_1 \in H^*(B_1),\ t \in H^*(A).$$

Then, an easy calculation using $\Delta = r^*(k^*)^{-1}\delta$ gives the formula of the lemma. □

Let us deal first with the easy case in which $q \not\equiv 1\ (p)$. If (A_1, ϵ_1) and (A_2, ϵ_2) are augmented graded $\mathbb{F}_p$-algebras, let us denote by $A_1 \vee A_2$ the augmented graded algebra defined as the kernel of $\epsilon_1 - \epsilon_2 \colon A_1 \oplus A_2 \to \mathbb{F}_p$.

Proposition 8.2. *If $q \not\equiv \pm 1\ (p)$ then $H^*(BG; \mathbb{F}_p) \cong \mathbb{F}_p$. If $q \equiv -1\ (p)$ then $H^*(BG; \mathbb{F}_p) \cong A \vee A$ with $A = \mathbb{F}_p[x_4] \otimes E(y_3)$ (subscripts denote degrees).*

Proof. This follows immediately from the Mayer-Vietoris exact sequence

$$\cdots \longrightarrow H^{*-1}(BT) \xrightarrow{\Delta} H^*(BG) \xrightarrow{\psi} H^*(BH_1) \oplus H^*(BH_2) \xrightarrow{\phi} H^*(BT) \longrightarrow \cdots$$

and Proposition 6.1. □

Of course, the relevant case is when $q \equiv 1\ (p)$. In this case, the computation of $H^*(BG; \mathbb{F}_p)$ will be made under the following **stability hypothesis**:

(SH) The prime p is large enough so that $ab(ab-4)$ is prime to p.

This hypothesis implies that the representation of the Weyl group W of G is of Type VI (cf. 5.2). We use the notations of section 7. In particular $S^W = \mathbb{F}_p[x_4, x_{2m}] \otimes E(y_3, y_{2m-1})$ and $P^W = \mathbb{F}_p[x_4, x_{2m}]$. We are ready for the main result of this paper:

Theorem 8.3. *Assume that $q \equiv 1\ (p)$ and the stability hypothesis (SH) is satisfied. Then $H^*(BG; \mathbb{F}_p)$ is a S^W-module with generators $1, \alpha_3, \alpha_4, J_{2m}, J_{2m+1}$ (subscripts denote degrees) and relations*

(1) $y_3\alpha_3 = 0,\ y_{2m-1}\alpha_3 = 0.$
(2) $2x_4\alpha_3 - y_3\alpha_4 = 0,\ mx_{2m}\alpha_3 - y_{2m-1}\alpha_4 = 0$
(3) $y_3J_{2m} - m\lambda x_{2m}\alpha_3 = 0,\ y_{2m-1}J_{2m} - 2m^2\mu x_4^{m-1}\alpha_3 = 0$
(4) $y_3J_{2m+1} + m\lambda x_{2m}\alpha_4 - 2x_4J_{2m} = 0,\ y_{2m-1}J_{2m+1} + 2m^2\mu x_4^{m-1}\alpha_4 - mx_{2m}J_{2m} = 0$

where the parameters λ and μ are as follows:

(1) *If $p \equiv 1\,(m)$, then $\lambda = \mu = 1$.*
(2) *If $p \equiv -1\,(m)$, then $\lambda = ab(ab-4)$ and $\mu = ab(ab-4)$ if $m \neq 2$ is even and $\mu = -1$ if m is odd.*

The algebra structure of $H^*(BG; \mathbb{F}_p)$ is determined by the above relations and the fact that all products between the generators $\alpha_3, \alpha_4, J_{2m}, J_{2m+1}$ vanish.

Proof. As discussed above, we have a Mayer-Vietoris exact sequence (coefficients in $\mathbb{F}_p$ are assumed)

$$\cdots \longrightarrow H^{*-1}(BT) \xrightarrow{\Delta} H^*(BG) \xrightarrow{\psi} H^*(BH_1) \oplus H^*(BH_2) \xrightarrow{\phi} H^*(BT) \longrightarrow \cdots$$

By 6.1 we know that $H^*(BH_i) \cong S^{\omega_i}$, $i = 1, 2$, where ω_1, ω_2 are the generators of the Weyl group W. Then, the kernel of ϕ is S^W and if we denote $\mathcal{S} = \mathrm{Coker}(\phi)$ as in section 7, we have a short exact sequence

$$0 \longrightarrow \mathcal{S} \xrightarrow{\overline{\Delta}} H^*(BG) \xrightarrow{\overline{\psi}} S^W \longrightarrow 0\ .$$

Notice that $\overline{\psi}$ is a ring homomorphism and S^W is a free graded algebra. Hence, we can choose a section σ of ϕ which is a ring homomorphism. This section turns $H^*(BG)$ into an S^W-module. Then, lemma 8.1 implies that Δ is S^W-linear. We have that the

above short exact sequence is an exact sequence of P^W-modules. Both $\mathcal{S}$ and S^W are free P^W-modules (Proposition 7.2) and so $H^*(BG)$ is a free P^W-module with basis

$$1, \sigma(y_3), \sigma(y_{2m-1}), \sigma(y_3)\sigma(y_{2m-1}),$$
$$\alpha_3 = \overline{\Delta}[\alpha], \alpha_4 = \overline{\Delta}[\alpha'], J_{2m+1} = \overline{\Delta}[J], J_{2m} = \overline{\Delta}[J'].$$

To complete the proof we should compute all products between these generators. First of all, notice that Δ is a connecting homomorphism which arises from a map $BG \to \Sigma BT$ and so all products in the image of Δ vanish. In S we have $y_3\alpha = y_{2m-1}\alpha = 0$ and this implies the relations (1) in $H^*(BG(k))$. Applying δ we get the relations (2). Relations (3) and (4) follow from the identities in S proven in Theorem 7.3 and the fact that $\gamma^2 - 4 = ab(ab-4)$ in this case. □

9. Comparison between BG and the fixed points by an Adams map

There is a well know result by Friedlander ([Fri76]) that relates the classifying space of a Chevalley group over a finite field k to the homotopy fixed points of and unstable Adams map from the classifying space of the corresponding compact connected Lie group, after completion at a prime different to the characteristic of k. More precisely, let p, ℓ be different primes, k a finite field of order q and characteristic ℓ, K a compact connected Lie group and $K(q)$ the Chevalley group over k of type K. Then, there is a homotopy pullback diagram

$$\begin{array}{ccc} BK(q)^\wedge_p & \longrightarrow & BK^\wedge_p \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ BK^\wedge_p & \xrightarrow{(1,\psi^q)} & BK^\wedge_p \times BK^\wedge_p \end{array}$$

where ψ^q is an unstable Adams map of exponent q. This result was generalized in [BM07] to the case in which K is a p-compact group. Then, $K(q)$ turns out to be a p-local finite group. In this final section we want to investigate if such a pullback diagram could also exist in the context of Kac-Moody groups.

Let K denote the unitary form of a connected, simply connected, not compact Kac-Moody group as in [Kac85], [Kit98], [BK02] and assume that K is of rank two, uniquely determined by two positive integers a, b with $ab > 4$. In the case of rank two, there is a satisfactory theory of Adams maps $\psi^q \colon BK \to BK$ developed in [AR03] and so, for a prime p and a prime power q with $p \nmid q$, it makes sense to consider the space X defined as the homotopy pullback

$$\begin{array}{ccc} X & \longrightarrow & BK^\wedge_p \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ BK^\wedge_p & \xrightarrow{(1,\psi^q)} & BK^\wedge_p \times BK^\wedge_p \end{array}$$

If Friedlander theorem were true for Kac-Moody groups, then this space X should have the same mod p cohomology as the classifying space of the discrete Kac-Moody group $G_{\mathcal{D}}(\mathbb{F}_q)$ associated to the data $\mathcal{D}$ consisting of the generalized Cartan matrix

$A = \left(\begin{smallmatrix} 2 & -a \\ -b & 2 \end{smallmatrix}\right)$ and the matrix $\left(\begin{smallmatrix} n_1 & n_2 \\ m_1 & m_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (see section 2). Interestingly enough, this is *not* true.

Let us choose the data a, b, p, q such that the following holds. p and q are odd and relatively prime. p does not divide $ab(ab-4)$, $q \not\equiv \pm 1\,(p)$ and $q^m \equiv 1\,(p)$ where m is the multiplicative order in $\mathbb{F}_{p^2}$ of the roots of $X^2 + (2-ab)X + 1$. It is easy to see that these choices are possible. Under these hypothesis, the mod p cohomology of the classifying space of the (discrete) Kac-Moody group G is trivial (Proposition 8.2).

The mod p cohomology of the classifying space of the unitary, connected, simply connected, Kac-Moody group $K = K(a, b)$ is computed in [ABKS05]:

$$H^*(BK; \mathbb{F}_p) = \mathbb{F}_p[x_4, y_{2m}] \otimes E(z_{2m+1}).$$

Let $\psi^q \colon BK \to BK$ be an unstable Adams map as defined in [AR03]. When restricted to a maximal torus of K, ψ^q is the q-power map. Let X be defined by the homotopy pullback above. We will show that the mod p cohomology of X does not vanish in positive degrees.

Consider the Serre spectral sequence in mod p cohomology for the fibration sequence

$$K \longrightarrow BK \xrightarrow{\Delta} BK \times BK.$$

The cohomology of the topological group K is as follows ([Kit98]):

$$H^*(K; \mathbb{F}_p) = E(\tilde{x}_3, \tilde{y}_{2m-1}) \otimes \Gamma[\tilde{z}_{2m}].$$

(Γ denotes a divided power algebra.) Then, it is obvious that the elements $\tilde{x}_3$ and $\tilde{y}_{2m-1}$ in the spectral sequence have to be transgressive to $x_4 \otimes 1 - 1 \otimes x_4$ and $y_{2m} \otimes 1 - 1 \otimes y_{2m}$, respectively. Then, pulling back by $(1, \psi^q)^*$ to the Serre spectral sequence for $K \longrightarrow X \longrightarrow BK$ we see that in this spectral sequence the element $\tilde{y}_{2m-1}$ is transgressive to $(1 - q^m)y_{2m} = 0$ and so it survives to $H^{2m-1}(X; \mathbb{F}_p)$.

A similar phenomenon happens also in the case $q \equiv \pm 1\,(p)$.

This discrepancy between classifying spaces of infinite Kac-Moody groups over finite fields and the homotopy fixed points of the corresponding Adams maps deserves further study.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Cerdanyola del Vallès, Spain.

E-mail address: aguade@mat.uab.cat

E-mail address: Albert.Ruiz@uab.cat